Cosserat Operators of Higher Order and Applications

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German Abstract

In der vorliegenden Arbeit werden Cosseratoperatoren höherer Ordnung auf beschränkten Gebieten $G \subset\subset \mathbb{R}^n$, $n \geq 2$ (mit genügend glattem Rand ∂G) untersucht. Dies sind (die Größe $m \in \mathbb{N}$ bezeichnet die Ordnung) Operatoren der Form (die Räume $B_0^{m-1,q}(G)$ bezeichnen die Räume der mittelwertfreien m-harmonischen $H_0^{m-1,q}(G)$-Funktionen)

$$Z_q^{(m)} - \frac{1}{2} Id : B_0^{m-1,q}(G) \to B_0^{m-1,q}(G), \quad 1 < q < \infty$$

wobei $Z_q^{(m)} := \operatorname{div} \circ \underline{T}_q^{(m)}$ (unterstrichene Größen bezeichnen Vektoren oder vektorwertige Operatoren) und $\underline{T}_q^{(m)}$ jedem $p \in H_0^{m-1,q}(G)$ die eindeutige Lösung $\underline{u} \in \underline{H}_0^{m,q}(G)$ der Funktionalgleichung

$$B_m [\underline{u}, \underline{\Phi}] = B_{m-1} [p, \operatorname{div} \underline{\Phi}] \text{ für alle } \underline{\Phi} \in \underline{\mathcal{C}}_0^\infty(G)$$

zuordnet. Die Bilinearformen $B_m [\cdot, \cdot]$ sind dabei folgendermaßen definiert: Für $\Phi \in H_0^{m,q}(G)$ und $\Psi \in H_0^{m,q'}(G)$ (wobei $\frac{1}{q} + \frac{1}{q'} = 1$) setzen wir

$$B_m [\Phi, \Psi] := \begin{cases} \langle \Delta^{\frac{m}{2}} \Phi, \Delta^{\frac{m}{2}} \Psi \rangle & \text{für gerades } m \\ \langle \nabla \Delta^{\frac{m-1}{2}} \Phi, \nabla \Delta^{\frac{m-1}{2}} \Psi \rangle & \text{für ungerades } m \end{cases} .$$

Für Vektoren $\underline{\Phi} \in \underline{H}_0^{m,q}(G)$ und $\underline{\Psi} \in \underline{H}_0^{m,q'}(G)$ setzen wir

$$B_m [\underline{\Phi}, \underline{\Psi}] := \sum_{i=1}^{n} B_m [\Phi_i, \Psi_i] .$$

Der Operator $\underline{T}_q^{(m)}$ ist also ein schwacher Lösungsoperator für die Differentialgleichungen

$$\Delta^m u_i = \partial_i \Delta^{m-1} p, \quad i = 1, \ldots, n$$

mit homogenen Randwerten für die u_i.

Mit einigem Aufwand (für einen groben Überblick des Vorgehens siehe Abschnitt 1) werden wir in dieser Arbeit zeigen können, dass die so definierten Cosseratoperatoren kompakt sind (siehe die Theoreme 6.5 und 9.11). Dies hat weitreichende Konsequenzen. Die wichtigsten davon sind:

- Wir bekommen eine Strukturaussage für die Räume $\underline{H}_0^{m,q}(G)$, nämlich die Gültigkeit der direkten Zerlegung

$$\underline{H}_0^{m,q}(G) = \{\underline{u} \in \underline{H}_0^{m,q}(G) | \quad \operatorname{div} \underline{u} = 0\} \oplus \underline{M}_q^{(m)}(G),$$

wobei $\underline{M}_q^{(m)}(G) := \underline{T}_q^{(m)}(H_{0,0}^{m-1,q}(G))$ (wir bezeichnen mit $H_{0,0}^{m-1,q}(G)$ den Raum der mittelwertfreien $H_0^{m-1,q}(G)$-Funktionen) und die Einschränkung des Operators div auf $\underline{M}_q^{(m)}(G)$ (mit Bild $H_{0,0}^{m-1,q}(G)$) eine stetige Inverse besitzt.

Diese Aussage ist eng verwandt mit dem folgenden Satz, der auf M. E. Bogovskii (siehe [4], [5]) zurückgeht:

Gegeben sei ein Gebiet $G \subset\subset \mathbb{R}^n$, $n \geq 2$ mit lokalem Lipschitzrand, und $1 < q < \infty$, $m \geq 0$. Dann gibt es eine Konstante $C = C(m, q, G)$, so dass für jedes $f \in H_0^{m,q}(G)$ mit

$$\int_G f \, dx = 0$$

es ein (nicht notwendigerweise eindeutig bestimmtes) $\underline{v} \in H_0^{m+1,q}(G)$ gibt mit

$$\operatorname{div} \underline{v} = f$$

und

$$\|\underline{v}\|_{m+1,q} \leq C \, \|f\|_{m,q} \, .$$

Zusätzlich kann man, falls $f \in \mathcal{C}_0^\infty(G)$, das Vektorfeld $\underline{v}$ aus $\underline{\mathcal{C}}_0^\infty(G)$ wählen.

Unter unseren zusätzlichen Voraussetzungen an das Gebiet G (nämlich Beschränktheit und eine gewisse Glattheit des Randes) gelingt es uns mit der Gültigkeit der obigen Zerlegung, zu vorgegebenem $f \in H_0^{m,q}(G)$ mit $\int_G f \, dx = 0$, die Gleichung $\operatorname{div} \underline{v} = f$ auf stetige Weise im Raum $\underline{M}_q^{(m)}(G)$ eindeutig zu lösen.

- Mit diesen Kenntnissen sind wir in der Lage, das folgende Stokes-ähnliche Problem zu behandeln: Zu vorgegebenem $F \in \left(\underline{H}_0^{m,q'}(G)\right)^\star$, finde ein $\underline{u} \in \underline{H}_0^{m,q}(G)$ und ein $p \in H_{0,0}^{m-1,q}(G)$, so dass

$$B_m \left[\underline{u}, \underline{\Phi}\right] + B_{m-1} \left[p, \operatorname{div} \underline{\Phi}\right] = F(\underline{\Phi}) \quad \text{für alle } \underline{\Phi} \in \underline{H}_0^{m,q'}(G)$$

und

$$\operatorname{div} \underline{u} = 0.$$

In seiner Arbeit [17] hat C. G. Simader dieses Problem im Hilbertraumfall für $m = 2$ untersucht. Ihm ist es gelungen, dieses Problem ohne Verwendung der Kompaktheit des entsprechenden Cosseratoperators zu lösen, indem er die Gültigkeit einer Divergenzungleichung (siehe unser Theorem 7.11) zeigte. Dabei spielt der oben erwähnte Satz von Bogovskii eine wesentliche Rolle.

Wir erhalten aus unserem Zugang über den Cosseratoperator nicht nur ein Lösbarkeitsresultat für obiges System, sondern zudem noch Regularitätsaussagen. Dabei stützt sich unser Vorgehen auf Ideen aus [13], wo ein einfacher und eleganter Zugang zur Regularität des Stokesschen Systems mit Hilfe des Cosseratoperators (der Ordnung 1) beschritten wird.

Contents

Part I

Cosserat Operators of Order Two

1 Introduction and Overview

Starting point for this work were questions arising from an investigation of the operator

$$\operatorname{div} : \underline{H}_0^{2,2}(G) \to H_{0,0}^{1,2}(G)$$

by Joachim Naumann (Humboldt-University Berlin), where $\underline{H}_0^{2,2}(G)$ is the space of vector fields on a bounded domain $G \subset \mathbb{R}^n$ with every component in $H_0^{2,2}(G)$ (underlinings are used throughout to mark objects as vector valued) and $H_{0,0}^{1,2}(G)$ denotes the space consisting of the $p \in H_0^{1,2}(G)$ satisfying the compatibility condition

$$\int_G p \, dx = 0,$$

which must be fulfilled for divergences of vector fields from $\underline{H}_0^{2,2}(G)$, as is easily seen by approximation and by Gauß' Divergence Theorem.
Looking at the operator div from above in this Hilbert space setting, the question for the adjoint operator $\operatorname{div}^\star : H_{0,0}^{1,2}(G) \to \underline{H}_0^{2,2}(G)$ arises naturally. If we equip the spaces $H_{0,0}^{1,2}(G)$ and $\underline{H}_0^{2,2}(G)$ with the inner products

$$\langle g, h \rangle_{H_{0,0}^{1,2}(G)} := \langle \nabla g, \nabla h \rangle_{L^2(G)} := \sum_{i=1}^{n} \langle \partial_i g, \partial_i h \rangle_{L^2(G)} \text{ and}$$

$$\langle \underline{u}, \underline{v} \rangle_{\underline{H}_0^{2,2}(G)} := \langle \Delta \underline{u}, \Delta \underline{v} \rangle_{L^2(G)} := \sum_{i=1}^{n} \langle \Delta u_i, \Delta v_i \rangle_{L^2(G)} \text{ respectively,}$$

we are searching (for a given $p \in H_{0,0}^{1,2}(G)$) a $\underline{v} \in \underline{H}_0^{2,2}(G)$ satisfying for every $\underline{\Phi} \in \underline{H}_0^{2,2}(G)$ the functional equation

$$\langle \Delta \underline{v}, \Delta \underline{\Phi} \rangle = \langle \nabla p, \nabla \operatorname{div} \underline{\Phi} \rangle. \tag{1}$$

(In the following, we will omit a detailed specification of $\langle \cdot, \cdot \rangle$ with the use of an index. As we only use it either in the L^2-sense or in the sense of the dual L^q-$L^{q'}$-pairing, the meaning of it should be clear from the context.)

This leads to the definition of the operator adjoint to div, namely $\underline{T} = \operatorname{div}^\star$ assigning to each $p \in H^{1,2}_{0,0}(G)$ the unique solution $\underline{v} \in \underline{H}^{2,2}_0(G)$ to the functional equation (1).
In Simader's paper [17], this operator is investigated further in the Hilbert space setting described above. The main results are:

- There is a subspace of $\underline{H}^{2,2}_0(G)$, called $\underline{M}^2(G)$, such that the restriction of div to $\underline{M}^2(G)$ is continuously invertible. We have $\underline{M}^2(G) = \underline{T}(H^{1,2}_{0,0}(G))$ and validity of the following orthogonal decomposition:

$$\underline{H}^{2,2}_0(G) = \underline{D}^2(G) \oplus \underline{M}^2(G),$$

where

$$\underline{D}^2(G) = \left\{ \underline{v} \in \underline{H}^{2,2}_0(G) : \quad \operatorname{div} \underline{v} = 0 \right\}.$$

- On $\underline{M}^2(G) \subset \underline{H}^{2,2}_0(G)$, we have $\|\nabla \operatorname{div} \cdot\|_2$ as an equivalent norm to $\|\Delta \cdot\|_2$. Furthermore, for every $p \in H^{1,2}_{0,0}(G)$ the following inequality is valid with a constant $C = C(G) > 0$:

$$\|\nabla p\|_2 \le C \sup_{0 \neq \underline{v} \in \underline{M}^2(G)} \frac{\langle \nabla p, \nabla \operatorname{div} \underline{v} \rangle}{\|\nabla \operatorname{div} \underline{v}\|_2}$$

- With this, treatment of the following Stokes-like system of fourth order becomes quite simple:
  For a given $F \in (\underline{H}^{2,2}_0(G))^\star$, find $\underline{u} \in \underline{H}^{2,2}_0(G)$ and $p \in H^{1,2}_{0,0}(G)$ such that

$$\begin{aligned} \langle \Delta \underline{u}, \Delta \underline{\Phi} \rangle + \langle \nabla p, \nabla \operatorname{div} \underline{\Phi} \rangle &= F(\underline{\Phi}) \text{ for all } \underline{\Phi} \in \underline{H}^{2,2}_0(G) \\ \text{and} \qquad \operatorname{div} \underline{u} &= 0. \end{aligned} \tag{2}$$

  A sketch of proof is given below for motivation.

In the first part, we will find analogous results to the first two of the above mentioned results in the Banach space setting, where $q \neq 2$, which means that we are looking at

$$\operatorname{div} : \underline{H}^{2,q}_0(G) \to H^{1,q}_{0,0}(G) \text{ and } \underline{T}_q : H^{1,q}_{0,0}(G) \to \underline{H}^{2,q}_0(G)$$

assigning to each $p \in H^{1,q}_{0,0}(G)$ the unique $\underline{v} \in \underline{H}^{2,q}_0(G)$ satisfying

$$\langle \Delta \underline{v}, \Delta \underline{\Phi} \rangle = \langle \nabla p, \nabla \operatorname{div} \underline{\Phi} \rangle \text{ for all } \underline{\Phi} \in \underline{H}^{2,q'}_0(G).$$

The heart of our approach is the generalization of an ansatz which was introduced by Crouzeix in [6] and already used by Weyers in [22] yielding results similar to the ones we search, but in a different setting. The main point for the proof of our central compactness Theorem 6.5 is showing that the operator under consideration (which is $Z_q - \frac{1}{2}Id$ restricted to a suited subspace $B_0^q(G) \subset H_{0,0}^{1,q}(G)$, namely the subspace of harmonic $H_{0,0}^{1,q}(G)$-functions, where $Z_q := \operatorname{div} \circ \underline{T_q}$), has it's image not only in $H_{0,0}^{1,q}(G)$ but even in $H^{2,q}(G)$ and that the mapping $Z_q - \frac{1}{2}Id$ (which defines what we call a "Cosserat operator") is even continuous with respect to these spaces. Then, compactness of the operator $Z_q - \frac{1}{2}Id$ is simply a direct consequence of the compact embedding from $H^{2,q}(G)$ into $H^{1,q}(G)$. Once we have validity of an inequality of the form

$$\left\| \left(Z_q - \frac{1}{2}Id \right)(p) \right\|_{2,q} \leq C \left\| \nabla p \right\|_q \text{ for all } p \in B_0^q(G) \cap H^{3,q}(G) \tag{3}$$

with a $C > 0$ depending only on G and q, this inequality carries over by an approximation argument to all $p \in B_0^q(G)$, see Theorems 5.1 and 6.4. So the situation is somewhat better than hoped for. Showing inequality (3) is the point where Crouzeix' idea is applying: At first, we construct a function $f \in H^{2,q}(G)$ which depends continuously on $p \in H_{0,0}^{1,q}(G)$ and assumes the boundary-values of $Z_q(p) - \frac{1}{2}p$ in the sense that

$$f - \left(Z_q(p) - \frac{1}{2}p \right) \in H_0^{2,q}(G).$$

Then we have Müller's variational inequality at hand which is valid for $H_0^{2,q}(G)$-functions and helps us showing inequality (3).
The construction of an $f \in H^{2,q}(G)$ which depends continuously on p and which assumes the boundary values of $Z_q(p) - \frac{1}{2}p$ (see Theorems 6.2 and 6.3) is the complicated part of the proof and the success of the made ansatz is at first sight not evident at all. For the reader of [6] and [22], the ansatz which is given there might be seemingly strange. Therefore, it was not easy to find the right generalization of the original Crouzeix-ansatz to our problem right away. However, having found the suitable generalization after some fiddling about it, the idea behind it comes more to light. Therefore, at the beginning of Section 6, we give a motivation for this ansatz for Weyers' case where we try to find a suitable candidate for f via a kind of product-ansatz with one factor consisting of the given dates p and $\underline{u} = \underline{\tilde{T}}_q(p)$ and the other factor consisting of "free" functions, which are to be found. After some calculation, rather reasonable requirements for the unspecified functions are found, and one is led quite naturally to the ansatz made by Crouzeix. The compactness Theorem 6.5 for the Cosserat operator makes it easy to prove the

generalizations for the first two of the above given main results by Simader, see our Theorems 7.6, 7.8, Remark 7.9 and Theorems 7.10 and 7.11.

At different points, for example the continuous dependency of f from p (Theorem 6.2 and 6.3) or the regularity Theorems 10.3, 10.4 and 10.6 for the Crouzeix-construction and for $Z_q^{(m)} - \frac{1}{2}Id$ in Part II, we use the important Theorems 6.1, 10.1 on Elliptic Regularity from Simader's [15]. The role of Theorems 6.1 and 10.1 must not be underestimated: The Crouzeix-ansatz and all the regularity theorems (even the one in Part II for a kind of generalized Stokes-problem, see Theorem 11.2) are proved using merely the regularity Theorems 6.1 and 10.1 for the uniformly strongly elliptic regular Dirichlet bilinear forms in the sense of [15], see Definition 2.11, which are associated to Δ^m (for a precise definition of these $B_m[\cdot,\cdot]$, see (4)). Especially for our Stokes-like-system in Part II, no results on elliptic systems need to be used, but only regularity for Δ^m and the regularity of the respective Cosserat operator (which is also proved using regularity for Δ^m). This beautiful and elegant approach to regularity is due to C. G. Simader and his [13].
In the second part, we generalize our procedure from the first part to higher orders, that is: We concentrate then on the operators

$$\operatorname{div} : \underline{H}_0^{m,q}(G) \to H_{0,0}^{m-1,q}(G)$$

and

$$\underline{T}_q^{(m)} : H_{0,0}^{m-1,q}(G) \to \underline{H}_0^{m,q}(G),$$

which assigns to each $p \in H_{0,0}^{m-1,q}(G)$ the unique $\underline{v} \in \underline{H}_0^{m,q}(G)$ satisfying

$$B_m[\underline{v}, \underline{\Phi}] = B_{m-1}[p, \operatorname{div}\underline{\Phi}] \ \text{ for all } \underline{\Phi} \in \underline{H}_0^{m,q'}(G),$$

where B_m and B_{m-1} are uniformly strongly elliptic regular Dirichlet bilinear forms $B_m[\cdot,\cdot]$ in the sense of [15] which are associated to Δ^m respectively Δ^{m-1}. This means that for $m \in \mathbb{N}$, $\Phi \in H_0^{m,q}(G)$ and $\Psi \in H_0^{m,q'}(G)$ we define

$$B_m[\Phi,\Psi] := \begin{cases} \langle \Delta^{\frac{m}{2}}\Phi, \Delta^{\frac{m}{2}}\Psi\rangle \text{ for even } m \\ \langle \nabla\Delta^{\frac{m-1}{2}}\Phi, \nabla\Delta^{\frac{m-1}{2}}\Psi\rangle \text{ for odd } m \end{cases}. \tag{4}$$

Once one has found the right generalization of the Crouzeix-ansatz to this situation, results similar to those from the first part can be derived quite easily from the respective compactness Theorem 9.10. This general account covers the case of our first part (for $m = 2$) and also Weyers' situation from [22] (for $m = 1$) in the case of bounded domains. Here, in the special case of Weyers' situation, we get weaker requirements for the regularity of ∂G as in [22], which is actually only a benefit of using the notion of the trace (see Section 4).

In the second part, we are also looking at the generalization of the Stokes-like system (2) treated by Simader in [17] to the Banach space setting and in higher orders, see Section 11. As a motivation for our account, we will give a brief sketch of the proof of solvability for the system (2):

- At first, the n scalar Dirichlet problems are solved: There exists a $\underline{w} \in \underline{H}_0^{2,2}(G)$ such that
$$\langle \Delta \underline{w}, \Delta \underline{\Phi} \rangle = F(\underline{\Phi}) \quad \forall \underline{\Phi} \in H_0^{2,2}(G).$$
- Then $\operatorname{div} \underline{w} =: \pi \in H_{0,0}^{1,2}(G)$ and thus we find a unique $\underline{v} \in \underline{M}^2(G)$ such that $\operatorname{div} \underline{v} = \pi$. Therefore we have $\underline{u} := w - v \in H_0^{2,2}(G)$ and $\operatorname{div} \underline{u} = 0$. We also have, as $\underline{v} \in \underline{M}^2(G)$, a $p \in H_{0,0}^{1,2}(G)$ such that
$$\langle \Delta \underline{v}, \Delta \underline{\Phi} \rangle = \langle \nabla p, \nabla \operatorname{div} \underline{\Phi} \rangle \quad \forall \underline{\Phi} \in \underline{H}_0^{2,2}(G).$$
- So, all in all, we have for $\underline{\Phi} \in \underline{H}_0^{2,2}(G)$:
$$F(\underline{\Phi}) = \langle \Delta \underline{w}, \Delta \underline{\Phi} \rangle = \langle \Delta \underline{u}, \Delta \underline{\Phi} \rangle + \langle \Delta \underline{v}, \Delta \underline{\Phi} \rangle =$$
$$= \langle \Delta \underline{u}, \Delta \underline{\Phi} \rangle + \langle \nabla p, \nabla \operatorname{div} \underline{\Phi} \rangle$$
and $\operatorname{div} \underline{u} = 0$.

This motivates us to consider for the fourth order Stokes' system the term $\langle \nabla p, \nabla \operatorname{div} \underline{\Phi} \rangle$ (which generalizes later in order m to what we denote with $B_{m-1}[p, \operatorname{div} \underline{\Phi}]$) as the natural candidate for the generalized pressure functional and we regard thus the system (2) as the appropriate generalization of the usual Stokes' system.
In [3], Amrouche and Girault looked at another way of generalizing Stokes' system: Their homogeneous version (i. e. with finding a solution vector field with boundary values zero and divergence zero) of a fourth-order Stokes'-like system reads
$$\Delta^2 \underline{u} + \nabla p = \underline{f},$$
$$\operatorname{div} \underline{u} = 0,$$
$$\underline{u} = 0 \text{ on } \partial G \text{ and } \frac{\partial \underline{u}}{\partial n} = 0 \text{ on } \partial G$$
and as a generalization to higher orders they suggest the system
$$\Delta^m \underline{u} + \nabla p = \underline{f}.$$

Contrary to this, the "classical" or "strong" form of our generalization reads

$$\Delta^m \underline{u} + \nabla \Delta^{m-1} p = \underline{f}$$

and the weak form we investigate is the following:
For $m \in \mathbb{N}$, given a functional $F \in \left(\underline{H}_0^{m,q'}(G) \right)^\star$, we are looking for an $\underline{u} \in \underline{H}_0^{m,q}(G)$ and a $p \in H_{0,0}^{m-1,q}(G)$ such that

$$B_m \left[\underline{u}, \underline{\Phi} \right] + B_{m-1} \left[p, \operatorname{div} \underline{\Phi} \right] = F(\underline{\Phi}) \quad \text{for all } \underline{\Phi} \in \underline{H}_0^{m,q'}(G)$$

and

$$\operatorname{div} \underline{u} = 0,$$

where $B_m [\cdot, \cdot]$ and $B_{m-1} [\cdot, \cdot]$ again are our bilinear forms representing Δ^m and Δ^{m-1}.
The reason to prefer this generalization to the generalization by Amrouche and Girault is it's connection to the operators div and $\underline{T}_q^{(m)}$. If we analyze the sketch of proof of solvability given above for the system in question, we see that the use of the operators div and $\underline{T}_q^{(m)}$ and knowledge about them plays the central role in the proof, resulting in a very elegant way of proving solvability (and regularity) for the investigated system. For regularity of their variant, Amrouche and Girault have to cite the very general and complicated theory of Agmon, Douglis and Nirenberg. However, with regularity for our system at hand, it would be easy to derive regularity theorems for the system of Amrouche and Girault.

2 Preliminaries

2.1 Notations

Throughout the whole first part of the paper (Sections 2 to 8) let $n \in \mathbb{N}$ with $n \geq 2$ and $G \subset \mathbb{R}^n$ denote a bounded domain (that is G is open and connected) with $\partial G \in \mathcal{C}^5$. $\partial G \in \mathcal{C}^5$ means that for every $p \in \partial G$ we find an open set $U \subset \mathbb{R}^n$ with $p \in U$ and a function $f \in \mathcal{C}^5(U)$ with $\nabla f(p) \neq 0$ and

$$\Omega \cap U = \{x \in U : f(x) > 0\} \text{ and } \partial\Omega \cap U = \{x \in U : f(x) = 0\}.$$

As $\nabla f(p) \neq 0$ and $\partial_i f$ $(i = 1, \dots n)$ is continuous in U, we find a smaller open set $V \subset U$ with $p \in V$ such that for a certain $i \in \{1, \dots, n\}$ we have $\partial_i f \neq 0$ in V. After a permutation of variables we may assume that $i = n$ and by the implicit function theorem we find a function $\Phi \in \mathcal{C}^5(\Delta)$ with

$\Delta =]p_1 - \delta, p_1 + \delta[\times \cdots \times]p_{n-1} - \delta, p_{n-1} + \delta[\subset \mathbb{R}^{n-1}$ for a sufficiently small $\delta > 0$ and an open set $W \subset V$ with $p \in W$ such that

$$\{x = (x', x_n) \in \partial G \cap W\} = \{x = (x', x_n) \in \mathbb{R}^n : x' \in \Delta \text{ and } x_n = \Phi(x')\} .$$

We can also see that we can take as W a set of the form

$$W = \bigcup_{x' \in \Delta} \{x'\} \times]\Phi(x') - \varepsilon, \Phi(x') + \varepsilon[$$

for a sufficiently small $0 < \varepsilon$ and that we can achieve that

$$W = \{x = (x', x_n) \in W : x_n > \Phi(x')\} \cup \{x = (x', x_n) \in W : x_n = \Phi(x')\} \cup \\ \cup \{x = (x', x_n) \in W : x_n < \Phi(x')\}$$

and either

$$W \cap \Omega = \{x = (x', x_n) \in W : x_n > \Phi(x')\}$$

or

$$W \cap \Omega = \{x = (x', x_n) \in W : x_n < \Phi(x')\} .$$

Such local representations of ∂G will be used later where we show claims locally and use a partition of unity to show the claim in general. Note that due to the boundedness of G, ∂G is a compact set and we thus can assume that ∂G is covered by finitely many open sets of the type of the above defined W.

Further let $1 < q < \infty$ and $q' := \frac{q}{q-1}$. For two sets $U, V \subset \mathbb{R}^n$ we use the notation $V \subset\subset U$ to denote that U and V are open sets, V is bounded and $\overline{V} \subset U$. By $|G|$ we denote the Lebesgue-measure of G.

For $f \in L^q(G)$ we write

$$\|f\|_q := \|f\|_{q,G} := \left(\int_G |f|^q \, dx \right)^{\frac{1}{q}} .$$

Regarding elements of $L^q(G)$ we tend to be a little sloppy and will not always distinguish between an element f of $L^q(G)$ which is by definition an equivalence class of functions with respect to the equivalence relation “equality almost everywhere” and a certain representative of this equivalence class. In this regard a statement like

$f \in L^q(G)$ has the property (P)

where (P) is a pointwise property means:

There is representative of f for which (P) is valid.

2.2 The Relevant Spaces

We now introduce the relevant spaces and the notations we use in conjunction with them:

Definition 2.1.

- *For* $k \in \mathbb{N}$ *we denote by* $H^{k,q}(G)$ *the usual Sobolev spaces of functions* $f \in L^q(G)$ *which possess for any multiindex* $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}_0)^n$ *with* $|\alpha| := \sum_{i=1}^n \alpha_i \leq k$ *a weak* α*-derivative in* $L^q(G)$*, that is a function* $f_\alpha \in L^q(G)$ *which admits partial integration, that is for every* $\Phi \in \mathcal{C}_0^\infty(G)$ *we have*

$$\int_G f D^\alpha \Phi \, dx = (-1)^{|\alpha|} \int_G f_\alpha \Phi \, dx,$$

  where D^α *is the derivative* $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$. *We often write* $D^\alpha f$ *for* f_α *and for* $|\alpha| = 1$ *with* $\alpha_i = 1$ *we write simply* $\partial_i f$ *for* $D^\alpha f$. *By the definition*

$$\|f\|_{k,q} := \|f\|_{k,q,G} := \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_q^q \right)^{\frac{1}{q}}$$

  for $f \in H^{k,q}(G)$ *the space* $H^{k,q}(G)$ *becomes a reflexive (we have throughout* $1 < q < \infty$*) Banach space and for* $q = 2$ *even a Hilbert space with inner product*

$$\langle f, g \rangle := \sum_{|\alpha| \leq k} \langle D^\alpha f, D^\alpha g \rangle_2 = \sum_{|\alpha| \leq k} \int_G D^\alpha f D^\alpha g \, dx$$

  where $\langle \cdot, \cdot \rangle_2$ *denotes the usual* L^2*-product. In case* $q \neq 2$*, we understand* $\langle \cdot, \cdot \rangle$ *as the* L^q*-*$L^{q'}$ *dual-pairing*

$$\langle f, g \rangle := \int_G fg \, dx, \text{ for } f \in L^q(G),\ g \in L^{q'}(G).$$

- *By* $H_0^{k,q}(G)$ *we denote the closure of* $\mathcal{C}_0^\infty(G)$ *in* $H^{k,q}(G)$ *with respect to the above defined norm.*

- *Due to the boundedness of* G *we can make use of the elementary Poincaré inequality:*

There is a constant $C_{Poincaré} = C_{Poincaré}(q, G) > 0$ such that

$$\|u\|_q \leq C_{Poincaré} \|\nabla u\|_q = C_{Poincaré} \left(\sum_{j=1}^{n} \|\partial_j u\|_q^q \right)^{\frac{1}{q}}$$

for all $u \in H_0^{1,q}(G)$.

By (for $k > 1$ iterated) application of the elementary Poincaré inequality we get norms which are equivalent to the above defined ones on the closed subspace $H_0^{k,q}(G)$ in $H^{k,q}(G)$. These are:

– $\|\cdot\|'_{1,q}$ *defined by*

$$\|u\|'_{1,q} := \|\nabla u\|_q := \left(\sum_{j=1}^{n} \|\partial_j u\|_q^q \right)^{\frac{1}{q}}.$$

– $\|\cdot\|'_{2,q}$ *defined by*

$$\|u\|'_{2,q} := \|\nabla \nabla u\|_q := \left(\sum_{j,k=1}^{n} \|\partial_j \partial_k u\|_q^q \right)^{\frac{1}{q}}.$$

– $\|\cdot\|'_{3,q}$ *defined by*

$$\|u\|'_{3,q} := \|\nabla \nabla \nabla u\|_q := \left(\sum_{j,k,l=1}^{n} \|\partial_j \partial_k \partial_l u\|_q^q \right)^{\frac{1}{q}}.$$

- *Beside the elementary Poincaré inequality, we also have a kind of Poincaré inequality which is valid for mean-value-free functions:*

  For every bounded domain $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^1$, $1 \leq q \leq \infty$, there exists a constant $C_{Poi} = C_{Poi}(G, q)$ such that

  $$\|u\|_{q,G} \leq C_{Poi} \|\nabla u\|_{q,G}$$

  holds for every $u \in H^{1,q}(G)$ satisfying $\int_G u \, dx = 0$.

  For a proof, see for example, [7], Theorem 1 in 5.8.1., page 275.

- *By an application of the Calderon-Zygmund estimate, one can show the following theorem (for a proof, see [18], page 280):*

For $1 < q < \infty$ there exists a constant $C_{CZ} = C_{CZ}(n,q) > 0$ such that for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we have:

$$\left(\sum_{j,k=1}^{n} \|\partial_j \partial_k u\|_q^q\right)^{\frac{1}{q}} \leq C_{CZ} \|\Delta u\|_q$$

Due to this estimate, we can employ new norms on $H_0^{2,q}(G)$, $H_0^{3,q}(G)$ which are equivalent to the norms $\|\cdot\|_{2,q}'$ and $\|\cdot\|_{3,q}'$ given above:

- $\|\cdot\|_{2,q}''$ *defined by*
$$\|u\|_{2,q}'' := \|\Delta u\|_q$$
- $\|\cdot\|_{3,q}''$ *defined by*
$$\|u\|_{3,q}'' := \|\nabla \Delta u\|_q := \left(\sum_{j=1}^{n} \|\partial_j \Delta u\|_q^q\right)^{\frac{1}{q}}.$$

In the following we will prefer the direct and more suggestive notations like $\|\nabla\Delta\cdot\|_q$ to the above defined ones like $\|\cdot\|_{3,q}''$.

- $H_{0,0}^{1,q}(G) := \left\{p \in H_0^{1,q}(G) : \quad \int_G p\, dx = 0\right\}$
- $\underline{H}_0^{2,q}(G) := \left(H_0^{2,q}(G)\right)^n$ *denotes the space of vector fields $\underline{v} = (v_1, \ldots, v_n)$ with components $v_i \in H_0^{2,q}(G)$. We will throughout use underlinings to mark a certain object as vector valued. However, we will use these underlinings also for constants $\in \mathbb{R}^+$, should they arise in a suited situation where it is appropriate to distinguish between a "vector-case" and a "non-vector-case".*

Remark 2.2. *For $\underline{u} \in \underline{H}_0^{2,q}(G)$, $p \in H_0^{1,q'}(G)$, we will often use the following formula:*

$$\langle \nabla \operatorname{div} \underline{u}, \nabla p\rangle = \langle \Delta \underline{u}, \nabla p\rangle$$

To show this formula, we approximate $\underline{u}$ with a sequence $(\underline{\Phi}_k) \subset \underline{\mathcal{C}}_0^\infty(G)$ with respect to the $\underline{H}^{2,q}(G)$-norm. Then we have

$$\langle \nabla \operatorname{div} \underline{u}, \nabla p\rangle = \lim_{k\to\infty} \langle \nabla \operatorname{div} \underline{\Phi}_k, \nabla p\rangle = -\lim_{k\to\infty} \langle \Delta \operatorname{div} \underline{\Phi}_k, p\rangle =$$

$$-\lim_{k\to\infty} \langle \operatorname{div} \Delta \underline{\Phi}_k, p\rangle = \lim_{k\to\infty} \langle \Delta \underline{\Phi}_k, \nabla p\rangle = \langle \Delta \underline{u}, \nabla p\rangle.$$

The following theorems about $H^{1,q}(G)$-functions will be used later:

Theorem 2.3. *For $G \subset \mathbb{R}^n$ open, $1 \leq q < \infty$ and $u \in H^{1,q}(G)$ let*

$$Z(u) := \{x \in G : \quad u(x) = 0\}.$$

Then for $i = 1, \ldots, n$ it is

$$\partial_i u(x) = 0 \text{ for almost every } x \in Z(u).$$

For a proof, see [16], Satz 6.15, pages 151-152.

Theorem 2.4. *Let $G \subset \mathbb{R}^n$ be open, $1 \leq q < \infty$ and $u \in H^{1,q}(G)$ (resp. $\in H_0^{1,q}(G)$). Then $|u|$, u_+, $u_- \in H^{1,q}(G)$ (resp. $\in H_0^{1,q}(G)$), where*

$$u_+(x) := \max\{u(x), 0\}$$

for almost every $x \in G$,

$$u_-(x) := \min\{u(x), 0\}$$

for almost every $x \in G$.
Further

- *for $i = 1, \ldots, n$*

$$(\partial_i |u|)(x) = \begin{cases} \partial_i u(x), \text{ for almost every } x \in G \text{ with } u(x) > 0 \\ 0, \text{ for almost every } x \in G \text{ with } u(x) = 0 \\ -\partial_i u(x), \text{ for almost every } x \in G \text{ with } u(x) < 0 \end{cases}$$

  i.e. $\partial_i |u| = \operatorname{sgn}(u)\partial_i u$,

$$(\partial_i u_+)(x) = \begin{cases} \partial_i u(x), \text{ for almost every } x \in G \text{ with } u(x) > 0 \\ 0, \text{ otherwise} \end{cases}$$

  and

$$(\partial_i u_-)(x) = \begin{cases} \partial_i u(x), \text{ for almost every } x \in G \text{ with } u(x) < 0 \\ 0, \text{ otherwise} \end{cases}.$$

- *the assignments $u \mapsto |u|$, $u \mapsto u_+$, $u \mapsto u_-$ from $H^{1,q}(G)$ to $H^{1,q}(G)$ (resp. from $H_0^{1,q}(G)$ to $H_0^{1,q}(G)$) are continuous with respect to these spaces.*

For a proof, see [16], Satz 6.17, pages 153-156.

2.3 Friedrichs' Mollification and it's Fundamental Properties

In this subsection we will only cite some of the fundamental properties of Friedrichs' mollification process, which we will need in the following. The respective proofs can be found in almost any book on partial differential equations, we simply refer to [16]. In the following, let $j \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $j \geq 0$, $j(x) = 0$ for $\|x\| \geq 1$ and $\int_{\mathbb{R}^n} j(x)\,dx = 1$. Such functions exist, a widely known example is the (only radially depending)

$$j(x) := \begin{cases} ce^{-\frac{1}{1-\|x\|^2}} \text{ for } \|x\| < 1 \\ 0 \text{ for } \|x\| \geq 1 \end{cases}$$

where $c > 0$ is the constant scaling j to satisfy $\int_{\mathbb{R}^n} j(x)\,dx = 1$. For $\varepsilon > 0$ we further define $j_\varepsilon(x) := \varepsilon^{-n} j(\frac{x}{\varepsilon})$. It is immediately seen that $j_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and by the transformation formula for integrals we see $\int_{\mathbb{R}^n} j_\varepsilon(x)\,dx = 1$.

Theorem 2.5. *Friedrichs' mollification*
Let $G \subset \mathbb{R}^n$ be open. For $f \in L^q(G)$ we define for $x \in \mathbb{R}^n$ the new function

$$f_\varepsilon(x) := \int_G j_\varepsilon(x-y) f(y)\,dy.$$

Then we have:

i) $f_\varepsilon \in \mathcal{C}^\infty(G)$ for all $\varepsilon > 0$.

ii) For all $\varepsilon > 0$ it is $f_\varepsilon \in L^q(G)$ and $\|f - f_\varepsilon\|_{q,G} \xrightarrow{\varepsilon \to 0} 0$.

iii) If $f \in H^{1,q}(G)$ and there is a compact set $K \subset G$ such that $f(x) = 0$ for almost every $x \in G \setminus K$, we have for every $0 < \varepsilon < \operatorname{dist}(K, \partial G)$:

$$(f)_\varepsilon \in \mathcal{C}_0^\infty(G)$$

and for every $x \in \mathbb{R}^n$, $0 < \varepsilon \in \mathbb{R}$:

$$(\partial_i f)_\varepsilon(x) = \partial_i (f_\varepsilon)(x),\ i = 1, \ldots, n$$

Concerning Friedrichs' mollification, we also state an important property of harmonic functions (which is indeed characterizing harmonic functions), for a proof, we refer to [14], Lemmas 2.5 and 2.6, pages 765, 766:

Theorem 2.6. *Let $G \subset \mathbb{R}^n$ be open, $G' \subset\subset G$ and u be harmonic in G. Then for every $0 < \varepsilon < \operatorname{dist}(G', \partial G)$ we have*

$$u_\varepsilon(x) = u(x)$$

for every $x \in G'$. In particular, $u \in \mathcal{C}^\infty(G)$.

2.4 Solvability of the Weak Dirichlet Problem in L^q with Homogeneous Boundary Conditions for Δ^2 and Δ^3

In this subsection we cite the important variational inequalities and solvability theorems which will be used in the following.

Theorem 2.7. *Müller's variational inequality in $H_0^{2,q}(G)$ (see [10], Hauptsatz, page 191):*
Let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in \mathcal{C}^2$, $1 < q < \infty$ with $q' := \frac{q}{q-1}$. Then there is a constant $C_{M,q} > 0$ depending only on G and q such that

$$\|\Delta u\|_q \leq C_{M,q} \sup_{0 \neq \Phi \in H_0^{2,q'}(G)} \frac{\langle \Delta u, \Delta \Phi \rangle}{\|\Delta \Phi\|_{q'}}$$

holds for all $u \in H_0^{2,q}(G)$.

For a proof, we refer to [10], pages 191-194.

In fact Müller proved this variational inequality not just for the case of bounded domains but also for exterior domains. We also have a vector-version of this variational inequality:

Theorem 2.8. *Müller's variational inequality in $\underline{H}_0^{2,q}(G)$:*
Let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in \mathcal{C}^2$, $1 < q < \infty$ with $q' := \frac{q}{q-1}$. Then there is a constant $\underline{C}_{M,q} > 0$ depending only on G and q such that

$$\|\Delta \underline{u}\|_q \leq \underline{C}_{M,q} \sup_{0 \neq \underline{\Phi} \in \underline{H}_0^{2,q'}(G)} \frac{\langle \Delta \underline{u}, \Delta \underline{\Phi} \rangle}{\|\Delta \underline{\Phi}\|_{q'}}$$

holds for all $\underline{u} \in \underline{H}_0^{2,q}(G)$.

Proof. We have for $\underline{u} \in \underline{H}_0^{2,q}(G)$:

$$\|\Delta \underline{u}\|_q = \left(\sum_{j=1}^{n} \|\Delta u_j\|_q^q \right)^{\frac{1}{q}}$$

and as for $j = 1, \ldots, n$ we have

$$\|\Delta u_j\|_q^q \leq C_{M,q}^q \left(\sup_{0 \neq \Phi \in H_0^{2,q'}(G)} \frac{\langle \Delta u_j, \Delta \Phi \rangle}{\|\Delta \Phi\|_{q'}} \right)^q$$

by Theorem 2.7, we find

$$\|\Delta \underline{u}\|_q \le C_{M,q} \left(\sum_{j=1}^{n} \left(\sup_{0 \ne \Phi \in H_0^{2,q'}(G)} \frac{\langle \Delta u_j, \Delta \Phi \rangle}{\|\Delta \Phi\|_{q'}} \right)^q \right)^{\frac{1}{q}}.$$

But we have for $j = 1, \dots, n$

$$\sup_{0 \ne \Phi \in H_0^{2,q'}(G)} \frac{\langle \Delta u_j, \Delta \Phi \rangle}{\|\Delta \Phi\|_{q'}} \le \sup_{0 \ne \underline{\Phi} \in \underline{H}_0^{2,q'}(G)} \frac{\langle \Delta \underline{u}, \Delta \underline{\Phi} \rangle}{\|\Delta \underline{\Phi}\|_{q'}}$$

and so we find

$$\|\Delta \underline{u}\|_q \le C_{M,q} \left(\sum_{j=1}^{n} \left(\sup_{0 \ne \underline{\Phi} \in \underline{H}_0^{2,q'}(G)} \frac{\langle \Delta \underline{u}, \Delta \underline{\Phi} \rangle}{\|\Delta \underline{\Phi}\|_{q'}} \right)^q \right)^{\frac{1}{q}} =$$

$$= \underbrace{C_{M,q} n^{\frac{1}{q}}}_{=:\underline{C}_{M,q}} \sup_{0 \ne \underline{\Phi} \in \underline{H}_0^{2,q'}(G)} \frac{\langle \Delta \underline{u}, \Delta \underline{\Phi} \rangle}{\|\Delta \underline{\Phi}\|_{q'}}.$$

□

Validity of Müller's variational inequalities for q and q' are equivalent to the unique solvability of the weak Dirichlet problem for Δ^2 in L^q and $L^{q'}$ with homogeneous boundary conditions. For a proof, we refer to [10], Lemma III.15. on page 164, but for the analogous problem (Theorem 2.17) for Δ^3 we will give a proof below, see Theorem 2.17.

Theorem 2.9. *Let F be a bounded linear functional $\in \left(H_0^{2,q'}(G)\right)^*$. Then there is exactly one $u \in H_0^{2,q}(G)$ with*

$$\langle \Delta u, \Delta \Phi \rangle = F(\Phi) \text{ for all } \Phi \in H_0^{2,q'}(G).$$

Moreover, the solution u satisfies

$$\|\Delta u\|_q < C_{\Delta^2} \|F\|_{\left(H_0^{2,q'}(G)\right)^*}$$

with a $C_{\Delta^2} = C_{\Delta^2}(q, G) > 0$.

For a proof, see [10], Lemma III.15., page 164.
In [10], Satz IV.1.1., page 195, Müller gives the following regularity result in a version for exterior domains. We state the theorem in another version for our domains which we will give a proof for:

Theorem 2.10. *Let $1 < q, s < \infty$ and $u \in H_0^{2,q}(G)$ satisfying*

$$\sup_{\Phi \in \mathcal{C}_0^\infty(G)} \frac{\langle \Delta u, \Delta \Phi \rangle}{\|\Delta \Phi\|_{s'}} < \infty. \tag{5}$$

Then $u \in H_0^{2,s}(G)$.

Proof. In the case $1 < s \leq q < \infty$ the statement $u \in H_0^{2,s}(G)$ is shown easily: For $s = q$ everything is clear and for $s < q$ the statement is merely a consequence of the boundedness of G and the Hölder inequality. In this case we do not even need the validity of assumption (5).
So look now at the case $1 < q < s < \infty$. Let first $0 \neq \Phi \in \mathcal{C}_0^\infty(G)$. By (5) we see that by

$$F(\Phi) := \langle \Delta u, \Delta \Phi \rangle$$

we have

$$|F(\Phi)| = |\langle \Delta u, \Delta \Phi \rangle| = \frac{|\langle \Delta u, \Delta \Phi \rangle|}{\|\Delta \Phi\|_{s'}} \|\Delta \Phi\|_{s'} \leq \sup_{0 \neq \Psi \in \mathcal{C}_0^\infty(G)} \frac{\langle \Delta u, \Delta \Psi \rangle}{\|\Delta \Psi\|_{s'}} \|\Delta \Phi\|_{s'} .$$

As $\mathcal{C}_0^\infty(G)$ is dense in $H_0^{2,s'}(G)$ with respect to the norm $\|\Delta \cdot\|_{s'}$, there is an unique linear and continuous extension $\tilde{F} \in \left(H_0^{2,s'}(G)\right)^*$ of F. By Theorem 2.9 we find an unique $v \in H_0^{2,s}(G)$ with

$$\langle \Delta v, \Delta \Phi \rangle = \tilde{F}(\Phi) \text{ for all } \Phi \in H_0^{2,s'}(G).$$

As $s > q$ we find that $v \in H_0^{2,q}(G)$, too. For all $\Phi \in \mathcal{C}_0^\infty(G)$ we find that:

$$\langle \Delta(v - u), \Delta \Phi \rangle = 0$$

and thus by the uniqueness in Theorem 2.9 we conclude $u = v \in H_0^{2,s}(G)$. □

For the solvability of the analogous problem to Theorem 2.9 for Δ^3 we have to refer to [15], Theorems 7.5. and 7.6., which apply not only to Δ^3 but to uniformly strongly elliptic regular Dirichlet bilinear forms of given order m in the sense of [15], Definitions 1.3 and 1.4, pages 14-16. This means:

Definition 2.11. *Let $G \subset\subset \mathbb{R}^n$ be open, $n, m \in \mathbb{N}$ with $n \geq 2$ and $m \geq 1$. Let for every $\alpha, \beta \in (\mathbb{N}_0)^n$ with $|\alpha|, |\beta| \leq m$ a complex-valued bounded measurable function $a_{\alpha,\beta}$ defined in G be given. For $\Phi, \Psi \in \mathcal{C}_0^\infty(G)$ let*

$$B[\Phi, \Psi] := \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} \langle a_{\alpha\beta} D^\alpha \Phi, D^\beta \Psi \rangle.$$

and

$$L_B := (-1)^m \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(\cdot) D^\alpha D^\beta.$$

Then B is called an uniformly strongly elliptic Dirichlet bilinear form of order m in G, if the differential operator L_B is uniformly strongly elliptic of order $2m$ in G, that is:

- *For every fixed $(l_1, \ldots, l_{n-1}) =: l \in \mathbb{R}^{n-1} \setminus \{0\}$ and every $x \in G$ the polynomial in $\tau \in \mathbb{C}$*

  $$P(\tau, l, x) := \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) l^{\alpha'+\beta'} \tau^{\alpha_n+\beta_n}, \quad \alpha = (\alpha', \alpha_n), \beta = (\beta', \beta_n)$$

  has exactly m roots with positive and m roots with negative imaginary part.

- *There exists a constant $E > 0$ such that*

  $$(-1)^m \operatorname{Re} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) l^{\alpha+\beta} \geq E \, |l|^{2m}$$

  holds for every $x \in G$ and $l \in \mathbb{R}^n$.

Moreover, by regularity of B, it is meant that the functions $a_{\alpha\beta}$ admit for $|\alpha| = |\beta| = m$ a continuous continuation to $\overline{G}$ and are bounded in G for $|\alpha| = |\beta| < m$.

Remark 2.12. *All the bilinear forms we will use in the following are defined for some $m \in \mathbb{N}$ by*

$$B_m [\Phi, \Psi] := \begin{cases} \langle \Delta^{\frac{m}{2}} \Phi, \Delta^{\frac{m}{2}} \Psi \rangle \text{ for even } m \\ \langle \nabla \Delta^{\frac{m-1}{2}} \Phi, \nabla \Delta^{\frac{m-1}{2}} \Psi \rangle \text{ for odd } m \end{cases}.$$

Now it is quickly seen that B_m defines an uniformly strongly elliptic regular Dirichlet bilinear form of order m in the above introduced sense:
We see that we can write

$$B_m[\Phi, \Psi] = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} \langle a_{\alpha\beta} D^\alpha \Phi, D^\beta \Psi \rangle$$

with all the $a_{\alpha\beta} \geq 0$ constant.
Next, we see that the differential operator L_B associated to the bilinear form $B = B_m$ has the form

$$L_B = (-1)^m (-1)^m \Delta^m = \Delta^m.$$

The requirements from Definition 2.11 are quickly verified:

- *The regularity assumptions on the $a_{\alpha\beta}$ are trivially fulfilled, for they are all constant.*
- *The polynomial $P(\tau, l, x)$ with $l \in \mathbb{R}^{n-1} \setminus \{0\}$, $\tau \in \mathbb{C}$ can be written as*
$$(-1)^m \left(\sum_{i=1}^{n-1} l_i^2 + \tau^2 \right)^m .$$
This polynomial has the same zeros τ as the polynomial $\sum_{i=1}^{n-1} l_i^2 + \tau^2$, but with m-times as much multiplicities. As the polynomial $\sum_{i=1}^{n-1} l_i^2 + \tau^2$ has real coefficients, the zeros occur in pairs of complex conjugates and as $l \neq 0$, there can be no real zeros. So the original polynomial must have one zero with positive imaginary part (with multiplicity m) and one zero with negative imaginary part (with multiplicity m).
- *Looking at*
$$(-1)^m \operatorname{Re}\left((-1)^m |l|^{2m}\right) = |l|^{2m} ,$$
we see that we can choose $E = 1$.

Having now verified that our $B_m[\cdot,\cdot]$ are admissible for Simader's theory from [15], we cite the important theorems from there which we are going to use in order to get our solvability statements:

Theorem 2.13. *(Compare [15], Theorem 7.5., page 129)*
Let $m \geq 1$ be an integer and let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded open set with boundary $\partial G \in \mathcal{C}^m$. Let $B[\Phi, \Psi]$ be an uniformly strongly elliptic regular Dirichlet bilinear form of order m and q, q' two real numbers with $1 < q, q' < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$.
Let
$$N_q := \left\{ w \in H_0^{m,q}(G) : \quad B[w, \Phi] = 0 \text{ for every } \Phi \in H_0^{m,q'}(G) \right\}$$
and let
$$N_{q'} := \left\{ z \in H_0^{m,q'}(G) : \quad B[\Psi, z] = 0 \text{ for every } \Psi \in H_0^{m,q}(G) \right\} .$$
Then $\dim N_q = \dim N_{q'} = d < \infty$. For $F \in \left(H_0^{m,q'}(G)\right)^\star$ the functional equation
$$B[u, \Phi] = F(\Phi) \text{ for every } \Phi \in H_0^{m,q'}(G)$$
has a solution $u \in H_0^{m,q}(G)$ if and only if $F(z) = 0$ for every $z \in N_{q'}$.
Particularly, in case of $d = 0$, the equation is uniquely solvable for arbitrary $F \in \left(H_0^{m,q'}(G)\right)^\star$.

To show in our case that $d = 0$, we use the following theorem:

Theorem 2.14. *(Compare [15], Theorem 7.6., page 131) Let $m \geq 1$ be an integer and let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded open set with boundary $\partial G \in \mathcal{C}^m$. Let $B[\Phi, \Psi]$ be an uniformly strongly elliptic regular Dirichlet bilinear form of order m, let q be a real number with $1 < q < \infty$ and $u \in H_0^{m,q}(G)$ such that*

$$B[u, \Phi] = 0 \text{ for every } \Phi \in \mathcal{C}_0^\infty(G).$$

Then $u \in H_0^{m,r}(G)$ for every $1 < r < \infty$.

Remark 2.15. *So in our case, where $m := 3$, $B[\Psi, \Phi] := \langle \nabla\Delta\Psi, \nabla\Delta\Phi \rangle$, look at an $u \in N_q$. Then for every $\Phi \in \mathcal{C}_0^\infty(G)$ we find*

$$\langle \nabla\Delta u, \nabla\Delta\Phi \rangle = 0$$

and with Theorem 2.14 we conclude that $u \in H_0^{3,2}(G)$ and $u \in N_2(G)$, too. Thus taking u itself as a testing function (which can be justfied by approximating u in the $H_0^{3,2}(G)$-sense by $\mathcal{C}_0^\infty(G)$-functions), we see that

$$\langle \nabla\Delta u, \nabla\Delta u \rangle = 0$$

and thus $u = 0$.

This leads us to the following solvability theorem:

Theorem 2.16. *Let F be a bounded linear functional $\in \left(H_0^{3,q'}(G)\right)^*$. Then there is exactly one $u \in H_0^{3,q}(G)$ satisfying*

$$\langle \nabla\Delta u, \nabla\Delta\Phi \rangle = F(\Phi) \text{ for all } \Phi \in H_0^{3,q'}(G). \tag{6}$$

Moreover, there is a $C_{\Delta^3} = C_{\Delta^3}(q, G) > 0$ such that for every $F \in \left(H_0^{3,q'}(G)\right)^*$ and u with (6) we have the estimate

$$\|\nabla\Delta u\|_q \leq C_{\Delta^3} \|F\|_{\left(H_0^{3,q'}(G)\right)^*}. \tag{7}$$

Proof. The existence of an unique $u \in H_0^{3,q}(G)$ satisfying

$$\langle \nabla\Delta u, \nabla\Delta\Phi \rangle = F(\Phi) \text{ for all } \Phi \in H_0^{3,q'}(G) \tag{8}$$

is a direct consequence of Theorem 2.13 and Remark 2.15. The only thing that remains to be shown is the existence of a $C_{\Delta^3} = C_{\Delta^3}(q, G)$ such that

$$\|\nabla\Delta u\|_q \leq C_{\Delta^3} \|F\|_{\left(H_0^{3,q'}(G)\right)^*}.$$

With Theorem 9.11 in [15] (see our Theorem 6.1 (with $m = 3$, $j = 0$)) we get with (8) the estimate

$$\|u\|_{3,q} \leq \gamma \left(\|F\|_{\left(H_0^{3,q'}(G)\right)^*} + \|u\|_q \right), \tag{9}$$

where γ is dependent only on q and G (note that m, j and B, as they were called in Theorem 6.1 are fixed here and n is already coded in G). In view of the equivalence of the norms $\|\nabla\Delta\cdot\|_q$ and $\|\cdot\|_{3,q}$ on $H_0^{3,q}(G)$, it is sufficient to show that for u we have an estimate of the form

$$\|u\|_q \leq C \|F\|_{\left(H_0^{3,q'}(G)\right)^*} \tag{10}$$

with a $C = C(q, G) > 0$. As we have

$$\|u\|_q \leq \|u\|_{3,q} ,$$

it suffices to show validity of an estimate of the form

$$\|u\|_{3,q} \leq C \|F\|_{\left(H_0^{3,q'}(G)\right)^*} \tag{11}$$

with a $C = C(q, G) > 0$. Then estimate (7) follows easily with (9).
Assume (11) were false. Then we could find a sequence $(F_\nu)_{\nu\in\mathbb{N}} \subset \left(H_0^{3,q'}(G)\right)^*$ and $(u_\nu)_{\nu\in\mathbb{N}} \subset H_0^{3,q}(G)$ with

$$\langle \nabla\Delta u_\nu, \nabla\Delta\Phi \rangle = F_\nu(\Phi) \text{ for all } \Phi \in \mathcal{C}_0^\infty(G),\ \nu \in \mathbb{N}$$

with

$$\|u_\nu\|_{3,q} = 1 \tag{12}$$

and

$$\|u_\nu\|_{3,q} > \nu \|F_\nu\|_{\left(H_0^{3,q'}(G)\right)^*} . \tag{13}$$

With (12) and (13) we conclude

$$F_\nu \xrightarrow{\left(H_0^{3,q'}(G)\right)^*} 0.$$

By (12) the sequence $(u_\nu)_{\nu\in\mathbb{N}} \subset H_0^{3,q}(G)$ is bounded in $H_0^{3,q}(G)$ and we can assume (by passing to a subsequence) without loss of generality that there is an $u \in H_0^{3,q}(G)$ with

$$u_\nu \xrightarrow{\text{weakly in } H_0^{3,q}(G)} u$$

and

$$u_\nu \xrightarrow{\text{strongly in } H_0^{2,q}(G)} u$$

by Rellich's compact embedding from $H_0^{3,q}(G)$ into $H_0^{2,q}(G)$, see for example [2], A6.1, pages 256, 257.
By $u_\nu \xrightarrow{\text{weakly in } H_0^{3,q}(G)} u$ and $F_\nu \xrightarrow{\left(H_0^{3,q'}(G)\right)^*} 0$ we see easily that

$$\langle \nabla\Delta u, \nabla\Delta\Phi \rangle = 0$$

for all $\Phi \in H_0^{3,q'}(G)$ and by the unique solvability already verified, we see $u = 0$. By the inequality (9) and the convergence of u_ν in $L^q(G)$ to 0, we see that

$$\|u_\nu - u_\mu\|_{3,q} \leq \gamma \left(\|F_\nu - F_\mu\|_{\left(H_0^{3,q'}(G)\right)^*} + \|u_\nu - u_\mu\|_q \right) \leq$$

$$\leq \gamma \left(\|F_\nu\|_{\left(H_0^{3,q'}(G)\right)^*} + \left\|F_\mu\right\|_{\left(H_0^{3,q'}(G)\right)^*} + \|u_\nu\|_q + \|u_\mu\|_q \right) \xrightarrow{\mu,\nu\to\infty} 0$$

and so (u_ν) is a Cauchy-sequence in $H_0^{3,q}(G)$ and thus has a limit $v \in H_0^{3,q}(G)$. But then (u_ν) converges also weakly in $H_0^{3,q}(G)$ to v and by uniqueness of the weak limit we have $u = v$ and thus (u_ν) converges strongly to $u = 0$. This, however, is a contradiction to $\|u_\nu\|_{3,q} = 1$ for all $\nu \in \mathbb{N}$. □

With Theorem 2.16 we also get a variational inequality:

Theorem 2.17. *1. There is a $C_V = C_V(q, G) > 0$ such that for all $u \in H_0^{3,q}(G)$ the following inequality is valid:*

$$\|\nabla\Delta u\|_q \leq C_V \sup_{0 \neq \Phi \in H_0^{3,q'}(G)} \frac{\langle \nabla\Delta u, \nabla\Delta\Phi \rangle}{\|\nabla\Delta\Phi\|_{q'}}$$

2. The validity of this variational inequality is equivalent to our solvability Theorem 2.16 in the following sense: If $G \subset \mathbb{R}^n$ is a domain such that the statement of the variational inequality is valid for $1 < q < \infty$ and q' with $\frac{1}{q} + \frac{1}{q'} = 1$ then also the solvability theorem is valid for q and q' and vice versa.

Proof. At first we will prove the statement of the variational inequality using the solvability Theorem 2.16, thus showing 1. and one part of the equivalence in 2.:
Let $u \in H_0^{3,q}(G)$ be arbitrary. Then by setting for $\Phi \in H_0^{3,q'}(G)$

$$F(\Phi) := \langle \nabla\Delta u, \nabla\Delta\Phi \rangle,$$

a bounded linear functional $F \in \left(H_0^{3,q'}(G)\right)^*$ is defined. By definition, we have

$$\langle \nabla\Delta u, \nabla\Delta\Phi\rangle = F(\Phi) \text{ for all } \Phi \in H_0^{3,q'}(G)$$

and thus by Theorem 2.16 we have

$$\|\nabla\Delta u\|_q \le C_{\Delta^3} \|F\|_{\left(H_0^{3,q'}(G)\right)^*} \le C_{\Delta^3} \sup_{\Phi\in H_0^{3,q'}(G),\, \|\Phi\|_{3,q'}\le 1} |F(\Phi)| \le$$

$$\le C_{\Delta^3} \sup_{0\ne\Phi\in H_0^{3,q'}(G)} \frac{\langle \nabla\Delta u, \nabla\Delta\Phi\rangle}{\|\Phi\|_{3,q'}}$$

and as for all $\Phi \in H_0^{3,q'}(G)$ we have

$$\|\nabla\Delta\Phi\|_{q'} = \left(\sum_{i=1}^n \|\partial_i\Delta\Phi\|_{q'}^{q'}\right)^{\frac{1}{q'}} \le \left(\sum_{i,j=1}^n \|\partial_i\partial_j\partial_j\Phi\|_{q'}^{q'}\right)^{\frac{1}{q'}} \le \|\Phi\|_{3,q'}$$

and $\|\Phi\|_{3,q'} = 0 \Leftrightarrow \|\nabla\Delta\Phi\|_{q'} = 0 \Leftrightarrow 0 = \Phi \in H_0^{3,q}(G)$ we find for all $0 \ne \Phi \in H_0^{3,q'}(G)$

$$\frac{|\langle \nabla\Delta u, \nabla\Delta\Phi\rangle|}{\|\Phi\|_{3,q'}} \le \frac{|\langle \nabla\Delta u, \nabla\Delta\Phi\rangle|}{\|\nabla\Delta\Phi\|_{q'}}$$

and thus

$$\|\nabla\Delta u\|_q \le C_{\Delta^3} \sup_{0\ne\Phi\in H_0^{3,q'}(G)} \frac{\langle \nabla\Delta u, \nabla\Delta\Phi\rangle}{\|\nabla\Delta\Phi\|_{q'}}.$$

So, validity of our solvability statement for q implies validity of our variational inequality for q.
To show the other implication in 2., we assume validity of our variational inequality for q and q'. Take a look at the set $T :=$

$$\left\{F \in \left(H_0^{3,q'}(G)\right)^* : \exists u \in H_0^{3,q}(G) : \; F(\Phi) = \langle \nabla\Delta u, \nabla\Delta\Phi\rangle \forall\Phi \in \mathcal{C}_0^\infty(G)\right\}.$$

By the variational inequality, we see that $T \subset \left(H_0^{3,q'}(G)\right)^*$ is a closed linear subspace: Taking a Cauchy-sequence $(F_\nu) \subset T \subset \left(H_0^{3,q'}(G)\right)^\star$ converging to an $F \in \left(H_0^{3,q'}(G)\right)^*$ with

$$u_\nu \in H_0^{3,q}(G) \text{ such that } F_\nu(\Phi) = \langle \nabla\Delta u_\nu, \nabla\Delta\Phi\rangle \text{ for all } \Phi \in \mathcal{C}_0^\infty(G),$$

existing by definition of T, we see by the variational inequality for q applied to Cauchy differences, that

$$\|\nabla\Delta(u_\nu - u_\mu)\|_q \leq C_V \sup_{0\neq\Phi\in H_0^{3,q'}(G)} \frac{\langle\nabla\Delta(u_\nu - u_\mu), \nabla\Delta\Phi\rangle}{\|\nabla\Delta\Phi\|_{q'}} =$$

$$= C_V \sup_{0\neq\Phi\in H_0^{3,q'}(G)} \frac{(F_\nu - F_\mu)(\Phi)}{\|\nabla\Delta\Phi\|_{q'}} \xrightarrow{\mu,\nu\to\infty} 0.$$

So, the sequence (u_ν) converges in $H_0^{3,q}(G)$ towards an element u. For any $\Phi \in \mathcal{C}_0^\infty(G)$, we have

$$F(\Phi) = \lim_{\nu\to\infty} F_\nu(\Phi) = \lim_{\nu\to\infty} \langle\nabla\Delta u_\nu, \nabla\Delta\Phi\rangle = \langle\nabla\Delta u, \nabla\Delta\Phi\rangle$$

and thus $F \in T$.
We now want to show that $T = \left(H_0^{3,q'}(G)\right)^*$: Assume that this were not so. Then by a consequence of the Hahn-Banach Theorem, we could find a functional $0 \neq H \in \left(H_0^{3,q'}(G)\right)^{**}$ with $H(F) = 0$ for all $F \in T$. But as $H_0^{3,q'}(G)$ is reflexive, we find an element $v \in H_0^{3,q'}(G)$ with $H(F) = F(v)$ for all $F \in \left(H_0^{3,q'}(G)\right)^*$. We find that for v we have therewith

$$\langle\nabla\Delta v, \nabla\Delta\Phi\rangle = 0 \text{ for all } \Phi \in H_0^{3,q}(G),$$

as every $\Phi \in H_0^{3,q}(G)$ defines an element $F_\Phi \in \left(H_0^{3,q'}(G)\right)^*$ by

$$F_\Phi(\Psi) := \langle\nabla\Delta\Phi, \nabla\Delta\Psi\rangle \text{ for all } \Psi \in H_0^{3,q'}(G)$$

and $F_\Phi \in T$. Then we have

$$\langle\nabla\Delta v, \nabla\Delta\Phi\rangle = F_\Phi(v) = H(F_\Phi) = 0$$

and thus by the variational inequality for q' we find:

$$\|\nabla\Delta v\|_{q'} \leq \tilde{C}_V \sup_{0\neq\Phi\in H_0^{3,q}(G)} \frac{\langle\nabla\Delta v, \nabla\Delta\Phi\rangle}{\|\nabla\Delta\Phi\|_q} = 0,$$

so $v = 0$ and thus $H = 0$, a contradiction. The uniqueness of the solution is shown easily: Assume that $u \in H_0^{3,q}(G)$ with

$$\langle\nabla\Delta u, \nabla\Delta\Phi\rangle = 0 \text{ for all } \Phi \in H_0^{3,q'}(G).$$

Then as above with the variational inequality for q we find that $u = 0$, so the solution must be uniquely determined in $H_0^{3,q}(G)$. The continuity of the solution process is a direct consequence of the variational inequality.

□

Remark 2.18. *The foregoing account to solvability and validity of a variational inequality for the problem related to Δ^3 could also have been used without problems for the problem of Δ^2. However, the cited theorems for the case of Δ^2 were used due to their generality (they also apply to the case of exterior domains) and the fact that their proof is more elementary than the proof of the theorems from Simader's Theory.*

2.5 A Decomposition of $H_0^{1,q}(G)$

Definition 2.19. *Between the spaces from Definition 2.1 (equipped with the respective norms) we have the following continuous linear mappings:*

- $\operatorname{div} : \underline{H}_0^{2,q}(G) \to H_{0,0}^{1,q}(G)$, $\underline{v} = (v_1, \ldots, v_n) \mapsto \sum_{i=1}^n \partial_i v_i$.
- $\underline{T}_q : H_{0,0}^{1,q}(G) \to \underline{H}_0^{2,q}(G), p \mapsto \underline{v}$ *where* $\underline{v}$ *is the unique element in* $\underline{H}_0^{2,q}(G)$ *satisfying*

$$\langle \Delta \underline{v}, \Delta \underline{\Phi} \rangle = \langle \nabla p, \nabla \operatorname{div} \underline{\Phi} \rangle \qquad \forall \underline{\Phi} \in \underline{H}_0^{2,q'}(G). \tag{14}$$

  The solvability of (14), the uniqueness of $\underline{v}$ *and the continuity of* $\underline{T}_q$ *are guaranteed by Theorem 2.9.*

- *By* $Z_q : H_{0,0}^{1,q}(G) \to H_{0,0}^{1,q}(G)$ *we denote the composition* $Z_q = \operatorname{div} \circ \underline{T}_q$.

We state a generalization of Weyl's Lemma which is valid for arbitrary open sets $G \subset \mathbb{R}^n$:

Weyl's Lemma 2.20. *Assume* $f \in L^1_{\text{loc}}(G)$ *satisfies*

$$\langle f, \Delta^m \Phi \rangle = 0 \text{ for every } \Phi \in \mathcal{C}_0^\infty(G). \tag{15}$$

Then $f \in \mathcal{C}^\infty(G)$ *and* $\Delta^m f = 0$.

Proof. A very elementary proof for the cases $m = 1, 2$ making big use of Friedrichs' mollification can be found in [14] (see Lemma 2.7, page 767 and Theorem 3.4, page 770). It can easily be generalized to $m \in \mathbb{N}$ by a simple induction argument, the first part already being done. Let $m \in \mathbb{N}$, $m > 1$ and the assumption hold for $m - 1$ and $m = 1$. As being $\mathcal{C}^\infty$ is a local property we can look at $x \in G$ arbitrary and it suffices to show that f is $\mathcal{C}^\infty$ in an open ball $B_r(x) \subset G$. So let now $x \in G$ be arbitrary, $r > 0$ be so small that $B_r(x) \subset\subset G$ and $\varepsilon > 0$ be so small that $B_{r+2\varepsilon}(x) \subset\subset G$. Because $\Phi_z(y) := j_\varepsilon(y - z)$ is for fixed $z \in B_r(x)$ a function with compact support in G, we see with equation (15) that we have

$$0 = \int_G f(y) \Delta_y^m j_\varepsilon(y - z) dy = \int_G f(y) \Delta_z^m j_\varepsilon(y - z) dy =$$

$$= \Delta^m \int_G j_\varepsilon(y-z) f(y) dy = \Delta^m f_\varepsilon(z)$$

and thus $\Delta^m f_\varepsilon = 0$ in $B_r(x)$. So we have shown for every sufficiently small $\varepsilon > 0$ that $\Delta^{m-1} f_\varepsilon$ is harmonic in $B_r(x)$. By the property of harmonic functions that they stay invariant under mollification with an only radially depending kernel (to be more precise: For h harmonic in G we have $h(z) = h_\varepsilon(z)$ for all $z \in G$ with $\operatorname{dist}(z, \partial G) < \varepsilon$) and the fact that for $z \in B_r(x)$ and $0 < \delta < \varepsilon$ we have

$$(f_\varepsilon)_\delta (z) = (f_\delta)_\varepsilon (z)$$

and

$$\partial_i (g_\varepsilon)(z) = (\partial_i g)_\varepsilon (z) \text{ for } g \in \mathcal{C}^1(G) \text{ and } i \in \{1, \ldots, n\}$$

used iteratively, we see that for $z \in B_r(x)$ we have with $0 < \delta < \varepsilon$ since $\Delta^{m-1}(f_\delta)$ and $\Delta^{m-1}(f_\varepsilon)$ are harmonic

$$\Delta^{m-1} f_\varepsilon(z) = \left(\Delta^{m-1} f_\varepsilon\right)_\delta (z) = \Delta^{m-1} (f_{\varepsilon,\delta})(z) = \Delta^{m-1} (f_{\delta,\varepsilon})(z) =$$

$$= \left(\Delta^{m-1} f_\delta\right)_\varepsilon (z) = \Delta^{m-1} f_\delta(z).$$

So we find for all $0 < \delta < \varepsilon$:

$$\Delta^{m-1}(f_\delta) = \Delta^{m-1}(f_\varepsilon) \text{ on } B_r(x)$$

Defining $g := \Delta^{m-1} f_\varepsilon$, we see that g is harmonic on $B_r(x)$ and that it is no restriction to assume $g \in \overline{\mathcal{C}}^\infty(B_r(x))$. We can find a $h \in \overline{\mathcal{C}}^\infty(B_r(x))$ with $\Delta^{m-1} h = g$: By classical theory we find a $h_1 \in \overline{\mathcal{C}}^\infty(B_r(x))$ with $\Delta h_1 = g$ with the representation formula

$$h_1(y) := -\int_{B_r(x)} S(y-z) g(z)\, dz,$$

where S denotes the fundamental solution to the Laplacian, see for example [16], Satz 4.5, page 102. Iterating this process, we finally reach our sought after h. Taking now a close look at $f - h$, we see that for $\Phi \in \mathcal{C}_0^\infty(G)$ we have

$$\langle f - h, \Delta^{m-1}\Phi\rangle = \langle f, \Delta^{m-1}\Phi\rangle - \langle h, \Delta^{m-1}\Phi\rangle = \lim_{\varepsilon\to 0}\langle f_\varepsilon, \Delta^{m-1}\Phi\rangle - \langle \Delta^{m-1} h, \Phi\rangle =$$

$$= \lim_{\varepsilon\to 0}\langle f_\varepsilon, \Delta^{m-1}\Phi\rangle - \langle g, \Phi\rangle = \lim_{\varepsilon\to 0}\left(\langle f_\varepsilon, \Delta^{m-1}\Phi\rangle - \langle \Delta^{m-1} f_\varepsilon, \Phi\rangle\right) = 0,$$

and thus by the induction hypothesis we conclude $f - h \in \mathcal{C}^\infty(B_r(x))$ and as $h \in \mathcal{C}^\infty(B_r(x))$ we also find $f \in \mathcal{C}^\infty(B_r(x))$. □

Definition 2.21. *We introduce the spaces*

- $A^q(G) := \{p \in H_0^{1,q}(G) : \quad p = \Delta s$ *for an* $s \in H_0^{3,q}(G)\}$
- $B^q(G) := \{p \in H_0^{1,q}(G) : \quad \int_G p\Delta^2\Phi\, dx = 0 \quad \forall \Phi \in \mathcal{C}_0^\infty(G)\}$

Furthermore we write $A_0^q(G) := A^q(G) \cap H_{0,0}^{1,q}(G)$ *and* $B_0^q(G) := B^q(G) \cap H_{0,0}^{1,q}(G)$. *Note that* $A^q(G) = A_0^q(G)$ *as can easily be seen by Gauß' divergence Theorem.*

In view of Weyl's Lemma 2.20 above we readily see that the space $B^q(G)$ is consisting exactly of the biharmonic $H_0^{1,q}(G)$-functions. In particular, every $h \in B^q(G)$ fulfills $h \in \mathcal{C}^\infty(G)$.

Theorem 2.22. $\mathcal{C}_{0,0}^\infty(G) := \mathcal{C}_0^\infty(G) \cap H_{0,0}^{1,q}(G)$ *is dense in* $H_{0,0}^{1,q}(G)$.

Proof. Let $p \in H_{0,0}^{1,q}(G)$ be arbitrary and $f \in \mathcal{C}_0^\infty(G)$ with $f \geq 0$ in G and $\int_G f\, dx = 1$. As $p \in H_0^{1,q}(G)$, we find a sequence $(p_\nu)_{\nu\in\mathbb{N}} \subset \mathcal{C}_0^\infty(G)$ with $\|(p - p_\nu)\|_{1,q} \xrightarrow{\nu\to\infty} 0$. Define $c_\nu := \int_G p_\nu\, dx$. Then with the Hölder inequality we get

$$|c_\nu| = \left|\int_G p_\nu\, dx\right| = \left|\int_G p_\nu - p\, dx\right| \leq \int_G |p_\nu - p|\, dx \leq |G|^{\frac{1}{q'}} \|p_\nu - p\|_q \to 0.$$

Let now $\tilde{p}_\nu := p_\nu - c_\nu \cdot f$. We then have $\tilde{p}_\nu$ mean-value-free and thus $\tilde{p}_\nu \in \mathcal{C}_{0,0}^\infty(G)$ and

$$\|\tilde{p}_\nu - p\|_{1,q} \leq |c_\nu|\, \|f\|_{1,q} + \|p_\nu - p\|_{1,q} \xrightarrow{\nu\to\infty} 0.$$

□

The weak solvability of the Dirichlet problem for Δ^3 with zero boundary data from Theorem 2.16 gives rise to a direct (if $q = 2$ orthogonal) decomposition of $H_0^{1,q}(G)$ and $H_{0,0}^{1,q}(G)$ similar to the decomposition of $L^q(G)$ obtained by Müller, see [10], Satz IV.2.1, page 201:

Theorem 2.23. *We have the direct decompositions*

$$H_0^{1,q}(G) = A^q(G) \oplus B^q(G) \tag{16}$$

$$H_{0,0}^{1,q}(G) = A_0^q(G) \oplus B_0^q(G) \tag{17}$$

These decompositions are orthogonal if $q = 2$.
If $p = \Delta s + h$ *according to this decomposition we find the estimate:*

$$\|\nabla\Delta s\|_q + \|\nabla h\|_q \leq C_D \|\nabla p\|_q \tag{18}$$

with a constant $C_D > 0$ *depending only on* G *and* q.

Proof. For $p \in H_0^{1,q}(G)$ there exists according to our Theorem 2.16 an unique $s \in H_0^{3,q}(G)$ satisfying

$$\langle \nabla \Delta s, \nabla \Delta \Phi \rangle = \langle \nabla p, \nabla \Delta \Phi \rangle \quad \forall \Phi \in H_0^{3,q'}(G) \tag{19}$$

and we have a constant $C_{\Delta^3} = C_{\Delta^3}(q, G)$ with $\|\nabla \Delta s\|_q \leq C_{\Delta^3} \|\nabla p\|_q$. Define $h := p - \Delta s \in H_0^{1,q}(G)$. Then we have for all $\Phi \in \mathcal{C}_0^\infty(G)$:

$$\langle h, \Delta^2 \Phi \rangle = -\langle \nabla h, \nabla \Delta \Phi \rangle = \langle \nabla p - \nabla \Delta s, \nabla \Delta \Phi \rangle = 0$$

that is $h \in B^q(G)$ and we have a representation $p = \Delta s + h$ as desired. The uniqueness of the representation of $p = \Delta s + h$ with $s \in H_0^{3,q}(G)$ and $h \in B^q(G)$ is due to the unique solvability of (19): Assume that $p = \Delta s_1 + h_1$ and $p = \Delta s_2 + h_2$. Then we have $\Delta(s_1 - s_2) = h_2 - h_1 \in B^q(G)$ is biharmonic and therewith

$$\langle \nabla \Delta (s_1 - s_2), \nabla \Delta \Phi \rangle = -\langle \Delta (s_1 - s_2), \Delta^2 \Phi \rangle = 0 \quad \forall \Phi \in \mathcal{C}_0^\infty(G)$$

so $s_1 = s_2$ and then $h_1 = p - \Delta s_1 = p - \Delta s_2 = h_2$. So the decomposition is direct and we have shown (16).
Furthermore, we see that for every $p \in H_0^{1,q}(G)$ we have $\Delta s \in H_{0,0}^{1,q}(G)$ and thus $h \in H_{0,0}^{1,q}(G)$ if and only if $p \in H_{0,0}^{1,q}(G)$ yielding (17).
To see that this decomposition is orthogonal in case $q = 2$ we note that if $h \in B^2(G)$, $\Delta s \in A^2(G)$ we find $\langle \nabla h, \nabla \Delta s \rangle = 0$ (through $H_0^{3,2}(G)$-approximation of s by $\mathcal{C}_0^\infty(G)$-functions and partially integrating).
Further we have for a given $p \in H_0^{1,q}(G)$ and $p = \Delta s + h$:

$$\|\nabla \Delta s\|_q \leq C_{\Delta^3} \|\nabla p\|_q \tag{20}$$

and

$$\|\nabla h\|_q = \|\nabla p - \nabla \Delta s\|_q \leq \|\nabla p\|_q + \|\nabla \Delta s\|_q \leq (C_{\Delta^3} + 1) \|\nabla p\|_q \tag{21}$$

Sticking (20) and (21) together we get:

$$\|\nabla \Delta s\|_q + \|\nabla h\|_q \leq \underbrace{(2C_{\Delta^3} + 1)}_{=:C_D} \|\nabla p\|_q .$$

□

The decomposition (17) and the operator Z_q defined in (2.19) are closely related and so (17) plays an important role in the study of Z_q. As a first insight we have:

Theorem 2.24. *Regarding the restrictions of Z_q to $A_0^q(G)$ and to $B_0^q(G)$, we get*

$$Z_q|_{A_0^q(G)} : A_0^q(G) \to A_0^q(G) \text{ and } Z_q(p) = p \quad \forall p \in A_0^q(G)$$
$$Z_q|_{B_0^q(G)} : B_0^q(G) \to B_0^q(G)$$

Proof. Let $p \in A_0^q(G)$. Then $p = \Delta s$ for an $s \in H_0^{3,q}(G)$ and we find $\underline{T}_q(p) = \nabla s$, for we have

$$\langle \Delta \nabla s, \Delta \underline{\Phi} \rangle = \langle \nabla \Delta s, \Delta \underline{\Phi} \rangle = \langle \nabla p, \Delta \underline{\Phi} \rangle \text{ for all } \underline{\Phi} \in \underline{H}_0^{2,q'}(G)$$

and so by uniqueness of the solution (Theorem 2.16) we have

$$\underline{T}_q(p) = \nabla s, \; Z_q(p) = \operatorname{div} \nabla s = \Delta s = p$$

For $p \in B_0^q(G)$ we have for $\Phi \in \mathcal{C}_0^\infty(G)$

$$\langle Z_q(p), \Delta^2 \Phi \rangle = \langle \operatorname{div} \underline{T}_q(p), \Delta^2 \Phi \rangle = -\langle \Delta \underline{T}_q(p), \nabla \Delta \Phi \rangle = -\langle \nabla p, \Delta \nabla \Phi \rangle =$$
$$= \langle p, \Delta^2 \Phi \rangle = 0$$

and so we conclude $Z_q(p) \in B_0^q(G)$. □

Regarding eigenvalues of Z_q we have due to our direct decomposition from Theorem 2.23 and Theorem 2.24 the following easy fact:

Theorem 2.25. *Suppose $\lambda \in \mathbb{R}$ and $p \in H_{0,0}^{1,q}(G)$ suffice $Z_q(p) = \lambda p$. Then we have $\lambda = 1$ or $p \in B_0^q(G)$.*

Proof. Assume that $Z_q(p) = \lambda p$. Applying the decomposition (17) from Theorem 2.23 to p we get $p = \Delta s + h$ with $s \in H_0^{3,q}(G)$ and $h \in B_0^q(G)$. So we have on the one hand

$$Z_q(p) = \lambda p = \lambda(\Delta s + h) = \underbrace{\lambda \Delta s}_{\in A_0^q(G)} + \underbrace{\lambda h}_{\in B_0^q(G)}$$

and on the other hand using Theorem 2.24 we have

$$Z_q(p) = Z_q(\Delta s + h) = Z_q(\Delta s) + Z_q(h) = \underbrace{\Delta s}_{\in A_0^q(G)} + \underbrace{Z_q(h)}_{\in B_0^q(G)}.$$

So by the directness of the decomposition (2.23) we have:

$$\lambda \Delta s = \Delta s \qquad \text{and} \qquad \lambda h = Z_q(h)$$

The first of these two equalities can only be satisfied if $\lambda = 1$ or $s = 0$. So we have shown: $\lambda = 1$ or $p = h \in B_0^q(G)$.

□

Note that the "or" in (2.25) is not an exclusive one. The question whether there are $p \in B_0^q(G)$ with $Z_q(p) = p$ will be examined later (see subsection 8.2). It will show up that there is a finite dimensional subspace of $B_0^q(G)$ of such elements and that the dimension of this subspace is only dependent on topological properties of G.

2.6 Another Decomposition of $H_0^{1,q}(G)$

In this paper we will make use of the following decomposition of $H_0^{1,q}(G)$ which has already been investigated by C. G. Simader in [17] for the case $q = 2$:

Theorem 2.26. *For $H_0^{1,q}(G)$ we have the direct decomposition:*

$$H_0^{1,q}(G) = H_{0,0}^{1,q}(G) \oplus \underbrace{\left\{g \in H_0^{1,q}(G) : \quad \langle \nabla g, \Delta \underline{\Phi} \rangle = 0 \text{ for all } \underline{\Phi} \in \underline{\mathcal{C}}_0^\infty(G)\right\}}_{=:N^q(G)} \tag{22}$$

Furthermore, for $\Psi \in H_0^{1,q}(G)$ with $\Psi = \Psi_0 + \Psi_1$ where $\Psi_0 \in H_{0,0}^{1,q}(G)$ and $\Psi_1 \in N^q(G)$, we have the estimate

$$\|\nabla \Psi_0\|_q + \|\nabla \Psi_1\|_q \leq C_d \|\nabla \Psi\|_q \tag{23}$$

with a constant $C_d = C_d(q, G) > 0$.
The space $N^q(G)$ is a one dimensional real vector space and independent of q.

Proof. Let at first q be fixed and $g \in N^q(G)$. Then ∇g satisfies

$$\langle \nabla g, \Delta \underline{\Phi} \rangle = 0 \text{ for all } \underline{\Phi} \in \underline{\mathcal{C}}_0^\infty(G).$$

By Weyl's Lemma 2.20 we conclude that for $i = 1, \dots, n$ we have $\partial_i g \in \mathcal{C}^\infty(G)$ and consequently

$$\Delta g = \sum_{i=1}^n \partial_i \partial_i g \in \mathcal{C}^\infty(G), \text{ too.}$$

Now we find that for $\underline{\Phi} \in \underline{\mathcal{C}}_0^\infty(G)$

$$0 = \langle \nabla g, \Delta \underline{\Phi} \rangle = -\langle g, \operatorname{div} \Delta \underline{\Phi} \rangle = -\langle \Delta g, \operatorname{div} \underline{\Phi} \rangle = \langle \nabla (\Delta g), \underline{\Phi} \rangle$$

and thus

$$\nabla(\Delta g) = 0,$$

so Δg is constant in G. Furthermore we find that g itself is in $\mathcal{C}^\infty(G)$, for if we define the constant value of Δg to be called k, we see that the function

$$f := g - \frac{k}{2n} |x|^2$$

satisfies $\Delta f = k - 2n\frac{k}{2n} = 0$ and by Weyl's Lemma 2.20 it follows that $f \in \mathcal{C}^\infty(G)$ and then $g = f + \frac{k}{2n} |x|^2 \in \mathcal{C}^\infty(G)$.
Regarding the functional equation

$$\langle \nabla h, \nabla \Phi \rangle = \langle 1, \Phi \rangle \text{ for all } \Phi \in H_0^{1,q'}(G), \tag{24}$$

where 1 denotes the function which is constant with value 1 on G, we get an unique $h \in H_0^{1,q}(G)$ solving the equation. Doing this for two different values q_1 and q_2 of q, we find functions $h_1 \in H_0^{1,q_1}(G)$, $h_2 \in H_0^{1,q_2}(G)$ with

$$\langle \nabla h_1, \nabla \Phi \rangle = \langle 1, \Phi \rangle \text{ for all } \Phi \in H_0^{1,q_1'}(G) \text{ and}$$

$$\langle \nabla h_2, \nabla \Phi \rangle = \langle 1, \Phi \rangle \text{ for all } \Phi \in H_0^{1,q_2'}(G).$$

As G is bounded, we see immediately that $h_1, h_2 \in H_0^{1,\min\{q_1,q_2\}}(G)$ and thus $h_2 - h_1 \in H_0^{1,\min\{q_1,q_2\}}(G)$, and with

$$\langle \nabla (h_2 - h_1), \nabla \Phi \rangle = 0 \text{ for all } \Phi \in \mathcal{C}_0^\infty(G)$$

we find that $h_2 - h_1 = 0$ or $h_2 = h_1$.
So we have $h \in H_0^{1,q}(G)$ for all $1 < q < \infty$, solving (24) for every $1 < q < \infty$.
By the unique solvability of (24) in $H_0^{1,q}(G)$ we see that for $g \in N^q(G)$ with $\Delta g = c$ in G it must be $g = ch$. We have shown that $N^q(G)$ is independent of q and one dimensional.
We further find for h like above: $h(x) \geq 0$ for almost every $x \in G$:
We first define h_- to be the negative part of h, that is

$$h_-(x) := h(x) \text{ if } h(x) < 0$$

$$h_-(x) := 0 \text{ otherwise.}$$

Then $h_- \in H_0^{1,2}(G)$ by Theorem 2.4 and

$$0 \leq \|\nabla h_-\|_2^2 = \langle \nabla h_-, \nabla h_- \rangle = \langle \nabla h, \nabla h_- \rangle = \langle 1, h_- \rangle \leq 0,$$

so we see $h_- = 0$. Because h is not the zero function, we further have

$$\int_G h \, dx > 0$$

and we can define

$$\tilde{h} := \frac{1}{\int_G h\,dx} h$$

with

$$\tilde{h} \geq 0, \text{ and } \int_G \tilde{h}\,dx = 1.$$

Now we can prove the decomposition (22): Let $\Psi \in H_0^{1,q}(G)$ be arbitrary. Then let $c_\Psi := \int_G \Psi\,dx$ and we have $\Psi_0 := \Psi - c_\Psi \tilde{h} \in H_{0,0}^{1,q}(G)$, $\Psi_1 := c_\Psi \tilde{h} \in N^q(G)$ and

$$\Psi_0 + \Psi_1 = \Psi - c_\Psi \tilde{h} + c_\Psi \tilde{h} = \Psi.$$

For the directness of the sum, suppose, $\Psi \in H_{0,0}^{1,q}(G) \cap N_q(G)$. Then $\Psi = \alpha \tilde{h}$ with $\alpha \in \mathbb{R}$ and because of

$$0 = \int_G \Psi\,dx = \int_G \alpha \tilde{h}\,dx = \alpha$$

we see $\Psi = 0$.
To show the estimate (23), we look at

$$\|\nabla \Psi_1\|_q = |c_\Psi| \underbrace{\left\|\nabla \tilde{h}\right\|_q}_{=:c=c(q,G)} = c \left| \int_G \Psi\,dx \right| \leq c \int_G |\Psi|\,dx \leq c \left(\int_G 1\,dx \right)^{\frac{1}{q'}} \|\Psi\|_q \leq$$

$$\leq \underbrace{c\,|G|^{\frac{1}{q'}}\,C_{\text{Poincaré}}(q,G)}_{=:C(q,G)} \|\nabla \Psi\|_q$$

So, we have

$$\|\nabla \Psi_0\|_q + \|\nabla \Psi_1\|_q = \|\nabla(\Psi - \Psi_1)\|_q + \|\nabla \Psi_1\|_q \leq$$

$$\leq \|\nabla \Psi\|_q + 2\,\|\nabla \Psi_1\|_q \leq \underbrace{(2C+1)}_{=:C_d(q,G)} \|\nabla \Psi\|_q.$$

□

Remark 2.27. *One can easily show that the decomposition from Theorem 2.26 is orthogonal in the case $q = 2$ by using Theorem 7.6, which is still to be shown:*
For let $r \in H_{0,0}^{1,2}(G)$ and $s \in N^2(G)$, we see that we find due to Theorem 7.6 a $\underline{v} \in \underline{H}_0^{2,2}(G)$ satisfying $\operatorname{div} \underline{v} = r$ and thus

$$\langle \nabla r, \nabla s \rangle = \langle \nabla \operatorname{div} \underline{v}, \nabla s \rangle.$$

Approximating $\underline{v}$ in the $H^{2,2}(G)$-norm by $(\underline{v}_k) \subset \underline{\mathcal{C}}_0^\infty(G)$, we see by partial integration that

$$\langle \nabla \operatorname{div} \underline{v}_k, \nabla s \rangle. = -\langle \Delta \operatorname{div} \underline{v}_k, s \rangle = -\langle \operatorname{div} \Delta \underline{v}_k, s \rangle = \langle \Delta \underline{v}_k, \nabla s \rangle = 0,$$

as $s \in N^2(G)$ and thus also

$$\langle \nabla r, \nabla s \rangle = 0.$$

3 Helpful Theorems

3.1 A Helpful Function

In [22] (Theorem 6.1) Weyers constructed to a given $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{k+1}$ a function $\zeta = \zeta(G) \in \mathcal{C}_0^k(\mathbb{R}^n)$ with $\zeta|_{\partial G} = 0$ and $\nabla \zeta|_{\partial G} = \underline{N}$ where $\underline{N}$ is the outward unit normal of G. So in our case of a fixed G with $\partial G \in \mathcal{C}^5$, we find a $\zeta = \zeta(G)$ according to Weyers with $\zeta \in \mathcal{C}_0^4(\mathbb{R}^n)$, $\zeta|_{\partial G} = 0$ and $\nabla \zeta|_{\partial G} = \underline{N}$. As $\zeta \in \mathcal{C}_0^4(\mathbb{R}^n)$ we find a constant $C_\zeta > 0$ with $\sup_{x \in \mathbb{R}^n} |D^\alpha \zeta(x)| < C_\zeta$ for all α with $|\alpha| \leq 4$.

3.2 The Theorems

The following theorems involve a function ζ meeting varying requirements. Our $\zeta(G)$ as defined in Subsection 3.1 satisfies all these requirements and is therefore admissible for each of these theorems. The following theorems are indeed needed only for the case "$\zeta = \zeta(G)$" and are tailored to the use of our ζ from Subsection 3.1. We state and prove the theorems for our fixed G with $\partial G \in \mathcal{C}^5$ although we even could weaken the requirements for ∂G.

Theorem 3.1. *For $p \in H^{1,q}(G)$ and $\zeta \in \overline{C}^1(G)$ it follows: $p\zeta \in H^{1,q}(G)$ and $\nabla (p\zeta) = \zeta \nabla p + p \nabla \zeta$.*

Proof. First of all $p\zeta$ is in $L^q(G)$ as $p \in L^q(G)$ and ζ is continuous and bounded in G. Let now $i \in \{1, \ldots, n\}$. We will show that $p\zeta$ has a weak ∂_i-derivative. Let $\Phi \in \mathcal{C}_0^\infty(G)$. Then according to the classical product rule we have:

$$\int_G (p\zeta) \partial_i \Phi \, dx = \int_G p \partial_i (\zeta \Phi) \, dx - \int_G p \partial_i \zeta \Phi \, dx =$$

and with $\Phi \in \mathcal{C}_0^\infty(G)$, we also have $\zeta \Phi \in \mathcal{C}_0^4(G)$. With $p \in H^{1,q}(G)$ it follows that this is equal to

$$-\int_G \partial_i p \zeta \Phi \, dx - \int_G p \partial_i \zeta \Phi \, dx = -\int_G (\zeta \partial_i p + p \partial_i \zeta) \Phi \, dx,$$

and $\zeta\partial_i p + p\partial_i\zeta \in L^q(G)$ because p and $\partial_i p$ are in $L^q(G)$ and $\zeta, \partial_i\zeta$ are continuous and bounded in G. $\square$

Theorem 3.2. *Let $\underline{u} \in \underline{H}_0^{2,q}(G)$ and $\zeta \in \overline{\mathcal{C}}^3(G)$. Then $\underline{u}\cdot\nabla\zeta = \sum_{i=1}^n u_i\partial_i\zeta \in H_0^{2,q}(G)$.*

Proof. Take a sequence $(\underline{u}_\nu)_{\nu\in\mathbb{N}} \subset \underline{\mathcal{C}}_0^\infty(G)$ with $\underline{u}_\nu \xrightarrow{\underline{H}^{2,q}(G)} \underline{u}$. Then we find that $(\underline{u}_\nu \cdot \nabla\zeta)_{\nu\in\mathbb{N}} \subset \mathcal{C}_0^2(G) \subset H_0^{2,q}(G)$ is a Cauchy-sequence in $H_0^{2,q}(G)$ because for two indices $\nu, \mu \in \mathbb{N}$ we find

$$\left\|\Delta(\underline{u}_\nu \cdot \nabla\zeta - \underline{u}_\mu \cdot \nabla\zeta)\right\|_q = \left\|\sum_{i=1}^n \Delta\left(\left(\underline{u}_\nu - \underline{u}_\mu\right)_i \partial_i\zeta\right)\right\|_q =$$

$$= \left\|\sum_{i=1}^n \Delta\left(\underline{u}_\nu - \underline{u}_\mu\right)_i \partial_i\zeta + 2\sum_{i,j=1}^n \partial_j\left(\underline{u}_\nu - \underline{u}_\mu\right)_i \partial_j\partial_i\zeta + \sum_{i=1}^n \left(\underline{u}_\nu - \underline{u}_\mu\right)_i \Delta\partial_i\zeta\right\|_q$$

$$\leq \left\|\sum_{i=1}^n \Delta\left(\underline{u}_\nu - \underline{u}_\mu\right)_i \partial_i\zeta\right\|_q + 2\left\|\sum_{i,j=1}^n \partial_j\left(\underline{u}_\nu - \underline{u}_\mu\right)_i \partial_j\partial_i\zeta\right\|_q +$$

$$+ \left\|\sum_{i=1}^n \left(\underline{u}_\nu - \underline{u}_\mu\right)_i \Delta\partial_i\zeta\right\|_q \leq$$

$$C_\zeta\left(\sum_{i=1}^n \underbrace{\left\|\Delta\left(\underline{u}_\nu - \underline{u}_\mu\right)_i\right\|_q}_{\xrightarrow{\mu,\nu\to\infty} 0} + 2\sum_{i,j=1}^n \underbrace{\left\|\partial_j\left(\underline{u}_\nu - \underline{u}_\mu\right)_i\right\|_q}_{\xrightarrow{\mu,\nu\to\infty} 0} + n\sum_{i=1}^n \underbrace{\left\|\left(\underline{u}_\nu - \underline{u}_\mu\right)_i\right\|_q}_{\xrightarrow{\mu,\nu\to\infty} 0}\right),$$

where C_ζ is a constant such that $\sup_{x\in\overline{G}} |D^\alpha\zeta(x)| < C_\zeta$ for all α with $|\alpha| \leq 3$. As $H_0^{2,q}(G)$ is complete, the Cauchy-sequence $(\underline{u}_\nu \cdot \nabla\zeta)_{\nu\in\mathbb{N}}$ converges to an element $v \in H_0^{2,q}(G)$. It is the Theorem of Riesz-Fischer (or better: An addendum to this Theorem found in any modern book on calculus) that allows us to pass to a subsequence of $(\underline{u}_\nu \cdot \nabla\zeta)_{\nu\in\mathbb{N}}$ which converges almost everywhere in G to v. But taking analogously a subsequence of $(\underline{u}_\nu)_{\nu\in\mathbb{N}}$ converging almost everywhere in G to $\underline{u}$ we see that it must be $v = \underline{u}\cdot\nabla\zeta \in H_0^{2,q}(G)$. $\square$

Theorem 3.3. *Let $f \in H_0^{1,q}(G) \cap H^{2,q}(G)$ with $\nabla f \in \underline{H}_0^{1,q}(G)$. Then $f \in H_0^{2,q}(G)$.*

Proof. Due to the compactness of $\overline{G}$ and the smoothness of ∂G we can find an $N \in \mathbb{N}$, open subsets $U_0, \dots U_N \subset \mathbb{R}^n$ with $U_0 \subset\subset G$, $\bigcup_{i=0}^N U_i \supset G$ such that for $i \geq 1$ we have

$$\partial G \cap U_i = \left\{ (x', \Phi_i(x')) : x' \in \Delta_i := \prod_{j=1}^{n-1}]x_{i,j} - \delta_i, x_{i,j} + \delta_i[\right\}$$

after an appropriate permutation of variables with a $\delta_i > 0$ for a certain $x_i = (x_{i,1}, \dots, x_{i,n}) \in \mathbb{R}^n$ and a certain $\Phi_i \in \mathcal{C}^5(\Delta_i)$. We can furthermore assume that for $i \geq 1$ we have

$$G \cap U_i = \bigcup_{x' \in \Delta_i} \{x'\} \times]\Phi_i(x'), \Phi_i(x') + \varepsilon_i[\tag{25}$$

or

$$G \cap U_i = \bigcup_{x' \in \Delta_i} \{x'\} \times]\Phi_i(x') - \varepsilon_i, \Phi_i(x')[\tag{26}$$

for an $\varepsilon_i > 0$.
We find a partition of unity subordinate to the covering $U_0, \dots, U_N$, that is we find for $i = 0, \dots, N$ functions $\Psi_i \in \mathcal{C}_0^\infty(U_i)$ with $0 \leq \Psi_i(x) \leq 1$ and $\sum_{i=0}^N \Psi_i(x) = 1$ for all $x \in G$. As $\text{supp}(\Psi_0) \subset U_0 \subset\subset G$ we have $\Psi_0 \in \mathcal{C}_0^\infty(G)$ and thus $\Psi_0 f \in H_0^{2,q}(G)$ can be shown easily by approximation as in the proof of Theorem 3.2. In the following we will show that also for arbitrary $i \geq 1$ we have $\Psi_i f \in H_0^{2,q}(G)$ which will yield that $f = \sum_{i=0}^N \Psi_i f \in H_0^{2,q}(G)$. We fix an $i \geq 1$ and suppress in the following the indices. Assume that for $U \cap G$ the case (25) applies, case (26) can be treated analogously. We define the function

$$z : \Delta \times]0, \varepsilon[\to G \cap U,\ x = (x', y) \mapsto (x', \Phi(x') + y)$$

and it can be seen by standard argumentation that with

$$\Psi f \in H_0^{1,q}(G \cap U) \cap H^{2,q}(G \cap U)$$

we also have

$$(\Psi f) \circ z \in H_0^{1,q}(\Delta \times]0, \epsilon[) \cap H^{2,q}(\Delta \times]0, \epsilon[)$$

and

$$\Psi f \in H_0^{2,q}(G \cap U) \Leftrightarrow (\Psi f) \circ z \in H_0^{2,q}(\Delta \times]0, \varepsilon[)\,.$$

With $\tilde{f} := (\Psi f) \circ z$ and $Q := \Delta \times]0, \varepsilon[$ we have $\tilde{f} \in H_0^{1,q}(Q) \cap H^{2,q}(Q)$ and $\nabla \tilde{f} \in \underline{H}_0^{1,q}(Q)$. Because of $\text{supp}(\Psi) \subset U$ we find an $\varepsilon' < \varepsilon, \delta$ with $\varepsilon' > 0$ such

that with $\Delta' := \prod_{j=1}^{n-1} \left]x_j' - \delta + \varepsilon', x_j' + \delta - \varepsilon'\right[\subset \Delta = \prod_{j=1}^{n-1} \left]x_j' - \delta, x_j' + \delta\right[$ we find $\tilde{f}(x', x_n) = 0$ for all (x', x_n) in $Q \setminus (\Delta' \times (]0, \varepsilon - \varepsilon'[))$.
As $\tilde{f} \in H_0^{1,q}(Q)$ we find a sequence $(u_\nu)_{\nu \in \mathbb{N}} \subset \mathcal{C}_0^\infty(Q)$ with $u_\nu \xrightarrow{H^{1,q}(Q)} \tilde{f}$. For a fixed $\nu \in \mathbb{N}$ we have by the Fundamental Theorem of Calculus for an arbitrary $x = (x', x_n) \in Q$ with $0 < x_n < \rho < \varepsilon$:

$$u_\nu(x', x_n) = u_\nu(x', x_n) - \underbrace{u_\nu(x', 0)}_{=0} = \int_0^{x_n} \partial_n u_\nu(x', t)\, dt$$

and thus

$$|u_\nu(x', x_n)|^q \leq (x_n)^{\frac{q}{q'}} \int_0^{x_n} |\partial_n u_\nu(x', t)|^q \; dt \leq \rho^{\frac{q}{q'}} \int_0^{\rho} |\partial_n u_\nu(x', t)|^q \; dt$$

after using the Hölder inequality. Integrating this over $\Delta \times]0, \rho[$ yields:

$$\int_{\Delta \times]0,\rho[} |u_\nu(x', x_n)|^q \; dx'\, dx_n \leq \int_\Delta \int_0^\rho \int_0^\rho |\partial_n u_\nu(x', t)|^q \, \rho^{q-1}\, dt\, dx_n\, dx' \leq$$

$$\leq \rho^q \, \|\partial_n u_\nu\|_{q, \Delta \times]0,\rho[}^q$$

and thus we have for every $0 < \rho < \varepsilon$

$$\|u_\nu\|_{q, \Delta \times]0,\rho[} \leq \rho \, \|\partial_n u_\nu\|_{q, \Delta \times]0,\rho[}$$

and finally

$$\|\tilde{f}\|_{q, \Delta \times]0,\rho[} \leq \rho \|\partial_n \tilde{f}\|_{q, \Delta \times]0,\rho[} \tag{27}$$

by approximation.
Doing the same with $\partial_n \tilde{f} \in H_0^{1,q}(Q)$ we get

$$\|\partial_n \tilde{f}\|_{q, \Delta \times]0,\rho[} \leq \rho \|{\partial_n}^2 \tilde{f}\|_{q, \Delta \times]0,\rho[} \tag{28}$$

and in the end sticking the two estimates (27) and (28) together

$$\|\tilde{f}\|_{q, \Delta \times]0,\rho[} \leq \rho^2 \|{\partial_n}^2 \tilde{f}\|_{q, \Delta \times]0,\rho[}. \tag{29}$$

Let now $\eta \in \mathcal{C}^\infty(\mathbb{R})$ be a function with $0 \leq \eta(t) \leq 1$ for all $t \in \mathbb{R}$, $\eta(t) = 0$ for all $t \in \mathbb{R}$ with $t \leq 1$ and $\eta(t) = 1$ for all $t \in \mathbb{R}$ with $t \geq 2$. Then $\operatorname{supp}(\eta'), \operatorname{supp}(\eta'') \subset [1, 2]$. Define for $k \in \mathbb{N}$: $\eta_k(t) := \eta(kt)$. Then $\eta_k'(t) = k\eta'(kt)$, $\eta_k''(t) = k^2 \eta''(kt)$, $\operatorname{supp}(\eta_k'), \operatorname{supp}(\eta_k'') \subset \left[\frac{1}{k}, \frac{2}{k}\right]$ and we can find a constant $c > 0$ such that $|\eta_k'(t)| \leq ck$ and $|\eta_k''(t)| \leq ck^2$ for all $t \in \mathbb{R}$.
Defining $\tilde{f}_k(x', x_n) := \tilde{f}(x', x_n)\eta_k(x_n)$ we see:

1. (i) $\partial_n(\tilde{f}_k(x', x_n)) = \partial_n \tilde{f}(x', x_n)\eta_k(x_n) + \tilde{f}(x', x_n)\eta_k'(x_n)$
2. (ii) $\partial_n\partial_n(\tilde{f}_k(x', x_n)) =$
$= \partial_n\partial_n \tilde{f}(x', x_n)\eta_k(x_n) + 2\partial_n \tilde{f}(x', x_n)\eta_k'(x_n) + \tilde{f}(x', x_n)\eta_k''(x_n)$

Now we are able to see that the sequence $(\tilde{f}_k)_{k\in\mathbb{N}}$ approximates $\tilde{f}$ with respect to the $H^{2,q}(Q)$-norm:

- We have
$$\left\|\tilde{f} - \tilde{f}_k\right\|_{q,Q} = \left\|\tilde{f}(1-\eta_k)\right\|_{q,Q} \to 0$$
by Lebesgue's Dominated Convergence Theorem as $(1-\eta_k)$ is converging almost everywhere pointwise to 0 for $k \to \infty$.

  Analogously we see for $j \neq n$:
$$\left\|\partial_j \tilde{f} - \partial_j \tilde{f}_k\right\|_{q,Q} = \left\|\partial_j \tilde{f}(1-\eta_k)\right\|_{q,Q} \to 0$$

- $$\left\|\partial_n \tilde{f} - \partial_n \tilde{f}_k\right\|_{q,Q} = \left\|\partial_n \tilde{f} - \partial_n \tilde{f}\eta_k - \tilde{f}\eta_k'\right\|_{q,Q} \leq$$
$$\leq \underbrace{\left\|\partial_n \tilde{f}(1-\eta_k)\right\|_{q,Q}}_{\to 0 \text{ as above}} + \left\|\tilde{f}\eta_k'\right\|_{q,Q}$$
and $\left\|\tilde{f}\eta_k'\right\|_{q,Q} \leq ck \left\|\tilde{f}\right\|_{q,\Delta\times]0,\frac{2}{k}[} \overset{(27)}{\leq} 2c \left\|\partial_n \tilde{f}\right\|_{q,\Delta\times]0,\frac{2}{k}[} \to 0$ with (27) used for $\rho = \frac{2}{k}$ by Lebesgue's Dominated Convergence Theorem.

  Arguing analogously (now for $\tilde{f}$ replaced by $\partial_j \tilde{f}$), we can see that for $j \neq n$ we have
$$\left\|\partial_j\partial_n \tilde{f} - \partial_j\partial_n \tilde{f}_k\right\|_{q,Q} \to 0.$$

- $$\left\|\partial_n\partial_n \tilde{f} - \partial_n\partial_n \tilde{f}_k\right\|_{q,Q} = \left\|\partial_n\partial_n \tilde{f} - \partial_n\partial_n \tilde{f}\eta_k - 2\partial_n \tilde{f}\eta_k' - \tilde{f}\eta_k''\right\|_{q,Q} \leq$$
$$\leq \underbrace{\left\|\partial_n\partial_n \tilde{f}(1-\eta_k)\right\|_{q,Q}}_{\to 0 \text{ as above}} + 2\left\|\partial_n \tilde{f}\eta_k'\right\|_{q,Q} + \left\|\tilde{f}\eta_k''\right\|_{q,Q}$$
The term $\left\|\partial_n \tilde{f}\eta_k'\right\|_{q,Q}$ can be treated as above:
$$\left\|\partial_n \tilde{f}\eta_k'\right\|_{q,Q} \leq ck \left\|\partial_n \tilde{f}\right\|_{q,Q} = ck \left\|\partial_n \tilde{f}\right\|_{q,\Delta\times]0,\frac{2}{k}[} \overset{(28)}{\leq}$$

$$\leq 2c \underbrace{\left\| \partial_n \partial_n \tilde{f} \right\|_{q,\Delta \times \left]0,\frac{2}{k}\right[}}_{\text{tends to zero by Lebesgue's Dominated Convergence Theorem}} \to 0$$

For the term $\left\| \tilde{f}\eta_k'' \right\|_{q,Q}$ we have:

$$\left\| \tilde{f}\eta_k'' \right\|_{q,Q} \leq ck^2 \left\| \tilde{f} \right\|_{q,\Delta \times \left]0,\frac{2}{k}\right[} \overset{(29)}{\leq} 4c \left\| \partial_n \partial_n \tilde{f} \right\|_{q,\Delta \times \left]0,\frac{2}{k}\right[}$$

which again converges to zero by Lebesgue's Dominated Convergence Theorem.

We note that for every $k > \frac{1}{\varepsilon - \varepsilon'}$ the function $\tilde{f}_k$ is zero outside the set $\Delta' \times \left(\left] \frac{1}{k}, \varepsilon - \varepsilon' \right[\right)$ as $\tilde{f} = 0$ outside of $\Delta' \times]0, \epsilon - \epsilon'[$ and $\eta_k(x_n) = 0$ for all x with $0 < x_n < \frac{1}{k}$. Therefore we find a $0 < \varepsilon_k < \frac{1}{2} \operatorname{dist} \left(\Delta' \times \left] \frac{1}{k}, \varepsilon - \varepsilon' \right[, \partial Q \right)$ such that the Friedrichs mollification $(\tilde{f}_k)_{(\varepsilon_k)}$ of $\tilde{f}_k$ satisfies $\left\| \tilde{f}_k - (\tilde{f}_k)_{(\varepsilon_k)} \right\|_{2,q,Q} < \frac{1}{k}$. This can be achieved since $\tilde{f}_k$ is zero outside of $\Delta' \times \left] \frac{1}{k}, \varepsilon - \varepsilon' \right[$ and thus differentiation and mollification commute, see Theorem 2.5. We further find that $(\tilde{f}_k)_{(\varepsilon_k)}$ has compact support in Q for ε_k was chosen to be smaller than $\frac{1}{2} \operatorname{dist} \left(\Delta' \times \left] \frac{1}{k}, \varepsilon - \varepsilon' \right[, \partial Q \right)$. The sequence $\left((\tilde{f}_k)_{(\varepsilon_k)} \right)_{k \in \mathbb{N}}$ converges in $H^{2,q}(Q)$ to $\tilde{f}$ because

$$\left\| \tilde{f} - (\tilde{f}_k)_{(\varepsilon_k)} \right\|_{2,q,Q} \leq \left\| \tilde{f} - \tilde{f}_k \right\|_{2,q,Q} + \left\| \tilde{f}_k - (\tilde{f}_k)_{(\varepsilon_k)} \right\|_{2,q,Q} \to 0$$

and we thus have $\tilde{f} \in H_0^{2,q}(Q)$ and we get $\Psi f \in H_0^{2,q}(G \cap U)$ and finally $f \in H_0^{2,q}(G)$.

□

Theorem 3.4. *Let* $p \in H^{2,q}(G) \cap H_0^{1,q}(G)$ *and* $\zeta \in \overline{\mathcal{C}}^2(G)$ *with* $\zeta|_{\partial G} = 0$. *Then* $p\zeta \in H_0^{2,q}(G)$.

Proof. At first we find with a calculation like in Theorem 3.2 that $p\zeta \in H^{2,q} \cap H_0^{1,q}(G)$. Furthermore we can apply the product rule (see Theorem 3.1) and get $\nabla (p\zeta) = \zeta \nabla p + p \nabla \zeta$. As $\zeta \in H_0^{1,q}(G)$ (see [22] Theorem 6.5, page 101 or our Theorem 4.7) we can see like in Theorem 3.2 that $\nabla (p\zeta) \in \underline{H}_0^{1,q}(G)$ and then with Theorem 3.3 we find: $p\zeta \in H_0^{2,q}(G)$. □

4 Some Facts about the Trace Operator

In this section we state the existence of the two trace operators which we will use. These trace operators allow us to talk about boundary values of $H^{1,q}(G)$- and $H^{2,q}(G)$-functions. As for our G the boundary ∂G is of Lebesgue-measure zero, it makes a priori no sense to talk in an L^r-sense about restrictions of such functions to the boundary. The following theorems give us the answer that there is a reasonable way in which we can associate to every such function a "boundary value": Although restricting an $u \in H^{1,q}(G)$ to ∂G obviously has no sense, any $u \in H^{1,q}(G)$ has the property that the restrictions to ∂G of elements of each sequence of $\overline{\mathcal{C}}^{\infty}(G)$-functions converging in $H^{1,q}(G)$-sense to u also converge in a space $L^r(\partial G)$ (which is still to be defined) to a specific function which is only depending on u (and not on the chosen sequence) and which thus in some sense generalizes the notion of boundary value.
In the following, we take a fixed set of charts $(\Delta_i, W_i, \Phi_i)_{i=1,\ldots,N}$ of ∂G in the sense of $\partial G \in \mathcal{C}^5$. Note that the following definitions are at first sight depending highly on the choice of charts. However, being a set of ∂G-measure zero does not depend on this choice and the defined norms on $L^r(\partial G)$, $H^{1,r}(\partial G)$ are in general different for different choices of charts, but how ever two choices are made, the corresponding norms are equivalent and the corresponding spaces do not depend on this choice, for details we refer to [11], chapitre 2, §4, chapitre 3, §1. We could also have made these definitions independent of the choice of charts by including the Gram's determinant-term. However, as this term is bounded from below and from above, we will simply ignore it, which results in different (but still equivalent, which is enough for us as we are only interested in the respective topologies) norms for different choices of charts.
The following definitions introduce the important spaces and state the basic facts which can also be read in [11], chapitre 2, §4, chapitre 3, §1, and in [12], Kapitel 2, 3, too.

Definition 4.1. *A subset $V \subset \partial G$ is called "of ∂G-measure zero" if and only if for every $i = 1, \ldots, n$ the set (again after an appropriate permutation of variables)*

$$\left\{x' \in \mathbb{R}^{n-1} : \text{ there is an } x_n \in \mathbb{R} \text{ such that } x = (x', x_n) \in W_i \cap V\right\} \subset \Delta_i$$

is a set of measure zero in $\mathbb{R}^{n-1}$.

Having now a concept of zero measure, we have again the possibility of saying "∂G-almost everywhere".

Definition 4.2. *The spaces* $L^r(\partial G)$*,* $1 < r < \infty$*.*
A function $f : \partial G \to \mathbb{R}$ *is said to be in* $L^r(\partial G)$ *if and only if for every* $i = 1, \dots, N$ *the function*

$$g_i : \Delta_i \to \mathbb{R},\, x' \mapsto f(x', \Phi_i(x'))$$

is in $L^r(\Delta_i)$*.*
Furthermore,

$$|f|_{L^r(\partial G)} := \left(\sum_{i=1}^{N} \|g_i\|_{r,\Delta_i}^r \right)^{\frac{1}{r}}$$

defines a norm on $L^r(\partial G)$*.*

Definition 4.3. *The spaces* $H^{1,r}(\partial G)$*,* $1 < r < \infty$*.*
A function $f : \partial G \to \mathbb{R}$ *is said to be in* $H^{1,r}(\partial G)$ *if and only if for every* $i = 1, \dots, N$ *the function*

$$g_i : \Delta_i \to \mathbb{R},\, x' \mapsto f(x', \Phi_i(x'))$$

is in $H^{1,r}(\Delta_i)$*.*
Furthermore,

$$|f|_{H^{1,r}(\partial G)} := \left(\sum_{i=1}^{N} \|g_i\|_{1,r,\Delta_i}^r \right)^{\frac{1}{r}}$$

defines a norm on $H^{1,r}(\partial G)$*.*

Having now introduced the important spaces, we can state the existence of the needed trace operators: Let in the following whenever a r is used in context of a trace operator this r be $r = r(q, n) := \frac{nq-q}{n-q}$ if $1 < q < n$ and $r > 1$ otherwise. Note that, for our purposes it would suffice, according to the book [2] (see there A6.6, page 265 and A6.10, page 270), to use $r = q$ in every case.

Theorem 4.4. *(Compare [12] Satz 2.4.1., page and [11] chapitre 2,théorème 4.2., page 84)*
Let $G \subset \mathbb{R}^n$ *be a bounded domain with Lipschitz-boundary. Then there exists exactly one linear continuous map* $Z^1 : H^{1,q}(G) \to L^r(\partial G)$ *with* $Z^1(u) = u|_{\partial G}$ *for all* $u \in \overline{\mathcal{C}}^\infty(G)$*.*

Theorem 4.5. *(Compare [12] Satz 3.1.3., page and [11] chapitre 2, théorème 4.11., page 89)*
Let $G \subset \mathbb{R}^n$ *be a bounded domain with Lipschitz-boundary. Then there exists exactly one linear continuous map* $Z^2 : H^{2,q}(G) \to H^{1,r}(\partial G)$ *with* $Z^2(u) = u|_{\partial G}$ *for all* $u \in \overline{\mathcal{C}}^\infty(G)$*.*

Remark 4.6. *Studying the proofs of Theorems 4.4 and 4.5 given in [12] and [11], one easily sees that the theorems can be modified in the following way: The linear continuous map Z^k fulfills even $Z^k(u) = u|_{\partial G}$ for all $u \in \overline{\mathcal{C}}^k(G)$, $k = 1, 2$.*

The above defined trace operator Z^1 gives us another characterization of the spaces $H_0^{1,q}(G)$ and $H_0^{2,q}(G)$:

Theorem 4.7.

$$H_0^{1,q}(G) = \left\{u \in H^{1,q}(G) : \, Z^1(u) = 0\right\}$$

For a proof of this Theorem, see [11], chapitre 2, théorème 4.10., pages 87, 88 or [12], Satz 2.6.3, pages 40-42.

Theorem 4.8.

$$H_0^{2,q}(G) = \left\{ u \in H^{2,q}(G) : \, Z^1(u) = 0 \text{ and } \sum_{i=1}^{n} Z^1(\partial_i u) N_i = 0 \right\},$$

where $\underline{N} := (N_1, \ldots, N_n)$ denotes the outward unit normal vector.

Proof. This theorem is just a combination of Theorem 4.7, our Theorem 4.10 and Theorem 3.3. For a different proof we refer to [11], chapitre 2, théorème 4.12., page 90 or [12], Satz 3.2.1, page 45. □

The following theorem tells us that the trace-operator behaves very much like a restriction with respect to special kinds of products:

Theorem 4.9. *Let $s \in H^{1,q}(G)$ and $f \in \overline{\mathcal{C}}^\infty(G)$. Then*

$$Z^1(fs)(x) = f(x) Z^1(s)(x)$$

for almost every $x \in \partial G$.

Proof. Let $(s_\nu)_{\nu \in \mathbb{N}} \subset \overline{\mathcal{C}}^\infty(G)$ be a sequence such that $\|s_\nu - s\|_{1,q} \to 0$. This is possible because G is bounded and has continuous boundary, see for example [9], 1.1.6, Theorem 2, page 14). Then, as f and $\partial_i f$, $i = 1, \ldots, n$ are bounded in G, we see that $fs \in H^{1,q}(G)$ and $(fs_\nu)_{\nu \in \mathbb{N}}$ is a Cauchy sequence in $H^{1,q}(G)$ converging to fs. Take a chart (Δ_i, W_i, Φ_i) and note that in this chart $fs_\nu(x', \Phi_i(x'))$ converges in $L^r(\Delta_i)$ to $fZ^1(s)$ as f is bounded and s_ν converges in $L^r(\Delta_i)$ to $Z^1(s)$, so $Z^1(fs)$ must be equal to $fZ^1(s)$. □

Theorem 4.10. *Let $s \in H_0^{1,q}(G) \cap H^{2,q}(G)$. Then we have*

$$Z^1(\nabla s)(x) := \big(Z^1(\partial_1 s)(x), \dots, Z^1(\partial_n s)(x)\big) = \lambda(x)\underline{N}(x)$$

for almost every $x \in \partial G$ with a function $\lambda \in L^r(\partial G)$.

Proof. The proof is done in three steps:

a) Localization by partition of unity

Let G be covered by finitely many open sets $U_0, \dots, U_N \subset \mathbb{R}^n$ such that $U_0 \subset\subset G$, ∂G is covered by $U_1, \dots, U_N$ and for $i = 1, \dots, N$ let $\Phi_i \in \overline{\mathcal{C}}^5(\Delta_i)$, such that after a permutation of coordinates we have

$$\partial G \cap U_i = \{(x', \Phi_i(x')) : x' \in \Delta_i\} \text{ and}$$

$$G \cap U_i = \bigcup_{x' \in \Delta_i} \{x'\} \times]\Phi_i(x'), \Phi_i(x') + \varepsilon_i[$$

or

$$G \cap U_i = \bigcup_{x' \in \Delta_i} \{x'\} \times]\Phi_i(x') - \varepsilon_i, \Phi_i(x')[$$

for real numbers $\varepsilon_i > 0$. We only consider the case

$$G \cap U_i = \bigcup_{x' \in \Delta_i} \{x'\} \times]\Phi_i(x'), \Phi_i(x') + \varepsilon_i[\tag{30}$$

in the following, the other one can be treated in the same manner. We find a partition of unity Ψ_i, $i = 0, \dots, N$ of G subordinate to the covering U_i, $i = 0, \dots, N$.

For $j \in \{1, \dots, n\}$ we find

$$Z^1(\partial_j s) = Z^1\left(\sum_{l=0}^{N} \Psi_l \partial_j s\right) = Z^1\left(\sum_{l=1}^{N} \Psi_l \partial_j s\right),$$

as $\operatorname{supp} \Psi_0 \subset\subset G$ and with Theorem 4.9 we see $Z^1(\Psi_0 \partial_j s) = 0$. Moreover, we also see that $Z^1(\partial_j \Psi_l \cdot s) = 0$ with Theorem 4.9 because $s \in H_0^{1,q}(G)$. So we get to

$$Z^1(\partial_j s) = Z^1\left(\sum_{l=1}^{N} \partial_j(\Psi_l s)\right)$$

and it suffices to show the claim only for functions of the form $\Psi_l s$. Moreover, it suffices to show the claim only locally, that is we can take $G \cap U_l$ as our new G, which we call G' and we are searching a function $\lambda \in L^r(\partial G \cap U_l)$ such that $Z^1(\partial_j(\Psi_l s)) = \lambda N_j$ almost everywhere on $\partial G \cap U_l$. In the following we will omit the now fixed index l.

b) Straightening of a local model

By smoothness of ∂G, we find a $\mathcal{C}^5$-diffeomorphism

$$g : Q := \Delta \times]0, \varepsilon[\to G',\ (x', x_n) \mapsto (x', \Phi(x') + x_n),$$

and without loss of generality, we can assume even $g : \overline{Q} \to \overline{G'}$. We further find with

$$\Psi s \in H_0^{1,q}(G') \cap H^{2,q}(G')$$

that

$$\tilde{s} := (\Psi s) \circ g \in H_0^{1,q}(Q) \cap H^{2,q}(Q).$$

In a point $p = (x', \Phi(x')) \in \partial G \cap U$ we have the tangent vectors $\underline{t}_1, \dots, \underline{t}_n$ to ∂G with

$$\underline{t}_1(p) = \left(1, 0, \dots, 0, \left(\frac{\partial \Phi}{\partial x_1}\right)(x')\right)$$

$$\underline{t}_2(p) = \left(0, 1, 0, \dots, 0, \left(\frac{\partial \Phi}{\partial x_2}\right)(x')\right)$$

$$\vdots$$

$$\underline{t}_{n-1}(p) = \left(0, \dots, 0, 1, \left(\frac{\partial \Phi}{\partial x_{n-1}}\right)(x')\right)$$

and with

$$\underline{\tilde{N}}(p) := \left(\frac{\partial \Phi}{\partial x_1}(x'), \dots, \frac{\partial \Phi}{\partial x_{n-1}}(x'), -1\right)$$

we see that

$$\underline{N} := \frac{\underline{\tilde{N}}}{\left\|\underline{\tilde{N}}\right\|}$$

is the outward unit normal vector to G in x because we are considering the case (30). Taking a function $\zeta \in \overline{\mathcal{C}}^{\infty}(G')$ we have for the directional derivative $D_{\underline{t}_i}\zeta$ of ζ in direction $\underline{t}_i$ in a point $p = (x', x_n = \Phi(x')) \in \partial G \cap U$, $i = 1, \dots, n-1$

$$D_{\underline{t}_i}\zeta(p) = \nabla\zeta \cdot \underline{t}_i(p) = \frac{\partial \zeta}{\partial x_i}(p) + \frac{\partial \Phi}{\partial x_i}(x')\frac{\partial \zeta}{\partial x_n}(p)$$

and for $\tilde{\zeta} := \zeta \circ g \in \overline{\mathcal{C}}^5(Q)$ we find for $i = 1, \dots, n-1$

$$\partial_i \tilde{\zeta}(x', 0) = \sum_{j=1}^{n} \frac{\partial \zeta}{\partial x_j}(g(x', 0))\frac{\partial g_j}{\partial x_i}(x', 0) =$$

$$= \frac{\partial \zeta}{\partial x_i}(g(x',0)) + \frac{\partial \zeta}{\partial x_n}(g(x',0)) \frac{\partial \Phi}{\partial x_i}(x'),$$

so

$$D_{\underline{t}_i} \zeta(g(x',0)) = \partial_i \tilde{\zeta}(x',0). \tag{31}$$

Taking an approximating sequence ζ_ν in $\overline{\mathcal{C}}^\infty(G')$ of Ψs with respect to $\|\cdot\|_{2,q}$, we see that $\tilde{\zeta_\nu} := \zeta_\nu \circ g$ is an approximating sequence in $\overline{\mathcal{C}}^5(Q)$ of $\tilde{s}$ with respect to $\|\cdot\|_{2,q}$ and we see by equation (31) applied to the approximating sequence and Theorem 4.4 that for almost every $p = (x', \Phi(x')) \in \partial G \cap U$ we have

$$Z^1(\partial_i \tilde{s})(x',0) = \sum_{j=1}^n Z^1 \left(\frac{\partial(\Psi s)}{\partial x_j} \right) (x', \Phi(x')) \left(\underline{t}_i\right)_j (x', \Phi(x')).$$

In the following, we will show that $Z^1(\partial_i \tilde{s}) = 0$ and thus we will find that in almost every point $p \in \partial G' \cap U$ it is

$$\sum_{j=1}^n Z^1 \left(\frac{\partial(\Psi s)}{\partial x_j} \right) (p) \left(\underline{t}_i\right)_j (p) = 0, \; i = 1, \dots, n-1$$

and thus in almost every point $p \in \partial G' \cap U$ we will then find a $\lambda(p) \in \mathbb{R}$ such that by the definition $(\nabla(\Psi s))_j(p) := Z^1(\partial_j(\Psi s))(p)$ for $p \in \partial G' \cap U$ we find:

$$\nabla(\Psi s)(p) = \lambda(p) \underline{N}(p)$$

It is easily seen that $\lambda = \frac{1}{N_n} \partial_n(\Psi s)$ is then a measurable function (in the $\partial G' \cap U$-sense) because $N_n \neq 0$ is with the help of Weyers' helpful function easily to be seen smooth enough, and because of $\|\underline{N}\| = 1$ we also have almost everywhere on $\partial G' \cap U$: $|\lambda(p)| = |\nabla(\Psi s)(p)|$ and thus $\lambda \in L^r(\partial G' \cap U)$.

c) The straight problem:

So we just have to show $Z^1(\partial_j \tilde{s})(x',0) = 0$ for $x' \in \Delta$. As $\tilde{s} \in H_0^{1,q}(Q) \cap H^{2,q}(Q)$ we find a sequence $(h_\nu)_{\nu \in \mathbb{N}} \subset \overline{\mathcal{C}}^\infty(Q)$ with $h_\nu \xrightarrow{H^{2,q}(Q)} \tilde{s}$. With the definition

$$f_\nu := h_\nu|_{\Delta \times \{0\}}$$

we see that

$$f_\nu \xrightarrow{L^r(\Delta)} 0$$

and

$$\partial_i f_\nu \xrightarrow{L^r(\Delta)} Z^1(\partial_i \tilde{s}), \; i = 1, \dots, n-1$$

because h_ν is also an approximating sequence for $\tilde{s}$ with respect to the $H^{1,q}(Q)$-norm and $\partial_i h_\nu$ is one in the same norm for $\partial_i \tilde{s}$.

But we also know that the f_ν converge in the $H^{1,r}(\Delta)$-norm to $Z^2(\tilde{s})$ and by choosing subsequences of f_ν converging almost everywhere on Δ, we conclude $Z^2(\tilde{s}) = 0$ and this means that $\partial_i f_\nu \xrightarrow{L^r(\Delta)} 0$. Because $\partial_i h_\nu$ is an $H^{1,q}(Q)$-norm approximating sequence for $\partial_i \tilde{s}$ we also have $\partial_i f_\nu \xrightarrow{L^r(\Delta)} Z^1(\partial_i \tilde{s})$ and thus $Z^1(\partial_i \tilde{s}) = 0$.

□

5 An Approximation Theorem

In this section we will find a theorem which will allow us in the next section to draw back from a $p \in B_0^q(G)$ to a $p \in B_0^q(G) \cap H^{3,q}(G)$.

Theorem 5.1. *Let $p \in B_0^q(G)$. Then there exists a sequence $(p_\nu)_{\nu\in\mathbb{N}} \subset B_0^q(G) \cap H^{3,q}(G)$ with*

$$\|p_\nu - p\|_{1,q} \to 0.$$

Proof. It suffices by Poincaré's Lemma to find a sequence $(p_\nu)_{\nu\in\mathbb{N}} \subset B_0^q(G) \cap H^{3,q}(G)$ with $\|\nabla p_\nu - \nabla p\|_q \to 0$. For p we find according to Theorem 2.22 a sequence $(g_\nu)_{\nu\in\mathbb{N}} \subset \mathcal{C}_{0,0}^\infty(G)$ with $\|\nabla g_\nu - \nabla p\|_q \to 0$. For every $\nu \in \mathbb{N}$ we can find according to Theorem 2.16 an unique $s_\nu \in H_0^{3,q}(G)$ satisfying

$$\langle \nabla\Delta s_\nu, \nabla\Delta\Phi\rangle = \langle \nabla g_\nu, \nabla\Delta\Phi\rangle \quad \text{for every } \Phi \in H_0^{3,q'}(G)$$

and we even see that $s_\nu \in H^{5,q}(G)$ by Theorem 6.1, as $\partial G \in \mathcal{C}^5$ and $g_\nu \in \mathcal{C}_{0,0}^\infty(G)$.
But this means that for $\nu \in \mathbb{N}$ we find that with $p_\nu := g_\nu - \Delta s_\nu \in H^{3,q}(G)$ we have

$$-\langle p_\nu, \Delta^2\Phi\rangle = \langle \nabla p_\nu, \nabla\Delta\Phi\rangle = \langle \nabla g_\nu - \nabla\Delta s_\nu, \nabla\Delta\Phi\rangle = 0 \text{ for every } \Phi \in \mathcal{C}_0^\infty(G)$$

and thus according to Weyl's Lemma 2.20 we conclude $p_\nu \in \mathcal{C}^\infty(G)$ with $\Delta^2 p_\nu = 0$. As $g_\nu \in \mathcal{C}_{0,0}^\infty(G)$ and $\Delta s \in H_{0,0}^{1,q}(G) \cap H^{3,q}(G)$, we have $p_\nu \in H_{0,0}^{1,q}(G) \cap H^{3,q}(G)$ and thus $p_\nu \in B_0^q(G) \cap H^{3,q}(G)$.
Now look at

$$\|\nabla p_\nu - \nabla p\|_q = \|\nabla g_\nu - \nabla\Delta s_\nu - \nabla p\|_q \le \underbrace{\|\nabla g_\nu - \nabla p\|_q}_{\to 0} + \|\nabla\Delta s_\nu\|_q$$

and we see by the variational inequality (Theorem 2.17) that with a constant $C_V > 0$

$$\|\nabla\Delta s_\nu\|_q \leq C_V \sup_{0\neq\Phi\in H_0^{3,q'}(G)} \frac{\langle\nabla\Delta s_\nu, \nabla\Delta\Phi\rangle}{\|\nabla\Delta\Phi\|_{q'}} = C_V \sup_{0\neq\Phi\in H_0^{3,q'}(G)} \frac{\langle\nabla g_\nu, \nabla\Delta\Phi\rangle}{\|\nabla\Delta\Phi\|_{q'}} =$$

$$\overset{p\in B_0^q(G)}{=} C_V \sup_{0\neq\Phi\in H_0^{3,q'}(G)} \frac{\langle\nabla g_\nu - \nabla p, \nabla\Delta\Phi\rangle}{\|\nabla\Delta\Phi\|_{q'}} \leq C_V C' \|\nabla g_\nu - \nabla p\|_q \to 0.$$

□

6 Compactness of $Z_q - \frac{1}{2}Id : B_0^q(G) \to B_0^q(G)$

In this section we will generalize a proof by Weyers which goes back to Crouzeix (see [22], [6]).
As Crouzeix's method is somehow exceptional, we will at first give a motivation for it in the easier case which Weyers examined:

6.1 A Little Motivation for Crouzeix's Method

First, we will fix notations and describe Weyers' problem for the case of bounded G:
For $1 < q < \infty$, $G \subset\subset \mathbb{R}^n$ with sufficient smooth boundary (the boundary smoothness requirements for G used by Weyers can be weakened as in our approach to $\partial G \in \mathcal{C}^4$, see Part II) we have the direct decomposition (see [22], Theorem 5.2, page 96)

$$L_0^q(G) = \tilde{A}^q(G) \oplus \tilde{B}_0^q(G),$$

where

$$L_0^q(G) := \left\{p \in L^q(G) : \int_G p\,dx = 0\right\},$$

$$\tilde{A}^q(G) := \left\{\Delta u : u \in H_0^{2,q}(G)\right\}$$

and

$$\tilde{B}_0^q(G) := \left\{h \in L_0^q(G) : \langle h, \Delta\Phi\rangle = 0 \quad \forall\Phi \in H_0^{2,q'}(G)\right\}.$$

The proof of this decomposition is similar to our decomposition from Lemma 2.23. Furthermore, Weyers investigated the operator $\underline{\tilde{T}}_q : L_0^q(G) \to \underline{H}_0^{1,q}(G)$ where $\underline{\tilde{T}}_q(p)$ is the unique element of $\underline{H}_0^{1,q}(G)$ such that

$$\langle\nabla\underline{\tilde{T}}_q(p), \nabla\underline{\Phi}\rangle = \langle p, \operatorname{div}\underline{\Phi}\rangle \text{ for all } \underline{\Phi} \in \underline{H}_0^{1,q'}(G).$$

Similarly to our notation, Weyers defined an operator $\tilde{Z}_q : L^q_0(G) \to L^q_0(G)$ by $\tilde{Z}_q := \operatorname{div} \circ \underline{\tilde{T}}_q$. It shows up in analogy to Theorem 2.24, that we have the following situation:

$$\tilde{Z}_q|_{\tilde{A}^q(G)} : \tilde{A}^q(G) \to \tilde{A}^q(G) \text{ and } \tilde{Z}_q(p) = p \quad \forall p \in \tilde{A}^q(G)$$
$$\tilde{Z}_q|_{\tilde{B}^q_0(G)} : \tilde{B}^q_0(G) \to \tilde{B}^q_0(G)$$

In Weyers' terms, the problem which is analogous to our problem here is to show that the operator $\left(\tilde{Z}_q - \frac{1}{2}Id\right)|_{\tilde{B}^q_0(G)}$ is compact.
Crouzeix's idea to show this is now given: Find for given $p \in \tilde{B}^q_0(G) \cap H^{2,q}(G)$, $\underline{u} := \underline{\tilde{T}}_q(p) \in \underline{H}^{1,q}_0(G) \cap \underline{H}^{3,q}(G)$ (we can get rid of the extra premise $p \in H^{2,q}(G)$ by approximation just like in Theorems 5.1, 6.4) a $v \in H^{1,q}(G)$ such that:

(i) there is a $C = C(q, G)$ such that

$$\|v\|_{1,q} \leq C \|p\|_q .$$

(ii) $v - \left(\operatorname{div} \underline{u} - \frac{1}{2}p\right) \in H^{1,q}_0(G)$.

If this can be achieved, the rest is not hard and done like in Theorem 6.3 with Müller's inequality (theorem 2.7) replaced by Simader's inequality (see [18], Chapter II, Theorem 1.1, page 44). We get to

$$\left\| \left(\tilde{Z}_q - \frac{1}{2}Id\right) p \right\|_{1,q} \leq C' \|p\|_q$$

and it shows up by approximation that the linear operator $\left(\tilde{Z}_q - \frac{1}{2}Id\right) : \tilde{B}^q_0(G) \to \tilde{B}^q_0(G)$ has its image indeed in $H^{1,q}(G)$ and is continuous with respect to these spaces. Compactness of $\left(\tilde{Z}_q - \frac{1}{2}Id\right)$ is then just a consequence of the compact embedding of $H^{1,q}(G)$ in $L^q(G)$.

Now we finally motivate Crouzeix's ansatz:
We stick to the model case where $p, \underline{u}$ have classical derivatives of the respective orders which are continuous up to the boundary. We could get rid of this assertion by using the idea of trace from Section 4, but this would not bring us any new insights and just make the whole procedure a little more complicated. Starting from our ingredients $p, \underline{u}$, we first notice that as the system of partial differential equations which links p and $\underline{u}$ (the classical formulation

would be $\Delta \underline{u} = \nabla p$) involves higher derivatives, the only possible chance of using this linking might be finding a $w \in H_0^{1,q}(G) \cap H^{2,q}(G)$ with an estimate $\|w\|_{2,q} \leq C \|p\|$ and defining v somehow by w's derivatives. One other way of looking at this first step is the idea of using the Theorem on Elliptic Regularity (Theorem 6.1). This is actually the way we will do it later, but in Weyers' case it can be avoided. As we need with this approach (here we have $m = 1$ in Theorem 6.1) a function in $H_0^{1,q}(G)$ to be able to apply Theorem 6.1, we nevertheless need to make this "step up" here, too, defining at first a function $w \in H_0^{1,q}(G) \cap H^{2,q}(G)$ in terms of $\underline{u}$ and p which is satisfying the above given estimate and then trying to define v via w's derivatives. For the connection between these two accounts, we refer the reader to our proof of Theorem 9.7 where both accounts are in some way present.
One very easy ansatz for w is trying to define

$$w = \sum_{i=1}^{n} g_i u_i + hp,$$

where g_i, $i = 1, \dots, n$ and h are sought after functions which shall be often enough continuously differentiable and the derivatives shall be continuous up to the boundary of G. As we are looking for a $w \in H_0^{1,q}(G)$ and $\underline{u} \in \underline{H}_0^{1,q}(G)$ the reasonable requirement for h is

$$h = 0 \text{ on } \partial G. \tag{32}$$

Making this ansatz, we can easily see:

$$\|w\|_q \leq C_1 \|p\|_q$$

As w shall be in $H_0^{1,q}(G)$, we have with Simader's variational inequality

$$\|\nabla w\|_q \leq C_S \sup_{\Phi \in H_0^{1,q'}(G)} \frac{\langle \nabla w, \nabla \Phi \rangle}{\|\nabla \Phi\|_{q'}} = C_S \sup_{\Phi \in H_0^{1,q'}(G)} \frac{\langle \Delta w, \Phi \rangle}{\|\nabla \Phi\|_{q'}} \leq$$
$$\leq C' \|\Delta w\|_q$$

using the Poincaré inequality.
With the use of a regularity theorem ([22], Theorem 7.6, page 110) stating that we have with a $C > 0$ an estimate of the form

$$\|\nabla u\|_{1,q,G} \leq C \left(\|\Delta u\|_{q,G} + \|u\|_{q,G} + \|\nabla u\|_{q,G} \right)$$

valid for every $u \in H_0^{1,q}(G) \cap H^{2,q}(G)$, the problem of showing the estimate

$$\|w\|_{2,q} \leq C \|p\|_q \tag{33}$$

with a constant $C > 0$ reduces to showing an estimate of the form

$$\|\Delta w\|_q \leq C \|p\|_q$$

with a constant $C > 0$.
Now taking a look at Δw, we see:

$$\Delta w = \sum_{i=1}^{n} \Delta g_i u_i + 2 \sum_{i,j=1}^{n} \partial_j g_i \partial_j u_i + \sum_{i=1}^{n} g_i \Delta u_i + \Delta h p +$$

$$+2 \sum_{j=1}^{n} \partial_j h \partial_j p + h \underbrace{\Delta p}_{=0, \text{ as } p \in \tilde{B}_0^q(G)}$$

We see that the terms

$$\sum_{i=1}^{n} \Delta g_i u_i, \; 2 \sum_{i,j=1}^{n} \partial_j g_i \partial_j u_i \text{ and } \Delta h p$$

do not pose us any problems as they contain at most first order derivatives of $\underline{u}$ and no derivatives of p. Using now the linking $\Delta \underline{u} = \nabla p$ between $\underline{u}$ and p, we have for the remaining two terms:

$$\sum_{i=1}^{n} g_i \Delta u_i + 2 \sum_{j=1}^{n} \partial_j h \partial_j p = \sum_{i=1}^{n} (g_i + 2\partial_i h)\, \partial_i p.$$

To find this term equal to zero, the plausible requirement is

$$g_i = -2\partial_i h, \; i = 1, \ldots, n. \tag{34}$$

Having established a w as we searched, we now define

$$v = \sum_{i=1}^{n} f_i \partial_i w,$$

where f_i, $i = 1, \ldots, n$ are sought after functions which shall be often enough continuously differentiable and the derivatives shall be continuous up to the boundary of G. Because of the validity of inequality (33), we have automatically (i) for v. Now we want to find further conditions on g_i, h and f_i which are ensuring the validity of (ii). On the boundary ∂G, we have:

$$v = \sum_{i=1}^{n} f_i \partial_i w = \sum_{i=1}^{n} f_i \partial_i \left(-2 \sum_{j=1}^{n} \partial_j h u_j + hp \right) =$$

$$= \sum_{i=1}^{n}\left(f_i\left(-2\sum_{j=1}^{n}\left(\partial_i\partial_j h \underbrace{u_j}_{=0 \text{ on } \partial G} + \partial_j h \partial_i u_j\right) + \partial_i h p + \underbrace{h}_{=0 \text{ on } \partial G} \partial_i p\right)\right) =$$

$$= \sum_{i=1}^{n}\left(f_i\left(-2\sum_{j=1}^{n}\partial_j h \partial_i u_j + \partial_i h p\right)\right) =$$

$$= -2\sum_{i,j=1}^{n} f_i \partial_j h \partial_i u_j + \sum_{i=1}^{n} f_i \partial_i h p$$

Our aim is this to be equal to

$$\operatorname{div} \underline{u} - \frac{1}{2}p = \sum_{j=1}^{n} \partial_j u_j - \frac{1}{2}p.$$

This can be achieved with the requirements

- $$-2\sum_{i,j=1}^{n} f_i \partial_j h \partial_i u_j = \sum_{j=1}^{n} \partial_j u_j.$$

  As $\underline{u} \in \underline{H}_0^{1,q}(G)$, we find for $j = 1, \dots, n$ a function λ_j with $\partial_i u_j = \lambda_j N_i$ on ∂G. This leads us to the requirement

  $$-2\sum_{i,j=1}^{n} f_i \partial_j h \lambda_j N_i = \sum_{j=1}^{n} \lambda_j N_j$$

  As

  $$\|\underline{N}\|^2 = \sum_{i=1}^{n} N_i^2 = 1,$$

  a reasonable try for f_i, $i = 1, \dots, n$ and $\partial_j h = \mu N_j$, $j = 1, \dots, n$ (with a suitable function μ on ∂G which can be found as $h = 0$ on ∂G) to fulfill this requirement seems to be

  $$f_i = N_i,\ i = 1, \dots, n \text{ and } \partial_j h = -\frac{1}{2}N_j,\ j = 1, \dots, n \qquad (35)$$

- and

  $$\sum_{i=1}^{n} f_i \partial_i h = -\frac{1}{2}, \qquad (36)$$

  which is automatically fulfilled with (35).

All in all we have the three conditions (32), (34) and (35) which are all valid if $h = -\frac{1}{2}\zeta$, $g_i = \partial_i \zeta$, $f_i = \partial_i \zeta$ where ζ is the function Weyers constructed with boundary values $\zeta = 0$ and $\nabla \zeta = \underline{N}$ on ∂G. This is for Weyers' problem exactly the ansatz of Crouzeix. So the somehow complicated and exotic construction by Crouzeix is not a kind of coincidence. In order to solve the problem there seems to be no other easy choice of defining v.

6.2 The Compactness of $Z_q - \frac{1}{2}Id : B_0^q(G) \to B_0^q(G)$

In the following we will need a differentiability theorem due to Christian G. Simader (see [15], Theorem 9.11., page 156) which we cite in the generality given in [15]:

Theorem 6.1. *Assume*

1. *that $m \geq 1$ and $j \geq 0$ (with $j \leq m$) are integers and that $1 < p, q < \infty$ are real numbers with $\frac{1}{p} + \frac{1}{q} = 1$,*
2. *that $G \subset \mathbb{R}^n$ is a bounded open set with boundary $\partial G \in \mathcal{C}^{m+j}$,*
3. *that B is an uniformly elliptic, j-smooth regular Dirichlet bilinear form of degree m in G,*
4. *that $F \in \left(H_0^{m-j,q}(G)\right)^*$ and $u \in H_0^{m,p}(G)$ such that*

$$B\,[u, \Phi] = F(\Phi) \qquad \text{for all } \Phi \in \mathcal{C}_0^\infty(G)$$

Then $u \in H_0^{m,p}(G) \cap H^{m+j,p}(G)$ and there is a constant $\gamma = \gamma(n, m, j, p, B, G) > 0$ such that

$$\|u\|_{m+j,p} \leq \gamma \left(\|F\|_{\left(H_0^{m-j,q}(G)\right)^*} + \|u\|_{0,p} \right)$$

In a first step we will show the important estimate under the extra assumption $p \in H^{3,q}(G)$:

Theorem 6.2. *Let $p \in B_0^q(G) \cap H^{3,q}(G)$, $\underline{u} := \underline{T}_q(p)$. Then*

$$w := \underline{u} \cdot \nabla \zeta - \frac{1}{4} p \zeta \in H_0^{2,q}(G) \cap H^{3,q}(G)$$

and there is a constant $C = C(G, q) > 0$ with

$$\|w\|_{3,q} \leq C \, \|\nabla p\|_q \, . \tag{37}$$

Proof. According to the Theorems 3.2 and 3.4 we have $w \in H_0^{2,q}(G) \cap H^{3,q}(G)$. We now want to apply Theorem 6.1 in a special case. We take $m = 2$, $j = 1$, the role of p in 6.1 is played by our q, $B[u, v] := \langle \Delta u, \Delta v \rangle$ and the role of u is played by our w. We now show that there is a continuous linear functional $F \in \left(H_0^{1,q'}(G)\right)^*$ such that $B[w, \Phi] = F(\Phi)$ for all $\Phi \in \mathcal{C}_0^\infty(G)$:
Let $\Phi \in \mathcal{C}_0^\infty(G)$ be arbitrary. Then we have

$$B[w,\Phi] = \langle \Delta w, \Delta\Phi \rangle = \langle \Delta \left(\underline{u} \cdot \nabla\zeta - \frac{1}{4} p\zeta \right), \Delta\Phi \rangle =$$

$$= \underbrace{\langle \Delta \left(\sum_{i=1}^{n} u_i \partial_i \zeta \right), \Delta\Phi \rangle}_{=:T_1} - \underbrace{\frac{1}{4} \langle \Delta (p\zeta), \Delta\Phi \rangle}_{=:T_2}$$

As there will appear in the following many terms which we have to drag along with us through the whole calculation, we introduce the following short notation: An expression $\langle A, B \rangle$ with A consisting of only up to second order derivatives of the u_i, up to first order derivatives of p and up to fourth order derivatives of ζ and B consisting of only up to first order derivatives of Φ will be called an expression "of type L". Such an expression defines for variable Φ a bounded linear functional on $H_0^{1,q'}(G)$ with its norm being dominated by a constant $C(A, B) > 0$ times $\|\nabla p\|_{q,G}$. In this context "$=_L$" means equality up to an additive expression of type L and our aim is to show that $B[w, \Phi]$ is for variable Φ of type L.
Let's look first at T_1:

$$T_1 = \langle \Delta \left(\sum_{i=1}^{n} u_i \partial_i \zeta \right), \Delta\Phi \rangle = \sum_{i=1}^{n} \langle \Delta u_i \partial_i \zeta, \Delta\Phi \rangle + 2 \sum_{i,j=1}^{n} \langle \partial_j u_i \partial_j \partial_i \zeta, \Delta\Phi \rangle +$$

$$+ \sum_{i=1}^{n} \langle u_i \Delta \partial_i \zeta, \Delta\Phi \rangle = \sum_{i=1}^{n} \langle \Delta u_i \partial_i \zeta, \Delta\Phi \rangle -$$

$$\underbrace{2 \sum_{i,j,k=1}^{n} \langle \partial_k \partial_j u_i \partial_j \partial_i \zeta + \partial_j u_i \partial_k \partial_j \partial_i \zeta, \partial_k \Phi \rangle}_{\text{of type L}} - \underbrace{\sum_{i,k=1}^{n} \langle \partial_k u_i \Delta \partial_i \zeta + u_i \Delta \partial_k \partial_i \zeta, \partial_k \Phi \rangle}_{\text{of type L}}$$

$$=_L \sum_{i=1}^{n} \langle \Delta u_i \partial_i \zeta, \Delta\Phi \rangle = \sum_{i=1}^{n} \langle \Delta u_i, \partial_i \zeta \Delta\Phi \rangle =$$

$$= \sum_{i=1}^{n} \langle \Delta u_i, \Delta(\partial_i \zeta \Phi) - 2 \sum_{j=1}^{n} \partial_j \partial_i \zeta \partial_j \Phi - \partial_i \Delta \zeta \Phi \rangle = \langle \Delta \underline{u}, \Delta \left(\Phi \nabla \zeta \right) \rangle$$

$$\underbrace{-2 \sum_{i,j=1}^{n} \langle \partial_i \partial_j \zeta \Delta u_i, \partial_j \Phi \rangle - \sum_{i=0}^{n} \langle \Delta u_i \partial_i \Delta \zeta, \Phi \rangle}_{\text{of type L}} =_L$$

$$=_L \langle \Delta \underline{u}, \Delta \left(\Phi \nabla \zeta \right) \rangle \overset{\underline{u}=\underline{T}_q(p)}{=} \langle \nabla p, \Delta \left(\nabla \zeta \Phi \right) \rangle$$

for $\nabla \zeta \Phi$ is a permitted testing function $\in \underline{H}_0^{2,q'}(G)$. Calculating further we get:

$$\sum_{i=1}^{n} \langle \partial_i p, \Delta \left(\partial_i \zeta \Phi \right) \rangle = \underbrace{\sum_{i=1}^{n} \langle \partial_i p, \Delta \partial_i \zeta \Phi \rangle}_{\text{of type L}} + 2 \underbrace{\sum_{i,j=1}^{n} \langle \partial_i p, \partial_j \partial_i \zeta \partial_j \Phi \rangle}_{\text{of type L}} +$$

$$+ \sum_{i=1}^{n} \langle \partial_i p, \partial_i \zeta \Delta \Phi \rangle =_L \sum_{i=1}^{n} \langle \partial_i p \partial_i \zeta, \Delta \Phi \rangle$$

For T_2 we find

$$4T_2 = \langle \Delta \left(p\zeta \right), \Delta \Phi \rangle = \langle \Delta p \zeta + 2 \sum_{j=1}^{n} \partial_j p \partial_j \zeta + p \Delta \zeta, \Delta \Phi \rangle = \langle \Delta p \zeta, \Delta \Phi \rangle +$$

$$+2 \sum_{j=1}^{n} \langle \partial_j p \partial_j \zeta, \Delta \Phi \rangle - \underbrace{\sum_{k=1}^{n} \langle \partial_k p \Delta \zeta + p \partial_k \Delta \zeta, \partial_k \Phi \rangle}_{\text{of type L}} =_L$$

$$=_L \langle \Delta p \zeta, \Delta \Phi \rangle + 2 \sum_{j=1}^{n} \langle \partial_j p \partial_j \zeta, \Delta \Phi \rangle.$$

With the results for T_1 and T_2 we can write:

$$B\left[w, \Phi \right] = T_1 - T_2 =_L \sum_{i=1}^{n} \langle \partial_i p \partial_i \zeta, \Delta \Phi \rangle - \frac{1}{4} \left(\langle \Delta p \zeta, \Delta \Phi \rangle + 2 \sum_{j=1}^{n} \langle \partial_j p \partial_j \zeta, \Delta \Phi \rangle \right)$$

$$= -\frac{1}{4} \langle \Delta p \zeta, \Delta \Phi \rangle + \frac{1}{2} \sum_{i=1}^{n} \langle \partial_i p \partial_i \zeta, \Delta \Phi \rangle$$

and we find that

$$-\frac{1}{4}\langle \Delta p\zeta, \Delta\Phi\rangle = -\frac{1}{4}\langle \Delta p, \zeta\Delta\Phi\rangle = -\frac{1}{4}\langle \Delta p, \Delta(\zeta\Phi) - 2\nabla\zeta\cdot\nabla\Phi - \Delta\zeta\Phi\rangle =$$

$$= \underbrace{-\frac{1}{4}\langle \Delta p, \Delta(\zeta\Phi)\rangle}_{=0 \text{ as } \zeta\Phi\in H_0^{2,q'}(G) \text{ and } p\in B_0^q(G)} + \frac{1}{2}\sum_{j=1}^{n}\langle \Delta p, \partial_j\zeta\partial_j\Phi\rangle +$$

$$+ \underbrace{\frac{1}{4}\langle \Delta p, \Delta\zeta\Phi\rangle}_{\text{after integrating partially this term is seen to be of type L}} =_L$$

$$=_L \frac{1}{2}\sum_{j,k=1}^{n}\langle \partial_k\partial_k p, \partial_j\zeta\partial_j\Phi\rangle = -\frac{1}{2}\sum_{j,k=1}^{n}\langle \partial_k p, \partial_k\partial_j\zeta\partial_j\Phi + \partial_j\zeta\partial_k\partial_j\Phi\rangle =$$

$$= \underbrace{-\frac{1}{2}\sum_{j,k=1}^{n}\langle \partial_k p\partial_k\partial_j\zeta, \partial_j\Phi\rangle}_{\text{of type L}} - \frac{1}{2}\sum_{j,k=1}^{n}\langle \partial_k p, \partial_j\zeta\partial_k\partial_j\Phi\rangle =_L$$

$$=_L -\frac{1}{2}\sum_{j,k=1}^{n}\langle \partial_k p, \partial_j\zeta\partial_k\partial_j\Phi\rangle = -\frac{1}{2}\sum_{j,k=1}^{n}\langle \partial_k p\partial_j\zeta, \partial_k\partial_j\Phi\rangle =$$

$$= \frac{1}{2}\sum_{j,k=1}^{n}\langle \partial_j\partial_k p\partial_j\zeta, \partial_k\Phi\rangle + \underbrace{\frac{1}{2}\sum_{j,k=1}^{n}\langle \partial_k p\partial_j\partial_j\zeta, \partial_k\Phi\rangle}_{\text{of type L}} =_L$$

$$=_L \frac{1}{2}\sum_{j,k=1}^{n}\langle \partial_j\partial_k p, \partial_j\zeta\partial_k\Phi\rangle = -\frac{1}{2}\sum_{j,k=1}^{n}\langle \partial_j p, \partial_k\partial_j\zeta\partial_k\Phi\rangle -$$

$$\frac{1}{2}\sum_{j,k=1}^{n}\langle \partial_j p, \partial_j\zeta\partial_k\partial_k\Phi\rangle = \underbrace{-\frac{1}{2}\sum_{j,k=1}^{n}\langle \partial_j p, \partial_k\partial_j\zeta\partial_k\Phi\rangle}_{\text{of type L}} - \frac{1}{2}\sum_{j=1}^{n}\langle \partial_j p\partial_j\zeta, \Delta\Phi\rangle =_L$$

$$=_L -\frac{1}{2}\sum_{j=1}^{n}\langle \partial_j p\partial_j\zeta, \Delta\Phi\rangle$$

and all together this means that

$$B\left[w, \Phi\right] = T_1 - T_2 =_L -\frac{1}{4}\langle \Delta p\zeta, \Delta\Phi\rangle + \frac{1}{2}\sum_{i=1}^{n}\langle \partial_i p, \partial_i\zeta\Delta\Phi\rangle =_L$$

$$=_L -\frac{1}{2}\sum_{j=1}^{n}\langle \partial_j p \partial_j \zeta, \Delta\Phi\rangle + \frac{1}{2}\sum_{j=1}^{n}\langle \partial_j p \partial_j \zeta, \Delta\Phi\rangle = 0$$

and $B[w,\cdot]$ itself is of type L. Theorem 6.1 yields $w \in H_0^{2,q}(G) \cap H^{3,q}(G)$ and a constant $\gamma > 0$ such that

$$\|w\|_{3,p} \leq \gamma \left(\|F\|_{\left(H_0^{1,q'}(G)\right)^*} + \|w\|_q \right)$$

where F is the element of $\left(H_0^{1,q'}(G)\right)^*$ belonging to $B[w,\cdot]$ in the sense of "being of type L". Thus there is a constant $C > 0$ such that

$$\|F\|_{\left(H_0^{1,q'}(G)\right)^*} \leq C \|\nabla p\|_q$$

and as

$$w = \underline{u} \cdot \nabla\zeta - \frac{1}{4} p\zeta$$

we have

$$\|w\|_q \leq \|\underline{u} \cdot \nabla\zeta\|_q + \frac{1}{4}\|p\zeta\|_q \leq C' \|\nabla p\|_q$$

and the desired estimate follows. □

Theorem 6.3. *Let $p \in B_0^q(G) \cap H^{3,q}(G)$, $\underline{u} := \underline{T}_q(p)$ and $w = \underline{u} \cdot \nabla\zeta - \frac{1}{4}p\zeta$ as in Theorem 6.2. Then we have*

$$v := \nabla w \cdot \nabla\zeta - \left(\operatorname{div}\underline{u} - \frac{1}{2}p\right) \in H_0^{2,q}(G)$$

and there is a constant $C = C(G,q) > 0$ with

$$\|v\|_{2,q} \leq C \|\nabla p\|_q . \tag{38}$$

Further we have

$$\operatorname{div}\underline{u} - \frac{1}{2}p \in H^{2,q}(G)$$

and there is a constant $C = C(G,q) > 0$ with

$$\left\|\operatorname{div}\underline{u} - \frac{1}{2}p\right\|_{2,q} \leq C \|\nabla p\|_q \tag{39}$$

Proof. As $w \in H_0^{2,q}(G) \cap H^{3,q}(G)$ we see that $\nabla w \cdot \nabla \zeta \in H_0^{1,q}(G) \cap H^{2,q}(G)$ and as $p \in B^q(G) \cap H^{3,q}(G)$ we see by elliptic regularity (see Theorem 6.1) $\underline{u} \in \underline{H}_0^{2,q}(G) \cap \underline{H}^{4,q}(G)$ and thus $\operatorname{div} \underline{u} - \frac{1}{2}p \in H_0^{1,q}(G) \cap H^{2,q}(G)$. In view of Theorem 3.3 we only need to show that $\nabla v \in \underline{H}_0^{1,q}(G)$. We have for $1 \leq k \leq n$ and almost every $x \in \partial G$ with the trace operator Z^1 as defined in Theorem 4.4:

$$Z^1(\partial_k(\nabla w \cdot \nabla \zeta))(x) = Z^1\left(\sum_{j=1}^n \partial_k \partial_j w \partial_j \zeta\right)(x) + \underbrace{Z^1\left(\sum_{j=1}^n \underbrace{\partial_j w \partial_k \partial_j \zeta}_{\in H_0^{1,q}(G)}\right)(x)}_{=0} =$$

$$= Z^1\left(\sum_{j=1}^n \partial_k \partial_j \left(\sum_{l=1}^n u_l \partial_l \zeta - \frac{1}{4}p\zeta\right) \partial_j \zeta\right)(x) =$$

$$= Z^1\left(\sum_{l,j=1}^n \partial_k \left(\partial_j u_l \partial_l \zeta + u_l \partial_j \partial_l \zeta\right) \partial_j \zeta - \frac{1}{4}\sum_{j=1}^n \partial_k \left(\partial_j p \zeta + p \partial_j \zeta\right) \partial_j \zeta\right)(x) =$$

$$= Z^1\left(\sum_{l,j=1}^n \left(\partial_k \partial_j u_l \partial_l \zeta \partial_j \zeta + \underbrace{\partial_j u_l \partial_k \partial_l \zeta \partial_j \zeta}_{\in H_0^{1,q}(G)} + \underbrace{\partial_k u_l \partial_j \partial_l \zeta \partial_j \zeta}_{\in H_0^{1,q}(G)} + \underbrace{u_l \partial_k \partial_j \partial_l \zeta \partial_j \zeta}_{\in H_0^{1,q}(G)}\right) - \right.$$

$$\left. \frac{1}{4}\sum_{j=1}^n \left(\underbrace{\partial_k \partial_j p \zeta \partial_j \zeta}_{\in H_0^{1,q}(G)} + \partial_j p \partial_k \zeta \partial_j \zeta + \partial_k p \partial_j \zeta \partial_j \zeta + \underbrace{p \partial_k \partial_j \zeta \partial_j \zeta}_{\in H_0^{1,q}(G)}\right)\right)(x) =$$

$$= Z^1\left(\sum_{l,j=1}^n \partial_k \partial_j u_l \partial_l \zeta \partial_j \zeta - \frac{1}{4}\sum_{j=1}^n \left(\partial_j p \partial_k \zeta \partial_j \zeta + \partial_k p \partial_j \zeta \partial_j \zeta\right)\right)(x) =$$

$$= \left(\sum_{l,j=1}^n \underbrace{Z^1(\partial_k \partial_j u_l)}_{=\lambda_{k,l} N_j} N_l N_j - \frac{1}{4}\sum_{j=1}^n Z^1(\partial_j p) N_k N_j - \frac{1}{4}\sum_{j=1}^n Z^1(\partial_k p) N_j^2\right)(x)$$

where $\lambda_{k,l} \in L^r(\partial G)$ can be found after application of Theorem 4.10 to the $H_0^{1,q}(G) \cap H^{2,q}(G)$-function $\partial_k u_l$ and thus

$$Z^1(\partial_k \partial_j u_l) = Z^1(\partial_j \partial_k u_l) = \lambda_{k,l} N_j.$$

This and application of Theorem 4.10 to p leads us to

$$\left(\sum_{j,k=1}^{n}\lambda_{k,l}N_jN_lN_j-\frac{1}{4}\sum_{j=1}^{n}\underbrace{Z^1(\partial_j p)}_{=:\mu N_j}N_kN_j-\frac{1}{4}\underbrace{Z^1(\partial_k p)}_{=\mu N_k}\right)(x)=$$

$$=\left(\sum_{l=1}^{n}\lambda_{k,l}N_l-\frac{1}{2}\mu N_k\right)(x).$$

With a similar calculation we find out that

$$Z^1\left(\partial_k\left(\operatorname{div}\underline{u}-\frac{1}{2}p\right)\right)(x)=Z^1\left(\partial_k\sum_{l=1}^{n}\partial_l u_l-\frac{1}{2}\partial_k p\right)(x)=$$

$$=Z^1\left(\sum_{l=1}^{n}\partial_k\partial_l u_l-\frac{1}{2}\partial_k p\right)(x)=\left(\sum_{l=1}^{n}Z^1\underbrace{(\partial_k\partial_l u_l)}_{=\lambda_{k,l}N_l}-\frac{1}{2}\underbrace{Z^1(\partial_k p)}_{=\mu N_k}\right)(x)$$

and altogether we have $Z^1(\nabla v)=0$. With Theorem 4.7,we conclude that $\nabla v\in\underline{H}_0^{1,q}(G)$ and with Theorem 3.3 it follows $v\in H_0^{2,q}(G)$ and we can prove the desired estimate (38) with the use of Müller's variational inequality (Theorem 2.7) and the fact that for every $\Phi\in\mathcal{C}_0^\infty(G)$ we have

$$\langle\Delta(\operatorname{div}\underline{u}-\frac{1}{2}p),\Delta\Phi\rangle=\langle\Delta\underbrace{\operatorname{div}\underline{u}}_{=Z_q(p)\in B_0^q(G)},\Delta\Phi\rangle-\frac{1}{2}\langle\Delta\underbrace{p}_{\in B_0^q(G)},\Delta\Phi\rangle=0$$

and thus this equality is even valid for every $\Phi\in H_0^{2,q'}(G)$.
We have with a $C_{M,q}>0$ by Müller's variational inequality:

$$\|\Delta v\|_q\leq C_{M,q}\sup_{0\neq\Phi\in H_0^{2,q'}(G)}\frac{\langle\Delta\left(\nabla w\cdot\nabla\zeta-(\operatorname{div}\underline{u}-\frac{1}{2}p)\right),\Delta\Phi\rangle}{\|\Delta\Phi\|_{q'}}=$$

$$=C_{M,q}\sup_{0\neq\Phi\in H_0^{2,q'}(G)}\frac{\langle\Delta\left(\nabla w\cdot\nabla\zeta\right),\Delta\Phi\rangle}{\|\Delta\Phi\|_{q'}}\leq C_{M,q}\|\Delta(\nabla w\cdot\nabla\zeta)\|_q\leq$$

$$\leq C_{M,q}C'\|w\|_{3,q,G}\leq C_{M,q}C'C\|\nabla p\|_q$$

To get (39), we noticed already that $\operatorname{div}\underline{u}-\frac{1}{2}p\in\underline{H}^{2,q}(G)$.
By the triangle inequality, we get

$$\left\|\operatorname{div}\underline{u}-\frac{1}{2}p\right\|_{2,q}\leq\left\|\nabla w\cdot\nabla\zeta-(\operatorname{div}\underline{u}-\frac{1}{2}p)\right\|_{2,q}+\|\nabla w\cdot\nabla\zeta\|_{2,q}.$$

Note that

$$\left\| \nabla w \cdot \nabla \zeta - (\operatorname{div} \underline{u} - \frac{1}{2} p) \right\|_{2,q} = \|v\|_{2,q}$$

and because of $v \in H_0^{2,q}(G)$, we have the estimate $\|v\|_{2,q} \leq C_1 \|\Delta v\|_q$ with C_1 according to Definition 2.1. By our calculation from above, we see that we can thus find a constant $C_2 > 0$ with

$$\|v\|_{2,q} \leq C_2 \|\nabla p\|_q .$$

Furthermore, we see that the term $\|\nabla w \cdot \nabla \zeta\|_{2,q}$ can be estimated against $C_3(\zeta) \|w\|_{3,q}$ and by Theorem 6.2, we have a respective estimate for this term. □

Theorem 6.4. *Let $p \in B_0^q(G)$, $u := \underline{T}_q(p)$. Then we have*

$$\operatorname{div} \underline{u} - \frac{1}{2} p \in H^{2,q}(G)$$

and there is a constant $C = C(G,q) > 0$ with

$$\left\| \operatorname{div} \underline{u} - \frac{1}{2} p \right\|_{2,q} \leq C \|\nabla p\|_q \tag{40}$$

Proof. In Theorem 6.3 we have shown the claim under the extra assumption that $p \in H^{3,q}(G)$. Approximate now p with a sequence $(p_\nu)_{\nu \in \mathbb{N}} \subset B_0^q(G) \cap H^{3,q}(G)$ with $\|p_\nu - p\|_{1,q} \to 0$ according to Theorem 5.1. Then the sequence $(\underline{u}_\nu)_{\nu \in \mathbb{N}}$ defined by $\underline{u}_\nu := \underline{T}_q(p_\nu)$ lies in $\underline{H}_0^{2,q}(G) \cap \underline{H}^{3,q}(G)$ and $\underline{u}_\nu \to \underline{u} := \underline{T}_q(p)$ in $\underline{H}^{3,q}(G)$, according to our Theorem on Elliptic Regularity.
With Theorem 6.3 we have for $\nu \in \mathbb{N}$:

$$\operatorname{div} \underline{u}_\nu - \frac{1}{2} p_\nu \in H^{2,q}(G)$$

Applying the inequality (39) we get for $\mu, \nu \in \mathbb{N}$:

$$\left\| \operatorname{div} \underline{u}_\mu - \frac{1}{2} p_\mu - \left(\operatorname{div} \underline{u}_\nu - \frac{1}{2} p_\nu \right) \right\|_{2,q} = \left\| \operatorname{div} \left(\underline{u}_\mu - \underline{u}_\nu \right) - \frac{1}{2} \left(p_\mu - p_\nu \right) \right\|_{2,q} \leq$$

$$\leq \left\| \nabla \left(p_\mu - p_\nu \right) \right\|_q \xrightarrow{\mu,\nu \to \infty} 0.$$

and so $\operatorname{div} \underline{u}_\nu - \frac{1}{2} p_\nu$ is a Cauchy-sequence in $H^{2,q}(G)$ and has a limit in $H^{2,q}(G)$. But after passing to subsequences with pointwise convergence almost everywhere we see that the limit must be equal to $\operatorname{div} \underline{u} - \frac{1}{2} p$.
The inequality (39) easily carries over by passing to the limit. □

Theorem 6.5. *The operator* $Z_q - \frac{1}{2}Id : B_0^q(G) \to B_0^q(G)$ *is a compact operator.*

Proof. We have the following situation:

$$B_0^q(G) \xrightarrow{Z_q - \frac{1}{2}Id} H^{2,q}(G) \xrightarrow{Id} H^{1,q}(G)$$

where the first arrow is a continuous map by Theorem 6.4 and the second arrow is a compact embedding, so the composition is compact, too. □

7 Consequences and Applications

In the preceding section we have proved the compactness of the operator $Z_q - \frac{1}{2}Id : B_0^q(G) \to B_0^q(G)$. This has far-ranging consequences. We first prove a regularity theorem for eigenvectors p of Z_q very similar to a regularity theorem by Weyers (see [22], Theorem 13.1, page 138):

Theorem 7.1. *Let* $\lambda \in \mathbb{R}$, $\lambda \neq \frac{1}{2}$ *and* $p \in B_0^q(G)$ *satisfying* $Z_q(p) = \lambda p$. *Then for every* $1 < r < \infty$:

$$p \in B_0^r(G) \text{ and } Z_r(p) = Z_q(p) = \lambda p$$

Proof. We have with $\underline{u} := \underline{T}_q(p)$:

$$\lambda p = \operatorname{div} \underline{u} \text{ which leads to } \left(\lambda - \frac{1}{2}\right) p = \operatorname{div} \underline{u} - \frac{1}{2}p, \text{ so}$$

$$p = \frac{1}{\lambda - \frac{1}{2}} \left(\operatorname{div} \underline{u} - \frac{1}{2}p\right) \in B_0^q(G) \cap H^{2,q}(G)$$

according to Theorem 6.4. We have three cases in each of which we will use the Sobolev Embedding Theorem (see for instance [2], Theorem 8.9, page 328 and Theorem 8.13, page 333):

1. $q > n$: In this case it follows by the Sobolev Embedding Theorem with

   $$p \in H^{2,q}(G)$$

   that $p \in \overline{\mathcal{C}}^1(G)$.

   It follows $p \in H^{1,r}_{0,0}(G)$ for every $1 < r < \infty$ because we have $p|_{\partial G} = 0$ by $p \in H_0^{1,q}(G)$, $p \in \overline{\mathcal{C}}^1(G)$, Theorem 4.7 and Remark 4.6 and thus with $p \in H^{1,r}(G)$ and $p|_{\partial G} = 0$ also $p \in H_0^{1,r}(G)$ by Theorem 4.7 and Remark 4.6.

We then find $p \in B_0^r(G)$ for every $1 < r < \infty$. Now we show $Z_r(p) = Z_q(p)$:

We have $\underline{T}_r(p) \in \underline{H}_0^{2,r}(G)$ and $\underline{T}_q(p) \in \underline{H}_0^{2,q}(G)$.

Applying Theorems 2.8 and 2.10 (in a vector-valued version which can be derived from Theorem 2.10 easily) we conclude that also $\underline{T}_r(p) \in \underline{H}_0^{2,q}(G)$ and we have for all $\underline{\Phi} \in \underline{\mathcal{C}}_0^\infty(G)$:

$$\langle \Delta(\underline{T}_r(p) - \underline{T}_q(p)), \Delta\underline{\Phi}\rangle = \langle \nabla(p - p), \Delta\underline{\Phi}\rangle = 0$$

and by density of $\underline{\mathcal{C}}_0^\infty(G)$ in $\underline{H}_0^{2,q'}(G)$ and the unique solvability in Theorem 2.9 in $\underline{H}_0^{2,q}(G)$ we can conclude that

$$\underline{T}_r(p) = \underline{T}_q(p).$$

2. $q < n$: In this case we define $q^\star := \frac{nq}{n-q} > q$ and we find by Sobolev's embedding theorem $p \in H_{0,0}^{1,q^\star}(G)$ and as p is still biharmonic we have $p \in B_0^{q^\star}(G)$. As above, we conclude with Theorem 6.4 again that also $p \in H^{2,q^\star}(G)$. If $q^\star > n$ we are in the first case and done. The rest is a simple induction argument, the induction step already given:

   We define recursively $q_0 := q$ and for $i > 1$: $q_i := (q_{i-1})^\star = \frac{nq_{i-1}}{n-q_{i-1}} > q_{i-1}$ as long as $q_{i-1} < n$. By the induction step given above for $i = 1$ above, we conclude $p \in H_{0,0}^{1,q_i}(G)$ and $p \in B_0^{q_i}(G)$. Should it occur that $q_i > n$, we are in the first case and done. Should it occur that $q_i = n$ for an i, then we are in the third case and done.

   Now we still have to show a statement about the q_i. We show inductively that $q_i = \frac{nq}{n-iq}$. This is obviously true for $i = 1$. For $i > 1$ we find

$$q_i = \frac{nq_{i-1}}{n - q_{i-1}} = \frac{n\frac{nq}{n-(i-1)q}}{n - \frac{nq}{n-(i-1)q}} = \frac{nq}{n-(i-1)q-q} = \frac{nq}{n-iq}$$

   We see that formally for $i \geq \frac{n-q}{q}$, we have $q_i \geq n$, so there must be a first $i_0 \in \mathbb{N}$ with $q_{i_0} \geq n$ and the induction given above stops somewhere and we finally reach the first or third case.

3. $q = n$: In this case we see by the boundedness of G and the Hölder inequality that $p \in B_0^r(G)$ with $r := \frac{3}{4}n > 1$. In analogy to the second case, we conclude that $q^\star = \frac{\frac{3}{4}n^2}{\frac{1}{4}n} = 3n > n$ and we are in the first case.

□

Theorem 7.2. *The operator $Z_q : B_0^q(G) \to B_0^q(G)$ is bijective.*

Proof. We first prove the injectivity of Z_q: Let $p \in B_0^q(G)$ with $Z_q(p) = 0$. Then p is an eigenvector of Z_q with eigenvalue 0 and with Theorem 7.1 we see $p \in B_0^2(G)$ and $Z_2(p) = 0$. With $\underline{u} := \underline{T}_2(p) \in \underline{H}_0^{2,2}(G)$ we thus have $\operatorname{div} \underline{u} = 0$ and for all $\underline{\Phi} \in \underline{H}_0^{2,2}(G)$ we have

$$\langle \Delta \underline{u}, \Delta \underline{\Phi} \rangle = \langle \nabla p, \nabla \operatorname{div} \underline{\Phi} \rangle$$

As $\underline{u} \in \underline{H}_0^{2,2}(G)$ is a permitted testing function, we see for $\underline{\Phi} = \underline{u}$:

$$\langle \Delta \underline{u}, \Delta \underline{u} \rangle = \langle \nabla p, \nabla \underbrace{\operatorname{div} \underline{u}}_{=0} \rangle = 0,$$

so $\underline{u} = 0$ and $\langle \nabla p, \Delta \underline{\Phi} \rangle = 0$ for all $\underline{\Phi} \in \underline{H}_0^{2,q}(G)$. We conclude that $p \in N^q(G) \cap H_{0,0}^{1,q}(G)$ (see the decomposition Theorem 2.26), so $p = 0$, as the decomposition is direct.
Because we can write

$$Z_q = \frac{1}{2} Id - \underbrace{\left(\frac{1}{2} Id - Z_q \right)}_{\text{compact}},$$

we see that Z_q is a Fredholm operator and thus by injectivity automatically bijective. □

Analogously to [14], Theorem 3.2., page 174 we find

Theorem 7.3. *The bijective operator $Z_q : B_0^q(G) \to B_0^q(G)$ is a homeomorphism.*

Proof. We already know that Z_q is continuous and bijective, so all that we need to show is that there is a constant $C_H = C_H(q, G) > 0$ such that for every $p \in B_0^q(G)$ the inequality

$$\|\nabla p\|_q \leq C_H \|\nabla Z_q(p)\|_q \tag{41}$$

is valid. Let's assume that (41) were not valid. Then we could find a sequence $(p_\nu)_{\nu \in \mathbb{N}} \subset B_0^q(G)$ such that

$$\|\nabla p_\nu\|_q = 1$$

and

$$\|\nabla Z_q(p_\nu)\|_q \to 0.$$

As the sequence $(p_\nu)_{\nu \in \mathbb{N}} \subset B_0^q(G)$ is bounded in $H^{1,q}(G)$, we can assume by passing to a subsequence without loss of generality that the sequence is weakly convergent to a $p \in B_0^q(G)$ (as $B_0^q(G) \subset H^{1,q}(G)$ is a closed linear subspace, it is weakly closed, too). By Theorem 6.5, we see that $Z_q - \frac{1}{2}Id$ is compact. With the representation

$$p_\nu - p_\mu = 2Z_q(p_\nu - p_\mu) - 2Z_q(p_\nu - p_\mu) + p_\nu - p_\mu$$

we find

$$\|\nabla(p_\nu - p_\mu)\|_q \leq 2\underbrace{\|\nabla(Z_q(p_\nu))\|_q}_{\xrightarrow{\nu\to\infty}0} + 2\underbrace{\|\nabla Z_q(p_\mu)\|_q}_{\xrightarrow{\mu\to\infty}0} +$$

$$+\,2\underbrace{\left\|\nabla\left(Z_q(p_\nu - p_\mu) - \frac{1}{2}(p_\nu - p_\mu)\right)\right\|_q}_{\xrightarrow{\nu,\mu\to\infty}0 \text{ as } Z_q - \frac{1}{2}Id \text{ is compact and } (p_\nu - p_\mu)\xrightarrow{\nu,\mu\to\infty}0 \text{ weakly}}.$$

So the sequence $(p_\nu)_{\nu\in\mathbb{N}} \subset B_0^q(G)$ is a Cauchy-sequence in $H^{1,q}(G)$ and thus converging strongly to a $p' \in B_0^q(G)$. It follows $p = p'$,

$$\|\nabla p\|_q = 1$$

and

$$Z_q(p) = \lim_{\nu\to\infty} Z_q(p_\nu) = 0.$$

By injectivity of Z_q we find $p = 0$, a contradiction. □

Remark 7.4. *In view of Theorem 7.2 and the decomposition from Theorem 2.23 and Theorem 2.24, we immediately see that Z_q is even bijective when viewed as a mapping from $H_{0,0}^{1,q}(G)$ on itself.*
Moreover, with Theorem 7.3 and the estimate (18) from Theorem 2.23 we easily see that Z_q is a homeomorphism when viewed as a mapping from $H_{0,0}^{1,q}(G)$ on itself.

Definition 7.5. *Set $\underline{M}_q(G) := \underline{T}_q\left(H_{0,0}^{1,q}(G)\right)$.*

Theorem 7.6. *For every $p \in H_{0,0}^{1,q}(G)$ there is exactly one $\underline{u} \in \underline{M}_q(G)$ with*

$$\operatorname{div} \underline{u} = p.$$

The in this way well defined function

$$\underline{D}_q : H_{0,0}^{1,q}(G) \to \underline{M}_q(G),\ p \mapsto \text{ the unique } \underline{u} \in \underline{M}_q(G) \text{ with } \operatorname{div} \underline{u} = p$$

is continuous.

Proof. According to Theorem 2.23, we decompose $p = p_b + p_0$ with $p_b \in B_0^q(G)$ and $p_0 = \Delta s \in A_0^q(G)$ for an $s \in H_0^{3,q}(G)$ and we have

$$\|\nabla \Delta s\|_q + \|\nabla p_b\|_q \leq C_D \|\nabla p\|_q .$$

As $Z_q : B_0^q(G) \to B_0^q(G)$ is bijective, we find exactly one $h \in B_0^q(G)$ with $Z_q(h) = \operatorname{div} \underline{T}_q(h) = p_b$ and by Theorem 7.3 we have with a constant C_H

$$\|\nabla h\|_q \leq C_H \|\nabla p_b\|_q$$

and with $\underline{u} := \underline{T}_q(h)$ we have found an $\underline{u} \in \underline{M}_q(G)$ with $\operatorname{div} \underline{u} = p_b$ which is satisfying

$$\|\Delta \underline{u}\|_q \leq C \|\nabla h\|_q \leq C C_H \|\nabla p_b\|_q \leq C C_H C_D \|\nabla p\|_q .$$

We also have

$$\|\nabla \Delta s\|_q \leq C_D \|\nabla p\|_q$$

and altogether

$$\|\Delta(\underline{u} + \nabla s)\|_q \leq C C_H C_D \|\nabla p\|_q + C_D \|\nabla p\|_q = (C C_H C_D + C_D) \|\nabla p\|_q .$$

So the linear assignment $p \mapsto \underline{D}_q(p) := \underline{u} + \nabla s$ from $H_{0,0}^{1,q}(G)$ to $\underline{M}_q(G) \subset \underline{H}_0^{2,q}(G)$ is continuous. We immediately see that

$$\operatorname{div}(\underline{D}_q(p)) = p,$$

because

$$\operatorname{div}(\underline{D}_q(p)) = \operatorname{div}(\underline{u} + \nabla s) = Z_q(h) + \Delta s = p_b + \Delta s = p_b + p_0 = p.$$

For the uniqueness, we assume we had two elements $\underline{v} = \underline{T}_q(p_1), \underline{w} = \underline{T}_q(p_2) \in \underline{M}_q(G)$ with

$$\operatorname{div} \underline{v} = \operatorname{div} \underline{w} = p.$$

If we take a look at $t := p_1 - p_2 \in H_{0,0}^{1,q}(G)$, we immediately see that

$$Z_q(t) = \operatorname{div}(\underline{v} - \underline{w}) = 0.$$

By Remark 7.4, we have bijectivity of Z_q and thus we find $t = 0$. □

Concerning $\underline{D}_q$, we can also make the following regularity statement:

Theorem 7.7. *Let* $1 < q, r < \infty$ *and* $\underline{f} \in \underline{M}_q(G)$ *with*

$$\operatorname{div} \underline{f} \in H_{0,0}^{1,r}(G).$$

Then we also have $\underline{f} \in \underline{M}_r(G)$.

Proof. The proof is rather simple: As $\operatorname{div} \underline{f} \in H^{1,r}_{0,0}(G)$, we can take a look at $\tilde{\underline{f}} := \underline{D}_r(\operatorname{div} \underline{f}) \in \underline{M}_r(G)$.
But by boundedness of G, as $\underline{M}_r(G) \supset \underline{M}_q(G)$ (in case $r < q$) or $\underline{M}_q(G) \supset \underline{M}_r(G)$ (in case $q < r$), we have

$$\underline{f} - \tilde{\underline{f}} \in \underline{M}_q(G)$$

or

$$\underline{f} - \tilde{\underline{f}} \in \underline{M}_r(G),$$

and as $\operatorname{div}(\underline{f} - \tilde{\underline{f}}) = 0$, we see that $\underline{f} = \tilde{\underline{f}} \in \underline{M}_r(G)$. □

With Theorem 7.6 we get a direct decomposition of $\underline{H}^{2,q}_0(G)$:

Theorem 7.8.

$$\underline{H}^{2,q}_0(G) = \underline{D}^{2,q}_0(G) \oplus \underline{M}_q(G),$$

where $\underline{D}^{2,q}_0(G) = \left\{\underline{v} \in \underline{H}^{2,q}_0(G) : \operatorname{div} \underline{v} = 0\right\}$.
We also have the estimate

$$\|\Delta \underline{v}_1\|_q + \|\Delta \underline{v}_2\|_q \leq C \|\Delta \underline{v}\|_q \tag{42}$$

with a constant $C = C(q, G) > 0$ *for every* $\underline{v} = \underline{v}_1 + \underline{v}_2 \in \underline{H}^{2,q}_0(G)$ *with* $\underline{v}_1 \in \underline{D}^{2,q}_0(G)$ *and* $\underline{v}_2 \in \underline{M}_q(G)$.

Proof. Let $\underline{v} \in \underline{H}^{2,q}_0(G)$ be arbitrary. Then we can define $p := \operatorname{div} \underline{v} \in H^{1,q}_{0,0}(G)$ and $\underline{w} := \underline{D}_q(p) \in \underline{M}_q(G)$. Then $\operatorname{div} \underline{v} = p = \operatorname{div} \underline{w}$ and so we have $v - w \in \underline{D}^{2,q}_0(G)$ and

$$v = \underbrace{(v - w)}_{\in \underline{D}^{2,q}_0(G)} + \underbrace{w}_{\in \underline{M}_q(G)}$$

and the decomposition is shown.
For the directness we see: If $\underline{v} \in \underline{D}^{2,q}_0(G) \cap \underline{M}_q(G)$, then we have $\operatorname{div} \underline{v} = 0$. As $\underline{v} \in \underline{M}_q(G)$, according to Theorem 7.6, we have one unique element $\underline{u}$ in $\underline{M}_q(G)$ with $\operatorname{div} \underline{u} = 0 = \operatorname{div} \underline{v}$ and as $0 \in \underline{M}_q(G)$ we have $\underline{v} = 0$.
We further see with Theorem 7.6 and $\underline{w} = \underline{D}_q(\operatorname{div} \underline{v})$ that

$$\|\Delta \underline{w}\|_q \leq C_{\underline{D}_q} \|\nabla \operatorname{div} \underline{v}\| \leq C_{\underline{D}_q} C_{CZ} \|\Delta \underline{v}\|_q ,$$

where $C_{\underline{D}_q} = C_{\underline{D}_q}(q, G) > 0$ is the constant existing by Theorem 7.6 such that

$$\left\|\Delta \underline{D}_q(p)\right\|_q \leq C_{\underline{D}_q} \|\nabla p\|_q$$

for all $p \in H^{1,q}_{0,0}(G)$.

Further we also have

$$\|\Delta(\underline{v}-\underline{w})\|_q \leq \|\Delta\underline{v}\|_q + \|\Delta\underline{w}\|_q \leq (1 + C_{\underline{D}_q} C_{CZ}) \|\Delta\underline{v}\|_q$$

All in all we see that estimate (42) is shown. □

Remark 7.9. *Using the fact that* $\underline{T}_q = \underline{D}_q \circ Z_q : H^{1,q}_{0,0}(G) \to \underline{M}_q(G)$ *is a homeomorphism we get with Theorem 2.23 even the refined decomposition*

$$\underline{H}^{2,q}_0(G) = \underline{D}^{2,q}_0(G) \oplus \underline{T}_q(A^q_0(G)) \oplus \underline{T}_q(B^q_0(G))$$

and with the estimate (18) we get also a refined estimate analogous to estimate (42).

The next Theorem provides us with a norm on $\underline{M}_q(G)$ which is equivalent to the norm $\|\Delta\cdot\|_q$:

Theorem 7.10. *There is a constant* $C_e = C_e(q, G) > 0$ *such that for every* $\underline{u} \in \underline{M}_q(G)$ *the inequality*

$$\|\Delta\underline{u}\|_q \leq C_e \|\nabla \operatorname{div} \underline{u}\|_q$$

is valid.

Proof. Let $\underline{u} \in \underline{M}_q(G)$ be arbitrary. Then we find a $p \in H^{1,q}_{0,0}(G)$ with $\underline{u} = \underline{T}_q(p)$. We have the estimate

$$\|\Delta\underline{u}\|_q \leq C \|\nabla p\|_q$$

with a $C = C(n, q, G) > 0$ by continuity of $\underline{T}_q$ and with Remark 7.4 we have a $C' = C'(n, q, G) > 0$ such that

$$\|\nabla p\|_q \leq C' \|\nabla Z_q(p)\|_q = C' \|\nabla \operatorname{div} \underline{u}\|_q$$

with $\underline{u} = \underline{T}_q(p)$ and the theorem is proved. □

We can now prove a divergence inequality:

Theorem 7.11. *There is a* $C_{\mathrm{div}} = C_{\mathrm{div}}(q, G) > 0$ *such that for every* $p \in H^{1,q}_{0,0}(G)$ *we have the estimate*

$$\|\nabla p\|_q \leq C_{\mathrm{div}} \sup_{0 \neq \underline{\Phi} \in \underline{M}_{q'}(G)} \frac{\langle \nabla p, \nabla \operatorname{div} \underline{\Phi} \rangle}{\|\nabla \operatorname{div} \underline{\Phi}\|_{q'}}$$

satisfied.

Proof. As $\underline{T}_q = \underline{D}_q \circ Z_q : H^{1,q}_{0,0}(G) \to \underline{M}_q(G)$ is a homeomorphism, we have a constant $C_{\underline{T}_q} > 0$ such that

$$\|\nabla p\|_q \leq C_{\underline{T}_q} \left\|\Delta \underline{T}_q(p)\right\|_q$$

and by defining $\underline{u} := \underline{T}_q(p)$ we find with Müller's variational inequality:

$$\|\Delta \underline{u}\|_q \leq \underline{C}_{M,q} \sup_{0 \neq \underline{\Phi} \in \underline{H}^{2,q'}_0(G)} \frac{\langle \Delta \underline{u}, \Delta \underline{\Phi}\rangle}{\|\Delta \underline{\Phi}\|_{q'}} = \underline{C}_{M,q} \sup_{0 \neq \underline{\Phi} \in \underline{H}^{2,q'}_0(G)} \frac{\langle \nabla p, \Delta \underline{\Phi}\rangle}{\|\Delta \underline{\Phi}\|_{q'}} =$$

$$= \underline{C}_{M,q} \sup_{0 \neq \underline{\Phi} \in \underline{H}^{2,q'}_0(G)} \frac{\langle \nabla p, \nabla \operatorname{div} \underline{\Phi}\rangle}{\|\Delta \underline{\Phi}\|_{q'}}$$

Using now for $\underline{\Phi} \in \underline{H}^{2,q'}_0(G)$ the decomposition of Theorem 7.8 and writing

$$\underline{\Phi} = \underline{\Phi}_0 + \underline{\Phi}_1,$$

where $\underline{\Phi}_0 \in \underline{D}^{2,q'}_0(G)$ and $\underline{\Phi}_1 \in \underline{M}_{q'}(G)$, we see by employing estimate (42) that

$$\|\Delta \underline{\Phi}_1\|_q \leq C' \|\Delta \underline{\Phi}\|_q$$

with $C' > 0$ and we also see that with $\|\Delta \underline{\Phi}_1\|_q = 0$ we have $\underline{\Phi} = \underline{\Phi}_0$ and thus $\langle \nabla p, \nabla \operatorname{div} \underline{\Phi}\rangle = 0$, so we have

$$\underline{C}_{M,q} \sup_{0 \neq \underline{\Phi} \in \underline{H}^{2,q'}_0(G)} \frac{\langle \nabla p, \nabla \operatorname{div} \underline{\Phi}\rangle}{\|\Delta \underline{\Phi}\|_{q'}} = \underline{C}_{M,q} \sup_{0 \neq \underline{\Phi} = \underline{\Phi}_0 + \underline{\Phi}_1 \in \underline{H}^{2,q'}_0(G), \underline{\Phi}_1 \neq 0} \frac{\langle \nabla p, \nabla \operatorname{div} \underline{\Phi}\rangle}{\|\Delta \underline{\Phi}\|_{q'}}.$$

By additionally using $\operatorname{div} \underline{\Phi}_0 = 0$, we get to

$$\|\nabla p\|_q \leq \frac{C_{\underline{T}_q} \underline{C}_{M,q}}{C'} \sup_{0 \neq \underline{\Phi} = \underline{\Phi}_0 + \underline{\Phi}_1 \in \underline{H}^{2,q'}_0(G), \underline{\Phi}_1 \neq 0} \frac{\langle \nabla p, \nabla \operatorname{div} \underline{\Phi}_1\rangle}{\|\Delta \underline{\Phi}_1\|_{q'}} \leq$$

$$\leq \frac{C_{\underline{T}_q} \underline{C}_{M,q}}{C'} \sup_{0 \neq \underline{\Phi} \in \underline{M}_{q'}(G)} \frac{\langle \nabla p, \nabla \operatorname{div} \underline{\Phi}\rangle}{\|\Delta \underline{\Phi}\|_{q'}}$$

and applying the fact from Theorem 7.10 that on $\underline{M}_{q'}(G)$ the norms $\|\Delta \cdot\|_q$ and $\|\nabla \operatorname{div} \cdot\|_q$ are equivalent, we have shown the theorem. □

8 Eigenvalues of Z_q

Now, as we have achieved the main aims, we take a look at the eigenvalues of Z_q.

8.1 General Statements

With Theorems 2.25 and 7.1 we have made first steps towards the study of eigenvalues/eigenvectors of Z_q. We have seen that for $\lambda \neq 1$ the only eigenvectors $p \in H^{1,q}_{0,0}(G)$ to the eigenvalue λ lie in $B^q_0(G)$. Having shown in Theorem 6.5 the compactness of $Z_q - \frac{1}{2}Id : B^q_0(G) \to B^q_0(G)$, the spectral Theorem for compact operators is the right tool for us. We could now try to use the general spectral theorem for compact operators, as it is for example given in [2], pages 377-380, but this would require some extra work, most of which would be introducing some new notation. However, by our Theorem 7.1 about regularity of the respective eigenfunctions $p \in B^q_0(G)$ with $Z_q(p) = \lambda p$, $\lambda \neq \frac{1}{2}$, we have seen that also $p \in B^2_0(G)$ is a eigenfunction with respect to Z_2 for λ, and thus, all we have to do is look at the Hilbert space case, which furthermore gives us more concrete information:

Theorem 8.1. *(Spectral theorem for compact operators in Hilbert spaces) Let H be a Hilbert space and $0 \neq A : H \to H$ be a compact hermitian operator. Then there exists a set $\{\Phi_k\}$ of orthonormal eigenelements of A, which is finite or countably infinite. We denote the eigenvalue for Φ_k with λ_k, so $A\Phi_k = \lambda_k\Phi_k$. In case of countable infinity, there is* $\lim_{k\to\infty} \lambda_k = 0$. *The eigenspaces*

$$E_\lambda := \{f \in H : \quad Af = \lambda f\}\,, \quad \lambda \neq 0$$

are finite dimensional vector spaces and furthermore we have the following representation for A:

$$Af = \sum_{k:\lambda_k \neq 0} \lambda_k \langle \Phi_k, f\rangle \Phi_k \text{ for every } f \in H.$$

Remark 8.2. *The operator $Z_2 - \frac{1}{2}Id : B^2_0(G) \to B^2_0(G)$ is easily seen to be hermitian. As Id of course is hermitian, all that is to show that Z_2 is hermitian. Let $r, s \in B^2_0(G) \subset H^{1,2}_{0,0}(G)$, $\underline{u} := \underline{T}_2(r)$, $\underline{v} := \underline{T}_2(s)$ and take a look at*

$$\langle \nabla Z_2(r), \nabla s\rangle = \langle \nabla \operatorname{div} \underline{u}, \nabla s\rangle = \langle \Delta \underline{u}, \nabla s\rangle.$$

As we have here $q = 2$, $\underline{\Phi} := \underline{u} \in \underline{H}^{2,2}_0(G)$ is in the following a permitted testing function and so by definition of $\underline{v} = \underline{T}_2(s)$, we see that

$$\langle \nabla s, \Delta \underline{\Phi}\rangle = \langle \Delta \underline{v}, \Delta \underline{\Phi}\rangle \quad \forall \underline{\Phi} \in \underline{H}^{2,2}_0(G)$$

and thus

$$\langle \nabla Z_2(r), \nabla s\rangle = \langle \Delta \underline{u}, \Delta \underline{v}\rangle.$$

By making the calculation from above again, this time for $\langle \nabla Z_2(s), \nabla r\rangle$, *we find that also*

$$\langle \nabla Z_2(s), \nabla r\rangle = \langle \Delta \underline{v}, \nabla r\rangle = \langle \Delta \underline{u}, \Delta \underline{v}\rangle$$

and thus for all $s, r \in B_0^2(G)$ *we have*

$$\langle \nabla Z_2(r), \nabla s\rangle = \langle \nabla r, \nabla Z_2(s)\rangle.$$

Remark 8.3. *Making the easy calculation*

$$Z_q(p) = \lambda p \Longleftrightarrow Z_q(p) - \frac{1}{2}p = \left(\lambda - \frac{1}{2}\right) p$$

we see that $\lambda \in \mathbb{R}$ *is an eigenvalue of* Z_q *if and only if* $\left(\lambda - \frac{1}{2}\right)$ *is an eigenvalue for* $Z_q - \frac{1}{2}Id$. *So, in order to study eigenvalues of* Z_q, *it suffices to study the eigenvalues of the (on* $B_0^q(G)$ *compact) operator* $Z_q - \frac{1}{2}Id$.

Theorem 8.4. *The set*

$$E := \left\{\lambda \in \mathbb{R} : \lambda \notin \left\{\frac{1}{2}, 1\right\},\ \text{there is a } 0 \neq p \in H_{0,0}^{1,q}(G) \text{ with } Z_q(p) = \lambda p\right\}$$

is finite or countably infinite. In the case of countably infiniteness, E has only one accumulation point (in $\overline{\mathbb{R}}$*) and this accumulation point is* $\frac{1}{2}$. *For every* $\lambda \in E$, *the respective eigenspace*

$$V_\lambda := \left\{p \in H_{0,0}^{1,q}(G) :\ Z_q(p) = \lambda p\right\}$$

is finite dimensional.

Proof. This is just a direct application of Theorem 8.1 to the operator $Z_2 - \frac{1}{2}Id$, additionally using Theorems 2.25, 7.1 and Remark 8.3. □

The following easy calculation shows the connection between eigenfunctions of Z_q to eigenvalues $\neq 0$ and elements $\underline{u} \in \underline{H}_0^{2,q}(G)$ satisfying

$$\langle \Delta \underline{u}, \Delta \underline{\Phi}\rangle = \mu \langle \nabla \operatorname{div} \underline{u}, \nabla \operatorname{div} \underline{\Phi}\rangle \text{ for all } \underline{\Phi} \in \underline{H}_0^{2,q'}(G)$$

for a $\mu \neq 0$.

Remark 8.5. *If we have a* $\lambda \in \mathbb{R} \setminus \{0\}$ *and a* $p \in H_{0,0}^{1,q}(G)$ *with* $Z_q(p) = \lambda p$, *we find with* $\underline{u} := \underline{T}_q(p)$:

$$\operatorname{div} \underline{u} = \lambda p, \text{ which is } p = \frac{1}{\lambda} \operatorname{div} \underline{u}$$

and thus for all $\underline{\Phi} \in \underline{H}_0^{2,q'}(G)$*:*

$$\langle \Delta \underline{u}, \Delta \underline{\Phi} \rangle = \langle \nabla p, \nabla \operatorname{div} \underline{\Phi} \rangle = \frac{1}{\lambda} \langle \nabla \operatorname{div} \underline{u}, \nabla \operatorname{div} \underline{\Phi} \rangle.$$

This also works in the other direction: If we have a $\mu \in \mathbb{R} \setminus \{0\}$ *and an* $\underline{u} \in \underline{H}_0^{2,q}(G)$ *with*

$$\langle \Delta \underline{u}, \Delta \underline{\Phi} \rangle = \mu \langle \nabla \operatorname{div} \underline{u}, \nabla \operatorname{div} \underline{\Phi} \rangle \text{ for all } \underline{\Phi} \in \underline{H}_0^{2,q'}(G),$$

we have

$$\underline{u} = \underline{T}_q(\mu \operatorname{div} \underline{u})$$

and we get after applying divergence to this with $p := \operatorname{div} \underline{u} \in H_{0,0}^{1,q}(G)$*:*

$$p = \mu Z_q(p), \text{ which is } Z_q(p) = \frac{1}{\mu} p.$$

8.2 The Eigenspace for $\lambda = 1$

The following characterization of the eigenspace for $\lambda = 1$ of the eigenvalue problem $Z_q(p) = \lambda p$ is inspired by Simader's and Weyers' characterization of the analogous problem for their Cosserat problem (see [19], chapters 5,6).
With Theorems 2.24 and 2.25 we already have a first statement about

$$E := \left\{ p \in H_{0,0}^{1,q}(G) : \; Z_q(p) = p \right\} :$$

We have $A_0^q(G) \subset E$, and the study of eigenvectors $p \in H_{0,0}^{1,q}(G)$ for $\lambda = 1$ reduces as in the case of $\lambda \neq 1$ to the study of eigenvectors $p \in B_0^q(G)$ for $\lambda = 1$.
To begin, we notice the following:
If $p \in B_0^q(G)$ is an eigenvector to the eigenvalue $\lambda = 1$, we find that $\operatorname{div} \underbrace{\underline{T}_q(p)}_{=:\underline{u}} = p$. By the regularity Theorem 7.1, we find that $p \in \overline{\mathcal{C}}^0(G) \cap H_{0,0}^{1,r}(G)$ and $\underline{u} \in \overline{\underline{\mathcal{C}}}^1(G) \cap \underline{H}_0^{2,r}(G)$ for all $1 < r < \infty$.
Now we make some simple calculation for $\underline{\Psi}, \underline{\Phi} \in \underline{\mathcal{C}}_0^\infty(G)$:
We have

$$R\left[\underline{\Psi}, \underline{\Phi}\right] := \frac{1}{2} \sum_{k,l,s=1}^{n} \langle \partial_k \partial_s \Psi_l - \partial_k \partial_l \Psi_s, \partial_k \partial_s \Phi_l - \partial_k \partial_l \Phi_s \rangle =$$

$$= \frac{1}{2} \sum_{k,l,s=1}^{n} \langle \partial_k \partial_s \Psi_l, \partial_k \partial_s \Phi_l - \partial_k \partial_l \Phi_s \rangle - \frac{1}{2} \sum_{k,l,s=1}^{n} \langle \partial_k \partial_l \Psi_s, \partial_k \partial_s \Phi_l - \partial_k \partial_l \Phi_s \rangle.$$

By interchanging the names of the variables s and l in the second summation, we see that all in all we have

$$\frac{1}{2}\sum_{k,l,s=1}^{n}\langle\partial_k\partial_s\Psi_l-\partial_k\partial_l\Psi_s,\partial_k\partial_s\Phi_l-\partial_k\partial_l\Phi_s\rangle=\sum_{k,l,s=1}^{n}\langle\partial_k\partial_s\Psi_l,\partial_k\partial_s\Phi_l-\partial_k\partial_l\Phi_s\rangle$$

for $\underline{\Psi},\underline{\Phi}\in\underline{\mathcal{C}}_0^{\infty}(G)$. By approximation, we get the validity of these expressions even for $\underline{\Psi}\in\underline{H}_0^{2,r}(G)$, $\underline{\Phi}\in\underline{H}_0^{2,r'}(G)$, for every $1<r<\infty$.
The statement $Z_q(p)=p$ means for $\underline{u}=\underline{T}_q(p)\in\underline{T}_q(B_0^q(G))$ that for every $\underline{\Phi}\in\underline{H}_0^{2,q'}(G)$ we have:

$$\langle\Delta\underline{u},\Delta\underline{\Phi}\rangle=\langle\nabla\operatorname{div}\underline{u},\Delta\underline{\Phi}\rangle$$

with $\operatorname{div}\underline{u}=p$.
Looking at the difference of these two expressions, we see by some partial integrations that

$$0=\langle\Delta\underline{u},\Delta\underline{\Phi}\rangle-\langle\nabla\operatorname{div}\underline{u},\Delta\underline{\Phi}\rangle=\sum_{k,l,s=1}^{n}\langle\partial_k^2u_l,\partial_s^2\Phi_l\rangle-\sum_{k,l,s=1}^{n}\langle\partial_s\partial_lu_l,\partial_k^2\Phi_s\rangle=$$

$$=\sum_{k,l,s=1}^{n}\langle\partial_k\partial_su_l,\partial_k\partial_s\Phi_l\rangle-\sum_{k,l,s=1}^{n}\langle\partial_k\partial_su_l,\partial_k\partial_l\Phi_s\rangle=$$

$$=\sum_{k,l,s=1}^{n}\langle\partial_k\partial_su_l,\partial_k\partial_s\Phi_l-\partial_k\partial_l\Phi_s\rangle,$$

which is by our calculation from above equal to $R\,[\underline{u},\underline{\Phi}]$.
As we have seen that $\underline{u}\in\underline{H}_0^{2,2}(G)$, we can plug in $\underline{u}$ itself as testing function $\underline{\Phi}$ and see from $R\,[\underline{u},\underline{u}]=0$ that for all k,l,s we must have $\partial_k\partial_su_l-\partial_k\partial_lu_s=0$. For fixed values of s and l, we see that the $H_0^{1,q}(G)$-function $u^{l,s}:=\partial_su_l-\partial_lu_s$ must satisfy $\nabla u^{l,s}=0$ and therefore by the Poincaré inequality $u^{l,s}$ must be equal to 0 on G.
Define

$$\underline{\tilde{u}}:=\begin{cases}\underline{u}(x),\text{ for }x\in G\\ 0,\text{ for }x\in\mathbb{R}^n\setminus G\end{cases}.$$

Then we have, as $\underline{u}\in\overline{\underline{\mathcal{C}}}^1(G)$ with $\underline{u}=0$ and $\partial_iu_j=0$ on ∂G a well defined vector field $\underline{\tilde{u}}\in\underline{\mathcal{C}}^1(\mathbb{R}^n)$ and we find by the simply connectedness of $\mathbb{R}^n$ by classical calculus a function $\phi\in\mathcal{C}^2(\mathbb{R}^n)$ such that $\nabla\phi=\underline{\tilde{u}}$ unique up to a constant. We also see that the restriction of $\nabla\Phi$ to G is in $H_0^{2,q}(G)$ as it is equal to $\underline{u}$ there and that $\underline{\tilde{u}}$ satisfies $\partial_s\tilde{u}_l(x)=\partial_l\tilde{u}_s(x)$ for every $x\ \in\mathbb{R}^n$. As

$\mathbb{R}^n \setminus \overline{G}$ is an open set with smooth compact boundary, we see that there are finitely many connected components $C_1, \dots, C_N$ of $\mathbb{R}^n \setminus \overline{G}$ (see for example [19], Lemma 5.1, page 185), and we see on any fixed C_i by definition of $\underline{\tilde{u}}$ that $\nabla\phi = 0$ on C_i and therefore we must have a constant $c_i \in \mathbb{R}$ with $\phi = c_i$ on C_i. This allows us to "norm" our ϕ the following way: By demanding it to be zero on a fixed C_{i_0} (there is always $N \geq 1$ and we fix i_0 to be the unbounded connected component of $\mathbb{R}^n \setminus \overline{G}$), we dispose of the above occurring uniqueness of ϕ only up to a constant.
Now we take a look at some helpful functions. If we take for $i = 1, \dots, N$, $i \neq i_0$ functions $\Psi_i \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ satisfying $\Psi_i = 1$ on (the bounded) $\overline{C_i}$ and $\Psi_i = 0$ on every C_j with $j \neq i$ we immediately see that $\underline{u}_i := \nabla\Psi_i \in \underline{H}_0^{2,q}(G) \cap \overline{\underline{\mathcal{C}}}^\infty(G)$ satisfies for every $\underline{\Phi} \in \underline{\mathcal{C}}_0^\infty(G)$

$$\langle \Delta\underline{u}_i, \Delta\underline{\Phi}\rangle = \langle \Delta\nabla\Psi_i, \Delta\underline{\Phi}\rangle = \langle \nabla \operatorname{div} \nabla\Psi_i, \Delta\underline{\Phi}\rangle = \langle \nabla \operatorname{div} \underline{u}_i, \Delta\underline{\Phi}\rangle$$

and thus with $p_i := \operatorname{div} \underline{u_i}$ we have $\underline{u_i} = \underline{T}_q(p_i)$ and thus $Z_q(p_i) = p_i$.
Further we define for Ψ_i a function $b_i \in H_0^{3,q}(G)$ by the unique solution of the problem

$$\langle \nabla\Delta b_i, \nabla\Delta g\rangle = \langle \nabla\Delta\Psi_i, \nabla\Delta g\rangle \text{ for all } g \in H_0^{3,q'}(G).$$

Having given a certain $p \in B_0^q(G) \cap E$ and $\underline{\tilde{u}}$ and ϕ constructed like above with $\phi = c_i$ on C_i, we find out that

$$f := \phi - \sum_{i=1, i\neq i_0}^{N} c_i(\Psi_i - b_i) \in H_0^{3,q}(G) :$$

As each b_i is in $H_0^{3,q}(G)$, we just need to show that $l := \phi - \sum_{i=1,i\neq i_0}^N c_i\Psi_i \in H_0^{3,q}(G)$. At first we see that $l \in \overline{\mathcal{C}}^2(G)$, as ϕ and each Ψ_i are. Further we see by definition of the c_i and Ψ_i that $l = 0$ on ∂G. Thus, $l \in H_0^{1,q}(G)$ by Theorem 4.7. Looking at $\nabla l = \nabla\phi - \sum_{i=1,i\neq i_0}^N c_i\nabla\Psi_i = \underline{u} - \sum_{i=1,i\neq i_0}^N c_i\nabla\Psi_i$, we see that $l \in H^{3,q}(G)$ and by Theorem 3.3, we get by looking at the boundary values of ∇l: $\nabla l \in H_0^{1,q}(G)$ and thus $l \in H_0^{2,q}(G)$. In analogy to this, we get with the help of Theorem 9.5 as $\underline{u} \in H_0^{2,q}(G)$ and each $\nabla\Psi_i \in H_0^{2,q}(G)$, $i \neq i_0$: $l \in H_0^{3,q}(G)$.
f is thus satisfying for every $g \in H_0^{3,q'}(G)$

$$\langle \nabla\Delta f, \nabla\Delta g\rangle = \langle \nabla\Delta\Phi, \nabla\Delta g\rangle - \sum_{i=1,i\neq i_0}^{N} c_i\langle \nabla\Delta(\Psi_i - b_i), \nabla\Delta g\rangle =$$

$$= \langle \Delta\underline{u}, \nabla\Delta g\rangle - \sum_{i=1,i\neq i_0}^{N} c_i\langle \nabla\Delta(\Psi_i - b_i), \nabla\Delta g\rangle,$$

where we see by $\operatorname{div} \underline{u} \in B_0^q(G)$ that $\langle \Delta \underline{u}, \nabla \Delta g \rangle = 0$ and for $i \neq i_0$ we have $\langle \nabla \Delta (\Psi_i - b_i), \nabla \Delta g \rangle = 0$ by the definition of the b_i. So by the variational inequality we finally get $f = 0$ and so

$$\phi = \sum_{i=1, i \neq i_0}^{N} c_i (\Psi_i - b_i).$$

As $\nabla \phi = \underline{u}$ and $p = \operatorname{div} \underline{u}$, we get $p = \operatorname{div} \nabla \Phi$ or

$$p = \sum_{i=1, i \neq i_0}^{N} c_i \Delta \left(\Psi_i - b_i \right).$$

All in all we see that

$$E = A_0^q(G) \oplus V(G), \tag{43}$$

where $V(G)$ is a finite dimensional vector space spanned by the elements $\Delta(\Psi_i - b_i)$, $i = 1, \ldots N$, $i \neq i_0$.
Moreover, we can show that the elements $\Delta(\Psi_i - b_i)$, $i = 1, \ldots N$, $i \neq i_0$ are linearly independent: To show this, we notice that due to the linear isomorphism $\operatorname{div} : \underline{T}_q(B_0^q(G)) \to B_0^q(G)$, we only have to show linear independence of the elements

$$\nabla(\Psi_i - b_i),\ i = 1, \ldots N,\ i \neq i_0.$$

Suppose there are $\lambda_i \in \mathbb{R}$, $i = 1, \ldots N$, $i \neq i_0$ such that

$$\sum_{i=1, i \neq i_0}^{N} \lambda_i \nabla (\Psi_i - b_i) = 0.$$

Then the function

$$g := \sum_{i=1, i \neq i_0}^{N} \lambda_i (\Psi_i - b_i) \in H^{1,q}(G)$$

must be constant: For $C := \int_G g \, dx$, we have $h := g - \frac{C}{|G|} \in H^{1,q}(G)$ mean-value-free and thus by the Poincaré inequality follows $h = 0$ which means that g is constant. Looking now at ∂C_{i_0}, we see that there each Ψ_i is zero and as $b_i \in H_0^{3,q}(G)$, the trace of b_i is zero on ∂C_{i_0}, too. But as g is constant, the trace of g on ∂C_{i_0} must be equal to this constant, too. Thus this constant must necessarily be 0. So now we have

$$\sum_{i=1, i \neq i_0}^{N} \lambda_i (\Psi_i - b_i) = 0.$$

On ∂C_j, $j \neq i_0$, we find $\Psi_i = \delta_{i,j}$ and the trace of each b_i vanishing. So we have the trace of g on ∂C_j equal to λ_j. But on the other hand it must be 0, so $\lambda_j = 0$, $j \neq i_0$ and we have linear independence.
We have proved the

Theorem 8.6. *Let G be a bounded domain with $\partial G \in \mathcal{C}^5$ and $N \geq 1$ be the number of connected components of $\mathbb{R}^n \setminus \overline{G}$. Then*

$$\left\{p \in H^{1,q}_{0,0}(G) : Z_q(p) = p\right\} = A^q_0(G) \oplus V(G),$$

where $V(G)$ is a finite dimensional real vector space of dimension $N-1$.

Part II
Cosserat Operators of Arbitrary Order and Study of Stokes-Like Systems Connected With Them

9 A Generalization of our Account to the Cosserat Spectrum to Arbitrary Orders

This section will give an outline on how to generalize our account to higher orders. In contrast to the first part, we do not assume a specific fixed regularity of ∂G and show the theorems for this fixed regularity but we will state with each theorem the regularity of ∂G we need for the proof.
At first we give an introduction to the general setting:

9.1 The General Situation

Let in the following $m \in \mathbb{N}$ be arbitrary but fixed, $G \subset\subset \mathbb{R}^n$, $1 < q < \infty$ and $1 < q' < \infty$ with $\frac{1}{q} + \frac{1}{q'} = 1$. For $u \in H^{m,q}_0(G)$, $v \in H^{m,q'}_0(G)$ define

$$B_m\left[u,v\right] := \begin{cases} \langle \Delta^{\frac{m}{2}} u, \Delta^{\frac{m}{2}} v\rangle \text{ for even } m \\ \langle \nabla\Delta^{\frac{m-1}{2}} u, \nabla\Delta^{\frac{m-1}{2}} v\rangle \text{ for odd } m \end{cases}.$$

First we state a theorem on solvability of the respective weak Dirichlet problem which can be proved using Simader's general theory from [15] in an analogous way as in our proofs of the theorems in Subsection 2.4:

Theorem 9.1. *Let $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^m$, $F \in \left(H_0^{m,q'}(G)\right)^\star$. Then there exists exactly one $u \in H_0^{m,q}(G)$ such that*

$$B_m\left[u, \Phi\right] = F(\Phi) \text{ for all } \Phi \in H_0^{m,q'}(G).$$

Further there exists a constant $C = C(m, q, G) > 0$ such that

$$\|u\|_{m,q} \leq C \, \|F\|_{\left(H_0^{m,q'}\right)^\star}.$$

Along with this solvability statement comes the following variational inequality:
There exists a $\tilde{C} = \tilde{C}(m, q, G) > 0$ such that for all $u \in H_0^{m,q}(G)$ we have

$$\|u\|_{m,q} \leq \tilde{C} \sup_{0 \neq \Phi \in H_0^{m,q'}(G)} \frac{B_m\left[u, \Phi\right]}{\|\Phi\|_{m,q'}}.$$

As in Section 2.5 we find a decomposition theorem which is a direct consequence of Theorem 9.1:

Theorem 9.2. *Let $m \in \mathbb{N}_0$ and $\partial G \in \mathcal{C}^{m+2}$ and set for $m = 0$: $H_0^{0,q}(G) := L^q(G)$. With the definitions*

$$H_{0,0}^{m,q}(G) := \left\{ f \in H_0^{m,q}(G) : \int_G f \, dx = 0 \right\},$$

$$A^{m,q}(G) := \left\{ f \in H_0^{m,q}(G) : f = \Delta s \text{ for an } s \in H_0^{m+2,q}(G) \right\},$$

$$B^{m,q}(G) := \left\{ f \in H_0^{m,q}(G) : \int_G f \Delta^{m+1} \Phi \, dx = 0 \text{ for all } \Phi \in \mathcal{C}_0^\infty(G) \right\}$$

and

$$A_0^{m,q}(G) := A^{m,q}(G) \cap H_{0,0}^{m,q}(G),\ B_0^{m,q}(G) := B^{m,q}(G) \cap H_{0,0}^{m,q}(G)$$

(we also have here again $A_0^{m,q}(G) = A^{m,q}(G)$ but we will write often the 0-index for consistency), we have the direct decompositions

$$H_0^{m,q}(G) = A^{m,q}(G) \oplus B^{m,q}(G)$$

and

$$H_{0,0}^{m,q}(G) = A_0^{m,q}(G) \oplus B_0^{m,q}(G).$$

Moreover, for an $f \in H_0^{m,q}(G)$ with the representation $f = \Delta s + p$ with $s \in H_0^{m+2,q}(G)$ and $p \in B^{m,q}(G)$, we have with a constant $C = C(G, q, m) > 0$:

$$\|\Delta s\|_{m,q} + \|p\|_{m,q} \leq C \, \|f\|_{m,q}$$

Proof. The proof is a straight-forward generalization of the proof of Theorem 2.23. However, as we need some more detailed information about this decomposition later in the proof of Theorem 11.2, we will give an outline of the proof there. □

Let in the following be $m \in \mathbb{N}$, $m \geq 1$. In analogy to our decomposition (22) from Theorem 2.26 we have in our general situation

Theorem 9.3. *For $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^m$, we have*

$$H_0^{m,q}(G) = H_{0,0}^{m,q}(G) \oplus N_m^q(G)$$

with

$$N_m^q(G) := \{g \in H_0^{m,q}(G) : B_m[g, \operatorname{div} \underline{\Phi}] = 0 \text{ for all } \underline{\Phi} \in \underline{\mathcal{C}}_0^\infty(G)\}.$$

The space $N_m^q(G)$ is a one dimensional real vector space generated by an element $h \in H_0^{m,q}(G) \cap \mathcal{C}^\infty(G)$ with

$$\Delta^m h(x) = 1 \text{ for every } x \in G$$

and $\int_G h\, dx \neq 0$.
Further we have $N_m^q(G) = N_m^r(G)$ for every $1 < r < \infty$.

Proof. The proof is a direct generalization of the proof to Theorem 2.26. □

Now we define the generalizations of the operators $\underline{T}_q$ and Z_q:
For $p \in H_{0,0}^{m-1,q}(G)$ define as $\underline{T}_q^{(m)}(p)$ the unique $\underline{u} \in \underline{H}_0^{m,q}(G)$ satisfying

$$B_m[\underline{u}, \underline{\Phi}] = B_{m-1}[p, \operatorname{div} \underline{\Phi}] \text{ for all } \underline{\Phi} \in \underline{H}_0^{m,q'}(G).$$

This $\underline{u}$ is found by application of Theorem 9.1 to the problems

$$B_m[u_i, \phi] = B_{m-1}[p, \partial_i \phi] \text{ for all } \phi \in H_0^{m,q'}(G),\ i = 1, \dots, n.$$

Theorem 9.1 also guarantees us the continuity of

$$\underline{T}_q^{(m)} : H_{0,0}^{m-1,q}(G) \to \underline{H}_0^{m,q}(G).$$

With $\underline{T}_q^{(m)}$ now defined, we define

$$Z_q^{(m)} := \operatorname{div} \circ \underline{T}_q^{(m)} : H_{0,0}^{m-1,q}(G) \to H_{0,0}^{m-1,q}(G)$$

which gets continuous by continuity of $\underline{T}_q^{(m)}$ and div. Regarding our decomposition from Theorem 9.2 we get an analogy to Theorem 2.24:

Theorem 9.4. *For $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+1}$, we have*

$$Z_q^{(m)}|_{A_0^{m-1,q}(G)} : A_0^{m-1,q}(G) \to A_0^{m-1,q}(G) \text{ and } Z_q^{(m)}(p) = p \quad \forall p \in A_0^{m-1,q}(G)$$
$$Z_q^{(m)}|_{B_0^{m-1,q}(G)} : B_0^{m-1,q}(G) \to B_0^{m-1,q}(G)$$

Proof. The proof is done as the proof of Theorem 2.24. □

Moreover, we need generalizations of our helpful theorems from Section 3, namely Theorems 3.1, 3.2, 3.3 and 3.4. The Theorems 3.1, 3.2 and 3.4 are easily seen to be generalizable, the generalization of Theorem 3.3 which we will need is the following:

Theorem 9.5. *Let $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^m$ and $f \in H_0^{m-1,q}(G) \cap H^{m,q}(G)$ satisfy*

$$D^\alpha f \in H_0^{1,q}(G) \text{ for every } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| = m-1.$$

Then $f \in H_0^{m,q}(G)$.

Proof. For the proof we will use the notation used in the proof to Theorem 3.3 except calling $\tilde{f}$ and $\tilde{f}_k$ now f and f_k. As in that proof, we can (using a localization process as in the proof of Theorem 3.3 via a partition of unity, this is were we need $\partial G \in \mathcal{C}^m$) reduce ourselves to the following case:

$$G = Q = \Delta \times]0, \varepsilon[\,,\ f \in H_0^{m-1,q}(Q) \cap H^{m,q}(Q) \text{ with}$$
$$D^\alpha f \in H_0^{1,q}(Q) \text{ for every } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| = m-1$$
$$\text{and } f(x) = 0 \text{ almost everywhere in } Q \setminus (\Delta' \times]0, \varepsilon - \varepsilon'[)$$

Now we have to show that $f \in H_0^{m,q}(Q)$. We can make up estimates similar to the ones made in the proof of Theorem 3.3:

- By approximating f with $\mathcal{C}_0^\infty(Q)$-functions u_ν in the $H^{m-1,q}(Q)$-sense as it is possible by $f \in H_0^{m-1,q}(Q)$, we can see like in the proof of Theorem 3.3 that with the multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\alpha_1, \ldots, \alpha_{n-1}, 0)$ with $|\alpha| \le m-1$ and $0 < \rho < \varepsilon$ we get the inequalities

$$\|D^\alpha f\|_{\Delta \times]0,\rho[,q} \le \rho^{\alpha_n} \left\|D^\beta f\right\|_{\Delta \times]0,\rho[,q}.$$

- Because of $D^\beta f \in H_0^{1,q}(Q)$ for every β with $|\beta| = m-1$, we see that for α with $|\alpha| = m$ and $\alpha_n \ge 1$, $\beta := (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n - 1)$ we get by approximating $D^\beta f$ by $\mathcal{C}_0^\infty$-functions in the $H^{1,q}(Q)$-sense for $0 < \rho < \varepsilon$ the inequality

$$\|D^\alpha f\|_{\Delta \times]0,\rho[,q} \le \rho \left\|D^\beta f\right\|_{\Delta \times]0,\rho[,q}.$$

- By sticking the two estimates from above together, we get for every multiindex α with $\alpha_n \neq 0$ and $|\alpha| \leq m$ the inequality

$$\|D^\alpha f\|_{\Delta\times]0,\rho[,q} \leq \rho^{\alpha_n} \left\|D^\beta f\right\|_{\Delta\times]0,\rho[,q}.$$

  This inequality obviously is valid for $\alpha_n = 0$, where $\beta = \alpha$, too.

Taking again a cut-off function $\eta \in \mathcal{C}^\infty(\mathbb{R})$ with $\eta(t) = 0$ for all $t \leq 1$ and $\eta(t) = 1$ for all $t \geq 2$ and setting $\eta_k(t) := \eta(kt)$, we see that $\operatorname{supp}(\eta_k') \subset \left[\frac{1}{k}, \frac{2}{k}\right]$ and all the higher derivatives of η_k have this property, too. We also find a constant $c = c(\eta, m) > 0$ such that

$$\left|\partial_n^l \eta_k(t)\right| \leq ck^l \text{ for all } l \leq m.$$

In the following, we will use a generalized Leibniz rule for the derivatives of $f\eta_k$, which is shown as in the classical account by iterated use of the product rule (our Theorem 3.1):

For $f \in H^{m,q}(Q)$, $g \in \overline{\mathcal{C}}^\infty(Q)$, $r \leq m$, we have $fg \in H^{m,q}(Q)$ and for $i = 1, \ldots, n$ we have

$$\partial_i^r(fg) = \sum_{l=0}^{r} \binom{r}{l} \partial_i^l f \partial_i^{r-l} g.$$

So, writing $f_k(x', x_n) := f(x', x_n)\eta_k(x_n)$ and letting $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multiindex with $|\alpha| \leq m$, $\beta := (\alpha_1, \ldots, \alpha_{n-1}, 0)$, we find because of

$$\partial_i f_k(x', x_n) = \partial_i f(x', x_n)\eta_k(x_n) \text{ for } i \neq n:$$

$$D^\alpha f_k = \partial_n^{\alpha_n}\left(D^\beta f_k\right) = \partial_n^{\alpha_n}\left(D^\beta f \eta_k\right) = \sum_{l=0}^{\alpha_n} \binom{\alpha_n}{l} \partial_n^l D^\beta f \partial_n^{\alpha_n - l}\eta_k$$

Then we see for every multiindex α with $\alpha_n > 0$, $|\alpha| \leq m$, $\beta = (\alpha_1, \ldots, \alpha_{n-1}, 0)$ that we have

$$\|D^\alpha f - D^\alpha f_k\|_q = \left\| D^\alpha f - \left(D^\alpha f \eta_k + \sum_{l=0}^{\alpha_n - 1} \binom{\alpha_n}{l} \partial_n^l D^\beta f \partial_n^{\alpha_n - l} \eta_k \right) \right\|_q \leq$$

$$\leq \underbrace{\|D^\alpha f - D^\alpha f \eta_k\|_q}_{\to 0 \text{ as in Theorem 3.3}} + \sum_{l=0}^{\alpha_n - 1} \binom{\alpha_n}{l} \left\|\partial_n^l D^\beta f \partial_n^{\alpha_n - l} \eta_k\right\|_q.$$

Looking at the right term, we find with our previous observations and the fact that $\operatorname{supp} \partial_n^{\alpha_n - l}\eta_k \subset \left[\frac{1}{k}, \frac{2}{k}\right]$ for $l < \alpha_n$:

$$\left\|\partial_n^l D^\beta f \partial_n^{\alpha_n-l}\eta_k\right\|_q^q = \int_Q \left| \partial_n^l D^\beta f \underbrace{\partial_n^{\alpha_n-l}\eta_k}_{|\cdot| \le C k^{\alpha_n - l}} \right|^q dx \le$$

$$\le C^q k^{q(\alpha_n-l)} \int_{\Delta' \times]0,\frac{2}{k}[} \left|\partial_n^l D^\beta f\right|^q dx = C^q k^{q(\alpha_n-l)} \left\|\partial_n^l D^\beta f\right\|^q_{\Delta' \times]0,\frac{2}{k}[,q} \le$$

$$\le C^q \underbrace{k^{q(\alpha_n-l)} \left(\frac{2}{k}\right)^{q(\alpha_n-l)}}_{=2^{q(\alpha_n-l)}} \underbrace{\|D^\alpha f\|^q_{\Delta'\times]0,\frac{2}{k}[,q}}_{\xrightarrow{0} \text{ by Lebesgue's Dominated Convergence Theorem}}$$

In the case where $\alpha_n = 0$, we have again

$$\|D^\alpha f_k - D^\alpha f\|_q = \|(1-\eta_k) D^\alpha f\|_q \to 0.$$

So we see that the sequence f_k approximates f in the $H^{m,q}(Q)$-norm. Furthermore, by construction, the f_k are all equal to 0 outside a compact set $S = S(k) \subset Q$. By mollifying the f_k with small enough mollification parameter ε_k, we can, as in the proof of Theorem 3.3, find $(f_k)_{\varepsilon_k} \in \mathcal{C}_0^\infty(Q)$ and $(f_k)_{\varepsilon_k} \to f$ in $H^{m,q}(Q)$. □

Furthermore, we will need in the following as a generalization of Theorem 5.1 the approximability of a $p \in B_0^{m-1,q}(G)$ with elements from $B_0^{m-1,q}(G) \cap H^{m+1,q}(G)$:

Theorem 9.6. *Let $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+3}$ and $p \in B_0^{m-1,q}(G)$. Then there exists a sequence $(p_\nu)_{\nu\in\mathbb{N}} \subset B_0^{m-1,q}(G) \cap H^{m+1,q}(G)$ with $\|p - p_\nu\|_{m-1,q} \to 0$.*

Proof. The proof can be done in analogy to the one from Theorem 5.1:
As in Theorem 2.22, we can show easily that $\mathcal{C}_{0,0}^\infty(G)$ is dense in $H_{0,0}^{m-1,q}(G)$ with respect to the $H^{m-1,q}(G)$-norm. So we find a sequence $(g_\nu)_{\nu\in\mathbb{N}} \subset \mathcal{C}_{0,0}^\infty(G)$ with $\|g_\nu - p\|_{m-1,q} \to 0$. As $\partial G \in \mathcal{C}^{m+1}$, we can solve for each $\nu \in \mathbb{N}$ the problem

$$B_{m+1}[s_\nu, \Phi] = B_{m-1}[g_\nu, \Delta\Phi] \text{ for all } \Phi \in H_0^{m+1,q'}(G)$$

with $s_\nu \in H_0^{m+1,q}(G)$. As $\partial G \in \mathcal{C}^{m+3}$ and $g_\nu \in \mathcal{C}_0^\infty(G)$, we find with Theorem 6.1 that we even have $s_\nu \in H^{m+3,q}(G)$. Looking now at $p_\nu := g_\nu - \Delta s_\nu$, we see that $p_\nu \in H_0^{m-1,q}(G) \cap H^{m+1,q}(G)$ and we find for $\Phi \in \mathcal{C}_0^\infty(G)$:

$$B_m[p_\nu,\Phi] = B_m[g_\nu,\Phi,] - B_m[\Delta s_\nu,\Phi] = B_m[g_\nu,\Phi,] + B_{m+1}[s_\nu,\Phi] =$$

$$= B_m[g_\nu,\Phi,] + B_{m-1}[g_\nu,\Delta\Phi] = B_m[g_\nu,\Phi,] - B_m[g_\nu,\Phi] = 0$$

and we have by Weyl's Lemma: $p_\nu \in \mathcal{C}^\infty(G)$ and $\Delta^m p_\nu = 0$. All in all we have $p_\nu \in B_0^{m-1,q}(G) \cap H^{m+1,q}(G)$. Now all left to show is that $\|p_\nu - p\|_{m-1,q} \to 0$. To see this, we look at

$$\|p_\nu - p\|_{m-1,q} = \|g_\nu - \Delta s_\nu - p\|_{m-1,q} \leq \|g_\nu - p\|_{m-1,q} + \|\Delta s_\nu\|_{m-1,q}$$

and note that as $s_\nu \in H_0^{m+1,q}(G)$, we have with the generalized variational inequality from Theorem 9.1

$$\|\Delta s_\nu\|_{m-1,q} \leq \|s_\nu\|_{m+1,q} \leq C \sup_{0\neq\Phi\in H_0^{m+1,q'}(G)} \frac{B_{m+1}[s_\nu,\Phi]}{\|\Phi\|_{m+1,q'}} =$$

$$= C \sup_{0\neq\Phi\in H_0^{m+1,q'}(G)} \frac{B_{m-1}[g_\nu,\Delta\Phi]}{\|\Phi\|_{m+1,q'}} = C \sup_{0\neq\Phi\in H_0^{m+1,q'}(G)} \frac{B_{m-1}[g_\nu - p,\Delta\Phi]}{\|\Phi\|_{m+1,q'}} \leq$$

$$\leq C\|g_\nu - p\|_{m-1,q} \to 0,$$

because we have for every $\Phi \in H_0^{m+1,q'}(G)$: $\|\Delta\Phi\|_{m-1,q'} \leq \|\Phi\|_{m+1,q'}$. So we have found an approximating sequence p_ν satisfying our requirements. □

9.2 The Generalization of Crouzeix's Ansatz

Now we are able to start the generalized Crouzeix approach. In the first version, Theorem 9.7, we are (as shows up by comparison with our special case $m = 2$ already inspected in Part I) too restrictive on the required regularity of ∂G. This is done in order to make classical calculation doable: For the calculations we will do, we will need a relatively high regularity of ζ. In the end, however, we will get rid of the too restrictive requirements by an approximation idea leading to a second statement, Theorem 9.8, with weaker requirements on ∂G.

Theorem 9.7. *Let $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{2m+2}$ and $p \in B_0^{m-1,q}(G) \cap H^{m,q}(G)$, $\underline{u} := \underline{T}_q^{(m)}(p)$ and let*

$$w := \underline{u} \cdot \nabla\zeta - \frac{1}{2m} p\zeta \in H_0^{m,q}(G),$$

where $\zeta \in \mathcal{C}_0^{2m+1}(\mathbb{R}^n)$ is Weyers' helpful function from Section 3.1.
Then $w \in H^{m+1,q}(G)$ and there is a constant $C = C(m,q,G)$ such that

$$\|w\|_{m+1,q} \leq C\,\|p\|_{m-1,q}\,.$$

Proof. That $w \in H_0^{m,q}(G)$ is clear by definition of w and the information we have about $\underline{u}$, ζ and p: Since $\underline{u} \in \underline{H}_0^{m,q}(G)$ and $\zeta \in \mathcal{C}_0^{2m+1}(\mathbb{R}^n)$, we have $\underline{u}\cdot\nabla\zeta \in \underline{H}_0^{m,q}(G)$ and with $p \in H_0^{m-1,q}(G) \cap H^{m,q}(G)$ and $\zeta = 0$ on ∂G, we see as in the proof of Theorem 3.4 that $p\zeta \in H_0^{m,q}(G)$ with use of Theorem 9.5. In the following we are making a mixture of our account from the motivation of Crouzeix' method and our proof from Theorem 6.2. As $p \in B_0^{m-1,q}(G)$, we see by Weyl's Lemma that $p \in \mathcal{C}^\infty(G)$. Then we see by an argument very similar to the one used already in the induction step of our proof of Weyl's Lemma (Lemma 2.20) that $\underline{u} \in \underline{\mathcal{C}}^\infty(G)$, too:
With $p \in \mathcal{C}^\infty(G)$ we conclude $\nabla p \in \underline{\mathcal{C}}^\infty(G)$. For $x \in G$ arbitrary and $r > 0$ such that $B_r(x) \subset\subset G$, we find by classical theory in analogy to our proof of Lemma 2.20 an $\underline{\tilde{u}} \in \overline{\underline{C}}^\infty(B_{r'}(x))$ with a $0 < r' < r$ such that $\Delta\underline{\tilde{u}} = \nabla p$ on $B_{r'}(x)$. With $\underline{v} := \underline{u} - \underline{\tilde{u}}$ we see for $\underline{\Phi} \in \underline{\mathcal{C}}_0^\infty(B_{r'}(x))$:

$$B_m\left[\underline{v},\underline{\Phi}\right] = B_m\left[\underline{u}-\underline{\tilde{u}},\underline{\Phi}\right] = B_m\left[\underline{u},\underline{\Phi}\right] + B_{m-1}\left[\Delta\underline{\tilde{u}},\underline{\Phi}\right] =$$

$$= B_m\left[\underline{u},\underline{\Phi}\right] + B_{m-1}\left[\nabla p,\underline{\Phi}\right] = B_m\left[\underline{u},\underline{\Phi}\right] - B_{m-1}\left[p,\operatorname{div}\underline{\Phi}\right] = 0$$

and thus by Weyl's Lemma we conclude that $\underline{v} \in \underline{\mathcal{C}}^\infty(B_{r'}(x))$ and then $\underline{u} = \underline{v} + \underline{\tilde{u}} \in \underline{\mathcal{C}}^\infty(B_{r'}(x))$.
Because of this and $\partial G \in \mathcal{C}^{2m+2}$, by which we find $\zeta \in \mathcal{C}_0^{2m+1}(\mathbb{R}^n)$, we conclude that

$$w := \underline{u}\cdot\nabla\zeta - \frac{1}{2m}p\zeta \in H_0^{m,q}(G) \cap \mathcal{C}^{2m}(G)$$

and we can calculate classically:

$$\Delta^m(w) = \Delta^m\left(\underline{u}\cdot\nabla\zeta - \frac{1}{2m}p\zeta\right) \tag{44}$$

Looking first at $\Delta^m\left(\underline{u}\cdot\nabla\zeta\right)$, we see that

$$\Delta^m\left(\underline{u}\cdot\nabla\zeta\right) = \Delta^m(\underline{u})\cdot\nabla\zeta + T, \tag{45}$$

where T stands for terms consisting of derivatives of order $< 2m$ of the u_i and derivatives of order $\leq 2m+1$ of ζ, where for each summand we have that the respective orders of derivatives of the u_i and ζ sum up to $2m+1$.

Looking at

$$\Delta^m \left(-\frac{1}{2m} p\zeta\right), \tag{46}$$

we see by induction that for $j \in \mathbb{N}$, $j \leq m$ we have

$$\Delta^j(p\zeta) = \left(\Delta^j p\right)\zeta + 2j\Delta^{j-1}\nabla p \cdot \nabla\zeta + T_j, \tag{47}$$

where T_j stands for terms containing derivatives of p of order $\leq 2(j-1)$ and derivatives of ζ of order $\leq 2j$, where for each summand we have the sum of the respective orders of derivatives equal to $2j$. For $j = 1$ this is clear by the formula

$$\Delta(p\zeta) = \Delta p\zeta + 2\nabla p \cdot \nabla\zeta + p\Delta\zeta.$$

The induction step from j to $j+1$ is just another application of this formula:

$$\Delta^{j+1}(p\zeta) = \Delta\left(\Delta^j(p\zeta)\right) = \Delta\left(\left(\Delta^j p\right)\zeta + 2j\Delta^{j-1}\nabla p \cdot \nabla\zeta + T_j\right) =$$

$$= \left(\Delta^{j+1}p\right)\zeta + 2\nabla\Delta^j p \cdot \nabla\zeta + 2j\left(\Delta^j\nabla p \cdot \nabla\zeta\right) + T_{j+1},$$

where T_{j+1} stands for an expression consisting of derivatives of p of order $\leq 2j$ and derivatives of ζ of order $\leq 2(j+1)$, where the orders of the derivatives sum up in each summand to $2(j+1)$. So we can conclude that we have

$$\Delta^m\left(-\frac{1}{2m}p\zeta\right) = -\frac{1}{2m}\underbrace{\Delta^m p}_{=0, \text{ as } p \in B_0^{m-1,q}(G)}\zeta - \Delta^{m-1}\nabla p \cdot \nabla\zeta + T, \tag{48}$$

where T denotes terms consisting of derivatives of p of order $\leq 2m-2$ and derivatives of ζ of order $\leq 2m$, where the respective orders of derivatives in each summand add up to $2m$.
By partially integrating we see that with $\underline{u} = \underline{T}_q^{(m)}(p)$ we get for all $\underline{\Phi} \in \underline{\mathcal{C}}_0^\infty(G)$: We have by definition of $\underline{u}$

$$B_m\left[\underline{u}, \underline{\Phi}\right] = B_{m-1}\left[p, \operatorname{div}\underline{\Phi}\right],$$

which leads to

$$\langle \Delta^m \underline{u}, \underline{\Phi}\rangle = \langle \nabla\Delta^{m-1}p, \underline{\Phi}\rangle \text{ for all } \underline{\Phi} \in \underline{\mathcal{C}}_0^\infty(G)$$

and it follows that $\Delta^m\underline{u} = \nabla\Delta^{m-1}p$, see for example [16], Satz 2.5 (4), page 33.
Using this, we see that the calculated terms $\Delta^m\underline{u}\cdot\nabla\zeta$ and $\nabla\Delta^{m-1}p\cdot\nabla\zeta$ from (45) and (48) cancel out and thus when calculating (44), we see that $\Delta^m w$ is a sum of products of derivatives of the u_i of order $\leq 2m-1$ and derivatives

of ζ of order $\leq 2m+1$ where in each such product the respective orders of the derivatives add up to $2m+1$ and products of derivatives of p of order $\leq 2m-2$ and derivatives of ζ of order $\leq 2m$ where in each such product the orders of the respective derivatives add up to $2m$.
So, with $\Phi \in \mathcal{C}_0^\infty(G)$ we see again by partial integration that

$$B_m[w,\Phi] = \pm\langle \Delta^m w, \Phi\rangle = \sum_{\substack{i=1,\dots,n\\ |\alpha|<2m\\ |\beta|+|\alpha|=2m+1}} \langle a_{\alpha,\beta,i} D^\alpha\ u_i D^\beta \zeta, \Phi\rangle +$$

$$+ \sum_{\substack{|\gamma|<2m-1\\ |\delta|+|\gamma|=2m}} \langle b_{\gamma,\delta} D^\gamma p D^\delta \zeta, \Phi\rangle,$$

with suitable $a_{\alpha,\beta,i}$ and $b_{\gamma,\delta} \in \mathbb{Z}$. So we can write $B_m[w,\Phi]$ as

$$\sum_{\substack{i=1,\dots,n\\ |\alpha|<2m\\ |\beta|+|\alpha|=2m+1}} \langle a_{\alpha,\beta,i} D^\alpha\ u_i, D^\beta \zeta\Phi\rangle + \sum_{\substack{|\gamma|<2m-1\\ |\delta|+|\gamma|=2m}} \langle b_{\gamma,\delta} D^\gamma p, D^\delta \zeta\Phi\rangle. \tag{49}$$

For the first sum in (49), we look at three cases:

i) Looking at a summand $\langle a_{\alpha,\beta,i} D^\alpha\ u_i, D^\beta\zeta\Phi\rangle$ from the first sum in (49) with $|\alpha| < 2m$ and $|\beta| + |\alpha| = 2m+1$, we see that in the cases where we have $|\alpha| > m$ we can make $|\alpha|-m$ partial integrations and get with an $\eta \leq \alpha$ of length $|\eta| = |\alpha| - m$ to

$$\pm\langle a_{\alpha,\beta,i} D^\varepsilon\ u_i, D^\eta\left(D^\beta\zeta\Phi\right)\rangle,$$

where $|\varepsilon| = m$, $|\eta| < m$, $\varepsilon + \eta = \alpha$ and $|\eta| + |\beta| = (|\alpha| - m) + (2m+1-|\alpha|) = m+1$.

ii) In the cases where $|\alpha| < m$, look at a summand from the first sum in (49) of the form

$$\langle a_{\alpha,\beta,i} D^\alpha\ u_i, D^\beta\zeta\Phi\rangle$$

with $|\alpha| \leq 2m-1$ and $|\alpha| + |\beta| = 2m+1$.

Then we have $|\beta| = 2m+1-|\alpha| > 2m+1-m = m+1$. In this case we write

$$\langle a_{\alpha,\beta,i} D^\alpha\ u_i, D^\beta\zeta\Phi\rangle = \langle a_{\alpha,\beta,i} D^\alpha\ u_i\Phi, D^\beta\zeta\rangle$$

and make for a multiindex γ with $\gamma \leq \beta$ and $|\gamma| = |\beta| - m - 2$ partial integrations such that

$$|\beta| - |\gamma| = m+2, \quad |\gamma| < 2m+1-(m+1) = m$$

thus arriving at

$$\pm\langle a_{\alpha,\beta,i} D^{\gamma}\left(D^{\alpha}\ u_i \Phi\right), D^{\eta}\zeta\rangle$$

with $|\eta| = |\beta| - |\gamma| = m+2$ and by carrying out the D^{γ}-differentiation, we get a $\mathbb{Z}$-linearcombination of terms of the form

$$\langle D^{\epsilon} u_i D^{\nu}\Phi, D^{\eta}\zeta\rangle$$

with

$$|\epsilon| \leq |\alpha| + |\gamma| = |\alpha| + |\beta| - (m+2) = |\alpha| + 2m + 1 - |\alpha| - m - 2 = m - 1,$$

$$|\nu| \leq |\gamma| \leq m-1 \text{ and } |\eta| = m+2.$$

iii) In the case where $|\alpha| = m$, we see that $|\beta| = 2m+1-m = m+1$.

Analogously, looking at a summand from the second sum in (49) of the form

$$\langle b_{\gamma,\delta} D^{\gamma} p, D^{\delta}\zeta\Phi\rangle,$$

with $|\gamma| < 2m-1$ and $|\delta| + |\gamma| = 2m$, we look at two cases:

i) We can make in the case where $|\gamma| > m-1$ some partial integrations. There are $|\gamma| - (m-1)$ partial integrations necessary to make the derivatives on the left side of order $m-1$. We thus get to a term of the form

$$\pm\langle b_{\gamma,\delta} D^{\mu}\ p, D^{\nu}\left(D^{\delta}\zeta\Phi\right)\rangle,$$

where $|\mu| = m-1$, $|\nu| = |\gamma| - (m-1) \leq m-1$ and $|\nu| + |\delta| = (|\gamma| - (m+1)) + (2m - |\gamma|) = m+1$.

ii) In the case where $|\gamma| \leq m-1$, looking at term of the form

$$\langle b_{\gamma,\delta} D^{\gamma} p, D^{\delta}\zeta\Phi\rangle = \langle b_{\gamma,\delta} D^{\gamma} p\Phi, D^{\delta}\zeta\rangle$$

from (49), where we have $|\delta| = 2m - |\gamma| \geq 2m - (m-1) = m+1$, we get with a multiindex ε with $\varepsilon \leq \delta$ and $|\varepsilon| = |\delta| - (m+1)$ after $|\varepsilon|$ partial integrations to

$$\pm\langle b_{\gamma,\delta} D^{\varepsilon}(D^{\gamma} p\Phi), D^{\nu}\zeta\rangle$$

with $|\nu| = |\delta| - |\epsilon| = |\delta| - (|\delta| - (m+1)) = m+1$ and

$$|\varepsilon| + |\gamma| = |\delta| - (m+1) + |\gamma| = 2m - (m+1) = m-1 \text{ and}$$

$$|\varepsilon| = |\delta| - (m+1) \leq 2m - (m+1) = m-1,$$

so this can be written as a $\mathbb{Z}$-linearcombination of terms of the form

$$\langle D^{\mu} p D^{\nu}\Phi, D^{\eta}\zeta\rangle,$$

where $|\mu| \leq m-1$, $|\nu| \leq m-1$ and $|\eta| = m+1$.

All together, reviewing all our inspected cases, we find an $N \in \mathbb{N}$ and numbers $a_{i,j}, b_i \in \mathbb{Z}, i = 1, \dots, N,\ j = 1, \dots, n$, such that with $\Phi \in \mathcal{C}_0^\infty(G)$ we have:

$$B_m[w, \Phi] = \sum_{i=1}^{N} \sum_{j=1}^{n} \langle a_{i,j} D^{\alpha_i} u_j, D^{\beta_i} \zeta D^{\gamma_i} \Phi \rangle + \sum_{i=1}^{N} \langle b_i D^{\delta_i} p, D^{\nu_i} \zeta D^{\mu_i} \Phi \rangle,$$

where $|\alpha_i| \leq m,\ |\beta_i|, |\nu_i| \leq m+2,\ |\delta_i|, |\gamma_i|, |\mu_i| \leq m-1$. So what we get is the fact that $B_m[w, \cdot]$ defines an element F from $\left(H_0^{m-1,q'}(G)\right)^\star$ with $\|F\|_{\left(H_0^{m-1,q'}(G)\right)^\star} \leq c \|p\|_{m-1,q}$, with a $c > 0$ depending on G, q, ζ, m and with an application of Theorem 6.1, we can conclude that

$$w \in H^{m+1,q}(G), \text{ and } \|w\|_{m+1,q} \leq C \|p\|_{m-1,q},$$

as $\|w\|_{0,p}$ can also be estimated against $\|p\|_{m-1,q}$ as is easily seen from the definition of w via $\underline{u}, p$ and ζ.

□

Theorem 9.8. *A weakening of the regularity requirements for ∂G in Theorem 9.7*
Let $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+3}$ and $p \in B_0^{m-1,q}(G) \cap H^{m,q}(G)$, $\underline{u} := \underline{T}_q^{(m)}(p)$ and let

$$w := \underline{u} \cdot \nabla \zeta - \frac{1}{2m} p\zeta \in H_0^{m,q}(G),$$

where $\zeta \in \mathcal{C}_0^{m+2}(\mathbb{R}^n)$ is Weyers' helpful function from Section 3.1.
Then $w \in H^{m+1,q}(G)$ and there is a constant $C = C(m, q, G)$ such that

$$\|w\|_{m+1,q} \leq C \|p\|_{m-1,q}.$$

Proof. Looking at the proof of Theorem 9.7, we see that we can take over the first part of the proof word by word. We also still have here

$$w := \underline{u} \cdot \nabla \zeta - \frac{1}{2m} p\zeta \in H_0^{m,q}(G).$$

Then we can take a look at $B_m[w, \Phi]$ for $\Phi \in \mathcal{C}_0^\infty(G)$. As, in our special setting $\partial G \in \mathcal{C}^{m+3}$, we can just assume ζ to be in $\mathcal{C}_0^{m+2}(\mathbb{R}^n)$, we can not proceed as in the proof of Theorem 9.7 and integrate partially, landing at the term $\langle \Delta^m w, \Phi \rangle$ and calculate $\Delta^m w$ classically. But we can do the following: By mollification of $\zeta \in \mathcal{C}_0^{m+2}(\mathbb{R}^n)$, we find a sequence $(\zeta_k)_{k \in \mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $D^\alpha \zeta_k \to D^\alpha \zeta$ uniformly in $\mathbb{R}^n$ for every multiindex α with $|\alpha| \leq m+2$. For $k \in \mathbb{N}$ we define

$$w_k := \underline{u} \cdot \nabla \zeta_k - \frac{1}{2m} p\zeta_k$$

and find that $w_k \xrightarrow{k\to\infty} w$ in $H^{m,q}(G)$. So, we find for $\Phi \in \mathcal{C}_0^\infty(G)$

$$B_m[w,\Phi] = \lim_{k\to\infty} B_m[w_k,\Phi]$$

and for each $B_m[w_k,\Phi]$ we can do the calculation from the proof of Theorem 9.7 (as we did not use any of the specific properties of ζ except it's differentiability) and find that

$$B_m[w_k,\Phi] = \sum_{i=1}^{N}\sum_{j=1}^{n}\langle a_{i,j}D^{\alpha_i}u_j, D^{\beta_i}\zeta_k D^{\gamma_i}\Phi\rangle + \sum_{i=1}^{N}\langle b_i D^{\delta_i}p, D^{\nu_i}\zeta_k D^{\mu_i}\Phi\rangle,$$

where $|\alpha_i| \le m$, $|\beta_i|, |\nu_i| \le m+2$, $|\delta_i|, |\gamma_i|, |\mu_i| \le m-1$ are multiindices and $i = 1,\ldots,N$ for an $N \in \mathbb{N}$. It is clear that N, the multiindices α_i, β_i, γ_i, δ_i, μ_i, ν_i and the $a_{i,j}, b_i \in \mathbb{Z}$ do not depend on our specific $k \in \mathbb{N}$. For example for the $a_{i,j}, b_i \in \mathbb{Z}$, we see that they emerge only from our "classical calculation" and applications of the product rule, and thus these $a_{i,j}, b_i$ are the same for every k.
Consequently, the right hand side tends for $k \to \infty$ to

$$\sum_{i=1}^{N}\sum_{j=1}^{n}\langle a_{i,j}D^{\alpha_i}u_j, D^{\beta_i}\zeta D^{\gamma_i}\Phi\rangle + \sum_{i=1}^{N}\langle b_i D^{\delta_i}p, D^{\nu_i}\zeta D^{\mu_i}\Phi\rangle,$$

due to the uniform convergence of the ζ_k to ζ. Then we are again in the situation we arrived at in the proof of Theorem 9.7 and the rest of the proof can be done as we did it there.

□

Theorem 9.9. *Let $\partial G \in \mathcal{C}^{m+3}$, $p \in B_0^{m-1,q}(G) \cap H^{m+1,q}(G)$. Then*

$$v := \nabla w \cdot \nabla\zeta - \left(\operatorname{div}\underline{u} - \frac{1}{2}p\right) \in H_0^{m,q}(G)$$

and we have

$$\operatorname{div}\underline{u} - \frac{1}{2}p \in H^{m,q}(G)$$

and there is a constant $C = C(m,G,q) > 0$ with

$$\left\|\operatorname{div}\underline{u} - \frac{1}{2}p\right\|_{m,q} \le C\,\|p\|_{m-1,q} \tag{50}$$

Proof. $v \in H^{m,q}(G) \cap H_0^{m-1,q}(G)$ can be seen easily: As $w \in H^{m+1,q}(G) \cap H_0^{m,q}(G)$ according to Theorem 9.8, $\zeta \in \mathcal{C}_0^{m+2}(\mathbb{R}^n)$, we see that $\nabla w \cdot \nabla \zeta \in H^{m,q}(G) \cap H_0^{m-1,q}(G)$. We also have $p \in H_0^{m-1,q}(G) \cap H^{m+1,q}(G)$ and by Theorem 6.1 (for an explicit application of Theorem 6.1 in this situation, see our Theorem 10.2), we also have $\underline{u} \in \underline{H}_0^{m,q}(G) \cap \underline{H}^{m+2,q}(G)$ and thus $\operatorname{div} \underline{u} \in H_0^{m-1,q}(G) \cap H^{m+1,q}(G)$. All in all, we get $v \in H^{m,q}(G) \cap H_0^{m-1,q}(G)$. We want to use now Theorem 9.5 to show that we even have $v \in H_0^{m,q}(G)$. So we have to show that for arbitrary $s_1, \ldots, s_{m-1} \in \{1, \ldots, n\}$ we have $Z^1\left(\partial_{s_1} \ldots \partial_{s_{m-1}} v\right) = 0$ almost everywhere on ∂G, where Z^1 denotes the trace operator from Theorem 4.4.

In the following we will make for the sake of clarity and readability the calculations as if the corresponding functions were continuously differentiable and all the upcoming derivatives continuous up to the boundary. Theorem 9.5 and generalizations of the Theorems 3.1, 3.2, 3.4 justify this way of calculation. Note that in the proofs to the Theorems 9.7, 9.8, we did not yet need $p \in H^{m+1,q}(G)$ but only $p \in H^{m,q}(G)$. However, in the following, $p \in H^{m+1,q}(G)$ is implicitly used: As in the following calculations there occur derivatives of p of order up to m and we have to be able to determine the trace of these derivatives, we have to assume here $p \in H^{m+1,q}(G)$.

So look for an $x \in \partial G$ at

$$\partial_{s_1} \ldots \partial_{s_{m-1}} v(x) = \partial_{s_1} \ldots \partial_{s_{m-1}} \left(\sum_{l=1}^{n} \partial_l w \partial_l \zeta - \sum_{i=1}^{n} \partial_i u_i + \frac{1}{2} p \right)(x) =$$

$$= \partial_{s_1} \ldots \partial_{s_{m-1}} \left(\sum_{l=1}^{n} \left(\partial_l \left(\sum_{r=1}^{n} u_r \partial_r \zeta - \frac{1}{2m} p \zeta \right) \partial_l \zeta \right) - \sum_{i=1}^{n} \partial_i u_i + \frac{1}{2} p \right)(x).$$

We first want to inspect for $x \in \partial G$

$$\partial_{s_1} \ldots \partial_{s_{m-1}} \left(\sum_{\substack{l=1 \\ r=1}}^{n} \partial_l u_r \partial_r \zeta \partial_l \zeta + u_r \partial_l \partial_r \zeta \partial_l \zeta \right)(x).$$

As $\underline{u} \in \underline{H}_0^{m,q}(G)$, we see that for $x \in \partial G$ we have $D^\alpha u_r(x) = 0$ for all α with $|\alpha| < m$ and thus this expression reduces to

$$\sum_{\substack{l=1 \\ r=1}}^{n} \partial_{s_1} \ldots \partial_{s_{m-1}} \partial_l u_r \partial_r \zeta \partial_l \zeta(x).$$

Writing now $\partial_t \zeta = N_t$ and using the fact that for m indices $l_0, l_1, \ldots, l_{m-1} \in \{0, \ldots, n\}$ we have $\partial_{l_1} \ldots \partial_{l_{m-1}} u_t \in H_0^{1,q} \cap H^{2,q}(G)$ and thus

$$\partial_{l_0} \partial_{l_1} \ldots \partial_{l_{m-1}} u_t = \lambda^t_{l_1, \ldots, l_{m-1}} N_{l_0} \text{ on } \partial G$$

for a suited function $\lambda^t_{l_1,\ldots,l_{m-1}}$ defined on ∂G. For the functions $\lambda^t_{l_1,\ldots,l_{m-1}}$ we note that we have the following fact (using the notation $\widehat{l_i}$ to denote the missing of the index l_i):

$$\lambda^t_{l_1,\ldots,l_{m-1}} N_{l_0} = \lambda^t_{l_0,\ldots,\widehat{l_i},\ldots l_{m-1}} N_i,\; i = 1,\ldots,n,$$

which is simply a direct consequence of

$$\partial_{l_0}\partial_{l_1}\ldots\partial_{l_{m-1}} u_t = \partial_{l_i}\partial_{l_0}\ldots\widehat{\partial_{l_i}}\ldots\partial_{l_{m-1}} u_t$$

and we get

$$\sum_{\substack{l=1\\r=1}}^{n} \partial_{s_1}\ldots\partial_{s_{m-1}}\partial_l u_r \partial_r\zeta\partial_l\zeta(x) = \sum_{\substack{l=1\\r=1}}^{n} \lambda^r_{s_1,\ldots,s_{m-1}} N_l N_r N_l(x) =$$

$$= \sum_{r=1}^{n} \lambda^r_{s_1,\ldots,s_{m-1}} N_r(x) = \sum_{r=1}^{n} \partial_r\partial_{s_1}\ldots\partial_{s_{m-1}} u_r(x) = \partial_{s_1}\ldots\partial_{s_{m-1}} \operatorname{div} \underline{u}(x).$$

So, all which is still to be shown is that on ∂G we have

$$-\frac{1}{2m}\partial_{s_1}\ldots\partial_{s_{m-1}}\sum_{l=1}^{n}\partial_l(p\zeta)\partial_l\zeta = -\frac{1}{2}\partial_{s_1}\ldots\partial_{s_{m-1}}(p).$$

Looking at the left side, we get

$$\partial_{s_1}\ldots\partial_{s_{m-1}}\sum_{l=1}^{n}\partial_l(p\zeta)\partial_l\zeta = \partial_{s_1}\ldots\partial_{s_{m-1}}\sum_{l=1}^{n}(\partial_l p\zeta\partial_l\zeta + p\partial_l\zeta\partial_l\zeta) =$$

$$= \partial_{s_1}\ldots\partial_{s_{m-1}}\sum_{l=1}^{n}\partial_l p\zeta\partial_l\zeta + \partial_{s_1}\ldots\partial_{s_{m-1}} p,$$

as $D^\alpha p D^\beta(\partial_l\zeta\partial_l\zeta) = 0$ on ∂G for all $|\alpha| \le m-2$, as $p \in H_0^{m-1,q}(G)$ and so, it remains to show that on ∂G

$$\partial_{s_1}\ldots\partial_{s_{m-1}}\sum_{l=1}^{n}\partial_l p\zeta\partial_l\zeta = (m-1)\partial_{s_1}\ldots\partial_{s_{m-1}} p. \tag{51}$$

Because of $\zeta|_{\partial G} = 0$ and $p \in H_0^{m-1,q}(G) \cap H^{m+1,q}(G)$, we see that on ∂G we have

$$\partial_{s_1}\ldots\partial_{s_{m-1}}\sum_{l=1}^{n}\partial_l p\zeta\partial_l\zeta = \sum_{l=1}^{n}\sum_{j=1}^{m-1}\partial_{s_1}\ldots\widehat{\partial_{s_j}}\ldots\partial_{s_{m-1}}\partial_l p\partial_{s_j}\zeta\partial_l\zeta$$

because all other terms resulting from applying $\partial_{s_1} \ldots \partial_{s_{m-1}}$ to $\partial_l p \zeta \partial_l \zeta$ have either the form $D^\alpha p D^\beta \zeta D^\gamma \zeta$ with $|\alpha| < m-1$ or $\beta = 0$.
We get

$$\sum_{l=1}^{n} \sum_{j=1}^{m-1} \partial_{s_1} \ldots \widehat{\partial_{s_j}} \ldots \partial_{s_{m-1}} \partial_l p \partial_{s_j} \zeta \partial_l \zeta = \sum_{l=1}^{n} \sum_{j=1}^{m-1} \partial_l \partial_{s_1} \ldots \widehat{\partial_{s_j}} \ldots \partial_{s_{m-1}} p N_{s_j} N_l.$$

By defining for the $m-2$ indices $l_1, \ldots, l_{m-2} \in \{1, \ldots, n\}$, $\mu_{l_1,\ldots,l_{m-2}}$ to be the function satisfying on ∂G for all l_0

$$\partial_{l_0} \partial_{l_1} \ldots \partial_{l_{m-2}} p = \mu_{l_1,\ldots,l_{m-2}} N_{l_0},$$

as it is possible by $\partial_{l_1} \ldots \partial_{l_{m-2}} p \in H_0^{1,q}(G)$ with Theorem 4.10, we can write this as

$$\sum_{l=1}^{n} \sum_{j=1}^{m-1} \mu_{s_1 \ldots \widehat{s_j} \ldots s_{m-1}} N_l N_{s_j} N_l = \sum_{j=1}^{m-1} \mu_{s_1 \ldots \widehat{s_j} \ldots s_{m-1}} N_{s_j}.$$

But we have on ∂G

$$\mu_{s_1 \ldots \widehat{s_j} \ldots s_{m-1}} N_{s_j} = \partial_{s_j} \partial_{s_1} \ldots \widehat{\partial_{s_j}} \ldots \partial_{s_{m-1}} p = \partial_{s_1} \ldots \partial_{s_{m-1}} p,$$

and thus

$$\sum_{j=1}^{m-1} \mu_{s_1 \ldots \widehat{s_j} \ldots s_{m-1}} N_{s_j} = \sum_{j=1}^{m-1} \partial_{s_1} \ldots \partial_{s_{m-1}} p = (m-1) \partial_{s_1} \ldots \partial_{s_{m-1}} p.$$

For the estimate (50), we can at first make use of the variational inequality from Theorem 9.1: As $v \in H_0^{m,q}(G)$, we find that

$$\|v\|_{m,q} \le C \sup_{0 \neq \Phi \in H_0^{m,q'}(G)} \frac{B_m[v, \Phi]}{\|\Phi\|_{m,q'}} =$$

$$= C \sup_{0 \neq \Phi \in H_0^{m,q'}(G)} \frac{B_m\left[\nabla w \cdot \nabla \zeta - \left(\operatorname{div} \underline{u} - \frac{1}{2} p\right), \Phi\right]}{\|\Phi\|_{m,q'}} =$$

$$= C \sup_{0 \neq \Phi \in H_0^{m,q'}(G)} \frac{B_m[\nabla w \cdot \nabla \zeta, \Phi]}{\|\Phi\|_{m,q'}},$$

as $\operatorname{div} \underline{u}$ and p are in $B_0^{m-1,q}(G)$ and the resulting term can be estimated against

$$CC' \|w\|_{m+1,q} \le CC'C'' \|p\|_{m-1,q},$$

according to Theorem 9.8 with the respective constant called C''.

With this fact, we can conclude

$$\left\|\operatorname{div}\underline{u}-\frac{1}{2}p\right\|_{m,q}=\left\|\nabla w\cdot\nabla\zeta-\left(\nabla w\cdot\nabla\zeta-\left(\operatorname{div}\underline{u}-\frac{1}{2}p\right)\right)\right\|_{m,q}\leq$$

$$\leq\|\nabla w\cdot\nabla\zeta\|_{m,q}+\left\|\nabla w\cdot\nabla\zeta-\left(\operatorname{div}\underline{u}-\frac{1}{2}p\right)\right\|_{m,q}\leq$$

$$\leq C'''\underbrace{\|w\|_{m+1,q}}_{\leq C''\|p\|_{m-1,q}}+CC'C''\|p\|_{m-1,q}\leq\tilde{C}\,\|p\|_{m-1,q}$$

and estimate (50) is shown. □

The rest of the account is easy again: As in Theorem 6.4 we can prove now with use of Theorem 9.9 and Theorem 9.6 the important

Theorem 9.10. *Let $\partial G\in\mathcal{C}^{m+3}$, $p\in B_0^{m-1,q}(G)$. Then we have*

$$\operatorname{div}\underline{u}-\frac{1}{2}p\in H^{m,q}(G)$$

and there is a constant $C=C(m,G,q)>0$ with

$$\left\|\operatorname{div}\underline{u}-\frac{1}{2}p\right\|_{m,q}\leq C\,\|p\|_{m-1,q}\tag{52}$$

Proof. The proof goes as follows: For $p\in B_0^{m-1,q}(G)$ we find according to Theorem 9.6 a sequence $p_\nu\in B_0^{m-1,q}(G)\cap H^{m+1,q}(G)$ with

$$\|p_\nu-p\|_{m-1,q}(G)\to 0.$$

Applying Theorem 9.9 to Cauchy differences of the p_ν, we get for the sequence $(\operatorname{div}\underline{u}_\nu-\frac{1}{2}p_\nu)$ with $(\underline{u}_\nu)=\underline{T}_q^{(m)}(p)$:

$$\left\|\operatorname{div}\underline{u}_\nu-\frac{1}{2}p_\nu-\left(\operatorname{div}\underline{u}_\nu-\frac{1}{2}p_\nu\right)\right\|_{m,q}\leq C\,\|p_\nu-p_\mu\|_{m-1,q}$$

and see that the sequence $(\operatorname{div}\underline{u}_\nu-\frac{1}{2}p_\nu)$ is a Cauchy-sequence in $H^{m,q}(G)$ and thus has a limit in $H^{m,q}(G)$. We see by passing to subsequences with pointwise convergence almost everywhere that this limit must be equal to $(\operatorname{div}\underline{u}-\frac{1}{2}p)\in H_0^{m-1,q}(G)$ and thus

$$\left(\operatorname{div}\underline{u}-\frac{1}{2}p\right)\in H^{m,q}(G).$$

For the estimate (52), we see that $\left\|\operatorname{div}\underline{u}_\nu-\frac{1}{2}p_\nu\right\|_{m,q}\to\left\|\operatorname{div}\underline{u}-\frac{1}{2}p\right\|_{m,q}$ and $\|p_\nu\|_{m-1,q}\to\|p\|_{m-1,q}$ and thus the estimate (50) carries over to this case and we have (52). □

Now we are able to prove the generalized compactness theorem and draw the important structural conclusions:

Theorem 9.11. *Let $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+3}$. The operator*

$$Z_q^{(m)} - \frac{1}{2} Id : B_0^{m-1,q}(G) \to B_0^{m-1,q}(G)$$

is a compact operator.

Proof. We have by Theorem 9.10 the fact that

$$Z_q^{(m)} - \frac{1}{2} Id : B_0^{m-1,q}(G) \to H^{m,q}(G) \cap B_0^{m-1,q}(G)$$

is continuous and by the compact embedding $H^{m,q}(G) \to H^{m-1,q}(G)$ we have the compactness of $Z_q^{(m)} - \frac{1}{2} Id : B_0^{m-1,q}(G) \to B_0^{m-1,q}(G)$. □

As in Theorem 7.1, we get a statement about regularity of eigenfunctions.

Theorem 9.12. *Let $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+3}$ and $\lambda \in \mathbb{R}$, $\lambda \neq \frac{1}{2}$ and $p \in B_0^{m-1,q}(G)$ satisfying $Z_q(p) = \lambda p$. Then for every $1 < r < \infty$:*

$$p \in B_0^{m-1,r}(G) \text{ and } Z_r(p) = Z_q(p) = \lambda p$$

Proof. The proof can be done in an analogous fashion to the one of Theorem 7.1. We can make an inductive proof based on the Sobolev Embedding Theorem and the fact that with an eigenfunction $p \in B_0^{m-1,q}(G)$ to $\frac{1}{2} \neq \lambda \in \mathbb{R}$ we have

$$p = \frac{1}{\lambda - \frac{1}{2}} \left(\operatorname{div} \underline{u} - \frac{1}{2} p \right) \in B_0^{m-1,q}(G) \cap H^{m,q}(G)$$

with $\underline{u} := \underline{T}_q^{(m)}(p)$ and Theorem 9.10. This simple observation assures us as in the proof of Theorem 7.1 gaining an order of derivatives in each inductive step compensating the loss of a derivative which is due to application of the Sobolev Embedding Theorem. □

Theorem 9.13. *Let $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+3}$. Then the operator $Z_q^{(m)}$: $B_0^{m-1,q}(G) \to B_0^{m-1,q}(G)$ is bijective.*

Proof. Again, as in the proof of Theorem 7.2, we see that $Z_q^{(m)} : B_0^{m-1,q}(G) \to B_0^{m-1,q}(G)$ is a Fredholm operator and thus, all we have to show is injectivity. For injectivity, assume $Z_q^{(m)}(p) = 0$ for a $p \in B_0^{m-1,q}(G)$. Then, as p is an eigenfunction of $Z_q^{(m)}$ for the eigenvalue 0, we conclude with Theorem 9.12 that we can assume that $q = 2$. Writing $\underline{u} = \underline{T}_2^{(m)}(p) \in \underline{H}_0^{m,2}(G)$ with

$$B_m[\underline{u}, \underline{\Phi}] = B_{m-1}[p, \operatorname{div} \underline{\Phi}] \text{ for all } \underline{\Phi} \in \underline{H}_0^{m,2}(G),$$

and $0 = Z_q^{(m)}(p) = \operatorname{div} \underline{u}$, we see that for $\underline{\Phi} := \underline{u}$ we get to

$$B_m[\underline{u}, \underline{u}] = 0$$

and thus it follows $\underline{u} = 0$.
We see thus that

$$B_{m-1}[p, \operatorname{div} \underline{\Phi}] = 0 \text{ for all } \underline{\Phi} \in \underline{H}_0^{m,2}(G)$$

and it follows $p \in H_{0,0}^{m-1,2}(G) \cap N_{m-1}^2(G)$ (see Theorem 9.3). Thus, by Theorem 9.3 we can conclude $p = 0$ and injectivity is shown. □

As in the case $m = 2$ (see Theorem 7.3), we find even continuity of $\left(Z_q^{(m)}\right)^{-1}$: $B_0^{m-1,q}(G) \to B_0^{m-1,q}(G)$:

Theorem 9.14. *Let $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+3}$. Then the operator $Z_q^{(m)}$: $B_0^{m-1,q}(G) \to B_0^{m-1,q}(G)$ is a homeomorphism.*

Proof. The proof is essentially the same as the proof of Theorem 7.3. □

Remark 9.15. *Regarding Theorem 9.4 we also quickly see that $Z_q^{(m)}$ is even a homeomorphism if seen as a mapping from $H_{0,0}^{m-1,q}(G)$ to $H_{0,0}^{m-1,q}(G)$.*

Definition 9.16. *Let $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+3}$. Then*

$$\underline{M}_q^{(m)}(G) := \underline{T}_q^{(m)}(H_{0,0}^{m-1,q}(G)).$$

As a generalization of Theorem 7.6 we arrive at

Theorem 9.17. *Let $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+3}$. For $p \in H_{0,0}^{m-1,q}(G)$ there is exactly one $\underline{u} \in \underline{M}_q^{(m)}(G)$ with*

$$\operatorname{div} \underline{u} = p.$$

The in this way well defined function

$$\underline{D}_q^{(m)} : H_{0,0}^{m-1,q}(G) \to \underline{M}_q^{(m)}(G),\ p \mapsto \text{ the unique } \underline{u} \in \underline{M}_q(G) \text{ with } \operatorname{div} \underline{u} = p$$

is continuous.

Proof. The proof goes like the proof of Theorem 7.6. □

With Theorem 9.17 now available, we get the accompanying decomposition:

Theorem 9.18. *Let $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+3}$. Then we have the direct decomposition*

$$\underline{H}_0^{m,q}(G) = \underline{D}_0^{m,q}(G) \oplus \underline{M}_q^{(m)}(G),$$

where

$$\underline{D}_0^{m,q}(G) := \{\underline{v} \in \underline{H}_0^{m,q}(G) :\ \operatorname{div} \underline{v} = 0\}$$

Proof. Again, the proof is a direct consequence of Theorem 9.17, as Theorem 7.8 was a direct consequence of Theorem 7.6. □

10 Some Regularity Theorems

In the following, we will need a kind of variant of Theorem 6.1, which can be found as Theorem 9.12 in [15], on the pages 157 and 158:

Theorem 10.1. *Assume*

(1) that $m \geq 1$ and $k \geq 0$ are integers and that $1 < p, q < \infty$ are real numbers with $\frac{1}{p} + \frac{1}{q} = 1$,

(2) that $G \subset \mathbb{R}^n$ is a bounded open set with boundary $\partial G \in \mathcal{C}^{2m+k}$,

(3) that B is an uniformly elliptic, $(m+k)$-smooth regular Dirichlet bilinear form of degree m in G,

(4) that $f \in H^{k,p}(G)$,

(5) that $u \in H_0^{m,p}(G)$ and that $B[u, \Phi] = \langle f, \Phi \rangle$ for all $\Phi \in \mathcal{C}_0^\infty(G)$.

Then $u \in H_0^{m,q}(G) \cap H^{2m+k,p}(G)$ and there is a constant $\gamma = \gamma(n, m, k, p, G, B)$ such that

$$\|u\|_{2m+k,p} \leq \gamma(\|f\|_{k,p} + \|u\|_{0,p})$$

Applying Theorems 6.1 and 10.1, we can show the following regularity theorem:

Theorem 10.2. *Let $k \in \mathbb{N}_0$, $G \subset\subset \mathbb{R}^n$ and $\partial G \in \mathcal{C}^{m+k+1}$. Then for $p \in B_0^{m-1,q}(G) \cap H^{m+k,q}(G)$ it is:*

$$\underline{T}_q^{(m)}(p) =: \underline{u} \in \underline{H}_0^{m,q}(G) \cap \underline{H}^{m+k+1,q}(G)$$

and we find a constant $C = C(m, k, q, G) > 0$ such that

$$\|\underline{u}\|_{m+k+1,q} \leq C \|p\|_{m+k,q}$$

Proof. The proof is only the definition of $\underline{u} = \underline{T}_q^{(m)}(p)$ and the regularity Theorems 6.1 and 10.1: As $\underline{u} = \underline{T}_q^{(m)}(p)$, we find that for every $\underline{\Phi} \in \underline{H}_0^{m,q'}(G)$ we have

$$B_m[\underline{u}, \underline{\Phi}] = B_{m-1}[p, \operatorname{div} \underline{\Phi}]$$

or equivalently, for every $i \in \{1, \ldots, n\}$ we have for all $\Phi \in H_0^{m,q'}(G)$

$$B_m[u_i, \Phi] = B_{m-1}[p, \partial_i \Phi].$$

So, in the case $k < m$, we find with $k+1$ partial integrations that $B_m[u_i, \Phi]$ can be written as a sum of terms of the form

$$\langle D^\alpha p, D^\beta \Phi \rangle,$$

where $|\alpha| = m - 1 + k + 1 = m + k$ and $|\beta| = m - k - 1$. So, $B_m[u_i, \Phi]$ defines for variable Φ an element F from $\left(H_0^{m-k-1,q'}(G)\right)^\star$ with

$$\|F\|_{\left(H_0^{m-k-1,q'}(G)\right)^\star} \leq C \|p\|_{m+k,q}$$

and Theorem 6.1 gives us

$$\|u_i\|_{m+k+1,q} \leq \gamma \left(\|F\|_{\left(H_0^{m-k-1,q'}(G)\right)^\star} + \|u_i\|_{0,q} \right) \leq C' \|p\|_{m+k,q}$$

and thus the Theorem.
For $k \geq m$, we use Theorem 10.1 instead:
After m partial integrations, we get that $B_m[u_i, \Phi]$ can be written as a sum of terms of the form

$$\langle D^\alpha p, \Phi \rangle,$$

where $|\alpha| = m-1+m = 2m-1$ and $D^\alpha p \in H^{m+k-(2m-1),q}(G) = H^{k-m+1,q}(G)$. So, according to Theorem 10.1, arguing analogously as above, we conclude that for every $i = 1, \ldots, n$ we have

$$u_i \in H_0^{m,q}(G) \cap H^{2m+(k-m+1),q}(G)$$

and we find a constant C'' such that

$$\|u_i\|_{m+k+1,q} \leq C'' \|p\|_{m+k,q}$$

and the desired estimate follows. □

With some calculations, we can see that we can apply the Theorems 6.1 and 10.1 to w as in the proof of Theorem 9.7 and get estimates for the higher derivatives of w. We begin with a version using Theorem 6.1. Let in the following be $m \geq 2$. The case $m = 1$ actually poses no difficulties, but due to formal reasons we look at it separately later in Theorem 10.5.

Theorem 10.3. *Let $m \in \mathbb{N}$, $m \geq 2$ and $k \in \mathbb{N}_0$, $k \leq m-2$ and $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+k+4}$, $p \in B_0^{m-1,q}(G) \cap H^{m+k,q}(G)$ and set $\underline{u} := \underline{T}_q^{(m)}(p) \in \underline{H}_0^{m,q}(G) \cap \underline{H}^{m+k+1,q}(G)$ (according to Theorem 10.2) and let ζ be Weyers' function $\in \mathcal{C}_0^{m+k+3}(\mathbb{R}^n)$. Further let*

$$w := \underline{u} \cdot \nabla\zeta - \frac{1}{2m} p\zeta \in H_0^{m,q}(G) \cap H^{m+k,q}(G).$$

Then we even find $w \in H^{m+k+2,q}(G)$ *and there is a constant* $C = C(m,k,q,G)$ *such that*

$$\|w\|_{m+k+2,q} \leq C\,\|p\|_{m+k,q}$$

Proof. We see that we can proceed as in Theorems 9.7, 9.8: All the assumptions except regularity from Theorem 9.7 are fulfilled, but with an argument as in the proof of Theorem 9.8, we can fix this.
Assuming first (we can get rid of this assumption later exactly as we did it in Theorems 9.7 and 9.8) that $\zeta \in \mathcal{C}_0^{2m+1}(\mathbb{R}^n)$ we can find with the calculation from the proof of Theorem 9.7 that for $\Phi \in \mathcal{C}_0^\infty(G)$ we have

$$B_m\,[w,\Phi] = \sum_{\substack{i=1,\ldots,n\\|\alpha|<2m\\|\beta|+|\alpha|=2m+1}} \langle a_{\alpha,\beta,i} D^\alpha\, u_i D^\beta \zeta, \Phi\rangle + \sum_{\substack{|\gamma|<2m-1\\|\delta|+|\gamma|=2m}} \langle b_{\gamma,\delta} D^\gamma p D^\delta \zeta, \Phi\rangle \quad (53)$$

with the $a_{\alpha,\beta,i}$, $b_{\gamma,\delta} \in \mathbb{Z}$. Looking at one term of the form

$$\langle D^\alpha u_i D^\beta \zeta, \Phi\rangle, \quad |\alpha| < 2m,\ |\beta| \leq 2m+1, |\alpha|+|\beta| = 2m+1, \quad (54)$$

we inspect the following three cases:

i) We see that in the cases where $|\alpha| \geq m+k+1$ we can (with an $\alpha' \leq \alpha$ of length $|\alpha| - (m+k+1)$) make

$$|\alpha'| = |\alpha| - (m+k+1) < 2m-k-k-1 = m-k-1$$

partial integrations and get to a term of the form

$$\pm\langle D^{\tilde{\alpha}} u_i, D^{\alpha'}(D^\beta \zeta \Phi)\rangle$$

with $\tilde{\alpha} + \alpha' = \alpha$ and $|\tilde{\alpha}| + |\alpha'| + |\beta| = 2m+1$, which can be written as a sum of terms of the form

$$\pm\langle D^{\tilde{\alpha}} u_i, D^\gamma \zeta D^\delta \Phi\rangle$$

with $|\tilde{\alpha}| = |\alpha| - |\alpha'| = |\alpha| - (|\alpha| - (m+k+1)) = m+k+1$,

$|\gamma| \leq |\beta| + |\alpha'| = 2m+1-|\alpha|+|\alpha|-(m+k+1) = 2m+1-(m+k+1) =$

$= m-k$ and

$$|\delta| \leq |\alpha'| = |\alpha| - (m+k+1) \leq 2m-1-m-k-1 = m-k-2.$$

ii) In the cases where in (54) we have $|\alpha| < m + k + 1$, look again at

$$\langle D^\alpha u_i, D^\beta \zeta\Phi\rangle = \langle D^\alpha u_i \Phi, D^\beta \zeta\rangle$$

with $|\beta| = 2m - 1 - |\alpha| > 2m + 1 - m - k - 1 = m - k$.

In the subcases where we find $|\beta| \leq m + k + 2$, we are again in a nice situation.

In the subcases where $|\beta| > m + k + 2$, we search a $\gamma \leq \beta$ with $|\gamma| = |\beta| - (m + k + 3)$ and make $|\gamma|$ partial integrations leading us to terms of the form

$$\langle D^\gamma \left(D^\alpha u_i \Phi\right), D^{\tilde{\beta}} \zeta\rangle$$

with $\left|\tilde{\beta}\right| = |\beta| - |\gamma| = |\beta| - (|\beta| - (m + k + 3)) = m + k + 3$, $|\gamma| + |\alpha| = |\beta| - (m + k + 3) + 2m + 1 - |\beta| = m - k - 2 \leq m + k + 1$ and $|\gamma| \leq 2m + 1 - (m + k + 3) = m - k - 2$.

So, reviewing all the preceding cases, we can rewrite every term of type (54) as a $\mathbb{Z}$-linearcombination of terms of the form

$$\langle D^\alpha u_i, D^\beta \zeta D^\gamma \Phi\rangle,$$

where $|\alpha| \leq m + k + 1$, $|\beta| \leq m + k + 3$ and $|\gamma| \leq m - k - 2$.
For variable Φ, each of these terms defines an element (which we call in the following F) from $\left(H_0^{m-(k+2),q'}(G)\right)^\star$ and as $\zeta \in \mathcal{C}_0^{m+k+3}(\mathbb{R}^n)$, we can find a constant $c > 0$ such that for every multiindex β with $|\beta| \leq m + k + 3$ we have $\left|D^\beta \zeta(x)\right| < c$ for all $x \in \mathbb{R}^n$ and as we have constants C_i' with $\|u_i\|_{m+k+1,q} \leq C_i' \|p\|_{m+k,q}$ according to Theorem 10.2, we have validity of an estimate of the form

$$\|F\|_{\left(H_0^{m-(k+2),q'}(G)\right)^\star} \leq C \|u_i\|_{m+k+1,q} \leq CC_i' \|p\|_{m+k,q}.$$

The terms of the form

$$\langle D^\gamma p D^\delta \zeta, \Phi\rangle \text{ with } |\gamma| < 2m - 1,\ |\delta| + |\gamma| = 2m \tag{55}$$

occurring in equation (53) can be treated in a similar way:

i) In the cases where $|\gamma| \geq m + k$, we can make with a $\nu \leq \gamma$ of length $|\nu| = |\gamma| - (m + k)$ the corresponding partial integrations and arrive at terms of the form

$$\pm\langle D^\eta p, D^\nu (D^\delta \zeta\Phi)\rangle$$

with $\eta + \nu = \gamma$, $|\eta| = |\gamma| - |\nu| = m + k$, $|\nu| = |\gamma| - (m + k)$ and $|\delta| = 2m - |\gamma|$.

By carrying out the D^ν-differentiation of the product $D^\delta \zeta \Phi$, we land at terms of the form

$$\pm\langle D^\eta p, D^\varepsilon \zeta D^\sigma \Phi\rangle$$

with

$$|\varepsilon| \leq |\nu| + |\delta| = |\gamma| - (m + k) + 2m - |\gamma| = m - k \leq m + k + 2$$

and

$$|\sigma| \leq |\nu| = \gamma - (m + k) \leq 2m - 2 - (m - k) = m - k - 2.$$

ii) In the cases where $|\gamma| < m + k$ in a term of the form (55), we have $|\delta| = 2m - |\gamma| > m - k$. In the subcases where $|\delta| \leq m + k + 2$, no problems occur, so look in the following at the subcases where we have $|\delta| > m + k + 2$. Here, we can make with a $\nu \leq \delta$ with $|\nu| = |\delta| - (m+k+2)$ the corresponding partial integrations to land at terms of the form

$$\pm\langle D^\nu(D^\gamma p\Phi), D^\varepsilon \zeta\rangle$$

with $\varepsilon + \nu = \delta$, $|\varepsilon| = |\delta| - |\nu| = m + k + 2 \leq m + k + 3$, $|\nu| \leq 2m - (m+k+2) = m - k - 2$ and $|\nu| + |\gamma| = |\delta| - (m+k+2) + 2m - |\delta| = m - k - 2 \leq m + k$.

So here, too, we can in every case transform every term of the form (55) into a $\mathbb{Z}$-linearcombination of terms of the form

$$\langle D^\alpha p D^\beta \zeta, D^\gamma \Phi\rangle$$

with $|\alpha| \leq m + k$, $|\beta| \leq m + k + 3$ and $|\gamma| \leq m - k - 2$. Arguing as in the first part of the proof for the terms involving the u_i, we can see here that these terms define for variable Φ elements from $\left(H_0^{m-(k+2),q'}(G)\right)^\star$ for each of which we have validity of an estimate of the $\left(H_0^{m-(k+2),q'}(G)\right)^\star$-norm against a constant times $\|p\|_{m+k,q}$.

All in all, we see that $B_m[w, \Phi]$ defines an element F from $\left(H_0^{m-(k+2),q}(G)\right)^\star$, too. Moreover, we also get for F an estimate as above.

With Theorem 6.1, we conclude that $w \in H^{m+k+2,q}(G)$ and

$$\|w\|_{m+k+2,q} \leq \gamma \left(C \|p\|_{m+k,q} + \|w\|_{0,q}\right).$$

As we can estimate $\|w\|_{0,q}$ against $\|p\|_{m+k,q}$ by it's definition via the u_i and p, we are done.
Concerning the regularity of ∂G needed, we see that we used $\zeta \in \mathcal{C}_0^{2m+1}(\mathbb{R}^n)$ as in the proof of Theorem 9.7 only to justify classical calculation and in the end of the calculation we arrive at terms involving derivatives of ζ only up to order $m+k+3$. As in our proof of Theorem 9.8, we can use here an approximation argument and get the statement even if we just assume $\partial G \in \mathcal{C}^{m+k+4}$. □

In order to get a variant of Theorem 10.3 for $k > m-2$, we simply have to use Theorem 10.1 instead of Theorem 6.1.

Theorem 10.4. *Let $m \in \mathbb{N}$, $m \geq 2$, $k \in \mathbb{N}$, $k > m-2$ and $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+k+4}$, $p \in B_0^{m-1,q}(G) \cap H^{m+k,q}(G)$ and set $\underline{u} := \underline{T}_q^{(m)}(p) \in \underline{H}_0^{m,q}(G) \cap \underline{H}^{m+k+1,q}(G)$ (according to Theorem 10.2) and let ζ be Weyers' function $\in \mathcal{C}_0^{m+k+3}(\mathbb{R}^n)$. Further let*

$$w := \underline{u} \cdot \nabla\zeta - \frac{1}{2m} p\zeta \in H_0^{m,q}(G) \cap H^{m+k,q}(G).$$

Then we even find $w \in H^{m+k+2,q}(G)$ and there is a constant $C = C(m,k,q,G)$ such that

$$\|w\|_{m+k+2,q} \leq C \|p\|_{m+k,q}$$

Proof. With the same procedure as in the proof of Theorem 10.3, we get again to

$$B_m[w,\Phi] = \sum_{\substack{i=1,\dots,n \\ |\alpha|<2m \\ |\beta|+|\alpha|=2m+1}} \langle a_{\alpha,\beta,i} D^\alpha\, u_i D^\beta \zeta, \Phi\rangle + \sum_{\substack{|\gamma|<2m-1 \\ |\delta|+|\gamma|=2m}} \langle b_{\gamma,\delta} D^\gamma p D^\delta \zeta, \Phi\rangle \quad (56)$$

for $\Phi \in \mathcal{C}_0^\infty(G)$. We see that the functions on the left side of the sums in (56)

$$a_{\alpha,\beta,i} D^\alpha\, u_i D^\beta \zeta$$

with $|\alpha| < 2m$, $|\beta| + |\alpha| = 2m+1$ and

$$b_{\gamma,\delta} D^\gamma p D^\delta \zeta$$

with $|\gamma| < 2m-1$, $|\delta| + |\gamma| = 2m$ have derivatives up to order $k-m+2$ in $L^q(G)$:

- As $|\alpha| \leq 2m-1$, we find $D^\alpha u_i$ to have derivatives of up to order $m+k+1-(2m-1) = k-m+2$ in $L^q(G)$.

- As $|\gamma| \leq 2m-2$, we find $D^\gamma p$ to have derivatives of up to order $m+k-(2m-2) = k-m+2$ in $L^q(G)$.
- As $|\beta| \leq 2m+1$, we find $D^\beta \zeta$ to have classical derivatives of up to order $m+k+3-(2m+1) = k-m+2$ bounded in $\overline{G}$.
- As $\zeta \in \overline{\mathcal{C}}^{m+k+3}(G)$ and $|\delta| \leq 2m$, we find $D^\delta \zeta$ to have classical derivatives of up to order $m+k+3-2m = k-m+3$ in $L^q(G)$ bounded in $\overline{G}$.

So, also the products (as ζ is classically differentiable) $D^\alpha\, u_i D^\beta \zeta$ and $D^\gamma p D^\delta \zeta$ are in $H^{k-m+2,q}(G)$. Thus, we have found an $f \in H^{k-m+2,q}(G)$ such that $B\,[w,\Phi] = \langle f, \Phi\rangle$ for all $\Phi \in \mathcal{C}_0^\infty(G)$.
Applying now Theorem 10.1 with $\tilde{k} := k-m+2$ playing the role of the k from Theorem 10.1, we conclude that if $\partial G \in \mathcal{C}^{2m+k-m+2} = \mathcal{C}^{m+k+2}$ (which is satisfied), we get

$$w \in H_0^{m,q}(G) \cap H^{m+k+2}(G) \text{ and there is a constant } \gamma \text{ such that}$$

$$\|w\|_{m+k+2,q} \leq \gamma(\|f\|_{k-m+2,q} + \|w\|_{0,q})$$

Again, we can estimate $\|f\|_{k-m+2,q}$ and $\|w\|_{0,q}$ because of the consistences of f and w against $\|\underline{u}\|_{m+k+1,q}$ (which can again be estimated against $\|p\|_{m+k,q}$) and $\|p\|_{m+k,q}$ and thus we get the estimate

$$\|w\|_{m+k+2,q} \leq C\gamma(\|p\|_{m+k,q})$$

with a suited constant $C > 0$ depending on G, m, k and q. □

In Theorems 10.3 and 10.4, we made the distinction between the cases $k \leq m-2$ and $k > m-2$ and applied the respective suited regularity Theorem 6.1 or 10.1 which resulted in ignoring the case $m = 1$. For $m = 1$, we have the following regularity Theorem, which is blending well with the regularity Theorem 10.3 and 10.4:

Theorem 10.5. *Let $k \in \mathbb{N}_0$ and $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{k+5}$, $p \in B_0^{0,q}(G) \cap H^{k+1,q}(G)$ and set $\underline{u} := \underline{T}_q^{(1)}(p) \in \underline{H}_0^{1,q}(G) \cap \underline{H}^{k+2,q}(G)$ (according to Theorem 10.2) and let ζ be Weyers' function $\in \mathcal{C}_0^{k+4}(\mathbb{R}^n)$. Further let*

$$w := \underline{u} \cdot \nabla \zeta - \frac{1}{2} p\zeta \in H_0^{1,q}(G) \cap H^{1+k,q}(G).$$

Then we even find $w \in H^{k+3,q}(G)$ and there is a constant $C = C(k,q,G)$ such that

$$\|w\|_{k+3,q} \leq C\,\|p\|_{k+1,q}$$

Proof. The proof goes the same way as the proof of Theorems 10.3, 10.4: Assuming first that ζ is smooth enough, we can do a classical calculation of Δw and find out that for $\Phi \in \mathcal{C}_0^\infty(G)$ we have

$$\langle \nabla w, \nabla \Phi \rangle = -\langle \Delta w, \Phi \rangle$$

and this can be written as

$$\sum_{\substack{i=1,\dots,n \\ |\alpha|\leq 1 \\ |\beta|+|\alpha|=3}} \langle a_{\alpha,\beta,i} D^\alpha \; u_i D^\beta \zeta, \Phi \rangle + \sum_{|\delta|=2} \langle b_\delta p D^\delta \zeta, \Phi \rangle, \tag{57}$$

with suited $a_{\alpha,\beta,i}$, $b_\delta \in \mathbb{Z}$ which is nothing but the representation (53) from Theorem 10.3 in the special case $m = 1$.
Looking at a summand of the form $\langle a_{\alpha,\beta,i} D^\alpha \; u_i D^\beta \zeta, \Phi \rangle$ of the left sum in (57), we see that the function $a_{\alpha,\beta,i} D^\alpha \; u_i D^\beta \zeta$ on left with $|\alpha| \leq 1$ and $|\beta| + |\alpha| = 3$ has weak derivatives in $L^q(G)$ up to order $k+1$, as with $u_i \in H^{k+2,q}(G)$ and $|\alpha| \leq 1$ we have $D^\alpha u_i \in H^{k+1,q}(G)$ and as $\zeta \in \mathcal{C}_0^{k+4}(\mathbb{R}^n)$ and $|\beta| \leq 3$ we have $D^\beta \zeta \in \mathcal{C}_0^{k+1}(\mathbb{R}^n)$.
Furthermore we have an estimate of the form

$$\left\| D^\alpha u_i D^\beta \zeta \right\|_{k+1,q} \leq C(\zeta) \left\| u_i \right\|_{k+2,q} \leq C(\zeta) \tilde{C} \left\| p \right\|_{k+1,q}$$

with $\tilde{C}$ according to Theorem 10.2.
Looking analogously at a summand of the form $\langle b_\delta p D^\delta \zeta, \Phi \rangle$ with $|\delta| = 2$, we see that $p \in H^{k+1,q}(G)$ and $D^\beta \zeta \in \mathcal{C}_0^{k+4-2}(\mathbb{R}^n)$ and thus $p D^\delta \zeta \in H^{k+1,q}(G)$ and we have also here an estimate of the form

$$\left\| p D^\delta \zeta \right\|_{k+1,q} \leq C(\zeta) \left\| p \right\|_{k+1,q}.$$

All in all, we see that with Theorem 10.1, we can conclude as in Theorem 10.4 that

$$w \in H_0^{1,q}(G) \cap H^{k+2,q}(G)$$

and validity of an estimate of the form

$$\left\| w \right\|_{k+2,q} \leq C \left\| p \right\|_{k+1,q}.$$

After this, we see with an approximation argument as already used repeatedly, that the smoothness of ζ which is needed is $\zeta \in \mathcal{C}_0^{k+4}(\mathbb{R}^n)$ and thus it suffices to demand from ∂G to be in $\mathcal{C}^{k+5}$. □

Theorem 10.6. *Let $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+k+4}$, and let $p \in B_0^{m-1,q}(G) \cap H^{m+k,q}(G)$. Then $Z_q^{(m)}(p) - \frac{1}{2}p \in H_0^{m-1,q}(G) \cap H^{m+k+1,q}(G)$ and with a constant $C = C(m,q,k,G) > 0$ we have*

$$\left\| Z_q^{(m)}(p) - \frac{1}{2}p \right\|_{m+k+1,q} \leq C \left\| p \right\|_{m+k,q}$$

Proof. With $v := \nabla w \cdot \nabla \zeta - (Z_q^{(m)}(p) - \frac{1}{2}p) \in H_0^{m,q}(G)$ as in Theorem 9.9, we see that from $v \in H^{m+k+1,q}(G)$, we could conclude $Z_q^{(m)}(p) - \frac{1}{2}p \in H^{m+k+1,q}(G)$ as $\nabla w \cdot \nabla \zeta \in H^{m+k+1,q}(G)$. This and a respective estimate is gained again by our Theorems on Elliptic Regularity 6.1, 10.1:
As we have $Z_q^{(m)}(p) - \frac{1}{2}p \in B_0^{m-1,q}(G) \cap H^{m,q}(G)$, we find

$$B_m \left[Z_q^{(m)}(p) - \frac{1}{2}p, \Phi \right] = 0 \text{ for all } \Phi \in \mathcal{C}_0^\infty(G)$$

and thus

$$B_m \left[v, \Phi \right] = B_m \left[\nabla w \cdot \nabla \zeta, \Phi \right].$$

As $\nabla w \cdot \nabla \zeta \in H^{m+k+1,q}(G)$, we can conclude with use of Theorems 6.1, 10.1 that $v \in H_0^{m,q}(G) \cap H^{m+k+1,q}(G)$ and that there are constants C, C' and C'' such that

$$\left\| v \right\|_{m+k+1,q} \leq C \left\| \nabla w \cdot \nabla \Phi \right\|_{m+k+1,q} \leq CC' \left\| w \right\|_{m+k+2,q} \leq CC'C'' \left\| p \right\|_{m+k,q},$$

according to Theorems 10.3, 10.4 and 10.5.
Then we notice:

$$\left\| Z_q^{(m)}(p) - \frac{1}{2}p \right\|_{m+k+1,q} \leq \left\| \nabla w \cdot \nabla \zeta - \left(Z_q^{(m)}(p) - \frac{1}{2}p \right) \right\|_{m+k+1,q} +$$

$$+ \left\| \nabla w \cdot \nabla \zeta \right\|_{m+k+1,q} \leq C_1 \left\| p \right\|_{m+k,q} + C_2 \left\| p \right\|_{m+k,q} \leq C \left\| p \right\|_{m+k,q}$$

□

We can now prove the following theorem, which will be very important in the next section, guaranteeing us the regularity for a Stokes-like system. The idea of proof of the following theorem and our account to regularity of our Stokes-like system is due to C. G. Simader, whose program from [13] is working fine here.

Theorem 10.7. *Let $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+k+3}$, $p \in B_0^{m-1,q}(G)$ and $\underline{u} = \underline{T}_q^{(m)}(p) \in \underline{H}_0^{m,q}(G)$.*

If $Z_q^{(m)}(p) = \operatorname{div} \underline{u} \in H^{m+k-1,q}(G)$ *then we find*

$$p \in B_0^{m-1,q}(G) \cap H^{m+k-1,q}(G), \quad \underline{u} \in \underline{H}_0^{m,q}(G) \cap \underline{H}^{m+k,q}(G)$$

and there exist constants C, $C' > 0$, *depending on* m, k, q *and* G *such that*

$$\|p\|_{m+k-1,q} \leq C \|\operatorname{div} \underline{u}\|_{m+k-1,q} \tag{58}$$

and

$$\|\underline{u}\|_{m+k,q} \leq C' \|\operatorname{div} \underline{u}\|_{m+k-1,q} \tag{59}$$

Proof. We will show the theorem by induction over k. Starting with $k = 0$, we do not have anything to show:
$p \in H^{m-1,q}(G)$ and $\underline{u} \in \underline{H}^{m,q}(G)$ are clear and the corresponding estimates (58) and (59) reduce to

$$\|p\|_{m-1,q} \leq C \|\operatorname{div} \underline{u}\|_{m-1.q}$$

and

$$\|\underline{u}\|_{m,q} \leq C' \|\operatorname{div} \underline{u}\|_{m-1.q},$$

which are clear by $\operatorname{div} \underline{u} = Z_q^{(m)}(p)$ and the fact that the operators $Z_q^{(m)} : B_0^{m-1,q}(G) \to B_0^{m-1,q}(G)$ and $\operatorname{div} : \underline{M}_q^{(m)}(G) \to H_{0,0}^{m-1,q}(G)$ are homeomorphisms by Theorems 9.14 and 9.17.
The inductive step: $k \to k+1$. Assume the claim to hold for k. With $\operatorname{div} \underline{u} \in H^{m+k,q}(G)$, we find $\operatorname{div} \underline{u} \in H^{m+k-1,q}(G)$ and thus by our inductive assumption:

$$p \in H^{m+k-1,q}(G).$$

Applying Theorem 10.6, we find that $Z_q^{(m)}(p) - \frac{1}{2}p \in H^{m+k,q}(G)$ and thus

$$p = -2\left(Z_q^{(m)} - \frac{1}{2}p\right) + 2 \underbrace{Z_q^{(m)}(p)}_{\operatorname{div} \underline{u} \in H^{m+k,q}(G)} \in H^{m+k,q}(G).$$

By Theorem 10.2, we then find $\underline{u} \in \underline{H}_0^{m,q}(G) \cap \underline{H}^{m+k+1,q}(G)$ and a constant C_1 such that

$$\|\underline{u}\|_{m+k+1,q} \leq C_1 \|p\|_{m+k,q}.$$

Regarding $\|p\|_{m+k,q}$, we notice with the triangle inequality

$$\|p\|_{m+k,q} \leq 2 \left\| Z_q^{(m)}(p) - \frac{1}{2}p \right\|_{m+k,q} + 2 \left\| Z_q^{(m)}(p) \right\|_{m+k,q}$$

and by Theorem 10.6 again, we see that with a $C_2 > 0$:

$$\left\| Z_q^{(m)}(p) - \frac{1}{2}p \right\|_{m+k,q} \le C_2 \|p\|_{m+k-1,q} \le C_2 C_3 \|\operatorname{div} \underline{u}\|_{m+k-1,q} ,$$

where C_3 is according to our inductive assumption.
So, what we get is

$$\|p\|_{m+k,q} \le 2C_2C_3 \|\operatorname{div} \underline{u}\|_{m+k-1,q} + 2 \|\operatorname{div} \underline{u}\|_{m+k,q} \le$$

$$\le (2C_2C_3 + 2) \|\operatorname{div} \underline{u}\|_{m+k,q} ,$$

which is (58) for $k+1$ and thus also

$$\|\underline{u}\|_{m+k+1,q} \le C_1(2C_2C_3 + 2) \|\operatorname{div} \underline{u}\|_{m+k,q} ,$$

which is (59) for $k+1$. □

Theorem 10.6 also allows us to find out another regularity-result for functions $p \in B_0^{(m-1),q}(G)$ satisfying the eigenvalue-relation

$$Z_q^{(m)}(p) = \lambda p, \quad \lambda \in \mathbb{R}, \, \lambda \ne \frac{1}{2},$$

saying that these p are as regular as ∂G "allows" them to be:

Theorem 10.8. *Let $m \in \mathbb{N}$, $m \ge 2$, $k \in \mathbb{N}$ and $G \subset\subset \mathbb{R}^n$ be a domain with $\partial G \in \mathcal{C}^{m+k+4}$ and $p \in B_0^{m-1,q}(G)$ with*

$$Z_q^{(m)}(p) = \lambda p$$

for a $\lambda \in \mathbb{R}$ with $\lambda \ne \frac{1}{2}$.
Then we also have

$$p \in H^{m+k+1,q}(G).$$

Proof. By Theorem 9.10, we get $Z_q(p) - \frac{1}{2}p = (\lambda - \frac{1}{2})p \in H^{m,q}(G)$ and thus $p \in H^{m,q}(G)$. Iterated application of Theorem 10.6 proves the theorem. □

11 A Stokes-Like System

In this section we will investigate the natural generalization of the Stokes-like system which was treated by Simader in [17] in the case $m = 2$, $q = 2$. As we are not in the Hilbert space setting from [17], we have to assume more regularity for ∂G than merely being Lipschitz. We will give solvability and regularity statements for our generalized Stokes-like system. At first, we state the problem we are investigating:

11.1 The Problem

Let $m \in \mathbb{N}$, $G \subset\subset \mathbb{R}^n$ be given with $\partial G \in \mathcal{C}^{m+3}$. Given a functional $F \in \left(\underline{H}_0^{m,q'}(G)\right)^\star$, we are looking for an $\underline{u} \in \underline{H}_0^{m,q}(G)$ and a $p \in H_{0,0}^{m-1,q}(G)$ such that

$$B_m\left[\underline{u}, \underline{\Phi}\right] + B_{m-1}\left[p, \operatorname{div} \underline{\Phi}\right] = F(\underline{\Phi}) \quad \text{for all } \underline{\Phi} \in \underline{H}_0^{m,q'}(G)$$

and

$$\operatorname{div} \underline{u} = 0.$$

11.2 The Solution and Regularity Theorems

With the general theorems from Sections 9 and 10 now available, we can easily derive unique solvability and regularity theorems for our generalized Stokes-like system.

Theorem 11.1. *Let $m \in \mathbb{N}$, $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+3}$ and let $F \in \left(\underline{H}_0^{m,q'}(G)\right)^\star$. Then there is exactly one pair $(\underline{u}, p) \in \underline{H}_0^{m,q}(G) \times H_{0,0}^{m-1,q}(G)$ such that*

$$B_m\left[\underline{u}, \underline{\Phi}\right] + B_{m-1}\left[p, \operatorname{div} \underline{\Phi}\right] = F(\underline{\Phi}) \quad \textit{for all } \underline{\Phi} \in \underline{H}_0^{m,q'}(G)$$

and

$$\operatorname{div} \underline{u} = 0.$$

Furthermore, we find a constant $C = C(m, q, G) > 0$ with

$$\|\underline{u}\|_{m,q} + \|p\|_{m-1,q} \leq C \|F\|_{\left(\underline{H}_0^{m,q'}(G)\right)^\star}.$$

Proof. Let's first show existence of $(\underline{u}, p)$. Given $F \in \left(\underline{H}_0^{m,q'}(G)\right)^\star$, we find a $\underline{v} \in \underline{H}_0^{m,q}(G)$ such that

$$B_m\left[\underline{v}, \underline{\Phi}\right] = F(\underline{\Phi}) \text{ for all } \underline{\Phi} \in \underline{H}_0^{m,q'}(G)$$

by Theorem 9.1 and we find $\|\underline{v}\|_{m,q} \leq C \|F\|_{\left(\underline{H}_0^{m,q'}(G)\right)^\star}$. As $\underline{v} \in \underline{H}_0^{m,q}(G)$, we see that $\operatorname{div} \underline{v} \in H_{0,0}^{m-1,q}(G)$ and find by Theorem 9.17 an unique $\underline{w} \in \underline{M}_q^{(m)}(G) = \underline{T}_q^{(m)}\left(H_{0,0}^{m-1,q}(G)\right)$ with $\operatorname{div} \underline{w} = \operatorname{div} \underline{v}$. With Theorem 9.17 we also see that we have

$$\|\underline{w}\|_{m,q} \leq C_1 \|\operatorname{div} \underline{v}\|_{m-1,q} \leq C_2 \|\underline{v}\|_{m,q} \leq C_2 C \|F\|_{\left(\underline{H}_0^{m,q'}(G)\right)^\star}.$$

So, for $\underline{u} := \underline{v} - \underline{w} \in \underline{H}_0^{m,q}(G)$, we have $\operatorname{div} \underline{u} = 0$ and

$$\|\underline{u}\|_{m,q} \leq \|\underline{v}\|_{m,q} + \|\underline{w}\|_{m,q} \leq C_2 C \|F\|_{\left(\underline{H}_0^{m,q'}(G)\right)^\star} + C \|F\|_{\left(\underline{H}_0^{m,q'}(G)\right)^\star} =$$

$$= (C_2 C + C) \|F\|_{\left(\underline{H}_0^{m,q'}(G)\right)^\star} .$$

For $p \in H_{0,0}^{m-1,q}(G)$ with $\underline{T}_q^{(m)}(p) = \underline{w}$, we also have by Theorems 9.1 and 9.17 that $\underline{T}_q^{(m)} : H_{0,0}^{m-1,q}(G) \to \underline{M}_q^{(m)}(G)$ is a homeomorphism and thus we find a constant $\tilde{C}$ such that $\|p\|_{m-1,q} \leq \tilde{C} \|\underline{w}\|_{m,q}$ and thus we also have with a constant C'

$$\|p\|_{m-1,q} \leq C' \|F\|_{\left(\underline{H}_0^{m,q'}(G)\right)^\star}$$

and the desired estimate is shown for $(\underline{u}, p)$. $(\underline{u}, p)$ is indeed a solution, for we see that for $\underline{\Phi} \in \underline{H}_0^{m,q'}(G)$ we have:

$$B_m [\underline{u}, \underline{\Phi}] + B_{m-1} [p, \operatorname{div} \underline{\Phi}] = B_m [\underline{v} - \underline{w}, \underline{\Phi}] + B_{m-1} [p, \operatorname{div} \underline{\Phi}] =$$

$$= \underbrace{B_m [\underline{v}, \underline{\Phi}]}_{=F(\underline{\Phi})} - B_m [\underline{w}, \underline{\Phi}] + \underbrace{B_{m-1} [p, \operatorname{div} \underline{\Phi}]}_{=B_m[\underline{w}, \underline{\Phi}]} = F(\underline{\Phi})$$

For uniqueness of the solution, we note that if we have two solution pairs $(\underline{u}_1, p_1)$, $(\underline{u}_2, p_2)$, the pair $(\underline{u} := \underline{u}_1 - \underline{u}_2, p := p_1 - p_2)$ is a solution to the problem with $F = 0$. This means $\operatorname{div} \underline{u} = 0$ and $\underline{u} = -\underline{T}_q^{(m)}(p)$, so it follows $Z_q^{(m)}(p) = \operatorname{div}(-\underline{u}) = 0$ and thus $p = 0$ and $\underline{u} = 0$ by injectivity of $Z_q^{(m)}$. $\square$

By Theorem 9.1, we can represent an element $F \in \left(\underline{H}_0^{m,q'}(G)\right)^\star$ by

$$B_m [\underline{v}, \underline{\Phi}] = F(\underline{\Phi}) \quad \forall \underline{\Phi} \in \underline{H}_0^{m,q'}(G)$$

with a $\underline{v} \in \underline{H}_0^{m,q}(G)$. We will show a regularity theorem stating the following: The regularity of the $\underline{v}$ belonging to F carries over to the regularities of $\underline{u}$ and p, the solutions of our problem:

Theorem 11.2. *Let $m \in \mathbb{N}$, $G \subset\subset \mathbb{R}^n$ with $\partial G \in \mathcal{C}^{m+k+3}$ and let $\underline{v} \in \underline{H}_0^{m,q}(G) \cap \underline{H}^{m+k,q}(G)$ be given. Then the (by Theorem 11.1 unique) pair $(\underline{u}, p) \in \underline{H}_0^{m,q}(G) \times H_{0,0}^{m-1,q}(G)$ satisfying*

$$B_m [\underline{u}, \underline{\Phi}] + B_{m-1} [p, \operatorname{div} \underline{\Phi}] = B_m [\underline{v}, \underline{\Phi}] \qquad \textit{for all } \underline{\Phi} \in \underline{H}_0^{m,q'}(G)$$

and

$$\operatorname{div} \underline{u} = 0$$

satisfies even $(\underline{u}, p) \in \underline{H}^{m+k,q}(G) \times H^{m+k-1,q}_{0,0}(G)$ *and we get the two estimates*

$$\|\underline{w}\|_{m+k,q} \leq C_1 \|\underline{v}\|_{m+k,q}$$

and

$$\|p\|_{m+k-1,q} \leq C_2 \|\underline{v}\|_{m+k,q} ,$$

where $C_1, C_2 > 0$ *are constants depending on* m, k, q *and* G.

Proof. For the proof, we simply try to go through the construction of the solution in the proof of Theorem 11.1 and show regularity at each step, using our already established regularity theorems from Section 10. We will give the corresponding objects the same name as in the proof of Theorem 11.1. As $\underline{v} \in \underline{H}^{m,q}_0(G) \cap \underline{H}^{m+k,q}(G)$, we can define $r := \operatorname{div} \underline{v} \in H^{m-1,q}_{0,0}(G) \cap H^{m+k-1,q}(G)$ and represent

$$r = \Delta s + t,$$

according to Theorem 9.2 with $s \in H^{m+1,q}_0(G) \cap H^{m+k+1,q}(G)$ and $t \in B^{m-1,q}_0(G)$. Further we have the estimate

$$\|\Delta s\|_{m,q} + \|t\|_{m,q} \leq C \|r\|_{m,q} .$$

The proof of this decomposition is nothing but use of the solvability statement 9.1, solving the problem

$$B_{m+1} [s, \Phi] = B_{m-1} [r, \Delta\Phi] \text{ for all } \Phi \in H^{m+1,q'}_0(G),$$

resulting at

$$B_m [\Delta s - r, \Phi] = 0 \text{ for all } \Phi \in \mathcal{C}^\infty_0(G)$$

and thus $(\Delta s - r) \in B^{m-1,q}_0(G)$. So, regularity of s is simply again elliptic regularity from Theorems 6.1 and 10.1 leading to

$$s \in H^{m+1,q}_0(G) \cap H^{m+k+1,q}(G)$$

and then we have also $t = r - \Delta s \in B^{m-1,q}_0(G) \cap H^{m+k-1,q}(G)$ and a constant C with

$$\|s\|_{m+k+1,q} \leq C \|r\|_{m+k-1,q} \leq C \|\underline{v}\|_{m+k,q}$$

and we also get

$$\|t\|_{m+k-1,q} = \|r - \Delta s\|_{m+k-1,q} \leq \|r\|_{m+k-1,q} + \|\Delta s\|_{m+k-1,q} \leq (1+C) \|\underline{v}\|_{m+k,q} .$$

Now we can find due to Theorem 9.17 a vector field $\underline{x} \in \underline{M}^{(m)}_q(G)$ with $\operatorname{div} \underline{x} = t \in B^{m-1,q}_0(G) \cap H^{m+k-1,q}(G)$. As further we have by $\underline{x} \in \underline{M}^{(m)}_q(G)$ an $f \in H^{m-1,q}_{0,0}(G)$ with $\underline{T}^{(m)}_q(f) = \underline{x}$, we find easily that $f \in B^{m-1,q}_0(G)$:

As we have $Z_q^{(m)}(f) = \operatorname{div} \underline{x} = t \in B_0^{m-1,q}(G)$, we see by Theorem 9.4 that f must also be in $B_0^{m-1,q}(G)$. Now we can apply Theorem 10.7 and conclude:

$$f \in H^{m+k-1,q}(G) \text{ and } \underline{x} \in H^{m+k,q}(G)$$

and

$$\|\underline{x}\|_{m+k,q} \leq C' \|\operatorname{div} \underline{x}\|_{m+k-1,q} = C' \|t\|_{m+k-1,q} \leq C'(1+C) \|\underline{v}\|_{m+k,q} ,$$

$$\|f\|_{m+k-1,q} \leq C'' \|\operatorname{div} \underline{x}\|_{m+k-1,q} \leq C''(1+C) \|\underline{v}\|_{m+k,q} .$$

□

The $\underline{w}$ from Theorem 11.1 must by uniqueness be equal to $\nabla s + \underline{x}$ and is thus also in $\underline{H}^{m+k,q}(G)$ and

$$\|\underline{w}\|_{m+k,q} = \|\nabla s + \underline{x}\|_{m+k,q} \leq \|\nabla s\|_{m+k,q} + \|\underline{x}\|_{m+k,q} \leq$$

$$\leq \|s\|_{m+k+1,q} + \|\underline{x}\|_{m+k,q} \leq (C + C'(1+C)) \|\underline{v}\|_{m+k,q}$$

and the p from Theorem 11.1 must be equal to $\Delta s + f \in H^{m+k-1,q}(G)$ and with an analogous calculation as above we get the estimate

$$\|p\|_{m+k-1,q} \leq (C + C''(1+C)) \|\underline{v}\|_{m+k,q} .$$

References

[1] S. Agmon, A. Douglis, L. Nirenberg *Estimates near the boundary for solutions of elliptical partial differential equations satisfying general boundary conditions II*, Comm. Pure Appl. Math., 17, pages 35-92, (1964)

[2] H. W. Alt, *Lineare Funktionalanalysis*, fifth, corrected edition, Springer Verlag, Berlin, Heidelberg, New York, (in German), (1992)

[3] C. Amrouche, V. Girault, *Problèmes généralisés de Stokes*, Portugaliae Mathematica 49, pages 463-503, (1992)

[4] M. E. Bogovskii, *Solution of the First Boundary Value Problem for the Equation of Continuity of an Incompressible Medium*, Dokl. Akad Nauk SSSR, **248**, (5), pages 1037-1040 (1979)
English Translation: Soviet Math Dokl., **20**, 1094-1098, (1979)

[5] M. E. Bogovskii, *Solution of Some Vector Analysis Problems Connected with Operators Div and Grad*, Trudy Seminar S. L. Sobolev #1, 80, Akademia Nauk SSSR, Sibirskoe Otdelnie Matematiki, Nowosibirsk, pages 5-40, (in Russian), (1980)

[6] M. Crouzeix, *On an operator related to the convergence of Uzawa's algorithm for the Stokes equation*, in: M.-O. Bristeau et al (ed.): Computational science for the 21st century, Wiley, Chichester, (1997)

[7] L. C. Evans, *Partial Differential Equations*, GSM Volume 19, AMS, (1997)

[8] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol. I: Linearized Steady Problems*, Springer Verlag, (1998)

[9] V. G. Maz'ja, *Sobolev Spaces*, Springer Verlag, (1985)

[10] R. Müller, *Das schwache Dirichletproblem in L^q für den Bipotentialoperator in beschränkten Gebieten und in Außengebieten*, Bayreuther Mathematische Schriften, Heft 49, pages 115-211 (in German), (1995)

[11] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Academia Prag, (in French), (1967)

[12] G. Schramm, *Spuren von Funktionen aus $W^{k,p}(\Omega)$ für beschränkte Gebiete und Außengebiete $\Omega \subset \mathbb{R}^n$*, diploma thesis (in German), (1991)

[13] C. G. Simader, *A new approach to the regularity of weak L^q-solutions of Stokes' and similar equations via the Cosserat operator*, In A. Sequeira and R. Rannacher, editors, Advances in Mathematical Fluid Mechanics. Dedicated to Giovanni Paolo Galdi on the Occasion of his 60th Birthday. Springer, (2009)

[14] C. G. Simader, *Mean value formulas, Weyl's lemma and Liouville Theorems for Δ^2 and Stokes' system*, Results in Math. 22, pages 761-780, (1992)

[15] C. G. Simader, *On Dirichlet's Boundary Value Problem: An L^p-Theory based on a Generalization of Gårding's Inequality*, Springer Verlag, (1972)

[16] C. G. Simader, *Partielle Differentialgleichungen. 1. Teil: Der Laplace-Operator*, Notes from a lecture held at the University of Bayreuth, (in German), (2003)

[17] C. G. Simader, *Weak L^2-solutions to a Stokes-like system of fourth order in bounded Lipschitz domains*, to appear 2010 in Applicable Analysis

[18] C. G. Simader, H. Sohr, *The Dirichlet problem for the Laplacian in bounded and unbounded domains*, Pitman Research Notes in Mathematics Series 360, Addison Wesley Longman, (1996)

[19] C. G. Simader, S. Weyers, *An operator related to the Cosserat spectrum. Applications*, Analysis (Munich) **26**, pages 169-198, (2006)

[20] H. Sohr, *The Navier-Stokes Equations: An Elementary Functional Analytic Approach*, Birkhäuser Basel, (2001)

[21] M. Stark, *Das schwache Dirichletproblem in L^q für das Differentialgleichungssystem $\Delta \underline{v} = \nabla p$ und die Lösung der Divergenzgleichung*, Bayreuther Mathematische Schriften, Heft 70, (in German), (2004)

[22] S. Weyers, *L^q-solutions to the Cosserat spectrum in bounded and exterior domains*, Analysis (Munich) **26**, (2006), pages 85-167