Generalized LMRD code bounds for constant dimension codes

Sascha Kurz, University of Bayreuth

***Abstract*—In random network coding so-called constant dimension codes (CDCs) are used for error correction and detection. Most of the largest known codes contain a lifted maximum rank distance (LMRD) code as a subset. For some special cases, Etzion and Silberstein have demonstrated that one can obtain tighter upper bounds on the maximum possible cardinality of CDCs if we assume that an LMRD code is contained [2]. The range of applicable parameters was partially extended by Heinlein in [4]. Here we fully generalize those bounds, which also sheds some light on recent constructions.**

***Index Terms*—Constant dimension codes, lifted maximum rank distance codes, network coding.**

I. Introduction

LET $V \cong \mathbb{F}_q^v$ be a v-dimensional vector space over the finite field $\mathbb{F}_q$ with q elements. By $\begin{bmatrix} V \\ k \end{bmatrix}$ we denote the set of all k-dimensional subspaces in V, where $0 \le k \le v$. The size of the so-called *Grassmannian* $\begin{bmatrix} V \\ k \end{bmatrix}$ is given by the q-binomial coefficient $\begin{bmatrix} v \\ k \end{bmatrix}_q := \prod_{i=1}^{k} \frac{q^{v-k+i}-1}{q^i-1}$. More generally, the set $P(V)$ of all subspaces of V forms a metric space with respect to the subspace distance defined by

$$\mathrm{d_s}(U,W) = \dim(U) + \dim(W) - 2\dim(U \cap W).$$

Coding theory on $P(V)$ is motivated by Kötter and Kschischang [8] via random network coding. For $\mathcal{C} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ we speak of a *constant dimension code* (CDC). By a $(v,N,d;k)_q$ code we denote a CDC in V with minimum (subspace) distance d and cardinality N. The corresponding maximum size is denoted by $A_q(v,d;k)$. In geometrical terms, a $(v,N,d;k)_q$ code $\mathcal{C}$ is a set of N k-dimensional subspaces of V, k-spaces for short, such that any $(k-d/2+1)$-space is contained in at most one element of $\mathcal{C}$. In other words, each two different codewords intersect in a subspace of dimension at most $k-d/2$. For two k-spaces U and W that have an intersection of dimension zero, we will say that they intersect trivially or are disjoint (since they do not share a common point, i.e., a 1-space). For the known lower and upper bounds on $A_q(v,d;k)$ we refer to the online tables http://subspacecodes.uni-bayreuth.de associated with the survey [5].

S. Kurz is with the Department of Mathematics, Physics, and Computer Science, University of Bayreuth, Bayreuth, GERMANY. email: sascha.kurz@uni-bayreuth.de

If a CDC contains an LMRD, see Section II for the definition, then the best known upper bound on the cardinality for the general case can be improved. Corresponding results have been obtained in [2], [4] for a restricted range of parameters. Here we remove the restriction and generalize those bounds to all parameters. To this end, we consider the so-called Anticode bound, which counts t-spaces that are contained in at most one codeword. We refine the approach by splitting the counts according by the dimension of the intersection with the special subspace that is disjoint to all codewords of the LMRD. This gives an integer linear programming problem, see Lemma 6, from which we conclude an explicit upper bound, see Corollary 7. Technically, we prove those results for the maximum number $B_q(v_1,v_2,d;k)$ of k-spaces in $\mathbb{F}_q^{v_1}$ with minimum subspace distance d such that there exists a v_2-space W which intersects every chosen k-space in dimension at least $d/2$, which is more general.

II. Preliminaries

In the following we will mainly consider the case $V = \mathbb{F}_q^v$ in order to simplify notation. We associate with a subspace $U \in \begin{bmatrix} V \\ k \end{bmatrix}$ a unique $k \times v$ matrix X_U in row reduced echelon form ($\texttt{rref}$) having the property that $\langle X_U \rangle = U$ and denote the corresponding bijection $\begin{bmatrix} \mathbb{F}_q^v \\ k \end{bmatrix} \to \{X_U \in \mathbb{F}_q^{k\times v} \mid \operatorname{rk}(X_U) = k, X_U \text{ is in } \texttt{rref}\}$ by τ.

For two matrices $A, B \in \mathbb{F}_q^{m\times n}$ we define the rank distance $\mathrm{d_r}(A,B) := \operatorname{rk}(A-B)$. A subset $\mathcal{M} \subseteq \mathbb{F}_q^{m\times n}$ is called a rank metric code.

Theorem 1: (see [3]) Let $m, n \ge d'$ be positive integers, q a prime power, and $\mathcal{M} \subseteq \mathbb{F}_q^{m\times n}$ be a rank metric code with minimum rank distance d'. Then, $\#\mathcal{M} \le q^{\max\{n,m\}\cdot(\min\{n,m\}-d'+1)}$.
Codes attaining this upper bound are called maximum rank distance (MRD) codes. They exist for all choices of parameters. Using an $m \times m$ identity matrix $I_{m\times m}$ as a prefix one obtains the so-called lifted MRD (LMRD) codes, i.e., the CDC

$$\left\{\tau^{-1}(I_{m\times m}|A) \mid A \in \mathcal{M}\right\} \subseteq \begin{bmatrix} \mathbb{F}_q^{m+n} \\ m \end{bmatrix},$$

where $(B|A)$ denotes the concatenation of the matrices B and A.

Theorem 2: [10, Proposition 4] For positive integers k, d, v with $k \le v$, $d \le 2\min\{k, v-k\}$, and d even, the size of a lifted MRD code $\mathcal{C} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ with minimum subspace distance d is given by

$$\#\mathcal{C} = q^{\max\{k,v-k\}\cdot(\min\{k,v-k\}-d/2+1)}.$$

So, for positive integers v, k, and d with $d \le 2k \le v$ and $d \equiv 0 \pmod 2$ we have

$$A_q(v,d;k) \ge q^{(v-k)\cdot(k-d/2+1)}.$$

For a $(v,\star,d;k)_q$ code $\mathcal{C}$ each $(k-d/2+1)$-space is contained in at most one element of $\mathcal{C}$, so that

$$A_q(v,d;k) \le \begin{bmatrix} v \\ k-d/2+1 \end{bmatrix}_q \Big/ \begin{bmatrix} k \\ k-d/2+1 \end{bmatrix}_q,$$

which is known as the *Anticode bound*. Analyzing the right hand side we obtain

$$1 \le \frac{A_q(v,d;k)}{q^{(v-k)\cdot(k-d/2+1)}} < 2, \qquad (1)$$

see e.g. [6, Proposition 8], noting that the upper bound is also valid if $2k > v$, i.e., $k > v-k$. We remark that the tightest known upper bounds for $A_q(v,d;k)$, where $d < 2k$, are obtained by a combination of the Johnson bound with divisible codes, see [7, Theorem 12] for the details.

We will also need to count the number of subspaces with certain intersection properties, see e.g. [6, Lemma 2]:

Lemma 3: Let $\overline{W}$ be a $\overline{w}$-space in $\mathbb{F}_q^{\overline{v}}$. The number of $\overline{u}$-spaces $\overline{U}$ in $\mathbb{F}_q^{\overline{v}}$ with $\dim(\overline{U} \cap \overline{W}) = \overline{s}$ is given by

$$q^{(\overline{w}-\overline{s})(\overline{u}-\overline{s})} \cdot \begin{bmatrix} \overline{w} \\ \overline{s} \end{bmatrix}_q \cdot \begin{bmatrix} \overline{v}-\overline{w} \\ \overline{u}-\overline{s} \end{bmatrix}_q$$

for all $0 \le \overline{s} \le \min\{\overline{u}, \overline{w}\}$.

Directly from the definition of the q-binomial coefficients we conclude $\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{bmatrix} a \\ a-b \end{bmatrix}_q$,

$$\begin{bmatrix} a+1 \\ b \end{bmatrix}_q \Big/ \begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{q^{a+1}-1}{q^{a-b+1}-1} \qquad (2)$$

and

$$\begin{bmatrix} a-1 \\ b \end{bmatrix}_q \Big/ \begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{q^{a-b}-1}{q^a-1}. \qquad (3)$$

As shown in e.g. [8, Lemma 4] we have

$$q^{(a-b)b} \le \begin{bmatrix} a \\ b \end{bmatrix}_q \le 4q^{(a-b)b} \le q^2 \cdot q^{(a-b)b}. \qquad (4)$$

III. Bounds for CDCs containing an LMRD subcode

Before we consider upper bounds we start with the constructive point of view.

Definition 4: Let $B_q(v_1, v_2, d; k)$ denote the maximum number of k-spaces in $\mathbb{F}_q^{v_1}$ with minimum subspace distance d such that there exists a v_2-space W which intersects every chosen k-space in dimension at least $d/2$, where $0 \le v_2 \le v_1$.

Theorem 5: If $k \le m \le v-k$, then we have

$$\begin{aligned} A_q(v,d;k) \quad &\ge \quad A_q(m,d;k) \cdot \left\lceil q^{(v-m)(k-d/2+1)} \right\rceil \\ &\quad +B_q(v, v-m, d; k). \end{aligned}$$

Proof. Let $k \le m \le v-k$ be an arbitrary integer, $\mathcal{C}$ be an $(m, N, d; k)_q$ code, where $N = A_q(m,d;k)$, and $\mathcal{M}$ an MRD of $k \times (v-m)$-matrices over $\mathbb{F}_q$ with rank distance $d/2$. With this, we set

$$\begin{aligned} \mathcal{C}' \quad &:= \quad \left\{\tau^{-1}(\tau(U)|A) \mid U \in \mathcal{C}, A \in \mathcal{M}\right\} \\ &\subseteq \quad \begin{bmatrix} \mathbb{F}_q^v \\ k \end{bmatrix}. \end{aligned}$$

The $(v-m)$-space W whose pivots in $\tau(W)$ are in the last $v-m$ coordinates is disjoint from all elements from $\mathcal{C}'$. Now let $\mathcal{C}'' \subseteq \begin{bmatrix} \mathbb{F}_q^v \\ k \end{bmatrix}$ be a CDC with minimum subspace distance d such that every codeword intersects W in dimension at least $d/2$, which has the maximum possible cardinality.

For each $U' \in \mathcal{C}'$ and each $U'' \in \mathcal{C}''$ we have $\dim(U' \cap U'') \le k - d/2$ since $\dim(U') = \dim(U'') = k$, $\dim(U' \cap W) = 0$, and $\dim(U'' \cap W) \ge d/2$. Thus, $\mathrm{d_s}(U', U'') \ge d$ and

$$\begin{aligned} A_q(v,d;k) &\ge \#\mathcal{C}' + \#\mathcal{C}'' \\ &= A_q(m,d;k) \cdot \left\lceil q^{(v-m)(k-d/2+1)} \right\rceil \\ &\quad + B_q(v, v-m, d; k). \end{aligned}$$

□

We remark that the construction of CDC $\mathcal{C}'$ is called *Construction D* in [9, Theorem 37]. If either $v_1 < k$ or $v_2 < \frac{d}{2}$, then we trivially have $B_q(v_1, v_2, d; k) = 0$. Similarly, if $v_1 \ge k$, $v_2 \ge \frac{d}{2}$, and $d > 2k$, then we also have $B_q(v_1, v_2, d; k) = 0$. We will call those parameters *trivial*. For (implicit) lower bounds for $B_q(v_1, v_2, d; k)$ we refer to [1], [11] and the references cited therein.

By refining the counting of $(k - d/2 + 1)$-spaces contained in codewords, underlying the presented argument for the Anticode bound, we obtain:

Lemma 6: As an abbreviation we set

$$\begin{aligned} t &:= k - \tfrac{d}{2} + 1, \\ \alpha(j) &: = (v_2 - j)(t - j), \text{ and} \\ b(i,j) &:= q^{(i-j)(t-j)} \cdot \begin{bmatrix} i \\ j \end{bmatrix}_q \cdot \begin{bmatrix} k-i \\ t-j \end{bmatrix}_q \end{aligned}$$

for all $1 \le j \le \min\{t, v_2\}$ and all

$$\max\{d/2, j\} \le i \le \min\{k, d/2 - 1 + j\}.$$

For non-trivial parameters we have

$$B_q(v_1, v_2, d; k) \le \sum_{i=\frac{d}{2}}^{\min\{k, v_2\}} a_i,$$

where the a_i are non-negative integers satisfying the constraints

$$\sum_{i=\max\{d/2,j\}}^{\min\{k,d/2-1+j\}} b(i,j) \cdot a_i \le q^{\alpha(j)} \begin{bmatrix} v_2 \\ j \end{bmatrix}_q \cdot \begin{bmatrix} v_1 - v_2 \\ t-j \end{bmatrix}_q \quad (5)$$

for all $1 \le j \le \min\{t, v_2\}$ and

$$\sum_{i=h}^{\min\{k,v_2\}} a_i \le A_q(v_2, 2(h-t+1); h) \qquad (6)$$

for all $\max\{t, d/2\} \le h \le \min\{k, v_2\}$.

PROOF. Let $V := \mathbb{F}_q^{v_1}$, W be a v_2-space in V, and $\mathcal{C}$ be a set of k-spaces in V that intersect W in dimension at least $\frac{d}{2}$ and has minimum subspace distance d. By a_i we denote the number of elements in $\mathcal{C}$ that have an intersection of dimension exactly i with W, so that

$$\#\mathcal{C} = \sum_{i=d/2}^{\min\{k,v_2\}} a_i.$$

We note that every t-space is contained in at most one element from $\mathcal{C}$.

Let $1 \le j \le \min\{t, v_2\}$ be arbitrary. First we count the number of t-spaces T in V such that $\dim(T \cap W) = j$. Applying Lemma 3 with $\overline{U} = T$, $\overline{u} = t$, $\overline{v} = v_1$, $\overline{W} = W$, $\overline{w} = v_2$, and $\overline{s} = j$ gives $q^{(v_2-j)(t-j)} \cdot \begin{bmatrix} v_2 \\ j \end{bmatrix}_q \cdot \begin{bmatrix} v_1 - v_2 \\ t-j \end{bmatrix}_q$ possibilities, which is the right hand side of Inequality (5). Now, consider a codeword $U \in \mathcal{C}$ with intersection dimension $i = \dim(U \cap W)$. Next we want to count those t-spaces T contained in U with $\dim(T \cap W) = j$. Applying Lemma 3 with $\overline{U} = T$, $\overline{u} = t$, $\overline{v} = k$, $\overline{W} = W \cap U$, $\overline{w} = i$, and $\overline{s} = j$ gives $q^{(i-j)(t-j)} \cdot \begin{bmatrix} i \\ j \end{bmatrix}_q \cdot \begin{bmatrix} k-i \\ t-j \end{bmatrix}_q = b(i,j)$ possibilities (if $\max\{1, t-k+i\} \le j \le \min\{i, t\}$). Since each such t-space T is contained in at most one codeword $U \in \mathcal{C}$, we obtain Inequality (5).

Given an integer $\max\{t, d/2\} \le h \le \min\{k, v_2\}$ we construct a CDC consisting of h-spaces from $\mathcal{C}$. To this end, we set $\mathcal{C}' = \{U \cap W : U \in \mathcal{C}, \dim(U \cap W) \ge h\}$, so that $\#\mathcal{C}' = \sum_{i=h}^{\min\{k,v_2\}} a_i$. Now let $\mathcal{C}''$ arise from $\mathcal{C}'$ by choosing an arbitrary h-subspace from each $U' \in \mathcal{C}'$ as codeword $U'' \in \mathcal{C}''$. By construction we have $\dim(A'' \cap B'') \le t - 1 = k - d/2$ for each pair of different codewords $A'', B'' \in \mathcal{C}''$, so that $d(A'', B'') \ge 2(h-t+1) \ge 2$ and $\#\mathcal{C}' = \#\mathcal{C}''$. Thus $\mathcal{C}''$ is a $(v_2, \#\mathcal{C}, 2(h-t+1); h)_q$ code and we obtain (6). □

For given parameters v_1, v_2, d, and k we can easily turn Lemma 6 into an integer linear programming formulation and solve it numerically. We can also conclude an explicit parametric upper bound:

Corollary 7: For non-trivial parameters we have

$$\begin{aligned} & B_q(v_1, v_2, d; k) \\ \le \; & A_q(v_2, (\Lambda+1)d - 2k; \Lambda d/2) \\ & + \sum_{l=1}^{\Lambda-1} q^{(v_2 - ld/2)(k-(l+1)d/2+1)} \cdot \begin{bmatrix} v_2 \\ ld/2 \end{bmatrix}_q \\ & \cdot \begin{bmatrix} v_1 - v_2 \\ k-(l+1)d/2+1 \end{bmatrix}_q \Big/ \begin{bmatrix} k - ld/2 \\ d/2 - 1 \end{bmatrix}_q , \end{aligned}$$

where $\Lambda := \lfloor 2k/d \rfloor$.

PROOF. We apply Lemma 6 and use the corresponding notation, i.e., we will use

$$B_q(v_1, v_2, d; k) \le \sum_{i=\frac{d}{2}}^{\min\{k, v_2\}} a_i$$

and upper bound the right hand side.

For $k < d$ we have $d/2 \ge k - d/2 + 1 = t$ so that we can apply Inequality (6) with $h = d/2$ to conclude the proposed upper bound for $\Lambda = 1$.

In the following we assume $k \ge d$, i.e., $\Lambda \ge 2$. From Equation (2) and Equation (3) we conclude

$$\frac{b(i+1, j)}{b(i, j)} = \frac{q^{k+1} - q^{t+i-j+1} - q^{k-i} + q^{t-j}}{q^{k+1-j} - q^{i-j+1} - q^{k-i} + 1}.$$

Using $i \le (d/2 - 1 + j) - 1 = k - t + j - 1$ and $i \le k-1$ we obtain

$$\begin{aligned} \frac{b(i+1, j)}{b(i, j)} & \ge \frac{q^{k+1} - q^k - q^{k-i} + q^{t-j}}{q^{k+1-j} - q^{k-j} - q^{k-i} + 1} \\ & \underset{t-j \ge 0}{\overset{j\ge 1}{\ge}} \frac{q^{k+1} - q^k - q^{k-i} + 1}{q^k - q^{k-1} - q^{k-i} + 1} \ge 1, \end{aligned}$$

i.e., the sequence $\left(b_{i,j}\right)_i$ is weakly monotonic increasing.

Next we want to apply Inequality (5) for special values of j. To this end, we use the parameterization $j = ld/2$ for $1 \le l < \Lambda$. Here we note that $\max\{d/2, j\} = ld/2$, due to $l \ge 1$, and $\min\{k, d/2 - 1 + j\} = (l+1)d/2 - 1$, due to $l \le \lfloor 2k/d \rfloor - 1$. With this, we have

$$\begin{aligned} \sum_{i=\max\{d/2, j\}}^{\min\{k, d/2-1+j\}} b(i,j) a_i & = \sum_{i=ld/2}^{(l+1)d/2-1} b(i, ld/2) a_i \\ & \ge \sum_{i=ld/2}^{(l+1)d/2-1} b(ld/2, ld/2) a_i \end{aligned}$$

for $j = ld/2$, where the latter inequality follows from $a_i \ge 0$ and the monotonicity of $\left(b_{i,j}\right)_i$. Thus, we conclude

$$\begin{aligned} & \sum_{i=ld/2}^{(l+1)d/2-1} b(ld/2, ld/2) \cdot a_i \\ \le \; & q^{(v_2 - ld/2)(t-ld/2)} \begin{bmatrix} v_2 \\ ld/2 \end{bmatrix}_q \begin{bmatrix} v_1 - v_2 \\ t - ld/2 \end{bmatrix}_q \end{aligned} \tag{7}$$

from Inequality (5) for $1 \le l < \Lambda$ and $j = ld/2$. Dividing Inequality (7) by $b(ld/2, ld/2)$ gives

$$\begin{aligned} & \sum_{i=ld/2}^{(l+1)d/2-1} a_i \le \\ & q^{(v_2 - ld/2)(t-ld/2)} \begin{bmatrix} v_2 \\ ld/2 \end{bmatrix}_q \begin{bmatrix} v_1 - v_2 \\ t - ld/2 \end{bmatrix}_q \Big/ \begin{bmatrix} k - ld/2 \\ d/2 - 1 \end{bmatrix}_q \end{aligned} \tag{8}$$

using $\begin{bmatrix} k - ld/2 \\ t - ld/2 \end{bmatrix}_q = \begin{bmatrix} k - ld/2 \\ d/2 - 1 \end{bmatrix}_q$. Since Inequality (6) with $h = \Lambda d/2$ gives

$$\sum_{i=\Lambda d/2}^{\min\{k, v_2\}} a_i \le A_q(v_2, (\Lambda+1)d - 2k; \Lambda d/2) \tag{9}$$

we can add the right hand side of Inequality (9) to the sum over the right hand side of Inequality (8) for $1 \le l < \Lambda$ to conclude the proposed upper bound. Note that the sum of the corresponding left hand sides equals

$$\sum_{l=1}^{\Lambda-1} \sum_{i=ld/2}^{(l+1)d/2-1} a_i + \sum_{i=\Lambda d/2}^{\min\{k, v_2\}} a_i = \sum_{i=d/2}^{\min\{k, v_2\}} a_i.$$

□

Applying Theorem 5 with $m = k$ gives a $(v, \star, d; k)_q$ code $\mathcal{C}$ with cardinality

$$q^{(v-k)\cdot(k-d/2+1)} + B_q(v, v-k, d; k).$$

Under the assumption that $\mathcal{C}$ contain a lifted MRD code as a subcode this is indeed the maximum possible cardinality:

Proposition 8: Let v, k, and $d/2$ be positive integers with $d \le 2k \le v$ and $\mathcal{C}$ be a $(v, \star, d; k)_q$ code that contains a lifted MRD code $\mathcal{C}'$ of cardinality $q^{(v-k)\cdot(k-d/2+1)}$ as a subcode. Then, we have $\#C \le q^{(v-k)\cdot(k-d/2+1)} + B_q(v, v-k, d; k)$.

PROOF. Let W be the $(v-k)$-space that is disjoint from all codewords of $\mathcal{C}'$. From e.g. [2, Lemma 4] we know that every $(k - d/2 + 1)$-space that is disjoint to W is contained in a codeword from $\mathcal{C}'$. Thus, the codewords in $\mathcal{C}\backslash\mathcal{C}'$ have to intersect W in dimension at least $d/2$. □

Corollary 9: Let v, k, and $d/2$ be positive integers with $d \le 2k \le v$ and $\mathcal{C}$ be a $(v, \star, d; k)_q$ code that contains a lifted MRD code $\mathcal{C}'$ of cardinality $q^{(v-k)\cdot(k-d/2+1)}$ as a subcode. Then, $\#C \le q^{(v-k)\cdot(k-d/2+1)} + A_q(v-k, (\Lambda+1)d - 2k; \Lambda d/2) + \sum_{l=1}^{\Lambda-1} q^{(v-k-ld/2)(k-(l+1)d/2+1)} \cdot \begin{bmatrix} v-k \\ ld/2 \end{bmatrix}_q \cdot \begin{bmatrix} k \\ (l+1)d/2-1 \end{bmatrix}_q / \begin{bmatrix} k-ld/2 \\ d/2-1 \end{bmatrix}_q$, where $\Lambda := \lfloor 2k/d \rfloor$.

PROOF. Apply Proposition 8 and Corollary 7. □

The cases $\Lambda \le 2$, i.e. $k < 3d/2$, cover [4, Theorem 1] as well as its predecessors [2, Theorem 10] and [2, Theorem 11]. For $\Lambda \ge 3$, i.e. $k \ge 3d/2$, Corollary 9 gives new upper bounds. As an example we consider the binary case $(v, d; k)_q = (12, 4; 6)_2$, where a CDC $\mathcal{C}$ that contains a lifted MRD code has to satisfy $\#\mathcal{C} \le 1\,321\,780\,637$, noting the best known general bounds $1\,212\,491\,081 \le A_2(12, 4; 6) \le 1\,816\,333\,805$.

Next we show that the upper bound of Corollary 7 for $B_q(v_1, v_2, d; k)$ is tight for $k < d$, i.e., those cases where the bound does not depend on v_1, provided that v_1 is sufficiently large.

Proposition 10: For non-trivial parameters we have $B_q(v_1, v_2, d; k) = A_q(v_2, 2d-2k; d/2)$ if $k < d$ and $v_1 \ge v_2 k$.

PROOF. Due to Corollary 7 it remains to construct a code $\mathcal{C}$ with cardinality $A_q(v_2, 2d-2k; d/2)$ that satisfies the conditions of Definition 4. To this end, let $W := \mathbb{F}_q^{v_2} \le \mathbb{F}_q^{v_2} \times \mathbb{F}_q^{v_1-v_2} =: V$ and let $\mathcal{F}$ be a $(v_2, N, 2d-2k; d/2)_q$ code of maximal size in W, i.e., $N = A_q(v_2, 2d-2k; d/2)$. If $d = 2$, then $k = 1 = d/2$, so that we can set $\mathcal{C} = \mathcal{F}$. Next, we assume $d \ge 4$ and set $t = k - d/2$. Let $\mathcal{P}$ be a partial t-spread in $\mathbb{F}_q^{v_1-v_2}$ of cardinality $A_q(v_1-v_2, 2t; t)$, so that (1) gives $\#\mathcal{P} \ge q^{v_1-v_2-k+d/2}$. Since

$$\#\mathcal{F} = A_q(v_2, 2d-2k; d/2) \overset{(1)}{\le} q \cdot q^{(v_2-d/2)(k-d/2+1)},$$

again using (1), we have $\#\mathcal{P} \ge \#\mathcal{F}$ if

$$v_1 - v_2 - k + d/2 \;\ge\; 1 + (v_2 - d/2)(k - d/2 + 1),$$

which is equivalent to

$$v_1 \;\ge\; v_2 + (v_2 - d/2 + 1)(k - d/2 + 1). \quad (10)$$

Since $d \ge 4$ and $k \ge 1$ the right hand side of (10) is at most $v_2 k$, so that $\#\mathcal{P} \ge \#\mathcal{F}$. For each $U \in \mathcal{F}$ we can choose a different element $f(U) \in \mathcal{P}$ and set $\mathcal{C} = \{U \times f(U) \mid U \in \mathcal{F}\}$, which has the desired properties of Definition 4 by construction. □

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