Analyzing Heterogeneous Agents Models Avoiding the Explicit Discretization of Stiff Equations*

(Enlarged Abstract)

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An interesting approach to analyze a number of important features of financial markets like bubbles and crashes is to build on market maker models with heterogeneous agents, typically fundamentalists F and chartists C. In this framework prices of assets depend on the aggregated demand of fundamentalists and chartists. Fundamentalists buy when assets are undervalued and sell when they are overvalued and chartists buy when prices rise and sell when they fall. More specifically and following a basic approach going back to [1] and [2] the log-price of an asset $p(t)$ is assumed to be linear in the demand of the agents, i.e., $p(t+1) = p(t) + (N_F(t)D_F(t) + N_C(t)D_C(t))/M$ (with $M > 0$) where D_F and D_C is the demand of the fundamentalists resp. chartists and N_F and N_C is the respective number of traders (trading volume). The demand function of the fundamentalists is assumed to be linear in the deviation of the log-price from the log-fundamental $f(t)$, i.e., $D_F(t) = F(f(t) - p(t))$ (with $F > 0$), while the demand function of the chartists is assumed to be linear in the trend of the log-price, i.e., $D_C(t) = C(p(t) - p(t-1))$ (with $C > 0$). Further work in this area include among others [3], [4], [5], [6], [7], and [8].

While it is in general taken as granted that fundamentalists stabilize markets by bringing asset prices back to their fundamental values, in these models this is not necessarily the case. In fact, when the number F of fundamentalists becomes large, oscillations occur whose amplitude grows exponentially in time when the number of fundamentalists exceeds a certain threshold. In this enlarged abstract we show that this instability phenomenon is due to the fact that these models contain a "hidden" explicit discretization of a stiff ordinary differential equation (ODE). Replacing this explicit discretization by an implicit one removes this artifact.

In order to concentrate on the key issues, in this abstract we consider a simple example of the model class, noting that our approach easily carries over to more involved models. We assume the log-fundamentals as well as the number of the traders to be constant and introduce weights

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of the respective types of traders $w_F = N_F F/M$ and $w_C = N_C C/M$, which leads to the following recurrence relation (seldom: difference equation) for the log-prices

$$p(t+1) = p(t) + w_F(f - p(t)) + w_C(p(t) - p(t-1)). \tag{1}$$

It is easy to see that $p^* = f$ is an equilibrium of the model. Figures 1 and 2 depict the vast spectrum of price dynamics that are generated by this explicit model under alternative parameters $f = 10$, $p(0) = 1$, as well as $(w_F, w_C) = (0.2, 0)$, $(1, 0)$, $(1.8, 0)$, $(2, 0)$, $(2.05, 0)$, and $(1.5, 0.5)$.

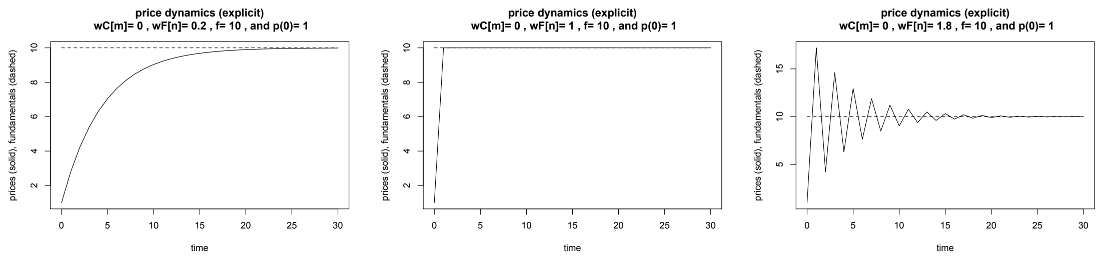

Figure 1: Price dynamics for the explicit model with varying weights w_F and $w_C = 0$.

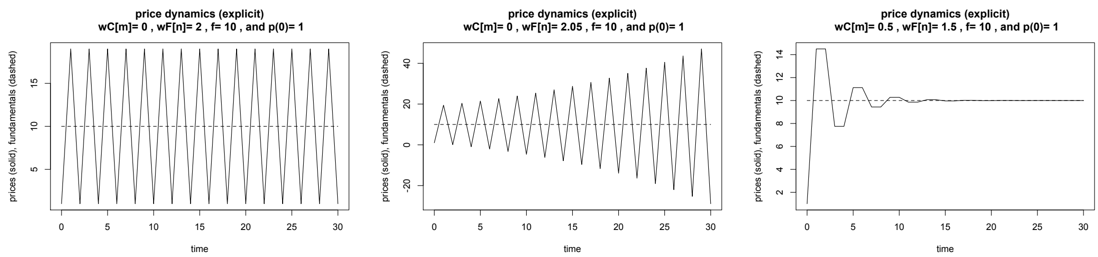

Figure 2: Price dynamics for the explicit model with varying weights w_F, w_C.

It turns out that even when there are no chartists, i.e., $w_C = 0$, $p^* = f$ is unstable if $w_F > 2$, cf. the unstable behavior of the model in the middle of Figure 2. That means, if there are "too many" fundamentalists, the price crosses the fundamental in each time step with exponentially increasing jump size. Of course, it cannot be excluded per se that asset prices jump beyond their fundamental value during the short-term adjustment to a shock, even if only fundamentalists are present. Yet, it still seems very plausible that in in the long run and in the absence of chartists prices should end up near their fundamental values. We therefore find it most likely that the model exhibits instability artifacts.

In order to support our simulations by analytical results, it is convenient to rewrite the second order equation (1) as

$$\begin{pmatrix} p_1(t+1) \\ p_2(t+1) \end{pmatrix} = \begin{pmatrix} 1 - w_F + w_C & -w_C \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} + \begin{pmatrix} w_F f \\ 0 \end{pmatrix},$$

which is a first order equation in two dimensions. The eigenvalues of the transition matrix are $\lambda_{1,2} = \frac{1}{2}\left(1 - w_F + w_C \pm \sqrt{w_C^2 - 2w_C w_F + w_F^2 - 2w_C - 2w_F + 1}\right)$. This allows us to calculate the values of $(w_F, w_C) \in \mathbb{R}^+ \times \mathbb{R}$ for which $p^* = f$ is stable. Figure 3 shows the region where the model is stable (actually, the figure shows the maximum of $|\lambda_{1,2}|$ depending on w_C and w_F—the model is stable when this maximum is smaller than one). Usually, this is done to determine the threshold of the fundamentalists to chartists ratio when this market model becomes unstable. However, in a model that contains instability artifacts, when the market becomes unstable, it is not clear whether too many chartists cause the instability or whether the instability is a modeling artifact. This leads us to the question how to model heterogenous agents without these effects.

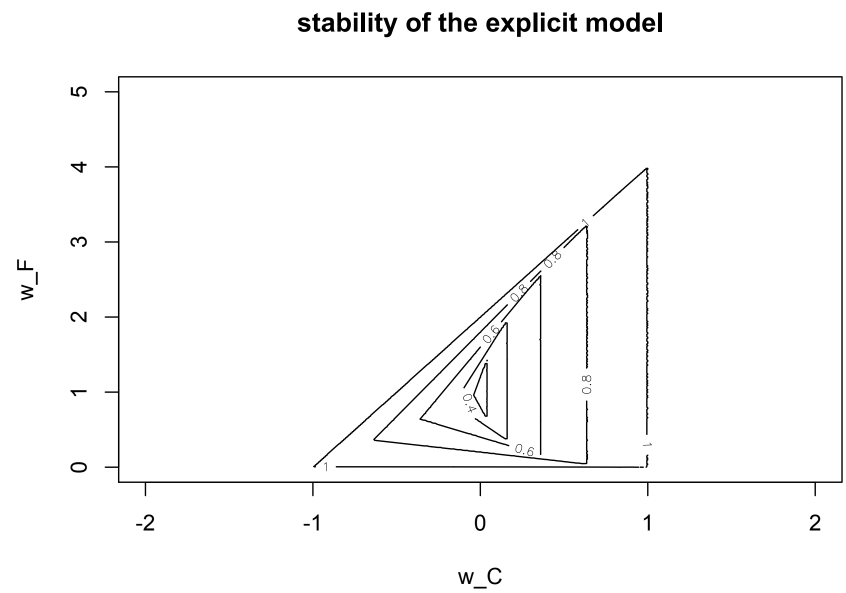

Figure 3: Parameters for which the explicit model (1) is stable.

The key to answer this question is the observation that the model (1) contains an explicit Euler discretization $p(t+h) = p(t) + hg(t, p(t))$ of a stiff ordinary differential equation (ODE) $\dot{p}(t) = g(t, p(t))$, see [9, Chapter 6] or [10]. Such a discretization is known to cause exactly the effects visible in Figure 2, cf. [9, Figure 6.3].

When setting $w_C = 0$ in Equation (1) we obtain $p(t+1) = p(t) + 1g(t, p(t))$ with $g(t, p(t)) =$

$w_F(f - p(t))$. This is exactly the explicit Euler discretization of the ODE

$$\dot{p}(t) = w_F(f - p(t)) \tag{2}$$

with mesh size $h = 1$.

A known remedy for the oscillatory instability is the use of an implicit discretization. Here we use an implicit Euler discretization for Equation (2) with mesh size $h > 0$, leading to $p(t+h) = p(t) + hw_F(f - p(t+h))$, i.e., $p(t+h) = \frac{p(t)+hw_F f}{1+hw_F}$. To this equation we add the demand for the chartists from Equation (1) (adjusted for $h > 0$). Thus, we have

$$p(t+h) = \frac{p(t) + hw_F f}{1 + hw_F} + hw_C(p(t) - p(t-h)). \tag{3}$$

In case of $h = 1$ this is $p(t+1) = \frac{p(t)+w_F f}{1+w_F} + w_C(p(t) - p(t-1))$. In Figures 4 and 5 we can see the simulations of the implicit model (3) using the same parameter values as in the simulations of the explicit model (1) in Figures 1 and 2. Note that the model is stable in absence of chartists even for $w_F > 2$, while at the same time it still allows for price overshoots and instability caused by an abundance of chartists, cf. Figure 5 and 6.

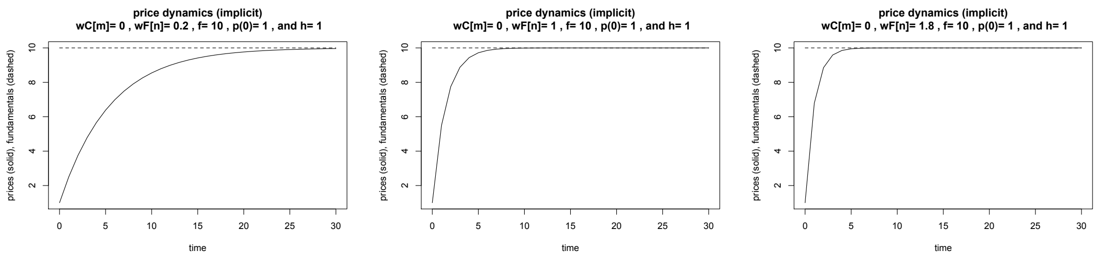

Figure 4: Price dynamics for the implicit model with varying weights w_F and $w_C = 0$ and mesh size $h = 1$.

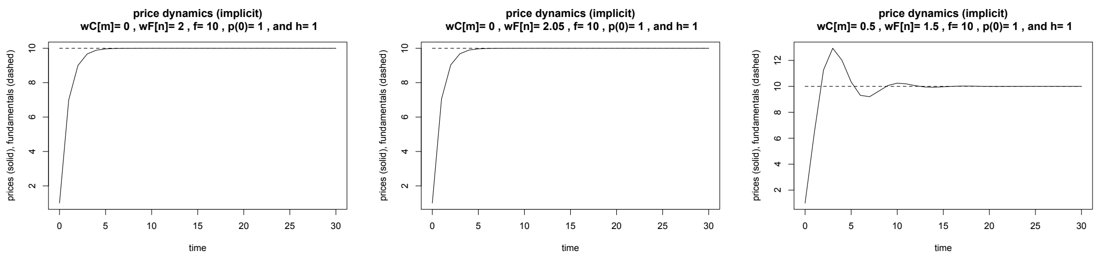

Figure 5: Price dynamics for the implicit model with varying weights w_F, w_C and mesh size $h = 1$.

Rewriting Equation (3) leads to $p(t+h) = \left(\frac{1}{1+hw_F} + hw_C\right) p(t) + \frac{hw_F f}{1+hw_F} - hw_C p(t-h)$ or, as a first order equation,

$$\begin{pmatrix} p_1(t+h) \\ p_2(t+h) \end{pmatrix} = \begin{pmatrix} \frac{1}{1+hw_F} + hw_C & -hw_C \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} + \begin{pmatrix} \frac{hw_F f}{1+hw_F} \\ 0 \end{pmatrix},$$

with eigenvalues

$$\lambda_{1,2} = \frac{h^2 w_C w_F + hw_C + 1 \pm \sqrt{h^4 w_C^2 w_F^2 + 2h^3 w_C^2 w_F - 4h^3 w_C w_F^2 + h^2 w_C^2 - 6h^2 w_C w_F - 2hw_C + 1}}{2(1+hw_F)}.$$

We analyze the influence of the mesh size h on the stability of the implicit model. Note that in the implicit model we can introduce $\tilde{w}_C = hw_C$ and $\tilde{w}_F = hw_F$, which eliminates all w_F, w_C, and h. In other words, the stability region scales with h, which is a known fact in the numerical analysis, see [9, Section 6.1.2]. In Figures 6, 7, and 8 the implicit model is simulated for the same weights $(w_F, w_C) = (4, 0)$, $(10, 1)$, and $(10, 1.5)$ but with varying mesh size $h = 1$, 0.25, and 0.1. We can see that a smaller mesh size stabilizes the model. In Figures 9, 10, and 11 the respective stability regions for the implicit model are shown.

In the ongoing work we introduce evolutionary rules for the numbers of chartists N_C and fundamentalists N_F. The results of this enlarged abstract will help us to distinguish between the following two effects: first, the instability of the model is caused by the fundamentalists due to some numerical artifacts and, second, the instability of the model is caused by a too large number (resp. weight) of the chartists compared to the fundamentalists. Clearly, the second effect is the one of interest.

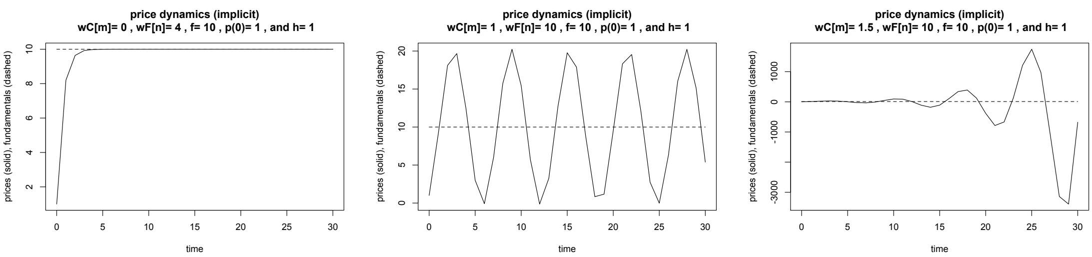

Figure 6: Price dynamics for the implicit model with varying weights w_F, w_C and mesh size $h = 1$.

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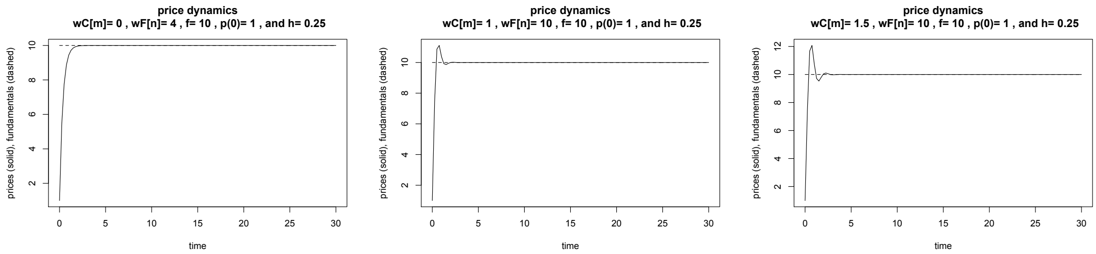

Figure 7: Price dynamics for the implicit model with varying weights w_F, w_C and mesh size $h = 0.25$.

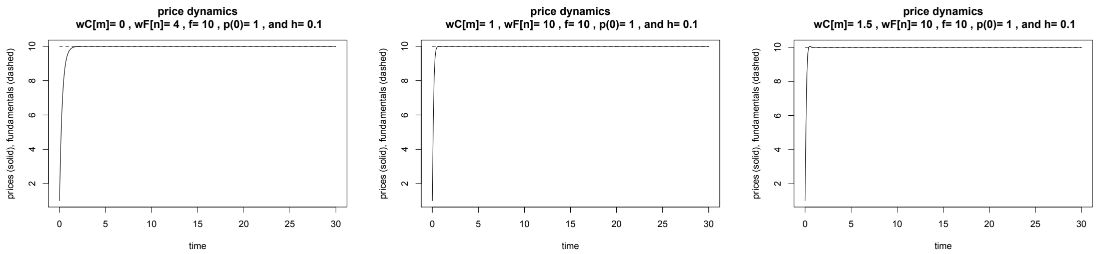

Figure 8: Price dynamics for the implicit model with varying weights w_F, w_C and mesh size $h = 0.1$.

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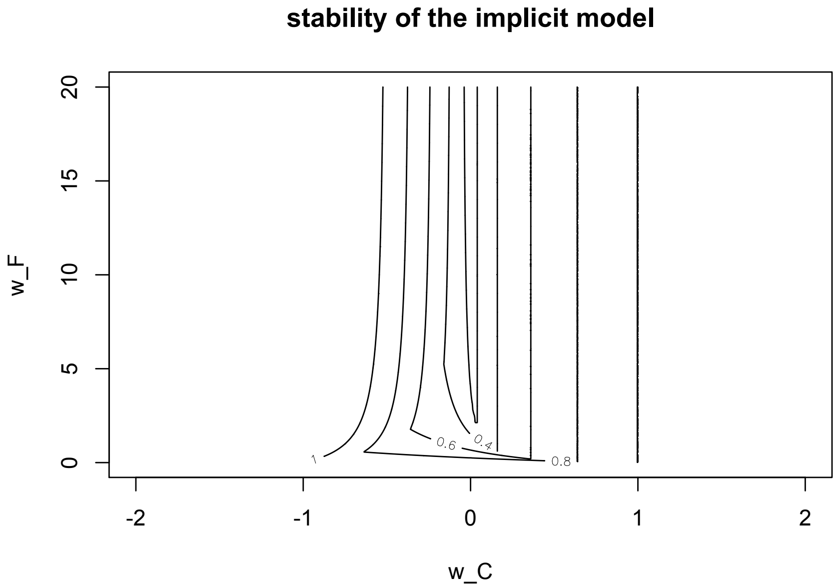

Figure 9: Parameters for which the implicit model (3) with mesh size $h = 1$ is stable.

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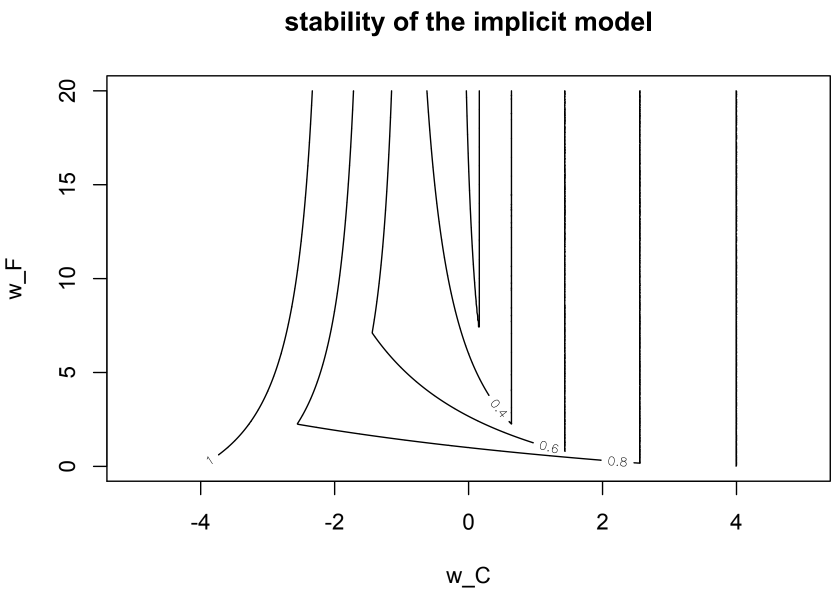

Figure 10: Parameters for which the implicit model (3) with mesh size $h = 0.25$ is stable.

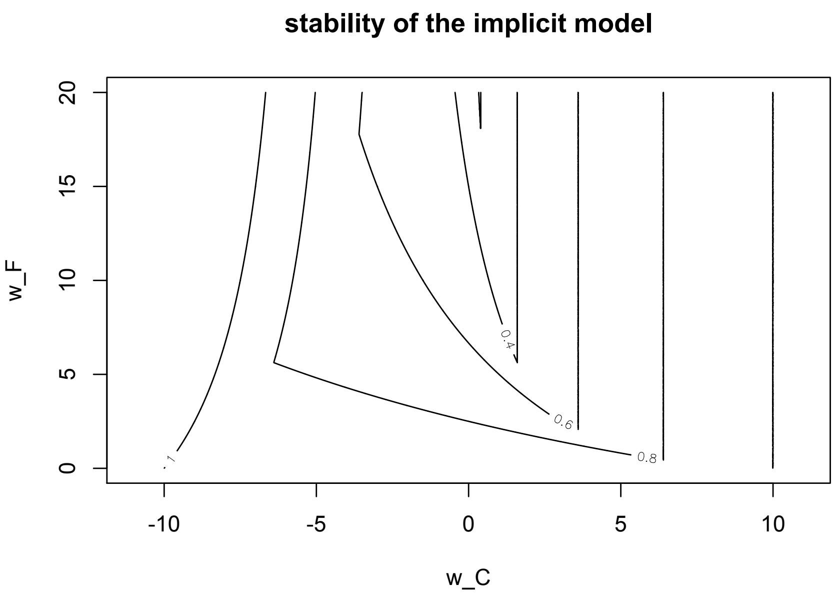

Figure 11: Parameters for which the implicit model (3) with mesh size $h = 0.1$ is stable.