Improving Heterogeneous Agent Models by Avoiding Explicit Discretizations of Stiff Equations

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Abstract We consider a standard heterogeneous agent model that is widely used to analyze price developments in financial markets. The model is linear in log-prices and, in its basic setting, populated by fundamentalists and chartists. These fundamentalists are typically believed to stabilize markets by bringing asset prices back to their fundamental values. However, we illustrate that in this type of model, this does not necessarily hold as—unintended and so far over-looked—instabilities might occur. As the number of fundamentalists increases and exceeds a specific threshold, oscillations occur whose amplitude might even grow exponentially over time.

We show that this instability phenomenon is due to a “hidden” explicit discretization of a stiff ordinary differential equation contained in the model. Replacing this explicit discretization by an implicit one removes this artifact, bringing the model’s prediction in line with standard theory. We extend our analysis and simulate markets with evolutionary rules, i.e., replicator dynamics, for the explicit as well as the implicit model. Overall, we find that our analytical results carry over to the extended model. Models based on explicit discretization are likely to overrate price instabilities and, in particular, bubbles and crashes and imply biased results in the empirical application of heterogeneous agent models.

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1 Introduction

In the last three decades, heterogeneous agent models (HAMs) have proven to be a very productive approach to analyze financial markets. Models of heterogeneous agents augmented by simple heuristic trading strategies are particularly suited to capture important financial market features such as technical trading, herding, overshooting prices, and bubbles. They are also well apt to replicate important stylized facts such as fat tales in return distributions, volatility clustering, and long-term memory. LeBaron (2006), Lux (2008), Chiarella et al. (2009), Dieci and He (2018), and Iori and Porter (2018), among others, survey this burgeoning field of research.

Despite their considerable heterogeneity, HAMs typically share several basic features that have already been present in the seminal papers by Day and Huang (1990) and Huang and Day (1993). Chief among them are (i) markets are cleared by a market maker who sets the next period's asset price with respect to the current excess demand of the different traders. (ii) Fundamentalists bet on a reduction in the current mispricing of assets, i.e., they buy when assets are undervalued and sell when they are overvalued relative to their fundamental value. (iii) There may be other types of (heterogeneous) agents, for example, those who trade on simple heuristic strategies that attempt to extract buying and selling signals from past price movements, e.g., momentum trading by chartists who buy when prices rise and sell when they fall. More generally, it is also characteristic for this approach to focus on the aggregate behavior of markets rather than to analyze specific transmission mechanisms.

Applying these three features to a market with fundamentalists and chartists, an asset's price movement can in general be described by an explicit difference equation of the type

$$p(t+1) = p(t) + w_F(f - p(t)) + w_C(p(t) - p(t-1)) \quad (1)$$

with the asset's log-price p, fundamentalists' weight in the market w_F, chartists' weight w_C, and log-fundamental-value f.

A closer inspection of this pricing equation reveals two important interrelated implications. Firstly, quite counterintuitive price developments might occur in such markets. For example, even in a market in which only fundamentalists trade, prices might oscillate with an exploding amplitude (see Figure 1). Technically, such an instability can occur if fundamentalists react very strongly to mispricing while at the same time, the market maker adjusts prices aggressively to the resulting excess demand. Please note that fundamentalists in this situation still buy at prices below and sell at prices above the fundamental value, as prices overshoot only in the subsequent period. Anyhow, such a pricing behavior is at odds with today's financial markets in which limit orders are an omnipresent feature. In a market in which only fundamentalists trade and limit orders are available, all the buy orders' limits are equal or below the fundamental value. In contrast, all sell orders' limits are at or above the fundamental value. As the orders are executed successively, we would expect prices to converge to their fundamental value, i.e., fundamentalists to stabilize the market.

Secondly, this behavior is akin to the effects that occur when a stiff differential equation is discretized using an explicit numerical scheme. Such effects appear, for instance, in reaction kinetics, see the work of Shampine and Gear (1979) and the references therein. Just as for HAMs, the models used in reaction kinetics describe the aggregate behavior of species—in this case in a chemical reaction—rather than the underlying molecular mechanisms. As in our simulations, spurious oscillations and instability artifacts can occur when specific parameters become large, and the underlying differential equation becomes stiff. This similarity is not a coincidence: as we will explain subsequently in greater detail, the term describing the price influence of the fundamentalists in the HAM is precisely in the form of an explicit Euler discretization. Since in chemical reaction kinetics (as well as in many other branches of science and engineering) using implicit discretizations is a well-known remedy to avoid these artifacts in case the underlying

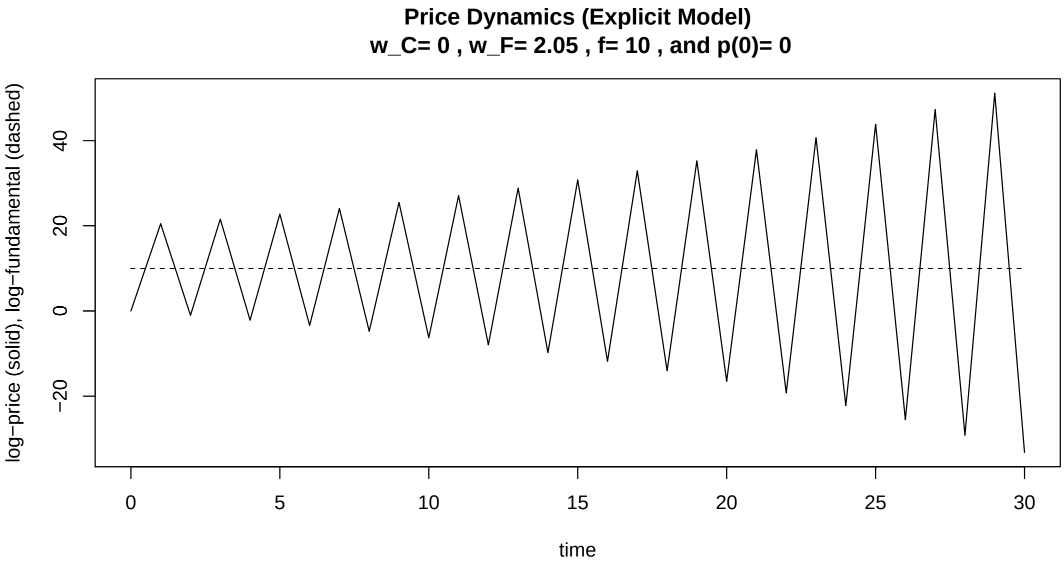

Figure 1: Oscillating and exploding prices in the explicit model with fundamentalists only ($w_F > 2$, $w_C = 0$).

equation is stiff, it seems straightforward to explore the same technique for HAMs.

When observing and interpreting overshooting prices and market instabilities in conventional HAMs, it is therefore not clear whether they are due to dynamic interactions of traders, which give rise to nonlinearities, or whether they are mere artifacts due to the specific form of modeling. The issue of unintended instability features due to the explicit discretization is omnipresent in the HAM literature and has so far not been addressed adequately.

In the following we analyze a class of HAMs that has been used as a very productive tool in the study of financial markets. Ultimately, this research builds on the basic model of Day and Huang (1990, in particular Equations (2), (5), (6), and (7)), which can in principle be subsumed under Equation (1) (see also Huang and Day, 1993; Tramontana et al., 2013). Three main fields of research have emerged over time, namely the analysis of the (in)stability of market equilibria, the interactions between different heterogeneous traders, and the calibration of market models using real-world data to replicate stylized facts.

Firstly, Day and Huang (1990) as well as Huang and Day (1993) use their market model of excess demand and price adjustment to study the (in)stability of equilibria. Dieci and Westerhoff

(2010), Tramontana et al. (2009, 2010), and Schmitt and Westerhoff (2014) focus on two stock markets (of two different countries), which are linked through a foreign exchange market. Among others, the authors analyze spill-over effects and their effect on market equilibria. Schmitt and Westerhoff (2017) analyze a similar set-up and additionally allow for the occurrence of sunspots.

Secondly, the interaction of heterogeneous agents is at the center of Brock and Hommes (1998) and Hommes and Wagener (2009) who analyze how different traders, e.g., trend chasers, contrarians, or fundamentalists, mutually influence each other when the model allows for evolutionary dynamics. They study the effects on the asset price dynamics and conclude that particularly when the intensity for switching the strategy is high, chaotic price movements may appear (Brock and Hommes, 1998) and that trend following can have destabilizing effects possibly leading to price bubbles (Hommes and Wagener, 2009). Lux (1995) and Kirman (1993) focus on trader interactions where the dynamics follow a probabilistic approach. Beja and Goldman (1980) analyze how the behavior of brokers and designated specialists affects the speed of price adjustments to changing conditions and find, e.g., that the existence of limit orders gives rise to price discrepancies affecting stock price dynamics.

Thirdly, researchers have increasingly analyzed to what degree HAMs are able to replicate important stylized facts of financial markets. Schmitt and Westerhoff (2019) consider a market with a market maker and several trader types whereby the calibration of the model is conducted via trial-and-error. Franke and Westerhoff (2016) use a herding model related to Lux (1995) to represent the switching between fundamentalist and chartist trading strategies and estimate the model parameters by a method of simulated moments. For evaluation they conduct a bootstrap procedure. Platt (2020) conduct an extensive comparison of different model calibration techniques.

As all of these important contributions to the analysis of (financial) markets make use of an explicit discretization, especially when modeling fundamentalists' strategies, they might contain so-called artifacts. The derived results are therefore not as general as they would be under an implicit discretization. Interestingly, Kukacka and Kristoufek (2020) make an analogous observation by building on a different approach, namely the analysis of the multifractal properties of agent-based models.

We contribute to the literature by showing how the standard approach in HAMs can give rise to counterintuitive asset price behavior. Based on both simulations and calculations, we show that in a standard HAM—even in financial markets that are solely populated by fundamentalists—explosive price developments can occur. We relate some market instabilities to the observation that the price equation in a standard HAM should be interpreted as the explicit discretization of a stiff ordinary differential equation. We propose an implicit discretization as a way to improve the performance of HAMs. In a complementary simulation, we generalize our analysis and compare the effects of the explicit and implicit discretization in a more complex financial market setup.

The remainder of the paper is organized as follows. Section 2 describes a standard HAM, while Section 3 provides evidence for instabilities in a financial market in which only fundamentalists trade. This ostensibly destabilizing role of fundamentalists is related to the explicit discretization of a stiff ordinary differential equation subsequently. Section 4 proposes the implict discretization as a remedy for the oscillatory instabilities, while Section 5 gives an economic interpretation to this seemingly technical procedure. Inspired by the cited literature, the simulation studies in Sections 6 and 7 augment the standard model to include the evolutionary development of the different trader types and compare the consequences of the two ways of discretization in a more general framework. Section 8 concludes.

2 Standard Market Model

A well-established approach to analyze important features of financial markets like bubbles and crashes is to build on market maker models with heterogeneous agents, typically fundamentalists F and chartists C. In such a framework asset prices depend on the aggregated excess demand of fundamentalists and chartists. Fundamentalists buy when assets are undervalued and sell when they are overvalued, while chartists buy when prices rise and sell when they fall. More specifically and following the basic approach going back to Day and Huang (1990) and Huang and Day (1993), the log-price of an asset $p(t)$ is assumed to be linear in the excess demand of the agents, i.e.,

$$p(t+1) = p(t) + (N_F(t)D_F(t) + N_C(t)D_C(t)) \cdot M^{-1} \tag{2}$$

(with $M > 0$) where D_F resp. D_C is the excess demand of a typical fundamentalist resp. chartist and N_F resp. N_C denotes the respective number of traders. The (excess) demand function of the fundamentalists is assumed to be linear in the deviation of the log-price from the log-fundamental $f(t)$, i.e.,

$$D_F(t) = F(f(t) - p(t)) \tag{3}$$

(with $F > 0$), while the (excess) demand function of the chartists is assumed to be linear in the trend of the log-price, i.e.,

$$D_C(t) = C(p(t) - p(t-1)) \tag{4}$$

(with $C > 0$). Since fundamentalists and chartists are the two types of traders who drive most stylized facts in financial markets, they are at the focus of most of the literature. We follow this approach in the analytical part of the paper. Subsequently, we generalize our analysis in Section 6 and introduce additional types of traders in our simulation study.

3 Discretization Artifact

While the basic model of Equations (2), (3), and (4) is able to replicate a number of stylized facts such as excess volatility, mean reversion, and high trading volume, it also exhibits important deficiencies. Notably, it implies a counterintuitive instability behavior. Since fundamentalists buy when assets are undervalued and sell when they are overvalued, these traders are associated with a market stabilizing behavior. Yet, as we show, the standard model becomes unstable when there are too many fundamentalists. One might argue that different fundamentalists do not coordinate their actions and therefore, prices might overshoot. However, in the basic model—even with only fundamentalists present—prices may not only overshoot but oscillate with an exploding amplitude.

Such price dynamics seem to be at odds with today's financial markets. Modern exchanges typically allow for limit orders. In contrast to market orders, which are of the "whatever it takes" style, limit orders allow to set a maximum/minimum price in advance. As fundamentalists base their trading strategy on the (expected) fundamental value $\phi(t) = exp(f(t))$, they have no reason to place a market order. Instead, they set, e.g., a buy limit order with fundamental value $\phi(t)$ as the maximum price if $\phi(t) > \rho(t)$, with asset price $\rho(t) = exp(p(t))$, and a sell limit order with $\phi(t)$ as the minimum price if $\phi(t) < \rho(t)$. It follows that with only fundamentalists trading, asset prices should not overshoot. Such overshootings are only caused by chartists or by the interaction of chartists and fundamentalists. As we show subsequently, models such as (2), (3), (4) do not adequately account for such stabilizing effects by fundamentalists.

To concentrate on the key issues, we consider a simple example of the standard model class in this section, noting that our approach easily carries over to more complex models (as we show subsequently in Section 6). We assume log-fundamentals and the number of traders, namely fundamentalists and chartists, to be constant and introduce weights of the respective types of traders, i.e., $w_F = N_F F/M$ and $w_C = N_C C/M$. This leads to the following recurrence relation or difference equation for log-prices

$$p(t+1) = p(t) + w_F(f - p(t)) + w_C(p(t) - p(t-1)). \tag{5}$$

Obviously, $p^* = f$ is an equilibrium of the model. Figure 2 depicts the vast spectrum of price dynamics that are generated by this so-called explicit model under the parameters $f = 10$, $p(0) = 1$ and varying weights $(w_F, w_C) = (0.2, 0)$, $(1, 0)$, $(1.8, 0)$, $(2, 0)$, $(2.05, 0)$, and $(1.8, 0.8)$.

It is evident from the fifth graph (bottom left) of Figure 2 that even when there are no chartists, i.e., $w_C = 0$, $p^* = f$ is unstable if $w_F > 2$. If there are "too many" fundamentalists, the price crosses the fundamental in each time step with exponentially increasing jump size. Of course, in real markets, it cannot be excluded per se that asset prices jump beyond their fundamental value during the short-term adjustment to a shock. However, if only fundamentalists are present, such a price behavior would seem to be counterintuitive, in the short as well as the long run, not the least due to limit orders. Instead, prices should end up in a neighborhood of their fundamental values. We, therefore, conclude that most likely, the model exhibits instability artifacts.

To complement our simulations with analytical results, it is convenient to rewrite the second-order equation (5) as a first-order equation in two dimensions:

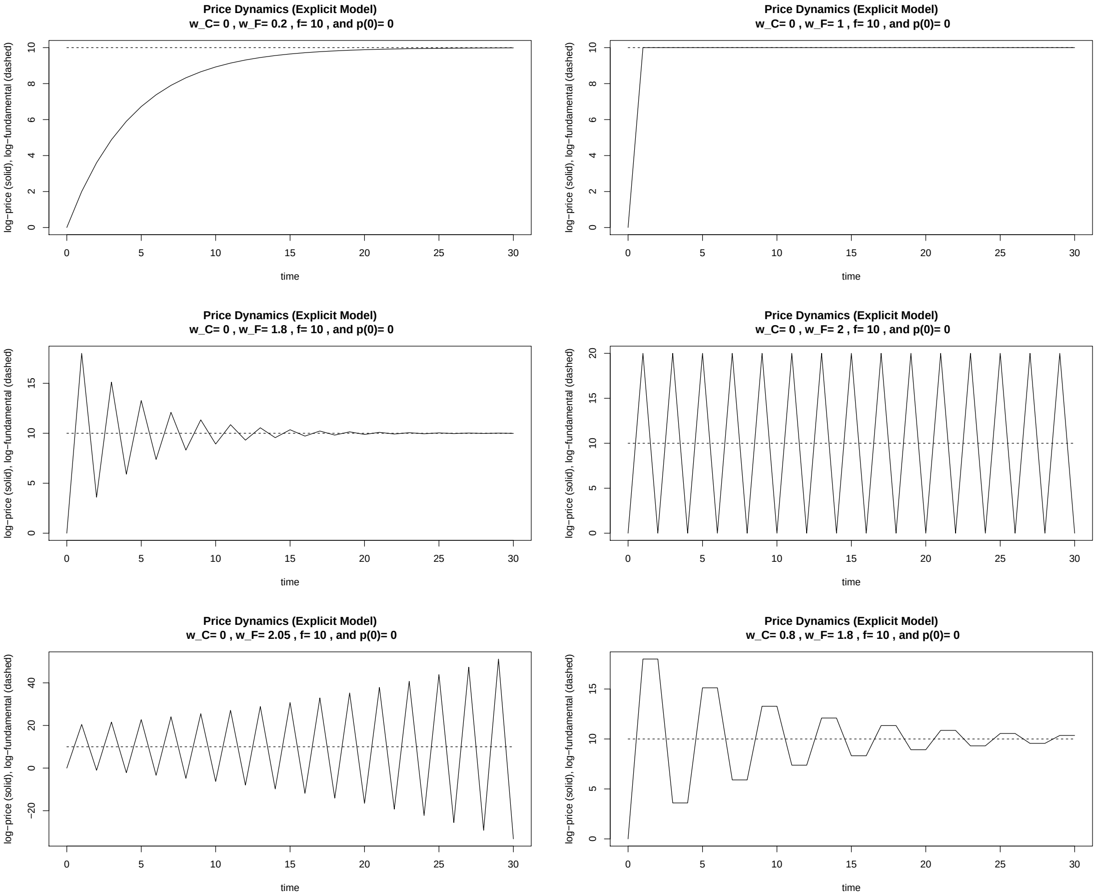

Figure 2: Price dynamics for the explicit model with fundamentalists only (first five graphs, $w_C = 0$ and $w_F \in \{0.2, 1, 1.8, 2, 2.05\}$) and weights $(w_C, w_F) = (0.8, 1.8)$ (sixth graph).

$$\begin{pmatrix} p_1(t+1) \\ p_2(t+1) \end{pmatrix} = \begin{pmatrix} 1 - w_F + w_C & -w_C \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} + \begin{pmatrix} w_F f \\ 0 \end{pmatrix}$$

The eigenvalues of the transition matrix are

$$\lambda_{1,2} = \frac{1}{2}\left(1 - w_F + w_C \pm \sqrt{w_C^2 - 2w_C w_F + w_F^2 - 2w_C - 2w_F + 1}\right).$$

This allows us to calculate the values of $(w_F, w_C) \in \mathbb{R}^+ \times \mathbb{R}$ for which $p^* = f$ is stable. Figure 3 shows the region where the model is stable (actually, the figure shows the maximum of $|\lambda_{1,2}|$ depending on w_C and w_F; the model is stable when this maximum is smaller than one).

With that, it is possible to determine the threshold of the fundamentalists to chartists ratio for which the market model becomes unstable.

However, in a model that contains instability artifacts, when the market becomes unstable, it is not clear whether this is due to a too large number of chartists or whether this is due to the model structure itself. As a consequence, results from stability analyses of such models have to be analyzed more closely and, more generally, the adequacy of such a modeling approach has to be scrutinized. This leads us to the question how to adequately model heterogeneous agents and avoid such structural artifacts.

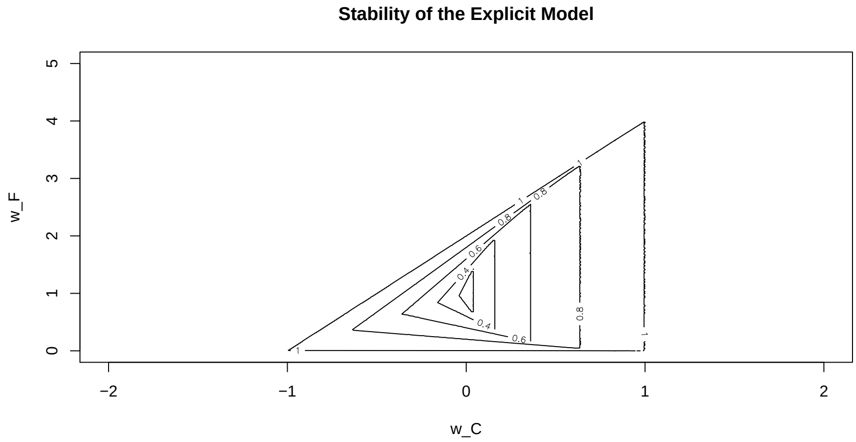

Figure 3: The explicit model (5) is stable for parameter combinations inside the “triangle.”

The key to answer this question is the observation that model (5) contains an explicit Euler discretization

$$p(t + h) = p(t) + hg(t, p(t))$$

of a stiff ordinary differential equation (stiff ODE)

$$\dot{p}(t) = g(t, p(t)),$$

see, e.g., Deuflhard and Bornemann (2012, Chapter 6) or Wanner and Hairer (1996). Such a discretization is known to cause exactly the effects visible in Figure 2 (cf. Deuflhard and Bornemann, 2012, Figure 6.3).

When setting $w_C = 0$ in Equation (5) we obtain

$$p(t+1) = p(t) + 1g(t, p(t))$$

with $g(t, p(t)) = w_F(f - p(t))$. This is exactly the explicit Euler discretization of the ODE

$$\dot{p}(t) = w_F(f - p(t)) \tag{6}$$

with step size $h = 1$.

4 Implicit Discretization

A well-known remedy to account for the oscillatory instability of stiff ODEs is to use an implicit discretization. Such an approach—avoiding oscillations by replacing the explicit discretization by an implicit one—is common in engineering sciences, mathematics, and physics (see, e.g., Deuflhard and Bornemann, 2012; Wanner and Hairer, 1996). However, there is a fundamental difference to the application in HAMs and, more generally, in economics: In physics, for example, the underlying mechanisms are typically modeled via differential equations. These equations have to be simulated and, thus, a discretization is necessary. The task is then to choose the right method for this discretization, such that the simulated behavior is close to the real behavior of the underlying differential equation. In our case, the financial market decisions are already modeled in discrete time, i.e., per se, there is no need to discretize and accordingly, there is no such thing as a wrong discretization technique. However, as shown above, the discrete-time model implies some counterintuitive behavior, which is in stark contrast to real market mechanisms (cf. limit orders). For this reason, we propose the following work-around: (i) find a differential equation such that the initial discrete-time model is an explicit discretization of this differential equation and (ii) calculate the implicit discretization of this continuous-time model.

The differential equation for which our original recurrence relation is an explicit discretization is given in Equation (6). As an implicit version, we use an implicit Euler discretization with step size $h > 0$, leading to

$$p(t+h) = p(t) + hw_F(f - p(t+h)),$$

i.e., $p(t+h) = \frac{p(t)+hw_F f}{1+hw_F}$. To this equation we add the demand of the chartists from Equation (5) (adjusted for $h > 0$). Thus, we have

$$p(t+h) = \frac{p(t) + hw_F f}{1 + hw_F} + hw_C(p(t) - p(t-h)). \tag{7}$$

In case of $h = 1$ this is

$$p(t+1) = \frac{p(t) + w_F f}{1 + w_F} + w_C(p(t) - p(t-1)). \tag{8}$$

We note that there is no reason to use an implicit discretization also for the chartists as we could not observe any artificial instabilities caused by this part of the model. In fact, using an implicit discretization for the chartists would be difficult, since the update rule cannot be interpreted as a discretization of an ordinary differential equation at all. Due to the dependence of the chartist dynamics on past prices, one would have to resort to so-called delay differential equations for this task. This is a technically quite involved procedure, which is unnecessary because of the lack of stiffness of this part of the model. We note that discretization methods that discretize only parts of an equation implicitly are also used in numerical analysis, see, e.g., the linearly implicit schemes described by Wanner and Hairer (1996).

Implicit discretization is used in other fields of economics, see e.g., a recent version of Nordhaus' Dynamic Integrated model of Climate and Economy (DICE, see, e.g., Nordhaus, 2017) and his 2018 Nobel Prize lecture (see Kellett et al., 2019, Footnote 4). However, Nordhaus (2017) does not elaborate on the choice of this type of discretization.

Figure 4 depicts the simulations of the implicit model (7) using the same parameter values as the simulations of the explicit model (5) in Figure 2. Note that the model is stable in the absence of chartists even for $w_F > 2$, while at the same time, it still allows for overshooting prices and instabilities caused by an abundance of chartists, cf. the graphs of Figures 4 and 5 with $w_C > 0$.

Rewriting Equation (7) leads to $p(t+h) = \left(\frac{1}{1+hw_F} + hw_C\right) p(t) + \frac{hw_F f}{1+hw_F} - hw_C p(t-h)$ or,

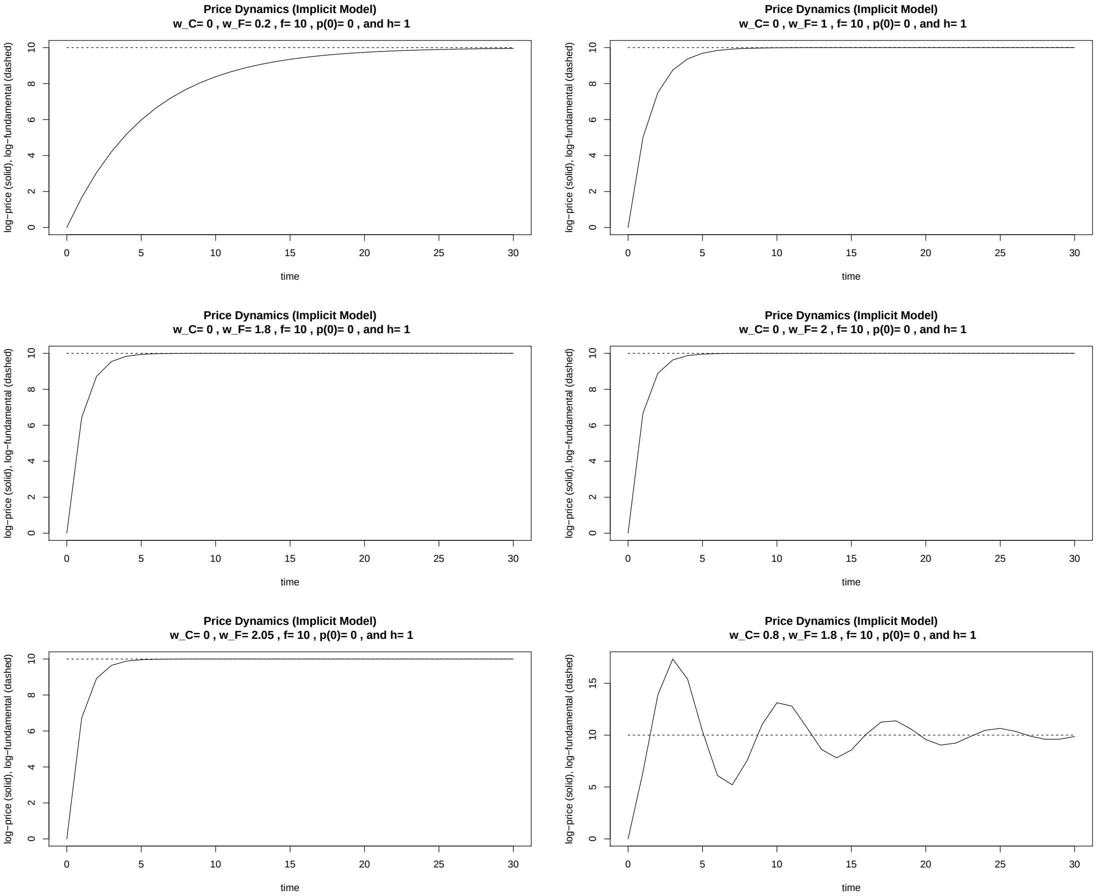

Figure 4: Price dynamics for the implicit model with varying weights w_F and $w_C = 0$ (first five plots) resp. $w_C = 0.8$ (sixth plot) and step size $h = 1$.

as a first-order equation,

$$\begin{pmatrix} p_1(t+h) \\ p_2(t+h) \end{pmatrix} = \begin{pmatrix} \frac{1}{1+hw_F} + hw_C & -hw_C \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} + \begin{pmatrix} \frac{hw_F f}{1+hw_F} \\ 0 \end{pmatrix}$$

with eigenvalues

$$\lambda_{1,2} = \frac{h^2 w_C w_F + hw_C + 1 \pm \sqrt{h^4 w_C^2 w_F^2 + 2h^3 w_C^2 w_F - 4h^3 w_C w_F^2 + h^2 w_C^2 - 6h^2 w_C w_F - 2hw_C + 1}}{2(1+hw_F)}.$$

Next, we analyze the influence of the step size h on the stability of the implicit model. Note that in the implicit model we can introduce $\tilde{w}_C = hw_C$ and $\tilde{w}_F = hw_F$, which eliminates all w_F, w_C, and h. In other words, the stability region scales with h, which is a well-known result in numerical analysis (see Deuflhard and Bornemann, 2012, Section 6.1.2). In Figures 5 and 6 the implicit model is simulated for the same weights $(w_F, w_C) = (4, 0)$, $(10, 1)$, and $(10, 1.5)$ but with varying step size $h = 1$, 0.25, and 0.1. We can see that a smaller step size stabilizes the model. In Figures 7, 8, and 9 the respective stability regions for the implicit model are shown.

With that, the question arises whether it is a reasonable behavior of the model that the stability region scales with the step size (more specifically with h^{-1}) since this implies that for $h \to 0$, the model becomes unconditionally stable. From an economic point of view, there are two explanations for this feature. On the one hand, the smaller $h > 0$ becomes, the faster the fundamentalists react to price changes, which should increase their stabilizing effect on the market. On the other hand, the smaller $h > 0$ is, the shorter the reference period $[t-h, t]$ becomes that the chartists use to calculate past gains or losses based on $p(t)$ and $p(t-h)$. Thus, these gains and losses, and consequently, the price changes caused by the chartists become smaller and smaller for shrinking h, hence, reducing their destabilizing effect on the market. In more mathematical terms, the term $hw_c(p(t) - p(t-h))$ tends to 0 faster than h, meaning that its effect after $N \sim 1/h$ time steps decreases with h even though the number of simulation steps N up to a given time T increases proportionally to $1/h$.

As a consequence, the time step $h > 0$ should not be chosen depending on the trading frequency. It could be very high in some of today's financial markets, e.g., in high-frequency trading, so that $h > 0$ would be very small. Instead, it should reflect the time horizon the chartists use for defining their trading strategy. When the trading frequency is of interest, i.e., when the time distance between two trades h and the chartists' time horizon are different, another parameter has to be added to the model. Although a very interesting aspect, it is beyond the scope of this work and, thus, the subject of future work. It is particularly important that the model produces plausible qualitative results also for relatively large time steps $h > 0$—and this is precisely what an implicit discretization achieves, cf. the discussion of A-stability and related concepts, e.g., in the work of Deuflhard and Bornemann (2012).

Taken together our analysis draws attention to the following two drivers of price instabilities: first, the instability of the model is caused by the fundamentalists due to some numerical artifacts

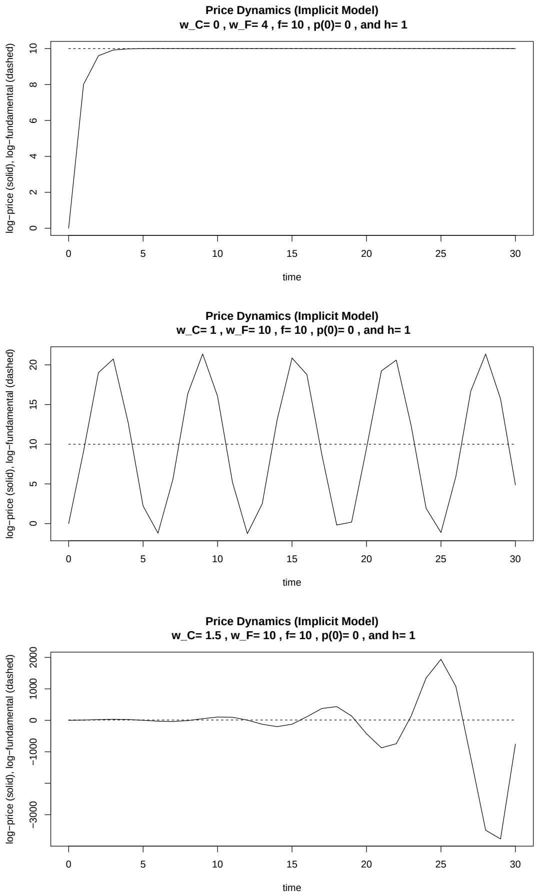

Figure 5: Price dynamics for the implicit model with varying weights w_F, w_C and step size $h = 1$.

and, second, the instability of the model is caused by a too large number (resp. weight) of chartists relative to fundamentalists. Clearly, the second effect is the one of interest when analyzing bubbles etc.

5 Economic Interpretation of the Implicit Pricing Rule

When reconsidering the explicit pricing rule (2) resp. (5), we can interpret the next period's log-price in a straightforward way as a function of the current log-price and a share of the excess demand of all traders, where the demand of the fundamentalists depends on the difference between the market price and the fundamental value. That is exactly how the model was con-

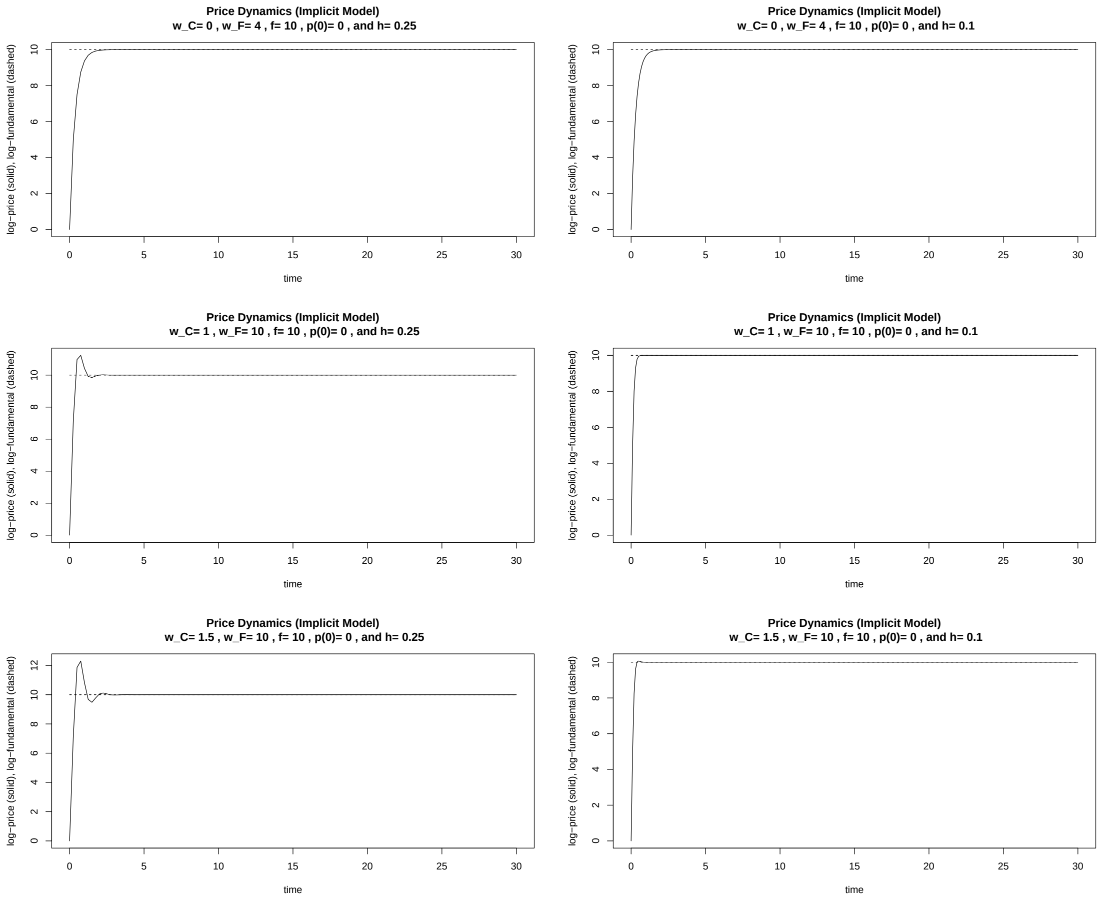

Figure 6: Price dynamics for the implicit model with varying weights w_F, w_C and step size $h = 0.25$ (graphs on the left) resp. $h = 0.1$ (graphs on the right).

structed in the first place. In contrast, a first look at the implicit pricing rule for $h = 1$ (8) does not yield such a straightforward interpretation. However, when rewriting Equation (8) to

$$
\begin{aligned}
p(t+1) &= \frac{p(t) + w_F f}{1 + w_F} + w_C(p(t) - p(t-1)) \\
&= \left(\frac{1}{1 + w_F} p(t) + \frac{w_F}{1 + w_F} f \right) + w_C(p(t) - p(t-1)),
\end{aligned}
$$

a new intuitive interpretation suggests itself. Note that $\lim_{w_F \to 0} \frac{1}{1+w_F} = 1$, $\lim_{w_F \to \infty} \frac{1}{1+w_F} = 0$, $\lim_{w_F \to 0} \frac{w_F}{1+w_F} = 0$, and $\lim_{w_F \to \infty} \frac{w_F}{1+w_F} = 1$. Next period's log-price is a convex combination of the log-fundamental-value and the current log-price depending on the weight of the fundamental-

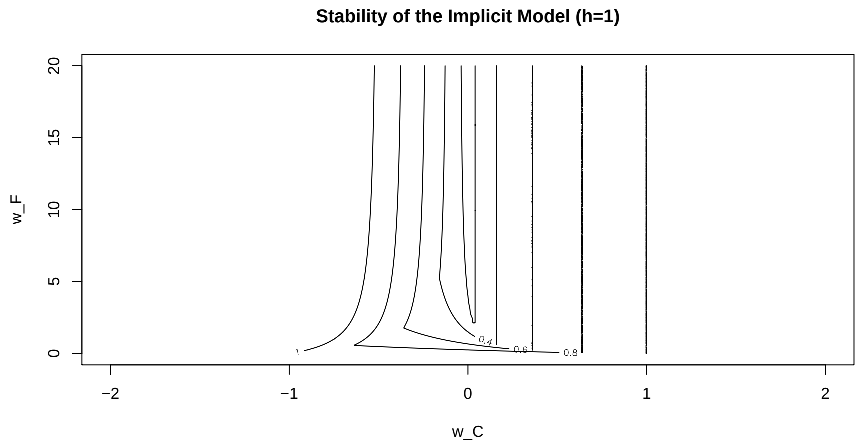

Figure 7: Parameters for which the implicit model (7) with step size $h = 1$ is stable.

ists plus a certain share of the excess demand of the chartists or, in general, of all traders except the fundamentalists. Thus, with the explicit pricing rule, the actions of the fundamentalists ("What do they do?") are modeled specifically. In contrast, in the case of the implicit pricing rule, the implications of the fundamentalists' trading ("What happens?") is reproduced—which is exactly in line with the qualifications "explicit" and "implicit." If we are interested in an adequate reproduction of the behavior of real-world systems—the reduced form—and not so much of the specific mechanics—the structural form—the implicit rather than the explicit model is the adequate choice.

To this end, let us again consider a market in which only fundamentalists trade. In the explicit model, when the weight of the fundamentalists is small, the price converges monotonically to the fundamental but never reaches it. When the respective weight is medium-sized, the price jumps to the fundamental value within one step and stays there. When the weight is high, the price oscillates around the fundamental value, maybe converges to it, but never reaches it. Finally, when the fundamentalists' weight is even higher, the price oscillates and explodes.

In the implicit model, the price converges to the fundamentals monotonically for all weights, and the larger w_F is, the closer the price is to the fundamental value after one time step. More precisely, the difference between this price after one time step and the fundamental value tends to 0 if w_F tends to infinity. In financial markets with limit orders, a constant fundamental

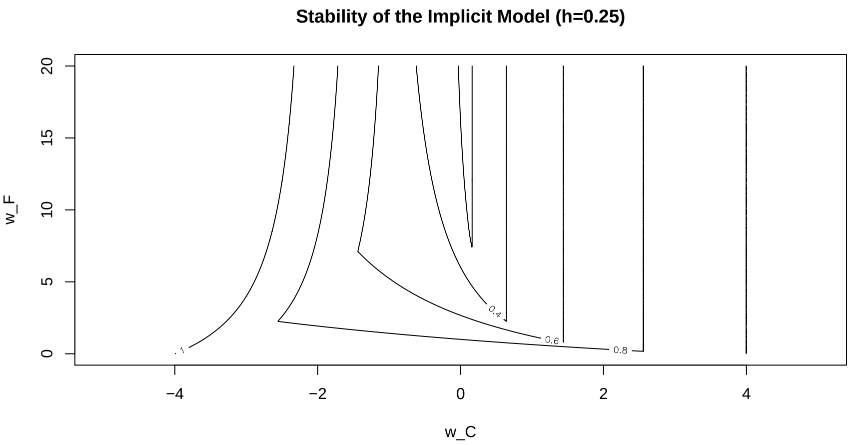

Figure 8: Parameters for which the implicit model (7) with step size $h = 0.25$ is stable.

value, and fundamentalist traders only, under one sufficiently large trade, the price jumps to the fundamental value in one step and stays there. When there are enough small trades, the price converges to the fundamental value monotonically and reaches it. When there are too few small trades, the price might at first move towards the fundamental value and then stay at some other level without jumping to the fundamental value.

Hence, qualitatively, the implicit model seems to capture the price behavior in today's financial markets much better than the explicit model. It reflects not only monotone convergence, which implies that asset prices do not jump across their fundamental values but also the fact that the resulting prices are the closer to their fundamental values the larger w_F is. One aspect that is not captured by the implicit model is that using a sufficiently large limit order, the price may reach the fundamental value in finite time. However, given that markets are subject to noise, uncertainty of the fundamental value, and perturbations by other trader types, this phenomenon is likely to occur only in highly idealized markets. Moreover, when considering prices, for instance, in USD or EUR, we can consider the fundamental value practically reached when $|\rho(t) - \phi(t)| < 0.005$ since there is a smallest unit. Thus, the fact that this particular phenomenon is not reflected in the implicit model is much less of a drawback than the artificial overshootings, oscillations, and instability of the classical explicit model. An interesting point for future work would be the development of a model with a so-called event-based timeline, but that is beyond

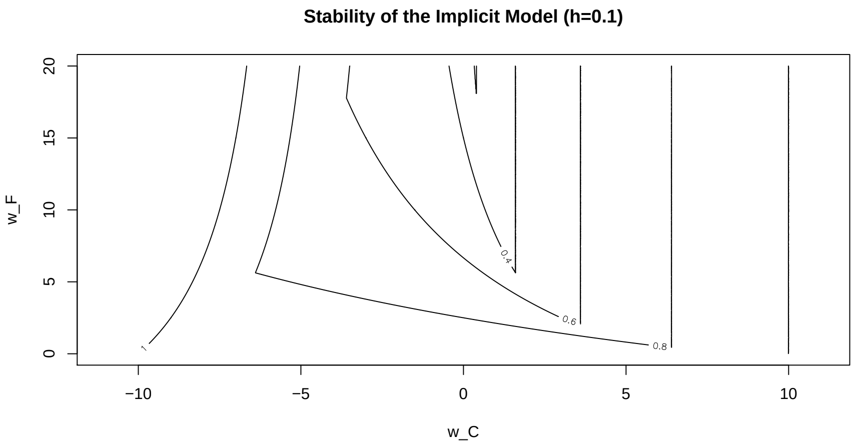

Figure 9: Parameters for which the implicit model (7) with step size $h = 0.1$ is stable.

the scope of this paper.

6 Evolutionary Rules for the Numbers of the Traders

To examine whether our insights also hold in more complex financial environments, we generalize our model along three dimensions in the next step. We allow for more types of traders, in particular, noise traders and sentimentalists, introduce an evolutionary mechanism that drives the distribution of the different types of traders in the market, and allow the fundamental value to be stochastic. Based on these generalizations, we develop two versions of an agent-based market model that differ only concerning the pricing rule. Model a) uses the explicit discretization (5) of the differential equation (6). In contrast, model b) makes use of the implicit one (7). With this approach, we should be able to better differentiate the effects of the two discretization techniques.

Firstly, we introduce additional trader types, namely noise traders and sentimentalists. Noise traders (N) trade more or less independently of market dynamics. This can be the case because they are the proverbial small traders without enough information about the market or because they are liquidity traders, i.e., traders who have to buy/sell specific amounts of the asset irrespective of the price and the fundamental, e.g., because they need it for hedging, for a mutual funds portfolio, or some external engagement. Sentimentalists (S) are a type of trader that switches with a certain probability or ratio to other, usually better performing strategies, i.e., they observe the profits of the other trader types and can switch in every period to their preferred strategies.

The demand functions of the basic trader types are:

$$D_N(t)/N \sim \mathcal{N}\left(h\left(\mu_N - \sigma_N^2/2\right), \sigma_N\sqrt{h}\right) \ i.i.d.,$$

$$D_F(t) = hF(f(t) - p(t)),$$

and

$$D_C(t) = hC(p(t) - p(t-1) \cdot 1_{t>0}.$$

with $t \in \{0, h, 2h, \ldots, T/h\}$, $T = dy$ the total number of trading days, d the number of trading days per year, y the total number of years under analysis, h the time step between two trades, $C, F, N > 0$ parameters modeling the aggressiveness of the respective trader group, f the log-fundamental-value, p the log-price (either modeled explicitly or implicitly), as well as $\mu_N > -1$ the trend and $\sigma_N > 0$ the volatility of the noise traders' demand.

Secondly, we introduce evolutionary growth rules for the share of traders as a second generalization (cf. Hommes, 2006). More specifically, we apply so-called exponential replicator dynamics. Thus, we fix the share of traders for chartists $N_C \geq 0$, fundamentalists $N_F \geq 0$, noise traders $N_N \geq 0$, and sentimentalists $N_S \geq 0$ such that $N_C + N_F + N_N + N_S = 1$. The overall share of a specific trading strategy is determined by the sentimentalists as they are the only group of traders that is allowed to switch the strategy. With a given initial distribution for the sentimentalists $N_{S_C}(0) \geq 0$, $N_{S_F}(0) \geq 0$, and $N_{S_N}(0) \geq 0$ (such that $N_{S_C}(0) + N_{S_F}(0) + N_{S_C}(0) = 1$) we define the numbers of the different sentimentalists' trading types via

$$N_{S_C}(t+1) = \frac{N_{S_C}(t)\exp(\beta U_C(t+1))}{N_{S_C}(t)\exp(\beta U_C(t+1)) + N_{S_F}(t)\exp(\beta U_F(t+1)) + N_{S_N}(t)\exp(\beta U_N(t+1))},$$

$$N_{S_F}(t+1) = \frac{N_{S_F}(t)\exp(\beta U_F(t+1))}{N_{S_C}(t)\exp(\beta U_C(t+1)) + N_{S_F}(t)\exp(\beta U_F(t+1)) + N_{S_N}(t)\exp(\beta U_N(t+1))},$$

and

$$N_{S_N}(t+1) = \frac{N_{S_N}(t)\exp(\beta U_N(t+1))}{N_{S_C}(t)\exp(\beta U_C(t+1)) + N_{S_F}(t)\exp(\beta U_F(t+1)) + N_{S_N}(t)\exp(\beta U_N(t+1))},$$

where $\beta > 0$ is a parameter controlling the speed of adaption (cf. Brock and Hommes, 1997) and

$$U_C(t+1) = D_C(t) \cdot (\exp(p(t+1) - p(t)) - 1),$$

$$U_F(t+1) = D_F(t) \cdot (\exp(p(t+1) - p(t)) - 1),$$

as well as

$$U_N(t+1) = D_N(t) \cdot (\exp(p(t+1) - p(t)) - 1)$$

describe the fitness of the trader groups. Taken together, at time t the share of the chartists is $N_C + N_S N_{S_C}(t) \in [N_C, N_C + N_S]$, the share of the fundamentalists is $N_F + N_S N_{S_F}(t) \in [N_F, N_F + N_S]$, while the share of the noise traders is $N_N + N_S N_{S_N}(t) \in [N_N, N_N + N_S]$. The sentimentalists do not have an own trading rule, but they are allowed to switch between the three basic trading rules F, C, and N. When $N_S = 1$, all traders can switch. However, note that $N_S = 1$ does not necessarily mean that there are no chartists, for example.

As a third extension, we allow the fundamental value of the asset to be stochastic. We define the fundamental value ϕ to fulfill the stochastic differential equation $d\phi(t) = \mu_F \phi(t)dt + \sigma_F \phi(t)dW(t)$, where W is a standard Brownian motion (Wiener process), $\mu_F > -1$ is the trend of the fundamental, and $\sigma_F > 0$ the volatility of the fundamental, i.e., the fundamental value, but not necessarily the price process, follows a geometric Brownian motion. We assume that the fundamental value is a stochastic process that can be observed perfectly by the fundamentalists, who base their demand at time t on $f(t) = \log(\phi(t))$. As an alternative, one could assume that there is a deterministic fundamental value that cannot be observed perfectly by the traders. However, the difference between these alternatives is negligible in our simulations; we do not concentrate on the mechanisms, but on the behavior. Hence, the pricing rules are for the implicit discretization

$$
\begin{aligned}
p_{imp}(t+h) =& \frac{p_{imp}(t) + hF(N_F + N_S N_{S_F}(t)) f(t) M^{-1}}{1 + hF(N_F + N_S N_{S_F}(t)) M^{-1}} \\
&+ hC(N_C + N_S N_{S_C}(t))(p_{imp}(t) - p_{imp}(t-h)) M^{-1} 1_{t>0} \\
&+ (N_N + N_S N_{S_N}(t)) D_N(t) M^{-1}
\end{aligned}
$$

and for the explicit discretization

$$
\begin{aligned}
p_{exp}(t+h) =& p_{exp}(t) + hF(N_F + N_S N_{S_F}(t))(f(t) - p_{exp}(t)) M^{-1} \\
&+ hC(N_C + N_S N_{S_C}(t))(p_{exp}(t) - p_{exp}(t-h)) M^{-1} 1_{t>0} \\
&+ (N_N + N_S N_{S_N}(t)) D_N(t) M^{-1}.
\end{aligned}
$$

Figure 10 depicts simulated price dynamics of four typical, qualitatively different cases: a hill-shaped price development, a temporary downward trend, a U-shaped price development, and a temporary upward trend. The simulations are based on the following parameter values, which were found by an extensive trail-and-error calibration. The parameters are chosen in such a way that the simulation is consistent with empirical stylized facts (cf. Hommes, 2006). There are $d = 250$ trading days per year and $y = 1$ year making a total of $T = 250$ trading days. The step size is set to $h = 1$, i.e. one trade per day. The initial values are $f(0) = p_{imp}(0) = p_{exp}(0) = \log(1) = 0$. The fundamental's trend is $\mu_F = 0.1h/d$ and its induced volatility is $\sigma_F = 0.03$. The noise traders' parameters are $\mu_N = 0.05$ and $\sigma_N = 0.5$. The shares of the traders are fixed to one quarter each, i.e., $N_F = N_C = N_N = N_S = 0.25$ while the sentimentalists are allowed to switch. Their initial shares are approximately one third each, i.e., $N_{S_F} = N_{S_C} = 0.33$ and $N_{S_N} = 0.34$. The scaling parameter for the market power and trading volume is set to $M = 1$ and the sentimentalists' switching velocity is defined as $\beta = 1$. The aggressiveness of the respective trading rule is $C = 2.1$, $F = 1.7$, and $N = 0.2$.

Additionally to the price dynamics of Simulation 1 to 4 in Figure 10 (fundamental value: solid, implicit price: dashed, and explicit price: dotted), the shares of the sentimentalists following fundamentalists (dashed), chartists (solid), or noise traders (dotted) are depicted in Figure 11 for the explicit model (fine) as well as for the implicit model (bold). Further, in Figure 12, the induced volatility of the fundamental σ_F (fine) and its sliding historical volatility (bold) with window size $m = 20$ are plotted (solid) together with the historical volatilities (with the same window size m) of the implicit model (dashed) and the explicit model (dotted). We observe that there are much more pikes in the price paths both for the implicit and the explicit model than in the fundamentals. Sometimes, the pikes in the implicitly modeled price correspond to pikes in the explicit model. However, there are as well pikes in the explicit model where no pikes in the implicit one are visible and vice versa. Additionally we mention that no trader type becomes extinct—a stylized fact market models should fulfill (Hommes, 2006; Kirman, 1993). This means that if there is no financial bubble in either model, the two models tend to follow a similar pattern.

In Simulation 1, in both models, the fundamentalists' rule is most profitable, and so their share (within the group of sentimentalists) increases. In Simulation 2, this is true for the chartists, and in Simulation 3, the noise traders' share is increasing. The latter point is remarkable since it has been argued that noise trading should be unprofitable because it is not rational—yet noise traders' profits, as well as trend followers' profits (Hommes, 2006), are considered to be an important stylized fact of financial markets (cf. De Long et al., 1987; Green and Heffernan, 2019). In Simulation 4, there are not only quantitative but also qualitative differences between the implicit and the explicit model. While fundamentalists gain under explicit discretization, they do not make profits under the implicit model. Note that in this model, the interrelation between the price dynamics and the successful type of trader is not limited to those cases shown, but these are only an illustrative selection. However, all in all, the simulations give reason to assume that, as common sense also suggests, chartists are more likely to win when trends are clear.

In Figure 12, we observe that in both models, price volatility is not only higher than the fundamental volatility but also clustered—two stylized facts market models should reproduce (Hommes, 2006). There are parameter settings, e.g., $\sigma_F = 0.1$, $C = N = 1$, $F = 10$, and all others as above, such that in all simulation runs price bubbles are generated under the explicit model, while no bubbles appear in the implicit model, cf. Figure 13, Simulation 5. Sets of parameters that generate bubbles under implicit discretization, but no bubble paths under the explicit model, are very rare—while there are many parameter settings leading to the opposite behavior (cf. Section 7). Under explicit discretization, more bubbles occur, preceded by higher volatilities.

Once implicit discretization is allowed for, there is a broader parameter space with bubble-and-crash free model specifications. That means when empirical data are used to estimate model parameters, those in the implicit model are potentially better, as those in the explicit model are biased. Especially when predictions or policy recommendations are to be made, it is preferable to use a larger parameter space, i.e., the implicit model (cf. Schröppel, 2018; Shiller, 1980)

Taken together, our results from the simple model typically carry over to the more general setting. Explicit discretization typically generates more unstable prices as well as price bubbles and crashes. In contrast, implicit discretization of the same underlying financial market model is associated with steadier price developments and, in particular, far fewer bubbles. In empirical analyses, models based on explicit discretization might lead to biased results as the calibration exercise to find the best fit to the stylized facts is restricted to a smaller parameter space than under implicit discretization. Thus, the choice of discretization—a seemingly technical issue only—is far from innocent but might have far-reaching implications for the analysis and interpretation of heterogeneous agent models.

7 Simulations

At first sight, the implicit and explicit modeling do not seem to differ substantially with respect to price dynamics, traders' success, and volatilities (see Figures 10, 11, and 12). However, as we have shown in the analytical investigations (Sections 3 and 4) the stability behavior differs significantly. In those cases where there appears no bubble in either model, the models behave similarly. However, the parameters' range in which the explicit model is stable is much smaller than the range in which the implicit model is stable. This means that if the parameters are adjusted to historical, real market developments, the space over which the implicit model is optimized is larger and therefore could deliver better results. For the range in which both models are stable, the simulation results are similar, so it is possible to switch from the explicit model

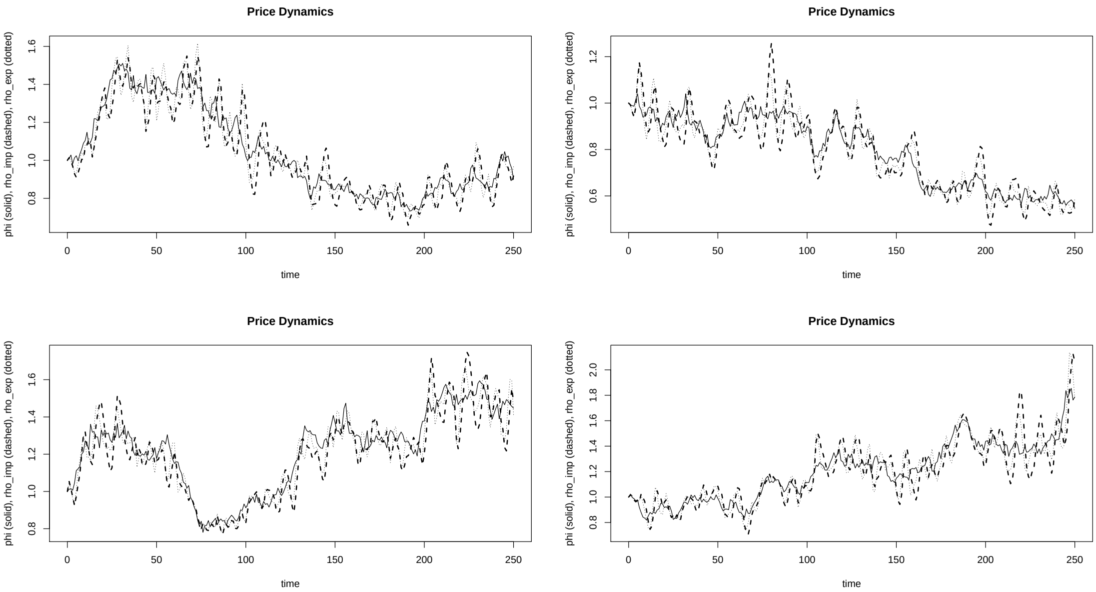

Figure 10: Simulations 1 (above left), 2 (above right), 3 (bottom left), and 4 (bottom right): fundamental value (solid), price path implicitly modeled (dashed) and explicitly modeled (dotted).

to the implicit one without restrictions. The differences between the explicit and the implicit model in the cases without any bubbles should therefore be at a minimum. In order to show that the implicit model has a larger stable range not only in theory, we perform an extensive simulation below with 1,000 runs.

In the simulation we use the setting of Section 6 including its pricing rules (explicit and implicit), its evolutionary rules, and, with few exceptions, its parameter specifications. In the simulation, we consider varying parameters $C = -10,\ -9.9,\ -9.8,\ \ldots,\ 9.9,\ 10$ and $F = 0,\ 0.1,\ 0.2,\ \ldots,\ 19.9,\ 20$ for the aggressiveness of the chartists resp. fundamentalists. In this way, we can see which parameter constellations lead to stable or unstable dynamics. Additionally, we increase the volatility of the fundamental values to $\sigma_F = 0.1$ and the aggressiveness of the noise traders to $N = 1$ to bring more variety to the simulation runs.

We simulated 1,000 fundamental value developements and performed the pricing and evolutionary rules for the two models and for all parameter pairs (C, F) in the mentioned range. In Figure 14 we see a countur plot of the numbers of bubbles in the explicit model for the varying parameters C and F. There is a small area for small F and C around zero where no bubbles

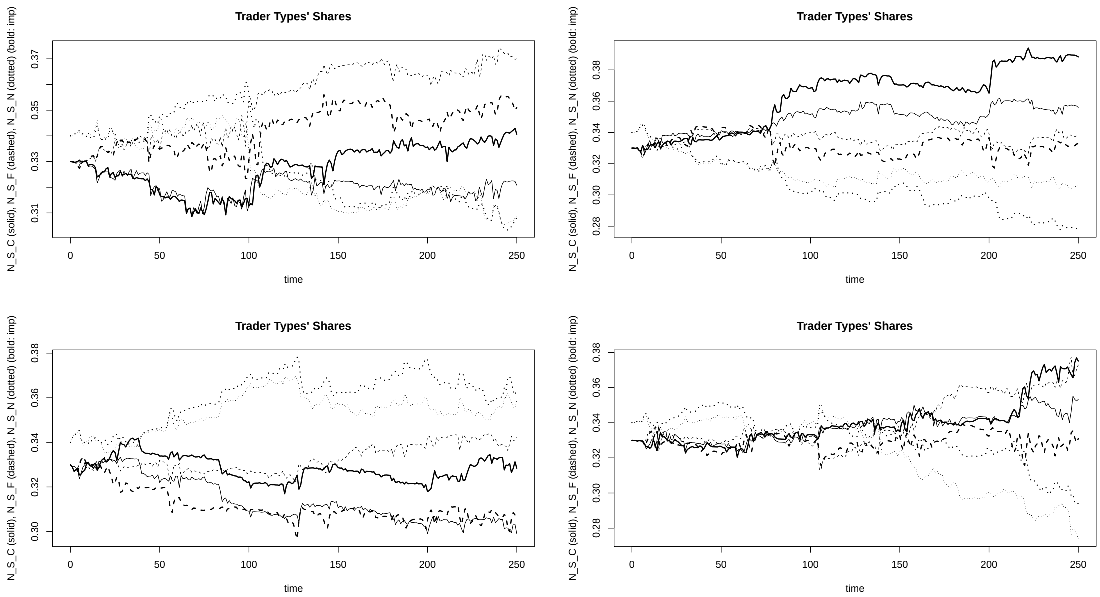

Figure 11: Simulations 1 (above left), 2 (above right), 3 (bottom left), and 4 (bottom right): shares of fundamentalists (dashed), chartists (solid), and noise traders (dotted); fine: explicit model, bold: implicit model.

occur. Outside this area there are bubbles in all of the 1,000 runs. In contrast, Figure 15 shows a contour plot of the numbers of bubbles in the implicit model for the varying parameters C and F. We define a bubble as a price development that tends to infinity. There are no bubbles for C around zero and all parameter values of F, which is perfectly in line with our analytical findings. Additionally, in Figures 16 and 17, we show the contour plots for the numbers of "excessive" bubbles in the corresponding model, i.e. for $\max\{0, \#\text{Bubbles in the explicit model} - \#\text{Bubbles in the implicit model}\}$ and $\max\{0, \#\text{Bubbles in the implicit model} - \#\text{Bubbles in the explicit model}\}$. Thus, we can easily observe the areas in which one model produces more bubbles than the other one. On the one hand, there is a very small area with C approximately between three and four and F around four, where the explicit model is more stable, on the other hand, the implicit model is more stable for C approximately between minus two and four and all F larger than some values between two and six. The fact that the explicit model is not dominated by the implicit one is not very significant. The standard literature on discretization techniques (Deuflhard and Bornemann, 2012; Wanner and Hairer, 1996) just states that in the case $C = 0$ the implicit model is more stable—as is the case in our simulation. It would be rather unusual if for

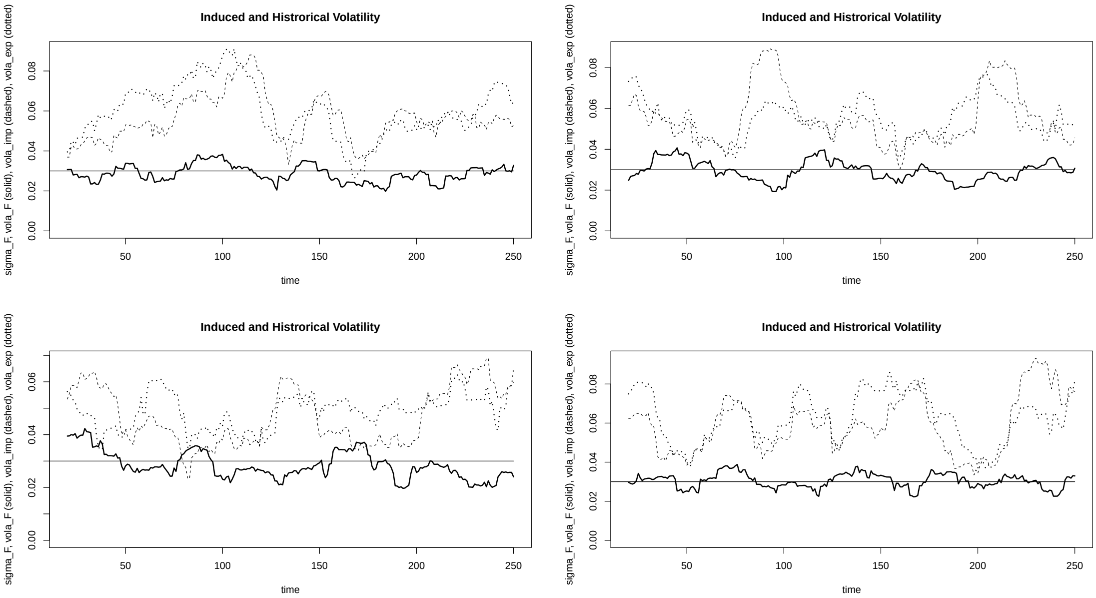

Figure 12: Simulations 1 (above left), 2 (above right), 3 (bottom left), and 4 (bottom right): induced (fine) and historical (bold) volatility of the fundamental value (solid), historical volatility of the implicitly modeled price path (dashed) and the explicitly model price path (dotted).

$C \neq 0$ there were no fluctuations in the results.

8 Conclusion

Heterogeneous agent models provide an interesting approach to analyze financial markets. Building on the interactions of different types of traders, in particular fundamentalists and chartists, HAMs have proven to be a very productive tool to analyze financial dynamics. However, when using these models, particular care should be taken to the specific modeling of the group of fundamentalist traders, a core element in this type of models. Following the standard approach in the literature by combining the excess demand of fundamentalists and chartists with a market maker mechanism can lead to quite counterintuitive price developments. Even in a simple market set up with fundamentalist traders only, market instabilities with exploding oscillatory prices might occur. We relate this odd market behavior to a seemingly technical issue, the explicit discretization of the price equation that is implied in a typically standard HAM analysis.

As a remedy to improve the HAM approach, we propose to instead use the implicit dis-

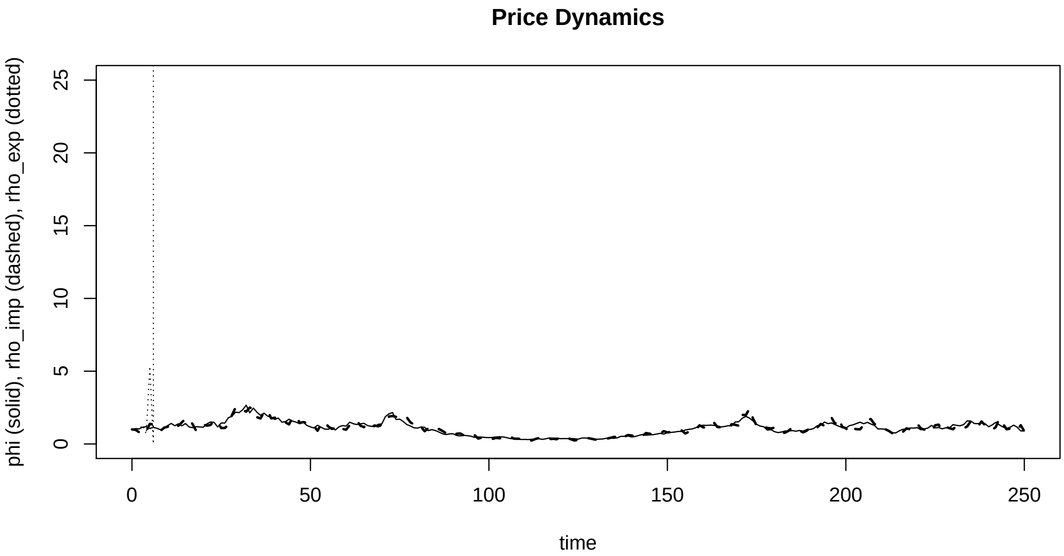

Figure 13: Simulation 5: Fundamental value (solid), price path implicitly modeled (dashed) and explicitly modeled (dotted). The explicitly modeled price path explodes.

cretization of the price equation as a straightforward, easy to implement procedure. Under this procedure, HAMs can be more trusted do be in line with today's financial markets, i.e., in particular, the presence of limit orders. Not accounting for this seemingly technical issue might have far-reaching implications. Researchers are likely to overestimate the occurrence of financial crashes and asset price bubbles. Also, when calibrating HAMs to replicate relevant empirical stylized facts, models based on implicit discretization incorporate a more extensive parameter space that should mitigate the issue of biased parameter values and improve the empirical fit of the models. Taken together, this should help to enhance HAMs' value as an instrument to design, analyze, and evaluate financial markets and related policies.

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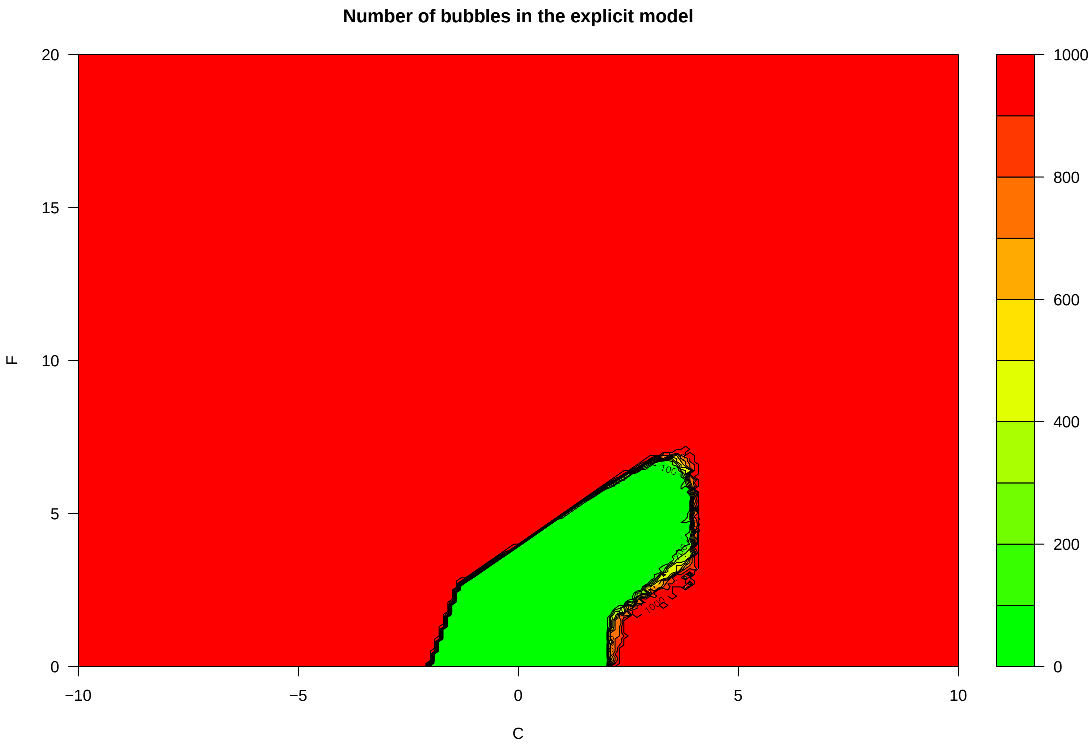

Figure 14: Number of bubbles in the explicit model from 0 (green) to 1,000 (red).

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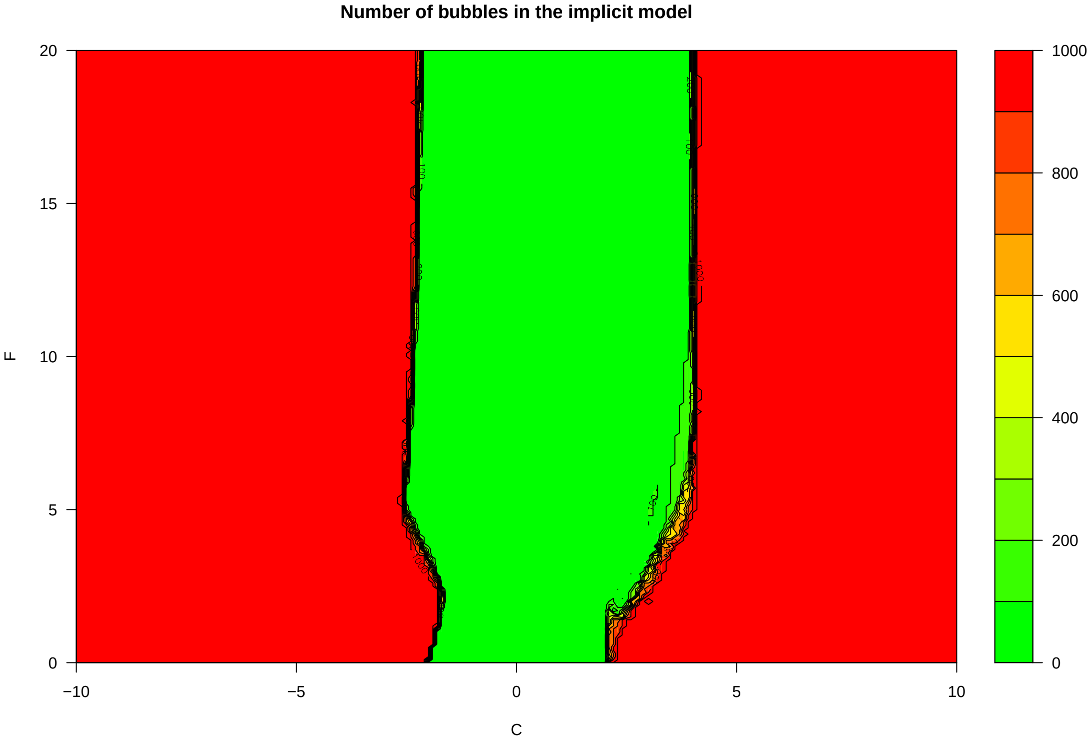

Figure 15: Number of bubbles in the implicit model from 0 (green) to 1,000 (red).

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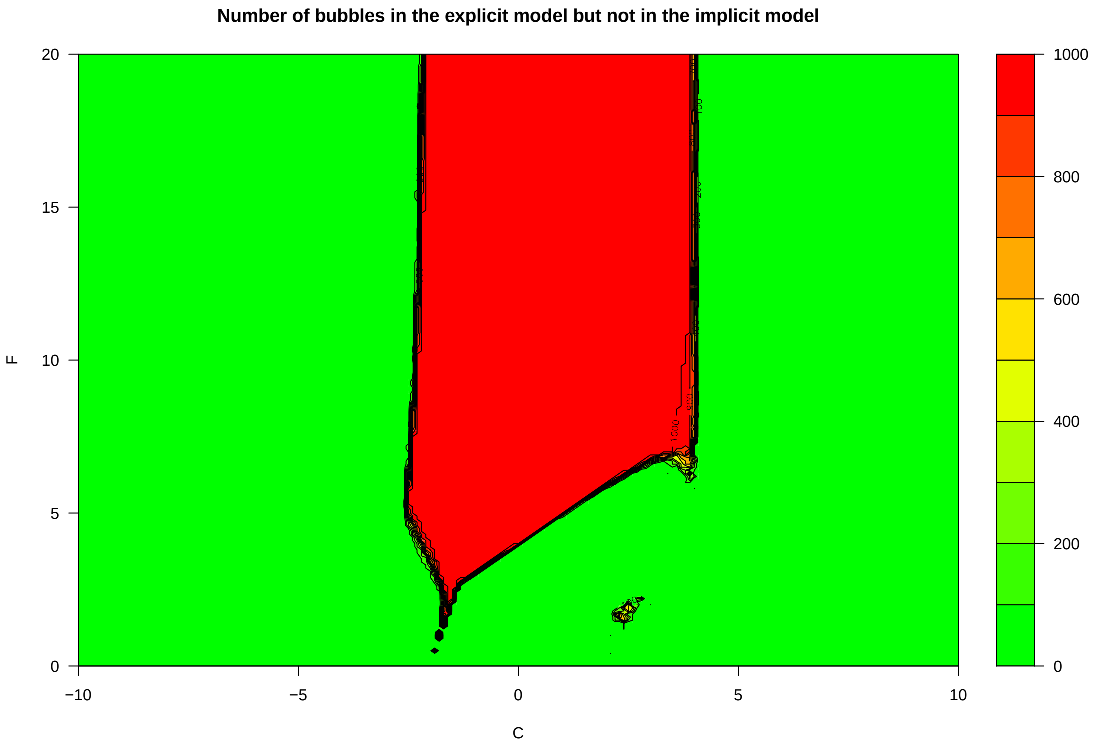

Figure 16: Number of excessive bubbles in the explicit model from 0 (green) to 1,000 (red).

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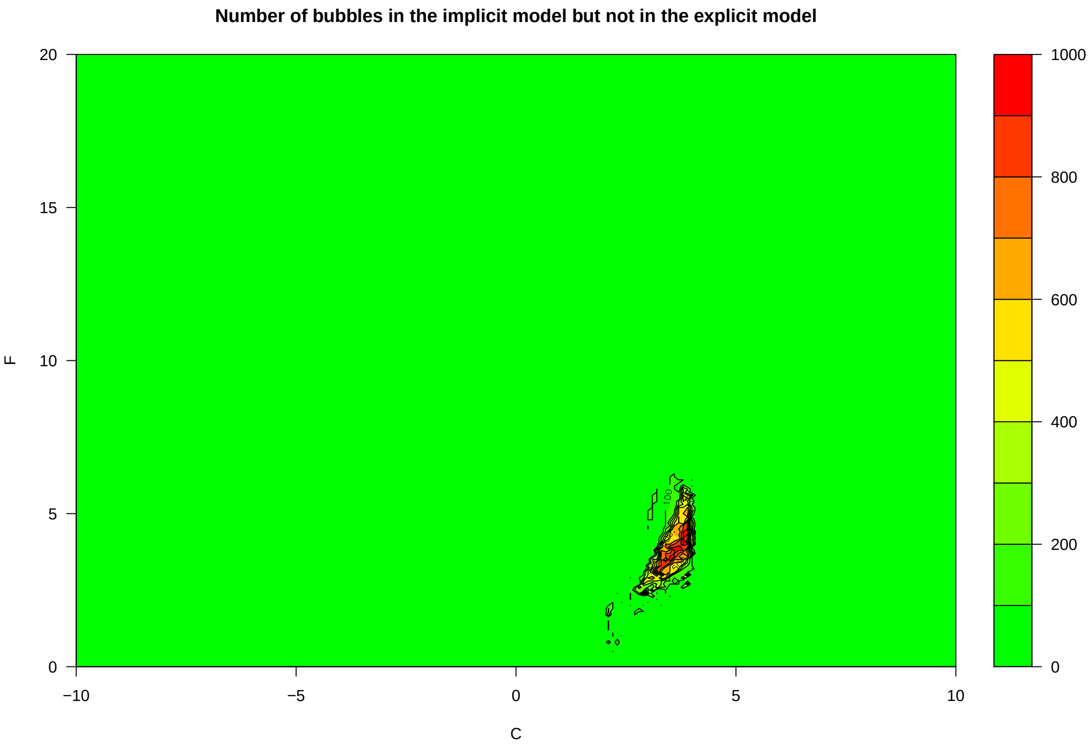

Figure 17: Number of excessive bubbles in the implicit model from 0 (green) to 1,000 (red).

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Appendix

Black-and-white plots for the print version when required.

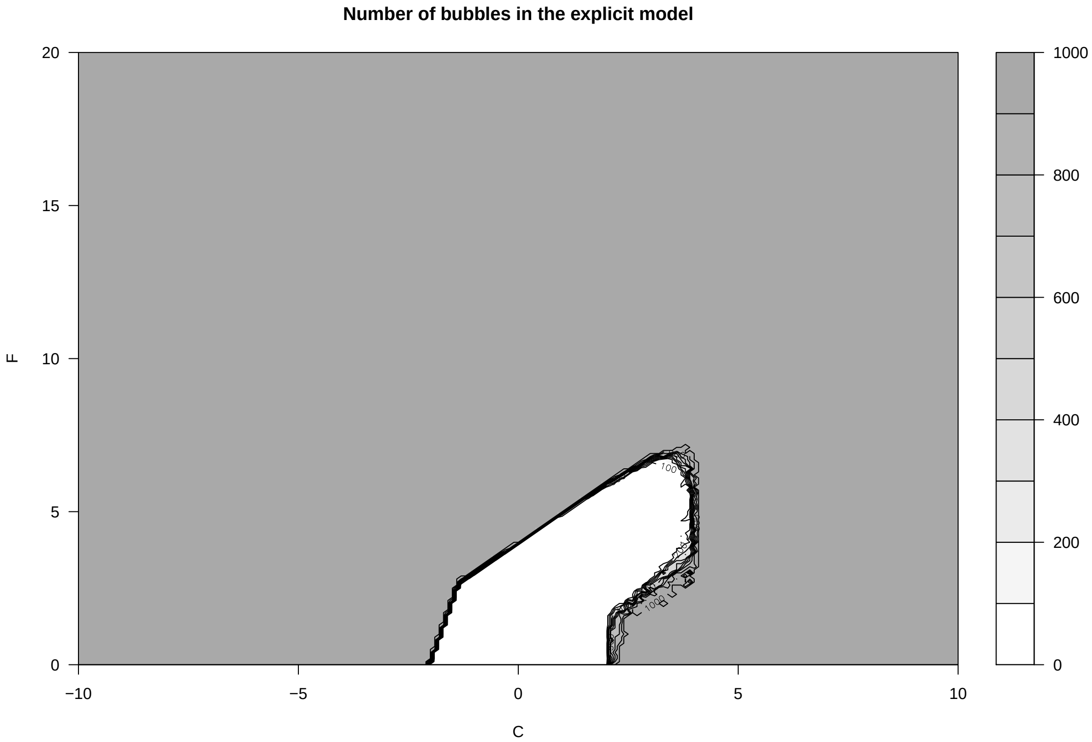

Figure 18: Number of bubbles in the explicit model from 0 (white) to 1,000 (gray).

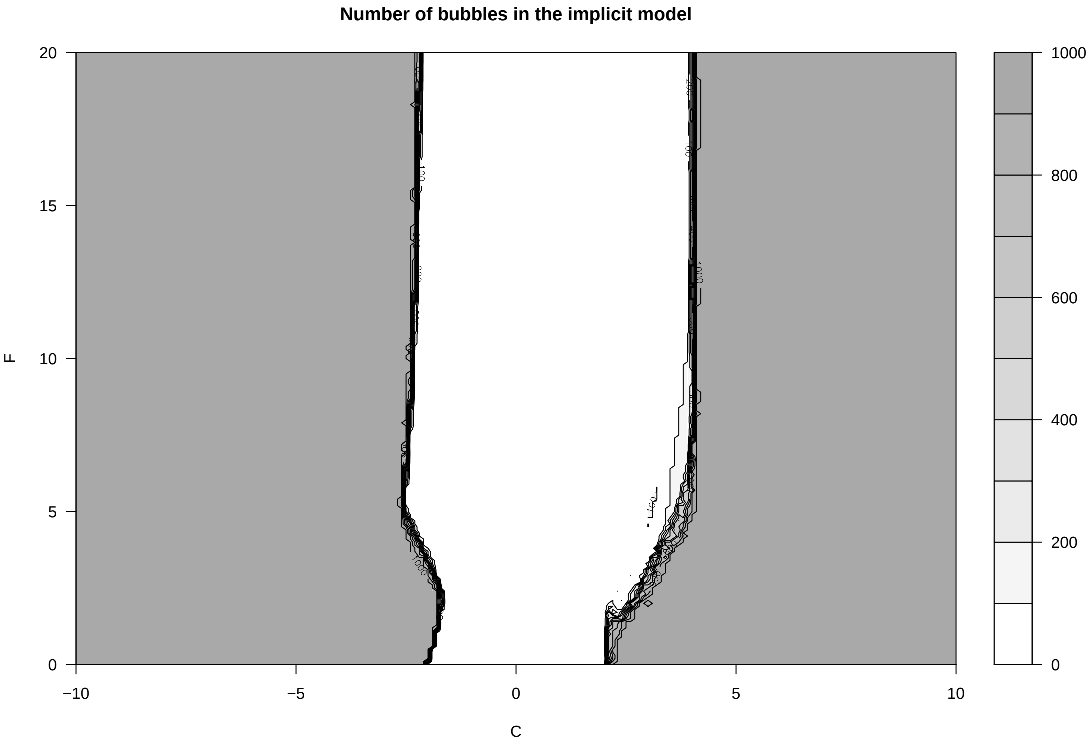

Figure 19: Number of bubbles in the implicit model from 0 (white) to 1,000 (gray).

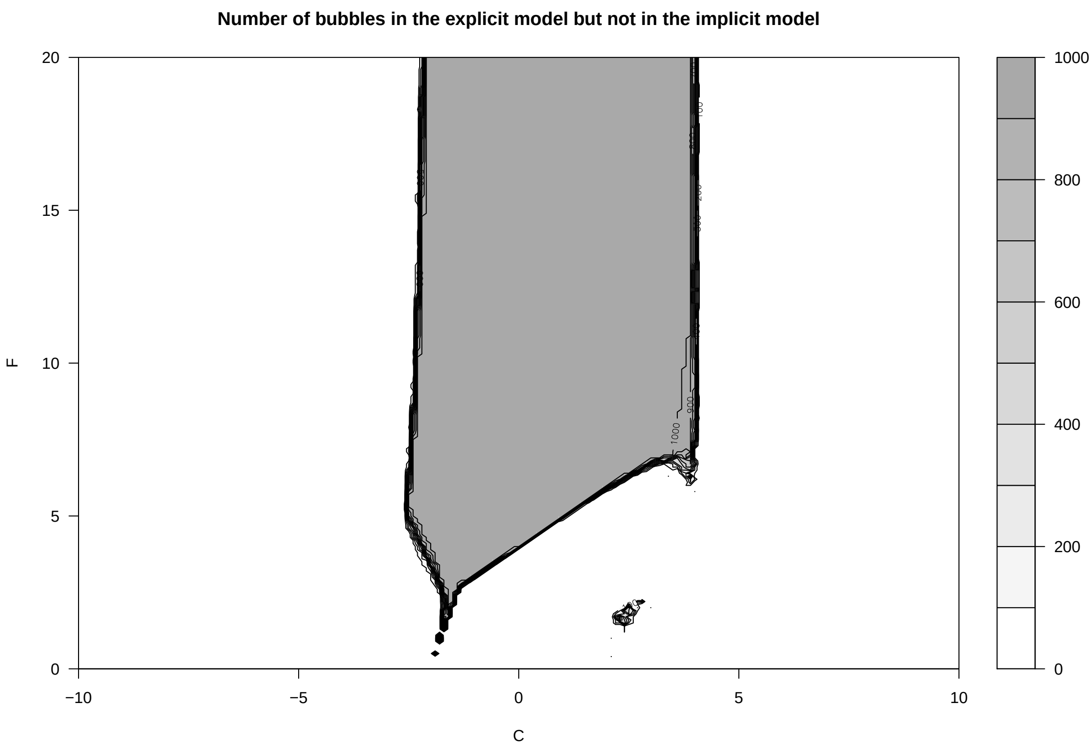

Figure 20: Number of excessive bubbles in the explicit model from 0 (white) to 1,000 (gray).

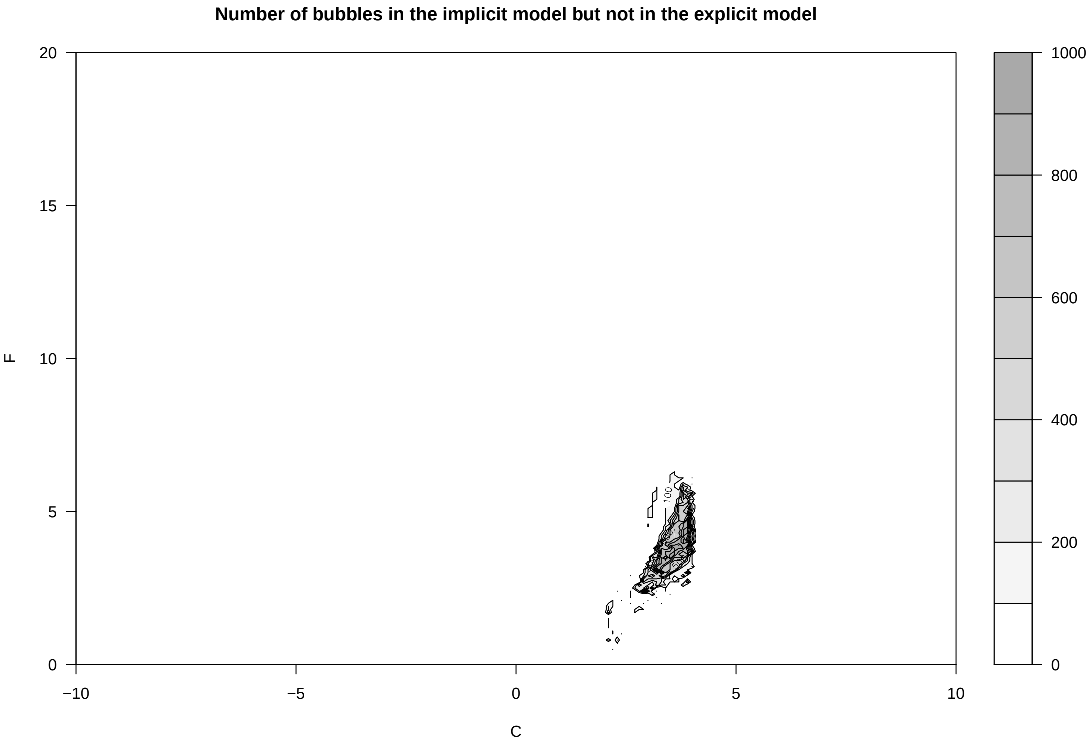

Figure 21: Number of excessive bubbles in the implicit model from 0 (white) to 1,000 (gray).