Why does strict dissipativity help in model predictive control?

Extended Abstract

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Abstract: During the last couple of years the theory of why and when Model Predictive Control (MPC) generates stable, feasible and near optimal closed-loop solutions has significantly matured. In this talk we give a survey about the contribution of the dissipativity concept in this line of research.

Keywords: Model Predictive Control, strict dissipativity, stability, near optimality, feasibility, detectability

1. INTRODUCTION

During the last couple of years the theory of why and when Model Predictive Control (MPC) generates stable, feasible and near optimal closed-loop solutions has significantly matured. In this talk we give a survey about the contribution of the dissipativity concept in this line of research.

2. PROBLEM FORMULATION

We present our results for discrete-time nonlinear control systems of the form

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0 \tag{1}$$

with $x(k) \in X$ and $u(k) \in U$ for normed vector spaces X and U. Most of the results in this talk hold in an analogous way for continuous time systems.

MPC then computes a control input $u_{MPC}(\cdot)$ by solving a sequence of optimal control problems on finite, overlapping time horizons. Here, the finite horizon optimal control problem is given as follows. For a given constraint set $\mathbb{Y} \subset X \times U$, a terminal constraints set $\mathbb{X}_f$, a stage cost $\ell : \mathbb{Y} \to \mathbb{R}$, a terminal cost $F : \mathbb{X}_f \to \mathbb{R}$, and a time horizon $N \in \mathbb{N}$ we define the finite horizon functional

$$J_N(x_0, u(\cdot)) := \sum_{k=0}^{N-1} \ell(x(k), u(k)) + F(x(N)), \tag{2}$$

where $x(\cdot)$ solves (1). Then we solve

$$\text{minimize}_{u(\cdot)} J_N(x_0, u(\cdot)) \tag{3}$$

subject to the constraints $(x(k), u(k)) \in \mathbb{Y}$ for all $k = 0, \dots, N-1$ and $x(N) \in \mathbb{X}_f$. We call a control $u(\cdot)$ *admissible* (for x_0) when these constraints are satisfied. Moreover, we set $\mathbb{X} := \{x \in X \,|\, \text{there is } u \in U \text{ with } (x, u) \in \mathbb{Y}\}$.

The pair $(\mathbb{X}_f, F)$ is referred to as *terminal condition* and in the trivial case $\mathbb{X}_f = \mathbb{X}$ and $F \equiv 0$ we refer to (2) as a problem *without terminal conditions*.

Associated to the optimal control problems (3) we define the *optimal value function*

$$V_N(x_0) := \inf_{u(\cdot) \text{ admissible}} J_N(x_0, u(\cdot))$$

and we call an admissible control $u^*(\cdot)$ *optimal* (for x_0), if $J_N(x_0, u^*(\cdot)) = V_N(x_0)$.

The corresponding MPC scheme then reads as follows (for much more detailed expositions we refer to Rawlings et al. (2017); Grüne and Pannek (2017)):

Given an initial condition $x_{MPC}(0) := \hat{x}_0 \in \mathbb{X}$ and an optimisation horizon $N \in \mathbb{N}$, for $n = 0, 1, 2, \dots$ we perform the following steps:

(1) Let $x_0 := x_{MPC}(n)$ denote the current state of the system.
(2) Solve the finite horizon optimal control problem (3) in order to obtain the optimal control sequence $u^*(\cdot)$.
(3) Apply the first element of the optimal control sequence $u^*(\cdot)$ as a feedback control value until the next time instant, i.e., set $u_{MPC}(n) := u^*(0)$ and $x_{MPC}(n+1) := f(x_{MPC}(n), u^*(0))$.
(4) Set $n := n + 1$ and go to Step 1.

Here, we consider general cost functions ℓ that do not need to have any a priori structure. This setting is typically termed *economic* MPC in the literature, although *general* MPC might be a more appropriate name.

When dealing with MPC, some of the central questions are:

- **Stability:** Does the MPC closed-loop solution exhibit stable behaviour?

- **Optimality:** Does the MPC closed-loop solution enjoy (approximate) optimality properties?
- **Feasibility:** Does the MPC closed-loop solution maintain the constraints?

As we will explain in the next section, a suitable dissipativity concept helps to give positive answers to all questions.

3. STRICT DISSIPATIVITY

The appropriate dissipativity concept is the following strict dissipativity notion. In this extended abstract we limit ourselves to strict dissipativity at an equilibrium $(x^e, u^e) \in \mathbb{Y}$ (i.e., $f(x^e, u^e) = x^e$). Extensions to periodic and general time-varying trajectories are possible and will be briefly explained in the talk.

Definition 3.1. The optimal control problem is called *strictly dissipative* at an equilibrium (x^e, u^e), if there exists a *storage function* $\lambda : \mathbb{X} \to \mathbb{R}$, bounded from below, and a function[1] $\alpha \in \mathcal{K}_\infty$ such that the inequality

$$\lambda(f(x, u)) \leq \lambda(x) + \ell(x, u) - \ell(x^e, u^e) - \alpha(\|x - x^e\|)$$

holds for all $(x, u) \in \mathbb{Y}$ with $f(x, u) \in \mathbb{X}$. Here, the function $s(x, u) = \ell(x, u) - \ell(x^e, u^e)$ is called the supply rate.

The optimal control problem is called *dissipative* if the above inequality holds with $\alpha \equiv 0$.

It follows immediately that if (not necessarily strict) dissipativity holds, then (x^e, u^e) is an optimal equilibrium, in the sense that $\ell(x^e, u^e) \leq \ell(\tilde{x}, \tilde{u})$ for all other equilibria $(\tilde{x}, \tilde{u}) \in \mathbb{Y}$.

The dissipativity notion for control systems was introduced by Willems (1972) in continuous time, the discrete time version used here is due to Byrnes and Lin (1994). It is interesting that *strict* dissipativity has not played a significant role in the literature until quite recently. The reason is that in the past the specific form of the suppy function often did not play a role. In this case, any dissipative system is also strictly dissipative; it suffices to replace $s(x, u)$ by $s(x, u) + \alpha(\|x - x^e\|)$. However, if the supply function s is linked to the cost function of the optimal control problem as in Definition 1, then it is not possible to modify it. In this sense, the application to MPC and optimal control are probably the main motivation for studying *strict* dissipativity.

4. STABILITY AND AVERAGED OPTIMALITY

The observation that dissipativity is beneficial for MPC was first made in Diehl et al. (2011), where it was observed that strict duality — which is nothing but strict dissipativity with a linear storage function — implies asymptotic stability of the optimal equilibrium for the MPC closed-loop under appropriate terminal conditions. This paper already contains the key idea of all dissipativity-based MPC stability results, namely the fact that the optimal value function of the optimal control problem with rotated cost

[1] As usual, we define $\mathcal{K}_\infty$ to be the space of continuous functions $\alpha : [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$ and α is strictly increasing to ∞.

$$\tilde{\ell}(x, u) = \ell(x, u) - \ell(x^e, u^e) + \lambda(x) - \lambda(f(x, u))$$

can be used as a Lyapunov function for the closed loop. The observation that this construction can be extended without additional effort from strict duality to strict dissipativity was then made in Angeli and Rawlings (2010).

The decisive contribution of the terminal condition in these papers lies in the fact that under this condition the optimal trajectories of the optimal control problems with cost ℓ and $\tilde{\ell}$, respectively, coincide. The properties of the terminal conditions needed for this were given in Amrit et al. (2011) and a special case was already used earlier in Angeli et al. (2009) in order to prove average optimality of the MPC closed-loop, i.e., that

$$\overline{J}_\infty(\hat{x}_0, u_{MPC}(\cdot)) = \inf_{u \text{ admissible}} \overline{J}_\infty(\hat{x}_0, u).$$

for $\overline{J}_\infty(x_0, u) := \lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell(x(k), u(k))$. We remark that, in contrast to most other results discussed here, for this proof strict dissipativity is not needed. However, it needs optimal operation of the system at the equilibrium (x^e, u^e), which under a controllability condition implies (non strict) dissipativity, see Müller (2014).

The fact that the optimal trajectories with cost ℓ and $\tilde{\ell}$ coincide is no longer the case for MPC without terminal conditions. However, as first observed in Grüne (2013), then refined in Grüne and Stieler (2014) and later streamlined in Chapter 8 of Grüne and Pannek (2017), the solutions are still very similar up to a certain time P. The reason for this is the so-called *turnpike property* in optimal control, which demands that the optimal trajectory most of the time stays near the optimal equilibrium. As noted in Grüne (2013), this property is implied by strict dissipativity under a reachability condition (conceptually similar results are much older and can be found, e.g., in Carlson et al. (1991)). Besides the possibility of building a Lyapunov function, its implication of the turnpike property is the second important feature of strict dissipativity.

As a consequence of this similarity, without terminal conditions we can still conclude near average optimality, i.e.,

$$\overline{J}_\infty(\hat{x}_0, u_{MPC}(\cdot)) = \inf_{u \text{ admissible}} \overline{J}_\infty(\hat{x}_0, u) + \varepsilon(N)$$

with $\varepsilon(N) \to 0$ as $N \to \infty$, and practical asymptotic stability of the closed loop, i.e., asymptotically stable behaviour outside a small neighbourhood of x^e, whose size also tends to 0 as N tends to infinity. This is due to the fact that the optimal value function for cost $\tilde{\ell}$ is still an approximate Lyapunov function for the MPC closed loop. These two properties hold provided the optimal value functions for different time horizons satisfy a uniform continuity condition at the optimal equilibrium x^e, which is needed in order to avoid that the small differences in the optimal trajectories cause large differences in the closed-loop behaviour.

5. TRANSIENT OPTIMALITY

While average optimality is a good measure to assess the performance of trajectories on very long time horizons, it does not tell much on short horizons. The reason is that a large cost on a short horizon contributes only very little to the average over a long horizon. Hence, a low average cost on a very long horizon does not allow for any conclusions about the cost on short horizons of the same trajectory. To this end, the concept of *transient optimality* is useful. Recall that under the strict dissipativity condition the closed-loop solutions converge to x^e (with appropriate terminal conditions) or to a small neighbourhood thereof (without terminal conditions). Hence, if we fix a sufficiently large time $K \in \mathbb{N}$ (that may be much larger than N), then we can find a small $\varepsilon > 0$ such that $\|x_{MPC}(n) - x^e\| \leq \varepsilon$ for all $n \geq K$. We can now compare the cost of this trajectory up to time K, i.e.,

$$J_K(\hat{x}_0, u_{MPC}(\cdot))$$

with the cost of all other trajectories that also end up in an ε-neighbourhood of x^e, i.e., with $V_K^{tr}(x_0) :=$

$$\inf\{J_K(x_0, u(\cdot)) \,|\, u(\cdot) \text{ admissible}, \|x(K) - x^e\| \leq \varepsilon\}.$$

It turns out that there exist two functions $\varepsilon_1(N), \varepsilon_2(K) \to 0$ as $N, K \to \infty$, such that

$$J_K(\hat{x}_0, u_{MPC}(\cdot)) \leq V_K^{tr}(\hat{x}_0) + \varepsilon_1(N) + \varepsilon_2(K)$$

in the case with terminal conditions and

$$J_K(\hat{x}_0, u_{MPC}(\cdot)) \leq V_K^{tr}(\hat{x}_0) + K\varepsilon_1(N) + \varepsilon_2(K)$$

in the case without terminal conditions. The former was proved in Grüne and Panin (2015) and the latter in Grüne and Stieler (2014); a unified treatment of both cases was later given in (Grüne and Pannek, 2017, Chapter 8).

6. FEASIBILITY

In all statements so far we have tacitly assumed that the solution $x_{MPC}(n)$ exists for all $n \geq 0$. However, this requires that in each sampling instance in Step (2) of the MPC scheme there exists an admissible control $u(\cdot)$ for the initial condition $x_0 = x_{MPC}(n)$. In this case, we call $x_0 = x_{MPC}(n)$ *feasible* and the question is thus whether $x_{MPC}(n)$ is feasible for all $n \geq 0$ In case of MPC with terminal conditions, feasibility for $x_{MPC}(n)$ follows if $x_{MPC}(n-1)$ is feasible — a property called *recursive feasibility* — provided the terminal constrained $\mathbb{X}_f$ is viable, i.e., for each $x \in \mathbb{X}_f$ there is $u \in U$ with $(x, u) \in \mathbb{Y}$ and $f(x, u) \in \mathbb{X}_f$, see, e.g., Mayne et al. (2000). This procedure and the related proofs are completely unrelated to dissipativity.

However, in the absence of terminal conditions, strict dissipativity or, more precisely, the turnpike property again play an important role. If we assume that the optimal equilibrium x^e lies in the interior of the state constraint set $\mathbb{X}$, then for all sufficiently large horizons N the turnpike property implies feasibility for all points that lies on the part of the optimal trajectory that approaches the turnpike. From this observation, it is then possible to conclude recursive feasibility, see Faulwasser and Bonvin (2015); Faulwasser et al. (2018).

7. CONCLUSION AND RECENT DEVELOPMENTS

Strict dissipativity allows to conclude a variety of desirable properties for the closed-loop system generated by MPC schemes with general cost functions. The two decisive features of strict dissipativity in the context of MPC are (i) that it allows to build a Lyapunov function for the closed-loop based on an optimal control problem with cost $\tilde{\ell}$ and (ii) that it implies the turnpike property.

This has motivated extensive studies about the nature of strict dissipativity. A very interesting connection for linear quadratic problems is that strict dissipativity is closely related to detectability properties, see Grüne and Guglielmi (2018), which in turn are again closely linked to the turnpike property in a very general infinite-dimensional evolution equation setting, see Grüne et al. (2019, 2020). This relation will also be explained in the talk.

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