BOUNDS FOR THE DIAMETER OF THE WEIGHT POLYTOPE

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ABSTRACT. A weighted game or a threshold function in general admits different weighted representations even if the sum of non-negative weights is fixed to one. Here we study bounds for the diameter of the corresponding weight polytope. It turns out that the diameter can be upper bounded in terms of the maximum weight and the quota or threshold. We apply those results to approximation results between power distributions, given by power indices, and weights.

1. INTRODUCTION

Consider a stock corporation whose shares are hold by three major stockholders owning 35%, 34%, and 17%, respectively. The remaining 14% are widely spread. Assuming that decisions are made by a simple majority rule, all three major stockholders have equal influence on the company's decisions, while the private shareholders have no say. To be more precise, any two major stockholders can adopt a proposal, while the private shareholders together with an arbitrary major stockholder need further affirmation. Such decision environments can be captured by means of weighted voting games. Formally, a weighted (voting) game consists of a set of players or voters $N = \{1, \dots, n\}$, a vector of non-negative weights $w = (w_1, \dots, w_n)$, and a positive quota q. A proposal is accepted if and only if the weight sum of its supporters meets or exceeds the quota.

Committees that decide between two alternatives have received wide attention. Von Neumann and Morgenstern introduced the notion of simple games, which is a super class of weighted games, in [21]. Examples of decision-making bodies that can be modeled as weighted games are the US Electoral College, the Council of the European Union, the UN Security Council, the International Monetary Fund or the Governing Council of the European Central Bank. Many applications seek to evaluate players' influence or power in simple or weighted games, see, e.g., [14]. The initial example illustrates that shares or weights can be a poor proxy for the distribution of power. Using the taxicab metric, i.e., the $\|\cdot\|_1$-distance, the corresponding distance between shares and relative power is $\left|0.35 - \frac{1}{3}\right| + \left|0.34 - \frac{1}{3}\right| + \left|0.17 - \frac{1}{3}\right| + |0.14 - 0| \approx$32.67%. If the weights add up to one, then we speak of relative or normalized weights. The insight that the power distribution differs from relative weights, triggered the invention of so-called power indices like the Shapley-Shubik index [19], the Penrose-Banzhaf index [2], or the nucleolus [18]. Due to the combinatorial nature of most of those power indices, qualitative assessments are technically demanding and large numbers of involved parties cause computational challenges [3].

One reason for the difference between relative weights and power is that a weighted game permits different representations. If there are two normalized representations whose weight vectors are at large distance then at least one of the relative weight vectors also has a large distance to the power distribution. So, here we study bounds for the diameter of the weight polytope, i.e., bounds for the maximal distance between two normalized vectors of the same weighted game. We will study those bounds in terms of the number of players, the relative quota, and the maximum relative weight in a given representation of the game.

Each weighted game, also called threshold function in threshold logic, admits a representation with integer weights. Bounds for the necessary magnitude of integer weighted are studied in the literature, see e.g. [1] and the references therein.

The remaining part of the paper is structured as follows. In Section 2 we give the necessary definitions for simple games, weighted games and the weight polytope. Worst case lower bounds on the diameter of the weight polytope are given in Section 3 and upper bounds are given in Section 4. Applications to approximation results for power indices are given in Section 5 before we draw a brief conclusion in Section 6. Some lengthy or more technical proofs are moved to an appendix.

2. THE WEIGHT POLYTOPE OF A WEIGHTED GAME

For a positive integer n let $N = \{1, \dots, n\}$ be the set of players. A *simple game* is a mapping $v\colon 2^N \to \{0, 1\}$ from the subsets of N to binary outcomes satisfying $v(\emptyset) = 0$, $v(N) = 1$, and $v(S) \le v(T)$ for all $\emptyset \subseteq S \subseteq T \subseteq N$. The interpretation in the context of binary voting systems is as follows. A subset $S \subseteq N$, also called coalition, is considered as the set of players that are in favor of a proposal, i.e., which vote “yes”. If $v(S) = 1$ we call coalition S winning and losing otherwise. By $\mathcal{W}(v)$ we denote the set of winning coalitions and by $\mathcal{L}(v)$ we denote the set of losing coalitions of v. If coalition S is winning but each proper subset is losing, then we call S minimal winning. Similarly, if S is losing but each proper superset of S is winning, then we call S maximal losing. By $\mathcal{W}^m(v)$ we denote the set of minimal winning and by $\mathcal{L}^m(v)$ we denote the set of maximal losing coalitions. $v(S)$ encodes the group decision, i.e., $v(S) = 1$ if the proposal is accepted and $v(S) = 0$ otherwise. So, these assumptions for a simple game are quite natural for a voting system with binary options in the input and output domain. The dual v^d of a simple game v is defined via $v^d(S) = v(N) - v(N\backslash S) = 1 - v(N\backslash S)$ for all $S \subseteq N$ and is a simple game itself. If $v(S) = v(S \cup \{i\})$ for all $S \subseteq N$, then we call player i a null player. Player i is a passer if $v(\{i\}) = 1$. Two players i and j are equivalent if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N\backslash\{i, j\}$.

A simple game v is called *weighted* if there exist weights $w \in \mathbb{R}^n_{\ge 0}$ and a quota $q \in \mathbb{R}_{>0}$ such that $v(S) = 1$ if and only if $w(S) := \sum_{i \in S} w_i \ge q$. From the conditions of a simple game we conclude $0 < q \le w(N)$. If $w(N) = 1$ we speak of normalized or relative weights, where $0 < q \le 1$. We denote the respective game by $v = [q; w]$ and refer to the pair $(q; w)$ as a weighted representation, i.e., we can have $[q; w] = [q'; w']$ but $(q; w) \neq (q'; w')$. The example from the introduction can, e.g., be represented by

$(51\%; 35\%, 34\%, 17\%, 14\%)$, $\left(\frac{1}{2}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$, or $(6; 4, 3, 3, 1)$, where the fourth player mimics the private shareholders.

Lemma 2.1. *If $(q; w)$ is a normalized representation of a weighted game v, then $(1 - q + \varepsilon; w)$ is a normalized representation of the dual game v^d for each $0 < \varepsilon < \min\{q - w(S) \mid S \in \mathcal{L}(v)\}$.*

Proof. For each losing coalition S of v^d the coalition $N \backslash S$ is winning in v, so that $w(N \backslash S) = 1 - w(S) \geq q$ and $w(S) \leq 1 - q < 1 - q + \varepsilon$. Now let S be a winning coalition of v^d, so that $N \backslash S$ is losing in v and $\varepsilon < q - w(N \backslash S) = q - 1 + w(S)$, which is equivalent to $w(S) > 1 - q + \varepsilon$. Since $\emptyset$ is a losing coalition in v we have $\varepsilon < q - w(\emptyset) = q$, so that $1 - q + \varepsilon < 1$. □

Note that $\min\{q - w(S) \mid S \in \mathcal{L}(v)\} > 0$.

Given a weighted game v, we call a weight vector $w \in \mathbb{R}_{\geq 0}$ feasible if there exists a quota $q \in \mathbb{R}_{>0}$ satisfying $v = [q; w]$. Obviously, such a quota exists iff the largest weight of a losing coalition is strictly smaller than the smallest weight of a winning coalition. Thus, c.f. [9, Lemma 3.2], the set of feasible normalized weight vectors is given by

$$\begin{aligned} & \left\{w \in \mathbb{R}^n_{\geq 0} \mid w(N) = 1, v(S) > v(T) \quad \forall S \in \mathcal{W}(v), T \in \mathcal{L}(v)\right\} \\ = & \left\{w \in \mathbb{R}^n_{\geq 0} \mid w(N) = 1, v(S) > v(T) \quad \forall S \in \mathcal{W}^m(v), T \in \mathcal{L}^m(v)\right\}. \end{aligned}$$

Note that these sets only depend on the game v and are non-empty for weighted games. Due to the involved strict inequalities we have to consider their closure in order to obtain polytopes.

Definition 2.2. For a weighted game v we define the weight polytope of v by

$$\mathsf{W}(v) = \left\{w \in \mathbb{R}^n_{\geq 0} \mid w(N) = 1, v(S) \geq v(T) \quad \forall S \in \mathcal{W}^m(v), T \in \mathcal{L}^m(v)\right\}$$

and call

$$\operatorname{diam}(\mathsf{W}(v)) = \max\left\{\|w - w'\|_1 \mid w, w' \in \mathsf{W}(v)\right\}$$

its diameter, where $\|x\|_1 := \sum_i |x_i|$.

As an example we consider the weighted game $v = [2; 1, 1, 1]$. For $w \in \mathsf{W}(v)$ the conditions $w(S) \geq w(T)$ for all $S \in \mathcal{W}^m(v)$ and all $T \in \mathcal{L}^m(v)$ read $w_1 + w_2 \geq w_3$, $w_1 + w_3 \geq w_2$, and $w_2 + w_3 \geq w_1$. The normalization $w(N) = 1$ can be used to eliminated w_3 via $w_3 = 1 - w_1 - w_2$. Finally, respecting $w \in \mathbf{R}^3_{\geq 0}$ gives

$$\mathsf{W}(v) = \left\{(w_1, w_2, 1 - w_1 - w_2) \mid 0 \leq w_1 \leq \frac{1}{2}, 0 \leq w_2 \leq \frac{1}{2}, w_1 + w_2 \geq \frac{1}{2}\right\}.$$

Since $w := \left(\frac{1}{2}, \frac{1}{2}, 0\right) \in \mathsf{W}(v)$ and $w' := \left(\frac{1}{2}, 0, \frac{1}{2}\right) \in \mathsf{W}(v)$, we have

$$\operatorname{diam}(\mathsf{W}(v)) \geq \|w - w'\|_1 = 1.$$

Indeed, it can be shown that $\|\tilde{w} - \hat{w}\|_1 \leq 1$ for all $\tilde{w}, \hat{w} \in \mathsf{W}(v)$, so that $\operatorname{diam}(\mathsf{W}(v)) = 1$ in our example.

For a simple game v the set $\mathsf{W}(v)$ is non-empty iff v is a so-called roughly weighted game, which is a relaxation of a weighted game. While also for a weighted game v not any element in $\mathsf{W}(v)$ can be completed by a suitable quota $q \in (0,1]$ to a normalized representation $(q;w)$, Definition 2.2 makes sense nevertheless since $\dim(\mathsf{W}(v)) = n-1$, see e.g. [9, Lemma 3.4], i.e., the weight polytope is full-dimensional. More concretely, for each weighted game v and each $\varepsilon \in \mathbb{R}_{>0}$ there are $w, w' \in \mathsf{W}(v)$ and $q, q' \in (0,1]$ such that $v = [q;w] = [q';w']$ and

$$\operatorname{diam}(\mathsf{W}(v)) - \varepsilon \le \|w - w'\|_1 \le \operatorname{diam}(\mathsf{W}(v)).$$

Given the indicated linear programming formulation, $\operatorname{diam}(\mathsf{W}(v))$ can be computed in polynomial time (in terms of the number of minimal winning and maximal losing coalitions). The same is true if we replace $\|\cdot\|_1$ by the maximum norm $\|x\|_\infty = \max\{x_i \mid 1 \le i \le n\}$ for $x \in \mathbb{R}^n$. We denote the corresponding diameter by $\operatorname{diam}^\infty(\mathsf{W}(v))$. For an arbitrary p-norm $\|x\|_p := \left(\sum_i x_i^p\right)^{1/p}$ with $1 < p < \infty$, we can obtain lower and upper bounds via $\|x\|_\infty \le \|x\|_p \le \|x\|_1$, so that we restrict ourselves to the corresponding two distance functions. The bound $\|x\|_\infty \le \|x\|_1$ can be slightly improved in our context.

Lemma 2.3. *For $w, w' \in \mathbb{R}^n_{\ge 0}$ with $\|w\|_1 = \|w'\|_1 = 1$, we have $\|w - w'\|_\infty \le \frac{1}{2}\|w - w'\|_1$.*

Proof. With $S := \{1 \le i \le n \mid w_i \le w'_i\}$ and $A := \sum_{i \in S}(w'_i - w_i)$, $B := \sum_{i \in N\setminus S}(w_i - w'_i)$, where $N = \{1,\dots,n\}$, we have $A - B = 0$ since $\|w\|_1 = \|w'\|_1$ and $w, w' \in \mathbb{R}^n_{\ge 0}$. Thus, $\|w - w'\|_1 = 2A$ and $\|w - w'\|_\infty \le \max\{A, B\} = A$. □ □

What can be said about $\operatorname{diam}(\mathsf{W}(v))$ and $\operatorname{diam}^\infty(\mathsf{W}(v))$ in general without solving the specific linear programs? Obviously, we have $\operatorname{diam}(\mathsf{W}(v)) \le 2$ and $\operatorname{diam}^\infty(\mathsf{W}(v)) \le 1$. These bounds are asymptotically attained for $n \ge 2$ and $v = [n;(1,\dots,1)]$, i.e., for any $0 < \varepsilon < \frac{1}{n}$ we can set $w = (1-(n-1)\cdot\varepsilon, \dots, \varepsilon)$, $w' = (\varepsilon, \dots, \varepsilon, 1-(n-1)\cdot\varepsilon)$, $q = q' = 1 - \varepsilon$ so that $v = [q;w] = [q';w']$, $\|w - w'\|_1 = 2\cdot(1 - n\varepsilon)$, and $\|w - w'\|_\infty = 1 - 2\varepsilon$. In other words, $(1,0,\dots,0), (0,\dots,0,1) \in \mathsf{W}([n;1,\dots,1])$ attain the desired distances. For the weighted game v with $n = 1$ players we have $\operatorname{diam}(\mathsf{W}(v)) = \operatorname{diam}^\infty(\mathsf{W}(v)) = 0$ since $\mathsf{W}(v) = \{(1)\}$.

In order to obtain tighter bounds for the diameter of the weight polytope we need more information besides the number of players. Given an exemplary normalized representation $(q;w)$, we study *key parameters* like the relative quota $q \in (0,1]$, i.e., the quota of a normalized representation, or the maximum relative weight $\Delta(w) := \|w\|_\infty \in (0,1]$, where we write Δ whenever w is clear from the context. Besides this, also more sophisticated invariants of weight vectors have been studied in applications. The so-called *Laakso-Taagepera index* a.k.a. *Herfindahl-Hirschman index*, c.f. [13], is used in Industrial Organization to measure the concentration of firms in a market, see, e.g., [4], and given by

$$L(w) = \left(\sum_{i=1}^{n} w_i\right)^2 / \sum_{i=1}^{n} w_i^2.$$

for $w \in \mathbb{R}^n_{\geq 0}$ with $w \neq 0$. In general we have $1 \leq L(w) \leq n$. If the weight vector w is normalized, then the formula simplifies to $L(w) = 1/\sum_{i=1}^{n} w_i^2$. Under the name "effective number of parties" the index is widely used in political science to measure party fragmentation, see, e.g., [12]. However, we observe the following relations between the maximum relative weight $\Delta = \Delta(w)$ and the Laakso-Taagepera index $L(w)$:

Lemma 2.4. *For $w \in \mathbb{R}^n_{\geq 0}$ with $\|w\|_1 = 1$, we have*

$$\frac{1}{\Delta} \leq \frac{1}{\Delta\left(1 - \alpha(1-\alpha)\Delta\right)} \leq L(w) \leq \frac{1}{\Delta^2 + \frac{(1-\Delta)^2}{n-1}} \leq \frac{1}{\Delta^2}$$

for $n \geq 2$, where $\alpha := \frac{1}{\Delta} - \left\lfloor \frac{1}{\Delta} \right\rfloor \in [0, 1)$. If $n = 1$, then $\Delta = L(w) = 1$.

Proof. Optimize $\sum\limits_{i=1}^{n} w_i^2$ with respect to the constraints $w \in \mathbb{R}^n$, $\|w\|_1 = 1$, and $\Delta(w) = \Delta$, see the appendix for the technical details. □

So, any lower or upper bound involving $L(w)$ can be replaced by a bound involving Δ instead. Since Δ has nicer analytical properties and requires less information on w, we stick to Δ in the following. We remark that there are similar inequalities for other indices measuring market concentration. Upper bounds on $\operatorname{diam}(\mathsf{W}(v))$, in terms of n, q, and Δ, will be given in Section 4 and worst case lower bounds for $\operatorname{diam}(\mathsf{W}(v))$ and $\operatorname{diam}^{\infty}(\mathsf{W}(v))$ will be given in Section 3.

3. Worst case lower bounds for the diameter of the weight polytope

For integers $1 \leq k \leq s$ and $t \geq 0$ we denote by $v_{k,s,t}$ the weighted game with s players of weight one, t players of weight zero, and a quota of k, i.e., $v_{k,s,t} = [k; 1, \ldots, 1, 0, \ldots 0]$. Players $1, \ldots, s$ are pairwise equivalent as well as players $s + 1, \ldots s + t$, which are null players. If $k = 1$, then each player $1 \leq i \leq s$ is a passer. First we study lower bounds for the diameter of those weighted games.

Lemma 3.1. *For integers $1 \leq k < s$ and $t \geq 0$ we have*

$$\operatorname{diam}(\mathsf{W}(v_{k,s,t})) \geq \max\left\{\tfrac{1}{10k}, \tfrac{1}{10(s-k)}\right\} \quad \textit{and} \quad \operatorname{diam}^{\infty}(\mathsf{W}(v_{k,s,t})) \geq \tfrac{1}{s}.$$

Proof. Let $S = \{1, \ldots s\}$ and $T = \{s + 1, \ldots, s + t\}$. We start with the lower bound for $\operatorname{diam}(\mathsf{W}(v_{k,s,t}))$. If s is even, then we set $S_1 = \{1, \ldots, s/2\}$, $S_0 = \emptyset$, and $S_{-1} = \{s/2 + 1, \ldots, s\}$. If s is odd, then we set $S_1 = \{1, \ldots, (s-1)/2\}$, $S_0 = \{(s + 1)/2\}$, and $S_{-1} = \{(s + 3)/2, \ldots, s\}$. Let $0 \leq \gamma \leq \frac{1}{s}$ be a parameter that we specify latter depending on further case differentiations. With this, we set $w_i = \frac{1}{s} + \gamma$ for all $i \in S_1$, $w_i = \frac{1}{s}$ for all $i \in S_0$, $w_i = \frac{1}{s} - \gamma$ for all $i \in S_{-1}$, $w_i = \bar{w}_i = 0$ for all $i \in T$, and

$\bar{w}_i = w_{s+1-i}$ for all $i \in S$. It is easily verified that $w \in \mathbb{R}_{\ge 0}^{s+t}$ and $\|w\|_1 = 1$. In order to conclude $w \in \mathsf{W}(v_{k,s,t})$ it suffices to check $w(U) + w(T) = w(U) \le w(V)$ for all $U, V \subseteq S$ with $|U| = k-1$ and $|V| = k$. Since $\bar{w}$ is a permutation of w, $w \in \mathsf{W}(v_{k,s,t})$ implies $\bar{w} \in \mathsf{W}(v_{k,s,t})$, so that

$$\operatorname{diam}(\mathsf{W}(v_{k,s,t})) \ge \|w - \bar{w}\|_1 = 2\gamma \cdot |S_1| = 2\gamma \cdot \left\lfloor \frac{s}{2} \right\rfloor \ge \frac{\gamma s}{2},$$

where we have used $s \ge 2$ for the last inequality.

If $k \le \frac{s+1}{2}$ we set $\gamma = \frac{1}{s(2k-1)} \le \frac{1}{s}$. For $U, V \subseteq S$ with $|U| = k-1$ and $|V| = k$ we have $w(U) \le (k-1) \cdot \left(\frac{1}{s} + \gamma\right)$ and $w(V) \ge k \cdot \left(\frac{1}{s} - \gamma\right)$ so that $w(U) \le w(V)$ and $\operatorname{diam}(\mathsf{W}(v_{k,s,t})) \ge \frac{1}{4k} \ge \frac{1}{10(s-k)}$.

If $k \ge \frac{s+2}{2}$ we set $\gamma = \frac{1}{s(2s+3-2k)} \le \frac{1}{s}$. For $U, V \subseteq S$ with $|U| = k-1$ and $|V| = k$ we have

$$w(U) \le \frac{s}{2} \cdot \left(\frac{1}{s} + \gamma\right) + \frac{1}{s} + \left(k - 1 - \frac{s}{2} - 1\right) \cdot \left(\frac{1}{s} - \gamma\right)$$

and

$$w(V) \ge \frac{s}{2} \cdot \left(\frac{1}{s} - \gamma\right) + \frac{1}{s} + \left(k - \frac{s}{2} - 1\right) \cdot \left(\frac{1}{s} + \gamma\right)$$

so that $w(U) \le w(V)$ and

$$\operatorname{diam}(\mathsf{W}(v_{k,s,t})) \ge \frac{\gamma s}{2} \ge \frac{1}{2(2(s-k)+3)} \overset{s-k\ge 1}{\ge} \frac{1}{10(s-k)} \ge \frac{1}{10k}.$$

Next we consider the lower bound for $\operatorname{diam}^\infty(\mathsf{W}(v_{k,s,t}))$. We set $\gamma = \frac{1}{2s}$, $w_1 = \bar{w}_2 = \frac{1}{s} + \gamma$, $w_2 = \bar{w}_1 = \frac{1}{s} - \gamma$, $w_i = \bar{w}_i = \frac{1}{s}$ for all $3 \le i \le s$, and $w_i = \bar{w}_i = 0$ for all $i \in T$. It is easily verified that $w \in \mathbb{R}_{\ge 0}^{s+t}$ and $\|w\|_1 = 1$. In order to conclude $w \in \mathsf{W}(v_{k,s,t})$ it suffices to check $w(U) + w(T) = w(U) \le w(V)$ for all $U, V \subseteq S$ with $|U| = k-1$ and $|V| = k$. The latter follows from $w(U) \le \frac{k-1}{s} + \gamma$ and $w(V) \ge \frac{k}{s} - \gamma$. Since $\bar{w}$ is a permutation of w, we also have $\bar{w} \in \mathsf{W}(v_{k,s,t})$, so that

$$\operatorname{diam}^\infty(\mathsf{W}(v_{k,s,t})) \ge \|w - \bar{w}\|_\infty = 2\gamma = \frac{1}{s}.$$

□

For the excluded cases $k = s$ we have:

Lemma 3.2. *For integers $s \ge 1$ and $t \ge 0$ with $t + s \ge 2$ we have*

$$\operatorname{diam}(\mathsf{W}(v_{s,s,t})) \ge \frac{2}{3} \quad \textit{and} \quad \operatorname{diam}^\infty(\mathsf{W}(v_{s,s,t})) \ge \frac{1}{3}.$$

Proof. Let $0 < \varepsilon < \frac{1}{s}$ be arbitrary. If $s \ge 2$ we choose $w_1 = \bar{w}_s = 1 - (s-1)\varepsilon$, $w_i = \bar{w}_{s+1-i} = \varepsilon$ for all $2 \le i \le s$, and $w_i = \bar{w}_i = 0$ for all $s+1 \le i \le s+t$. We can easily check $w, \bar{w} \in \mathsf{W}(v_{s,s,t})$. Since $\|w - \bar{w}\|_1 = 2 \cdot (1 - s\varepsilon)$ and $\|w - \bar{w}\|_\infty = 1 - s\varepsilon$ we have $\operatorname{diam}(\mathsf{W}(v_{s,s,t})) \ge \frac{2}{3}$ and $\operatorname{diam}^\infty(\mathsf{W}(v_{s,s,t})) \ge \frac{1}{3}$ using $\varepsilon < \frac{2}{3s}$.

If $s = 1$ then we consider $w = (1, 0, 0, \ldots, 0) \in \mathsf{W}(v_{1,1,t})$ and $\bar{w} = (\frac{2}{3}, \frac{1}{3}, 0, \ldots, 0) \in \mathsf{W}(v_{1,1,t})$. Thus, $\operatorname{diam}(\mathsf{W}(v_{1,1,t})) \geq \|w - \bar{w}\|_1 = \frac{2}{3}$ and $\operatorname{diam}^\infty(\mathsf{W}(v_{1,1,t})) \geq \|w - \bar{w}\|_\infty = \frac{1}{3}$. □

Next we show that for a given relative quota $q \in (0, 1]$ or a given maximum relative weight $\Delta \in (0, 1]$ we can construct a weighted game v, for any suitably large number of players, with matching representation such that $\operatorname{diam}(\mathsf{W}(v))$ is lower bounded by a positive constant independent of q or Δ. Actually, we construct two representations of the same weighted game and give a lower bound for the distance between the two normalized weight vectors.

Lemma 3.3. *For each $q \in (0, 1]$ there exists a weighted game $v = [q; w] = [q; \bar{w}]$ with $n \geq 2$ players, where $w, \bar{w} \in \mathbb{R}^n_{\geq 0}$, and $\|w\|_1 = \|\bar{w}\|_1 = 1$, such that $\|w - \bar{w}\|_\infty \geq \frac{1}{3}$ and $\|w - \bar{w}\|_1 \geq \frac{2}{3}$.*

Proof. We give general constructions for different ranges of q:

- $\frac{2}{3} < q \leq 1$: $w = \left(\frac{2}{3}, \frac{1}{3}, 0, \ldots, 0\right)$, $\bar{w} = \left(\frac{1}{3}, \frac{2}{3}, 0, \ldots, 0\right)$;
- $\frac{1}{3} < q \leq \frac{2}{3}$: $w = \left(\frac{2}{3}, \frac{1}{3}, 0, \ldots, 0\right)$, $\bar{w} = (1, 0, \ldots, 0)$;
- $0 < q \leq \frac{1}{3}$: $w = \left(\frac{2}{3}, \frac{1}{3}, 0, \ldots, 0\right)$, $\bar{w} = \left(\frac{1}{3}, \frac{2}{3}, 0, \ldots, 0\right)$.

□

Lemma 3.4. *Let $\Delta \in (0, 1]$ and $n \geq \frac{1}{\Delta} + 1$. There exist $w, \bar{w} \in \mathbb{R}^n_{\geq 0}$, $q, \bar{q} \in (0, 1]$ with $\|w\|_1 = \|\bar{w}\|_1 = 1$, $\Delta(w) = \Delta$, $[q; w] = [\bar{q}, \bar{w}]$, and $\frac{1}{2} \cdot \|w - \bar{w}\|_1 \geq \|w - \bar{w}\|_\infty \geq \frac{1}{7}$.*

Proof. We set $s = \left\lfloor \frac{1}{\Delta} \right\rfloor \geq 1$ and $t = n - s \geq 1$, since $n \geq \frac{1}{\Delta} + 1 \geq s + 1$. For $w = (\Delta, \ldots, \Delta, 1 - s\Delta, 0, \ldots, 0) \in \mathbb{R}^n_{\geq 0}$, with s entries being equal to Δ, we have $\Delta(w) = \Delta$ and $[q; w] = v_{s,s,t}$ for $0 < q = s\Delta \leq 1$. Due to Lemma 3.2 we have $\operatorname{diam}^\infty(\mathsf{W}(v_{s,s,t})) \geq \frac{1}{3}$, so that the triangle inequality implies the existence of a vector $w' \in \mathsf{W}(v_{s,s,t})$ with $\|w - w'\|_\infty \geq \frac{1}{6}$. If w' is on the boundary of $\mathsf{W}(v_{s,s,t})$ we slightly perturb w' to $\bar{w}$ in the interior of $\mathsf{W}(v_{s,s,t})$ and complete it to a representation $(\bar{q}, \bar{w})$ with $\bar{q} \in (0, 1]$, $[q; w] = [\bar{q}, \bar{w}]$, and $\|w - \bar{w}\|_\infty \geq \frac{1}{7}$. The inequality $\frac{1}{2} \cdot \|w - \bar{w}\|_1 \geq |w - \bar{w}\|_\infty$ follows from Lemma 2.3. □

By a tailored construction we can obtain a slightly more general result:

Lemma 3.5. *For each $\Delta \in (0, 1)$ there exists a weighted game $v = [q; w] = [q; \bar{w}]$ with $n \geq \frac{4}{3\Delta} + 6$ players, where $q \in (0, 1)$, $w, \bar{w} \in \mathbb{R}^n_{\geq 0}$, $\Delta(w) = \Delta(\bar{w}) = \Delta$, and $\|w\|_1 = \|\bar{w}\|_1 = 1$, such that $\|w - \bar{w}\|_1 \geq \frac{2}{3}$ and $\|w - \bar{w}\|_\infty \geq \Delta/2$.*

Proof. If $\Delta \geq \frac{2}{3}$, we can consider a weighted game with two passers and $n - 2$ null players. One representation is given by $q = 1 - \Delta$ and $w = (\Delta, 1 - \Delta, 0, \ldots, 0)$. Of course we can swap the weights of the first two players and obtain a second representation given by quota q an weight vector $\bar{w} = (1 - \Delta, \Delta, 0, \ldots, 0)$. With this, we compute $\|w - \bar{w}\|_1 = 2 \cdot (2\Delta - 1) \geq \frac{2}{3}$ and $\|w - \bar{w}\|_\infty = 2\Delta - 1 \geq \Delta/2$.

If $0 < \Delta < \frac{2}{3}$, we define an integer $a := \left\lfloor\frac{2}{3\Delta}\right\rfloor \ge 1$ and consider a weighted game with $2a$ passers and $n-2a$ null players. One representation is given by $q = \Delta/2$, $w_{2i-1} = \Delta$, $w_{2i} = \Delta/2$ for $1 \le i \le a$, $w_{2a+1} = w_{2a+3} = w_{2a+5} = \frac{1}{3} - \frac{a\Delta}{2} \ge 0$, $w_{2a+2} = w_{2a+4} = w_{2a+6} = 0$, and $w_i = 0$ for all $2a+7 \le i \le n$. By assumption we have $n \ge \frac{4}{3\Delta} + 6 \ge 2a+6$ and the first $2a$ players are obviously passers. By checking $0 \le \frac{1}{3} - \frac{a\Delta}{2} < \frac{\Delta}{2}$ we conclude that the remaining players are null players and have a non-negative weight. By construction, the weights of the n players sum up to one. Changing the weights of player $2i-1$ and player $2i$ for $1 \le i \le a$ does not change the game so that we obtain a second representation with quota q and weights $\bar{w}_{2i} = \Delta$, $\bar{w}_{2i-1} = \Delta/2$ for $1 \le i \le a$, $\bar{w}_{2a+2} = \bar{w}_{2a+4} = \bar{w}_{2a+6} = \frac{1}{3} - \frac{a\Delta}{2} \ge 0$, $w_{2a+1} = w_{2a+3} = w_{2a+4} = \bar{w}_{2a+1} = \bar{w}_{2a+2} = \bar{w}_{2a+3} = 0$, and $\bar{w}_i = 0$ for all $2a+7 \le i \le n$. With this, we have $\|w-\bar{w}\|_1 = a\Delta + 2 - 3a\Delta = 2(1-a\Delta) \ge \frac{2}{3}$ and $\|w-\bar{w}\|_\infty = \Delta/2$. □

For each $w, \bar{w} \in \mathbb{R}^n_{\ge 0}$ with $\Delta(w) = \Delta(\bar{w})$, we obviously have $\|w-\bar{w}\|_\infty \le \Delta(w)$. So, a constant lower bound for the $\|\cdot\|_\infty$-distance can only exist if we slightly weaken the assumptions as done in Lemma 3.4.

In some applications only weighted games with a quota of at least one half are considered, which clashes with some of our constructions in the proofs of the previous lemmas. However, by considering the dual of a given weighted game we can turn a quota below one half to a quota above one half, see Lemma 2.1. So, instead of small quotas we get large quotas.

So, either knowing the relative quota or the maximum relative weight is not sufficient in order to deduce a non-constant upper bound on the diameter of the weight polytope for a suitably large number of players. However, as we will see in the next section, knowing the relative quota and the maximum relative weight is indeed sufficient for such an upper bound, see Theorem 4.4. Our next aim is to show that this upper bound is tight up to a constant.

Lemma 3.6. *For each $0 < q < 1$, $0 < \Delta \le 1$, and each integer $n \ge \frac{1}{\Delta} + 2$ there exist weight vectors $w, \bar{w} \in \mathbb{R}^n_{\ge 0}$ with $\|w\|_1 = \|\bar{w}\|_1 = 1$, $\Delta(w) = \Delta$ and a quota $0 < \bar{q} \le 1$ with $[q; w] = [\bar{q}; \bar{w}]$ such that*

$$\|w-\bar{w}\|_1 \ge \frac{1}{200} \cdot \min\left\{2, \frac{4\Delta}{\min\{q, 1-q\}}\right\}.$$

Under the same assumptions there exist weight vectors $w, \bar{w} \in \mathbb{R}^n_{\ge 0}$ with $\|w\|_1 = \|\bar{w}\|_1 = 1$, $\Delta(w) = \Delta$ and a quota $0 < \bar{q} \le 1$ with $[q; w] = [\bar{q}; \bar{w}]$ such that $\|w-\bar{w}\|_\infty \ge \frac{\Delta}{5}$.

Proof. We set $a = \left\lfloor\frac{1}{\Delta}\right\rfloor \ge 1$ and choose the unique integer b with $b\Delta < q$ and $(b+1)\Delta \ge q$. With this we set $k = b+1 \ge 1$ and $w = (\Delta, \dots, \Delta, 1-a\Delta, 0, \dots, 0)$, where $0 \le 1-a\Delta < \Delta$, so that $w \in \mathbb{R}^n_{\ge 0}$ and $\|w\|_1 = 1$. If $b\Delta + (1-a\Delta) < q$ we set $s = a$ and $s = a+1$ otherwise, so that $[q; w] = v_{k,s,n-s}$. Note that $n - s \ge 1$.

If $k = s$, then Lemma 3.2 gives $\operatorname{diam}(\mathsf{W}(v_{s,s,t})) \ge \frac{2}{3}$, so that the triangle inequality implies the existence of a vector $w' \in \mathsf{W}(v_{s,s,t})$ with $\|w-w'\|_1 \ge \frac{1}{3}$. If $k < s$, then

Lemma 3.1 gives $\operatorname{diam}(\mathsf{W}(v_{k,s,t})) \geq \max\left\{\frac{1}{10k}, \frac{1}{10(s-k)}\right\}$, so that the triangle inequality implies the existence of a vector $w' \in \mathsf{W}(v_{k,s,t})$ with

$$\|w - w'\|_1 \geq \max\left\{\frac{1}{20k}, \frac{1}{20(s-k)}\right\} = \frac{1}{20s} \cdot \frac{1}{\min\left\{\frac{k}{s}, \frac{s-k}{s}\right\}}.$$

In the following we make several case distinctions for the subcase $k < s$.

If $k = 1$ or $s - k = 1$, then $\|w - w'\|_1 \geq \frac{1}{20}$. In the following we assume $k \geq 2$ and $s - k \geq 2$. By construction we have $\frac{k}{2} \leq (k-1)\Delta < q$, $k\Delta \geq q$, and $(s-1)\Delta \leq 1$, so that $k < \frac{2q}{\Delta}$, $\frac{s-k}{2}\Delta \leq (s-1)\Delta - k\Delta \leq 1 - q$ and $s - k \leq \frac{2(1-q)}{\Delta}$.

If $k \leq s - k$, i.e., $2k \leq s$, then $q \leq \frac{1}{2}$ and

$$\|w - w'\|_1 \geq \frac{1}{20s} \cdot \frac{1}{\min\left\{\frac{k}{s}, \frac{s-k}{s}\right\}} = \frac{1}{20k} \geq \frac{1}{40} \cdot \frac{\Delta}{q} = \frac{1}{40} \cdot \frac{\Delta}{\min\{q, 1-q\}}.$$

If $k > s - k$, i.e., $2k > s$, then $q > \frac{1}{2}$ and

$$\|w - w'\|_1 \geq \frac{1}{20s} \cdot \frac{1}{\min\left\{\frac{k}{s}, \frac{s-k}{s}\right\}} = \frac{1}{20(s-k)} \geq \frac{1}{40} \cdot \frac{\Delta}{1-q} = \frac{1}{40} \cdot \frac{\Delta}{\min\{q, 1-q\}}.$$

Thus,

$$\|w - w'\|_1 \geq \frac{1}{160} \cdot \min\left\{2, \frac{4\Delta}{\min\{q, 1-q\}}\right\}$$

in all cases. If w' is on the boundary of $\mathsf{W}(v_{k,s,n-s})$, then we slightly perturb w' to $\bar{w}$ in the interior of $\mathsf{W}(v_{k,s,n-s})$ and choose a quota $\bar{q} \in (0, 1]$ such that $[\bar{q}; \bar{w}] = v_{k,s,n-s}$. This gives the statement for the $\|\cdot\|_1$-distance, if the pertubation is small enough to be covered by our decrease of the factor $\frac{1}{160}$ to $\frac{1}{200}$.

For the $\|\cdot\|_\infty$-distance we choose w with $[q; w] = v_{k,s,n-s}$ as above. If $k = s$, then Lemma 3.2 gives $\operatorname{diam}^\infty(\mathsf{W}(v_{s,s,t})) \geq \frac{1}{3}$, so that the triangle inequality implies the existence of a vector $w' \in \mathsf{W}(v_{s,s,t})$ with $\|w - w'\|_\infty \geq \frac{1}{6}$. If $k < s$, then Lemma 3.1 gives $\operatorname{diam}^\infty(\mathsf{W}(v_{k,s,t})) \geq \frac{1}{s}$, so that the triangle inequality implies the existence of a vector $w' \in \mathsf{W}(v_{k,s,t})$ with $\|w - w'\|_\infty \geq \frac{1}{2s}$. For $s = 1$ this gives $\|w - w'\|_\infty \geq \frac{1}{2}$. For $s \geq 2$ we have $s \leq \frac{2}{\Delta}$ so that $\|w - w'\|_\infty \geq \frac{\Delta}{4}$. Since $\Delta \leq 1$ we have $\|w - w'\|_\infty \geq \frac{\Delta}{4}$ in all cases, so that the stated result follows possibly by a perturbation. $\square$

4. Upper bounds for the diameter of the weight polytope

Before we start to upper bound $\operatorname{diam}(\mathsf{W}(v))$ in terms of Δ and q, we provide a slightly more general result.

Lemma 4.1. *Let $w \in \mathbb{R}^n_{\geq 0}$ with $\|w\|_1 = 1$ for an integer $n \in \mathbb{N}_{>0}$ and $0 < q < 1$. For each $x \in \mathbb{R}^n_{\geq 0}$ with $\|x\|_1 = 1$ and $x(S) = \sum_{s \in S} x_s \geq q$ for every winning coalition S of $[q; w]$, we have*

$$\|w - x\|_1 \leq \frac{2\Delta}{\min\{q + \Delta, 1 - q\}} \leq \frac{2\Delta}{\min\{q, 1 - q\}},$$

where $\Delta = \Delta(w)$.

Proof. Consider a winning coalition T such that $x(T)$ is minimal and invoke $x(T) \geq q$, see the appendix for the technical details. $\square$

From Lemma 4.1 we can directly conclude:

Corollary 4.2. *Let* $w, \bar{w} \in \mathbb{R}^n_{\geq 0}$ *with* $\|w\|_1 = \|\bar{w}\|_1 = 1$ *for an integer* $n \in \mathbb{N}_{>0}$ *and* $0 < q, \bar{q} < 1$. *If* $[q; w] = [\bar{q}; \bar{w}]$, *then we have*

$$\|w - \bar{w}\|_1 \leq \max\left\{\frac{2\Delta(w)}{\min\{q, 1-q\}}, \frac{2\Delta(\bar{w})}{\min\{\bar{q}, 1-\bar{q}\}}\right\} \leq \frac{2\Delta(w)}{\min\{q, 1-q\}} + \frac{2\Delta(\bar{w})}{\min\{\bar{q}, 1-\bar{q}\}}.$$

Unfortunately, this does not allow us to derive an upper bound of $\|w - \bar{w}\|_1$ which only depends on q and $\Delta(w)$. However, we can obtain the following analog of Lemma 4.1 for losing instead of winning coalitions.

Lemma 4.3. *Let* $w \in \mathbb{R}^n_{\geq 0}$ *with* $\|w\|_1 = 1$, $\Delta = \Delta(w)$, *and* $0 < q < 1$. *For each* $x \in \mathbb{R}^n_{\geq 0}$ *with* $\|x\|_1 = 1$ *and* $x(S) = \sum_{s \in S} x_s \leq q$ *for every losing coalition* S *of* $[q; w]$, *we have*

$$\|w - x\|_1 \leq \frac{4\Delta}{\min\{q, 1-q\}}.$$

Moreover, if $q > \Delta$, *then* $\|w - x\|_1 \leq \frac{2\Delta}{\min\{q-\Delta, 1-q+\Delta\}} \leq \frac{2\Delta}{\min\{q-\Delta, 1-q\}}$.

Proof. Consider a losing coalition T such that $x(T)$ is maximal and invoke $x(T) \leq q$. Technical details are provided in the appendix. $\square$

Theorem 4.4. *Let* $w, \bar{w} \in \mathbb{R}^n_{\geq 0}$ *with* $\|w\|_1 = \|\bar{w}\|_1 = 1$, $\Delta = \Delta(w)$, *and* $0 < q, \bar{q} < 1$. *If* $[q; w] = [\bar{q}; \bar{w}]$, *then we have*

$$\|w - \bar{w}\|_1 \leq \min\left\{2, \frac{4\Delta}{\min\{q, 1-q\}}\right\} \leq \frac{4\Delta}{\min\{q, 1-q\}},$$

i.e., $\operatorname{diam}(\mathsf{W}([q; w])) \leq \frac{4\Delta(w)}{\min\{q, 1-q\}}$. *Moreover, if* $q > \Delta$, *then we have*

$$\|w - \bar{w}\|_1 \leq \frac{2\Delta}{\min\{q - \Delta, 1-q\}}.$$

Proof. In Section 2 we have observed $\|w - \bar{w}\|_1 \leq 2$. If $\bar{q} \geq q$, then $\bar{w}(S) \geq \bar{q} \geq q$ for every winning coalition S of $[q; w]$. Here, we can apply Lemma 4.1. Otherwise we have $\bar{w}(T) < \bar{q} < q$ for every losing coalition T of $[q; w]$ and Lemma 4.3 applies. $\square$

As an example we consider the normalized weight vector $w = \frac{1}{120} \cdot (15, 14, \ldots, 1)$ and the quota $\frac{3}{5}$. Let $(\bar{q}; \bar{w})$ be another normalized representation of the weighted game $[q; w]$, then the first bound gives $\|w - \bar{w}\|_1 \leq \frac{5}{4}$. Since $\Delta = \frac{1}{8} > q$, also the second bound applies yielding $\|w - \bar{w}\|_1 \leq \frac{5}{8}$. We remark that for this specific example the diameter $\operatorname{diam}(\mathsf{W}([q; w]))$ is much smaller than $\frac{5}{8}$.

5. APPLICATIONS

A power index φ is a mapping from the set of weighted games on n players into $\mathbb{R}^n_{\ge 0}$. We call φ efficient if $\|\varphi(v)\|_1 = 1$ for all weighted games v. The difference $\|w - \varphi([q;w])\|_1$ between relative weights and the corresponding power distribution is studied in the literature, see e.g. [5, 11, 16]. Lemma 4.1 is a generalization of [11, Lemma 1]: if φ is the nucleolus, see e.g. [18], and $0 < q < 1$ then

$$\|w - \varphi([q;w])\|_1 \le \frac{2\Delta(w)}{\min\{q, 1-q\}} \tag{1}$$

for all $w \in \mathbb{R}^n_{\ge 0}$ with $\|w\|_1 = 1$. From Theorem 4.4 we directly conclude:

Corollary 5.1. *Let $w \in \mathbb{R}^n_{\ge 0}$ with $\|w\|_1 = 1$ and $0 < q < 1$. If an efficient power index φ permits the existence of a quota $q' \in (0,1)$ such that $[q'; \varphi([q;w])] = [q;w]$, i.e., the power vector of the given weighted game can be completed to a representation of the same game, then*

$$\|w - \varphi([q;w])\|_1 \le \frac{4\Delta(w)}{\min\{q, 1-q\}}.$$

Representation compatibility of φ for $[q;w]$ is automatically satisfied for the modified nucleolus (modiclus) [20], minimum sum representation index [6] or one of the power indices based on averaged representations [8] for all weighted games and for the Penrose-Banzhaf index for all spherically separable simple games [7]. The theorem also applies to the bargaining model for weighted games analyzed in [17], cf. [15]. It is unknown whether there exists a constant $c \in \mathbb{R}_{>0}$ such that

$$\|w - \mathrm{SSI}([q;w])\|_1 \le \frac{c\Delta(w)}{\min\{q, 1-q\}}. \tag{2}$$

holds for the Shapley-Shubik index SSI and all $w \in \mathbb{R}^n_{\ge 0}$ with $\|w\|_1 = 1$ and $0 < q < 1$. For the Penrose-Banzhaf index such a constant c can not exist, see [10, Proposition 2].

For the other direction we have:

Lemma 5.2. *Let $n \in \mathbb{N}_{>0}$, $q, \bar{q} \in (0,1]$, $w, \bar{w} \in \mathbb{R}^n_{\ge 0}$ with $\|w\|_1 = \|\bar{w}\|_1 = 1$ and $[q;w] = [\bar{q};\bar{w}]$, $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ and φ be a mapping from the set of weighted games (on n players) into $\mathbb{R}^n_{\ge 0}$, then we have*

$$\max\left\{\|w - \varphi([q;w])\|, \|\bar{w} - \varphi([\bar{q};\bar{w}])\|\right\} \ge \frac{\|w - \bar{w}\|}{2}.$$

Proof. Using the triangle inequality yields $\|w - \varphi([q;w])\| + \|\bar{w} - \varphi([\bar{q};\bar{w}])\| \ge \|w - \bar{w}\|$ from which we can conclude the stated inequality. □

Proposition 5.3. *Let φ be a mapping from the set of weighted games (on n players) into $\mathbb{R}^n_{\ge 0}$.*

(i) *For each $q \in (0,1]$ and each integer $n \ge 2$ there exists a weighted game $[q;w]$, where $w \in \mathbb{R}^n_{\ge 0}$ and $\|w\|_1 = 1$, such that $\|w - \varphi([q;w])\|_1 \ge \frac{1}{3}$ and $\|w - \varphi([q;w])\|_\infty \ge \frac{1}{6}$.*

(ii) *For each* $\Delta \in (0,1)$ *and each integer* $n \ge \frac{4}{3\Delta}+6$ *there exists a weighted game* $[q;w]$*, where* $q \in (0,1]$*,* $w \in \mathbb{R}^n_{\ge 0}$*,* $\|w\|_1 = 1$*, and* $\Delta(w) = \Delta$*, such that* $\|w-\varphi([q;w])\|_1 \ge \frac{1}{3}$*, and* $\|w-\varphi([q;w])\|_\infty \ge \Delta/4$.

Proof. Combine Lemma 5.2 with lemmas 3.3 and 3.5. □

Proposition 5.4. *Let* φ *be a mapping from the set of weighted games (on* n *players) into* $\mathbb{R}^n_{\ge 0}$*. For each* $q \in (0,1)$*,* $\Delta \in (0,1]$*, there exist* $w, \bar{w} \in \mathbb{R}^n_{\ge 0}$*,* $\bar{q} \in (0,1]$ *with* $\|w\|_1 = \|\bar{w}\|_1 = 1$*,* $\Delta(w) = \Delta$*,* $[q;w] = [\bar{q};\bar{w}]$*, and*

$$\|\bar{w}-\varphi([\bar{q};\bar{w}])\|_1 \ge \frac{1}{200}\cdot\min\left\{2, \frac{4\Delta}{\min\{q,1-q\}}\right\}.$$

Proof. We construct w as in the proof of Lemma 3.6 and choose integers k, s, and t such that $[q;w] = v_{k,s,t}$. In the proof of Lemma 3.6 we have actually verified

$$\operatorname{diam}(\mathsf{W}([q;w])) \ge \frac{1}{80}\cdot\min\left\{2, \frac{4\Delta}{\min\{q,1-q\}}\right\} =: \Lambda.$$

Now choose $w', w'' \in \mathsf{W}([q;w])$ with $\|w'-w''\|_1 \ge \Lambda$. By the triangle inequality we have either $\|w'-\varphi([q;w])\|_1 \ge \Lambda/2$ or $\|w''-\varphi([q;w])\|_1 \ge \Lambda/2$. By choosing $\bar{w}$ as w' or w'' and eventually moving it into the interior of $\mathsf{W}([q;w])$ we obtain the stated result. □

So, upper bounds for the $\|\cdot\|_1$-distance between normalized weights and a power distribution, as in Inequality (1 or Inequality (2) are tight up to the constant c if only the normalized quota and the normalized maximum weight are taken into account.

6. Conclusion

In this paper we have introduced the concept of the diameter of the weight polytope of a weighted game. This number measures how diverse two different normalized weight vectors, representing the same given game, can be. In Theorem 4.4 we have shown that

$$\operatorname{diam}(\mathsf{W}([q;w])) \le \min\left\{2, \frac{4\Delta}{\min\{q,1-q\}}\right\} \le \frac{4\Delta}{\min\{q,1-q\}},$$

for any $q \in (0,1)$ and any $w \in \mathbb{R}^n_{\ge 0}$ with $\|w\|_1 = 1$. Lemma 3.6 certifies that this upper bound is in general, i.e., in the worst case, tight up to a constant. (This paper traded smaller constants for easier proofs.) The super-exponential growth of the number of weighted games (see [22]) indicates that this is not the case for the majority of weighted games. Thus, it would be interesting to determine other parameters of a representation of a weighted game that permit tight upper bounds on the diameter of the corresponding weight polytope. Another possible line for future research is to consider games with a priori unions, spatial games, or games with restricted communication.

As shown in Section 5, there are connections to approximations of power indices by weight vectors. Proposition 5.4 gives a partial explanation for the conditions of the main theorem of [16] on a limit result for the Shapley-Shubik index. Moreover, for a general power index it shows that upper bounds for the $\|\cdot\|_1$-distance between normalized weights and a power distribution, taking only the normalized quota and the normalized maximum weight into account, as in Corollary 5.1, would be tight up to a constant.

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APPENDIX A. DELAYED PROOFS

Proof. (Lemma 2.4)
For $n = 1$, we have $w_1 = 1$, $\Delta(w) = 1$, $\alpha = 0$, and $L(w) = 1$, so that we assume $n \geq 2$ in the remaining part of the proof. For $w_i \geq w_j$ consider $a := \frac{w_i+w_j}{2}$ and $x := w_i - a$, so that $w_i = a + x$ and $w_j = a - x$. With this we have $w_i^2 + w_j^2 = 2a^2 + 2x^2$ and $(w_i + y)^2 + (w_j - y)^2 = 2a^2 + 2(x + y)^2$. Let us assume that $w^\star$ minimizes $\sum_{i=1}^n w_i^2$ under the conditions $w \in \mathbb{R}_{\geq 0}$, $\|w\|_1 = 1$, and $\Delta(w) = \Delta$. (Since the target function is continuous and the feasible set is compact and non-empty, a global minimum indeed exists.) W.l.o.g. we assume $w_1^\star = \Delta$. If there are indices $2 \leq i, j \leq n$ with $w_i^\star > w_j^\star$, i.e., $x > 0$ in the above parameterization, then we may choose $y = -x$. Setting $w_i' := w_i^\star + y = a = \frac{w_i^\star + w_j^\star}{2}$, $w_j' := w_j^\star - y = a = \frac{w_i^\star + w_j^\star}{2}$, and $w_h' := w_h^\star$ for all $1 \leq h \leq n$ with $h \notin \{i, j\}$, we have $w' \in \mathbb{R}^n_{\geq 0}$, $\|w'\|_1 = 1$, $\Delta(w') = \Delta$, and $\sum_{h=1}^n (w_h')^2 = \sum_{h=1}^n (w_h^\star)^2 - x^2$. Since this contradicts the minimality of $w^\star$, we have $w_i^\star = w_j^\star$ for all $2 \leq i, j \leq n$, so that we conclude $w_i^\star = \frac{1-\Delta}{n-1}$ for all $2 \leq i \leq n$ from $1 = \|w^\star\|_1 = \sum\limits_{h=1}^n w_h^\star$. Thus, $L(w) \leq 1/\left(\Delta^2 + \frac{(1-\Delta)^2}{n-1}\right)$, which is tight. Since $\Delta \leq 1$ and $n \geq 2$, we have $1/\left(\Delta^2 + \frac{(1-\Delta)^2}{n-1}\right) \leq \frac{1}{\Delta^2}$, which is tight if and only if $\Delta = 1$, i.e., $n - 1$ of the weights have to be equal to zero.

Now, let us assume that w maximizes $\sum_{i=1}^n w_i^2$ under the conditions $w \in \mathbb{R}_{\geq 0}$, $\|w\|_1 = 1$, and $\Delta(w) = \Delta$. (Due to the same reason a global maximum indeed exists.) Due to $1 = \|w\|_1 \leq n\Delta$ we have $0 < \Delta \leq 1/n$, where $\Delta = 1/n$ implies $w_i = \Delta$ for all $1 \leq i \leq n$. In that case we have $L(w) = n$ and $\alpha = 0$, so that the stated lower bounds for $L(w)$ are valid. In the remaining cases we assume $\Delta > 1/n$. If there would exist two indices $1 \leq i, j \leq n$ with $w_i \geq w_j$, $w_i < \Delta$, and $w_j > 0$, we may strictly increase the target function by moving weight from w_j to w_i (this corresponds to choosing $y > 0$), by an amount small enough to still satisfy the constraints $w_i \leq \Delta$ and $w_j \geq 0$. Since $\Delta > 0$, we can set $a := \lfloor 1/\Delta \rfloor \geq 0$ with $a \leq n - 1$ due to $\Delta > 1/n$. Thus, for a maximum solution, we have exactly a weights that are equal to Δ, one weight that is equal to $1 - a\Delta \geq 0$ (which may indeed be equal to zero), and $n - a - 1$ weights that are equal to zero. With this and $a\Delta = 1 - \alpha\Delta$ we have $\sum_{i=1}^n w_i^2 = a\Delta^2(1 - a\Delta)^2 = \Delta - \alpha\Delta^2 + \alpha^2\Delta^2 = \Delta(1 - \alpha\Delta + \alpha^2\Delta) = \Delta\left(1 - \alpha(1 - \alpha)\Delta\right) \leq \Delta$. Here, the latter inequality is tight if and only if $\alpha = 0$, i.e., $1/\Delta \in \mathbb{N}$. $\square$

Proof. (Lemma 4.1)
We set $N = \{1,\dots,n\}$, $w(U) = \sum_{u\in U} w_u$ and $x(U) = \sum_{u\in U} x_u$ for each $U \subseteq N$. Let $S^+ = \{i \in N \mid x_i > w_i\}$ and $S^- = \{i \in N \mid x_i \le w_i\}$, i.e., S^+ and S^- partition the set N of players. We have $w(S^+) < 1$ since $w(S^+) < x(S^+) \le x(N) = 1$, so that $w(S^-) > 0$. Define $0 \le \delta \le 1$ by $x(S^-) = (1-\delta)w(S^-)$. We have

$$x(S^+) = 1 - x(S^-) = w(S^+) + w(S^-) - (1-\delta)w(S^-) = w(S^+) + \delta w(S^-) \quad (3)$$

and

$$\|w - x\|_1 = \big(x(S^+) - w(S^+)\big) + \big(w(S^-) - x(S^-)\big) = 2\delta w(S^-). \quad (4)$$

Generate a set T by starting at $T = \emptyset$ and successively add a remaining player i in $N\backslash T$ with minimal x_i/w_i, where all players j with $w_j = 0$ are the worst ones. Stop if $w(T) \ge q$. By construction T is a winning coalition of $[q; w]$ with $w(T) < q + \Delta$, since the generating process did not stop earlier and $w_j \le \Delta(w)$ for all $j \in N$.

If $w(S^-) \ge q$, we have $T \subseteq S^-$ and $x(T)/w(T) \le x(S^-)/w(S^-) = 1 - \delta$. Multiplying by $w(T)$ and using $w(T) < q + \Delta$ yields

$$x(T) \le (1-\delta)w(T) < (1-\delta)(q+\Delta) = (1-\delta)q + (1-\delta)\Delta. \quad (5)$$

Since $x(T) \ge q$, as T is a winning coalition, we conclude $\delta < \Delta/(q+\Delta)$. Using this and $w(S^-) < 1$ in Equation (4) yields

$$\|w - x\|_1 < \frac{2\Delta}{q+\Delta} < \frac{2\Delta}{q}. \quad (6)$$

If $w(S^-) < q$, we have $S^- \subseteq T$, $x(T) = x(S^-) + x(T\backslash S^-)$, $w(T\backslash S^-) > 0$, and $w(S^+) > 0$. Since $T\backslash S^- \subseteq S^+$, $x(T\backslash S^-)/w(T\backslash S^-) \le x(S^+)/w(S^+)$, so that

$$\begin{aligned}
x(T) &= x(S^-) + x(T\backslash S^-) \le (1-\delta)w(S^-) + \frac{x(S^+)}{w(S^+)} \cdot \big(w(T) - w(S^-)\big) \\
&\le (1-\delta)w(S^-) + \frac{x(S^+)}{w(S^+)} \cdot \big(q + \Delta - w(S^-)\big) \\
&= q + \frac{x(S^+)\Delta - (1-q)\delta w(S^-)}{w(S^+)} \\
&\le q + \frac{\Delta - (1-q)\delta w(S^-)}{w(S^+)}.
\end{aligned}$$

Since $x(T) \ge q$, we conclude $(1-q)\delta w(S^-) \le \Delta$, so that $\|w - x\|_1 \le \frac{2\Delta}{1-q}$. □

Proof. (Lemma 4.3)
If $q \le 2\Delta$, then $\frac{4\Delta}{\min\{q,1-q\}} \ge \frac{4\Delta}{q} \ge 2 \ge \|x - w\|_1$, so that we can assume $q > \Delta$.

Using the notation from the proof of Lemma 4.1, we have $x(S^+) = w(S^+) + \delta w(S^-)$ and $\|w - x\|_1 = 2\delta w(S^-)$.

Generate T by starting at $T = \emptyset$ and successively add a remaining player i in $N\backslash T$ with maximal x_i/w_i, where all players j with $w_j = 0$ are taken in the first rounds, as long as $w(T) + w_i < q$. By construction T is a losing coalition of $[q; w]$ with $q - \Delta \le w(T) < q$, since the generating process did not stop earlier.

If $w(S^+) \ge q$, we have $T \subseteq S^+$ and $x(T)/w(T) \ge x(S^+)/w(S^+) = 1 + \frac{\delta w(S^-)}{w(S^+)} \ge 1 + \delta w(S^-)$. Multiplying by $w(T)$ and using $w(T) \ge q - \Delta$ yields

$$x(T) \ge \big(1 + \delta w(S^-)\big)\, w(T) \ge \big(1 + \delta w(S^-)\big)\,(q - \Delta) = (q - \Delta) + \delta w(S^-)(q - \Delta).$$

Since $x(T) \le q$, as T is a losing coalition, we conclude $\delta w(S^-) \le \Delta/(q - \Delta)$, so that $\|w - x\|_1 < \frac{2\Delta}{q-\Delta}$.

If $w(S^+) < q$, we have $S^+ \subseteq T$, $x(T) = x(S^+) + x(T\backslash S^+)$, $w(T\backslash S^+) > 0$, and $w(S^-) > 0$. Since $T\backslash S^+ \subseteq S^-$, $x(T\backslash S^+)/w(T\backslash S^+) \ge x(S^-)/w(S^-)$, so that

$$\begin{aligned}
x(T) &= x(S^+) + x(T\backslash S^+) \ge w(S^+) + \delta w(S^-) + \frac{x(S^-)}{w(S^-)} \cdot \big(w(T) - w(S^+)\big) \\
&\ge w(S^+) + \delta w(S^-) + (1 - \delta) \cdot \big(q - \Delta - w(S^+)\big) \\
&= \delta w(S^-) + q - \Delta - \delta q + \delta\Delta + \delta w(S^+) = q - \Delta + \delta(1 - q + \Delta).
\end{aligned}$$

Since $x(T) \le q$, $\delta \le \frac{\Delta}{1-q+\Delta}$, so that $\|w - x\|_1 \le \frac{2\Delta}{1-q+\Delta}$ due to $w(S^-) \le 1$.

So, for $q > \Delta$ we have $\|w - x\|_1 \le \frac{2\Delta}{\min\{q-\Delta, 1-q+\Delta\}} \le \frac{2\Delta}{\min\{q-\Delta, 1-q\}}$. In order to show $\|w - x\|_1 \le \frac{4\Delta}{\min\{q, 1-q\}}$ it remains to consider the case $q \le 1 - q$. For $q > 2\Delta$, see the start of the proof, we have $\|w - x\|_1 \le \frac{2\Delta}{\min\{q-\Delta, 1-q\}} \le \frac{2\Delta}{q-\Delta} \le \frac{4\Delta}{q} \le \frac{4\Delta}{\min\{q, 1-q\}}$. $\square$

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