Optimal Control Problems Governed by Nonlinear Partial Differential Equations and Inclusions

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Abstract

The focus of this thesis lies on examining the solvability of optimal control problems constrained by nonlinear partial differential equations (PDE) and inclusions (PDI). There exist statements on the existence of solutions for optimal control problems with linear and semi-linear PDEs with monotone parts. The theory for non-monotone PDEs resp. the related optimal control problems is, to the authorś knowledge, incomplete regarding important issues. This concerns particularly the case of PDEs containing mappings, which only satisfy boundedness conditions on restricted sets. The optimal control problem considered first is characterized by a Laplace equation with Dirichlet boundary conditions:

$$\begin{aligned} -\Delta y + g \circ y &= f \text{ in } \Omega \\ y &= 0 \text{ on } \Gamma \end{aligned} \tag{PDE1}$$

Under the decisive assumption of the existence of so called sub- and supersolutions for (PDE1) and by introducing a truncation operator we can define an auxiliary problem which is characterized by a pseudomonotone operator. Thereby the solution theory for pseudomonotone operators of Brézis (1968) is applicable. Moreover, starting with the definitions of sub- und supersolution it can be shown, that every solution of the auxiliary problem is a solution of the original problem. The choice of a new optimal control problem which substitutes the original optimal control problem is again governed by the properties of the auxiliary operator. The equivalence of the auxiliary problem to the original problem and the existence of at least one solution can be shown. The technique of applying the Theorem of Lax-Milgram on a linearized problem can be adapted to the semi-linear non-monotone case. This procedure is already known from the theory of semi-linear monotone problems.

For optimal control problems with quasi-linear differential equations, different methods are required. As in the semi-linear case, the property of pseudomonotonicity plays a key role in proving the existence of a solution of the quasi-linear PDE. In the proof of the existence of a solution for the optimal control problem other properties of the auxiliary operator are exploited. In the elliptic case operators which satisfy the S_+-property are important. In order to utilize this property, a transformation of the operator to some coercive auxiliary operator is necessary. For this reason a term is added, which penalizes the deviation from the admissible set of states. This term is characterized by a factor, which is derived explicitly in this work.

The proof of the existence of a solution of the optimal control problem with parabolic equations is based on the definition of an auxiliary operator, coercivity and the S_+-property of operators.

The set of solutions of the considered PDE is compact, but the number of solutions and the situation to each other is unknown. This leads to difficulties in deriving necessary optimality conditions. For this reason a direct approach to solve the optimal control problem with semi-linear PDEs is introduced. It is assumed, that the state constraints coincide with the sub- and the supersolution of the PDE with the upper and lower boundary of the control variable. Using an auxiliary operator, this assumption allows the formulation of an equivalent optimal control problem without pointwise state constraints. Through semi-discretization we can define a family of optimal control problems on a finite dimensional state-space. Existence of a subsequence of solutions of these optimal control problems which converges to a solution of the original problem is shown.
Another important class of optimal control problems include differential inclusions which are described by multivalued operators. Quasi-linear elliptic inclusions are examined under global as well as local boundedness conditions. Under the assumption of global boundedness the properties of pseudomonotonicity and coercivity for a multivalued auxiliary operator are proven. The existence of at least one solution for the original inclusion follows from the application of a result from Hu and Papageorgiou (1997) on the auxiliary problem. The existence of at least one solution of the optimal control problem is proven by exploiting the coercivity of the multivalued auxiliary operator and the S_+-property of the non-multivalued part of this mapping.
In the case of multivalued mappings of Clarke's gradient type, the existence of at least one solution of the optimal control problem can be shown under local boundedness conditions. Elliptic as well as parabolic quasi-linear inclusions are considered. The proof is again based on coercivity and the S_+-property of the related auxiliary operators and the embedding properties of the spaces.

Zusammenfassung

Gegenstand dieser Arbeit sind Untersuchungen zur Lösbarkeit von Optimalsteuerungsproblemen, welche durch nichtlineare partielle Differentialgleichungen (PDG) und Inklusionen (PDI) restringiert sind. Die Berücksichtigung nichtlinearer Terme spielt dabei die zentrale Rolle. Während für lineare und semilineare PDGs mit monotonen Anteilen Aussagen über die Existenz von Lösungen der zugehörigen Optimalsteuerungsprobleme bekannt sind, ist, nach Kenntnis der Autorin, die Theorie für nichtmonotone PDGs bzw. deren Optimalsteuerungsprobleme für wichtige Fragestellungen unvollständig. Dies betrifft insbesondere den Fall von PDGs, welche Abbildungen enthalten, die nur auf einem begrenzten Bereich Beschränktheitsbedingungen erfüllen.
Das zunächst betrachtete Optimalsteuerungsproblem wird durch eine Laplacegleichung mit Dirichlet-Randbedingung charakterisiert:

$$\begin{aligned} -\Delta y + g \circ y &= f \text{ in } \Omega \\ y &= 0 \text{ auf } \Gamma \end{aligned} \qquad \text{(PDG1)}$$

Unter der maßgeblichen Annahme der Existenz von sogenannten Sub- und Superlösungen für (PDG1) kann mit Hilfe eines Abschneideoperators ein Hilfsproblem formuliert werden, welches durch einen pseudomonotonen Operator beschrieben wird. Das ermöglicht die direkte Anwendung der Lösungstheorie für pseudomonotone Operatoren von Brézis (1968). Zudem kann ausgehend von den Definitionen für Sub- und Superlösung gezeigt werden, dass jede Lösung des Hilfsproblems auch eine Lösung des ursprünglichen Problems darstellt.
Auch bei der Wahl eines neuen Optimalsteuerungsproblems, welches das eigentliche Optimalsteuerungsproblem ersetzt, sind die Eigenschaften des Hilfsoperators maßgebend. In diesem Zusammenhang kann die Äquivalenz des Hilfsproblems zum ursprünglichen Problem und die Existenz mindestens einer Lösung nachgewiesen werden. Auf diesen semilinearen, aber möglicherweise nichtmonotonen Fall lässt sich die Technik - die Anwendung des Satzes von Lax-Milgram auf ein linearisiertes Problem - übertragen. Dieses Vorgehen ist bereits aus der Theorie semilinearer monotoner Probleme bekannt.
Für Optimalsteuerungsprobleme mit quasilinearen Differentialgleichungen müssen andere Methoden gesucht werden. Während zum Nachweis der Existenz einer Lösung der quasilinearen PDG wie im semilinearen Fall die Pseudomonotonie-Eigenschaft eine zentrale Rolle einnimmt, werden im Existenzbeweis einer Lösung für das Optimalsteuerungsproblem andere Besonderheiten des Hilfsoperators ausgenutzt. Im elliptischen Fall kommt dabei den Operatoren, welche die S_+-Eigenschaft besitzen, eine große Bedeutung zu. Um diese Eigenschaft gezielt ausschöpfen zu können, wird zunächst eine Überführung des Operators in einen koerzitiven Hilfsoperator nötig. Aus diesem Grund wird ein Term addiert, der die

Abweichung vom zulässigen Zustandsbereich bestraft. Dieser Term wird durch einen Faktor charakterisiert, welcher in dieser Arbeit hergeleitet und explizit angegeben wird.

Der Nachweis der Existenz einer Lösung des Optimalsteuerungsproblems mit parabolischen Gleichungen basiert wie im elliptischen Fall auf der Konstruktion eines Hilfsoperators, der Koerzitivität und der S_+-Eigenschaft von Operatoren.

Bei der Herleitung notwendiger Optimalitätsbedingungen liegt die Schwierigkeit darin, dass die Lösungsmenge der PDG zwar kompakt ist, die Anzahl und die Lage der Lösungen zueinander jedoch unbekannt ist. Aus diesem Grund wird zur numerischen Bestimmung einer Lösung des Optimalsteuerungsproblems mit semilinearen PDGs ein direkter Ansatz vorgestellt. Dabei wird vorausgesetzt, dass die Zustandsschranken Sub- und Superlösung einer PDG mit unterer bzw. oberer Schranke für die Steuerung als rechter Seite darstellen. Unter Verwendung eines Hilfsoperators ermöglicht diese Annahme die Formulierung eines äquivalenten Optimalsteuerungsproblems ohne punktweise Zustandsbeschränkungen. Mittels Semi-Diskretisierung lässt sich eine Schar von Optimalsteuerungsproblemen über einem endlich dimensionalen Zustandsraum definieren. Es wird gezeigt, dass mindestens eine Teilfolge der Lösungen dieser Optimalsteuerungsprobleme existiert, die gegen eine Lösung des ursprünglichen Problems konvergiert.

Eine weitere wichtige Klasse von Optimalsteuerungsproblemen enthält Differentialinklusionen, welche durch mengenwertige Operatoren beschrieben werden. Im Fall quasilinearer elliptischer Inklusionen werden dabei Probleme sowohl unter globalen als auch unter lokalen Beschränktheitsbedingungen betrachtet. Unter der Annahme globaler Beschränkungen werden die Eigenschaft der Pseudomonotonie und der Koerzitivität für einen mengenwertigen Hilfsoperator nachgewiesen. Die Existenz mindestens einer Lösung der ursprünglichen Inklusion folgt dann aus der Anwendung eines Resultates von Hu and Papageorgiou (1997) auf das Hilfsproblem. Der Beweis der Existenz mindestens einer Lösung des Optimalsteuerungsproblems erfolgt unter Ausnutzung der Koerzitivität des mengenwertigen Hilfsoperators und der S_+-Eigenschaft des nicht mengenwertigen Anteils dieser Abbildung.

Auch unter lokalen Beschränktheitsbedingungen wird die Existenz mindestens einer Lösung des Optimalsteuerungsproblems gezeigt, wobei in diesem Fall ausschließlich mengenwertige Abbildungen in Form des Clarke-Gradienten betrachtet werden. Dabei werden sowohl elliptische als auch parabolische quasilineare Inklusionen zugelassen. Der ausgeführte Existenzbeweis baut erneut auf den Merkmalen der Koerzitivität und der S_+-Eigenschaft der entsprechenden Hilfsoperatoren bzw. den Einbettungseigenschaften der Räume auf.

Contents

1 Introduction

Several systems with physical or biological origin can be described by partial differential equations (PDEs). In an easy case, a linear model is a sufficiently precise simplification of reality. The solution theory for linear PDEs is in general well known and is based on the Theorem of Lax-Milgram. More sophisticated techniques are required in the nonlinear case. Let us consider the following nonlinear problem with Dirichlet boundary conditions which appears, for example, in plasma physics:

$$
\begin{aligned}
-\Delta y - \alpha \frac{\exp(-y)}{\left(\int_\Omega \exp(-y)\, d\lambda_\Omega\right)^p} &= 0 \quad \text{on } \Omega \\
y &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\tag{1}
$$

Here, α and p denote positive real numbers. For $p = 1$, this PDE plays an important role in kinetics and is called Poisson-Boltzmann equation. A discussion of this model can be found in Bavaud (1991).
In the case $p = 0$, equation (1) includes a nonlinear, but monotone term. This allows us to apply the solution theory for monotone operators. We refer to Zarantonello (1960). The solution theory for nonlinear and non-monotone problems differs from the linear and nonlinear monotone coercive case. Even if solvability is warranted, a lack of uniqueness appears for non-monotone operators. In Carrillo (1998) it is shown that for $p > 2$ and α smaller than some critical value the boundary value problem (1) possesses exactly two solutions.
Other PDEs including nonlinear and non-monotone terms can be found in Takeuchi (2001) in which the multiplicity of solutions of the problem

$$
\begin{aligned}
-\Delta_p y &= \lambda y^{q-1} \left(1 - y^r\right) \text{ on } \Omega \\
y &= 0 \text{ on } \partial\Omega
\end{aligned}
\tag{2}
$$

is shown. Here Δ_p describes the p-Laplacian, whereas $2 < p < q$, $\lambda > 0$ and $r > 0$ are constants. Moreover, the model of a molten carbonate fuel cell contains a quadratic term, see Sternberg (2006).
A powerful tool to handle this kind of PDEs is the method of sub- and supersolutions. Theoretical results can be found, e.g., in the monographs Pao (1992), Heikkilä and Lakshmikantham (1994) and Carl et al. (2007).
Regarding optimal control problems, differences in the theory of the corresponding PDEs have to be taken into account. In Casas (1986) optimal control problems including linear elliptic PDEs with state constraints have been considered. An overview about optimal control problems constrained by linear and semi-linear elliptic and parabolic equations is given in Tröltzsch (2009). Therein, for problems with linear PDEs, the related operator is assumed to be bounded and

coercive. In the case of a Laplace equation with Neumann boundary conditions the corresponding operator fails to be coercive. Linear, non-coercive operators lead to so called singular problems. In Lions (1985) the existence of multiple solutions for such a PDE is shown. Nevertheless, for the optimal control problem with boundary control and a well chosen objective functional the optimal control-state pair is unique. A singular optimality system is stated for this problem.
In the papers Papageorgiou (1991), Papageorgiou (1993) and Halidias and Papageorgiou (2002), the existence of a solution for optimal control problems of nonlinear and non-monotone type is proven. Therein the Carathéodory function which describes the nonlinear term is assumed to be bounded on the whole real axis. This condition is not satisfied in the above example (1) with $p > 2$.
Optimal control problems with quasi-linear PDEs of the form

$$\begin{aligned} -div(Ay\,\nabla y) + Fy &= u \text{ on } \Omega \\ y &= 0 \text{ on } \partial\Omega \end{aligned} \tag{3}$$

are considered in Casas et al. (1995) and Casas and Tröltzsch (2008). A and F are Nemytskii operators related to some Carathéodory functions. For the mapping F a monotonicity condition is assumed in both papers. The existence of a solution and necessary optimality conditions are proven.
Another important class of optimal control problems contains partial differential inclusions (PDIs) which can be described by multivalued mappings. As shown in Smirnov (2002), differential inclusions are often generated by differential equations with discontinuous right hand side. One of these situations is the following:

$$\begin{aligned} -\Delta_p y &\ni \lambda \partial F y \text{ on } \Omega \\ y &= 0 \text{ on } \partial\Omega \end{aligned} \tag{4}$$

Here ∂F denotes Clarke's subdifferential of a locally Lipschitz mapping F. In Dai and Liu (2009) it is shown that for some $\lambda > 0$ there exist multiple solutions of (4). Similar results for Neumann boundary conditions are obtained in Dai (2009). In Bauwe (2007) the sub-supersolution technique has been applied on stochastic PDIs including maximal-monotone mappings.
In the monographs Hu and Papageorgiou (1997) and Hu and Papageorgiou (2000) results for a large variety of PDIs and adjusted optimal control problems are proven. Nevertheless, strong boundedness conditions for nonlinear terms are assumed.
Also in the works of Lukaszewicz and Ton (1994) and Ton (1996) optimal control problems with constraints of PDIs haven been considered.

The present work includes examinations for partial differential equations and

inclusions with non-monotone and nonlinear terms and for the corresponding optimal control problems. The main focus lies on proving the existence of solutions for the state constrained optimal control problems.
In the first chapter the Laplace equation with a non-monotone nonlinearity described by a Nemytskii operator will be considered. Dirichlet boundary conditions are assumed. To prove the existence of at least one solution of the partial differential equation a boundedness condition for the nonlinear term on a so called sub-supersolution interval has to be assumed, see Carl et al. (2007). By truncation techniques a pseudomonotone auxiliary operator is obtained. Now the work of Brézis (1968) can be applied. The properties of the auxiliary operator are also useful in proving the existence of the corresponding optimal control problem, where the states have to be constrained on the interval on which the boundedness of the nonlinearity is warranted.
In Theorem 2.14 numerical results for problems in which the state constraints coincide with the sub-supersolution interval are stated. We examine the properties of problems which are discretized with respect to the state variable. These so called "first discretize - then optimize" methods converge under a regularization assumption. The crucial technique for proving the existence of a solution is solving the unconstrained auxiliary optimal control problem. Since the auxiliary problem yields only solutions in the admissible set of states, this constraint can be omitted.
General boundary conditions for elliptic equations are admitted in the following Chapter 3.
In Papageorgiou (1991), the existence of an optimal control-state pair for a Lagrange type optimal control problem is proven. The considered partial differential equation includes non-monotone, but bounded nonlinearities which are described by a Carathéodory function. The boundedness of the Carathéodory function is assumed on the whole real axis whereas in this work we will only assume boundedness on a restricted set. This weakened assumption is decisive for nonlinear terms as y^2 or $\alpha \exp(-y)(\int_\Omega \exp(-y)\, d\lambda_\Omega)^{-p}$ with $p > 2$ from the above example (1) in which the hypotheses of Papageorgiou (1991) are not satisfied. The property of pseudomonotonicity plays a key role in this work as well as in the paper Papageorgiou (1991).
Under a boundedness assumption for the nonlinear term on a given set of the states, the statements obtained for the previous optimal control problem can be adapted for results of a greater class of pointwise state-constrained optimal control problems. Moreover, for a restricted class of PDEs the assumption of a two-sided pointwise constraint can be substituted by a condition including an upper or a lower bound. In this case stronger hypotheses for the leading operator and the nonlinear term are supposed.

In Chapters 5 and 6 results for the parabolic case are derived in which general boundary conditions are admitted. A boundedness condition is again only assumed on a certain state-interval in contrast to the hypotheses in the works Papageorgiou (1993) and Halidias and Papageorgiou (2002).
In the last part of this thesis differential inclusions of the form

$$\mathcal{A}y + Gy + \mathcal{M}y \ni f \text{ in } V^\star \tag{5}$$

and corresponding optimal control problems are examined. Here, $\mathcal{A}$ denotes an operator satisfying the Leray-Lions conditions and G is some nonlinear mapping bounded on a given interval. The operator $\mathcal{M}$ forms a multivalued mapping which can be assumed to be, e.g., a subdifferential of some locally Lipschitz functional. A solution of (5) exists, if there is some so called selector $m \in \mathcal{M}$ with

$$\mathcal{A}y + Gy + my = f \text{ in } V^\star. \tag{6}$$

At first, the multivalued mapping satisfies a bounded global growth. Later, the multivalued mapping is formed by a subdifferential of some locally Lipschitz functional. For this case only a local growth condition is assumed.
The examinations for stationary inclusions are adapted to the evolutionary case in the ensuing section.
The appendix includes some basics referring to Sobolev theory and nonlinear and multivalued analysis.
Finally, some open questions are formulated in the conclusion.

We agree on the following denotations in the whole thesis: We assume that $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, is a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$. The Lebesgue measure on Ω is denoted by λ_Ω and λ_Γ is the corresponding boundary measure on Γ. For $L^p(\Omega, \lambda_\Omega)$ resp. $L^p(\Gamma, \lambda_\Gamma)$, $1 < p < \infty$, we write shortly $L^p(\Omega)$ resp. $L^p(\Gamma)$. The set $L^p_+(\Omega)$ contains all positive functions of $L^p(\Omega)$.
Let f_1 and f_2 be some λ_Ω-measurable functions. The notation $f_1(x) \leq f_2(x)$ $\lambda_\Omega(dx)$-a.e. is abbreviated by $f_1 \leq f_2$. The inclusion $f \in [f_1, f_2]$ means $f_1 \leq f \leq f_2$ in which f_1, f_2, f are λ_Ω-measurable functions on Ω with $f_1 \leq f_2$.
For appropriate mappings F, G, we write FG instead of $F \circ G$.
We denote the operator norm of A by $\|A\|_{Op}$ and the indicator function on the set M by I_M.

2 Optimal Control Problems for Semi-linear Elliptic PDEs with Dirichlet Boundary Conditions

2.1 Existence of Solutions for Optimal Control Problems

In this chapter we consider optimal control problems including a semi-linear PDE with a (possibly) non-monotone mapping. We concentrate on the Laplace equation with a non-monotone summand and Dirichlet boundary conditions. In later presentations these restrictions are weakened. The ideas of handling optimal control problems including nonlinear and non-monotone PDEs are demonstrated in detail in this section.
We agree on the abbreviations V and V_0 for the Sobolev spaces $W^{1,2}(\Omega)$ resp. $W_0^{1,2}(\Omega)$.

2.1.1 Motivation

For a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ and a sufficient smooth function $f\colon \Omega \to \mathbb{R}$ we consider the boundary value problem (BVP) in the strong formulation:

Problem 1. *Find some $y \in C^2(\Omega) \cap C(\bar{\Omega})$ such that*

$$\begin{aligned} -\Delta y + g \circ y &= f \ \textit{in } \Omega \\ y &= 0 \ \textit{on } \Gamma \end{aligned} \tag{BVP1}$$

Choose any test function $\phi \in C_0^\infty(\Omega)$. We assume that y solves (BVP1) and apply the integration by parts formula

$$\begin{aligned} \int_\Omega (-\Delta y + g \circ y)\, \phi \, d\lambda_\Omega = &- \int_\Gamma \partial_\nu y \, \phi \, d\lambda_\Gamma + \int_\Omega \nabla y \, \nabla \phi \, d\lambda_\Omega \\ &+ \int_\Omega (g \circ y)\, \phi \, d\lambda_\Omega, \end{aligned}$$

where ∂_ν denotes the normal derivative. From $\phi \in C_0^\infty(\Omega)$ we infer

$$\int_\Omega \nabla y \, \nabla \phi \, d\lambda_\Omega + \int_\Omega (g \circ y)\, \phi \, d\lambda_\Omega = \int_\Omega f \, \phi \, d\lambda_\Omega. \tag{7}$$

Varying ϕ in $C_0^\infty(\Omega)$ yields the weak representation. Since $C_0^\infty(\Omega)$ is dense in V_0, this formula can be regarded as an equation in the dual space ${V_0}^\star$. The considered technique leads to an operator equation which we are going to formulate in the next subsection.

2.1.2 Operator Equation

In the following we introduce the Laplace operator in the generalized form and the Nemytskii operator G referring to g.

Definition 2.1. *a) The Laplacian* $-\Delta\colon V_0 \to {V_0}^\star$, $y \mapsto -\Delta y$ *is given by*

$$\langle -\Delta y, \phi \rangle := \int_\Omega \nabla y \, \nabla \phi \, d\lambda_\Omega \qquad \text{for all } \phi \in V_0.$$

b) Let $g\colon \mathbb{R} \to \mathbb{R}$ *be continuous. The related Nemytskii operator* G *is induced by*

$$Gy := g \circ y.$$

Let now $f \in {V_0}^\star$. Denote by $\tau_{V_0,2}$ the compact embedding from V_0 into $L^2(\Omega)$ and by ${\tau_{V_0,2}}^\star$ its adjoint. The isomorphism from $L^2(\Omega)$ to $L^2(\Omega)^\star$ is denoted by i. Using the embedding operators, the term ${\tau_{V_0,2}}^\star i G \tau_{V_0,2}$ represents a mapping from V_0 to ${V_0}^\star$. With these notations, the formulation (7) of the boundary value problem can be written as:

Problem 2. *Find some* $y \in V_0$ *such that*

$$-\Delta y + ({\tau_{V_0,2}}^\star i G \tau_{V_0,2}) y = f \ \text{ in } {V_0}^\star \tag{BVP2}$$

To ensure that this equation is well defined, we have to assume $(G\tau_{V,2})y \in L^2(\Omega)$.

Definition 2.2 (solution)**.** *The function* $y \in V_0$ *is called a solution of Problem (BVP2) if* $(G\tau_{V_0,2})y \in L^2(\Omega)$ *and*

$$\int_\Omega \nabla y \, \nabla \phi \, d\lambda_\Omega + \langle ({\tau_{V_0,2}}^\star i G \tau_{V_0,2}) y, \phi \rangle = \langle f, \phi \rangle \text{ for all } \phi \in V_0.$$

Remark 2.1. *As stated in Zeidler (1990b), p. 1027, the embedding* $V_0 \hookrightarrow L^2(\Omega)$ *is compact. Hence the embedding operator* $\tau_{V_0,2}$ *and its adjoint* ${\tau_{V_0,2}}^\star$ *are strongly continuous (see Theorem VI.4.8.2 (Schauder) in Dunford and Schwartz (1957)). By Lemma 8.6 the isomorphism* $i\colon L^2(\Omega) \to L^2(\Omega)^\star$ *is weakly continuous. The combined strongly continuous mapping* ${\tau_{V_0,2}}^\star i\colon L^2(\Omega) \to {V_0}^\star$, $y \mapsto {\tau_{V_0,2}}^\star i y$ *is given by*

$$\langle {\tau_{V_0,2}}^\star i y, \phi \rangle = (y, \phi)_{L^2(\Omega)}.$$

Let the mapping $\gamma_2\colon V \to L^2(\Gamma)$ denote the trace operator. In order to guarantee a solution for (BVP2) we assume the existence of a sub- and a supersolution as defined in:

Definition 2.3 (subsolution). *The function $\underline{y} \in V$ is called a subsolution of Problem (BVP2) if $(G\tau_{V,2})\underline{y} \in L^2(\Omega)$, $\gamma_2\underline{y} \leq 0$ on Γ and*

$$\int_\Omega \nabla\underline{y}\,\nabla\phi\,d\lambda_\Omega + \langle(\tau_{V_0,2}{}^\star iG\tau_{V,2})\underline{y},\phi\rangle \leq \langle f,\phi\rangle \text{ for all } \phi \in V_0 \cap L^2_+(\Omega).$$

Definition 2.4 (supersolution). *The function $\overline{y} \in V$ is called a supersolution of Problem (BVP2) if $(G\tau_{V,2})\overline{y} \in L^2(\Omega)$, $\gamma_2\overline{y} \geq 0$ on Γ and*

$$\int_\Omega \nabla\overline{y}\,\nabla\phi\,d\lambda_\Omega + \langle(\tau_{V_0,2}{}^\star iG\tau_{V,2})\overline{y},\phi\rangle \geq \langle f,\phi\rangle \text{ for all } \phi \in V_0 \cap L^2_+(\Omega).$$

In the following lemma a certain kind of transitivity is shown for sub- and supersolutions.

Lemma 2.2. *Let $u_1 \in L^2(\Omega)$. Then it holds:*

a) Let the function $\underline{y} \in V$ be a subsolution of (BVP2) with right hand side $f = \tau_{V_0,2}{}^\star iu_1 \in V_0{}^\star$. We consider an arbitrary function $u \in L^2(\Omega)$ with $u_1 \leq u$ and the corresponding element $\tau_{V_0,2}{}^\star iu \in V_0{}^\star$. Then $\underline{y}$ is a subsolution of the boundary value problem (BVP2) with right hand side $\tau_{V_0,2}{}^\star iu$, too.

b) Let the function $\overline{y} \in V$ be a supersolution of (BVP2) with right hand side $f = \tau_{V_0,2}{}^\star iu_1 \in V_0{}^\star$. We consider an arbitrary function $u \in L^2(\Omega)$ with $u_1 \geq u$ and the corresponding element $\tau_{V_0,2}{}^\star iu \in V_0{}^\star$. Then $\overline{y}$ is a supersolution of the boundary value problem (BVP2) with right hand side $\tau_{V_0,2}{}^\star iu$, too.

Proof. a) From the definition of subsolution it follows for every ϕ in $V_0 \cap L^2_+(\Omega)$:

$$\begin{aligned}\int_\Omega \nabla\underline{y}\,\nabla\phi\,d\lambda_\Omega + \langle\tau_{V_0,2}{}^\star iG\tau_{V,2}\underline{y},\phi\rangle &\leq \langle\tau_{V_0,2}{}^\star iu_1,\phi\rangle = (u_1,\tau_{V_0,2}\phi)_{L^2(\Omega)} \\ &= \int_\Omega u_1\,\tau_{V_0,2}\phi\,d\lambda_\Omega \leq \int_\Omega u\,\tau_{V_0,2}\phi\,d\lambda_\Omega \\ &= (u,\tau_{V_0,2}\phi)_{L^2(\Omega)} = \langle\tau_{V_0,2}{}^\star iu,\phi\rangle.\end{aligned}$$

b) is along the lines of a). □

Remark 2.3. *i) Let $\mathcal{U}_{ad} := [\underline{u},\overline{u}] \cap L^2(\Omega)$ with $\underline{u}$, $\overline{u} \in L^2(\Omega)$, $\underline{u} \leq \overline{u}$ and assume that there exist*

- *a subsolution for (BVP2) with right hand side $\tau_{V,2}{}^\star i\underline{u}$ and*
- *a supersolution for (BVP2) with right hand side $\tau_{V,2}{}^\star i\overline{u}$.*

Applying Lemma 2.2 yields the existence of a sub- and a supersolution of (BVP2) for every $\tau_{V_0,2}{}^\star iu$ with $u \in \mathcal{U}_{ad}$.

ii) *Consider the mapping* $B\colon L^2(\Omega) \to L^2(\Omega)$, $u \mapsto Bu$. *We assume that there exist some* u_1, $u_2 \in [\underline{u}, \overline{u}] \cap L^2(\Omega)$ *with*

$$\begin{aligned} Bu_1 &\le Bu \quad \text{for all } u \in [\underline{u}, \overline{u}] \cap L^2(\Omega) \text{ and} \\ Bu_2 &\ge Bu \quad \text{for all } u \in [\underline{u}, \overline{u}] \cap L^2(\Omega). \end{aligned}$$

Moreover, suppose the existence of

- *a subsolution for (BVP2) with right hand side* $\tau_{V_0,2}{}^\star iBu_1$ *and*
- *a supersolution for (BVP2) with right hand side* $\tau_{V_0,2}{}^\star iBu_2$.

Applying Lemma 2.2 yields the existence of a sub- and a supersolution of (BVP2) for every $\tau_{V_0,2}{}^\star iBu$ *with* $u \in \mathcal{U}_{ad}$.

The following lemma is due to Carl et al. (2007).

Lemma 2.4. *Let* $\underline{y}$, $\overline{y} \in L^2(\Omega)$ *with* $\underline{y} \le \overline{y}$.

a) *The Laplacian* $-\Delta\colon V_0 \to V_0{}^\star$, $y \mapsto -\Delta y$ *is a continuous, bounded and strongly monotone mapping.*

b) *Assume that there exists some* $k_G \in L^2_+(\Omega)$ *with*

$$|Gy(x)| \le k_G(x) \ \lambda_\Omega(dx)\text{-a.e. for all } y \in [\underline{y}, \overline{y}] \cap L^2(\Omega). \tag{8}$$

Then the restricted operator $G\colon [\underline{y}, \overline{y}] \cap L^2(\Omega) \to L^2(\Omega)$ *is continuous.*

c) *The truncation operator* $T\colon L^2(\Omega) \to L^2(\Omega)$ *given by*

$$Ty(x) := \begin{cases} \overline{y}(x) & \text{if} \quad y(x) > \overline{y}(x) \\ y(x) & \text{if} \quad \underline{y}(x) \le y(x) \le \overline{y}(x) \\ \underline{y}(x) & \text{if} \quad y(x) < \underline{y}(x) \end{cases} \tag{9}$$

is continuous and bounded.

Proof. a): See Lemma 2.111 in Carl et al. (2007).
b): Choose any sequence $(y_m)_{m\in\mathbb{N}} \subseteq [\underline{y}, \overline{y}]$ with $y_m \to y_0$ in $L^2(\Omega)$. Obviously $y_0 \in [\underline{y}, \overline{y}]$. To show continuity of G assume that $(Gy_m)_{m\in\mathbb{N}}$ does not converge to Gy_0 in $L^2(\Omega)$. Given $\epsilon > 0$, choose a subsequence $(Gy_m)_{m\in M\subseteq\mathbb{N}}$ with

$$\|Gy_m - Gy_0\|_{L^2(\Omega)} > \epsilon \text{ for all } m \in M. \tag{10}$$

By the Riesz's Theorem there is a subsequence $(y_{m'})_{m\in M'\subseteq M}$ which converges λ_Ω-a.e.. The continuity of the Carathéodory function g implies that the sequence $(Gy_{m'} - Gy_0)_{m\in M}$ converges to 0 λ_Ω-a.e.. According assumption (8) it holds

$$|Gy_{m'}(x) - Gy_0(x)| \le 2k_G(x) \qquad \lambda_\Omega(dx)\text{-a.e.}.$$

With Lebesgue's Theorem we can infer

$$\|Gy_{m'} - Gy_0\|_{L^2(\Omega)} \to 0,$$

contradicting (10).
c): See Lemma 2.89 in Carl et al. (2007) resp. Lemma 1.22 in Heinonen et al. (1993). □

Remark 2.5. *Since the mappings $\tau_{V_0,2}$ and $\tau_{V_0,2}{}^\star i$ are strongly continuous (see Remark 2.1), $\tau_{V_0,2}{}^\star iG\tau_{V_0,2}\colon [\underline{y}, \overline{y}] \cap V_0 \to V_0{}^\star$ is strongly continuous, too.*

The following theorem refers to Theorem 3.4 in Carl et al. (2007).

Theorem 2.6. *Let $\underline{y}$ and $\overline{y}$ be a sub- and a supersolution of (BVP2) satisfying $\underline{y} \le \overline{y}$ and assume that there exists some $k_G \in L^2_+(\Omega)$ with*

$$|Gy(x)| \le k_G(x)\ \lambda_\Omega(dx)\text{-a.e. for all } y \in [\underline{y}, \overline{y}] \cap L^2(\Omega). \tag{11}$$

Then there exists at least one solution of (BVP2) which lies in $[\underline{y}, \overline{y}]$.

For proving Theorem 2.6 an auxiliary problem is introduced in Carl et al. (2007):
Find some $y \in V_0$ such that $(GT\tau_{V_0,2})y \in L^2(\Omega)$ and

$$-\Delta y + (\tau_{V_0,2}{}^\star iGT\tau_{V_0,2})y = f \quad \text{in } V_0{}^\star, \tag{A-BVP2}$$

where T is the truncation operator defined in Lemma 2.4.
It is shown that every solution of (A-BVP2) is a solution of (BVP2) under the assumption that $\underline{y}$ resp. $\overline{y}$ is a sub- resp. a supersolution.
For $f \in V_0{}^\star$ we denote the set of all solutions of (BVP2) within $[\underline{y}, \overline{y}]$ by $\mathcal{S}(f)$. Due to Theorem 3.10 in Carl et al. (2007) it holds:

Theorem 2.7. *$\mathcal{S}(f)$ is compact in V_0.*

2.1.3 Optimal Control Problem

For $\underline{y}$, $\overline{y} \in V$ and $\underline{u}$, $\overline{u} \in L^2(\Omega)$ with $\underline{y} \le \overline{y}$ and $\underline{u} \le \overline{u}$ let the mappings $R\colon [\underline{y}, \overline{y}] \cap L^2(\Omega) \to \mathbb{R}$ and $Q\colon [\underline{u}, \overline{u}] \cap L^2(\Omega) \to \mathbb{R}$ be convex and continuous. We assume an objective functional J of the form $J(y, u) := (R\tau_{V_0,2})y + Qu$ and introduce the abbreviation $\mathcal{U}_{ad} := [\underline{u}, \overline{u}] \cap L^2(\Omega)$. With these definitions we can formulate the following optimal control (OC) problem.

Problem 3.

$$\begin{aligned} \min\quad & J(y, u) \\ \textit{s.t.}\quad & -\Delta y + (\tau_{V_0,2}{}^\star iG\tau_{V_0,2})y = \tau_{V_0,2}{}^\star iu \quad \textit{in } V_0{}^\star \\ \textit{and}\quad & u \in [\underline{u}, \overline{u}] \cap L^2(\Omega) \\ & y \in V_0 \end{aligned}$$

Since the continuity of G is only guaranteed on $[\underline{y}, \overline{y}]$, we introduce an additional pointwise state constraint.

Problem 4.

$$\begin{aligned} \min \quad & J(y,u) \\ s.t. \quad & -\Delta y + (\tau_{V_0,2}{}^\star i G \tau_{V_0,2}) y = \tau_{V_0,2}{}^\star i u \quad in\ V_0{}^\star \\ and \quad & u \in [\underline{u}, \overline{u}] \cap L^2(\Omega) \\ & y \in [\underline{y}, \overline{y}] \cap V_0 \end{aligned} \tag{OC-BVP2}$$

Now, the main theorem about the existence of a solution for the nonlinear optimal control problem (OC-BVP2) is stated.

Theorem 2.8. *Let $\underline{y}$ be a subsolution of (BVP2) with $f = \tau_{V_0,2}{}^\star i\underline{u}$ and $\overline{y}$ be a supersolution of (BVP2) with $f = \tau_{V_0,2}{}^\star i\overline{u}$ and suppose $\underline{y} \leq \overline{y}$. We assume* (11). *Then the optimal control problem* (OC-BVP2) *has a solution* (y,u).

Proof. Since the assumptions of Theorem 2.6 are satisfied for every $u \in [\underline{u}, \overline{u}] \cap L^2(\Omega)$, there exists at least one $y \in [\underline{y}, \overline{y}]$ for every $u \in \mathcal{U}_{ad}$ which solves (BVP2). Let $(y_m, u_m)_{m\in\mathbb{N}}$ be the infimal sequence with

$$\lim_{m\to\infty} J(y_m, u_m) = \inf_{u\in\mathcal{U}_{ad}, y\in\mathcal{S}(\tau_{V_0,2}{}^\star iu)} J(y,u).$$

Since $L^2(\Omega)$ is reflexive and $[\underline{u}, \overline{u}] \cap L^2(\Omega)$ is nonempty, convex, bounded and closed in $L^2(\Omega)$, the set $[\underline{u}, \overline{u}] \cap L^2(\Omega)$ is weakly sequentially compact. Thus, there exist a weakly convergent subsequence $(u_m)_{m\in M}$, $M \subseteq \mathbb{N}$, and a weak limit $u_0 \in [\underline{u}, \overline{u}] \cap L^2(\Omega)$:

$$u_m \rightharpoonup u_0 \text{ in } L^2(\Omega). \tag{12}$$

By the strong continuity of $\tau_{V_0,2}{}^\star i$ we can infer

$$\tau_{V_0,2}{}^\star i u_m \to \tau_{V_0,2}{}^\star i u_0 \text{ in } V_0{}^\star. \tag{13}$$

By assumption, the sequence $(z_m)_{m\in M}$ with $z_m := (G\tau_{V_0,2})y_m$ is bounded in $L^2(\Omega)$. Hence there exist a weakly convergent subsequence $(z_m)_{m\in M}$ (w.l.o.g. $M = M'$) and a weak limit $z_0 \in L^2(\Omega)$. It follows $\tau_{V_0,2}{}^\star i z_m \to \tau_{V_0,2}{}^\star i z_0$ in $V_0{}^\star$. We consider the linearized boundary value problem

$$-\Delta y = Z_m := \tau_{V_0,2}{}^\star i(u_m - z_m) \text{ in } V_0{}^\star,$$

which is solved by $y = y_m$. Lax-Milgram's Theorem (see, e.g., Aufgabe V.6.18 in Alt (2006)) yields that the mapping $Z_m \mapsto y, V_0{}^\star \to V_0$ is linear and continuous.

Hence y_m converges to some limit y_0 in V_0 and with Lemma 2.4 we can observe that

$$-\Delta y_m \to -\Delta y_0 \text{ in } V_0^\star. \tag{14}$$

From the convergence of y_m to y_0 in V_0 we can infer

$$\tau_{V_0,2} y_m \to \tau_{V_0,2} y_0 \text{ in } L^2(\Omega). \tag{15}$$

Since $G\colon [\underline{y}, \overline{y}] \cap L^2(\Omega) \to L^2(\Omega)$ and $\tau_{V_0,2}{}^\star i$ are continuous mappings (see Lemma 2.4), we have

$$(\tau_{V_0,2}{}^\star i G \tau_{V_0,2}) y_m \to (\tau_{V_0,2}{}^\star i G \tau_{V_0,2}) y_0 \text{ in } V_0^\star. \tag{16}$$

By (13)

$$-\Delta y_m + (\tau_{V_0,2}{}^\star i G \tau_{V_0,2}) y_m = \tau_{V_0,2}{}^\star i u_m \to \tau_{V_0,2}{}^\star i u_0 \text{ in } V_0^\star.$$

The convergence results in (14) and (16) imply

$$-\Delta y_m + (\tau_{V_0,2}{}^\star i G \tau_{V_0,2}) y_m \to -\Delta y_0 + (\tau_{V_0,2}{}^\star i G \tau_{V_0,2}) y_0 \text{ in } V_0^\star.$$

We obtain the equality $-\Delta y_0 + (\tau_{V_0,2}{}^\star i G \tau_{V_0,2}) y_0 = \tau_{V_0,2}{}^\star i u_0$.
Because Q is convex and continuous, it is lower semi-continuous and the level set $\{u \in L^2(\Omega) \,|\, Qu \leq \lambda\}$ is convex and closed for any $\lambda \in \mathbb{R}$. For convex sets it holds closedness if and only if the set is weakly closed, see Theorem 3.12 in Rudin (1991). This implies the weakly lower semi-continuity of Q. Therefore by (12) and (15) we derive

$$\begin{aligned}\lim_{m\to\infty} J(y_m, u_m) &= \lim_{m\to\infty} (R\tau_{V_0,2}) y_m + \lim_{m\to\infty} Q u_m = (R\tau_{V_0,2}) y_0 + \liminf_{m\to\infty} Q u_m \\ &\geq (R\tau_{V_0,2}) y_0 + Q u_0 = J(y_0, u_0)\end{aligned}$$

and conclude that the pair (y_0, u_0) solves Problem (OC-BVP2). □

2.2 Examples

In Tröltzsch (2009) hypothesis 4.14, including boundedness, locally Lipschitz and monotonicity conditions, is assumed for the existence of a solution for a semi-linear optimal control problem (see Theorem 4.15 in Tröltzsch (2009) with Neumann boundary conditions). Only the case of monotone, locally Lipschitz functions g is considered. For this kind of problems the corresponding boundary value problem is uniquely solvable. The following examples show two cases for which Theorem 4.15 Tröltzsch (2009) is not applicable.

Example 2.9.

$$\begin{aligned}
\min \quad & \int_\Omega y\, d\lambda_\Omega + \int_\Omega u\, d\lambda_\Omega \\
s.t. \quad & -\Delta y + \sqrt{y} = u \ in\ [0,1] \\
and \quad & 0 \le u \le 1, \qquad \underline{y} \le y \le \overline{y} \\
& y(0) = y(1) = 0
\end{aligned}$$

The mapping $\sqrt{\cdot}$ *is not locally Lipschitz around 0. Choose the subsolution* $\underline{y} = 0$ *and the supersolution* $\overline{y} = 1$*. The solution is obviously* $u_0 = y_0 = 0$*.*

Example 2.10.

$$\begin{aligned}
\min \quad & \int_\Omega |y - y_\Omega|^2\, d\lambda_\Omega + \int_\Omega |u|^2\, d\lambda_\Omega \qquad with\ y_\Omega(\cdot) = \sin(\cdot) \\
s.t. \quad & -\Delta y + y^2 - y - y_\Omega^2 = u \ in\ [-\pi,\pi] \\
and \quad & 0 \le u \le 1, \qquad \underline{y} \le y \le \overline{y} \\
& y(-\pi) = y(\pi) = 0
\end{aligned}$$

Choose the subsolution $\underline{y} = y_\Omega$ *and the supersolution* $\overline{y} = 10$*. Since* $\underline{y}, \overline{y} \in C^2([-\pi,\pi])$ *we can use the strong formulation:*

$$\begin{aligned}
&-\Delta\underline{y} + \underline{y}^2 - \underline{y} - y_\Omega^2 = \sin(\cdot) + \sin^2(\cdot) - \sin(\cdot) - \sin^2(\cdot) = 0 \\
&-\Delta\overline{y} + \overline{y}^2 - \overline{y} - y_\Omega^2 = 90 - \sin^2(\cdot) > 1
\end{aligned}$$

The obvious solution is given by $u_0 = 0$ *and* $y_0 = y_\Omega$ *even though* $y_1 = -y_\Omega$ *is a solution of the partial differential equation with right hand side* $u_0 = 0$*, too.*

2.3 Approximation of Solutions for Optimal Control Problems

We are interested in the numerical approximation of a solution of the optimal control problem (OC-BVP2). Direct methods, also known as "first discretize - then optimize" techniques, are based on the transcription of the original optimal control problem by a sequence of problems in a finite dimensional space. The optimization is made for the finite dimensional problem. The aim of the present section is to show the convergence of the sequence of solutions of semi-discretized optimal control problems to one solution of the original problem. The proof is based on the techniques used in Theorem 2.8 and Galerkin convergence arguments.

Let $\underline{y}$ be a subsolution of (BVP2) with $f = \tau_{V_0,2}{}^\star i\underline{u}$ and $\overline{y}$ be a supersolution of

(BVP2) with $f = \tau_{V_0,2}{}^\star i\overline{u}$. We denote with (y_0, u_0) a solution of (OC-BVP2) and with $\{w_1, w_2, \ldots\}$ a basis of V_0. For $n \in \mathbb{N}$ we set

$$V_0{}^n := \operatorname{lin}\{w_1, \ldots, w_n\}, \tag{17}$$

$$P_n\colon V_0 \to V_0 \text{ the orthogonal projection onto } V_0{}^n, \tag{18}$$

$$\tau_{V_0{}^n,V_0}\colon V_0{}^n \to V_0 \text{ the linear continuous embedding of } V_0{}^n \text{ into } V_0 \text{ and} \tag{19}$$

$$\tau_{V_0{}^n,V_0}{}^\star\colon V_0{}^\star \to V_0{}^{n\star} \text{ its adjoint.} \tag{20}$$

Since $-\Delta\colon V_0 \to V_0{}^\star$ is strongly monotone (see Lemma 2.4), the mapping

$$-\Delta_n\colon V_0{}^n \to V_0{}^{n\star},\ y \mapsto -\tau_{V_0{}^n,V_0}{}^\star \Delta \tau_{V_0{}^n,V_0} y$$

is strongly monotone as well. For all $y \in V_0{}^n$ it holds

$$\langle -\Delta_n y, y\rangle = \langle -\Delta \tau_{V_0{}^n,V_0} y, \tau_{V_0{}^n,V_0} y\rangle \geq c\|\tau_{V_0{}^n,V_0} y\|_{V_0}^2 \ = c\|\tau_{V_0{}^n,V_0}\|_{Op}^2 \,\|y\|_{V_0{}^n}^2 .$$

In the following we write $\tau_{V_0{}^n,2}$ for the combined mapping $\tau_{V_0,2} \circ \tau_{V_0{}^n,V_0}$. Observe that $\tau_{V_0{}^n,V_0}{}^\star \circ \tau_{V_0,2}{}^\star = \tau_{V_0{}^n,2}{}^\star$ and $\langle \tau_{V_0,2}{}^\star iy, w\rangle = \langle \tau_{V_0{}^n,2}{}^\star iy, w\rangle$ for all $y \in L^2(\Omega)$, $w \in V_0{}^n$.
In the proof of Theorem 2.6 it was shown that every solution of the auxiliary problem (A-BVP2) is a solution of (BVP2) under the assumption that $\underline{y}$ resp. $\overline{y}$ is a sub- resp. a supersolution. For Problem (A-BVP2) the theory of pseudomonotone operators can be applied. Therefore, it suggests itself to use the Galerkin method for approximating the solution of the unconstrained problem (A-BVP2) and not for (BVP2) itself.

Problem 5. *Find some $y \in V_0{}^n$ such that*

$$\langle -\Delta_n y + (\tau_{V_0{}^n,2}{}^\star iGT\tau_{V_0{}^n,2})y, w_i\rangle = \langle f, w_i\rangle \textit{ for all } i = 1, \ldots, n \quad y \in V_0{}^n \tag{BVP2-n}$$

We denote the set of all solutions of (BVP2-n) with $\mathcal{S}_n(f)$ and remark that these solutions do not necessarily lie in $[\underline{y}, \overline{y}]$.

Theorem 2.11. *Assume* (11). *There exists at least one solution of (BVP2-n).*

Proof. As shown in Carl et al. (2007) the operator $-\Delta_n + \tau_{V_0{}^n,2}{}^\star iGT\tau_{V_0{}^n,2}$ defined on $V_0{}^n$ is pseudomonotone, bounded and coercive. Hence we can apply Theorem 27.A (b) in Zeidler (1990b). □

Regularity Assumption 2.12. *There exists a subsequence $(y_n)_{n\in M}$, $M \subseteq \mathbb{N}$, with $y_n \in \mathcal{S}_n(\tau_{V_0,2}{}^\star iu_0)$, $n \in M$, and $\tau_{V_0,2} y_n \to \tau_{V_0,2} y_0$ in $L^2(\Omega)$.*

For every $n \in \mathbb{N}$ we now define the optimal control problem in which y varies over $V_0{}^n$. The objective functional J is defined as before.

Problem 6.

$$\begin{aligned} &\min && J(y,u) && \text{(OC-BVP2-n)}\\ &\textit{s.t.} && \langle -\Delta_n y + (\tau_{{V_0}^n,2}{}^\star iGT\tau_{{V_0}^n,2})y, w_i\rangle = \langle \tau_{{V_0}^n,2}{}^\star iu, w_i\rangle \textit{ for all } i=1,\dots,n\\ &\textit{and} && u \in [\underline{u},\overline{u}] \cap L^2(\Omega)\\ & && y \in {V_0}^n \end{aligned}$$

Theorem 2.13. *Assume* (11)*. Then the optimal control problem* (OC-BVP2-n) *has a solution* (y,u)*.*

Proof. The same arguments as in Theorem 2.8 stay valid. For any infimal sequence $(y_n)_{n\in\mathbb{N}}$ the sequence $((GT\tau_{{V_0}^n,2})y_n)_{n\in\mathbb{N}}$ is bounded due to assumption (11). □

Theorem 2.14 (Approximation Theorem)**.** *Assume* (11) *and suppose Regularity Assumption 2.12. Let the pair* (y_n,u_n) *be the solution of (OC-BVP2-n) for any* $n\in\mathbb{N}$*. Then there exists a subsequence* $(y_n,u_n)_{n\in M\subseteq\mathbb{N}}$ *which converges to some* $(\tilde{y}_0,\tilde{u}_0)$ *in* $V_0\times(L^2(\Omega))_w$*, where* $(L^2(\Omega))_w$ *denotes the space* $L^2(\Omega)$ *equipped with the weak topology. The pair* $(\tilde{y}_0,\tilde{u}_0)$ *solves (OC-BVP2).*

Proof. Let (y_0,u_0) be a solution of Problem (OC-BVP2) and $y_{0,n}$ be a solution of (BVP2-n) with right hand side $\tau_{{V_0}^n,2}{}^\star iu_0$ for any $n\in\mathbb{N}$. Due to Assumption 2.12 there exists a weakly convergent subsequence $(y_{0,n})_{n\in M\subseteq\mathbb{N}}$ such that $y_{0,n}$ solves (BVP2-n) referring to $f=\tau_{{V_0}^n,2}iu_0$. Choose a sequence $(y_n,u_n)_{n\in\mathbb{N}}$, where (y_n,u_n) is a solution of (OC-BVP2-n) for any $n\in\mathbb{N}$. Then it holds

$$J(y_n,u_n)\le J(y_{0,n},u_0).$$

The continuity of R implies

$$\lim_{n\to\infty} J(y_{0,n},u_0) = (R\tau_{V_0,2})y_0 + Qu_0 = J(y_0,u_0).$$

Now we consider the associated subsequence $(y_n,u_n)_{n\in M}$. As shown in the proof of Theorem 2.8 we obtain a subsequence $(u_n)_{n\in M}$ (w.l.o.g. $M'=M$) and a function $\tilde{u}_0\in[\underline{u},\overline{u}]\cap L^2(\Omega)$ with

$$u_n \rightharpoonup \tilde{u}_0 \text{ in } L^2(\Omega). \tag{21}$$

The following arguments are similar to those in the proof of Theorem 2.8.
Because the sequence $(z_n)_{n\in M}$ with $z_n := (GT\tau_{V_0,2})y_n$ is bounded in $L^2(\Omega)$ (assumption (11)), there exist a weakly convergent subsequence $(z_n)_{n\in M}$ and a corresponding weak limit $z_0\in L^2(\Omega)$. It holds $\tau_{V_0,2}{}^\star i(u_n-z_n)\to\tau_{V_0,2}{}^\star i(\tilde{u}_0-z_0)$ in ${V_0}^\star$. We consider the linearized Galerkin equations

$$\langle -\Delta y, w_i\rangle = \langle -\Delta_n y, w_i\rangle = \langle \tau_{{V_0}^n,2}{}^\star i(u_n-z_n), w_i\rangle$$

for all $i = 1, \dots, n$ and $y \in {V_0}^n$, which are solved by $y = y_n$. Now we can apply Lemma 8.8, since $-\Delta$ is strongly monotone (see Lemma 2.4). There exists some $\tilde{y}_0 \in V_0$ with

$$y_n \rightarrow \tilde{y}_0 \text{ in } V_0. \tag{22}$$

The mapping ${\tau_{V_0,2}}^\star iGT\tau_{V_0,2}\colon V_0 \rightarrow {V_0}^\star$ is strongly continuous (see Lemma 2.4), hence it is

$$({\tau_{V_0,2}}^\star iGT\tau_{V_0,2})y_n \rightarrow ({\tau_{V_0,2}}^\star iGT\tau_{V_0,2})\tilde{y}_0 \text{ in } {V_0}^\star. \tag{23}$$

Choose any $\phi \in V_0$ and fix some $n_0 \in \mathbb{N}$. Then we have

$$\langle -\Delta y_n + ({\tau_{V_0,2}}^\star iGT\tau_{V_0,2})y_n, P_{n_0}\phi\rangle = \langle {\tau_{V_0,2}}^\star iu_n, P_{n_0}\phi\rangle \rightarrow \langle {\tau_{V_0,2}}^\star i\tilde{u}_0, P_{n_0}\phi\rangle.$$

With Lemma 2.4 and (23) we get:

$$\langle -\Delta y_n + ({\tau_{V_0,2}}^\star iGT\tau_{V_0,2})y_n, P_{n_0}\phi\rangle \rightarrow \langle -\Delta \tilde{y}_0 + ({\tau_{V_0,2}}^\star iGT\tau_{V_0,2})\tilde{y}_0, P_{n_0}\phi\rangle.$$

We obtain

$$\langle -\Delta \tilde{y}_0 + ({\tau_{V_0,2}}^\star iGT\tau_{V_0,2})\tilde{y}_0, P_{n_0}\phi\rangle = \langle {\tau_{V_0,2}}^\star i\tilde{u}_0, P_{n_0}\phi\rangle \text{ for all } n_0 \in \mathbb{N}.$$

With $P_n\phi \rightarrow \phi$ in V_0 we have

$$\langle -\Delta \tilde{y}_0 + ({\tau_{V_0,2}}^\star iGT\tau_{V_0,2})\tilde{y}_0, P_n\phi\rangle \rightarrow \langle -\Delta \tilde{y}_0 + ({\tau_{V_0,2}}^\star iGT\tau_{V_0,2})\tilde{y}_0, \phi\rangle$$
$$\text{and } \langle {\tau_{V_0,2}}^\star i\tilde{u}_0, P_n\phi\rangle \rightarrow \langle {\tau_{V_0,2}}^\star i\tilde{u}_0, \phi\rangle$$

and therefore

$$-\Delta \tilde{y}_0 + ({\tau_{V_0,2}}^\star iGT\tau_{V_0,2})\tilde{y}_0 = {\tau_{V_0,2}}^\star i\tilde{u}_0 \text{ in } {V_0}^\star.$$

Together with (21) and (22) it holds:

$$J(\tilde{y}_0, \tilde{u}_0) \leq \varliminf_{n\rightarrow\infty} J(y_n, u_n) \leq \varliminf_{n\rightarrow\infty} J(y_{0,n}, u_0) = J(y_0, u_0).$$

□

Remark 2.15. *In general we can not show that $\tilde{y}_0$ resp. $\tilde{u}_0$ and y_0 resp. u_0 are equal. The same situation appears for the boundary value problem itself. For the Galerkin approximation of the solution of any pseudomonotone, bounded and coercive operator equation, we only can show the weak convergence of a subsequence to one solution (see Theorem 27.A in Zeidler (1990b)).*

3 Optimal Control Problems for Semi-linear Elliptic PDEs with General Boundary Conditions

3.1 Existence of Solutions for Optimal Control Problems

3.1.1 Motivation

Let Γ_1 and Γ_2 be a disjoint decomposition of the Lipschitz boundary Γ. Denote

$$V := \{y \in W^{1,2}(\Omega) | \, \gamma_2 y = 0 \text{ on } \Gamma_2\} \subseteq W^{1,2}(\Omega),$$

where the topology is induced by $W^{1,2}(\Omega)$ and V is closed, see Lemma 8.9. It holds

$$W_0^{1,2}(\Omega) \subseteq V \subseteq W^{1,2}(\Omega).$$

The embeddings $\tau_{W^{1,2}(\Omega),2}$ and $\tau_{V,2}$ of $W^{1,2}(\Omega)$ resp. V in $L^2(\Omega)$ are compact, since the embedding of $W^{1,2}(\Omega)$ in $L^2(\Omega)$ is compact, see Lemma 8.1.
For a Carathéodory function $g \colon \Omega \times \mathbb{R} \to \mathbb{R}$ and a sufficient smooth function $f \colon \Omega \to \mathbb{R}$ we consider a mixed boundary value problem in the strong formulation:

Problem 7. *Find some $y \in C^2(\Omega) \cap C(\bar{\Omega})$ such that*

$$\begin{aligned} -\Delta y + g \circ (id, y) &= f \text{ in } \Omega \\ \partial_\nu y + y &= 0 \text{ on } \Gamma_1 \\ y &= 0 \text{ on } \Gamma_2 \end{aligned} \tag{BVP3}$$

The mapping $id \colon \Omega \to \Omega$, $x \mapsto x$ denotes the identity.
Choose any $\phi \in C^\infty(\Omega)$ with $\phi = 0$ on Γ_2. We assume that y solves Problem (BVP3), apply the integration by parts formula and use that $\partial_\nu y + y = 0$ on Γ_1:

$$\begin{aligned} \int_\Omega (-\Delta y + g \circ (id, y)) \, \phi \, d\lambda_\Omega &= -\int_{\Gamma_1} \partial_\nu y \, \phi \, d\lambda_\Gamma - \int_{\Gamma_2} \partial_\nu y \, \phi \, d\lambda_\Gamma \\ &\quad + \int_\Omega \nabla y \, \nabla \phi \, d\lambda_\Omega + \int_\Omega (g \circ (id, y)) \, \phi \, d\lambda_\Omega. \\ &= -\int_{\Gamma_1} \partial_\nu y \, \phi \, d\lambda_\Gamma + \int_\Omega \nabla y \, \nabla \phi \, d\lambda_\Omega \\ &\quad + \int_\Omega (g \circ (id, y)) \, \phi \, d\lambda_\Omega + \int_{\Gamma_2} y \, \phi \, d\lambda_\Gamma. \end{aligned}$$

Since $\phi = 0$ on Γ_2 we obtain the following condition:

$$\int_\Omega \nabla y \, \nabla \phi \, d\lambda_\Omega + \int_\Omega (g \circ (id, y)) \, \phi \, d\lambda_\Omega + \int_\Gamma y \, \phi \, d\lambda_\Gamma = \int_\Omega f \, \phi \, d\lambda_\Omega. \tag{24}$$

3.1.2 Operator Equation

We assume that V is some arbitrary closed subset of the space $W^{1,2}(\Omega)$ satisfying $W_0^{1,2}(\Omega) \subseteq V \subseteq W^{1,2}(\Omega)$. For $f \in V^\star$ and the operators $\mathcal{A}\colon V \to V^\star$, $y \mapsto \mathcal{A}y$ and G, specified below, we consider the following nonlinear operator equation:

Problem 8. *Find some $y \in V$ such that*

$$\mathcal{A}y + ({\tau_{V,2}}^\star i G \tau_{V,2})y = f \text{ in } V^\star \tag{BVP4}$$

Definition 3.1 (solution)**.** *The function $y \in V$ is called solution of (BVP4) if $(G\tau_{V,2})y \in L^2(\Omega)$ and*

$$\langle \mathcal{A}y + ({\tau_{V,2}}^\star i G \tau_{V,2})y, \phi\rangle = \langle f, \phi\rangle \text{ for all } \phi \in V.$$

To derive existence results for Problem (BVP4), we formulate the following conditions:

(H$\mathcal{A}$) There exists some symmetric bilinear form

$$a\colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{R},\ (y,z) \mapsto a(y,z)$$

with $a(y, y^+) = a(y^+, y^+)$ for all $y \in W^{1,2}(\Omega)$ such that the operator

$$\mathcal{A}\colon V \to V^\star,\ y \mapsto a(y,\cdot)$$

is continuous and strongly monotone.

(HG) There exists some Carathéodory function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ such that the operator G is the corresponding Nemytskii operator given by

$$Gy\,(x) = g(x, y(x)).$$

We remark that for $y \in W^{1,2}(\Omega)$ it follows $y^+ \in W^{1,2}(\Omega)$, see Lemma 1.19 in Heinonen et al. (1993).

Example 3.1. *Let $A = (A_{ij})_{i,j=1,\ldots N}$ be a symmetric matrix with components in $L^\infty(\Omega)$ and assume that A is uniformly positive definite, i.e. there exists some positive $\alpha_0 \in \mathbb{R}$:*

$$\zeta' A(x)\zeta \geq \alpha_0 \left|\zeta\right|^2 \ \lambda_\Omega(dx)\text{-a.e. for all } \zeta \in \mathbb{R}^N, \tag{25}$$

i.e. the uniform ellipticity condition (compare, e.g., Tröltzsch (2009) p.30) is satisfied:

$$\sum_{i,j=1}^{N} A_{ij}(x)\zeta_i\zeta_j \geq \alpha_0 \left|\zeta\right|^2 \ \lambda_\Omega(dx)\text{-a.e. for all } \zeta \in \mathbb{R}^N.$$

Let Γ_1 and Γ_2 be a disjoint decomposition of the Lipschitz boundary Γ and choose $V := \{y \in W^{1,2}(\Omega) |\, \gamma_2 y = 0 \text{ on } \Gamma_2\}$. A general linear elliptic differential operator $\mathcal{A}\colon V \to V^\star$, $y \mapsto \mathcal{A}y$ is given by

$$\langle \mathcal{A}y, \phi\rangle := \int_\Omega \nabla y'\, A\, \nabla\phi\, d\lambda_\Omega + \int_\Omega c_0\, y\, \phi\, d\lambda_\Omega + \int_\Gamma \alpha\, \gamma_2 y\, \gamma_2\phi\, d\lambda_\Gamma \quad \phi \in V, \tag{26}$$

where $c_0 \in L^\infty(\Omega)$, $c_0 \geq 0$ on Ω, $\alpha \in L^\infty(\Gamma)$ and $\alpha \geq 0$ on Γ_2 and one of the following conditions is satisfied:

i) $\lambda_\Gamma(\Gamma_2) > 0$

ii) $\int_\Omega c_0^2\, d\lambda_\Omega + \int_\Gamma \alpha^2\, d\lambda_\Gamma > 0$

Analogously to Satz 2.7 in Tröltzsch (2009) we obtain that the general linear elliptic differential operator $\mathcal{A}$ is continuous and strongly monotone. The symmetry of the corresponding bilinear form is obvious. Moreover, it holds $a(y, y^+) = a(y^+, y^+)$ for all $y \in W^{1,2}(\Omega)$.

Proof. A is symmetric and uniformly positive definite. Thus there exists some matrix $B = (B)_{i,j=1,\ldots N}$ with components in $L^\infty(\Omega)$ and

$$A = B'B.$$

We show at first that $\mathcal{A}$ is continuous. Since the trace operator $\gamma_2\colon W^{1,2}(\Omega) \to L^2(\Gamma)$, $y \mapsto \gamma_2 y$ is bounded (see Lemma 8.3), it holds:

$$\begin{aligned}
|\langle \mathcal{A}y, \phi\rangle| &\leq \left|\int_\Omega \nabla y'\, A\, \nabla\phi\, d\lambda_\Omega\right| + \left|\int_\Omega c_0\, y\, \phi\, d\lambda_\Omega\right| + \left|\int_\Gamma \alpha\, \gamma_2 y\, \gamma_2\phi\, d\lambda_\Gamma\right| \\
&\leq \left|\int_\Omega (B\,\nabla y)'\,(B\nabla\phi)\, d\lambda_\Omega\right| + \|c_0\|_{L^\infty(\Omega)}\, \|y\|_{L^2(\Omega)}\, \|\phi\|_{L^2(\Omega)} \\
&\quad + \|\alpha\|_{L^\infty(\Gamma)}\, \|\gamma_2 y\|_{L^2(\Gamma)}\, \|\gamma_2\phi\|_{L^2(\Gamma)} \\
&\leq \|B\|^2_{L^\infty(\Omega)}\, \|y\|_{W^{1,2}(\Omega)}\, \|\phi\|_{W^{1,2}(\Omega)} \\
&\quad + \|c_0\|_{L^\infty(\Omega)}\, \|y\|_{W^{1,2}(\Omega)}\, \|\phi\|_{W^{1,2}(\Omega)} \\
&\quad + \|\alpha\|_{L^\infty(\Gamma)}\, \|\gamma_2\|^2_{Op}\, \|y\|_{W^{1,2}(\Omega)}\, \|\phi\|_{W^{1,2}(\Omega)} \\
&\leq c\, \|y\|_{W^{1,2}(\Omega)}\, \|\phi\|_{W^{1,2}(\Omega)}\,,
\end{aligned}$$

where

$$c := \|B\|^2_{L^\infty(\Omega)} + \|c_0\|_{L^\infty(\Omega)} + \|\alpha\|_{L^\infty(\Gamma)}\, \|\gamma_2\|^2_{Op}$$

and $\|B\|_{L^\infty(\Omega)} := \max_{i,j=1,\ldots,N} \|B_{i,j}\|_{L^\infty(\Omega)}$. Therefore, we get

$$\begin{aligned}\|\mathcal{A}\|_{Op} &= \sup_{y\in V} \frac{\|\mathcal{A}y\|_{W^{1,2}(\Omega)^\star}}{\|y\|_{W^{1,2}(\Omega)}} = \sup_{y\in V}\sup_{\phi\in V} \frac{|\langle \mathcal{A}y, \phi\rangle|}{\|y\|_{W^{1,2}(\Omega)}\|\phi\|_{W^{1,2}(\Omega)}} \\ &\leq \sup_{y\in V}\sup_{\phi\in V} \frac{c\,\|y\|_{W^{1,2}(\Omega)}\|\phi\|_{W^{1,2}(\Omega)}}{\|y\|_{W^{1,2}(\Omega)}\|\phi\|_{W^{1,2}(\Omega)}} \leq c.\end{aligned}$$

Now we show that $\mathcal{A}$ is strongly monotone. For the next argumentation we follow Tröltzsch (2009), p.29. Since i) or ii) holds and

$$\|y\|_\sim^2 := \|\nabla y\|_{L^2(\Omega)}^2 + \|\gamma_2 y\|_{L^2(\Gamma)}^2$$

defines an equivalent norm on $W^{1,2}(\Omega)$ (see Carl et al. (2007) p.39), there exists some $c > 0$ so that

$$\begin{aligned}\langle \mathcal{A}y, y\rangle &= \int_\Omega \nabla y' A \nabla y \, d\lambda_\Omega + \int_\Omega c_0\, y^2 \, d\lambda_\Omega + \int_\Gamma \alpha\, \gamma_2 y^2 \, d\lambda_\Gamma \\ &\geq \alpha_0 \int_\Omega |\nabla y|^2 \, d\lambda_\Omega + \int_\Omega c_0\, y^2 \, d\lambda_\Omega + \int_\Gamma \alpha\, \gamma_2 y^2 \, d\lambda_\Gamma \\ &\geq c\,\|y\|_{W^{1,2}(\Omega)}^2 .\end{aligned}$$

We applied (25). For the derivation of $c > 0$ see p. 29, 30 in Tröltzsch (2009). Since $\mathcal{A}$ is linear, the last inequality is equivalent to the property of strong monotonicity of $\mathcal{A}$. Moreover, by $I_{\{y>0\}}\nabla y = \nabla y^+$ and $(\gamma_2 y)^+ = \gamma_2 y^+$ (see Example 2.86 in Carl et al. (2007) or Theorem 2.8 in Chipot (2000)) it holds

$$\begin{aligned}a(y, y^+) &= \int_\Omega \nabla y'\, A\, \nabla y^+ \, d\lambda_\Omega + \int_\Omega c_0\, y\, y^+ \, d\lambda_\Omega + \int_\Gamma \alpha\, y\, y^+ \, d\lambda_\Gamma \\ &= \int_{\{y>0\}} \nabla y'\, A\, \nabla y^+ \, d\lambda_\Omega + \int_{\{y>0\}} c_0\, y\, y^+ \, d\lambda_\Omega \\ &\quad + \int_{\{\gamma_2 y>0\}} \alpha\, \gamma_2 y\, \gamma_2 y^+ \, d\lambda_\Gamma \\ &= \int_{\{y>0\}} \nabla y^{+\prime}\, A\, \nabla y^+ \, d\lambda_\Omega + \int_{\{y>0\}} c_0\, y^+\, y^+ \, d\lambda_\Omega \\ &\quad + \int_{\{\gamma_2 y>0\}} \alpha\, \gamma_2 y^+\, \gamma_2 y^+ \, d\lambda_\Gamma \\ &= \int_\Omega \nabla y^{+\prime}\, A\, \nabla y^+ \, d\lambda_\Omega + \int_\Omega c_0\, y^+\, y^+ \, d\lambda_\Omega + \int_\Gamma \alpha\, \gamma_2 y^+\, \gamma_2 y^+ \, d\lambda_\Gamma \\ &= a(y^+, y^+) \text{ for all } y \in W^{1,2}(\Omega).\end{aligned}$$

□

The definitions of sub- and supersolution follow analogously to the previous section.

Definition 3.2 (subsolution)**.** *The function* $\underline{y} \in W^{1,2}(\Omega)$ *is called subsolution of (BVP4) if* $(G\tau_{W^{1,2}(\Omega),2})\underline{y} \in L^2(\Omega)$, $(\underline{y} - y)^+ \in V$ *for all* $y \in V$ *and*

$$a(\underline{y}, \phi) + \langle(\tau_{V,2}{}^{\star} iG\tau_{W^{1,2}(\Omega),2})\underline{y}, \phi\rangle \leq \langle f, \phi\rangle \text{ for all } \phi \in V \cap L^2_+(\Omega). \tag{27}$$

Definition 3.3 (supersolution)**.** *The function* $\overline{y} \in W^{1,2}(\Omega)$ *is called supersolution of (BVP4) if* $(G\tau_{W^{1,2}(\Omega),2})\overline{y} \in L^2(\Omega)$, $(y - \overline{y})^+ \in V$ *for all* $y \in V$ *and*

$$a(\overline{y}, \phi) + \langle(\tau_{V,2}{}^{\star} iG\tau_{W^{1,2}(\Omega),2})\overline{y}, \phi\rangle \geq \langle f, \phi\rangle \text{ for all } \phi \in V \cap L^2_+(\Omega). \tag{28}$$

Remark 3.2. *The condition "*$(y - \overline{y})^+ \in V$ *for all* $y \in V$*" replaces the condition "*$\gamma_2\overline{y} \geq 0$ *on* Γ*" in the case of Dirichlet boundary conditions.*
Let $V := \{y \in W^{1,2}(\Omega) |\, \gamma_2 y = 0 \text{ on } \Gamma_2\}$ *be chosen as in Section 3.1.1, where* $\Gamma_1 \cup \Gamma_2$ *is a disjoint decomposition of the Lipschitz boundary* Γ*. By* $\overline{y}, y \in W^{1,2}(\Omega)$ *we have* $(y - \overline{y})^+ \in W^{1,2}(\Omega)$*, see Lemma 1.19 in Heinonen et al. (1993). Moreover,* $\gamma_2\overline{y} \geq 0$ *and* $\gamma_2 y = 0$ *on* Γ_2 *imply that* $\gamma_2(y - \overline{y}) \leq 0$ *on* Γ_2*. Hence it holds* $\gamma_2(y - \overline{y})^+ = 0$ *on* Γ_2 *and therefore* $(y - \overline{y})^+ \in V$*.*

The following theorem states an existence result for Problem (BVP4) and is a slight generalization of Theorem 3.4 in Carl et al. (2007). In the proof arguments of Carl et al. (2007) and Bauwe (2007) are applied.

Theorem 3.3. *Assume* (H$\mathcal{A}$) *and* (HG)*. Let* $\underline{y}$ *and* $\overline{y}$ *be a sub- and a supersolution of (BVP4) that satisfy* $\underline{y} \leq \overline{y}$ *and assume that there exists some* $k_G \in L^2_+(\Omega)$ *with*

$$|(Gy)(x)| \leq k_G(x) \;\; \lambda_\Omega(dx)\text{-a.e. for all } y \in [\underline{y}, \overline{y}] \cap L^2(\Omega). \tag{29}$$

Then there exists at least one solution of (BVP4) which lies in $[\underline{y}, \overline{y}]$*.*

Proof. We consider the following auxiliary problem

$$\mathcal{A}y + (\tau_{V,2}{}^{\star} iGT\tau_{V,2})y = f \quad \text{ in } V^{\star}, \tag{A-BVP4}$$

where T is the truncation operator defined in Lemma 2.4. Since T and G are continuous (see Lemma 2.4) and $\tau_{V,2}$ and $\tau_{V,2}{}^{\star} i \colon L^2(\Omega) \to V^{\star}$, $y \mapsto \tau_{V,2}{}^{\star} iy$ with

$$\langle \tau_{V,2}{}^{\star} iy, \phi\rangle = (y, \phi)_{L^2(\Omega)}$$

are strongly continuous mappings (see Lemma 8.7), the operator $\tau_{V,2}{}^{\star} iGT\tau_{V,2}$ is strongly continuous. Observe that the strong monotonicity of $\mathcal{A}$ implies the pseudomonotonicity of the operator $\mathcal{A} + \tau_{V,2}{}^{\star} iGT\tau_{V,2}$, see Proposition 27.7 (d) in Zeidler (1990b).

The mapping $\mathcal{A} + \tau_{V,2}{}^{\star} iGT\tau_{V,2}$ is coercive, since $\mathcal{A}$ is strongly monotone and $\tau_{V,2}{}^{\star} iGT\tau_{V,2}$ is bounded uniformly w.r.t. y due to (29):

$$\langle \mathcal{A}y + (\tau_{V,2}{}^{\star} iGT\tau_{V,2})y, y\rangle \geq c\|y\|_V^2 \; - \|k_G\|_{L^2(\Omega)} \, \|y\|_V \, ,$$

where c is some positive constant. $\mathcal{A}$ is linear and continuous and $\tau_{V,2}{}^{\star} iGT\tau_{V,2}$ is bounded uniformly w.r.t. y. Thus, the operator $\mathcal{A} + \tau_{V,2}{}^{\star} iGT\tau_{V,2}$ is bounded. Now we can apply Theorem 27.A in Zeidler (1990b) and obtain that the auxiliary problem (A-BVP4) has a solution.
We show that every solution of (A-BVP4) lies in $[\underline{y}, \overline{y}]$. Let y be any solution of (A-BVP4) which is equivalent to

$$\langle \mathcal{A}y + (\tau_{V,2}{}^{\star} iGT\tau_{V,2})y, \phi\rangle = \langle f, \phi\rangle \quad \text{for all } \phi \in V.$$

Subtracting the inequality (28) for the supersolution yields

$$a((y - \overline{y}), \phi) + \langle (\tau_{V,2}{}^{\star} iGT\tau_{V,2})y - (\tau_{V,2}{}^{\star} iGT\tau_{W^{1,2}(\Omega),2})\overline{y}, \phi\rangle \leq 0 \tag{30}$$

for all $\phi \in V \cap L^2_+(\Omega)$. For $\phi := (y - \overline{y})^+ \in V \cap L^2_+(\Omega)$ we have

$$\langle (\tau_{V,2}{}^{\star} iGT\tau_{V,2})y - (\tau_{V,2}{}^{\star} iGT\tau_{W^{1,2}(\Omega),2})\overline{y}, (y - \overline{y})^+\rangle = 0,$$

since $((\tau_{V,2}{}^{\star} iGT\tau_{V,2})y - (\tau_{V,2}{}^{\star} iGT\tau_{W^{1,2}(\Omega),2})\overline{y})|_{\{(y-\overline{y})^+>0\}} = 0$. Testing (30) with $(y - \overline{y})^+ \in V$ and making use of (H$\mathcal{A}$) the last equation implies

$$\begin{aligned} 0 &\leq c\|(y - \overline{y})^+\|^2_{W^{1,2}(\Omega)} \leq \langle \mathcal{A}(y - \overline{y})^+, (y - \overline{y})^+\rangle = a((y - \overline{y})^+, (y - \overline{y})^+) \\ &= a((y - \overline{y}), (y - \overline{y})^+) \leq 0 \end{aligned}$$

and hence $\|(y - \overline{y})^+\|^2_{W^{1,2}(\Omega)} = 0$, i.e. $y \leq \overline{y}$.
The proof of $\underline{y} \leq y$ follows with the same arguments. Since every solution of the auxiliary problem (A-BVP4) lies in $[\underline{y}, \overline{y}]$, it solves (BVP4), too. □

By strengthening condition (29), the continuity of every solution of (BVP5) can be shown.

Proposition 3.4 (Continuity result)**.** *Let $u \in L^r(\Omega)$, $r > \max\{N/2, 2\}$ and r' its conjugate exponent. We consider the following*

Problem 9. *Find some $y \in W^{1,2}(\Omega)$ such that $g \circ (id, y) \in L^r(\Omega)$ and*

$$\mathcal{A}y + \tau_{W^{1,2}(\Omega),L^{r'}(\Omega)}{}^{\star} i\, g \circ (id, y) = \tau_{W^{1,2}(\Omega),L^{r'}(\Omega)}{}^{\star} iu \text{ in } W^{1,2}(\Omega)^{\star}, \tag{BVP5}$$

where $\mathcal{A}$ is the elliptic operator defined in (3.1) and $i\colon L^r(\Omega) \to L^{r'}(\Omega)^{\star}$ denotes the isomorphism. We assume that the coefficients A_{ij}, $i, j = 1, \dots N$, are in $C^1_0(\bar{\Omega})$ and $\int_\Omega c_0^2 \, d\lambda_\Omega > 0$. Let $\underline{y}$ and $\overline{y}$ be some corresponding sub- and supersolution with $\underline{y} \leq \overline{y}$. In addition, we suppose that inequality (29) holds with k_G in $L^r_+(\Omega)$. Then there exists at least one solution of Problem (BVP5) which has a representation lying in $[\underline{y}, \overline{y}] \cap C(\bar{\Omega})$.

Proof. Applying Theorem 3.3 yields the existence of a solution $y \in [\underline{y}, \overline{y}] \cap L^2(\Omega)$. Hence it holds

$$\mathcal{A}y = \tau_{W^{1,2}(\Omega), L^{r'}(\Omega)}{}^{\star} i(u - g \circ (id, y)),$$

where the right hand side $u - g \circ (id, y)$ lies in $L^r(\Omega)$. The statement $y \in C(\bar{\Omega})$ follows from Theorem 3.1 in Casas (1993). □

3.1.3 Optimal Control Problem

For $f \in V^{\star}$, we denote the set of all solutions of (BVP4) which lie in $[\underline{y}, \overline{y}]$ by $\mathcal{S}(f)$. Let $\underline{y}$, $\overline{y} \in W^{1,2}(\Omega)$ and $\underline{u}$, $\overline{u} \in L^2(\Omega)$ with $\underline{y} \leq \overline{y}$ and $\underline{u} \leq \overline{u}$. Assume the following properties of the objective functional J.

(HJ) $J\colon \big([\underline{y}, \overline{y}] \cap V\big) \times \big([\underline{u}, \overline{u}] \cap L^2(\Omega)\big) \to \mathbb{R}$ satisfies the condition:
From $y_n \to y$ in V with $(y_n)_{n\in\mathbb{N}} \subset [\underline{y}, \overline{y}] \cap V$ and $u_n \rightharpoonup u$ in $L^2(\Omega)$ with $(u_n)_{n\in\mathbb{N}} \subset [\underline{u}, \overline{u}] \cap L^2(\Omega)$ it follows $J(y, u) \leq \liminf_{n\to\infty} J(y_n, u_n)$.

Now we consider the following optimal control problem:

Problem 10.

$$\begin{aligned} \min \quad & J(y, u) && \text{(OC-BVP4)} \\ \textit{s.t.} \quad & \mathcal{A}y + (\tau_{V,2}{}^{\star} i G \tau_{V,2}) y = \tau_{V,2}{}^{\star} i u \quad \textit{in } V^{\star} \\ \textit{and} \quad & u \in [\underline{u}, \overline{u}] \cap L^2(\Omega) \\ & y \in [\underline{y}, \overline{y}] \cap V \end{aligned}$$

According to Theorem 2.8 in Section 2 it holds:

Theorem 3.5. *Assume* (H$\mathcal{A}$)*,* (HG)*,* (29) *and* (HJ)*. Let $\underline{y}$ be a subsolution of (BVP4) with $f = \tau_{V,2}{}^{\star} i \underline{u}$ and $\overline{y}$ be a supersolution of (BVP4) with $f = \tau_{V,2}{}^{\star} i \overline{u}$ and suppose $\underline{y} \leq \overline{y}$. Then the optimal control problem (OC-BVP4) has a solution (y, u).*

Proof. The same arguments as in the proof of Theorem 2.8 hold. The properties of the embeddings are the same, since V forms as well as $W_0^{1,2}(\Omega)$ a closed subspace of $W^{1,2}(\Omega)$. Applying Lax-Milgram's Theorem on $\mathcal{A}$, the inverse $\mathcal{A}^{-1}$ exists and is continuous. Thus, for any sequence $(Z_m)_{m\in M\subseteq\mathbb{N}}$ with $Z_m \to Z$ in $V^{\star}$ we obtain that the sequence $(y_m)_{m\in M}$ with $y_m := \mathcal{A}^{-1} Z_m$ converges to $y := \mathcal{A}^{-1} Z$ in V. □

3.2 Approximation of Solutions for Optimal Control Problems

Let $\underline{y}$ be a subsolution of (BVP4) with $f = \tau_{V,2}{}^{\star}i\underline{u}$ and $\overline{y}$ be a supersolution of (BVP4) with $f = \tau_{V,2}{}^{\star}i\overline{u}$ and suppose $\underline{y} \leq \overline{y}$. We denote with (y_0, u_0) a solution of (OC-BVP4) and with $\{w_1, w_2, \ldots\}$ some basis of V. For $n \in \mathbb{N}$ we set

$$V^n := \operatorname{lin}\{w_1, \ldots, w_n\}, \tag{31}$$

$$P_n \colon V \to V \text{ the orthogonal projection onto } V^n, \tag{32}$$

$$\tau_{V^n,V} \colon V^n \to V \text{ the linear continuous embedding of } V^n \text{ into } V \text{ and} \tag{33}$$

$$\tau_{V^n,V}{}^{\star} \colon V^{\star} \to V^{n\star} \text{ its adjoint.} \tag{34}$$

In the following we write $\tau_{V^n,2}$ for the combined mapping $\tau_{V,2} \circ \tau_{V^n,V}$. Observe that $\tau_{V^n,V}{}^{\star} \circ \tau_{V,2}{}^{\star} = \tau_{V^n,2}{}^{\star}$ and $\langle \tau_{V,2}{}^{\star}iy, w\rangle = \langle \tau_{V^n,2}{}^{\star}iy, w\rangle$ for all $y \in L^2(\Omega)$, $w \in V^n$.

Since $\mathcal{A} \colon V \to V^{\star}$ is strongly monotone by assumption (H$\mathcal{A}$), the mapping

$$\mathcal{A}_n \colon V^n \to V^{n\star},\ y \mapsto \tau_{V^n,V}{}^{\star}\mathcal{A}\tau_{V^n,V}y$$

is strongly monotone as well:

$$\langle \mathcal{A}_n y, y\rangle = \langle \mathcal{A}\tau_{V^n,V}y, \tau_{V^n,V}y\rangle \geq c\|\tau_{V^n,V}y\|_V^2 = c\|y\|_{V^n}^2 \quad \text{for all } \ y \in V^n.$$

With these definitions we can formulate the following semi-discretized problem:

Problem 11. *Find some $y \in V^n$ such that*

$$\langle \mathcal{A}_n y + (\tau_{V^n,2}{}^{\star}iGT\tau_{V^n,2})y, w_i\rangle = \langle f, w_i\rangle \text{ for all } i = 1, \ldots, n \quad y \in V^n \tag{BVP4-n}$$

We denote the set of all solutions of (BVP4-n) with $\mathcal{S}_n(f)$.

Theorem 3.6. *Assume* (H$\mathcal{A}$), (HG) *and* (29). *There exists at least one solution of (BVP4-n).*

The proof is analogous to Theorem 2.11.

Regularity Assumption 3.7. *There exists a subsequence $(y_n)_{n\in M}$, $M \subseteq \mathbb{N}$, with $y_n \in \mathcal{S}_n(\tau_{V,2}{}^{\star}iu_0)$, $n \in M$, and $\tau_{V,2}y_n \to \tau_{V,2}y_0$ in $L^2(\Omega)$.*

Problem 12.

$$\begin{aligned} \min \quad & J(y,u) && \text{(OC-BVP4-n)} \\ \textit{s.t.} \quad & \langle \mathcal{A}_n y + (\tau_{V^n,2}{}^{\star}iGT\tau_{V^n,2})y, w_i\rangle = \langle \tau_{V^n,2}{}^{\star}iu, w_i\rangle \text{ for all } i = 1, \ldots, n \\ \textit{and} \quad & u \in [\underline{u}, \overline{u}] \subseteq L^2(\Omega) \\ & y \in V^n \end{aligned}$$

Theorem 3.8. *Assume* (H$\mathcal{A}$), (HG), (29) *and* (HJ). *Then the optimal control problem* (OC-BVP4-n) *has a solution* (y, u).

According to Theorem 2.14 in Section 2 it holds

Theorem 3.9 (Approximation Theorem)**.** *Assume* (H$\mathcal{A}$), (HG), (29) *and* (HJ) *and suppose Regularity Assumption* (3.7)*. Let the pair* (y_n, u_n) *be a solution of (OC-BVP4-n) for any* $n \in \mathbb{N}$*. Then there exists a subsequence* $(y_n, u_n)_{n \in M \subseteq \mathbb{N}}$ *which converges to some* $(\tilde{y}_0, \tilde{u}_0)$ *in* $V \times (L^2(\Omega))_w$*, where* $(L^2(\Omega))_w$ *denotes the space* $L^2(\Omega)$ *equipped with the weak topology. The pair* $(\tilde{y}_0, \tilde{u}_0)$ *solves (OC-BVP4).*

The proofs of the last two theorems are analogous to Section 2.

4 Optimal Control Problems for Quasi-linear Elliptic PDEs

4.1 Existence of Solutions for Optimal Control Problems

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary Γ. We assume that V is some closed subspace of $W^{1,p}(\Omega)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, with $W_0^{1,p}(\Omega) \subseteq V \subseteq W^{1,p}(\Omega)$. The embedding of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ is compact.
Let $a\colon W^{1,p}(\Omega) \times W^{1,p}(\Omega) \to \mathbb{R}$ be the semi-linear form given by

$$a(y, \phi) = \sum_{i=1}^{N} \int_\Omega a_i(id, y, \nabla y) \frac{\partial \phi}{\partial x_i} \, d\lambda_\Omega, \tag{35}$$

where $a_i\colon \Omega \times \mathbb{R} \times \mathbb{R}^N$, $i = 1, \ldots, N$ denote the coefficient functions. The mapping $\mathcal{A}\colon V \to V^\star$, $y \mapsto a(y, \cdot)$ denotes the associated operator. We suppose the following assumptions including the Leray-Lions conditions for the functions a_i which guarantee important properties of the operator $\mathcal{A}$ (compare, e.g., Theorem 2.109 in Carl et al. (2007)):

(H1) Carathéodory and Growth Condition: Every coefficient a_i, $i = 1, \cdots, N$ satisfies the Carathéodory conditions, i.e.

- $a_i(\cdot, s, \zeta)$ is measurable for all $(s, \zeta) \in \mathbb{R} \times \mathbb{R}^N$ and
- $a_i(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$.

There exist some constant $c_0 > 0$ and a function $k_0 \in L^q_+(\Omega)$ with

$$|a_i(x, s, \zeta)| \leq k_0(x) + c_0(|s|^{p-1} + |\zeta|^{p-1}).$$

(H2) Monotonicity Type Condition: The coefficients satisfy the monotonicity condition w.r.t. ζ, i.e.

$$\sum_{i=1}^{N}(a_i(x,s,\zeta)-a_i(x,s,\zeta'))(\zeta_i-\zeta_i')>0$$

for a.e. $x\in\Omega$, for all $s\in\mathbb{R}$ and all ζ, $\zeta'\in\mathbb{R}^N$ with $\zeta\neq\zeta'$.

(H3) Coercivity Type Condition: There exist some constant $c_1>0$ and a function $k_1\in L^1(\Omega)$ with

$$\sum_{i=1}^{N}a_i(x,s,\zeta)\zeta_i\geq c_1\,|\zeta|^p-k_1(x)$$

for a.e. $x\in\Omega$, for all $s\in\mathbb{R}$ and all $\zeta\in\mathbb{R}^N$.

(HG) There exists some Carathéodory function $g\colon\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ such that the operator G is the corresponding Nemytskii operator given by

$$Gy(x)=g(x,y(x),\nabla y(x)),\quad y\in W^{1,p}(\Omega).$$

Remark 4.1. *An operator $\mathcal{A}$ which satisfies these Leray-Lions conditions (H1)-(H3) is not necessary coercive. For example, the p-Laplacian*

$$-\Delta_p\colon V\to V^\star,\ y\mapsto\int_\Omega|\nabla y|^{p-2}\,\nabla y\,\nabla\cdot\,d\lambda_\Omega$$

satisfies (H3) and is uniformly monotone for $V=W_0^{1,p}(\Omega)$. Hence it is coercive for $2\leq p<\infty$ in this case. If $W_0^{1,p}(\Omega)\subsetneq V=W^{1,p}(\Omega)$, then the p-Laplacian is not coercive: Choose a sequence $(y_n)_{n\in\mathbb{N}}\subset\mathbb{R}$ with $|y_n|\to\infty$ in $\mathbb{R}$. Then

$$\lim_{n\to\infty}\frac{\|-\Delta_p y_n\|_{W^{1,p}(\Omega)}}{\|y_n\|_{W^{1,p}(\Omega)}}=0.$$

But in this case an auxiliary problem can be formulated which is described by a coercive operator. Therefore, a penalty term has to be introduced which requires the existence of a sub- and supersolution, see the definition below. Moreover, every solution of the auxiliary problem lies in the interval of the sub- and supersolution. Thus it is a solution of the original problem, too.

Let the isomorphism i and the embedding operator $\tau_{V,p}$ be defined as in Subsection 8.2. Now we examine the solvability of the following quasi-linear PDE for $f \in V^\star$:

Problem 13. *Find some $y \in V$ such that*

$$\mathcal{A}y + (\tau_{V,p}{}^\star iG)y = f \text{ in } V^\star \tag{BVP6}$$

Definition 4.1 (solution)**.** *The function $y \in V$ is called a solution of Problem (BVP6) if $Gy \in L^q(\Omega)$ and*

$$\langle \mathcal{A}y + (\tau_{V,p}{}^\star iG)y, \phi\rangle = \langle f, \phi\rangle \text{ for all } \phi \in V.$$

As in the previous chapters, the terms sub- and supersolution play a key role in the proof of the existence of some solution for (BVP6).

Definition 4.2 (subsolution)**.** *The function $\underline{y} \in W^{1,p}(\Omega)$ is called a subsolution of Problem (BVP6) if $G\underline{y} \in L^q(\Omega)$, $(\underline{y} - y)^+ \in V$ for all $y \in V$ and*

$$a(\underline{y}, \phi) + \langle(\tau_{V,p}{}^\star iG)\underline{y}, \phi\rangle \le \langle f, \phi\rangle \text{ for all } \phi \in V \cap L^p_+(\Omega).$$

Definition 4.3 (supersolution)**.** *The function $\overline{y} \in W^{1,p}(\Omega)$ is called a supersolution of Problem (BVP6) if $G\overline{y} \in L^q(\Omega)$, $(y - \overline{y})^+ \in V$ for all $y \in V$ and*

$$a(\overline{y}, \phi) + \langle(\tau_{V,p}{}^\star iG)\overline{y}, \phi\rangle \ge \langle f, \phi\rangle \text{ for all } \phi \in V \cap L^p_+(\Omega).$$

Now the existence of a solution of (BVP6) is stated.

Theorem 4.2. *Assume (H1) - (H3) and* (HG). *Let $\underline{y}$ and $\overline{y}$ be a sub- and a supersolution of (BVP6) satisfying $\underline{y} \le \overline{y}$ and assume that there exist some $k_G \in L^q_+(\Omega)$ and some constant $c_G > 0$ with*

$$|g(x, s, \zeta)| \le k_G(x) + c_G\,|\zeta|^{p-1} \text{ for all } s \in [\underline{y}(x), \overline{y}(x)] \;\; \lambda_\Omega(dx)\text{-a.e. and } \zeta \in \mathbb{R}^N. \tag{36}$$

Then there exists at least one solution of (BVP6) which lies in $[\underline{y}, \overline{y}]$.

For the proof we refer to Theorem 3.17 in Carl et al. (2007), in which the proof is given for $V = W^{1,p}_0(\Omega)$. The case for arbitrary V is along the same lines. The next lemma yields some technical results and generalizes the statement from Lemma 2.4 c).

Lemma 4.3. *a) Let $a, b \in W^{1,p}(\Omega)$ with $a \le b$. The truncation operator $T\colon L^p(\Omega) \to L^p(\Omega)$ resp. $T\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ given by*

$$Ty(x) := \begin{cases} b(x) & \text{if} \quad y(x) > b(x) \\ y(x) & \text{if} \quad a(x) \le y(x) \le b(x) \\ a(x) & \text{if} \quad y(x) < a(x) \end{cases} \tag{37}$$

is continuous and bounded.

b) *The Nemytskii operator* $\mathfrak{B}\colon L^p(\Omega) \to L^q(\Omega)$ *given by the Carathéodory function* $\mathfrak{b}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ *with*

$$\mathfrak{b}(x,s) = \begin{cases} (s - b(x))^{p-1} & \text{if} \quad s > b(x) \\ 0 & \text{if} \quad a(x) \le s \le b(x) \\ -(a(x) - s)^{p-1} & \text{if} \quad s < a(x) \end{cases} \tag{38}$$

is continuous and bounded. Moreover, the inequalities

$$|\mathfrak{b}(x,s)| \le c_{p,1}(|a(x)| + |b(x)|)^{p-1} + c_{p,1}\,|s|^{p-1} \tag{39}$$

and

$$\int_\Omega \mathfrak{b}(id, y) y \, d\lambda_\Omega \ge c_{\mathfrak{B}} \|y\|^p_{L^p(\Omega)} - C_{\mathfrak{B}} \tag{40}$$

hold for all $y \in L^p(\Omega)$*, where* $c_{\mathfrak{B}} := \frac{1}{c_{p,2}} - \epsilon$ *and* $0 < \epsilon < \frac{1}{c_{p,2}}$ *is arbitrary. The constants* $c_{p,1}$ *and* $c_{p,2}$ *are defined by*

$$c_{p,1} = \begin{cases} 2, & 1 < p \le 2 \\ 2^{p-1}, & 2 < p < \infty \end{cases} \tag{41}$$

and

$$c_{p,2} = \begin{cases} 1, & 1 < p < 2 \\ 2^{p-2}, & 2 \le p < \infty \end{cases}. \tag{42}$$

$C_{\mathfrak{B}}$ *denotes a constant given by*

$$C_{\mathfrak{B}} := \left(4\left(\frac{1}{c_{p,2}} - \epsilon\right) + 2C_\epsilon\right)\left(\|a\|^p_{L^p(\Omega)} + \|b\|^p_{L^p(\Omega)}\right), \tag{43}$$

where $C_\epsilon := (\epsilon p)^{-q/p}\frac{1}{q}$.

For the proof of a) we refer to Lemma 2.89 in Carl et al. (2007) resp. Lemma 1.22. in Heinonen et al. (1993). The last part of the lemma is proven in the Appendix.

In the following, the operators T and $\mathfrak{B}$ are defined with $a := \underline{y}$ and $b := \overline{y}$. In the proof of Theorem 3.17 in Carl et al. (2007) it is shown that every solution of the equation

$$\mathcal{A}_T y + \lambda(\tau_{V,p}{}^\star i \mathfrak{B} \tau_{V,p}) y + (\tau_{V,p}{}^\star i G T) y = f \text{ in } V^\star \tag{A-BVP6}$$

solves (BVP6) as well. Here, λ denotes a positive constant and the operator $\mathcal{A}_T$ is given by

$$\langle \mathcal{A}_T y, \phi \rangle := \sum_{i=1}^N \int_\Omega a_i(id, Ty, \nabla y) \frac{\partial \phi}{\partial x_i} \, d\lambda_\Omega. \tag{44}$$

In Theorem 2.109 in Carl et al. (2007) it is shown that under the boundedness condition (36) the operator $\mathcal{A}_T + (\tau_{V,p}{}^\star iGT)$ satisfies the S_+-property which plays a key role in the later proof of the existence of a solution for the optimal control problem. We remind the definition of the S_+-property, see, e.g., Zeidler (1990b) 27.1.:

$$y_n \rightharpoonup y \text{ and } \limsup \langle \mathcal{A}y_n - \mathcal{A}y, y_n - y\rangle \le 0 \text{ imply } y_n \to y.$$

For example, every uniformly monotone operator has the S_+-property, see example 27.2. in Zeidler (1990b). Every p-Laplacian, $1 < p < \infty$, defined by $\Delta_p\colon W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^\star$, $y \mapsto div(|\nabla y|^{p-2}\nabla y)$, is uniformly monotone. We refer to Carl et al. (2007).
Let $\underline{y}$, $\overline{y} \in V$ and $\underline{u}$, $\overline{u} \in L^q(\Omega)$ with $\underline{y} \le \overline{y}$ and $\underline{u} \le \overline{u}$. Assume the following properties on the objective functional J.

(HJ) $J\colon \big([\underline{y}, \overline{y}] \cap V\big) \times ([\underline{u}, \overline{u}] \cap L^q(\Omega)) \to \mathbb{R}$ satisfies the condition:
From $y_n \to y$ in V with $(y_n)_{n\in\mathbb{N}} \subset [\underline{y}, \overline{y}] \cap V$ and $u_n \rightharpoonup u$ in $L^q(\Omega)$ with $(u_n)_{n\in\mathbb{N}} \subset [\underline{u}, \overline{u}] \cap L^q(\Omega)$ it follows $J(y,u) \le \liminf_{n\to\infty} J(y_n, u_n)$.

With these definitions we can formulate the following optimal control problem.

Problem 14.

$$\begin{aligned} \min \quad & J(y,u) \\ s.t. \quad & \mathcal{A}y + \tau_{V,p}{}^\star iGTy = \tau_{V,p}{}^\star iu \quad in\ V^\star \\ and \quad & u \in [\underline{u}, \overline{u}] \cap L^q(\Omega) \\ & y \in [\underline{y}, \overline{y}] \cap V \end{aligned} \tag{OC-BVP6}$$

For $f \in V^\star$ we denote the set of all solutions of (BVP6) lying in $[\underline{y}, \overline{y}]$ with $\mathcal{S}(f)$. In the later proof of the existence of a solution for Problem (OC-BVP6) we make use of the following statement.

Lemma 4.4. *Let X be some Banach space and $A\colon X \to X^\star$ a coercive operator, i.e.*

$$\lim_{\|y\|_X \to\infty} \frac{\langle Ay, y\rangle}{\|y\|_X} \to \infty.$$

Moreover, assume that the equations

$$Ay_n = f_n,\ n \in \mathbb{N} \tag{45}$$

are satisfied for some given sequences $(y_n)_{n\in\mathbb{N}} \subset X$ and $(f_n)_{n\in\mathbb{N}} \subset X^\star$, where the sequence $(f_n)_{n\in\mathbb{N}}$ is assumed to be bounded in $X^\star$. Then $(y_n)_{n\in\mathbb{N}}$ is bounded.

Proof. Assume that the sequence $(y_n)_{n\in\mathbb{N}}$ is unbounded. Then there exists some subsequence $(y_{n'})_{n'\in M}$, $M \subset \mathbb{N}$, such that $\lim_{n'\to\infty} \|y_{n'}\|_X = \infty$ and $\|y_{n'}\|_X > 0$ for all $n' \in M$. By (45) we obtain

$$\|f_{n'}\|_{Op} \|y_{n'}\|_X \geq \langle f_{n'}, y_{n'} \rangle = \langle A y_{n'}, y_{n'} \rangle = \|y_{n'}\|_X \frac{\langle A y_{n'}, y_{n'} \rangle}{\|y_{n'}\|_X}$$

and hence by the coercivity of A

$$\|f_{n'}\|_{Op} \geq \frac{\langle A y_{n'}, y_{n'} \rangle}{\|y_{n'}\|_X} \to \infty,$$

which is a contradiction to the assumption of boundedness for the sequence $(f_{n'})_{n'\in M}$. □

Now we derive an existence result analogously to Theorem 2.8. In contrast to the case when $\mathcal{A}$ is a linear operator, we can not apply the Theorem of Lax-Milgram. Instead we use a technique which, for example, has been applied in the proof of Theorem 4.31 in Carl et al. (2007).

Theorem 4.5. *Let $\underline{y}$ be a subsolution of (BVP6) with $f = \tau_{V,p}{}^{\star} i \underline{u}$ and $\overline{y}$ be a supersolution of (BVP6) with $f = \tau_{V,p}{}^{\star} i \overline{u}$. We suppose $\underline{y} \leq \overline{y}$ and assume (H1) - (H3),* (HG), (36) *and* (HJ). *Then the optimal control problem* (OC-BVP6) *has a solution (y, u).*

Proof. We define the operator $\mathcal{A}_T \colon V \to V^\star$ as in (44) and consider the following related auxiliary problem:

$$\begin{aligned} \min \quad & J(y,u) && \text{(A-OC-BVP6)} \\ \text{s.t.} \quad & \mathcal{A}_T y + \lambda(\tau_{V,p}{}^{\star} i \mathfrak{B} \tau_{V,p}) y + (\tau_{V,p}{}^{\star} i G T) y = \tau_{V,p}{}^{\star} i u \text{ in } V^\star \\ \text{and} \quad & u \in [\underline{u}, \overline{u}] \cap L^q(\Omega) \\ & y \in [\underline{y}, \overline{y}] \cap V \end{aligned}$$

The constant $\lambda > 0$ satisfies

$$\lambda > \frac{(\epsilon q)^{-p/q} c_G}{p c_{\mathfrak{B}}}, \text{ where } \epsilon < \frac{c_1}{c_G}. \tag{46}$$

Since every state in $[\underline{y}, \overline{y}]$ solves (BVP6) if and only if it solves (A-BVP6), every solution of the optimal control problem (A-OC-BVP6) is a solution of the optimal control problem (OC-BVP6). As shown in the proof of Theorem 3.17 in Carl et al. (2007), the equation (A-BVP6) is solved by at least one $y \in V$ satisfying $y \in [\underline{y}, \overline{y}]$. Hence for every $u \in [\underline{u}, \overline{u}] \cap L^q(\Omega)$ there exists at least one $y \in [\underline{y}, \overline{y}]$

solving (A-BVP6). Let $(y_m, u_m)_{m\in\mathbb{N}}$ be the infimal sequence of Problem (A-OC-BVP6) with

$$\lim_{m\to\infty} J(y_m, u_m) = \inf_{u\in[\underline{u},\overline{u}]\cap L^q(\Omega), y\in\mathcal{S}_A(\tau_{V,p}{}^\star iu)} J(y,u),$$

where $\mathcal{S}_A(f)$ denotes the set of all solutions of (A-BVP6) lying in $[\underline{y}, \overline{y}]$ with right hand side f.
Since $L^q(\Omega)$ is reflexive and $[\underline{u}, \overline{u}] \cap L^q(\Omega)$ is nonempty, convex, bounded and closed in $L^q(\Omega)$, the set $[\underline{u}, \overline{u}] \cap L^q(\Omega)$ is weakly sequentially compact. Thus there exist a weakly convergent subsequence $(u_m)_{m\in M}$, $M \subseteq \mathbb{N}$ and a weak limit $u_0 \in L^q(\Omega)$, i.e.

$$u_m \rightharpoonup u_0 \text{ in } L^q(\Omega). \tag{47}$$

The weak closedness of $[\underline{u}, \overline{u}] \cap L^q(\Omega)$ implies $u_0 \in [\underline{u}, \overline{u}] \cap L^q(\Omega)$. By the strong continuity of $\tau_{V,p}{}^\star i$ we can infer

$$\tau_{V,p}{}^\star i u_m \to \tau_{V,p}{}^\star i u_0 \text{ in } V^\star. \tag{48}$$

As shown in the proof of Theorem 3.17 in Carl et al. (2007) the mapping $\mathcal{A}_T + \lambda(\tau_{V,p}{}^\star i\mathfrak{B}\tau_{V,p}) + \tau_{V,p}{}^\star iGT\colon V \to V^\star$ is coercive. Hence the sequence $(y_m)_{m\in M}$ is bounded in V since $(\tau_{V,p}{}^\star i u_m)_{m\in M}$ is bounded, see Lemma 4.4. By the Eberlein-Smulian Theorem there exists some subsequence $(y_m)_M$ which is weakly convergent in V:

$$y_m \rightharpoonup y_0 \text{ in } V. \tag{49}$$

The limit $\tau_{V,p} y_0$ lies in $[\underline{y}, \overline{y}] \cap L^p(\Omega)$ since the embedding $\tau_{V,p}$ is compact and the set $[\underline{y}, \overline{y}] \cap L^p(\Omega)$ is closed. By Theorem 2.109 in Carl et al. (2007) the operator $\mathcal{A}_T + \tau_{V,p}{}^\star iGT$ satisfies the S_+-property. If we can show

$$\limsup \langle (\mathcal{A}_T + \tau_{V,p}{}^\star iGT) y_m, y_m - y_0 \rangle \le 0,$$

then it follows the strong convergence $y_m \to y_0$ in V.
The convergence in (49) yields $\tau_{V,p} y_m \to \tau_{V,p} y_0$ in $L^p(\Omega)$. The continuity of the operator $\mathfrak{B}\colon L^p(\Omega) \to L^q(\Omega)$ (see Lemma 4.3) and (47) imply

$$\begin{aligned}\langle (\mathcal{A}_T + \tau_{V,p}{}^\star iGT) y_m, y_m - y_0 \rangle &= \langle \tau_{V,p}{}^\star i u_m - \lambda \tau_{V,p}{}^\star i\mathfrak{B}\tau_{V,p} y_m, y_m - y_0 \rangle \\ &= \int_\Omega \left(u_m - \lambda \mathfrak{B}\tau_{V,p} y_m\right) \tau_{V,p}(y_m - y_0)\, d\lambda_\Omega \\ &\to 0\end{aligned}$$

and hence $y_m \to y_0$ in V. By the continuity properties of $\mathcal{A}_T$, $\mathfrak{B}$ and $\tau_{V,p}{}^\star iGT$, see Theorem 2.109 in Carl et al. (2007) and Lemma 4.3, we obtain

$$\begin{aligned}&\mathcal{A}_T y_m + \lambda(\tau_{V,p}{}^\star i\mathfrak{B}\tau_{V,p})y_m + (\tau_{V,p}{}^\star iGT)y_m \\ &\qquad \to \mathcal{A}_T y_0 + \lambda(\tau_{V,p}{}^\star i\mathfrak{B}\tau_{V,p})y_0 + (\tau_{V,p}{}^\star iGT)y_0 \text{ in } V^\star.\end{aligned}$$

Therefore, by (47) and (HJ) it holds

$$J(y_0, u_0) \le \liminf_{m\to\infty} J(y_m, u_m).$$

□

Remark 4.6. *The constant λ has to be sufficiently large. If λ satisfies condition* (46)*, the coercivity of the operator defining equation (A-BVP6) is guaranteed. This can be seen by the relations*

$$\begin{aligned}\langle \mathcal{A}_T y, y\rangle &\ge c_1 \|\nabla y\|^p_{L^p(\Omega)} - \|k_1\|_{L^1(\Omega)} \\ \langle (\tau_{V,p}{}^\star i\mathfrak{B}\tau_{V,p})y, y\rangle &\ge c_{\mathfrak{B}} \|y\|^p_{L^p(\Omega)} - C_{\mathfrak{B}} \\ \langle (\tau_{V,p}{}^\star iGT)y, y\rangle &\le \|k_G\|_{L^q(\Omega)} \|y\|_{L^p(\Omega)} + c_G \|\nabla T y\|^{p-1}_{L^p(\Omega)} \|y\|_{L^p(\Omega)} \\ &\le \|k_G\|_{L^q(\Omega)} \|y\|_{L^p(\Omega)} + c_G c_p \| \left|\nabla \underline{y}\right| + |\nabla \overline{y}| \,\|^{p-1}_{L^p(\Omega)} \|y\|_{L^p(\Omega)} \\ &\quad + c_G \|\nabla y\|^{p-1}_{L^p(\Omega)} \|y\|_{L^p(\Omega)} \\ &\le \left(\|k_G\|_{L^q(\Omega)} + c_G c_p \| \left|\nabla \underline{y}\right| + |\nabla \overline{y}| \,\|^{p-1}_{L^p(\Omega)} \right) \|y\|_{L^p(\Omega)} \\ &\quad + c_G \epsilon \|\nabla y\|^p_{L^p(\Omega)} + c_G C_\epsilon \|y\|^p_{L^p(\Omega)} ,\end{aligned}$$

with $\epsilon < \frac{c_1}{c_G}$ and $C_\epsilon := (\epsilon q)^{-p/q} \frac{1}{p}$ and

$$c_p = \begin{cases} 1, & 1 < p \le 2 \\ 2, & 2 < p < 3 \\ 2^{p-2}, & 3 \le p < \infty \end{cases} . \tag{50}$$

These inequalities imply the following relation:

$$\begin{aligned}&\langle (\mathcal{A}_T + \tau_{V,p}{}^\star i\mathfrak{B}\tau_{V,p} + \tau_{V,p}{}^\star iGT)y, y\rangle \\ &\quad \ge (c_1 - c_G\epsilon)\|\nabla y\|^p_{L^p(\Omega)} + (\lambda c_{\mathfrak{B}} - c_G C_\epsilon)\|y\|^p_{L^p(\Omega)} - \|k_1\|_{L^1(\Omega)} - C_{\mathfrak{B}} \\ &\qquad - \left(\|k_G\|_{L^q(\Omega)} + c_G c_p \| \left|\nabla \underline{y}\right| + |\nabla \overline{y}| \,\|^{p-1}_{L^p(\Omega)} \right) \|y\|_{L^p(\Omega)} .\end{aligned}$$

4.2 Existence of Solutions for Extended Optimal Control Problems

In (OC-BVP6) we have considered pointwise state constraints described by the sub- and supersolution. We now admit generalized pointwise state constraints.

The boundedness assumption for G on $[\underline{y}, \overline{y}]$, see (36), has to be modified according to a condition referring to the considered admissible states. Let $Y := [y_1, y_2]$ with y_1, $y_2 \in W^{1,p}(\Omega)$ and $y_1 \leq y_2$ and denote $Y(x) := [y_1(x), y_2(x)]$ for $x \in \Omega$. The optimal control problem we are now interested in has the following form:

Problem 15.

$$\begin{aligned} \min \quad & J(y,u) \\ s.t. \quad & \mathcal{A}y + \tau_{V,p}{}^{\star} iGy = \tau_{V,p}{}^{\star} iu \quad in\ V^{\star} \\ and \quad & u \in [\underline{u}, \overline{u}] \cap L^q(\Omega) \\ & y \in Y \cap V \end{aligned} \tag{OC-BVP6-G}$$

Assumption (HJ) has to be adapted to the set Y.

(HJ) $J\colon (Y \cap V) \times ([\underline{u}, \overline{u}] \cap L^q(\Omega)) \to \mathbb{R}$ satisfies the condition:
From $y_n \to y$ in V with $(y_n)_{n\in\mathbb{N}} \subset Y \cap V$ and $u_n \rightharpoonup u$ in $L^q(\Omega)$ with $(u_n)_{n\in\mathbb{N}} \subset [\underline{u}, \overline{u}] \cap L^q(\Omega)$ it follows $J(y,u) \leq \liminf_{n\to\infty} J(y_n, u_n)$.

We now state the existence theorem for Problem (OC-BVP6-G).

Theorem 4.7. *We suppose (H1) - (H3),* (HG) *and* (HJ)*. Assume that there exists at least one $u \in [\underline{u}, \overline{u}] \cap L^q(\Omega)$ such that (BVP6) has a solution $y \in Y \cap V$. Moreover, we assume that there exist some $k_G \in L^q_+(\Omega)$ and some constant $c_G > 0$ such that*

$$|g(x,s,\zeta)| \leq k_G(x) + c_G\,|\zeta|^{p-1} \ \ \textit{for all } s \in Y(x) \ \ \lambda_\Omega(dx)\textit{-a.e. and } \zeta \in \mathbb{R}^N. \tag{51}$$

Then the optimal control problem (OC-BVP6-G) *has at least one solution pair (y,u).*

Proof. We set the truncation operator $\overline{T}\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ resp. $\overline{T}\colon L^p(\Omega) \to L^p(\Omega)$ given by

$$\overline{T}y(x) := \begin{cases} y_2(x) & \text{if} \quad y(x) > y_2(x) \\ y(x) & \text{if} \quad y_1(x) \leq y(x) \leq y_2(x). \\ y_1(x) & \text{if} \quad y(x) < y_1(x) \end{cases}$$

The Nemytskii operator $\overline{\mathfrak{B}}\colon L^p(\Omega) \to L^q(\Omega)$ is given by the Carathéodory function $\overline{\mathfrak{b}}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ with

$$\overline{\mathfrak{b}}(x,s) = \begin{cases} (s - y_2(x))^{p-1} & \text{if} \quad s > y_2(x) \\ 0 & \text{if} \quad y_1(x) \leq s \leq y_2(x). \\ -(y_1(x) - s)^{p-1} & \text{if} \quad s < y_1(x) \end{cases}$$

By assumption, there exists at least one solution of (BVP6). Hence there exists for at least one $u \in [\underline{u}, \overline{u}] \cap L^q(\Omega)$ a solution $y \in Y \cap V$ of the equation

$$\mathcal{A}_{\overline{T}} y + \overline{\lambda}\,\overline{\mathfrak{B}} y + (\tau_{V,p}{}^{\star} i G \overline{T}) y = \tau_{V,p}{}^{\star} i u \text{ in } V^{\star}, \tag{52}$$

where $\mathcal{A}_{\overline{T}}$ is defined analogously to (44) and $\overline{\lambda} > 0$ satisfies (46). Moreover, every solution of (52) with $y \in Y$ is a solution of (BVP6). Hence the problems (OC-BVP6-G) and

$$\begin{aligned} \min \quad & J(y,u) && \text{(A-OC-BVP6-G)} \\ \text{s.t.} \quad & \mathcal{A}_{\overline{T}} y + \overline{\lambda} (\tau_{V,p}{}^{\star} i \overline{\mathfrak{B}} \tau_{V,p}) y + (\tau_{V,p}{}^{\star} i G \overline{T}) y = \tau_{V,p}{}^{\star} i u \text{ in } V^{\star} \\ \text{and} \quad & u \in [\underline{u}, \overline{u}] \cap L^q(\Omega) \\ & y \in Y \cap V \end{aligned}$$

are equivalent. The operator $\mathcal{A}_{\overline{T}} + \tau_{V,p}{}^{\star} i G \overline{T} \colon V \to V^{\star}$ is bounded and continuous. The mapping $\mathcal{A}_{\overline{T}} + \overline{\lambda} \tau_{V,p}{}^{\star} i \overline{\mathfrak{B}} \tau_{V,p} + \tau_{V,p}{}^{\star} i G \overline{T}$ is coercive and satisfies the S_+-property, see Theorem 2.109 in Carl et al. (2007). The further proof is now along the lines of Theorem 4.5. □

Remark 4.8. *i) There may exist solutions $y \in V$ of equation (52) with $u \in [\underline{u}, \overline{u}] \cap L^q(\Omega)$ which do not lie in $[y_1, y_2]$. In the case that y_1 and y_2 are sub- resp. supersolutions, it holds $y \in [y_1, y_2]$.*

ii) For $\underline{u}$, $\overline{u} \in L^q(\Omega)$ resp. $L^q(\Gamma)$ let $U = [\underline{u}, \overline{u}] \cap L^q(\Omega)$ or $U = [\underline{u}, \overline{u}] \cap L^q(\Gamma)$ (in the case of no Dirichlet boundary conditions) and consider the optimal control problem

$$\begin{aligned} \min \quad & J(y,u) && (53) \\ \textit{s.t.} \quad & \mathcal{A} y + \tau_{V,p}{}^{\star} i G y = \tau_{V,p}{}^{\star} B u \quad \textit{in } V^{\star} \\ \textit{and} \quad & u \in U \\ & y \in Y \cap V. \end{aligned}$$

Here, the operator $B \colon U \to L^p(\Omega)^{\star}$ is assumed to be weakly continuous. The existence of an optimal pair can be proven analogously to Theorem 4.7.

We now consider the case of one-sided pointwise state constraints. Let $Y = (-\infty, y_1]$ resp. $Y = [y_1, \infty)$ with $y_1 \in W^{1,p}(\Omega)$ and introduce the abbreviations $Y(x) := (-\infty, y_1(x)]$ resp. $Y(x) := [y_1(x), \infty)$. The optimal control problem with one-sided pointwise state constraints reads as follows.

Problem 16.

$$\begin{aligned} \min \quad & J(y,u) && \text{(OC-BVP6-G2)} \\ \textit{s.t.} \quad & \mathcal{A} y + \tau_{V,p}{}^{\star} i G y = \tau_{V,p}{}^{\star} i u \quad \textit{in } V^{\star} \\ \textit{and} \quad & u \in [\underline{u}, \overline{u}] \cap L^q(\Omega) \\ & y \in Y \cap V \end{aligned}$$

Under appropriate assumptions, the existence for a solution of (OC-BVP6-G2) can be proven.

Theorem 4.9. *We suppose (H1), (H2),* (HG) *and* (HJ). *Assume that there exists at least one* $u \in [\underline{u}, \overline{u}] \cap L^q(\Omega)$ *such that (BVP6) has a solution* $y \in Y \cap V$. *Moreover, we assume that there exist some positive constants* c_1 *and* C_1 *with*

$$\langle \mathcal{A}y, y\rangle \geq c_1 \|y\|_V^p - C_1 \text{ for all } y \in V \tag{54}$$

and some $k_G \in L^q_+(\Omega)$ *such that*

$$|g(x, s, \zeta)| \leq k_G(x) \text{ for all } s \in Y(x) \;\; \lambda_\Omega(dx)\text{-a.e. and } \zeta \in \mathbb{R}^N. \tag{55}$$

Then the optimal control problem (OC-BVP6-G2) *has at least one solution pair* (y, u).

Proof. We set the one-sided truncation operator $\overline{T}\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ resp. $\overline{T}\colon L^p(\Omega) \to L^p(\Omega)$ given by - if $Y = (-\infty, y_1]$

$$\overline{T}y(x) := \begin{cases} y_1(x) & \text{if} \quad y(x) > y_1(x) \\ y(x) & \text{if} \quad y(x) \leq y_1(x) \end{cases}$$

or if $Y = [y_1, \infty)$

$$\overline{T}y(x) := \begin{cases} y(x) & \text{if} \quad y(x) > y_1(x) \\ y_1(x) & \text{if} \quad y(x) \leq y_1(x). \end{cases}$$

By assumption, there exists at least one solution of (BVP6). Hence there exists for at least one $u \in [\underline{u}, \overline{u}] \cap L^q(\Omega)$ a solution $y \in Y \cap V$ of

$$\mathcal{A}y + ({\tau_{V,p}}^\star iG\overline{T})y = {\tau_{V,p}}^\star iu \text{ in } V^\star. \tag{56}$$

We show that the operator $\mathcal{A} + {\tau_{V,p}}^\star iG\overline{T}$ satisfies the coercivity condition of Lemma 4.4. We obtain by (54) and (55) the inequality

$$\langle (\mathcal{A} + {\tau_{V,p}}^\star iG\overline{T})y, y\rangle \geq c_1 \|y\|_V^p - C_1 - \|k_G\|_{L^q(\Omega)} \|y\|_{L^p(\Omega)}.$$

The last part of the proof is along the lines of Theorem 4.7. □

Remark 4.10. *i) For example, the absolute values of the nonlinear and non-monotone functions* $y \mapsto y^2$ *on the interval* $[-1, 1]$ *and* $y \mapsto \exp(y) - 5|y| - 5y$ *on* $(-\infty, 1]$ *are bounded by a constant.*

ii) Every solution of the auxiliary PDE related to (56) *is a solution of (BVP6), if there exists a positive constant* c_a *such that the semi-linear form a defined in* (35) *is satisfying the inequality*

$$a(y, (y-z)^+) - a(z, (y-z)^+) \geq c_a \|(y-z)^+\|^p_{L^p(\Omega)}$$

for all $y,\, z \in W^{1,p}(\Omega)$.

5 Optimal Control Problems with Semi-linear Evolution Equations

5.1 Existence of Solutions for Initial Boundary Value Problems

In contrast to the previous chapters, we now consider equations depending on a time variable defined on an interval $[0, T] \subset \mathbb{R}$, $0 < T < \infty$. The approach is along the lines of the elliptic equations although the embedding properties are different. We introduce the following denotations:

$$Q := \Omega \times (0, T)$$
$$\Sigma := \Gamma \times (0, T)$$

5.1.1 Motivation

Let $\Gamma_1 \cup \Gamma_2$ be a disjoint decomposition of the Lipschitz boundary Γ. For a Carathéodory function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and a sufficient smooth function $f\colon \Omega \to \mathbb{R}$ we consider the initial boundary value problem:

Problem 17. *Find some $y \in C^{2,1}(Q) \cap C(\bar{Q})$ such that*

$$\begin{aligned} y_t - \Delta y + g \circ (id, y) &= f \ \text{ in } Q \qquad \text{(Par1)} \\ y &= 0 \ \text{ on } \Gamma_1 \times (0, T) \\ \partial_\nu y + y &= 0 \ \text{ on } \Gamma_2 \times (0, T) \\ y(\cdot, 0) &= y_0 \ \text{ on } \Omega \end{aligned}$$

By observing that every test function satisfies $\phi = 0$ on $\Gamma_1 \times (0, T)$ we obtain the following formulation:

$$\int_Q y_t \phi \, d\lambda_Q + \int_Q \nabla y \, \nabla \phi \, d\lambda_Q + \int_Q (g \circ (id, y))\phi \, d\lambda_Q + \int_{\Gamma_2 \times (0,T)} y\phi \, d\lambda_\Sigma = \int_Q f\phi \, d\lambda_Q. \tag{57}$$

5.1.2 Operator Equations

Let V be some closed subspace of $W^{1,p}(\Omega)$ with $W_0^{1,p}(\Omega) \subseteq V \subseteq W^{1,p}(\Omega)$ and $2 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. We introduce the abbreviations

$$W := \{y \in L^p(T; W^{1,p}(\Omega)) |\, y_t \in L^q(T; W^{1,p}(\Omega)^\star)\} \text{ and}$$
$$W_V := \{y \in L^p(T; V) |\, y_t \in L^q(T; V^\star)\},$$

where y_t denotes the generalized derivative, see Definition 8.3. Due to the Theorem of Lions and Aubin (see Example 8.18), W and W_V are compactly embedded into $L^p(Q)$.
Since V is a closed subspace of $W^{1,p}(\Omega)$, for every functional $f \in V^\star$ there exists some linear and continuous extension $\bar{f}$ on $W^{1,p}(\Omega)$, i.e. $\bar{f} \in W^{1,p}(\Omega)^\star$, due to the Theorem of Hahn-Banach, see, e.g., Zeidler (1986).
We denote the domain of the operator L by

$$D(L) := \{v \in L^p(T;V) \,|\, v_t \in L^q(T;V^\star), v(0) = 0\}$$

and define $L\colon D(L) \subset L^p(T;V) \to L^q(T;V^\star)$, $y \mapsto Ly$ by

$$\langle Ly, \phi\rangle_{L^q(T;V^\star)} := \int_0^T \langle y_t(t), \phi(t)\rangle_{V^\star}\, \lambda_{[0,T]}(dt), \quad \phi \in L^p(T;V). \tag{58}$$

The space $L^q(T;V^\star)$ can be identified with $L^p(T;V)^\star$, see Lemma 8.13. Let $f \in L^q(T;V^\star)$ and $\mathfrak{A}\colon L^p(T;V) \to L^q(T;V^\star)$ be an operator specified below. With these definitions the initial boundary value problem reads as follows:

Problem 18. *Find some $y \in D(L)$ such that*

$$Ly + \mathfrak{A}y = f \text{ in } L^q(T;V^\star) \tag{Par2}$$

5.1.3 Properties of the Operators

In this section we state the properties of the operator L and specify conditions for the operator $\mathfrak{A}$ which guarantee the existence of a solution for the operator equation (Par2).

Proposition 5.1. *The linear operator $L\colon D(L) \subset W_V \to L^q(T;V^\star)$, $y \mapsto Ly$ is densely defined, closed and maximal monotone.*
This is true also if the initial condition $y(0) = y_0$ is replaced by a periodicity condition of the form $y(0) = y(T)$.

Proof. The proposition is stated in Zeidler (1990b) (Proposition 32.10) and in Carl et al. (2007) (Lemma 2.149). □

The operator $\mathfrak{A}$ can be defined by a family of time-dependent elliptic operators $\{\mathcal{A}(t)\colon V \to V^\star \,|\, t \in [0,T]\}$ in the following way:

$$\mathfrak{A}(y)(t) := \mathcal{A}(t)y(t), \quad t \in [0,T]. \tag{59}$$

We now state hypotheses for the time-dependent operators $\mathcal{A}(t)$:

(H1) $\|\mathcal{A}(t)y\|_{V^\star} \leq c_0\|y\|_V^{p-1} + k_0(t)$ for all $y \in V$ and $t \in [0,T]$ with some positive constant c_0 and $k_0 \in L^q([0,T])$.

(H2) $\mathcal{A}(t)$ is continuous for all $t \in [0, T]$.

(H3) The function $t \mapsto \langle \mathcal{A}(t)y, v\rangle$ is measurable on $(0, T)$ for all $y, v \in V$.

(H4) $\langle \mathcal{A}(t)y, y\rangle \geq c_1 \|y\|_V^p - k_1(t)$ for all $y \in V$ and $t \in [0, T]$ with some constant $c_1 > 0$ and function $k_1 \in L^1([0, T])$.

Remark 5.2. *If there exist a constant $c > 0$ and functions k_1 in $L^q([0, T])$ and k_2 in $L^1([0, T])$ satisfying the inequality*

$$\langle \mathcal{A}(t)y, y\rangle \geq c\|y\|_V^p - k_1(t)\|y\|_V - k_2(t)$$

then there exist some $\tilde{c} > 0$ and some function $\tilde{k}_1$ in $L^1([0, T])$ with

$$\langle \mathcal{A}(t)y, y\rangle \geq \tilde{c}\|y\|_V^p - \tilde{k}_1(t).$$

For example, choose $\tilde{c} := \frac{c}{2}$ and $\tilde{k}_1(t) := k_1(t) \sqrt[p-1]{\frac{2k_1(t)}{c}} - k_2(t)$ (w.l.o.g. assume $k_1 \geq 0$). Consider the two cases $\|y\|_V \geq \frac{c\|y\|_V^p}{2k_1(t)}$ and $\|y\|_V \leq \frac{c\|y\|_V^p}{2k_1(t)}$.

The next theorem shows the link between the properties of $\mathcal{A}(t)$ for fixed $t \in [0, T]$ and the operator $\mathfrak{A}$ defined in (59).

Theorem 5.3. *a) Assume that hypotheses (H1)-(H3) are satisfied. Then the mapping $\mathfrak{A}\colon L^p(T; V) \to L^q(T; V^\star)$ is continuous.*

b) Assume that the conditions (H1) and (H3) hold and that the operators $\mathcal{A}(t)$ are demi-continuous for all $t \in [0, T]$. Then $\mathfrak{A}\colon L^p(T; V) \to L^q(T; V^\star)$ is demi-continuous.

c) Assume that the conditions (H1)-(H4) hold. If $\mathcal{A}(t)$ is pseudomonotone for all $t \in [0, T]$, then $\mathfrak{A}\colon L^p(T; V) \to L^q(T; V^\star)$ is pseudomonotone with respect to $D(L)$.

d) Assume that the conditions (H1)-(H4) hold. If $\mathcal{A}(t)$ satisfies the S_+-property for all $t \in [0, T]$, then $\mathfrak{A}\colon L^p(T; V) \to L^q(T; V^\star)$ satisfies the S_+-property with respect to $D(L)$.

e) Assume that (H1) and (H3) hold. Then $\mathfrak{A}\colon L^p(T; V) \to L^q(T; V^\star)$ is bounded.

Proof. For the proofs of c) and d) see Theorem 2. in Berkovits and Mustonen (1996). The proof of b) is stated in Lemma 1, Berkovits and Mustonen (1996), and the proof of a) is similar. The last statement e) is given in Theorem 2.153 in Carl et al. (2007). □

Example 5.4. *In the case $p = 2$, the linear operators $\mathcal{A}(t)$, $t \in [0, T]$, defined in Example 3.1 satisfy (H1)-(H4) if the constants are independent of time t.*

We now derive some useful technical results. Analogously to Lemma 2.4 we obtain:

Lemma 5.5. *Let a, b be functions in $L^q(Q)$ resp. $L^p(T;W^{1,p}(\Omega))$ with $a \leq b$.*

a) Assume that there exists some $k \in L^q_+(Q)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, with

$$|G\tau_{W,p} y(x,t)| \leq k(x,t) \ \lambda_Q(d(x,t))\text{-a.e. for all } y \in [a,b] \cap L^p(Q).$$

Then $G\colon [a,b] \cap L^p(Q) \to L^q(Q)$ is continuous.

b) The truncation operator $T\colon L^p(Q) \to L^p(Q)$, $y \mapsto Ty$ resp. the operator $T\colon L^p(T;W^{1,p}(\Omega)) \to L^p(T;W^{1,p}(\Omega))$ given by

$$Ty(x,t) := \begin{cases} b(x,t) & \text{if} \quad y(x,t) > b(x,t) \\ y(x,t) & \text{if} \quad a(x,t) \leq y(x,t) \leq b(x,t) \\ a(x,t) & \text{if} \quad y(x,t) < a(x,t) \end{cases} \tag{60}$$

is continuous and bounded.

c) The Nemytskii operator $\mathfrak{B}\colon L^p(Q) \to L^q(Q)$ given by the Carathéodory function $\mathfrak{b}\colon Q \times \mathbb{R} \to \mathbb{R}$ with

$$\mathfrak{b}(x,t,s) = \begin{cases} (s - b(x,t))^{p-1} & \text{if} \quad s > b(x,t) \\ 0 & \text{if} \quad a(x,t) \leq s \leq b(x,t) \\ -(a(x,t) - s)^{p-1} & \text{if} \quad s < a(x,t) \end{cases} \tag{61}$$

is continuous and bounded. Moreover, the inequalities

$$|\mathfrak{b}(x,t,s)| \leq c_{p,1}(|a(x,t)| + |b(x,t)|)^{p-1} + c_{p,1}\,|s|^{p-1} \tag{62}$$

and

$$\int_Q \mathfrak{b}(id,y)y \, d\lambda_Q \geq c_{\mathfrak{B}} \|y\|^p_{L^p(Q)} - C_{\mathfrak{B}} \tag{63}$$

hold for all $y \in L^p(Q)$, where $c_{\mathfrak{B}} := \frac{1}{c_{p,2}} - \epsilon$ and $0 < \epsilon < \frac{1}{c_{p,2}}$ is arbitrary. The constants $c_{p,1}$ and $c_{p,2}$ are defined by

$$c_{p,1} = \begin{cases} 2, & 1 < p \leq 2 \\ 2^{p-1}, & 2 < p < \infty \end{cases} \tag{64}$$

and

$$c_{p,2} = \begin{cases} 1, & 1 < p < 2 \\ 2^{p-2}, & 2 \leq p < \infty \end{cases} . \tag{65}$$

$C_{\mathfrak{B}}$ denotes a constant given by

$$C_{\mathfrak{B}} := \left(4\left(\frac{1}{c_{p,2}} - \epsilon\right) + 2C_\epsilon\right)\left(\|a\|^p_{L^p(Q)} + \|b\|^p_{L^p(Q)}\right), \tag{66}$$

where $C_\epsilon := (\epsilon p)^{-q/p}\frac{1}{q}$.

The proof of a) is analogous to Lemma 2.4. For the proof of b) we refer to Lemma 2.89 in Carl et al. (2007) and Theorem 5.3 a). The operator $T\colon L^p(T;W^{1,p}(\Omega)) \to L^p(T;W^{1,p}(\Omega))$ is continuous, too. This is, for example, shown in Proposition 3.1 in Kandilakis and Papageorgiou (1998). The techniques to show c) are similar to the elliptic case.

5.1.4 Semi-linear Evolution Equations

We now concentrate on the case $p = 2$. As we have seen in Problem (Par2) we are dealing with mappings of the form $L + \mathfrak{A}$, where L is defined on $D(L)$. The following assumptions are made:

(H$\mathfrak{A}$) There exist a family $\{a(t)\colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{R} |\, t \in [0,T]\}$ containing symmetric bilinear forms and related operators $\mathcal{A}(t)\colon V \to V^\star$, $y \mapsto a(t)(y,\cdot)$ which satisfy the conditions:

(1) $\|\mathcal{A}(t)y\|_{V^\star} \leq c_0\|y\|_V + k_0(t)$ for all $y \in V$ and $t \in [0,T]$ with some positive constant c_0 and $k_0 \in L^2([0,T])$.

(2) $\mathcal{A}(t)$ is continuous for all $t \in [0,T]$.

(3) The function $t \mapsto \langle \mathcal{A}(t)y, v\rangle$ is measurable on $(0,T)$ for all $y, v \in V$.

(4) $\langle \mathcal{A}(t)y, y\rangle \geq c_1\|y\|_V^2 - k_1(t)$ for all $y \in V$ and $t \in [0,T]$ with some constant $c_1 > 0$ and function $k_1 \in L^1([0,T])$.

(5) The relation

$$a(t)(y, y^+) = a(t)(y^+, y^+)$$

holds for all $t \in [0,T]$ and for all $y \in W^{1,2}(\Omega)$.

(HG) The mapping $g\colon Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and the operator G is the corresponding Nemytskii operator given by

$$Gy\,(x,t) = g(x,t,y(x,t)).$$

Remark 5.6. *The assumption* (H$\mathfrak{A}$)(5) *is used to show that the truncated auxiliary problem yields solutions only within the sub-supersolution interval.*

We define the operator $\mathfrak{A}\colon L^2(T;V) \to L^2(T;V^\star)$ by the family of time-dependent elliptic operators in the following way:

$$(\mathfrak{A}y)(t) := \mathcal{A}(t)y(t), \quad t \in [0,T].$$

From the family of time-dependent mappings $a(t)$ we deduce the related bilinear form $\mathfrak{a}\colon L^2(T;W^{1,2}(\Omega)) \times L^2(T;W^{1,2}(\Omega)) \to \mathbb{R}$ by the definition

$$\mathfrak{a}(y,\phi) := \langle \tilde{\mathfrak{A}} y, \phi\rangle,$$

where $\tilde{\mathfrak{A}}\colon L^2(T;W^{1,2}(\Omega)) \to L^2(T;W^{1,2}(\Omega)^\star)$ and $(\tilde{\mathfrak{A}}y)(t) := a(t)(y(t),\cdot)$ for $t \in [0,T]$. Now we consider the following differential equation in the weak formulation:

Problem 19. *Find some $y \in D(L)$ such that*

$$Ly + \mathfrak{A}y + (\tau_{L^2(T;V),2}{}^\star iG\tau_{L^2(T;V),2})y = f \quad \text{in } L^2(T;V^\star) \tag{Par3}$$

Definition 5.1 (solution)**.** *The function $y \in D(L)$ is called a solution of Problem (Par3) if $(G\tau_{L^2(T;V),2})y \in L^2(Q)$ and*

$$\langle Ly + \mathfrak{A}y + (\tau_{L^2(T;V),2}{}^\star iGT\tau_{L^2(T;V),2})y, \phi\rangle = \langle f, \phi\rangle$$

for all test functions $\phi \in L^2(T;V)$.

In order to derive existence results for (Par3), we introduce the definitions of sub- and supersolution.

Definition 5.2 (subsolution)**.** *The function $\underline{y} \in W$ is called a subsolution of Problem (Par3) if*

i) $(G\tau_{W,2})\underline{y} \in L^2(Q)$,

ii) $(\underline{y} - y)^+ \in L^2(T;V)$ *and* $(\underline{y} - y)^+(0) = 0$ *for all* $y \in D(L)$ *and*

iii) $\langle y_t, \phi\rangle + \mathfrak{a}(\underline{y},\phi) + \langle(\tau_{L^2(T;V),2}{}^\star iGT\tau_{L^2(T;W^{1,2}(\Omega)),2})\underline{y}, \phi\rangle \leq \langle f,\phi\rangle$ *for all test functions* $\phi \in L^2(T;V) \cap L^2_+(Q)$.

Definition 5.3 (supersolution)**.** *The function $\overline{y} \in W$ is called a supersolution of Problem (Par3) if*

i) $(G\tau_{W,2})\overline{y} \in L^2(Q)$,

ii) $(y - \overline{y})^+ \in L^2(T;V)$ *and* $(y - \overline{y})^+(0) = 0$ *for all* $y \in D(L)$ *and*

iii) $\langle y_t, \phi\rangle + \mathfrak{a}(\overline{y},\phi) + \langle(\tau_{L^2(T;V),2}{}^\star iGT\tau_{L^2(T;W^{1,2}(\Omega)),2})\overline{y}, \phi\rangle \geq \langle f,\phi\rangle$ *for all test functions* $\phi \in L^2(T;V) \cap L^2_+(Q)$.

The following theorem yields an existence result for (Par3). The proof is based on techniques used in Carl et al. (2007).

Theorem 5.7. *Suppose* (H$\mathfrak{A}$) *and* (HG). *Let* $\underline{y}$ *and* $\overline{y}$ *be a sub- and a supersolution of (Par3) satisfying* $\underline{y} \le \overline{y}$ *and assume that there exists some* $k_G \in L^2_+(Q)$ *with*

$$|(Gy)(x,t)| \le k_G(x,t) \quad \lambda_Q(d(x,t))\text{-a.e. for all } y \in [\underline{y}, \overline{y}] \cap L^2(Q). \tag{67}$$

Then there exists at least one solution of (Par3) which lies in $[\underline{y}, \overline{y}]$.

Proof. We consider the following auxiliary problem:
Find some $y \in D(L)$ such that

$$Ly + \mathfrak{A}y + (\tau_{L^2(T;V),2}{}^{\star} iGT\tau_{L^2(T;V),2})y = f \quad \text{ in } L^2(T;V^\star), \tag{A-Par3}$$

where T is the truncation operator relating to $\underline{y}$ and $\overline{y}$ which is defined in Lemma 5.5. The mapping GT is continuous, see Lemma 5.5. We show that the operator $\mathfrak{A} + \tau_{L^2(T;V),2}{}^{\star} iGT\tau_{L^2(T;V),2}$ is pseudomonotone w.r.t. $D(L)$, continuous, coercive and bounded. Then we can apply Theorem 8.22 and obtain that the auxiliary problem (A-Par3) has a solution.
By proving the conditions (H1)-(H4), the pseudomonotonicity w.r.t. $D(L)$ follows from Theorem 5.3 c).
(H1) is valid since it is $\|k_G(t)\|_{L^2(\Omega)} \in L^2([0,T])$ and for all $y \in V$

$$\begin{aligned} \| (\mathcal{A}(t) + \tau_{V,2}{}^{\star} iG_tT_t\tau_{V,2})\, y\|_{V^\star} &\le \|\mathcal{A}(t)y\|_{V^\star} + \|\tau_{V,2}{}^{\star} iG_tT_t\tau_{V,2}y\|_{V^\star} \\ &\le c_0\|y\|_V^{p-1} + k_0(t) + \|k_G(t)\|_{L^2(\Omega)}\,, \end{aligned}$$

where the definitions $G_ty\,(x) := g(x,t,y(x))$, $y \in V$, and

$$T_ty(x) := \begin{cases} \overline{y}(x,t) & \text{if} \quad y(x) > \overline{y}(x,t) \\ y(x) & \text{if} \quad \underline{y}(x,t) \le y(x) \le \overline{y}(x,t) \\ \underline{y}(x,t) & \text{if} \quad y(x) < \underline{y}(x,t) \end{cases} \tag{68}$$

hold. We remark that it is $\left(\tau_{L^2(T;V),2}{}^{\star} iGT\tau_{L^2(T;V),2}y\right)(t) = \left(\tau_{V,2}{}^{\star}\tilde{i}G_tT_t\tau_{V,2}\right)y(t)$ for all $y \in L^2(T;V)$ with the isomorphism $\tilde{i}\colon L^2([0,T]) \to L^2([0,T])^\star$. For fixed $t \in (0,T)$ we can apply Lemma 2.4 and obtain the continuity of the Nemytskii operator G_tT_t on $L^2(\Omega)$. This implies (H2).
The measurability of $t \mapsto \langle(\mathcal{A}(t) + \tau_{V,2}{}^{\star} iG_tT_t\tau_{V,2})\,y, v\rangle$ for all y, $v \in V$, i.e. (H3), is given since $t \mapsto \int_\Omega g(id,t,y)v\, d\lambda_\Omega$ is measurable due to Fubini's Theorem.
The last condition (H4) is proven by

$$\langle(\mathcal{A}(t) + \tau_{V,2}{}^{\star} iG_tT_t\tau_{V,2})\,y, y\rangle \ge c_1\|y\|_V^p - \|k_G(t)\|_{L^2(\Omega)}\, \|y\|_{L^2(\Omega)}\,.$$

Continuity is implied by Theorem 5.3 a), since the conditions (H1)-(H3) are satisfied, compare Remark 5.2

We show that the operator $\mathfrak{A} + \tau_{L^2(T;V),2}{}^\star iGT\tau_{L^2(T;V),2}$ is coercive. Due to (67) it holds

$$\begin{aligned}\langle \tau_{L^2(T;V),2}{}^\star iGT\tau_{L^2(T;V),2}y, y\rangle &= \int_Q GT\tau_{L^2(T;V),2}y\, \tau_{L^2(T;V),2}y\, d\lambda_Q \\ &\geq -\|GT\tau_{L^2(T;V),2}y\|_{L^2(Q)}\, \|\tau_{L^2(T;V),2}y\|_{L^2(Q)} \\ &\geq -\|k_G\|_{L^2(Q)}\, \|\tau_{L^2(T;V),2}y\|_{L^2(Q)} \\ &\geq -\|k_G\|_{L^2(Q)}\, \|\tau_{L^2(T;V),2}\|_{Op}\, \|y\|_{L^2(T;V)}\, .\end{aligned}$$

Together with the strongly monotonicity of $\mathfrak{A}$ this yields

$$\begin{aligned}\langle \mathfrak{A}y + (\tau_{L^2(T;V),2}{}^\star iGT\tau_{L^2(T;V),2})y, y\rangle \geq{}& c\|y\|^2_{L^2(T;V)} \\ &- \|k_G\|_{L^2(Q)}\, \|\tau_{L^2(T;V),2}\|_{Op}\, \|y\|_{L^2(T;V)}\, .\end{aligned}$$

$\mathfrak{A}$ is linear and continuous and $\tau_{L^2(T;V),2}{}^\star iGT\tau_{L^2(T;V),2}$ is bounded uniformly w.r.t. y. Thus, the mapping $\mathfrak{A} + \tau_{L^2(T;V),2}{}^\star iGT\tau_{L^2(T;V),2}$ is bounded.
We show that every solution of (A-Par3) lies in $[\underline{y}, \overline{y}]$. Let y be any solution of (A-Par3), i.e. that the equation

$$\langle Ly + \mathfrak{A}y + (\tau_{L^2(T;V),2}{}^\star iGT\tau_{L^2(T;V),2})y, \phi\rangle = \langle f, \phi\rangle$$

holds for all $\phi \in L^2(T;V)$. Subtracting the inequality (28) for the supersolution yields

$$\begin{aligned}\langle y_t - \overline{y}_t, \phi\rangle + \mathfrak{a}(y - \overline{y}, \phi) + \langle (\tau_{L^2(T;V),2}{}^\star iGT\tau_{L^2(T;V),2})y, \phi\rangle& \\ - \langle (\tau_{L^2(T;V),2}{}^\star iGT\tau_{L^2(T;V),2})\overline{y}, \phi\rangle &\leq 0\end{aligned} \tag{69}$$

for all $\phi \in L^2(T;V) \cap L^2_+(Q)$. By definition it holds that $(y - \overline{y})^+ \in L^2(T;V) \cap L^2_+(Q)$ and $(y - \overline{y})^+(0) = 0$. We are now testing (69) with $\phi := (y - \overline{y})^+$. For the first term, the generalized integration by parts formula, compare Proposition 8.16 or Remark 2.145 in Carl et al. (2007), implies

$$\begin{aligned}\langle y_t - \overline{y}_t, (y - \overline{y})^+\rangle &= \frac{1}{2}\left(\|(y - \overline{y})^+(T)\|^2_{L^2(Q)} - \|(y - \overline{y})^+(0)\|^2_{L^2(Q)}\right) \\ &= \frac{1}{2}\|(y - \overline{y})^+(T)\|^2_{L^2(Q)} \geq 0.\end{aligned}$$

Since $\mathfrak{A}$ satisfies assumption (H$\mathfrak{A}$), it holds

$$\begin{aligned}c_1\|(y - \overline{y})^+\|^2_{L^2(T;V)} &= \int_0^T \|(y(t) - \overline{y}(t))^+\|^2_V\, \lambda_{[0,T]}(dt) \\ &\leq \int_0^T \langle \mathcal{A}(t)(y(t) - \overline{y}(t))^+, (y(t) - \overline{y}(t))^+\rangle\, \lambda_{[0,T]}(dt) \\ &= \mathfrak{a}(y - \overline{y}, (y - \overline{y})^+).\end{aligned}$$

The equality

$$\left((GT\tau_{L^2(T;V),2})y - (GT\tau_{L^2(T;V),2})\overline{y}\right)|_{\{(y-\overline{y})^+>0\}} = 0$$

implies that the term

$$\langle(\tau_{L^2(T;V),2}{}^{\star}iGT\tau_{L^2(T;V),2})y - (\tau_{L^2(T;V),2}{}^{\star}iGT\tau_{L^2(T;V),2})\overline{y}, (y-\overline{y})^+\rangle$$

is equal 0. Therefore, we obtain together with the inequality (69) that

$$\begin{aligned} 0 \le c_1\|(y-\overline{y})^+\|^2_{L^2(T;V)} &\le \mathfrak{a}(y-\overline{y},(y-\overline{y})^+) \\ &\le \langle y_t - \overline{y}_t, (y-\overline{y})^+\rangle + \mathfrak{a}(y-\overline{y},(y-\overline{y})^+) \le 0, \end{aligned}$$

which implies $(y-\overline{y})^+ = 0$, i.e. $y \le \overline{y}$.
The proof of $\underline{y} \le y$ follows with the same arguments. □

5.2 Existence of Solutions for Optimal Control Problems

After proving the existence of at least one solution of Problem (Par3) with fixed right hand side, we are going to show the solvability of the related optimal control problem. We denote the set of all solutions of (Par3) lying in $[\underline{y},\overline{y}] \cap L^2(Q)$ with $\mathcal{S}(f)$. Let $\underline{y}$, $\overline{y} \in W$ and $\underline{u}$, $\overline{u} \in L^2(Q)$ with $\underline{y} \le \overline{y}$ and $\underline{u} \le \overline{u}$. Assume the following properties on the objective functional J.

(HJ) $J\colon \left([\underline{y},\overline{y}] \cap L^2(Q)\right) \times \left([\underline{u},\overline{u}] \cap L^2(Q)\right) \to \mathbb{R}$ satisfies:
From $y_n \to y$ in $L^2(T;V)$ with $(y_n)_{n\in\mathbb{N}} \subset [\underline{y},\overline{y}] \cap L^2(T;V)$ and $u_n \rightharpoonup u$ in $L^2(Q)$ with $(u_n)_{n\in\mathbb{N}} \subset [\underline{u},\overline{u}] \cap L^2(Q)$ it follows
$J(y,u) \le \liminf_{n\to\infty} J(y_n,u_n)$.

With the previous definitions, the optimal control problem reads as follows.

Problem 20.

$$\begin{aligned} \min \quad & J(y,u) && \text{(OC-Par3)} \\ \textit{s.t.} \quad & Ly + \mathfrak{A}y + (\tau_{L^2(T;V),2}{}^{\star}iGT\tau_{L^2(T;V),2})y = \tau_{L^2(T;V),2}{}^{\star}iu \ \textit{in}\ L^2(T;V^{\star}) \\ \textit{and} \quad & u \in [\underline{u},\overline{u}] \cap L^2(Q) \\ & y \in [\underline{y},\overline{y}] \cap D(L) \end{aligned}$$

The existence of at least one solution of (OC-Par3) is shown in the proof of the next Theorem.

Theorem 5.8. *Let $\underline{y}$ be a subsolution of (Par3) with $f = \tau_{L^2(T;V),2}{}^{\star}i\underline{u}$ and $\overline{y}$ be a supersolution of (Par3) with $f = \tau_{L^2(T;V),2}{}^{\star}i\overline{u}$. Assume $\underline{y} \le \overline{y}$ and the hypotheses* (H$\mathfrak{A}$)*,* (HG)*,* (67) *and* (HJ)*. Then the optimal control problem* (OC-Par3) *has at least one solution (y,u).*

Proof. The first part of the proof is analogous to Theorem 2.8.
Let $(y_m, u_m)_{m\in\mathbb{N}}$ be the infimal sequence with

$$\lim_{m\to\infty} J(y_m, u_m) = \inf_{u\in[\underline{u},\overline{u}]\cap L^2(Q), y\in\mathcal{S}(\tau_{L^2(T;V),2}{}^\star iu)} J(y, u).$$

Then we know that there exist a weakly convergent subsequence $(u_m)_{m\in M}$, $M \subseteq \mathbb{N}$, and a weak limit $u_0 \in L^2(Q)$ with

$$u_m \rightharpoonup u_0 \text{ in } L^2(Q) \quad \text{and} \quad \tau_{L^2(T;V),2}{}^\star iu_m \rightharpoonup \tau_{L^2(T;V),2}{}^\star iu_0 \text{ in } L^2(T; V^\star) \tag{70}$$

due to the weak continuity of $\tau_{L^2(T;V),2}{}^\star i$, see Lemma 8.21. (In Theorem 2.8 we obtained here strong convergence because V is compactly embedded into $L^2(\Omega)$. But this fact does not imply that the embedding of $L^2(T; V)$ in $L^2(T; L^2(\Omega))$ is compact.)
Under (67) there exist a weakly convergent subsequence $(z_m)_{m\in M}$ (w.l.o.g. $M = M'$) with $z_m := (G\tau_{L^2(T;V),2})y_m$ and a weak limit $z_0 \in L^2(Q)$. We obtain $\tau_{L^2(T;V),2}{}^\star i(u_m - z_m) \rightharpoonup \tau_{L^2(T;V),2}{}^\star i(u_0 - z_0)$ in $L^2(T; V^\star)$.
We consider the linearized boundary value problem

$$Ly + \mathfrak{A}y = Z_m := \tau_{L^2(T;V),2}{}^\star i(u_m - z_m) \text{ in } L^2(T; V^\star),$$

which is solved by $y = y_m$. Theorem 23.A in Zeidler (1990a) shows that the solution of this linear problem depends continuously on the data. Hence we get

$$y_m \rightharpoonup y_0 \text{ in } W_V. \tag{71}$$

The linearity and continuity of $\mathfrak{A}$ (see Theorem 5.3 a)) imply

$$\mathfrak{A}y_m \rightharpoonup \mathfrak{A}y_0 \text{ in } L^2(T; V^\star). \tag{72}$$

Since $W_V \hookrightarrow L^2(Q)$ is compact, see Example 8.18, it holds

$$\tau_{L^2(T;V),2}y_m \to \tau_{L^2(T;V),2}y_0 \text{ in } L^2(Q). \tag{73}$$

It follows by Lemma 5.5 and Lemma 8.21 that

$$(\tau_{L^2(T;V),2}{}^\star iG\tau_{L^2(T;V),2})y_m \to (\tau_{L^2(T;V),2}{}^\star iG\tau_{L^2(T;V),2})y_0 \text{ in } L^2(T; V^\star). \tag{74}$$

Due to $\|Ly_m\|_{L^2(T;V^\star)} \leq \|y_m\|_{W_V}$ we know from the Eberlein-Smulian Theorem that there exists some convergent subsequence with $Ly_m \rightharpoonup l$ in $L^2(T; V^\star)$ for some $l \in L^2(T; V^\star)$. As stated in Proposition 5.1, the mapping L is closed, i.e. the set

$$Gr\,(L) := \{(x, y)|\, x \in D(L),\, Lx = y\} \tag{75}$$

is closed in $L^2(T;V) \times L^2(T;V^\star)$. By the linearity of L we obtain that the set $Gr\,(L)$ is convex and hence weakly closed. This implies that from (71) and $Ly_m \rightharpoonup l$ in $L^2(T;V^\star)$ it follows $y_0 \in D(L)$ and $l = Ly_0$, i.e.

$$Ly_m \rightharpoonup Ly_0 \text{ in } L^2(T;V^\star). \tag{76}$$

Together we obtain from (70), (72), (73), (74) and (76) the weak convergence in $L^2(T;V^\star)$:

$$\begin{aligned} Ly_m + \mathfrak{A}y_m + &(\tau_{L^2(T;V),2}{}^\star iG\tau_{L^2(T;V),2})y_m \rightharpoonup \\ &Ly_0 + \mathfrak{A}y_0 + (\tau_{L^2(T;V),2}{}^\star iG\tau_{L^2(T;V),2})y_0 = \tau_{L^2(T;V),2}{}^\star iu_0. \end{aligned}$$

Due to the assumptions on J we obtain by (70) and (72)

$$J(y_0,u_0) \leq \liminf_{n\to\infty} J(y_n,u_n).$$

□

6 Optimal Control Problems with Quasi-linear Evolution Equations

6.1 Existence of Solutions for Optimal Control Problems

In the previous Section 5 semi-linear parabolic equations have been considered. Now, we examine the quasi-linear case. Assume that V is some closed subspace of $W^{1,p}(\Omega)$, $2 \leq p < \infty$ with $W_0^{1,p}(\Omega) \subseteq V \subseteq W^{1,p}(\Omega)$. The mapping i denotes the isomorphism from $L^q(Q)$ into $L^p(Q)^\star$ and the operator $\tau_{L^p(T;V),p}$ is the embedding from $L^p(T;V)$ into $L^p(Q)$. Let V, W, W_V and L be defined as in the former section.
For coefficient functions $a_i\colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N$, $i = 1,\dots,N$ let the semi-linear forms $a(t)\colon W^{1,p}(\Omega) \times W^{1,p}(\Omega) \to \mathbb{R}$, $t \in [0,T]$ and $\mathfrak{a}\colon L^p(T;W^{1,p}(\Omega)) \times L^p(T;W^{1,p}(\Omega)) \to \mathbb{R}$ be given by

$$a(t)(y,v) := \int_\Omega \sum_{i=1}^N a_i(id,t,y,\nabla y)\frac{\partial v}{\partial x_i}\,d\lambda_\Omega, \quad y,v \in W^{1,p}(\Omega) \text{ and}$$

$$\mathfrak{a}(y,v) := \int_Q \sum_{i=1}^N a_i(id,y,\nabla y)\frac{\partial v}{\partial x_i}\,d\lambda_Q, \quad y,v \in L^p(T;W^{1,p}(\Omega)).$$

We introduce the associated operators

$$\begin{aligned} &\mathcal{A}(t)\colon V \to V^\star,\ y \mapsto a(t)(y,\cdot) \qquad \text{and} \\ &\mathfrak{A}\colon L^p(T;V) \to L^q(T;V^\star),\ y \mapsto \mathfrak{a}(y,\cdot). \end{aligned}$$

In the following we impose the Leray-Lions conditions on the coefficient functions a_i:

(H1) Carathéodory and Growth Condition: Every coefficient a_i satisfies the Carathéodory conditions, i.e.

- $a_i(\cdot,\cdot,s,\zeta)$ is measurable for all $(s,\zeta) \in \mathbb{R} \times \mathbb{R}^N$ and
- $a_i(x,t,\cdot,\cdot)$ is continuous for a.e. $(x,t) \in Q$.

There exist some constant $c_0 > 0$ and a function $k_0 \in L^q_+(Q)$ with

$$|a_i(x,t,s,\zeta)| \leq k_0(x,t) + c_0(|s|^{p-1} + |\zeta|^{p-1}).$$

(H2) Monotonicity Type Condition: The coefficients satisfy the monotonicity condition with respect to ζ

$$\sum_{i=1}^{N}(a_i(x,t,s,\zeta) - a_i(x,t,s,\zeta'))(\zeta_i - \zeta'_i) > 0$$

for a.e. $(x,t) \in Q$, for all $s \in \mathbb{R}$ and all $\zeta, \zeta' \in \mathbb{R}^N$ with $\zeta \neq \zeta'$.

(H3) Coercivity Type Condition: There exist some constant $c_1 > 0$ and a function $k_1 \in L^1(Q)$ with

$$\sum_{i=1}^{N} a_i(x,t,s,\zeta)\zeta_i \geq c_1 \, |\zeta|^p - k_1(x,t).$$

(HG) There exists some Carathéodory function $g\colon Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that the operator G is the corresponding Nemytskii operator given by

$$Gy(x,t) = g(x,t,y(x,t),\nabla y(x,t)), \quad y \in L^p(T;W^{1,p}(\Omega)).$$

Now we examine the solvability of the following quasi-linear PDE for $f \in L^q(T;V^\star)$:

Problem 21. *Find some $y \in D(L)$ such that*

$$Ly + \mathfrak{A}y + (\tau_{L^p(T;V),p}{}^\star iG)y = f \text{ in } L^q(T;V^\star) \tag{Par4}$$

Definition 6.1 (solution)**.** *The function $y \in D(L)$ is called a solution of Problem (Par4) if $Gy \in L^q(Q)$ and*

$$\langle Ly + \mathfrak{A}y + (\tau_{L^p(T;V),p}{}^\star iGT)y, \phi\rangle = \langle f, \phi\rangle$$

for all test functions $\phi \in L^p(T;V)$.

Definition 6.2 (subsolution)**.** *The function* $\underline{y} \in W$ *is called a subsolution of Problem (Par4) if*

i) $G\underline{y} \in L^q(Q)$,

ii) $(\underline{y} - y)^+ \in L^p(T;V)$ *and* $(\underline{y} - y)^+(0) = 0$ *a.e. on* Ω *for all* $y \in D(L)$ *and*

iii) $\langle \underline{y}_t, \phi \rangle + \mathfrak{a}(\underline{y}, \phi) + \langle (\tau_{L^p(T;V),p}{}^\star iGT)\underline{y}, \phi \rangle \leq \langle f, \phi \rangle$ *for all test functions* $\phi \in L^p(T;V) \cap L^p_+(Q)$.

Definition 6.3 (supersolution)**.** *The function* $\overline{y} \in W$ *is called a supersolution of Problem (Par4) if*

i) $G\overline{y} \in L^q(Q)$,

ii) $(y - \overline{y})^+ \in L^p(T;V)$ *and* $(y - \overline{y})^+(0) = 0$ *a.e. on* Ω *for all* $y \in D(L)$ *and*

iii) $\langle \overline{y}_t, \phi \rangle + \mathfrak{a}(\overline{y}, \phi) + \langle (\tau_{L^p(T;V),p}{}^\star iGT)\overline{y}, \phi \rangle \geq \langle f, \phi \rangle$ *for all test functions* $\phi \in L^p(T;V) \cap L^p_+(Q)$.

Theorem 6.1. *Let* $\underline{y}$ *and* $\overline{y}$ *be a sub- and a supersolution of (Par4) satisfying* $\underline{y} \leq \overline{y}$ *and suppose (H1)-(H3), (HG). Assume that there exist some* $k_G \in L^q_+(Q)$ *and some constant* $c_G > 0$ *with*

$$|g(x,t,s,\zeta)| \leq k_G(x,t) + c_G\,|\zeta|^{p-1} \tag{77}$$

for all $s \in [\underline{y}(x,t), \overline{y}(x,t)]$ $\lambda_Q(d(x,t))$*-a.e. and* $\zeta \in \mathbb{R}^N$*. Then there exists at least one solution of (Par4) which lies in* $[\underline{y}, \overline{y}]$.

For the proof we refer to Theorem 3.37 in Carl et al. (2007) in which the proof is given for $V = W_0^{1,p}(\Omega)$. The case for arbitrary V is along the same lines.
The next lemma yields a technical result.
Let the constant $\lambda > 0$ satisfy the inequality

$$\lambda > \frac{(\epsilon q)^{-p/q} c_G}{p c_{\mathfrak{B}}}, \text{ where } \epsilon < \frac{c_1}{c_G}. \tag{78}$$

For the constant $c_{\mathfrak{B}}$ see Lemma 5.5 c). We introduce the operator $\mathfrak{A}_T\colon L^p(T;V) \to L^q(T;V^\star)$:

$$\mathfrak{A}_T y(x,t) := -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x,t,Ty(x,t), \nabla y(x,t)).$$

In the proof of Theorem 3.37 in Carl et al. (2007) it is shown that every solution of the equation

$$Ly + \mathfrak{A}_T y + \lambda(\tau_{L^p(T;V),p}{}^\star i\mathfrak{B}\tau_{L^p(T;V),p})y + (\tau_{L^p(T;V),p}{}^\star iGT)y = f \quad \text{in } L^q(T;V^\star) \tag{A-Par4}$$

solves (Par4) as well.
Let $\underline{y}$, $\overline{y} \in W$ and $\underline{u}$, $\overline{u} \in L^q(Q)$ with $\underline{y} \le \overline{y}$ and $\underline{u} \le \overline{u}$. Assume the following properties on the objective functional J.

(HJ) $J\colon \big([\underline{y},\overline{y}] \cap L^p(T;V)\big) \times ([\underline{u},\overline{u}] \cap L^q(Q)) \to \mathbb{R}$ satisfies:
From $y_n \to y$ in $L^p(T;V)$ with $(y_n)_{n\in\mathbb{N}} \subset [\underline{y},\overline{y}] \cap L^p(T;V)$ and $u_n \rightharpoonup u$ in $L^q(Q)$ with $(u_n)_{n\in\mathbb{N}} \subset [\underline{u},\overline{u}] \cap L^q(Q)$ it follows $J(y,u) \le \liminf_{n\to\infty} J(y_n,u_n)$.

With these definitions the optimal control problem reads as follows.

Problem 22.

$$\begin{aligned} \min \quad & J(y,u) \\ \textit{s.t.} \quad & Ly + \mathfrak{A}y + (\tau_{L^p(T;V),p}{}^\star iG)y = \tau_{L^p(T;V),p}{}^\star iu \textit{ in } L^q(T;V^\star) \\ \textit{and} \quad & u \in [\underline{u},\overline{u}] \cap L^q(Q) \\ & y \in [\underline{y},\overline{y}] \cap D(L) \end{aligned} \tag{OC-Par4}$$

For $f \in L^q(T;V^\star)$ we denote the set of all solutions of (Par4) lying in $[\underline{y},\overline{y}]$ with $\mathcal{S}(f)$. In the later proof of the existence of a solution for Problem (OC-Par4) we make use of the following statement.

Lemma 6.2. *Let $A\colon L^p(T;V) \to L^q(T;V^\star)$ be a coercive operator, i.e.*

$$\lim_{\|y\|_{L^p(T;V)}\to\infty} \frac{\langle Ay, y\rangle}{\|y\|_{L^p(T;V)}} \to \infty.$$

Moreover, assume that the equations

$$Ly_n + Ay_n = f_n,\ n \in \mathbb{N} \tag{79}$$

are satisfied for some given sequences $(y_n)_{n\in\mathbb{N}} \subset D(L) \subset L^p(T;V)$ and $(f_n)_{n\in\mathbb{N}} \subset L^q(T;V^\star)$, where the sequence $(f_n)_{n\in\mathbb{N}}$ is assumed to be bounded in $L^q(T;V^\star)$. Then $(y_n)_{n\in\mathbb{N}}$ is bounded in $L^p(T;V)$.

Proof. Assume that the sequence $(y_n)_{n\in\mathbb{N}}$ is unbounded. Then there exists some subsequence $(y_{n'})_{n'\in M}$, $M \subset \mathbb{N}$, such that it holds $\lim_{n'\to\infty} \|y_{n'}\|_{L^p(T;V)} = \infty$ and $\|y_{n'}\|_{L^p(T;V)} > 0$ for all $n' \in M$. By (79) and

$$\langle Ly_{n'}, y_{n'}\rangle = \frac{1}{2}\left(\|y_{n'}(T)\|_{L^2(\Omega)} - \|y_{n'}(0)\|_{L^2(\Omega)}\right)$$

(see, e.g., Remark 2.145 in Carl et al. (2007) we obtain

$$\begin{aligned}\|f_{n'}\|_{Op}\,\|y_{n'}\|_{L^p(T;V)} &\geq \langle f_{n'}, y_{n'}\rangle = \langle Ly_{n'} + Ay_{n'}, y_{n'}\rangle \\ &= \left(\frac{\|y_{n'}(T)\|_{L^2(\Omega)}}{2\|y_{n'}\|_{L^p(T;V)}} + \frac{\langle Ay_{n'}, y_{n'}\rangle}{\|y_{n'}\|_{L^p(T;V)}}\right)\|y_{n'}\|_{L^p(T;V)}\end{aligned}$$

and hence by the coercivity of A

$$\|f_{n'}\|_{Op} \geq \frac{\|y_{n'}(T)\|_{L^2(\Omega)}}{2\|y_{n'}\|_{L^p(T;V)}} + \frac{\langle Ay_{n'}, y_{n'}\rangle}{\|y_{n'}\|_{L^p(T;V)}} \to \infty,$$

which is a contradiction to the assumption of boundedness for the sequence $(f'_n)_{n'\in M}$. □

Now we derive an existence result analogously to Theorem 5.8.

Theorem 6.3. *Let $\underline{y}$ be a subsolution of (Par4) with $f = \tau_{L^p(T;V),p}{}^\star i\underline{u}$ and $\overline{y}$ be a supersolution of (Par4) with $f = \tau_{L^p(T;V),p}{}^\star i\overline{u}$. Assume $\underline{y} \leq \overline{y}$ and suppose (H1)-(H3), (HG),* (77) *and* (HJ). *Then the optimal control problem* (OC-Par4) *has a solution (y, u).*

Proof. We consider the following related auxiliary problem:

$$\begin{aligned}\min\quad & J(y,u) && \text{(A-OC-Par4)}\\ \text{s.t.}\quad & Ly + \mathfrak{A}_T y + \lambda(\tau_{L^p(T;V),p}{}^\star i\mathfrak{B}\tau_{L^p(T;V),p})y + (\tau_{L^p(T;V),p}{}^\star iGT)y \\ & \qquad = \tau_{L^p(T;V),p}{}^\star iu \text{ in } L^q(T;V^\star)\\ \text{and}\quad & u \in [\underline{u},\overline{u}] \cap L^q(Q)\\ & y \in [\underline{y},\overline{y}] \cap D(L)\end{aligned}$$

Since every state in $[\underline{y},\overline{y}]$ solves (Par4) if and only if it solves (A-Par4), every solution of the optimal control problem (A-OC-Par4) is a solution of the optimal control problem (OC-Par4). As shown in the proof of Theorem 3.17 in Carl et al. (2007), the equation (A-Par4) is solved by at least one $y \in L^p(T;V)$ satisfying $y \in [\underline{y},\overline{y}]$. Hence for every $u \in [\underline{u},\overline{u}] \cap L^q(Q)$ there exists at least one $y \in [\underline{y},\overline{y}] \cap D(L)$ solving (A-Par4). Let $(y_m, u_m)_{m\in\mathbb{N}}$ be the infimal sequence of Problem (A-OC-Par4) with

$$\lim_{m\to\infty} J(y_m, u_m) = \inf_{u\in[\underline{u},\overline{u}]\cap L^q(Q),\, y\in\mathcal{S}_A(\tau_{L^p(T;V),p}{}^\star iu)} J(y,u),$$

where $\mathcal{S}_A(f)$ denotes the set of all solutions of (A-Par4) lying in $[\underline{y}, \overline{y}] \cap D(L)$ with right hand side $f \in L^q(T; V^\star)$.
There exist a weakly convergent subsequence $(u_m)_{m\in M}$, $M \subseteq \mathbb{N}$, and $u_0 \in [\underline{u}, \overline{u}] \cap L^q(Q)$ with

$$u_m \rightharpoonup u_0 \text{ in } L^q(Q) \text{ and } \tau_{L^p(T;V),p}{}^\star i u_m \rightharpoonup \tau_{L^p(T;V),p}{}^\star i u_0 \text{ in } L^q(T; V^\star). \quad (80)$$

Due to the choice of λ, see (78), the mapping $\mathfrak{A}_T + \lambda \tau_{L^p(T;V),p}{}^\star i \mathfrak{B} \tau_{L^p(T;V),p} + \tau_{L^p(T;V),p}{}^\star i G T$ is coercive:

$$\begin{aligned}
&\langle (\mathfrak{A}_T + \tau_{L^p(T;V),p}{}^\star i \mathfrak{B} \tau_{L^p(T;V),p} + \tau_{L^p(T;V),p}{}^\star i G T) y, y\rangle \\
&\quad \geq (c_1 - c_G \epsilon) \|\nabla y\|^p_{L^p(Q)} + (\lambda c_{\mathfrak{B}} - c_G C_\epsilon) \|y\|^p_{L^p(Q)} - \|k_1\|_{L^1([0,T])} - C_{\mathfrak{B}} \\
&\qquad - \left(\|k_G\|_{L^q(Q)} + c_G c_p \| \left|\nabla \underline{y}\right| + |\nabla \overline{y}| \|^{p-1}_{L^p(Q)} \right) \|y\|_{L^p(Q)}
\end{aligned}$$

with $\epsilon < \frac{c_1}{c_G}$ and $C_\epsilon := (\epsilon q)^{-p/q} \frac{1}{p}$. The constant c_p is defined in (50). The sequence $(y_m)_{m\in M}$ is bounded in $L^p(T; V)$ since $(\tau_{L^p(T;V),p}{}^\star i u_m)_{m\in M}$ is bounded, see Lemma 6.2. Thus, there exists some subsequence $(y_m)_{m\in M}$ which is weakly convergent to some y_0 in $L^p(T; V)$:

$$y_m \rightharpoonup y_0 \text{ in } L^p(T; V). \quad (81)$$

The limit $\tau_{L^p(T;V),p} y_0$ lies in the weakly closed set $[\underline{y}, \overline{y}] \cap L^p(Q)$. The operators $\mathfrak{A}_T$, $\tau_{L^p(T;V),p}{}^\star i G T$ and $\tau_{L^p(T;V),p}{}^\star i \mathfrak{B} \tau_{L^p(T;V),p}$ are bounded mappings from $L^p(T; V)$ in $L^q(T; V^*)$, see Theorem 5.3 e). By the equation

$$\begin{aligned}
L y_m = &\tau_{L^p(T;V),p}{}^\star i u_m - \mathfrak{A}_T y_m - \lambda(\tau_{L^p(T;V),p}{}^\star i \mathfrak{B} \tau_{L^p(T;V),p}) y_m \\
&- (\tau_{L^p(T;V),p}{}^\star i G T) y_m
\end{aligned}$$

and the boundedness of the right hand side, the boundedness of the sequence $(L y_m)_{m\in M}$ in $L^q(T; V^\star)$ follows. Hence there exists a weakly convergent subsequence $(L y_m)_{m\in M}$ in $L^q(T; V^\star)$. The linear operator L is closed. This implies the weak closedness of L. Therefore, $L y_m$ converges weakly to $L y_0$ in $L^q(T; V^\star)$ and it holds that $(y_m)_{m\in M}$ is bounded in W_V. The embedding of W_V into $L^p(Q)$ is compact, see Example 8.18. Thus, we obtain

$$\tau_{L^p(T;V),p} y_m \to \tau_{L^p(T;V),p} y_0 \text{ in } L^p(Q). \quad (82)$$

We show that $\mathfrak{B}$ satisfies the properties (H1)-(H3) in Section 5.1.3. By Lemma 5.5 b) it holds for all $y \in V$

$$\|\tau_{L^p(T;V),p}{}^\star i \mathfrak{b}(id, t, y)\|_{V^\star} \leq c_{p,1} \| \left|\underline{y}(id, t)\right| + |\overline{y}(id, t)| \|^{p-1}_{L^p(\Omega)} + c_{p,1} \|y\|^{p-1}_{L^p(\Omega)}.$$

For fixed $t \in (0, T)$ we can apply Lemma 4.3 and obtain the continuity of the Nemytskii operator $\mathfrak{B}_t$ on $L^p(\Omega)$, where we define $\mathfrak{B}_t y\,(x) := \mathfrak{b}(x, t, y(x))$ for

$y \in V$.
The measurability of $t \mapsto \langle (\tau_{V,p}{}^{\star} i \mathfrak{B}_t \tau_{V,p})\, y, v\rangle$ for all $y,\, v \in V$, i.e. (H3), is given since $t \mapsto \int_\Omega \mathfrak{b}(id, t, y) v \, d\lambda_\Omega$ is measurable due to Fubini's Theorem.
Theorem 5.3 a) implies that the operator

$$\mathfrak{A}_T + \lambda \tau_{L^p(T;V),p}{}^{\star} i \mathfrak{B} \tau_{L^p(T;V),p} + \tau_{L^p(T;V),p}{}^{\star} iGT \colon L^p(T;V) \to L^q(T;V^\star)$$

is continuous.
For the underlying operators of the mapping $\mathfrak{A}_T + \tau_{L^p(T;V),p}{}^{\star} iGT$ conditions (H1)-(H4) in Section 5.1.3 hold. Applying Theorem 2.109 in Carl et al. (2007) and Theorem 5.3 d) shows that the operator $\mathfrak{A}_T + \tau_{L^p(T;V),p}{}^{\star} iGT$ satisfies the S_+-property w.r.t. $D(L)$. Hence we obtain from

$$\begin{aligned}
&\langle \mathfrak{A}_T y_m + \tau_{L^p(T;V),p}{}^{\star} iGT y_m, y_m - y_0 \rangle \\
&= \langle \tau_{L^p(T;V),p}{}^{\star} i u_m - \lambda \tau_{L^p(T;V),p}{}^{\star} i \mathfrak{B} \tau_{L^p(T;V),p} y_m, y_m - y_0 \rangle \\
&= \int_Q (u_m - \lambda \mathfrak{B} \tau_{L^p(T;V),p} y_m)\, \tau_{L^p(T;V),p}(y_m - y_0)\, d\lambda_Q \to 0
\end{aligned}$$

the convergence

$$y_m \to y_0 \text{ in } L^p(T;V).$$

Together with $Ly_m \rightharpoonup Ly_0$ in $L^q(T;V^\star)$ the convergence

$$\begin{aligned}
&\big(L + \mathfrak{A}_T + \tau_{L^p(T;V),p}{}^{\star} i \mathfrak{B} \tau_{L^p(T;V),p} + \tau_{L^p(T;V),p}{}^{\star} iGT\big)\, y_m \\
&\rightharpoonup \big(L + \mathfrak{A}_T + \tau_{L^p(T;V),p}{}^{\star} i \mathfrak{B} \tau_{L^p(T;V),p} + \tau_{L^p(T;V),p}{}^{\star} iGT\big)\, y_0
\end{aligned}$$

holds in $L^q(T;V^\star)$. Now we can conclude that

$$\big(L + \mathfrak{A}_T + \tau_{L^p(T;V),p}{}^{\star} i \mathfrak{B} \tau_{L^p(T;V),p} + \tau_{L^p(T;V),p}{}^{\star} iGT\big)\, y_0 = \tau_{L^p(T;V),p}{}^{\star} i u_0.$$

Therefore, by (80), (82) and (HJ), we get that

$$J(y_0, u_0) \leq \liminf_{m \to \infty} J(y_m, u_m).$$

□

6.2 Existence of Solutions for Extended Optimal Control Problems

In (OC-Par4) we have considered pointwise state constraints described by the sub- and supersolution. We now admit generalized pointwise state constraints independent of the sub- and supersolution. The boundedness condition of G on

$[\underline{y}, \overline{y}]$, see (77), and the assumption on the objective functional J, see (HJ), have to be adjusted appropriately. Let $Y := [y_1, y_2]$ with y_1, $y_2 \in L^p(T; W^{1,p}(\Omega))$ and $y_1 \leq y_2$ and denote $Y(x,t) := [y_1(x,t), y_2(x,t)]$ for $(x,t) \in \Omega \times [0,T]$. Assume $\underline{u}$, $\overline{u} \in L^q(Q)$. The optimal control problem we are now interested in has the following form:

Problem 23.

$$\begin{aligned} \min \quad & J(y,u) && \text{(OC-Par4-G)} \\ \textit{s.t.} \quad & Ly + \mathfrak{A}y + (\tau_{L^p(T;V),p}{}^{\star} iG)y = \tau_{L^p(T;V),p}{}^{\star} iu \textit{ in } L^q(T;V^{\star}) \\ \textit{and} \quad & u \in [\underline{u}, \overline{u}] \cap L^q(Q) \\ & y \in Y \cap D(L) \end{aligned}$$

Assumption (HJ) is now replaced by:

(HJ) $J\colon (Y \cap L^p(T;V)) \times ([\underline{u}, \overline{u}] \cap L^q(Q)) \to \mathbb{R}$ satisfies the condition:
From $y_n \to y$ in $L^p(T;V)$ with $(y_n)_{n\in\mathbb{N}} \subset Y \cap L^p(T;V)$ and
$u_n \rightharpoonup u$ in $L^q(Q)$ with $(u_n)_{n\in\mathbb{N}} \subset [\underline{u}, \overline{u}] \cap L^q(Q)$ it follows
$J(y,u) \leq \liminf_{n\to\infty} J(y_n, u_n)$.

Theorem 6.4. *Suppose (H1)-(H3), (HG) and* (HJ)*. We assume that there exists at least one $u \in [\underline{u}, \overline{u}] \cap L^q(\Omega)$ such that (Par4) has a solution $y \in Y \cap D(L)$ and that there exist some $k_G \in L^q_+(Q)$ and some positive constant c_G with*

$$|g(x,t,s,\zeta)| \leq k_G(x,t) + c_G\, |\zeta|^{p-1} \tag{83}$$

for all $s \in Y(x,t)$ $\lambda_Q(d(x,t))$-a.e. and $\zeta \in \mathbb{R}^N$. Then the optimal control problem (OC-Par4-G) *has at least one solution (y,u).*

Proof. The truncation operator $\overline{T}\colon L^p(T;W^{1,p}(\Omega)) \to L^p(T;W^{1,p}(\Omega))$ resp. the operator $\overline{T}\colon L^p(Q) \to L^p(Q)$ is defined by

$$\overline{T}y(x,t) := \begin{cases} y_2(x,t) & \text{if} \quad y(x,t) > y_2(x,t) \\ y(x,t) & \text{if} \quad y_1(x,t) \leq y(x,t) \leq y_2(x,t). \\ y_1(x,t) & \text{if} \quad y(x,t) < y_1(x,t) \end{cases} \tag{84}$$

The Nemytskii operator $\overline{\mathfrak{B}}\colon L^p(Q) \to L^q(Q)$ is given by the Carathéodory function $\overline{\mathfrak{b}}\colon Q \times \mathbb{R} \to \mathbb{R}$ with

$$\overline{\mathfrak{b}}(x,t,s) = \begin{cases} (s - y_2(x,t))^{p-1} & \text{if} \quad s > y_2(x,t) \\ 0 & \text{if} \quad y_1(x,t) \leq s \leq y_2(x,t). \\ -(y_1(x,t) - s)^{p-1} & \text{if} \quad s < y_1(x,t) \end{cases} \tag{85}$$

By assumption, there exists at least one solution of (Par4). Hence there exists for at least one $u \in [\underline{u}, \overline{u}] \cap L^q(Q)$ a solution $y \in Y \cap D(L)$ of

$$\begin{aligned} &Ly + \mathfrak{A}_{\overline{T}} y + \overline{\lambda}({\tau_{L^p(T;V),p}}^\star i \overline{\mathfrak{B}} \tau_{L^p(T;V),p}) y \\ &\qquad + ({\tau_{L^p(T;V),p}}^\star i G \overline{T}) y = {\tau_{L^p(T;V),p}}^\star i u \text{ in } L^q(T;V^\star), \end{aligned} \tag{86}$$

where $\overline{\lambda} > 0$ satisfies

$$\overline{\lambda} > \frac{(\epsilon q)^{-p/q} c_G}{p c_{\overline{\mathfrak{B}}}}, \text{ with } \epsilon < \frac{c_1}{c_G}.$$

Moreover, every solution of (86) with $y \in Y$ is a solution of (Par4). Hence the problems (OC-Par4-G) and

$$\begin{aligned} \min \quad & J(y,u) \\ \text{s.t.} \quad & Ly + \mathfrak{A}_{\overline{T}} y + \overline{\lambda}({\tau_{L^p(T;V),p}}^\star i \overline{\mathfrak{B}} \tau_{L^p(T;V),p}) y \\ & \quad + ({\tau_{L^p(T;V),p}}^\star i G \overline{T}) y = {\tau_{L^p(T;V),p}}^\star i u \text{ in } L^q(T;V^\star) \\ \text{and} \quad & u \in [\underline{u}, \overline{u}] \cap L^q(Q) \\ & y \in Y \cap D(L) \end{aligned} \tag{A-OC-Par4-G}$$

are equivalent. The operator

$$\mathfrak{A}_{\overline{T}} + {\tau_{L^p(T;V),p}}^\star i G \overline{T} \colon L^p(T;V) \to L^q(T;V^\star)$$

is coercive and continuous. The further proof is along the lines of Theorem 6.3. □

Remark 6.5. *For $\underline{u}$, $\overline{u} \in L^q(Q)$ resp. $L^q(\Sigma)$ let $U = [\underline{u}, \overline{u}] \cap L^q(Q)$ or $U = [\underline{u}, \overline{u}] \cap L^q(\Sigma)$ (in the case of no Dirichlet boundary conditions) and consider the optimal control problem*

$$\begin{aligned} \min \quad & J(y,u) \\ \textit{s.t.} \quad & Ly + \mathfrak{A} y + ({\tau_{L^p(T;V),p}}^\star i G) y = {\tau_{L^p(T;V),p}}^\star B u \;\; \textit{in } L^q(T;V^\star) \\ \textit{and} \quad & u \in U \\ & y \in Y \cap D(L). \end{aligned} \tag{87}$$

Here, the operator $B \colon U \to L^p(Q)^\star$ is assumed to be weakly continuous. The existence of an optimal pair can be proven analogously to Theorem 6.4.

We now consider the case of one-sided pointwise state constraints. Let $Y \in \{(-\infty, y_1], [y_1, \infty)\}$ with $y_1 \in L^p(T; W^{1,p}(\Omega))$ and denote $Y(x,t) := (-\infty, y_1(x,t)]$ resp. $Y(x,t) := [y_1(x,t), \infty)$. The corresponding optimal control problem reads as follows.

Problem 24.

$$\begin{aligned} \min \quad & J(y,u) && \text{(OC-Par4-G2)} \\ s.t. \quad & Ly + \mathfrak{A}y + (\tau_{L^p(T;V),p}{}^{\star} iG)y = \tau_{L^p(T;V),p}{}^{\star} iu \ in \ L^q(T;V^\star) \\ and \quad & u \in [\underline{u}, \overline{u}] \cap L^q(Q) \\ & y \in Y \cap D(L) \end{aligned}$$

Under appropriate assumptions, the existence of a solution for (Par4) can be proven.

Theorem 6.6. *Suppose (H1)-(H2), (HG) and* (HJ)*. We assume that there exists at least one* $u \in [\underline{u}, \overline{u}] \cap L^q(Q)$ *such that (Par4) has a solution* $y \in Y \cap D(L)$*. Moreover, we assume that there exist some constant* $c_1 > 0$ *and some* $k_1 \in L^1([0,T])$ *with*

$$\langle \mathcal{A}(t)y, y\rangle \geq c_1 \|y\|_V^p - k_1(t) \ \textit{for all} \ y \in V \ t \in [0,T] \tag{88}$$

and some $k_G \in L_+^q(\Omega)$ *such that*

$$|g(x,t,s,\zeta)| \leq k_G(x,t) \ \textit{for all} \ s \in Y(x,t) \ \lambda_Q(d(x,t))\textit{-a.e. and} \ \zeta \in \mathbb{R}^N. \tag{89}$$

Then the optimal control problem (OC-Par4-G2) *has at least one solution pair* (y,u)*.*

Proof. We set the one-sided truncation operator $\overline{T}\colon L^p(Q) \to L^p(Q)$ given by - if $Y = (-\infty, y_1]$

$$\overline{T}y(x,t) := \begin{cases} y_1(x,t) & \text{if} \quad y(x,t) > y_1(x,t) \\ y(x,t) & \text{if} \quad y(x,t) \leq y_1(x,t) \end{cases}$$

or if $Y = [y_1, \infty)$

$$\overline{T}y(x,t) := \begin{cases} y(x,t) & \text{if} \quad y(x,t) > y_1(x,t) \\ y_1(x,t) & \text{if} \quad y(x,t) \leq y_1(x,t). \end{cases}$$

By assumption, there exists at least one solution of (Par4). Hence there exists for at least one $u \in [\underline{u}, \overline{u}] \cap L^q(Q)$ a solution $y \in Y \cap D(L)$ of

$$Ly + \mathfrak{A}y + (\tau_{L^p(T;V),p}{}^{\star} iG\overline{T})y = \tau_{L^p(T;V),p}{}^{\star} iu \text{ in } L^q(T;V^\star). \tag{90}$$

We show that the operator

$$\mathfrak{A} + \tau_{L^p(T;V),p}{}^{\star} iG\overline{T}\colon L^p(T;V) \to L^q(T;V^\star)$$

satisfies the condition of Lemma 6.2. By assumptions (88) and (89), we obtain the relation

$$\begin{aligned} \langle (\mathfrak{A} + \tau_{L^p(T;V),p}{}^{\star} iG\overline{T})y, y\rangle \geq c_1 \|y\|_{L^p(T;V)}^p & - \|k_1\|_{L^1([0,T])} \\ & - \|k_G\|_{L^q(Q)} \|y\|_{L^p(Q)} . \end{aligned}$$

The last part of the proof is along the lines of Theorem 6.4. □

7 Optimal Control Problems with Multivalued Variational Equations

Let V be some closed subspace of $W^{1,p}(\Omega)$ with $W_0^{1,p}(\Omega) \subseteq V \subseteq W^{1,p}(\Omega)$, $1 < p < \infty$ and q its conjugate exponent. As shown in Lemma 8.7 the mapping ${\tau_{V,p}}^\star i\colon L^q(\Omega) \to V^\star$, $y \mapsto ({\tau_{V,p}}^\star i)y$ is strongly continuous. In what follows, by $P_c(X)$, X some Banach space, we will denote the family of nonempty closed and convex subsets of X.
Let M be some subset of $W^{1,p}(\Omega)$. We denote the set $\{\tau_{V,p} m |\, m \in M\}$ with $\tau_{V,p} M$.

7.1 Elliptic Inclusions

7.1.1 Global Growth Condition

7.1.1.1 Existence of Solutions for Multivalued Variational Equations

In contrast to the previous chapters we are now considering an inclusion problem instead of equality. Let $f \in V^\star$.

Problem 25. *Find some $y \in V$ such that*

$$\mathcal{A}y + ({\tau_{V,p}}^\star iG)y + ({\tau_{V,p}}^\star i\mathcal{M}\tau_{V,p})y \ni f \text{ in } V^\star, \qquad \text{(M\_Ell1)}$$

where $\mathcal{A}$ and G are operators and $\mathcal{M}$ is a multivalued mapping specified below.

Definition 7.1 (solution)**.** *The function $y \in V$ is called solution of the inclusion problem (M_Ell1) if there is a function $w \in L^q(\Omega)$ such that*

i) $Gy \in L^q(\Omega)$,

ii) $w \in \mathcal{M}\tau_{V,p} y$ *and*

iii) $\mathcal{A}y + ({\tau_{V,p}}^\star iG)y + {\tau_{V,p}}^\star iw = f$ *in* $V^\star$.

The Leray-Lions conditions (H1)-(H3) of Section 4 are assumed for the coefficient functions $a_i\colon \Omega \times \mathbb{R} \times \mathbb{R}^N$, $i = 1, \ldots, N$, where the definitions

$$a(y, \phi) = \sum_{i=1}^{N} \int_\Omega a_i(id, y, \nabla y) \frac{\partial \phi}{\partial x_i}\, d\lambda_\Omega$$

and $\mathcal{A}\colon V \to V^\star$, $y \mapsto a(y, \cdot)$ hold.

Moreover, we suppose the following hypotheses for $\mathcal{M}$.

(H$\mathcal{M}$) $\mathcal{M}\colon L^p(\Omega) \to P_c(L^q(\Omega))$, $y \mapsto \mathcal{M}y$ is a multifunction such that

(1) $\mathcal{M}$ is weakly closed.

(2) There exist some $c_{\mathcal{M}} > 0$ and $k_{\mathcal{M}} \in L^q_+(\Omega)$ such that for all $y \in L^p(\Omega)$ and for all $w \in \mathcal{M}y$ the inequality

$$|w(x)| \le k_{\mathcal{M}}(x) + c_{\mathcal{M}} \, |y(x)|^{p-1} \quad \lambda_\Omega(dx)\text{-a.e.}$$

holds.

(3) There exists some constant $c_{\mathcal{M},2} > 0$ such that for all y_1, $y_2 \in V$ with $y_1 < y_2$ it holds

$$w_1 - w_2 \le c_{\mathcal{M},2}(y_2 - y_1)^{p-1}$$

for all $w_1 \in \mathcal{M}y_1$ and $w_2 \in \mathcal{M}y_2$.

Example 7.1. *Let us consider a locally Lipschitz mapping $j\colon \mathbb{R} \to \mathbb{R}$. Clarke's generalized gradient is given by*

$$\partial j(s) := \{\zeta \in \mathbb{R} | \, j^\circ(s;r) \ge r \ \text{ for all } r \in \mathbb{R}\},$$

where j° denotes the generalized directional derivative, see , e.g., Clarke (1983) or Motreanu and Rădulescu (2003). We define the functional $J\colon L^p(\Omega) \to \mathbb{R}$ by

$$J(y) := \int_\Omega j \circ y \, d\lambda_\Omega \quad y \in L^p(\Omega).$$

By the Aubin-Clarke Theorem, see ,e.g., Theorem 1.3 in Motreanu and Rădulescu (2003), it holds $\partial J \subset \partial j(y)$ for all $y \in L^p(\Omega)$. This property plays an important role in the proof of the existence of a solution for the inclusion problem

$$\mathcal{A}y + (\tau_{V,p}{}^\star iG)y + \partial j(y) \ni f \ \text{ in } V^\star,$$

which is considered in Section 4.2 of Carl et al. (2007). The chain rule implies for $\partial J\colon L^p(\Omega) \to 2^{L^q(\Omega)}$ the identity $\partial J \tau_{V,p} = \tau_{V,p}{}^\star \partial J \tau_{V,p}\colon V \to 2^{V^\star}$ see, e.g., Corollary 2.180 in Carl et al. (2007). The mapping ∂J satisfies, under the conditions (H1) and (H2) on p.155, 156 in Carl et al. (2007), hypothesis (H$\mathcal{M}$)*, see Proposition 2.171 in Carl et al. (2007).*

The following lemma is analogous to Lemma 4.16 in Carl et al. (2007).

Lemma 7.2. *Under* (H$\mathcal{M}$) *the multifunction $\tau_{V,p}{}^\star i\mathcal{M}\tau_{V,p}\colon V \to 2^{V^\star}$ is pseudomonotone in the sense of Definition 8.7.*

Proof. For every $y \in V$ the set $\mathcal{M}\tau_{V,p}y$ is nonempty, closed and convex. By the linearity of $\tau_{V,p}{}^\star$ and i, the set $\tau_{V,p}{}^\star i\mathcal{M}\tau_{V,p}y$ is nonempty and convex, too. Choose a sequence $(w_n)_{n\in\mathbb{N}} \subset \tau_{V,p}{}^\star i\mathcal{M}\tau_{V,p}y$ with $w_n \to w$ in $V^\star$. Then there exist elements $z_n \in \mathcal{M}\tau_{V,p}y$ with $w_n = \tau_{V,p}{}^\star i z_n$ for all n. Applying Minkowski's Inequality and (H$\mathcal{M}$) (2) shows that the sequence $(z_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^q(\Omega)$:

$$\begin{aligned}\|z_n\|_{L^q(\Omega)} &\leq \|k_\mathcal{M}\|_{L^q(\Omega)} + c_\mathcal{M}\| \, |\tau_{V,p}y|^{p-1} \, \|_{L^q(\Omega)} \\ &= \|k_\mathcal{M}\|_{L^q(\Omega)} + c_\mathcal{M}\|\tau_{V,p}y\|^{p-1}_{L^p(\Omega)} \text{ for all } n.\end{aligned}$$

This implies the existence of some subsequence which converges weakly in $L^q(\Omega)$ to some z and it holds $w = \tau_{V,p}{}^\star iz$. Since the set $\mathcal{M}\tau_{V,p}y$ is closed and convex, it is weakly closed and therefore $z \in \mathcal{M}\tau_{V,p}y$. This implies $w \in \tau_{V,p}{}^\star i\mathcal{M}\tau_{V,p}y$.
We show that $\tau_{V,p}{}^\star i\mathcal{M}\tau_{V,p}$ is generalized pseudomonotone in the sense of Definition 8.8. The closedness of the sets $\tau_{V,p}{}^\star i\mathcal{M}\tau_{V,p}y$, $y \in V$, has already been proven. Choose a sequence $(y_n)_{n\in\mathbb{N}} \subset V$ with $y_n \rightharpoonup y$ in V, hence $\tau_{V,p}y_n \to \tau_{V,p}y$ in $L^p(\Omega)$. Moreover, we assume $w_n \rightharpoonup w$ in $V^\star$ with $w_n \in \tau_{V,p}{}^\star i\mathcal{M}\tau_{V,p}y_n$. Then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in \mathcal{M}\tau_{V,p}y_n$ and $\tau_{V,p}{}^\star iz_n = w_n$. Since

$$\|z_n\|_{L^q(\Omega)} \leq \|k_\mathcal{M}\|_{L^q(\Omega)} + c_\mathcal{M}\|\tau_{V,p}y_n\|^{p-1}_{L^p(\Omega)} \text{ for all } n,$$

there exist some subsequence $(z_{n'})_{n'\in M\subseteq\mathbb{N}}$ and some $z \in L^q(\Omega)$ with $z_{n'} \rightharpoonup z$ in $L^q(\Omega)$. Due to the hypothesis that $\mathcal{M}$ is weakly closed, we have that $z \in \mathcal{M}\tau_{V,p}y$. By the uniqueness of the weak limit we obtain $w = \tau_{V,p}{}^\star iz \in \tau_{V,p}{}^\star i\mathcal{M}\tau_{V,p}y$. Since V resp. $V^{\star\star}$ is a dense subset of $L^p(\Omega)$ resp. $L^p(\Omega)^{\star\star}$ and the sequence $(iz_n)_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega)^\star$, we can apply Proposition 21.23 (g) in Zeidler (1990a) and obtain for the whole sequence $(z_n)_{n\in\mathbb{N}}$ that

$$iz_n \rightharpoonup iz \text{ in } L^p(\Omega)^\star.$$

It follows by Proposition 21.23 (k) in Zeidler (1990a) that

$$\begin{aligned}\langle w_n, y_n\rangle_{V^\star} &= \langle \tau_{V,p}{}^\star iz_n, y_n\rangle_{V^\star} \\ &= \langle iz_n, \tau_{V,p}y_n\rangle_{L^p(\Omega)^\star} \to \langle iz, \tau_{V,p}y\rangle_{L^p(\Omega)^\star} = \langle w, y\rangle_{V^\star}.\end{aligned}$$

By using Proposition 8.24, the pseudomonotonicity is now proven. □

Remark 7.3. *For every $y \in V$ the set $\tau_{V,p}{}^\star i\mathcal{M}\tau_{V,p}y$ is even weakly closed. This can be seen immediately by replacing the convergent sequence $(w_n)_{n\in\mathbb{N}}$ by a weakly convergent sequence in the proof of Lemma 7.2.*

Definition 7.2 (subsolution)**.** *The function $\underline{y} \in W^{1,p}(\Omega)$ is called subsolution of the inclusion problem (M_Ell1) if there is a function $\underline{w} \in L^q(\Omega)$ such that* *i)* $G\underline{y} \in L^q(\Omega)$,

ii) $(\underline{y} - y)^+ \in V$ *for all* $y \in V$,

iii) $\underline{w} \in \mathcal{M}\tau_{W^{1,p}(\Omega),p}\underline{y}$ *and*

iv) $a(\underline{y}, \cdot) + (\tau_{V,p}{}^\star iG)\underline{y} + \tau_{V,p}{}^\star i\underline{w} \leq f$ *in* $V^\star$.

Definition 7.3 (supersolution)**.** *The function* $\overline{y} \in W^{1,p}(\Omega)$ *is called supersolution of the inclusion problem (M_Ell1) if there is a function* $\overline{w} \in L^q(\Omega)$ *such that*

i) $G\overline{y} \in L^q(\Omega)$,

ii) $(y - \overline{y})^+ \in V$ *for all* $y \in V$,

iii) $\overline{w} \in \mathcal{M}\tau_{W^{1,p}(\Omega),p}\overline{y}$ *and*

iv) $a(\overline{y}, \cdot) + (\tau_{V,p}{}^\star iG)\overline{y} + \tau_{V,p}{}^\star i\overline{w} \geq f$ *in* $V^\star$.

Theorem 7.4. *Let* $\underline{y}$ *and* $\overline{y}$ *be a sub- and a supersolution of (M_Ell1) that satisfies* $\underline{y} \leq \overline{y}$ *and suppose (H1)-(H3) and* (HG) *of Section 4 and* (H$\mathcal{M}$)*. Assume that there exist a constant* $c_G > 0$ *and* $k_G \in L^q_+(\Omega)$ *with*

$$|g(x, s, \zeta)| \leq k_G(x) + c_G\, |\zeta|^{p-1} \quad \text{for all } s \in [\underline{y}(x), \overline{y}(x)] \;\; \lambda_\Omega(dx)\text{-a.e. and } \zeta \in \mathbb{R}^N. \tag{91}$$

Then there exists at least one solution of (M_Ell1) which lies in $[\underline{y}, \overline{y}]$.

Proof. We make use of the penalization operator $\mathfrak{B}\colon L^p(\Omega) \to L^p(\Omega)$, $y \mapsto \mathfrak{b}(id, y)$ which is the Nemytskii operator generated by $\mathfrak{b}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ with

$$\mathfrak{b}(x, z) := \begin{cases} (z - \overline{y}(x))^{p-1} & \text{if} \quad z > \overline{y}(x) \\ 0 & \text{if} \quad \underline{y}(x) \leq z \leq \overline{y}(x) \\ -(\underline{y}(x) - z)^{p-1} & \text{if} \quad z < \underline{y}(x) \end{cases} . \tag{92}$$

As shown in Lemma 4.3 there exist some positive constants $c_{\mathfrak{B}}$, $C_{\mathfrak{B}}$ with

$$\int_\Omega \mathfrak{b}(id, y)\, y\, d\lambda_\Omega \geq c_{\mathfrak{B}} \|y\|^p_{L^p(\Omega)} - C_{\mathfrak{B}} \quad \text{for all } y \in L^p(\Omega). \tag{93}$$

As in the previous chapters we define the semi-linear form

$$a_T\colon W^{1,p}(\Omega) \times W^{1,p}(\Omega) \to \mathbb{R}$$

by

$$a_T(y, \phi) := \sum_{i=1}^N \int_\Omega a_i(id, Ty, \nabla y) \frac{\partial \phi}{\partial x_i}\, d\lambda_\Omega$$

and the operator $\mathcal{A}_T\colon V \to V^\star$, $y \mapsto a_T(y,\cdot)$. The operator T is the continuous truncation operator defined in Lemma 4.3.
With these denotations we can consider the following auxiliary problem with $f \in V^\star$:
Find some $y \in V$ such that

$$\mathcal{A}_T y + (\tau_{V,p}{}^\star iGT)y + \tau_{V,p}{}^\star i\mathcal{M}\tau_{V,p}y + \lambda(\tau_{V,p}{}^\star i\mathfrak{B}\tau_{V,p})y \ni f \quad \text{in } V^\star, \tag{A-M-Ell1}$$

where λ satisfies

$$\lambda > \max\{c_{\mathfrak{B}}^{-1}(c_G C_{\epsilon_G} + c_{\mathcal{M}}), c_{\mathcal{M},2}\}$$

with $C_{\epsilon_G} := (\epsilon_G p)^{-q/p}\frac{1}{q}$ and $\epsilon_G < \frac{c_1}{c_G}$. Since the hypotheses (H1), (H2) and (HG) of Section 4, the assumption (91) and the results of Lemma 4.3 hold, we can apply Theorem 2.109 in Carl et al. (2007). For this reason the multivalued auxiliary operator $\mathcal{A}_T + \tau_{V,p}{}^\star iGT + \lambda\tau_{V,p}{}^\star i\mathfrak{B}\tau_{V,p}\colon V \to V^\star$ is continuous, bounded and pseudomonotone. The boundedness and pseudomonotonicity of $\tau_{V,p}{}^\star i\mathcal{M}\tau_{V,p}$, see Lemma 7.2, implies that $\mathcal{A}_T + \tau_{V,p}{}^\star iGT + \tau_{V,p}{}^\star i\mathcal{M}\tau_{V,p} + \tau_{V,p}{}^\star i\mathfrak{B}\tau_{V,p}$ is pseudomonotone, see Theorem 2.124 (ii) in Carl et al. (2007). We show coercivity which follows by the assumptions on $\mathcal{A}$, (HG) of Section 4, (H$\mathcal{M}$) and (93):
For all $w \in \mathcal{M}\tau_{V,p}y$ it holds

$$\begin{aligned}
&\langle \mathcal{A}_T y + (\tau_{V,p}{}^\star iGT)y + w + \lambda(\tau_{V,p}{}^\star i\mathfrak{B}\tau_{V,p})y, y\rangle \\
&\quad \geq c_1\|\nabla y\|^p_{L^p(\Omega)} - \|k_1\|_{L^1(\Omega)} - c_G\epsilon_G\|\nabla y\|^p_{L^p(\Omega)} - c_G C_{\epsilon_G}\|y\|^p_{L^p(\Omega)} \\
&\qquad - \left(\|k_G\|_{L^q(\Omega)} + c_G c_p\|\,|\nabla y_1| + |\nabla y_2|\,\|^{p-1}_{L^p(\Omega)}\right)\|y\|_{L^p(\Omega)} \\
&\qquad - \|k_{\mathcal{M}}\|_{L^q(\Omega)}\|y\|_{L^p(\Omega)} - c_{\mathcal{M}}\|y\|^p_{L^p(\Omega)} + \lambda c_{\mathfrak{B}}\|y\|^p_{L^p(\Omega)} - \lambda C_{\mathfrak{B}} \\
&\quad = (c_1 - c_G\epsilon_G)\|\nabla y\|^p_{L^p(\Omega)} + (\lambda c_{\mathfrak{B}} - c_G C_{\epsilon_G} - c_{\mathcal{M}})\|y\|^p_{L^p(\Omega)} \\
&\qquad - \left(\|k_G\|_{L^q(\Omega)} + c_G c_p\|\,|\nabla y_1| + |\nabla y_2|\,\|^{p-1}_{L^p(\Omega)} + \|k_{\mathcal{M}}\|_{L^q(\Omega)}\right)\|y\|_{L^p(\Omega)} \\
&\qquad - \|k_1\|_{L^1(\Omega)} - \lambda C_{\mathfrak{B}},
\end{aligned}$$

where $c_1 > 0$ and $k_1 \in L^1(\Omega)$ exist due to hypothesis (H3) of Section 4 and c_p defined as in (50). Due to the choice of ϵ_G and λ, the coefficients of the two first summands are positive. Applying Theorem 8.25 yields the existence of a solution of (A-M-Ell1).
We show that every solution y lies in $[\underline{y}, \overline{y}]$. Let $w \in \mathcal{M}\tau_{V,p}y$ be the element corresponding to the solution. By definition of the supersolution $\overline{y}$ there exists some $\overline{w} \in \mathcal{M}\tau_{W^{1,p}(\Omega),p}\overline{y}$ with

$$a_T(\overline{y},\cdot) + (\tau_{V,p}{}^\star iGT)\overline{y} + \overline{w} + \lambda(\tau_{V,p}{}^\star i\mathfrak{B}\tau_{V,p})\overline{y} \geq f \quad \text{in } V^\star.$$

Subtracting the last inequality from (A-M-Ell1) and testing with $(y - \overline{y})^+ \in V \cap L^p_+(\Omega)$ yields:

$$a_T(y - \overline{y}, (y - \overline{y})^+) + \langle ((\tau_{V,p}{}^\star iGT)y - (\tau_{V,p}{}^\star iG)\overline{y}) + (w - \overline{w}), (y - \overline{y})^+ \rangle \leq 0.$$

Since $\mathcal{A}_T$ satisfies the Leray-Lions condition (H2) of Section 4 we have already seen that it holds:

$$a_T(y - \overline{y}, (y - \overline{y})^+) \geq 0.$$

In addition we have $\langle (\tau_{V,p}{}^\star iGT)y - (\tau_{V,p}{}^\star iG)\overline{y}, (y - \overline{y})^+ \rangle = 0$. With (H$\mathcal{M}$) we obtain

$$\begin{aligned}
\langle w - \overline{w}, (y - \overline{y})^+ \rangle &= \int_{\{y > \overline{y}\}} (w - \overline{w})(y - \overline{y})^+ \, d\lambda_\Omega \\
&\geq \int_{\{y > \overline{y}\}} -c_{\mathcal{M},2}(y - \overline{y})^{p-1}(y - \overline{y})^+ \, d\lambda_\Omega \\
&= \int_\Omega -c_{\mathcal{M},2}(y - \overline{y})^{+p} \, d\lambda_\Omega = -c_{\mathcal{M},2} \|(y - \overline{y})^+\|^p_{L^p(\Omega)} .
\end{aligned}$$

The definition of $\mathfrak{B}$ shows

$$\begin{aligned}
\langle (\tau_{V,p}{}^\star i\mathfrak{B}\tau_{V,p})y, (y - \overline{y})^+ \rangle &= \int_{\{y > \overline{y}\}} \mathfrak{b}(id, y)(y - \overline{y})^+ \, d\lambda_\Omega \\
&= \int_{\{y > \overline{y}\}} (y - \overline{y})^{p-1}(y - \overline{y})^+ \, d\lambda_\Omega \\
&= \int_\Omega (y - \overline{y})^{+p} \, d\lambda_\Omega.
\end{aligned}$$

This yields the following result with $\lambda > c_{\mathcal{M},2}$

$$\begin{aligned}
0 &\leq (\lambda - c_{\mathcal{M},2}) \|(y - \overline{y})^+\|^p_{L^p(\Omega)} \\
&\leq a_T(y - \overline{y}, (y - \overline{y})^+) + \langle w - \overline{w}, (y - \overline{y})^+ \rangle \\
&\quad + \langle \lambda(\tau_{V,p}{}^\star i\mathfrak{B}\tau_{V,p})y, (y - \overline{y})^+ \rangle \\
&\leq 0
\end{aligned}$$

and hence $(y - \overline{y})^+ = 0$. This implies $y \leq \overline{y}$.
The proof of $\underline{y} \leq y$ follows with the same arguments. □

7.1.1.2 Existence of Solutions for Optimal Control Problems In this section we consider optimal control problems with pointwise two-sided state constraints. Let $Y = [y_1, y_2]$ with y_1, $y_2 \in W^{1,p}(\Omega)$ and $y_1 \leq y_2$ and write $Y(x) := [y_1(x), y_2(x)]$. We denote the set of all solutions of (M_Ell1) lying in Y with $\mathcal{S}(f)$, where $f \in V^\star$. Let $\underline{u}$, $\overline{u} \in L^q(\Omega)$ and assume that the mapping $J\colon (Y \cap V) \times ([\underline{u}, \overline{u}] \cap L^q(\Omega)) \to \mathbb{R}$ satisfies (HJ) of Section 4.2. The generalized optimal control problem reads as follows.

Problem 26.

$$\begin{aligned} \min \quad & J(y,u) && \text{(OC-M-Ell1)} \\ s.t. \quad & \mathcal{A}y + (\tau_{V,p}{}^{\star} iG)y + (\tau_{V,p}{}^{\star} i\mathcal{M}\tau_{V,p})y \ni \tau_{V,p}{}^{\star} iu \quad \textit{in } V^{\star} \\ \textit{and} \quad & u \in [\underline{u}, \overline{u}] \cap L^q(\Omega) \\ & y \in Y \cap V \end{aligned}$$

For the later proof of the existence of some solution for (OC-M-Ell1), we need a generalization of Lemma 4.4.

Lemma 7.5. *Let X be some Banach space and $A\colon X \to 2^{X^{\star}}$ a multivalued mapping which is coercive in the sense of Definition 8.9. Moreover, assume that the inclusions*

$$Ay_n \ni f_n,\ n \in \mathbb{N} \tag{94}$$

are satisfied for some given sequences $(y_n)_{n\in\mathbb{N}} \subset X$ and $(f_n)_{n\in\mathbb{N}} \subset X^{\star}$, where the sequence $(f_n)_{n\in\mathbb{N}}$ is assumed to be bounded. Then $(y_n)_{n\in\mathbb{N}}$ is bounded in X.

Proof. Assume that the sequence $(y_n)_{n\in\mathbb{N}}$ is unbounded. Then there exists some subsequence $(y_{n'})_{n'\in M}$, $M \subset \mathbb{N}$, such that $\lim_{n'\to\infty} \|y_{n'}\|_X = \infty$ and $\|y_{n'}\|_X > 0$ for all $n' \in M$. Let $v_n \in Ay_n$. By (94) we obtain

$$\|f_{n'}\|_{Op} \|y_{n'}\|_X \geq \langle f_{n'}, y_{n'}\rangle = \langle v_{n'}, y_{n'}\rangle = \|y_{n'}\|_X \frac{\langle v_{n'}, y_{n'}\rangle}{\|y_{n'}\|_X}$$

and hence by the coercivity of A

$$\|f_{n'}\|_{Op} \geq \frac{\langle v_{n'}, y_{n'}\rangle}{\|y_{n'}\|_X} \geq \inf_{w_{n'}\in Ay_{n'}} \frac{\langle w_{n'}, y_{n'}\rangle}{\|y_{n'}\|_X} \to \infty,$$

which is a contradiction to the assumption of boundedness for the sequence $(f'_n)_{n'\in M}$. □

Theorem 7.6. *Suppose (H1)-(H3),* (HG) *and* (HJ) *of Section 4 resp. 4.2 and* (H$\mathcal{M}$)*. Assume that there exists at least one $u \in [\underline{u}, \overline{u}] \cap L^q(\Omega)$ such that (M_Ell1) has a solution in Y. Suppose that there exist a constant $c_G > 0$ and a function $k_G \in L^q_+(\Omega)$ satisfying*

$$|g(x,s,\zeta)| \leq k_G(x) + c_G\,|\zeta|^{p-1} \quad \textit{for all } s \in Y(x)\ \ \lambda_\Omega(dx)\textit{-a.e. and } \zeta \in \mathbb{R}^N.$$

Then the optimal control problem (OC-M-Ell1) *has a solution (y,u).*

Proof. The proof is analogous to Theorem 4.7 resp. Theorem 4.9.
We consider the following auxiliary problem:

$$\begin{aligned} \min\quad & J(y,u) && \text{(A-OC-M-Ell1)} \\ \text{s.t.}\quad & \mathcal{A}_{\overline{T}}\, y + \overline{\lambda}\tau_{V,p}{}^{\star} i \overline{\mathfrak{B}} \tau_{V,p}\, y + \tau_{V,p}{}^{\star} i G \overline{T}\, y + \tau_{V,p}{}^{\star} i \mathcal{M} \tau_{V,p}\, y \ni \tau_{V,p}{}^{\star} i u \text{ in } V^{\star} \\ \text{and}\quad & u \in [\underline{u}, \overline{u}] \cap L^q(\Omega) \\ & y \in Y \cap V \end{aligned}$$

Here, the definitions of the operators $\overline{T}$, $\mathcal{A}_{\overline{T}}$, $\overline{\mathfrak{B}}$ are analogous to Theorem 7.4. The constant $\overline{\lambda} > 0$ satisfies the inequality

$$\overline{\lambda} > c_{\overline{\mathfrak{B}}}^{-1}(c_G C_{\epsilon_G} + c_{\mathcal{M}})$$

with $C_{\epsilon_G} := (\epsilon_G p)^{-q/p}\frac{1}{q}$ and $\epsilon_G < \frac{c_1}{c_G}$.
By assumption, there exists at least one admissible pair (y,u). Let $(y_m, u_m)_{m\in\mathbb{N}}$ be the infimal sequence with

$$\lim_{m\to\infty} J(y_m, u_m) = \inf_{u\in[\underline{u},\overline{u}]\cap L^q(\Omega),\, y\in\mathcal{S}(\tau_{V,p}{}^{\star} i u)} J(y,u).$$

Then we know that there exist a weakly convergent subsequence $(u_m)_{m\in M}$, $M \subseteq \mathbb{N}$, and a $u_0 \in [\underline{u},\overline{u}]\cap L^q(\Omega)$ with $u_m \rightharpoonup u_0$ in $L^q(\Omega)$ and $\tau_{V,p}{}^{\star} i u_m \rightharpoonup \tau_{V,p}{}^{\star} i u_0$ in $V^{\star}$. Let $w_m \in \mathcal{M}\tau_{V,p} y_m$ be the corresponding sequence. Analogously to the proof of Theorem 7.4 it can be shown that the multivalued auxiliary operator $\mathcal{A}_{\overline{T}} + \overline{\lambda}\tau_{V,p}{}^{\star} i \overline{\mathfrak{B}}\tau_{V,p} + \tau_{V,p}{}^{\star} i G\overline{T} + \tau_{V,p}{}^{\star} i \mathcal{M}\tau_{V,p} \colon V \to 2^{V^{\star}}$ is coercive in the sense of Definition 8.9. This property implies with Lemma 7.5 that there exists some weakly convergent subsequence $(y_m)_{m\in M}$ with limit y_0 in V. The inequality

$$\|w_m\|_{L^q(\Omega)} \le \|k_{\mathcal{M}}\|_{L^q(\Omega)} + c_{\mathcal{M}}\|\tau_{V,p} y_m\|_{L^p(\Omega)}^{p-1} \text{ for all } m,$$

allows passing to a weakly convergent subsequence $(w_m)_{m\in M}$ with limit w_0 in $L^q(\Omega)$. As supposed in (H$\mathcal{M}$), $\mathcal{M}$ is weakly closed and hence $w_0 \in \mathcal{M}\tau_{V,p} y_0$. It follows

$$\begin{aligned} &\langle (\mathcal{A}_{\overline{T}} + \tau_{V,p}{}^{\star} i G\overline{T}) y_m, y_m - y_0\rangle \\ &\quad = \langle \tau_{V,p}{}^{\star} i u_m - \overline{\lambda}\tau_{V,p}{}^{\star} i\overline{\mathfrak{B}}\tau_{V,p} y_m - \tau_{V,p}{}^{\star} i w_m, y_m - y_0\rangle \to 0. \end{aligned}$$

By Theorem 2.109 in Carl et al. (2007) the operator $\mathcal{A}_{\overline{T}} + \tau_{V,p}{}^{\star} i G\overline{T}$ satisfies the S_+-property, therefore $y_m \to y_0$ in V. From the continuity of the operators the convergence

$$\begin{aligned} &\mathcal{A}_{\overline{T}} y_m + \overline{\lambda}(\tau_{V,p}{}^{\star} i\overline{\mathfrak{B}}\tau_{V,p}) y_m + (\tau_{V,p}{}^{\star} i G\overline{T}) y_m + \tau_{V,p}{}^{\star} i w_m \\ &\qquad \to \mathcal{A}_{\overline{T}} y_0 + \overline{\lambda}(\tau_{V,p}{}^{\star} i\overline{\mathfrak{B}}\tau_{V,p}) y_0 + (\tau_{V,p}{}^{\star} i G\overline{T}) y_0 + \tau_{V,p}{}^{\star} i w_0 \text{ in } V^{\star} \end{aligned}$$

can be deduced. Hence, the inclusion (M_Ell1) holds for the limits y_0, w_0 and u_0. We obtain

$$J(y_0, u_0) \leq \liminf_{m\to\infty} J(y_m, u_m).$$

□

7.1.2 Local Growth Condition

7.1.2.1 Existence of Solutions for Multivalued Variational Equations

The growth condition on the multivalued mapping $\mathcal{M}$ in hypothesis (H$\mathcal{M}$) was assumed to be of global character. In this section we consider the case of a multivalued mapping defined as Clarke's generalized gradient of some locally Lipschitz mapping $j\colon \Omega\times\mathbb{R}\to\mathbb{R}$. We are in the situation of Section 4.3 of Carl et al. (2007). Let $\mathfrak{g}\colon \Omega\times\mathbb{R}\to\mathbb{R}$ be a measurable and locally bounded function and assume that j is the primitive of $\mathfrak{g}$, i.e.

$$j(x,s) = \int_0^s \mathfrak{g}(x,\tau)\,\lambda(d\tau).$$

Clarke's generalized gradient is given by

$$\partial j(x,s) := \{\zeta\in\mathbb{R}|\, j^\circ(x,s;r)\geq r \ \text{ for all } r\in\mathbb{R}\}.$$

Here, j° denotes the generalized directional derivative. For further details on generalized derivatives we refer to Clarke (1983) and Motreanu and Rădulescu (2003).
We assume that the operator $\mathcal{A}\colon V\to V^\star$ defines a Leray-Lions operator satisfying hypotheses (H1)-(H3) of Section 4. The mapping $a\colon W^{1,p}(\Omega)\times W^{1,p}(\Omega)\to\mathbb{R}$ denotes the corresponding semi-linear form. With these definitions we are now considering the following inclusion problem.

Problem 27. *Find some $y\in V$ such that*

$$\mathcal{A}y + ({\tau_{V,p}}^\star iG)y + {\tau_{V,p}}^\star\partial j(id,\tau_{V,p}y) \ni f \ \text{ in } V^\star \qquad \text{(M\_Ell2)}$$

Definition 7.4 (solution)**.** *The function $y\in V$ is called solution of the inclusion problem (M_Ell2) if there is a function $w\in L^q(\Omega)$ such that*

i) $Gy\in L^q(\Omega)$,

ii) $w(x)\in\partial j(x,y(x))\ \lambda_\Omega(dx)$*-a.e. and*

iii) $\mathcal{A}y + ({\tau_{V,p}}^\star iG)y + {\tau_{V,p}}^\star iw = f$ *in* $V^\star$.

Definition 7.5 (subsolution)**.** *The function* $\underline{y} \in W^{1,p}(\Omega)$ *is called subsolution of the inclusion problem (M_Ell2) if there is a function* $\underline{w} \in L^q(\Omega)$ *such that*

i) $G\underline{y} \in L^q(\Omega)$,

ii) $(\underline{y} - y)^+ \in V$ *for all* $y \in V$,

iii) $\underline{w}(x) \in \partial j(x, \underline{y}(x))$ $\lambda_\Omega(dx)$*-a.e. and*

iv) $a(\underline{y}, \cdot) + (\tau_{V,p}{}^\star iG)\underline{y} + \tau_{V,p}{}^\star i\underline{w} \leq f$ *in* $V^\star$.

Definition 7.6 (supersolution)**.** *The function* $\overline{y} \in W^{1,p}(\Omega)$ *is called supersolution of the inclusion problem (M_Ell2) if there is a function* $\overline{w} \in L^q(\Omega)$ *such that*

i) $G\overline{y} \in L^q(\Omega)$,

ii) $(y - \overline{y})^+ \in V$ *for all* $y \in V$,

iii) $\overline{w}(x) \in \partial j(x, \overline{y}(x))$ $\lambda_\Omega(dx)$*-a.e. and*

iv) $a(\overline{y}, \cdot) + (\tau_{V,p}{}^\star iG)\overline{y} + \tau_{V,p}{}^\star i\overline{w} \geq f$ *in* $V^\star$.

Instead of hypothesis (H$\mathcal{M}$) of the previous Section 7.1.1, we assume:

(Hj) The function $\mathfrak{g}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(1) $\mathfrak{g}$ is measurable and $\mathfrak{g}(x, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is locally bounded $\lambda_\Omega(dx)$-a.e..

(2) There exist some constant $\alpha > 0$ and some function $k_j \in L^q_+(\Omega)$ such that for all $s \in [\underline{y}(x) - 2\alpha, \overline{y}(x) + 2\alpha]$ it holds

$$|\mathfrak{g}(x, s)| \leq k_j(x) \ \lambda_\Omega(dx)\text{-a.e..}$$

(3) There exists some constant $c_j > 0$ such that for all s_1, $s_2 \in \mathbb{R}$ with $\underline{y}(x) - \alpha < s_1 < s_2 < \overline{y}(x) + \alpha$ it holds

$$\mathfrak{g}(x, s_1) - \mathfrak{g}(x, s_2) \leq c_j(s_2 - s_1) \ \lambda_\Omega(dx)\text{-a.e..}$$

As shown in Remark 4.30 in Carl et al. (2007), condition (Hj)(1) implies

$$\partial j(x, s) = [\underline{\mathfrak{g}}(x, s), \overline{\mathfrak{g}}(x, s)], \quad s \in \mathbb{R} \ \lambda_\Omega(dx)\text{-a.e.,} \tag{95}$$

where

$$\underline{\mathfrak{g}}(x, s) := \lim_{\epsilon \downarrow 0} \operatorname{ess\,inf}\{\mathfrak{g}(x, t) | \ |t - s| < \epsilon\} \text{ and}$$
$$\overline{\mathfrak{g}}(x, s) := \lim_{\epsilon \downarrow 0} \operatorname{ess\,sup}\{\mathfrak{g}(x, t) | \ |t - s| < \epsilon\}.$$

Under these assumptions, in Theorem 4.31 in Carl et al. (2007) the existence of a solution of (M_Ell2) is proven.

Theorem 7.7. *Let $\underline{y}$ and $\overline{y}$ be a sub- and a supersolution of (M_Ell2) which satisfy $\underline{y} \leq \overline{y}$ and suppose the hypotheses (H1)-(H3) and* (HG) *of Section 4.1 and* (Hj)*. Moreover, we assume that there exist a constant $c_G > 0$ and a function $k_G \in L^q_+(\Omega)$ satisfying*

$$|g(x,s,\zeta)| \leq k_G(x) + c_G\,|\zeta|^{p-1} \quad \text{for all } s \in [\underline{y}(x), \overline{y}(x)] \;\; \lambda_\Omega(dx)\text{-a.e. and } \zeta \in \mathbb{R}^N.$$

Then there exists at least one solution of (M_Ell2) which lies in $[\underline{y}, \overline{y}]$.

7.1.2.2 Existence of Solutions for Optimal Control Problems In the following we are interested in a statement about the solvability of the optimal control problem formulated below. We assume pointwise state constraints with boundaries y_1, $y_2 \in W^{1,p}(\Omega)$ and $Y := [y_1, y_2]$ as defined in paragraph 7.1.1.2.

Problem 28.

$$\begin{aligned} \min \quad & J(y,u) && \text{(OC-M-Ell2)} \\ s.t. \quad & \mathcal{A}y + (\tau_{V,p}{}^\star iG)y + \tau_{V,p}{}^\star \partial j(id, \tau_{V,p} y) \ni \tau_{V,p}{}^\star iu \quad \text{in } V^\star \\ \text{and} \quad & u \in [\underline{u}, \overline{u}] \cap L^q(\Omega) \\ & y \in Y \cap V \end{aligned}$$

The previous assumptions on j have to be modified according to the admissible set Y.

(Hj') The function $\mathfrak{g}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(1) $\mathfrak{g}$ is measurable and $\mathfrak{g}(x,\cdot)\colon \mathbb{R} \to \mathbb{R}$ is locally bounded $\lambda_\Omega(dx)$-a.e..

(2) There exists some function $k_j \in L^q_+(\Omega)$ and some constant $\alpha > 0$ such that for all $s \in [y_1(x) - \alpha, y_2(x) + \alpha]$ it holds

$$|\mathfrak{g}(x,s)| \leq k_j(x) \;\; \lambda_\Omega(dx)\text{-a.e..}$$

The truncation operator $\overline{T}$ and the penalization operator $\overline{\mathfrak{B}}$ are introduced as in the previous paragraph 7.1.1.2. Due to assumption (Hj')(2) and (95), every $w \in \partial j(id, y)$ with $y \in Y$ is an element of $L^q(\Omega)$. Hence, the term $\tau_{V,p}{}^\star i \partial j(id, \overline{T}\tau_{V,p} y)$ is well defined.

Theorem 7.8. *Suppose the hypotheses (H1)-(H3),* (HG) *of Section 4.1,* (HJ) *of Section 4.2 and* (Hj')*. Assume that there exists at least one $u \in [\underline{u}, \overline{u}] \cap L^q(\Omega)$ such that (M_Ell2) has a solution in $Y \cap V$. Suppose that there exist a constant $c_G > 0$ and a function $k_G \in L^q_+(\Omega)$ satisfying*

$$|g(x,s,\zeta)| \leq k_G(x) + c_G\,|\zeta|^{p-1} \quad \text{for all } s \in Y(x) \;\; \lambda_\Omega(dx)\text{-a.e. and } \zeta \in \mathbb{R}^N. \tag{96}$$

Then the optimal control problem (OC-M-Ell2) *has a solution* (y,u).

Proof. We consider the following auxiliary problem

$$\mathcal{A}_{\overline{T}}y + \tau_{V,p}{}^{\star}iG\overline{T}y + \tau_{V,p}{}^{\star}i\partial j(id, \overline{T}\tau_{V,p}y) + \overline{\lambda}\tau_{V,p}{}^{\star}i\overline{\mathfrak{B}}\tau_{V,p}y \ni f \text{ in } V^{\star}, \tag{A-M-Ell2}$$

and the corresponding optimal control problem:

$$\begin{aligned}
\min \quad & J(y,u) && \text{(A-OC-M-Ell2)}\\
\text{s.t.} \quad & \mathcal{A}_{\overline{T}}\, y + \tau_{V,p}{}^{\star}iG\overline{T}\, y + \tau_{V,p}{}^{\star}i\partial j(id, \overline{T}\tau_{V,p}y) + \overline{\lambda}\tau_{V,p}{}^{\star}i\overline{\mathfrak{B}}\tau_{V,p}\, y \ni \tau_{V,p}{}^{\star}iu \text{ in } V^{\star}\\
\text{and} \quad & u \in [\underline{u}, \overline{u}] \cap L^q(\Omega)\\
& y \in Y \cap V
\end{aligned}$$

Here, the definitions of the operators $\overline{T}$, $\mathcal{A}_{\overline{T}}$, $\overline{\mathfrak{B}}$ are analogous to Theorem 7.6. The constant $\overline{\lambda} > 0$ satisfies

$$\overline{\lambda} > c_{\overline{\mathfrak{B}}}^{-1} c_G C_{\epsilon_G}$$

with $C_{\epsilon_G} := (\epsilon_G p)^{-q/p}\frac{1}{q}$ and $\epsilon_G < \frac{c_1}{c_G}$. By assumption, there exists at least one admissible pair (y,u) which solves (M_Ell2) and hence (A-M-Ell2), too. We denote the set of solutions of the equation (A-M-Ell2) lying in Y with $\mathcal{S}(f)$. Let $(y_m, u_m)_{m\in\mathbb{N}}$ be the infimal sequence with

$$\lim_{m\to\infty} J(y_m, u_m) = \inf_{u\in[\underline{u},\overline{u}]\cap L^q(\Omega), y\in\mathcal{S}(\tau_{V,p}{}^{\star}iu)} J(y,u).$$

Then there exist a weakly convergent subsequence $(u_m)_{m\in M}$, $M \subseteq \mathbb{N}$, and some function $u_0 \in [\underline{u}, \overline{u}] \cap L^q(\Omega)$ with $u_m \rightharpoonup u_0$ in $L^q(\Omega)$ and $\tau_{V,p}{}^{\star}iu_m \rightharpoonup \tau_{V,p}{}^{\star}iu_0$ in $V^{\star}$. Let $w_m \in \partial j(id, \overline{T}\tau_{V,p}y_m)$ be the corresponding sequence which is due to (Hj')(2) bounded in $L^q(\Omega)$.
We show coercivity of the operator

$$\mathcal{A}_{\overline{T}} + \tau_{V,p}{}^{\star}iG\overline{T} + \tau_{V,p}{}^{\star}i\partial j(id, \overline{T}\tau_{V,p}\,) + \overline{\lambda}\tau_{V,p}{}^{\star}i\overline{\mathfrak{B}}\tau_{V,p},$$

which follows by the assumptions (H3) and (HG) of Section 4.1, (Hj'), (96) and (93): For all $w \in \partial j(id, \overline{T}\tau_{V,p}y)$ it holds

$$\begin{aligned}
&\langle \mathcal{A}_{\overline{T}}y + \tau_{V,p}{}^{\star}iG\overline{T}y + w + \overline{\lambda}(\tau_{V,p}{}^{\star}i\overline{\mathfrak{B}}\tau_{V,p})y, y\rangle\\
&\quad \geq c_1\|\nabla y\|^p_{L^p(\Omega)} - \|k_1\|_{L^1(\Omega)} - c_G\epsilon_G\|\nabla y\|^p_{L^p(\Omega)} - c_G C_{\epsilon_G}\|y\|^p_{L^p(\Omega)}\\
&\qquad - \left(\|k_G\|_{L^q(\Omega)} + c_G c_p\|\,|\nabla y_1| + |\nabla y_2|\,\|^{p-1}_{L^p(\Omega)}\right)\|y\|_{L^p(\Omega)}\\
&\qquad - \|k_j\|_{L^q(\Omega)}\|y\|_{L^p(\Omega)} + \overline{\lambda}c_{\overline{\mathfrak{B}}}\|y\|^p_{L^p(\Omega)} - \overline{\lambda}C_{\overline{\mathfrak{B}}}\\
&\quad = (c_1 - c_G\epsilon_G)\|\nabla y\|^p_{L^p(\Omega)} + (\overline{\lambda}c_{\overline{\mathfrak{B}}} - c_G C_{\epsilon_G})\|y\|^p_{L^p(\Omega)}\\
&\qquad - \left(\|k_G\|_{L^q(\Omega)} + c_G c_p\|\,|\nabla y_1| + |\nabla y_2|\,\|^{p-1}_{L^p(\Omega)} + \|k_j\|_{L^q(\Omega)}\right)\|y\|_{L^p(\Omega)}\\
&\qquad - \|k_1\|_{L^1(\Omega)}\|y\|_{L^p(\Omega)} - \overline{\lambda}C_{\overline{\mathfrak{B}}},
\end{aligned}$$

where c_p is defined as in (50). Due to the choice of ϵ_G and $\overline{\lambda}$, the coefficients of the two first summands are positive. Applying Lemma 7.5 allows passing to some weakly convergent subsequence $(y_m)_{m\in M}$ with limit y_0 in V. Moreover, it holds, due to the boundedness of $(w_m)_{m\in M}$ in $L^q(\Omega)$, that there exists some weakly convergent subsequence in $L^q(\Omega)$. We denote the weak limit with w_0. It follows $w_0 \in \partial j(id, y_0(\cdot))$ from the upper semi-continuity of the mapping $s \mapsto \partial j(x,s) = [\underline{\mathfrak{g}}(x,s), \overline{\mathfrak{g}}(x,s)]$, compare Carl et al. (2007), p. 178. Since the operator $\mathcal{A}_{\overline{T}} + \tau_{V,p}{}^{\star} i G \overline{T}$ satisfies the S_+-property, the strong convergence of $(y_m)_{m\in M}$ in V follows. The further proof is along the lines of Theorem 7.6. □

7.2 Parabolic Inclusions

7.2.1 Existence of Solutions for Multivalued Variational Equations

In Section 6 we have considered evolution equations. We now consider differential inclusions for the parabolic case. Let $2 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let V, W, W_V, L and $D(L)$ be defined as in Section 6. We are now in the situation of Section 4.5 of Carl et al. (2007). For the operator $\mathfrak{A}\colon L^p(T;V) \to L^q(T;V^\star)$ defined by coefficient functions a_i, $i = 1,\ldots,N$, we assume the Leray-Lions conditions (H1)-(H3) of Section 6.1 which coincide with the assumptions (A1)-(A3) stated in Carl et al. (2007), p. 191. Let the mapping $s \mapsto j(\cdot,\cdot,s)$ with $j\colon \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be locally Lipschitz. The assumptions on the Clarke's gradient of j and the Nemytskii operator G are stated in (H1) and (H2), p. 192 and 193 in Carl et al. (2007). We assume a structure of the next differential inclusion as introduced in Chapter 4.5 in Carl et al. (2007). Let $f \in L^q(T;V^\star)$.

Problem 29. *Find some $y \in D(L)$ such that*

$$Ly + \mathfrak{A}y + (\tau_{L^p(T;V),p}{}^{\star} i G)y + \tau_{L^p(T;V),p}{}^{\star} i \partial j(id, \tau_{L^p(T;V),p} y) \ni f \quad \text{in } L^q(T;V^\star) \qquad \text{(M\_Par)}$$

Definition 7.7 (solution)**.** *The function $y \in D(L)$ is called solution of the inclusion problem (M_Par) if $Gy \in L^q(Q)$ and there is a function $w \in L^q(Q)$ such that*

i) $w \in \partial j(id, \tau_{L^p(T;V),p} y)$

ii) $Ly + \mathfrak{A}y + (\tau_{L^p(T;V),p}{}^{\star} i G)y + \tau_{L^p(T;V),p}{}^{\star} w = f$ *in* $L^q(T;V^\star)$

Definition 7.8 (subsolution)**.** *The function $\underline{y} \in W$ is called subsolution of the inclusion problem (M_Par) if $G\underline{y} \in L^q(Q)$ and there is a function $\underline{w} \in L^q(Q)$ such that*

i) $(\underline{y} - y)^+ \in L^p(T;V)$ *for all* $y \in L^p(T;V)$

ii) $\underline{w} \in \partial j(id, \tau_{L^p(T;V),p}\underline{y})$

iii) $\underline{y}_t + \mathfrak{a}(\underline{y}, \cdot) + (\tau_{L^p(T;V),p}{}^\star iG)\underline{y} + \tau_{L^p(T;V),p}{}^\star \underline{w} \leq f$ *in* $L^q(T;V^\star)$

Definition 7.9 (supersolution)**.** *The function* $\overline{y} \in W$ *is called supersolution of the inclusion problem (M_Par) if* $G\overline{y} \in L^q(Q)$ *and there is a function* $\overline{w} \in L^q(Q)$ *such that*

i) $(y - \overline{y})^+ \in L^p(T;V)$ *for all* $y \in L^p(T;V)$

ii) $\overline{w} \in \partial j(id, \tau_{L^p(T;V),p}\overline{y})$

iii) $\overline{y}_t + \mathfrak{a}(\overline{y}, \cdot) + (\tau_{L^p(T;V),p}{}^\star iG)\overline{y} + \tau_{L^p(T;V),p}{}^\star \overline{w} \geq f$ *in* $L^q(T;V^\star)$

The following theorem is stated in Carl et al. (2007), Theorem 4.46.

Theorem 7.9. *Let* $\underline{y}$ *and* $\overline{y}$ *be a sub- and a supersolution of (M_Par) that satisfies* $\underline{y} \leq \overline{y}$. *Suppose (H1)-(H3) and* (HG) *of Section 6.1 and the assumption (H1) on j formulated on p. 192 in Carl et al. (2007). Assume that there exists some* $k_G \in L^q_+(Q)$ *with*

$$|g(x,t,s)| \leq k_G(x,t) \text{ for all } s \in [\underline{y}(x,t), \overline{y}(x,t)] \ \lambda_Q(d(x,t))\text{-a.e..} \tag{97}$$

Then there exists at least one solution of (M_Par) which lies in $[\underline{y}, \overline{y}]$.

7.2.2 Existence of Solutions for Optimal Control Problems

Now we admit general two-sided pointwise state constraints. Let Y be defined as in Chapter 6. We denote the set of all solutions of (M_Par) lying in Y with $\mathcal{S}(f)$. Assume that the mapping $J\colon (Y \cap L^p(T;V)) \times ([\underline{u}, \overline{u}] \cap L^q(Q)) \to \mathbb{R}$ satisfies (HJ) of Section 6.2.

Problem 30.

$$\begin{aligned}
\min \quad & J(y,u) && \text{(OC-M-Par)}\\
s.t. \quad & Ly + \mathfrak{A}y + (\tau_{L^p(T;V),p}{}^\star iG)y \\
& \quad + \tau_{L^p(T;V),p}{}^\star i\partial j(id, \tau_{L^p(T;V),p}y) \ni \tau_{L^p(T;V),p}{}^\star iu \quad in\ L^q(T;V^\star)\\
and \quad & u \in [\underline{u}, \overline{u}] \cap L^q(Q)\\
& y \in Y \cap D(L)
\end{aligned}$$

For the later proof of the existence of some solution for OC-M-Par, a generalization of Lemma 6.2 is needed.

Lemma 7.10. *Let $A\colon L^p(T;V) \to 2^{L^q(T;V^\star)}$ be a multivalued mapping which is coercive in the sense of Definition 8.9. Moreover, assume that the inclusions*

$$Ly_n + Ay_n \ni f_n,\ n \in \mathbb{N} \tag{98}$$

are satisfied for some given sequences $(y_n)_{n\in\mathbb{N}} \subset D(L) \subset L^p(T;V)$ and $(f_n)_{n\in\mathbb{N}} \subset L^q(T;V^\star)$, where the sequence $(f_n)_{n\in\mathbb{N}}$ is assumed to be bounded in $L^q(T;V^\star)$. Then $(y_n)_{n\in\mathbb{N}}$ is bounded in $L^p(T;V)$.

Proof. Assume that the sequence $(y_n)_{n\in\mathbb{N}}$ is unbounded. Then there exists some subsequence $(y_{n'})_{n'\in M}$, $M \subset \mathbb{N}$, such that it holds $\lim_{n'\to\infty} \|y_{n'}\|_{L^p(T;V)} = \infty$ and $\|y_{n'}\|_{L^p(T;V)} > 0$ for all $n' \in M$. By (98) we obtain

$$\begin{aligned}\|f_{n'}\|_{Op}\,\|y_{n'}\|_{L^p(T;V)} &\geq \langle f_{n'}, y_{n'}\rangle = \langle Ly_{n'} + v_{n'}, y_{n'}\rangle \\ &= \left(\frac{\|y_{n'}(T)\|_{L^p(\Omega)}}{2\|y_{n'}\|_{L^p(T;V)}} + \frac{\langle v_{n'}, y_{n'}\rangle}{\|y_{n'}\|_{L^p(T;V)}}\right)\|y_{n'}\|_{L^p(T;V)}\end{aligned}$$

for all $v_{n'} \in Ay_{n'}$ and hence by the coercivity of A

$$\begin{aligned}\|f_{n'}\|_{Op} &\geq \frac{\|y_{n'}(T)\|_{L^p(\Omega)}}{2\|y_{n'}\|_{L^p(T;V)}} + \frac{\langle v_{n'}, y_{n'}\rangle}{\|y_{n'}\|_{L^p(T;V)}} \\ &\geq \frac{\|y_{n'}(T)\|_{L^p(\Omega)}}{2\|y_{n'}\|_{L^p(T;V)}} + \inf_{w_{n'}\in Ay_{n'}} \frac{\langle w_{n'}, y_{n'}\rangle}{\|y_{n'}\|_X} \to \infty,\end{aligned}$$

which is a contradiction to the assumption on the boundedness of the sequence $(f_n')_{n'\in M}$. □

The previous assumptions on j have to be modified according to the admissible set Y.

(Hj') The function $j\colon Q \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:
(1) $j(\cdot,\cdot,s)\colon Q \to \mathbb{R}$ is measurable for all $s \in \mathbb{R}$.
(2) $j(x,t,\cdot)\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz.
(3) There exists some function $k_j \in L^q_+(Q)$ such that for all $s \in Y(x,t)$ and all w with $w(x,t) \in \partial j(x,t,s)$ it holds

$$|w(x,t,s)| \leq k_j(x,t)\ \lambda_Q(d(x,t))\text{-a.e..}$$

Now we derive an existence result as in the proof of Theorem 5.8.

Theorem 7.11. *Suppose (H1)-(H3) and* (HJ) *of Section 6.2 and* (Hj')*. Assume that there exists at least one $u \in [\underline{u}, \overline{u}] \cap L^q(\Omega)$ such that (M_Par) has a solution*

$y \in Y \cap D(L)$. *We assume that there exist some* $k_G \in L^q_+(Q)$ *and some* $c_G > 0$ *with*

$$|g(x,t,s,\zeta)| \leq k_G(x,t) + c_G\,|\zeta|^{p-1} \tag{99}$$

for all $s \in Y(x,t)$ $\lambda_Q(d(x,t))$*-a.e. and* $\zeta \in \mathbb{R}^N$. *Then the optimal control problem* (OC-M-Par) *has at least one solution* (y,u).

Proof. We consider the following auxiliary problem

$$\begin{aligned} Ly + \mathfrak{A}_{\overline{T}}y + \tau_{L^p(T;V),p}{}^{\star} iG\overline{T}y + \tau_{L^p(T;V),p}{}^{\star} i\partial j(id, \overline{T}\tau_{L^p(T;V),p}y) \\ + \overline{\lambda}(\tau_{L^p(T;V),p}{}^{\star} i\overline{\mathfrak{B}}\tau_{L^p(T;V),p})y \ni f \text{ in } L^q(T;V^\star), \end{aligned} \tag{A-M-Par}$$

and the corresponding optimal control problem:

$$\begin{aligned} \min \quad & J(y,u) && \text{(A-OC-M-Par)} \\ \text{s.t.} \quad & Ly + \mathfrak{A}_{\overline{T}}y + \tau_{L^p(T;V),p}{}^{\star} iG\overline{T}y + \tau_{L^p(T;V),p}{}^{\star} i\partial j(id, \overline{T}\tau_{L^p(T;V),p}y) \\ & \quad + \overline{\lambda}(\tau_{L^p(T;V),p}{}^{\star} i\overline{\mathfrak{B}}\tau_{L^p(T;V),p})y \ni \tau_{L^p(T;V),p}{}^{\star} iu \text{ in } L^q(T;V^\star) \\ \text{and} \quad & u \in [\underline{u}, \overline{u}] \cap L^q(Q) \\ & y \in Y \cap D(L) \end{aligned}$$

Here, the definitions of the operators $\overline{T}$, $\mathfrak{A}_{\overline{T}}$ and $\overline{\mathfrak{B}}$ introduced in the proof of Theorem 6.4 hold. The constant $\overline{\lambda} > 0$ satisfies the inequality

$$\overline{\lambda} > c_{\mathfrak{B}}^{-1}(c_G C_{\epsilon_G} + c_j)$$

with $C_{\epsilon_G} := (\epsilon_G p)^{-q/p}\frac{1}{q}$ and $\epsilon_G < \frac{c_1}{c_G}$. By assumption, there exists at least one admissible pair (y,u) which solves (M_Par) and hence (A-M-Par), too. We denote the set of solutions of (A-M-Par) lying in Y with $\mathcal{S}(f)$, $f \in L^q(T;V^\star)$. Let $(y_m,u_m)_{m\in\mathbb{N}}$ be the infimal sequence with

$$\lim_{m\to\infty} J(y_m,u_m) = \inf_{u\in[\underline{u},\overline{u}]\cap L^q(\Omega),\, y\in\mathcal{S}(\tau_{L^p(T;V),p}{}^{\star} iu)} J(y,u).$$

There exist a weakly convergent subsequence $(u_m)_{m\in M}$, $M \subseteq \mathbb{N}$, and some function $u_0 \in [\underline{u},\overline{u}] \cap L^q(Q)$ with $u_m \rightharpoonup u_0$ in $L^q(Q)$ and $\tau_{L^p(T;V),p}{}^{\star} iu_m \rightharpoonup \tau_{L^p(T;V),p}{}^{\star} iu_0$ in $L^q(T;V^\star)$. Let $w_m \in \partial j(id, \overline{T}\tau_{L^p(T;V),p}y_m)$ be the corresponding sequence which is due to assumption (Hj') bounded in $L^q(Q)$.
We show coercivity of the operator

$$\mathfrak{A}_{\overline{T}} + \tau_{L^p(T;V),p}{}^{\star} iG\overline{T} + \tau_{L^p(T;V),p}{}^{\star} i\partial j(id, \overline{T}\tau_{L^p(T;V),p}\ ) + \overline{\lambda}\tau_{L^p(T;V),p}{}^{\star} i\overline{\mathfrak{B}}\tau_{L^p(T;V),p},$$

which follows by the assumptions on $\mathfrak{A}$, (HG) of Section 6.1, (Hj'), (99) and Lemma 5.5:
For all $w \in \partial j(id, \overline{T}\tau_{L^p(T;V),p} y)$ it holds

$$\begin{aligned}
&\langle \mathfrak{A}_{\overline{T}} y + (\tau_{L^p(T;V),p}{}^{\star} i G \overline{T}) y + w + \overline{\lambda}(\tau_{L^p(T;V),p}{}^{\star} i \overline{\mathfrak{B}} \tau_{L^p(T;V),p}) y, y \rangle \\
&\quad \geq (c_1 - c_G \epsilon_G) \|\nabla y\|^p_{L^p(\Omega)} + (\overline{\lambda} c_{\overline{\mathfrak{B}}} - c_G C_{\epsilon_G}) \|y\|^p_{L^p(Q)} \\
&\qquad - \Big(\|k_G\|_{L^q(Q)} + c_G c_p \| \, |\nabla y_1| + |\nabla y_2| \, \|^{p-1}_{L^p(Q)} + \|k_j\|_{L^q(Q)} \Big) \|y\|_{L^p(Q)} \\
&\qquad - \|k_1\|_{L^1(Q)} - \overline{\lambda} C_{\overline{\mathfrak{B}}}.
\end{aligned}$$

The constant c_p is defined in (50). Applying Lemma 7.10 allows passing to some weakly convergent subsequence $(y_m)_{m \in M}$ with limit y_0 in $L^p(T;V)$. By the estimation

$$\begin{aligned}
\|L y_m\|_{L^q(T;V^\star)} &\leq \|\mathfrak{A}_{\overline{T}} y_m\|_{L^q(T;V^\star)} + c\|w_m\|_{L^q(Q)} + \overline{\lambda} c \|\overline{\mathfrak{B}} \tau_{L^p(T;V),p} y_m\|_{L^q(Q)} \\
&\quad + c\|G\overline{T} y_m\|_{L^q(Q)} \, , \; c := \|\tau_{L^p(T;V),p}{}^{\star} i\|_{Op} \, ,
\end{aligned}$$

the boundedness of the sequence $(Ly_m)_{m \in M}$ in $L^q(T;V^\star)$ follows from the assumptions (H1) of Section 6.1, (Hj'), (99) and Lemma 5.5.
Due to the boundedness of $(w_m)_{m \in M}$ in $L^q(Q)$, see assumption (Hj'), there exists some weakly convergent subsequence with limit w_0 in $L^q(Q)$. Moreover, observing that the embedding of W_V into $L^p(Q)$ is compact yields the (strong) convergence of the sequence $(\tau_{L^p(T;V),p} y_m)_{m \in M}$ in $L^p(Q)$. By the continuity of $\overline{T}$ we obtain $\overline{T}\tau_{L^p(T;V),p} y_m \to \overline{T}\tau_{L^p(T;V),p} y_0$ in $L^p(Q)$. It follows $w_0 \in \partial j(id, \overline{T}\tau_{L^p(T;V),p} y_0)$ by the upper semi-continuity of the generalized directional derivative and a Lebesgue's point argument, compare Carl et al. (2007), p. 206.
The last part of the proof is along the lines of Theorem 7.6. □

8 Appendix

8.1 General Results

Lemma 8.1. *Let Y and Z be real Banach spaces and X a closed subspace of Y. Assume that the embedding $\tau_1 \colon Y \to Z$ is compact. Then the embedding $\tau_2 \colon X \to Z$ is compact, too.*

Proof. Let $(x_n)_{n \in \mathbb{N}} \subset X$ be a sequence with

$$x_n \rightharpoonup x \quad \text{in } X.$$

Due to the compact embedding of Y in Z it holds the convergence

$$\tau_2 x_n = \tau_1 x_n \to \tau_1 x = \tau_2 x \text{ in } Z.$$

□

Theorem 8.2. *Let Y and Z be real Banach spaces and $A \colon Y \to Z$ be a linear operator. A is continuous if and only if it is weakly continuous.*

For the proof see, e.g., Proposition 4.2 in Morrison (2001).

8.2 Embedding Theorems

Let Ω be a bounded domain with Lipschitz boundary Γ. We denote the trace operator with $\gamma_p \colon W^{1,p}(\Omega) \to L^p(\Gamma)$, where $1 \leq p < \infty$. The continuity of γ_p is shown in Zeidler (1990b), p. 1026 and p. 1029.

Lemma 8.3. *For each p, the trace operator γ_p is linear and continuous.*

The mapping $\tau_{W^{1,p}(\Omega),q} \colon W^{1,p}(\Omega) \to L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$, denotes the embedding operator.

Assumption 8.4 (Embedding assumption)**.** *For dimension $N \in \mathbb{N}$ of the bounded domain Ω and exponent $1 < p < \infty$ we assume*

(i) $2N \leq Np + p$ *or*

(ii) $2N < Np + p$.

Remark 8.5. *Under assumption* (i) *resp.* (ii) *the embedding operator $\tau_{W^{1,p}(\Omega),q}$ is continuous resp. compact. This can be seen in, e.g., Zeidler (1990b), p. 1028 or Adams and Fournier (2003). If $N \leq p$, then* (ii) *is satisfied. If $N > p$, then*

$$2N < Np + p \quad \Leftrightarrow \quad N - p < N(p-1) \quad \Leftrightarrow \quad q = \frac{p}{p-1} < \frac{Np}{N-p},$$

i.e. q is smaller than the critical exponent $(Np)/(N-p)$. For $p \geq 2$, (ii) *is satisfied for every $N \in \mathbb{N}$.*

Lemma 8.6. *Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *The isomorphism* $i\colon L^q(\Omega) \to L^p(\Omega)^\star$, $y \mapsto \int_\Omega y \cdot d\lambda_\Omega$ *is weakly continuous.*

Proof. We show that i is continuous. Then the statement is proven by applying Theorem 8.2. Assuming $y_n \to y$ in $L^q(\Omega)$ implies

$$\|iy_n - iy\|_{L^p(\Omega)^\star} = \sup_{\phi \in L^p(\Omega)} \frac{\left|\int_\Omega (y_n - y)\,\phi\, d\lambda_\Omega\right|}{\|\phi\|_{L^p(\Omega)}} \le \|y_n - y\|_{L^q(\Omega)} \to 0.$$

□

Lemma 8.7. *Let* $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *and let* X *be a closed subset of* $W^{1,p}(\Omega)$. *Denote by* $i\colon L^q(\Omega) \to L^p(\Omega)^\star$ *the isomorphism defined in Lemma 8.6. Then the mapping* ${\tau_{X,p}}^\star i\colon L^q(\Omega) \to X^\star$, $y \mapsto ({\tau_{X,p}}^\star i)y$ *is linear and strongly continuous.*

Proof. The embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, see, e.g., Zeidler (1990b), p. 1027. In Lemma 8.1 it is shown that the embedding $X \hookrightarrow L^p(\Omega)$ is compact, too. Hence the embedding operator $\tau_{X,p}$ and its adjoint ${\tau_{X,p}}^\star$ are strongly continuous, see Theorem VI.4.8.2 (Schauder) in Dunford and Schwartz (1957). The statement is proven since the isomorphism $i\colon L^q(\Omega) \to L^p(\Omega)^\star$ is weakly continuous, see Lemma 8.6. □

8.3 Elliptic Case

Lemma 8.8. *Let* X *be a separable infinite-dimensional Hilbert space, the set* $\{w_1, \ldots\}$ *a countable basis in* X *and* $X_m := \operatorname{lin}\{w_1, \ldots, w_m\}$. *Let* $(z_m)_{m\in\mathbb{N}}$ *with* $z_m \to z$ *in* $X^\star$. *Assume that* $(y_m)_{m\in\mathbb{N}}$ *is a sequence in* X *with* $y_m \in X_m$ *and*

$$\langle Ay_m, \phi\rangle = \langle z_m, \phi\rangle \quad \textit{for all } \phi \in X_m \quad \textit{for all } m \in \mathbb{N}, \tag{100}$$

where $A\colon X \to X^\star$ *is a linear, strongly monotone and continuous operator. Then there exists an element* y *with* $y_m \to y$ *in* X.

Proof. For the proof we follow Theorem 21.G in Zeidler (1990a).
Since the sequence $(z_m)_{m\in\mathbb{N}}$ converges in $X^\star$, there exists some constant $b > 0$ with $\|z_m\|_{X^\star} \le b$ for all $m \in \mathbb{N}$.
Let $P_m\colon X \to X$, $\phi \mapsto P_m\phi$ be the orthogonal projection on X_m and denote with ${P_m}^\star\colon X^\star \to X^\star$, $\phi \mapsto {P_m}^\star\phi$ its adjoint, where it holds that ${P_m}^\star X^\star = \{y^\star \in X^\star | \langle y^\star, y\rangle = 0, y \in (I - P_m)X\}$ (see Lemma VI.3.2.3 in Dunford and Schwartz (1957)). By the strong monotonicity of A we can derive the strong monotonicity of ${P_m}^\star A\colon X_m \to {X_m}^\star$:

$$\langle {P_m}^\star Ay, y\rangle = \langle Ay, P_m y\rangle = \langle Ay, y\rangle \ge c\|y\|_X^2 \quad \text{for all } y \in X_m. \tag{101}$$

By Lax-Milgram's Theorem, the operator equations

$$\begin{aligned} Ay &= z \quad \text{in } X^\star, \quad y \in X \\ P_m{}^\star A y_m &= P_m{}^\star z_m \quad \text{in } X_m^\star, \quad y_m \in X_m \end{aligned} \tag{102}$$

have unique solutions $y \in X$ resp. $y_m \in X_m$. If $m \geq j$, then it follows from (102)

$$\langle A y_m, w_j \rangle = \langle z_m, w_j \rangle \text{ and} \tag{103}$$

$$\langle A y_m, y_m \rangle = \langle z_m, y_m \rangle. \tag{104}$$

By (101) we have:

$$c \|y_m\|_X^2 \leq \langle A y_m, y_m \rangle = \langle z_m, y_m \rangle \leq \|z_m\|_{X^\star} \|y_m\|_X \leq b \|y_m\|_X .$$

This yields the a priori estimate

$$c \|y_m\|_X \leq b,$$

i.e. the sequence $(y_m)_{m \in \mathbb{N}}$ is bounded. Since X is reflexive, there exists a weakly convergent subsequence with

$$y_{m'} \rightharpoonup \tilde{y} \quad \text{in } X.$$

By (103) and $z_m \to z$ in $X^\star$,

$$\langle A y_{m'} - z_{m'}, \phi \rangle \to 0 \quad \text{for all } \phi \in \bigcup_m X_m.$$

The linearity and continuity of A implies the weak continuity, see Lemma 8.2. Hence it holds $A y_{m'} \rightharpoonup A\tilde{y}$ in $X^\star$ and it follows the boundedness of the sequence $(A y_m)_{m \in \mathbb{N}}$. Since $\bigcup_m X_m$ is dense in X, we can apply Proposition 21.23 (g) in Zeidler (1990a) and obtain that

$$A y_{m'} \to z \quad \text{in } X^\star.$$

This implies the equation $A\tilde{y} = z$ in $X^\star$ and the identity $\tilde{y} = y$. The weak limit is the same for all weakly convergent subsequences of $(y_m)_{m \in \mathbb{N}}$. Thus, we get with Proposition 21.23 (i) in Zeidler (1990a) that

$$y_m \rightharpoonup y \quad \text{in } X.$$

It follows from

$$\begin{aligned} c \|y_m - y\|_X^2 &\leq \langle A y_m - Ay, y_m - y \rangle \\ &= \langle z_m, y_m \rangle - \langle A y_m, y \rangle - \langle Ay, y_m - y \rangle \to 0 \end{aligned}$$

the convergence $y_m \to y \quad$ in X. □

The following lemma gives an example of a closed subset of the Sobolev space $W^{1,p}(\Omega)$.

Lemma 8.9. *Let $\Gamma_1 \subseteq \Gamma$ be measurable and*

$$V := \{v \in W^{1,p}(\Omega) |\, \gamma_p v = 0 \text{ on } \Gamma_1\} \subseteq W^{1,p}(\Omega),$$

where $1 \leq p < \infty$. Then V is a closed subset of $W^{1,p}(\Omega)$.

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in V with $v_n \to v$ in $W^{1,p}(\Omega)$. The trace operator γ_p is continuous, see Lemma 8.3. Therefore, the convergence $\gamma_p v_n \to \gamma_p v$ in $L^p(\Gamma)$ holds. Because of $(v_n)_{n\in\mathbb{N}} \subset V$, we have $\gamma_p v_n = 0$ on Γ_1 for every $n \in \mathbb{N}$. It follows $\gamma_p v = 0$ on Γ_1 by

$$\begin{aligned}\|(\gamma_p v) I_{\Gamma_1}\|_{L^p(\Gamma)} &\leq \|(\gamma_p v_n) I_{\Gamma_1}\|_{L^p(\Gamma)} + \|(\gamma_p v - \gamma_p v_n) I_{\Gamma_1}\|_{L^p(\Gamma)} \\ &= \|(\gamma_p v - \gamma_p v_n) I_{\Gamma_1}\|_{L^p(\Gamma)} \\ &\leq \|\gamma_p v - \gamma_p v_n\|_{L^p(\Gamma)} \to 0,\end{aligned}$$

where I_{Γ_1} denotes the indicator function on Γ_1. □

Now we proof the inequalities stated in Lemma 4.3.

Lemma 8.10. *a) For $1 < p < \infty$ with the definition of $\mathfrak{b}$ in (38) it holds for all $s \in \mathbb{R}$:*

$$|\mathfrak{b}(x,s)| \leq c_{p,1}(|a(x)| + |b(x)|)^{p-1} + c_{p,1}|s|^{p-1} \qquad \lambda_\Omega(dx)\text{-}a.e., \tag{105}$$

where

$$c_{p,1} := \begin{cases} 2, & 1 < p \leq 2 \\ 2^{p-1}, & 2 < p < \infty \end{cases}.$$

b) The inequality

$$\int_\Omega \mathfrak{b}(id, y) y \, d\lambda_\Omega \geq \left(\frac{1}{c_{p,2}} - \epsilon\right) \|y\|^p_{L^p(\Omega)} - C_{\mathfrak{B}} \tag{106}$$

is valid for any $0 < \epsilon < \frac{1}{c_{p,2}}$, where

$$c_{p,2} := \begin{cases} 1, & 1 < p \leq 2 \\ 2^{p-2}, & 2 \leq p < \infty \end{cases}, \tag{107}$$

$$C_{\mathfrak{B}} := \left(4\left(\frac{1}{c_{p,2}} - \epsilon\right) + 2C_\epsilon\right)\left(\|a\|^p_{L^p(\Omega)} + \|b\|^p_{L^p(\Omega)}\right) \tag{108}$$

and

$$C_\epsilon = (\epsilon p)^{-q/p}\frac{1}{q}. \tag{109}$$

Proof. a): We fix some $s \in \mathbb{R}$ and some $x \in \Omega$ and write shortly a resp. b for $a(x)$ resp. $b(x)$.

Case $1 < p \leq 2$

Case $b \leq 0 \leq a$

$$\begin{aligned} (b-s)^{p-1} &\leq |s|^{p-1}, \quad s < b \\ 0 &\leq |s|^{p-1}, \quad b \leq s \leq a \\ (s-a)^{p-1} &\leq |s|^{p-1}, \quad a < s \end{aligned},$$

i.e. $|\mathfrak{b}(x,s)| \leq |s|^{p-1}$

Case $0 < b \leq a$

$$\begin{aligned} (b-s)^{p-1} &\leq 2\,|s|^{p-1}, \quad s < b,\ |s| > b \\ (b-s)^{p-1} &\leq 2\,|b|^{p-1}, \quad s < b,\ |s| \leq b \\ 0 &\leq |s|^{p-1}, \quad b \leq s \leq a \\ (s-a)^{p-1} &\leq |s|^{p-1}, \quad a < s \end{aligned},$$

i.e. $|\mathfrak{b}(x,s)| \leq 2\,|b|^{p-1} + 2\,|s|^{p-1}$

Case $b \leq a < 0$ is similar to the previous case:

$$|\mathfrak{b}(x,s)| \leq 2\,|a|^{p-1} + 2\,|s|^{p-1}.$$

Conclusion

$$|\mathfrak{b}(x,s)| \leq 2(|a| + |b|)^{p-1} + 2\,|s|^{p-1}$$

Case $2 < p < \infty$

By definition we have

$$|\mathfrak{b}(x,s)| \leq |b-s|^{p-1} + |s-a|^{p-1}.$$

Since

$$|b-s|^{p-1} \leq 2^{p-2}(|b|^{p-1} + |s|^{p-1}) \quad \text{and} \quad |s-a|^{p-1} \leq 2^{p-2}(|s|^{p-1} + |a|^{p-1})$$

by Jensen's Inequality we have

$$|\mathfrak{b}(x,s)| \leq 2^{p-1}(|a|^{p-1} + |b|^{p-1}) + 2^{p-1}\,|s|^{p-1}$$

and with $p > 2$ it follows that

$$|\mathfrak{b}(x,s)| \leq 2^{p-1}(|a| + |b|)^{p-1} + 2^{p-1}\,|s|^{p-1}.$$

b): By definition it holds

$$\int_\Omega \mathfrak{b}(id,y)y\,d\lambda_\Omega = \int_{y>b}(y-b)^{p-1}y\,d\lambda_\Omega + \int_{a\leq y\leq b} 0\,d\lambda_\Omega - \int_{y<a}(a-y)^{p-1}y\,d\lambda_\Omega.$$

Case $y \geq 0$

Case $1 < p < 2$

Assume $y > b$. For $y < |b|$ it holds

$$|y - b|^{p-1} \geq 0 \geq |y|^{p-1} - |b|^{p-1}.$$

In the case $y > |b|$, the inequality

$$|y - b|^{p-1} \geq |y|^{p-1} - |b|^{p-1}$$

can be shown by applying the Fundamental Theorem of Calculus (FTC) and observing that the derivative of the mapping $x \mapsto x^{p-1}$ is monotone decreasing. Using Young's Inequality with $\epsilon > 0$ and C_ϵ defined in (109) yields

$$|b|^{p-1} y \leq \epsilon y^p + C_\epsilon \left(|b|^{p-1}\right)^q = \epsilon y^p + C_\epsilon |b|^p.$$

This implies with $\epsilon > 0$ the inequality

$$\begin{aligned}\int_{y>b} (y-b)^{p-1} y \, d\lambda_\Omega &\geq \int_{y>b} |y|^{p-1} y \, d\lambda_\Omega - \int_{y>b} |b|^{p-1} |y| \, d\lambda_\Omega \\ &\geq (1-\epsilon) \int_{y>b} y^p \, d\lambda_\Omega - C_\epsilon \|b\|^p_{L^p(\Omega)}.\end{aligned}$$

Furthermore we have

$$\begin{aligned}\int_{a\leq y\leq b} 0 \, d\lambda_\Omega &\geq \int_{a\leq y\leq b} |y|^p \, d\lambda_\Omega - \int_{a\leq y\leq b} |b|^p \, d\lambda_\Omega \\ &\geq \int_{a\leq y\leq b} |y|^p \, d\lambda_\Omega - \int_{a\leq y\leq b} |a|^p \, d\lambda_\Omega - \int_{a\leq y\leq b} |b|^p \, d\lambda_\Omega.\end{aligned}$$

For $y < a$ it holds

$$|a - y|^{p-1} \leq |a|^{p-1} \leq 2\,|a|^{p-1} - |y|^{p-1}$$

and hence

$$\begin{aligned}\int_{y<a} (a-y)^{p-1} y \, d\lambda_\Omega &= -\int_{y<a} |a-y|^{p-1} |y| \, d\lambda_\Omega \\ &\geq \int_{u<a} |y|^{p-1} \, d\lambda_\Omega - 2\int_{y<a} |a|^{p-1} |y| \, d\lambda_\Omega \\ &\geq \int_{u<a} |y|^{p-1} \, d\lambda_\Omega - 2\int_{y<a} |a|^p \, d\lambda_\Omega.\end{aligned}$$

Consequently it holds

$$\begin{aligned}&\int_\Omega \mathfrak{b}(id, y) y \, d\lambda_\Omega \\ &\geq (1-\epsilon)\,\|y\|^p_{L^p(\Omega)} - (2\,(1-\epsilon) + C_\epsilon)\left(\|a\|^p_{L^p(\Omega)} + \|b\|^p_{L^p(\Omega)}\right).\end{aligned}$$

<u>Case $2 \leq p < \infty$</u>
We obtain from Jensen's Inequality that

$$|y-b|^{p-1} \geq \frac{1}{c_{p,2}} |y|^{p-1} - |b|^{p-1}$$

and hence

$$\int_{y>b} (y-b)^{p-1} y \, d\lambda_\Omega \geq \frac{1}{c_{p,2}} \int_{y>b} y^p \, d\lambda_\Omega - \int_{y>b} |b|^{p-1} \, y \, d\lambda_\Omega.$$

Using Young's Inequality with $\epsilon > 0$ and C_ϵ defined in (109) yields

$$|b|^{p-1} \, y \leq \epsilon y^p + C_\epsilon \, |b|^p$$

and we get with $\epsilon > 0$ that

$$\int_{y>b} (y-b)^{p-1} y \, d\lambda_\Omega \geq \left(\frac{1}{c_{p,2}} - \epsilon \right) \int_{y>b} y^p \, d\lambda_\Omega - C_\epsilon \int_{y>b} |b|^p \, d\lambda_\Omega.$$

Furthermore we have

$$\begin{aligned} \int_{a \leq y \leq b} 0 \, d\lambda_\Omega &= \int_{a \leq y \leq b} y^p \, d\lambda_\Omega - \int_{a \leq y \leq b} y^p \, d\lambda_\Omega \\ &\geq \int_{a \leq y \leq b} y^p \, d\lambda_\Omega - \left(\int_{a \leq y \leq b} |a|^p \, d\lambda + \int_{a \leq y \leq b} |b|^p \, d\lambda_\Omega \right). \end{aligned}$$

Lastly

$$\int_{y<a} (a-y)^{p-1} y \, d\lambda_\Omega \leq \int_{y<a} (2a^{p-1} - y^{p-1}) y \, d\lambda_\Omega \leq 2 \int_{y<a} a^p \, d\lambda - \int_{y<a} y^p \, d\lambda_\Omega$$

and consequently it is for $\epsilon < \frac{1}{c_{p,2}}$

$$\begin{aligned} &\int_\Omega \mathfrak{b}(id, y) y \, d\lambda_\Omega \\ &\geq \left(\frac{1}{c_{p,2}} - \epsilon \right) \|y\|^p_{L^p(\Omega)} - \left(2 \left(\frac{1}{c_{p,2}} - \epsilon \right) + C_\epsilon \right) \left(\|a\|^p_{L^p(\Omega)} + \|b\|^p_{L^p(\Omega)} \right). \end{aligned}$$

Case $y \leq 0$

<u>Case $1 < p < 2$</u>
For $y > b$ it holds

$$|y-b|^{p-1} \leq |b|^{p-1} \leq 2 \, |b|^{p-1} - |y|^{p-1}$$

and hence

$$\begin{aligned}\int_{y>b}(y-b)^{p-1}y\,d\lambda_\Omega &= -\int_{y>b}|y-b|^{p-1}|y|\;d\lambda_\Omega\\ &\geq \int_{u>b}|y|^{p-1}\;d\lambda_\Omega - 2\int_{y>b}|b|^{p-1}|y|\;d\lambda_\Omega\\ &\geq \int_{u>b}|y|^{p-1}\;d\lambda_\Omega - 2\int_{y>b}|b|^{p}\;d\lambda_\Omega.\end{aligned}$$

Furthermore we have

$$\begin{aligned}\int_{a\leq y\leq b}0\,d\lambda_\Omega &\geq \int_{a\leq y\leq b}|y|^p\;d\lambda_\Omega - \int_{a\leq y\leq b}|a|^p\;d\lambda_\Omega\\ &\geq \int_{a\leq y\leq b}|y|^p\;d\lambda_\Omega - \left(\int_{a\leq y\leq b}|a|^p\;d\lambda_\Omega + \int_{a\leq y\leq b}|b|^p\;d\lambda_\Omega\right).\end{aligned}$$

Assume $y<a$. For $|y|<|a|$ it holds

$$|a-y|^{p-1}\geq 0\geq |y|^{p-1}-|a|^{p-1}.$$

In the case $y>|a|$, the inequality

$$|a-y|^{p-1}\geq |y|^{p-1}-|a|^{p-1}.$$

can be shown by applying the FTC and observing that the derivative of the mapping $x\mapsto x^{p-1}$ is monotone decreasing. Using Young's Inequality with $\epsilon>0$ and C_ϵ defined in (109) yields

$$|a|^{p-1}y\leq \epsilon y^p + C_\epsilon |a|^p$$

and we get with $\epsilon>0$ that

$$\begin{aligned}-\int_{y<a}(a-y)^{p-1}y\,d\lambda_\Omega &= \int_{y<a}(a-y)^{p-1}|y|\;d\lambda_\Omega\\ &\geq \int_{y<a}|y|^{p-1}y\,d\lambda_\Omega - \int_{y<a}|a|^{p-1}|y|\;d\lambda_\Omega\\ &\geq (1-\epsilon)\int_{y<a}y^p\,d\lambda_\Omega - C_\epsilon\int_{y<a}|a|^p\;d\lambda_\Omega.\end{aligned}$$

Consequently it holds

$$\begin{aligned}&\int_\Omega \mathfrak{b}(id,y)y\,d\lambda_\Omega\\ &\geq (1-\epsilon)\,\|y\|^p_{L^p(\Omega)} - (2\,(1-\epsilon)+C_\epsilon)\left(\|a\|^p_{L^p(\Omega)} + \|b\|^p_{L^p(\Omega)}\right).\end{aligned}$$

<u>Case $2 \leq p < \infty$</u>
For $c_{p,2}$ defined in (107) we obtain that

$$\begin{aligned}\int_{y>b} (y-b)^{p-1} y \, d\lambda_\Omega &= -\int_{y>b} |y-b|^{p-1} |y| \, d\lambda_\Omega \\ &\geq -2 \int_{u>b} |b|^{p-1} |y| \, d\lambda_\Omega + \int_{y>b} |y|^p \, d\lambda_\Omega \\ &\geq -2 \int_{y>b} |b|^p \, d\lambda_\Omega + \int_{y>b} |y|^p \, d\lambda_\Omega .\end{aligned}$$

Furthermore it holds

$$\begin{aligned}\int_{a\leq y\leq b} 0 \, d\lambda_\Omega &= \int_{a\leq y\leq b} |y|^p \, d\lambda_\Omega - \int_{a\leq y\leq b} |y|^p \, d\lambda_\Omega \\ &\geq \int_{a\leq y\leq b} |y|^p \, d\lambda_\Omega - \left(\int_{a\leq y\leq b} |a|^p \, d\lambda_\Omega + \int_{a\leq y\leq b} |b|^p \, d\lambda_\Omega \right).\end{aligned}$$

Lastly

$$-\int_{y<a} (a-y)^{p-1} y \, d\lambda_\Omega = \int_{y<a} (a+|y|)^{p-1} |y| \, d\lambda_\Omega$$

and since

$$(a+|y|)^{p-1} \geq \frac{1}{c_{p,2}} |y|^{p-1} - |a|^{p-1}$$

we have

$$-\int_{y<a} (a-y)^{p-1} y \, d\lambda_\Omega \geq \frac{1}{c_{p,2}} \int_{y<a} |y|^p \, d\lambda_\Omega - \int_{y<a} |a|^{p-1} |y| \, d\lambda_\Omega .$$

Using Young's Inequality yields

$$|a|^{p-1} |y| \leq \epsilon y^p + C_\epsilon |a|^p$$

This implies

$$-\int_{y<a} (a-y)^{p-1} y \, d\lambda_\Omega \geq \left(\frac{1}{c_{p,2}} - \epsilon \right) \int_{y<a} |y|^p \, d\lambda_\Omega - C_\epsilon \int_{y<a} |a|^p \, d\lambda_\Omega$$

and hence

$$\begin{aligned}&\int \mathfrak{b}(id, y) y \, d\lambda_\Omega \\ &\geq \left(\frac{1}{c_{p,2}} - \epsilon \right) \|y\|_{L^p(\Omega)}^p - \left(2 \left(\frac{1}{c_{p,2}} - \epsilon \right) + C_\epsilon \right) \left(\|a\|_{L^p(\Omega)}^p + \|b\|_{L^p(\Omega)}^p \right).\end{aligned}$$

Conclusion

By definition it holds

$$\int \mathfrak{b}(id,y)\,y\,d\lambda_\Omega = \int_{y>b}(y-b)^{p-1}y\,d\lambda_\Omega + \int_{a\le y\le b} 0\,d\lambda_\Omega - \int_{y<a}(a-y)^{p-1}y\,d\lambda_\Omega.$$

By the results from above, splitting in positive and negative part yields

$$\begin{aligned}\int_\Omega \mathfrak{b}(id,y)y\,d\lambda_\Omega &= \int_{y\le 0}\mathfrak{b}(id,y)y\,d\lambda_\Omega + \int_{y\ge 0}\mathfrak{b}(id,y)y\,d\lambda_\Omega\\ &\ge \left(\frac{1}{c_{p,2}}-\epsilon\right)\|y\|^p_{L^p(\Omega)}\\ &\quad - \left(4\left(\frac{1}{c_{p,2}}-\epsilon\right)+2C_\epsilon\right)\left(\|a\|^p_{L^p(\Omega)}+\|b\|^p_{L^p(\Omega)}\right).\end{aligned}$$

□

8.4 Parabolic Case

Throughout this chapter we will assume that T is some positive constant. For the following definition see, e.g., Definition 23.1. in Zeidler (1990a).

Definition 8.1. *Let X be some Banach space. For $1 \le p < \infty$, the space $L^p(T;X)$ consists of all measurable functions $y\colon [0,T]\to X$ for which it holds*

$$\|y\|^p_{L^p(T;X)} := \int_0^T \|y(t)\|^p_X\ \lambda_{[0,T]}(dt) < \infty.$$

Here, a function $y\colon[0,T]\to X$ is called measurable, if there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of functions $s_n(t) = \sum_{k=1}^{m_n} I_{A_{k,n}}(t)\,w_{k,n}$, $A_{k,n}\in\sigma([0,T])$, $w_{k,n}\in X$, which satisfy $s_n(t)\to y(t)$ $\lambda_{[0,T]}(dt)$-a.e. in X, compare Denkowski et al. (2003). $\sigma([0,T])$ denotes the Borel-σ-Algebra on $[0,T]$.

Remark 8.11. *The spaces $L^p(Q)$ and $L^p(T;L^p(\Omega))$ can be identified. On the one hand, (strong) measurability of $y\colon[0,T]\to L^p(\Omega)$ is equivalent to weak measurability by the separability of $L^p(\Omega)$, see Pettis' Theorem, e.g., in Denkowski et al. (2003). For any $F\in L^p(\Omega)^\star$ with the associated element $f\in L^q(\Omega)$, the measurability of the mapping $t\mapsto \int f\,y(t)\,d\lambda_\Omega = Fy(t)$ follows by Fubini's Theorem.*
On the other hand, for any measurable mapping $y\colon[0,T]\to L^p(\Omega)$ there exists a representative $\tilde{y}$ of y such that the mapping $Q\to\mathbb{R}$, $(t,\omega)\mapsto\tilde{y}(t)(\omega)$ is measurable. The proof consists of two parts. At first we prove as shown in Hille and Phillips (1957), p. 69, (for the Riemann-Stieltjes integrals) that for every $y\in C(T,L^p(\Omega))(\subset L^p(T;L^p(\Omega)))$ there exists a mapping $z\colon Q\to\mathbb{R}$ which is

$\sigma(Q)$-$\sigma(\mathbb{R})$-measurable with $z(t,\cdot) = y(t)$ in $L^p(\Omega)$ $\lambda_{[0,T]}(dt)$-a.e.. In the following step the same result is shown under the assumption $y \in L^p(T; L^p(\Omega))$.
Let $y_0 \colon Q \to \mathbb{R}$ be a representation of y with $y_0(t,\cdot) = y(t)$ in $L^p(\Omega)$ $\lambda_{[0,T]}(dt)$-a.e..
For $n \in \mathbb{N}$ let $0 = t_0, \dots, t_n = T$ be a decomposition of $[0,T]$ and define

$$z_n \colon Q \to \mathbb{R},\ (t,w) \mapsto y_0(I_{[t_0,t_1]}(t)\, t_0 + \sum_{i=1}^{n-1} I_{(t_i,t_{i+1}]}(t)\, t_i, w).$$

The mappings

$$Q \to \{t_0, \cdots, t_n\} \times \Omega,\ (t,w) \mapsto (I_{[t_0,t_1]}(t)\, t_0 + \sum_{i=1}^{n-1} I_{[t_i,t_{i+1}]}(t), w) \text{ and }$$
$$\{t_0, \cdots, t_n\} \times \Omega \to \mathbb{R},\ (t_i, w) \mapsto y_0(t_i, w)$$

are measurable due to the measurability of $y(t) \colon \Omega \to \mathbb{R}$ for fixed $t \in [0,T]$, finity of the set $\{t_0, \cdots, t_n\}$ and $\sigma(Q) = \sigma([0,T]) \otimes \sigma(\Omega)$. Hence the $\sigma(Q)$-$\sigma(\mathbb{R})$-measurability of z_n follows. By the continuity of y_0 resp. y it holds

$$\int_\Omega |z_n(t,\cdot) - z_m(t,\cdot)|^p \, d\lambda_\Omega \to 0$$

and there exists a constant $M > 0$ with $\|z_n(t,\cdot) - z_m(t,\cdot)\|_{L^p(\Omega)} < 2M$ for all $t \in [0,T]$. Hence by Fubini's and Lebesgue's Theorem the convergence

$$\int_Q |z_n - z_m|^p \, d\lambda_Q = \int_{[0,T]} \int_\Omega |z_n(t,\cdot) - z_m(t,\cdot)|^p \, d\lambda_\Omega \lambda_{[0,T]}(dt) \to 0$$

is given. Completeness of $L^p(Q)$ implies the existence of some $z_0 \in L^p(Q)$ with $z_0(t,\cdot) = y(t)$ in $L^p(\Omega)$ $\lambda_{[0,T]}(dt)$-a.e. since

$$0 \leftarrow \int_Q |z_n - z_0|^p \, d\lambda_Q = \int_{[0,T]} \int_\Omega |y_0(t_n(t),\cdot) - z_0(t,\cdot)|^p \, d\lambda_\Omega \, \lambda_{[0,T]}(dt)$$

with $t_n(t) := I_{[t_0,t_1]}(t)\, t_0 + \sum_{i=1}^{n-1} I_{(t_i,t_{i+1}]}(t)$.
Now we assume $y \in L^p(T; L^p(\Omega))$. By Proposition 23.2 (c) in Zeidler (1990a) the space $C(T, L^p(\Omega))$ is dense in $L^p(T; L^p(\Omega))$. Therefore, there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset C(T, L^p(\Omega))$ such that

$$y_n \to y \quad \text{in } L^p(T; L^p(\Omega)).$$

Moreover, there exists a corresponding sequence $(\tilde{y}_n)_{n\in\mathbb{N}} \subset L^p(Q)$ where

$$y_n(t) = \tilde{y}_n(t,\cdot) \quad \text{in } L^p(\Omega)\ \lambda_{[0,T]}(dt)\text{-a.e. for all } n \in \mathbb{N}$$

are $\sigma(Q)$-$\sigma(\mathbb{R})$-measurable mappings. Since $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(T;L^p(\Omega))$ it holds:

$$\int_Q |\tilde{y}_n - \tilde{y}_m|^p \, d\lambda_Q = \int_{[0,T]} \|y_n(t) - y_m(t)\|^p_{L^p(\Omega)} \, \lambda_{[0,T]}(dt) \to 0.$$

Therefore, $(\tilde{y}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(Q)$ and by completeness of $L^p(Q)$ there exists some $\tilde{y} \in L^p(Q)$.
From $Y\colon [0,T] \to L^p(\Omega)$, $t \mapsto \tilde{y}(t,\cdot) \in L^p(T;L^p(\Omega))$ it follows

$$\int_{[0,T]} \|y_n(t) - Y(t)\|^p_{L^p(\Omega)} \, \lambda_{[0,T]}(dt) = \int_Q |\tilde{y}_n - \tilde{y}|^p \, d\lambda_Q \to 0$$

and hence $y_n \to Y$ in $L^p(T;L^p(\Omega))$. This implies $y = Y$ in $L^p(T;L^p(\Omega))$.

The well known Hölder Inequality for L^p-spaces can be formulated for vector-valued functions.

Lemma 8.12. *Let V be some Banach space, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the Hölder Inequality*

$$\int_0^T |\langle v(t), y(t)\rangle_V| \, \lambda_{[0,T]}(dt) \\ \leq \left(\int_0^T \|v(t)\|^q_{V^\star} \, \lambda_{[0,T]}(dt)\right)^{\frac{1}{q}} \left(\int_0^T \|y(t)\|^p_V \, \lambda_{[0,T]}(dt)\right)^{\frac{1}{p}}$$

holds for all $y \in L^p(T;V)$ and $v \in L^q(T;V^\star)$.

For the proof we refer to Proposition 23.6 in Zeidler (1990a).
The Hölder Inequality is applied in the proof of the following lemma which states that the space $L^p(T;V)^\star$ can be identified with $L^q(T;V^\star)$, see Proposition 23.7 in Zeidler (1990a).

Lemma 8.13. *Let V be some reflexive and separable Banach space, $1 < p < \infty$ and q its conjugate exponent. Then:*

a) $L^p(T;V)$ is reflexive and separable.

b) For every $v \in L^q(T;V^\star)$ there exists an unique element $\bar{v} \in L^p(T;V)^\star$ with

$$\langle \tau v, y\rangle_{L^p(T;V)} = \int_0^T \langle v(t), y(t)\rangle_V \, \lambda_{[0,T]}(dt) \quad \text{for all } y \in L^p(T;V). \tag{110}$$

c) For every $\bar{v} \in L^p(T;V)^\star$ *there exists an unique element* $v \in L^q(T;V^\star)$ *with*

$$\|\bar{v}\|_{L^p(T;V)^\star} = \|v\|_{L^q(T;V^\star)}$$

and (110).

By the last lemma, there exists a linear bijective isometric mapping from $L^p(T;V)^\star$ to $L^q(T;V^\star)$, see Convention 23.8 in Zeidler (1990a). In the following we will identify $L^p(T;V)^\star$ with $L^q(T;V^\star)$.

Definition 8.2 (Evolution Triple)**.** *An evolution triple* $(V, H, V^\star)$ *is defined by*

i) $(V, \|\cdot\|_V)$ *is a real, separable and reflexive Banach space.* $(H, (\cdot,\cdot)_H)$ *is a Hilbert space.*

ii) V *is dense in* H *and* V *is continuously embedded into* H*.*

iii) The mapping $\tau\colon H \to V^\star$, $y \mapsto \tau y$, *given by*

$$\langle \tau y, \phi \rangle_{V^\star} := (y, \phi)_H \quad \text{for } \phi \in V,$$

is linear and continuous.

Example 8.14. *For a bounded region* $\Omega \subseteq \mathbb{R}^N$ *with Lipschitz boundary* Γ, $\Gamma_1 \subseteq \Gamma$ *and* $2 \leq p \leq \infty$, *the following spaces form an evolution triple:*

$$V := \{y \in W^{1,p}(\Omega) \,|\, \gamma_p y = 0 \text{ on } \Gamma_1\} \quad \text{and} \quad H := L^2(\Omega).$$

Lemma 8.9 shows that V *is closed in* $W^{1,p}(\Omega)$*. Thus,* V *forms a real, separable and reflexive Banach space. Since the embedding* $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ *is continuous (see the Sobolev Embedding Theorem, e.g., Theorem 4.12 in Adams and Fournier (2003)), it holds the same for* V*. As* $W_0^{1,p}(\Omega)$ *is dense in* $L^2(\Omega)$*, this follows for* $V \supset W_0^{1,p}(\Omega)$*, too.*

Definition 8.3. *Let* V, X *be Banach spaces,* $y \in L^1(T;V)$ *and* $w \in L^p(T;X)$, $1 < p < \infty$*. The function* w *is called generalized derivative of* y *in* $[0,T]$*, written* $w = y_t$*, if it holds:*

$$\int_0^T y(t)\,\phi_t(t)\,\lambda_{[0,T]}(dt) = -\int_0^T w(t)\,\phi(t)\,\lambda_{[0,T]}(dt) \text{ for all } \phi \in C_0^\infty((0,T)).$$

Theorem 8.15. *Let* $V \subseteq H \subseteq V^\star$ *be an evolution triple and* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $1 < p < \infty$ *and* $0 < T < \infty$*. Then the generalized derivative* $y_t \in L^q(T;V^\star)$ *exists if and only if there is a mapping* $w \in L^q(T;V^\star)$ *such that*

$$\int_0^T (y(t), v)_H\,\phi_t(t)\,\lambda_{[0,T]}(dt) = -\int_0^T \langle w(t), v\rangle_V\,\phi(t)\,\lambda_{[0,T]}(dt)$$

for all $v \in V$, $\phi \in C_0^\infty((0,T))$*. The generalized derivative* $y_t = w$ *is uniquely defined.*

For the proof see Theorem 2.139 in Carl et al. (2007). The next proposition is stated in Zeidler (1990a), Proposition 23.23.

Proposition 8.16. *Let $V \subseteq H \subseteq V^\star$ be an evolution triple and let $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p < \infty$. Then it follows:*

i) The set

$$W := \{y \in L^p(T;V) \,|\, y_t \in L^q(T;V^\star)\}$$

forms a real Banach space with the norm

$$\|y\|_W := \|y\|_{L^p(T;V)} + \|y_t\|_{L^q(T;V^\star)}\,.$$

ii) The embedding $W \hookrightarrow C([0,T],H)$ is continuous.

iii) Let $y, v \in W$ and t, s with $0 \le s \le t \le T$. Then the following generalized integration by parts formula holds

$$\langle y(t), v(t)\rangle_H - \langle y(s), v(s)\rangle_H = \int_{[s,t]} \langle y_t(\tau), v(\tau)\rangle_V + \langle v_t(\tau), y(\tau)\rangle_V \, \lambda_{[0,T]}(d\tau),$$

where the values $y(t)$ and $v(t)$ are the values of the continuous functions $y,\, v\colon [0,T] \to H$.

In the next lemma a well known result of Lions and Aubin about compact embedding is stated.

Lemma 8.17 (Lions-Aubin)**.** *Let B_0, B, B_1 be reflexive Banach spaces with $B_0 \subseteq B \subseteq B_1$ and assume that B_0 is compactly embedded in B and that B is embedded continuously in B_1. Let $1 < p < \infty$, q its conjugate exponent and define W by*

$$W := \{y \in L^p(T;B_0) \mid y_t \in L^q(T;B_1)\}.$$

Then W is compactly embedded into $L^p(T;B)$.

Example 8.18. *Assume* (i) *in Assumption 8.4 for $2 \le p < \infty$ and let V be some closed subspace of $W^{1,p}(\Omega)$ with $W^{1,p}_0(\Omega) \subseteq V \subseteq W^{1,p}(\Omega)$. Then it holds that the embedding $V \hookrightarrow L^p(\Omega)$ is compact and that the mapping $L^p(\Omega) \hookrightarrow V^\star$ is continuous. Applying Lemma 8.17 shows that the set*

$$W := \{y \in L^p(T;V) \,|\, y_t \in L^q(T;V^\star)\}$$

is compactly embedded into $L^p(T;L^p(\Omega))$, which can be identified with $L^p(Q)$, see Remark 8.11.

Lemma 8.19. *In the situation of example 8.18, the embedding* $\tau_{L^p(T;V),p}$ *from* $L^p(T;V)$ *into* $L^p(Q)$ *is continuous.*

The proof can be found in Proposition 23.2 (h), Zeidler (1990a).

Lemma 8.20. *The trace operator*

$$\gamma_p\colon W \to L^p(\Sigma),\, y \mapsto \gamma_p y$$

with $2 \le p < \infty$ *is compact.*

For the proof we refer to Proposition 2.143 in Carl et al. (2007).

Remark 8.21. *The isomorphism* $i\colon L^q(Q) \to L^p(Q)^\star$, $\frac{1}{p}+\frac{1}{q}=1$, *is continuous (see Lemma 8.6) and the adjoint mapping* ${\tau_{L^p(T;V),p}}^\star$ *as well by Lemma 8.19 and Theorem VI.4.8.2 (Schauder) in Dunford and Schwartz (1957). It follows that* ${\tau_{L^p(T;V),p}}^\star \circ i\colon L^q(Q) \to L^q(T;V^\star)$ *is continuous.*

The property of pseudomonotonicity plays as in the stationary case an important role.

Definition 8.4. *Let* X *be a reflexive Banach space and let* $L\colon D(L) \subseteq X \to X^\star$ *be linear, closed, maximal monotone and* $D(L)$ *dense in* X. *A mapping* $A\colon X \to X^\star$ *is called pseudomonotone with respect to* $D(L)$ *if for any sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $D(L)$ *with* $x_n \rightharpoonup x$ *in* X, $Lx_n \rightharpoonup Lx$ *in* $X^\star$ *and* $\limsup\langle Ax_n, x_n - x\rangle \le 0$ *it follows* $\lim\langle Ax_n, x_n - x\rangle = 0$ *and* $Ax_n \rightharpoonup Ax$ *in* $X^\star$.

In order to derive an existence result for pseudomonotone operators, we suppose the following hypotheses.

(H1) $(V, H, V^\star)$ is an evolution triple.

(H2) $\mathfrak{A}\colon L^p(T;V) \to L^q(T;V^\star)$ is bounded, demi-continuous, coercive and pseudomonotone with respect to $D(L)$.

Let $f \in L^q(T;V^\star)$ and let L be the mapping defined in (58). A general evolution equation reads as follows.

Problem 31. *Find some* $y \in D(L)$ *such that*

$$Ly + \mathfrak{A}y = f \text{ in } L^q(T;V^\star). \tag{Par3}$$

For this equation an existence result is proven in Berkovits and Mustonen (1996) (Theorem 1).

Theorem 8.22. *Assume (H1) and (H2). Then problem (Par3) has a solution for every* $f \in L^q(T;V^\star)$.

8.5 Multivalued Mappings

We now state some basic definitions of properties of set-valued mappings. We refer to Smirnov (2002), Carl et al. (2007) and Aubin and Cellina (1984). For a detailed survey of multivalued analysis see, e.g., Aubin and Frankowska (1990) or Hu and Papageorgiou (1997).

Definition 8.5. *Let X be a real Banach space and $A\colon X \to 2^{X^\star}$ a multivalued mapping. We define the domain of A by*

$$D\,(A) := \{x \in X |\, Ax \neq \emptyset\}$$

and its graph by

$$Gr\,(A) := \{(x, x^\star) \subset X \times X^\star |\, x^\star \in Ax\}.$$

We declare the following notation:

$$P_c(X) := \{M \subset X |\, M \text{ is nonempty, closed and convex}\}.$$

Definition 8.6 (Monotone Operators)**.** *Let X be some Banach space and M a subset of X. The mapping $A\colon M \subseteq X \to 2^{X^\star}$ is called*

i) monotone if

$$\langle y^\star - v^\star, y - v\rangle \geq 0 \;\; \textit{for all} \;\; (y^\star, y), (v^\star, v) \in Gr\,(A)$$

ii) maximal monotone if A is monotone and it follows from $(y, y^\star) \in M \times X^\star$ and

$$(y, y^\star) \in M \times X^\star : \;\; \langle y^\star - v^\star, y - v\rangle \geq 0 \;\; \textit{for all} \;\; (v, v^\star) \in Gr\,(A),$$

that $(y, y^\star) \in Gr\,(A)$

Remark 8.23. *As shown in Carl et al. (2007), p.47, a single valued map*

$$A\colon D(A) \subseteq M \to X^\star$$

can be identified with a multivalued map $\tilde{A}\colon M \to 2^{X^\star}$ by

$$\tilde{A}y := \begin{cases} \{Ay\} & \textit{if} \quad y \in D(A) \\ \emptyset & \textit{otherwise} \end{cases} .$$

Thus, the operator A is maximal monotone if A is monotone and the condition

$$(y, y^\star) \in M \times X^\star : \langle y^\star - Av, y - v\rangle \geq 0 \; \textit{for all} \; v \in D(A)$$

implies $y \in D(A)$ and $y^\star = Ay$. For example, this property is satisfied by the operator L defined in (58)*.*

For the next definition compare Chapter 2 in Naniewicz and Panagiotopoulos (1994). The term upper semi-continuity can be found, e.g., in Carl et al. (2007).

Definition 8.7. *Let X be a reflexive Banach space. The operator $A\colon X \to 2^{X^\star}$ is called pseudomonotone if the following conditions hold:*

i) *For each $y \in X$ the set $Ay \subset 2^{X^\star}$ is nonempty, bounded, convex and closed in $X^\star$.*

ii) *A is upper semi-continuous from each finite-dimensional subspace of X to the weak topology on $X^\star$.*

iii) *If $(y_n)_{n\in\mathbb{N}} \subset X$ with $y_n \rightharpoonup y$ in X and if $w_n \in Ay_n$ such that*

$$\limsup \langle w_n, y_n - y\rangle \leq 0,$$

then for all $v \in X^\star$ there exists an element $w = w(v) \in Ay$ with

$$\liminf \langle w_n, y_n - v\rangle \geq \langle w, y - v\rangle.$$

A weakened condition yields the notion of generalized pseudomonotonicity.

Definition 8.8. *Let X be a reflexive Banach space. The operator $A\colon X \to 2^{X^\star}$ is called generalized pseudomonotone if the following condition holds:*
If $(y_n)_{n\in\mathbb{N}} \subset X$ with $y_n \rightharpoonup y$ in X and if $w_n \in Ay_n$ with $w_n \rightharpoonup w$ in $X^\star$ such that

$$\limsup \langle w_n, y_n - y\rangle \leq 0,$$

then w lies in Ay and it holds

$$\langle w_n, y_n\rangle \to \langle w, y\rangle.$$

In Proposition 6.11 in Hu and Papageorgiou (1997) or Proposition 2.2 in Naniewicz and Panagiotopoulos (1994) sufficient conditions for pseudomonotone operators are stated and proven:

Proposition 8.24. *Let X be a reflexive Banach space. If $A\colon X \to P_c(X^\star)$ is bounded and generalized pseudomonotone, then A is pseudomonotone.*

The definition of coercivity for the multivalued case is stated, compare, e.g., Definition 3.1.4., p. 302, in Hu and Papageorgiou (1997).

Definition 8.9. *Let X be a reflexive Banach space. The operator $A\colon X \to 2^{X^\star}$ is called coercive, if either $D(A)$ is bounded or A is unbounded and*

$$\inf_{w\in Ay} \frac{\langle w, y\rangle}{\|y\|_X} \to \infty \quad \text{as } \|y\|_X \to \infty$$

The important result for the solvability of differential inclusions is stated in the next theorem which is proven in Theorem 2.6 and Remark 2.7 in Naniewicz and Panagiotopoulos (1994) and Corollary 3.6.30 in Hu and Papageorgiou (1997), p. 372.

Theorem 8.25. *Let X be some reflexive Banach space and let $A\colon X \to 2^{X^\star}$ be a pseudomonotone and coercive multivalued mapping. Then $Ra\,(A) = X^\star$.*

For parabolic inclusions with pseudomonotone operators an existence result is stated in Theorem 2.33 in Hu and Papageorgiou (2000).

9 Conclusion and Outlook

In this thesis we have considered the solvability of optimal control problems, which are constrained by nonlinear PDEs and PDIs. Existence results for optimal control problems including monotone nonlinearities have been proven, e.g., in Tröltzsch (2009), Casas et al. (1995) and Casas and Tröltzsch (2008). In contrast to the methods used in the monotone case, we have examined the application of sub- and supersolution techniques for dealing with optimal control problems which are constrained by nonlinear non-monotone PDEs. In the Chapters 2, 3 and 5 we have assumed the existence of sub- and supersolutions of the PDE with right hand side being the lower and the upper bound of the control. If the sub- and supersolution coincide with the pointwise state constraints, several properties of the optimal control problem could be proven. On the one hand, the substituting auxiliary problem characterized by the truncation operator can be defined without pointwise state constraints. On the other hand, the existence of a solution for the PDE with arbitrary admissible right hand side is given and hence the non-emptiness of the admissible set referring to the optimal control problem follows and has not to be assumed. This property of the sub- and supersolution is exploited in the Chapters 2, 3 and 4.1 for optimal control problems with elliptic PDEs and in 5 and 6.1 for optimal control problems with parabolic PDEs.
The quasi-linear case investigated in Chapters 4 and 6 has required different tools. If the leading operator fails to be strongly monotone, a penalty term secures the coercivity of an auxiliary operator. For elliptic PDEs, coercivity and the S_+-property of the auxiliary operator play a crucial role in replacing the linearizing step, which is in the semi-linear case an important tool in the proof. As we have seen in Chapters 5 and 6, for optimal control problems with parabolic PDEs the embedding properties, coercivity of the auxiliary operator and the S_+-property of a part of the auxiliary operator are dominant to show the existence of at least one solution. The main difference in contrast to the works of Papageorgiou (1991), Papageorgiou (1993) and Halidias and Papageorgiou (2002) is the boundedness assumption. In contrast to these works we have supposed boundedness properties only on a restricted set characterized by pointwise state constraints, compare, e.g., (11) of Chapter 2. As well as in Papageorgiou (1991), Papageorgiou (1993) and Halidias and Papageorgiou (2002), the nonlinear terms may be non-monotone.
The introduced methods have been applied to problems with two-sided and one-sided pointwise state constraints in this work. In the case of one-sided pointwise state constraints, we have made additional restrictions for the leading operator $\mathcal{A}$. The operator $\mathcal{A}$ was assumed to be strongly monotone, where for optimal control problems with two-sided pointwise state constraints, Leray-Lions conditions have been sufficient.

Moreover, in Chapter 7 the class of optimal control problems constrained by quasilinear elliptic and parabolic PDIs has been considered. Supposing proper hypotheses, in particular global assumptions of boundedness for the multivalued term, the pseudomonotonicity of the PDI describing multivalued auxiliary operator has been shown. This is the essential property for applying a result of Hu and Papageorgiou (1997), see Theorem 7.4. By making use of the S_+-property and the coercivity of the auxiliary operator the existence of at least one solution for the optimal control problem has been proven in Section 7.1.1.

The case of elliptic and parabolic inclusions of Clarke's gradient type has been examined under local assumptions of boundedness, compare Sections 7.1.2 and 7.2. New results presented in Carl et al. (2007) on the existence of a solution for the PDI have been applied and form a basis of the proof of the existence of a solution for the optimal control problem.

The derivation of necessary and sufficient optimality conditions is well known for some PDEs and their corresponding optimal control problems. Not only for the linear and semi-linear case results have been obtained (compare Tröltzsch (2009)), but also for optimal control problems with a PDE of quasi-linear type, see Casas et al. (1995) and Casas and Tröltzsch (2008). The PDEs, considered in this work, often yield multivalued control-to-state mappings. The derivation of necessary optimality conditions is much more difficult in this case and remains open.

In Sections 2.3 and 3.2 we have seen that under certain assumptions a solution of the optimal control problem can be approximated by solutions of optimal control problems with a finite dimensional space of the state variable. For the proof it was assumed that the lower and upper bound of the state variable are formed by a sub- resp. a supersolution. In this case an optimal control problem including pointwise state constraints is equivalent to some auxiliary problem without pointwise state constraints. It was shown, that there exists a sequence of problems defined on finite dimensional spaces for the state variable which have at least one solution. Under a regularity assumption, the convergence of a subsequence to some solution of the original optimal control problem was proven. If the existence of a solution for every semi-discretized optimal control problem can be warrented, then the assumption on the existence of a sub- and a supersolution is dispensable.

In the paper Papageorgiou (1992) another method for the derivation of necessary optimality conditions is introduced. The PDE constraining the optimal control problem is not assumed to be uniquely solvable. The technique is based on a penalty term measuring the derivation from some fixed solution of the considered optimal control problem. In my opinion applying this method on the auxiliary problem is promising. Problems may occur since the truncation operator fails to be differentiable. To avoid this situation, mollifier functions have to be included. Another important class of optimal control problems is constrained by variational inequalities. The methods applied in this thesis may be helpful in dealing with these problems.

References

Robert A. Adams and John J. F. Fournier (2003): *Sobolev spaces. 2nd ed.* Pure and Applied Mathematics 140. Academic Press, New York.

Hans Wilhelm Alt (2006): *Linear functional analysis. An application oriented introduction. (Lineare Funktionalanalysis. Eine anwendungsorientierte Einführung.)* 5th revised ed.,Springer, Berlin.

Jean-Pierre Aubin and Arrigo Cellina (1984): *Differential inclusions. Set-valued maps and viability theory.* Grundlehren der Mathematischen Wissenschaften 264. Springer, Berlin.

Jean-Pierre Aubin and Hélène Frankowska (1990): *Set-valued analysis.* Systems and Control: Foundations and Applications 2. Birkhäuser, Boston.

Anne Bauwe (2007): *Stochastische Differentialinklusionen.* PhD thesis, Martin-Luther-Universität Halle-Wittenberg.

François Bavaud (1991): *Equilibrium properties of the vlasov functional: The generalized poisson-boltzmann-emden equation. Reviews of Modern Physics*, 63 (1). URL `http://dx.doi.org/10.1103/RevModPhys.63.129`.

Juha Berkovits and Vesa Mustonen (1996): *Monotonicity methods for nonlinear evolution equations.* Nonlinear Anal., Theory Methods Appl. 27(12), 1397–1405.

Haïm Brézis (1968): *Équations et inéquations non linéaires dans les espaces vectoriels en dualité.* Ann. Inst. Fourier 18(1), 115–175.

Siegfried Carl, Vy Khoi Le and Dumitru Motreanu (2007): *Nonsmooth variational problems and their inequalities. Comparison principles and applications.* Springer Monographs in Mathematics. Springer, New York.

José A. Carrillo (1998): *On a nonlocal elliptic equation with decreasing nonlinearity arising in plasma physics and heat conduction.* Nonlinear Anal., Theory Methods Appl. 32(1), 97–115.

Eduardo Casas (1986): *Control of an elliptic problem with pointwise state constraints.* SIAM J. Control Optimization 24, 1309–1318.

Eduardo Casas (1993): *Boundary control of semilinear elliptic equations with pointwise state constraints.* SIAM J. Control Optimization 31(4), 993–1006.

Eduardo Casas and Fredi Tröltzsch (2008): *Optimality conditions for a class of optimal control problems with quasilinear elliptic equations.* DFG-Forschungszentrum MATHEON, Preprint 474.

Eduardo Casas, Luis A. Fernández, and Jiongmin Yong (1995): *Optimal control of quasilinear parabolic equations.* Proc. R. Soc. Edinb., Sect. A 125(3), 545–565.

Michel Chipot (2000): *Elements of nonlinear analysis.* Birkhäuser Advanced Texts. Birkhäuser, Basel.

Frank H. Clarke (1983): *Optimization and nonsmooth analysis.* Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York

Guowei Dai (2009): *Three solutions for a Neumann-type differential inclusion problem involving the $p(x)$-Laplacian.* Nonlinear Anal., Theory Methods Appl. 70(10), 3755–3760.

Guowei Dai and Wulong Liu (2009): *Three solutions for a differential inclusion problem involving the $p(x)$-Laplacian.* Nonlinear Anal., Theory Methods Appl. 71(11), 5318–5326.

Zdzisław Denkowski, Stanisław Migórski and Nikolas S. Papageorgiou (2003): *An introduction to nonlinear analysis. Theory.* Dordrecht: Kluwer Academic/Plenum Publishers. xv.

Nelson Dunford and Jacob T. Schwartz (1957): *Linear Operators I - General Theory.* Wiley-Interscience, New York.

Nikolaos Halidias and Nikolaos S. Papageorgiou (2002): *Optimal control problems for a class of nonlinear evolution equations.* Period. Math. Hung. 45(1-2), 43–63.

Seppo Heikkilä and Vangipuram Lakshmikantham (1994): *Monotone iterative techniques for discontinuous nonlinear differential equations.* Pure and Applied Mathematics (New York, Marcel Dekker) 181. Marcel Dekker, New York.

Juha Heinonen, Tero Kilpeläinen, and Olli Martio (1993): *Nonlinear potential theory of degenerate elliptic equations.* Oxford Mathematical Monographs. Clarendon Press, Oxford.

Einar Hille and Ralph S. Phillips (1957). *Functional analysis and semigroups. Rev. ed.* Colloquium Publications. 31. Providence, R. I. American Mathematical Society (AMS), XII.

Shouchuan Hu and Nikolaos S. Papageorgio (2000): *Handbook of multivalued analysis. Volume II: Applications.* Mathematics and its Applications 500. Kluwer Academic Publishers, Dordrecht.

Shouchuan Hu and Nikolaos S. Papageorgiou (1997): *Handbook of multivalued analysis. Volume I: Theory.* Mathematics and its Applications 419. Kluwer Academic Publishers, Dordrecht.

Dimitrios A. Kandilakis and Nikolaos S. Papageorgiou (1998): *Nonlinear periodic parabolic problems with nonmonotone discontinuities.* Proc. Edinb. Math. Soc., II. Ser. 41(1), 117–132.

Jacques-Louis Lions (1985): *Some questions in the optimal control of distributed systems.* Russ. Math. Surv. 40(4), 57–72.

Grzegorz Lukaszewicz and Bui An Ton (1994): *On some differential inclusions and their applications.* Journal of mathematical sciences, the University of Tokyo 1(2), 369–391. URL `http://ci.nii.ac.jp/naid/110000071203/en/`.

Terry J. Morrison (2001): *Functional analysis. An introduction to Banach space theory.* Pure and Applied Mathematics. A Wiley-Interscience Series of Texts, Monographs, and Tracts, Chichester.

Dumitru Motreanu and Vicenţiu Rădulescu (2003): *Variational and non-variational methods in nonlinear analysis and boundary value problems.* Nonconvex Optimization and Its Applications 67. Kluwer Academic Publishers, Dordrecht.

Zdzisław Naniewicz and Panagiotis D. Panagiotopoulos (1994): *Mathematical theory of hemivariational inequalities and applications.* Pure and Applied Mathematics, Marcel Dekker 188. Marcel Dekker, New York.

Chia-Ven Pao (1992): *Nonlinear parabolic and elliptic equations.* Plenum Press, New York.

Nikolaos S. Papageorgiou (1991): *An existence theorem for a class of nonlinear elliptic optimal control problems.* Commentat. Math. Univ. Carol. 32(2), 273–279.

Nikolaos S. Papageorgiou (1992): *On the optimal control of strongly nonlinear evolution equations.* J. Math. Anal. Appl., 164(1), 83–103.

Nikolaos S. Papageorgiou (1993): *Optimal control of nonlinear evolution equations with nonmonotone nonlinearities.* J. Optimization Theory Appl. 77(3), 643–660.

Walter Rudin (1991). *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, 2nd edition.

Georgi V. Smirnov (2002): *Introduction to the theory of differential inclusions.* Graduate Studies in Mathematics 41. American Mathematical Society, Providence.

Kati Sternberg (2006): *Simulation, Optimale Steuerung und Sensitivitätsanalyse einer Schmelzkarbonat-Brennstoffzelle mithilfe eines partiellen differential-algebraischen dynamischen Gleichungssystems*. PhD thesis, Universität Bayreuth.

Shingo Takeuchi (2001): *Multiplicity results for a degenerate elliptic equation with logistic reaction.* J. Differ. Equations 173(1), 138–144.

Bui An Ton (1996): *On an optimal control problem for a parabolic inclusion.* Nagoya mathematical journal 143, 195–217. URL `http://ci.nii.ac.jp/naid/110000023501/en/`.

Fredi Tröltzsch (2009): *Optimal control of partial differential equations. Theory, procedures, and applications. (Optimale Steuerung partieller Differentialgleichungen. Theorie, Verfahren und Anwendungen.).* Vieweg, Wiesbaden.

Eduardo H. Zarantonello (1960): *Solving functional equations by contractive averaging.* Technical Summary Report No. 160. Mathematical Research Center, Madison, Wisconsin.

Eberhard Zeidler (1990): *Nonlinear functional analysis and its applications. II/A: Linear monotone operators. Transl. from the German by the author and by Leo F. Boron.* Springer-Verlag, New York.

Eberhard Zeidler (1990): *Nonlinear functional analysis and its applications. II/B: Nonlinear monotone operators. Transl. from the German by the author and by Leo F. Boron.* Springer-Verlag, New York.

Eberhard Zeidler (1986): *Nonlinear functional analysis and its applications. I: Fixed-point theorems. Transl. from the German by Peter R. Wadsack.* Springer-Verlag, New York.