Die vorliegende Abhandlung

„Performance and Effects of Linear Feedback Stock Trading Strategies“

von

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Abriss (deutsch)

In effizienten Märkten können Händler zukünftige Entwicklungen nicht so prognostizieren, dass sie einen überdurchschnittlichen Gewinn erwarten können. Ob diese Hypothese gilt oder nicht, wird in der volkswirtschaftlichen Literatur stark diskutiert. Die vorliegende Arbeit untersucht eine eng damit verwandte Fragestellung, nämlich ob es möglich ist, durch Regelungstechniken Portfolios so zu kontrollieren, dass überhöhte Gewinne erwartet werden können, ohne dass dazu in die Zukunft geblickt werden muss.

In dieser Arbeit wird die Literatur der effizienten Märkte diskutiert, eine Einführung in die stochastische Analysis – wie sie für das Verständnis der modernen Finanzmathematik nötig ist – gegeben, die Literatur der sogenannten regelungsbasierten Handelsstrategien vorgestellt und Beispiele dazu gezeigt. Da es für die Analyse regelungsbasierter Handelsstrategien notwendig ist, wird im Laufe der Arbeit ein neues stochastisches Fubini-Theorem bewiesen, das es erlaubt, unter gewissen Voraussetzungen (z. B. in Erwartung linearer Integrator) Erwartungswert und Itō-Integral (für Semimartingale) zu vertauschen. Im Hauptteil der Arbeit werden bekannte Eigenschaften einer speziellen Strategie, der sogenannten SLS-Strategie, bei der gleichzeitig lang und kurz angelegt und dann auf die besser arbeitende Seite umgeschichtet wird, auf viel allgemeinere Marktmodelle als in der entsprechenden Literatur erweitert. Abschließend werden die Auswirkungen solcher Handelsregeln auf die Finanzmarktstabilität untersucht und die Ergebnisse diskutiert.

Abstract (English)

In efficient markets, it is not possible for any trader to look into the future in such a way that the trader can expect excess returns. Whether this hypothesis is true or false is highly discussed in the economics literature. The work at hand investigates a very close topic: Is is possible to construct predictable trading rules—by use of control techniques—that let the trader expect an excess return.

In this work, we summarize the literature on efficient markets, introduce the reader to the mathematical field of stochastic analysis, which is needed to understand which trading rules are allowed and which are not, review the literature on feedback trading, and give several examples. Because it is important for the analysis of feedback-backed stock trading strategies, we prove a new Fubini-type theorem, which allows to switch the expectation operator and the Itō integral (for semimartingales), if some conditions are fulfilled like in expectation linear integrators. In the main part, the properties of control-based trading rules known from the literature—like the robust positive expectation property—are extended to much more general market models. The thesis is concluded by an analysis of the impact of feedback-based trading strategies to market stability and a discussion of the obtained results.

Keywords

Asset Trading
Chartists
Control Theory
Demand Function
Efficient Market Hypothesis
Feedback-based Trading Rules
Financial Bubble
Financial Crisis
Fubini
Fundamentalists
Geometric Brownian Motion (GBM)
Heterogeneous Agent Model (HAM)
Hypothesis of Efficient Markets
Itō Integral
Joint Hypothesis Problem
Linear Feedback Trading
Market Efficiency
Market Maker Model
Merton's Jump Diffusion Model (MJDM)
Momentum Strategies
Product Integral
Risk
Robust Positive Expectation Property
Simultaneously Long Short Trading (SLS)
Stochastic Analysis
Stochastic Differential Equation (SDE)
Stochastic Fubini-type Theorem
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Chapter 1

Introduction

Trading means buying and selling assets. However, throughout this work we mostly rely on the case where one single asset with price p is traded. Generally, a trader tries to make a profit through buying the asset when it is cheap and selling it when it is expensive. In this work, we neglect additional gains, e.g. through dividends, since these gains would in average cause the same amount of trading losses (otherwise one could make money by buying the asset right before the dividends are payed out and selling right afterwards) and are not applicable to all kinds of assets. But in fact, the trader is facing the problem that a maximization of the trading gain is impossible if the market is arbitrage free. We assume that a trader cannot look into the future, thus, the trader has to estimate the future price by some heuristic rule.

The trader has several possibilities for choosing this rule, i.e. the trading strategy, e.g., relying on private (insider) information (insider; however, insider trading usually is not allowed) or public information. The latter one can be information on the asset's underlying firm (fundamentalist or fundamental analyst) or only on past price data (chartist or technical analyst). Another strategy is to trade fully random (noise trader). Insiders and fundamentalists believe that the stock price converges in a long run to or into a small environment around the fundamental price of the stock, i.e. the real price of the whole firm divided by the number of stocks issued or the fair price of the stocks calculated by use of (estimated) future payoffs. Insiders just have more information about the real value of the firm or about future payoffs. Note that in this work we do not discuss how fundamentalists calculate the fundamental value—they just know it. Trend followers, for example, think that the future stock price is higher than the current price if the price has risen in the past and vice versa, i.e., trend followers are chartists. An overview of these types of traders classified to which information is available and, thus, potentially used by the traders is given in Tab. 1.1.

The questions whether and which traders are able to make a profit on average are discussed in the context of efficient markets. The efficient market hypothesis states in its strong version that no trader is able to make money on average, in its semi-strong version that only with private information one can make money, and in its weak version that only with private information or information on fundamentals one can make money.

Depending on the definition of the efficient market hypothesis, risk and/or trading and information costs are taken into account. In all versions of the efficient market hypothesis chartists are not able to make money. Although most academic papers state that at least the semi-strong version has to be true and although it is reasonable that traders with more information should be able to make more money on average than traders with less information, Avramov et al. (2017) empirically find for single assets (i.e. not for indices) that technical analysts act better than fundamental analysts. For this reason we further analyze this topic.

trader type	private information	fundamentals (public)	charts (public)
insider	yes	yes	yes
fundamentalist	no	yes	yes
chartist	no	no	yes
noise trader	no	no	no

Table 1.1: Different types of traders classified to available or potentially used information

Much of the discussion on market efficiency, technical trading, and beating the market follows the idea that a trader (i) has to find a predictable pattern, like "higher returns at the beginning of January," (ii) has to construct a trading strategy exploiting this pattern, and (iii) has to test this new strategy against randomly selected broad index buy-and-hold strategies (Malkiel, 1973). However, a new strand of research, mainly in engineering sciences and mathematical control theory, goes another way: Assume task (i) can be skipped and trading strategies can be constructed directly. These strategies usually are model-free and do neither use predictions of patterns nor estimations of parameters like trends. In short and using the terminology of the control community: They are constructed to be robust against the price. Instead of task (iii), which relies on real market data, (performance) properties are proven mathematically. This way, the overfitting problem (cf. Bailey et al., 2014) is avoided.

Whether it is possible to estimate future prices and whether this can result in trading gains, is discussed in the broad literature on efficient markets in economics, which is reviewed in Chap. 2. After that, in Chap. 3, we shortly summarize the mathematical field of stochastic analysis, which investigates how price processes and trading strategies have to look like for being mathematical sound. Chapter 4 provides some new finding in the field of stochastic processes and probability theory. We prove that under specific assumptions it is allowed to switch the expectation operator and the Itō integral (cf. Fubini's theorem), use this for calculating the expected value of stochastic differential equations, and generalize Wald's lemma to the product case.

In Chap. 5, feedback trading is defined and feedback trading strategies are constructed. After that, in Chaps. 6, 7, and 8, we give a literature review on feedback trading, discuss market requirements, and analyze a small example model. In the main part of this work, the performance (Chap. 9) and the effects (Chap. 10) of these feedback-based trading rules are analyzed analytically. The work is concluded by a discussion of these results, especially of the performance results of a specific feedback trading rule, the so-called simultaneously long short (SLS) strategy, in the context of efficient markets, in Chap. 11.

To sum up, the questions answered in the work at hand are “What is feedback trading?,” “Does feedback trading perform well?,” “Is feedback trading in contradiction to market efficiency?,” “Which market assumptions are needed to prove specific performance properties?,” and “Does feedback trading affect market stability?” Additionally, we give (literature) overviews to the topics market efficiency, stochastic analysis, and feedback trading. Some theorems in the stochastics part are proven. Here, we mention that some proofs are not carried out in the work at hand and instead there are references given. Some proofs are done analogously to proofs that can be found in the related work. In this case, the references are given and there is no LaTeX proof environment used. The proofs that are written down in LaTeX proof environments are originally done by the author of this thesis. However, some proofs are not in LaTeX proof environments although they are done by the author—in this case, the reference (for the theorem or for the whole section) links to papers of the author.

Chapter 2

Hypothesis of Efficient Markets

The market efficiency hypothesis in its strong version states that no trader can make money on average. In its semi-strong version it states that only insiders make money on average and in its weak version that only insiders and fundamentalists, who are traders that know the fundamental value (loosely spoken: the real value) of the assets, can make money on average. In all versions, technical traders (technical analysts, chartists) cannot expect a profit. While in the 1970's this hypothesis was highly accepted (Fama, 1965, 1970), later on, it was highly criticized, and defended (Malkiel, 1989, 2005). Much of the critics concerned so-called predictable patterns, for example the January effect, i.e. high positive returns in the first two weeks of January. The defenders of the market efficiency hypothesis have several arguments against this critics, e.g., that patterns will self-destroy once published or that small possible gains will vanish when trading costs have to be payed.

Additionally, there is the so-called joint hypothesis problem which states that market efficiency and the used market model have to be tested nearly always simultaneously. That means, if the test fails, no one knows whether the market is not efficient or whether the model used is not sufficient. A second point of critics on the critics is the distinction between statistical inefficiency and economical inefficiency. The first one means that one can construct a test for showing that there are, for example, predictable patterns. The second one means that a trader has to be able to exploit this. When the costs for getting the information of an inefficient market behavior are high, the time range where markets are inefficient are very short, and/or the trading costs are not close to zero, it might happen that even if there are statistical inefficiencies, the traders are not able to make or to expect a profit out of these inefficient market behaviors.

And the last point to defend the market efficiency hypothesis we mention is that even if one can construct a strategy with “too high” returns, e.g., by taking into account some external variables, it may be that these variables are better ratios for measuring risk. When introducing risk-adjusted returns, excess returns are no contradiction to efficient markets when they go hand in hand with excess risk. In this chapter, we briefly discuss market efficiency, its critics, and its defense.

Because there is a very broad literature on this topic and there are also a lot of excellent and famous overviews we refer the interested reader to these overviews (e.g. Fama, 1991; Malkiel, 2003). Besides the definition and discussion of market efficiency, we discuss some topics where definitions are not clear, focused on the discussion of the simultaneously long short (SLS) strategy at the end of this work.

2.1 The Strong Version

In its strong version, the market efficiency states that all information is reflected in the price. That means, no "sophisticated" trader, even no "insider," who has private information, performs on average better than a simple buy-and-hold trader. That means, when there is no change in the fundamental value, all price movements are fully random without any trend. Mathematically spoken, the price process is a random walk around its fundamental value and it is not possible to predict future changes in the fundamentals. A little bit weaker and maybe closer to markets is the assumption that only nearly all information is incorporated in the price. But the costs for getting the missing information and for trading the asset are higher than the possible gain of exploiting this information (Fama, 1991).

2.2 The Semi-Strong Version

The semi-strong version of the market efficiency hypothesis states that all public information is reflected in the price. That means "insider trading" may be profitable, which is widely accepted. For example, the findings on the effects of Value Line rank changes are a sign that insider trading may be profitable (Stickel, 1985) (summarized in Fama, 1991). As mentioned in the introduction, both insiders and fundamentalists believe that the stock price goes to the fundamental value in a long run.

Under the assumption of the semi-strong version, all public information is immediately incorporated in the asset price. The word "immediately" has to be understood in an averaged sense, i.e., markets may overreact to new information or underreact and markets may reflect information too early or too late, but on average all these effects are balancing out (Fama, 1995). In other words, fundamental value analysis, i.e., trying to calculate the fundamental or intrinsic value (simplistic: the real value), is on average not profitable at all, because an asset's actual price is at any point of time the best estimate for the fundamental value (based on public information). Fundamentalists can make profit if they find relevant information faster and rate the effects to the fundamental values under analysis better. Thus, all fundamentalists try to be as fast and as accurate as possible, thereby adjusting prices instantaneously to the intrinsic values. Since no one knows who is the fastest and the best, on average fundamentalists cannot expect excess gains.

2.3 The Weak Version

Last, the weak version of market efficiency states that insider trading as well as fundamental analysis may be profitable but technical analysis is not. That means, no one can use past returns to predict future ones. Also in this version, chartists cannot make money on average, markets have no memory, and patterns do not exist. Or, even a little bit weaker, when there exists a dependence of past and future returns, these anomalies are so small that they are not exploitable.

To sum up, in all versions of the market efficiency hypothesis, for chartists it is not possible to make money on average. Because on the other hand there is a lot of literature on the profitability of technical trading and there are numerous fund managers who rely on such strategies, the question whether markets are efficient or not is always considered to be empirical. That means, chartist fund managers are challenged to provide statistics that their strategies outperform random-selected buy-and-hold strategies. Hereafter, we summarize a selection of common critics to the market efficiency hypothesis and state some arguments of the defenders of the hypothesis against these critics.

2.4 Discussion of the Efficient Market Hypothesis

One strand of critics to the market efficiency hypothesis relies on predictable patterns. With statistical or data science methods, patterns, i.e., on average recurring behaviors of stock market prices were found: the Monday effect (lower returns on Mondays; Cross (1973); French (1980)), the month effect (higher returns at the last day of the month; Ariel (1987)), the holiday effect (higher returns at the day before a holiday; Ariel (1990)), and the most famous January effect (higher returns in January and even higher returns in the first five days of January; Keim (1983); Roll (1983)). These critics attack the weak version of the efficient market hypothesis.

But, following Malkiel (2003), predictable patterns will self-destroy once published. Exemplary for the January effect: If the January effect exists, traders would buy at the last days of December and sell at the very beginning of January. That means, the pattern would move a few days. Observing this, traders would buy and sell again a few days earlier, and so on. At the end, the January effect would be destroyed. A second attack to this strand of critics is that the effects of (predictable) patterns are too small to exploit them (Lakonishok and Smidt, 1988), especially when trading costs are considered. This last argument can be generalized: Only because there is a statistical inefficiency (i.e. predictability in returns, which is shown by use of data science methods) that does not mean that a trader can make profit of it, when the effect and the power of the statistic is small relative to additional costs. That means, economical inefficiency had to be shown by trading performance statistics.

Another strand of critics to market efficiency is that stock returns may be predictable using some external variables, for example, dividend yields (D/P; Rozeff (1984); Shiller (1984)), earning per price ratios (E/P; Campbell and Shiller (1988)), or the firms' size (Banz, 1981). These studies are related to the semi-strong version of the hypothesis of efficient markets. But, as summarized by Fama (1991), these dependencies are either too small to exploit them (especially when trading costs are taken into account) or, like in the case of the size effect, they have another reason: Taking into account some external variables with predictive power may just mean that these variables are better ratios for measuring risk. That means, if someone finds external variables which are positively correlated with expected future returns of an asset, these variables are probably correlated with the risk of these assets. When assuming that investing in riskier assets should result in higher expected profits, the existence of such variables is no contradiction to market efficiency.

As mentioned above, the definition of market efficiency is not clear at all. Despite the statistical inefficiency vs. economical inefficiency problem, one can find statements like "traders cannot expect excess returns" as well as "traders can only expect excess returns when they accept excess risk" in the literature. The definition of market efficiency is not unique. So, often the term risk-adjusted gains is used. Here, the next problem arises: How to measure risk? Often the Capital Asset Pricing Model's (CAPM's) β or the standard deviation is used. We come back to this problem in Chaps. 9 and 11 again.

Next, we explain a few more problems that occur when discussing market efficiency. First, all empirical findings concerning market efficiency (in all versions) might be results of data-dredging (also known as p-hacking), i.e., the results might be found by use of data-mining techniques searching for significant p-values without causality or an underlying hypothesis. That means, when doing enough tests with enough data, it is very likely that eventually examples for and against all test hypotheses concerning efficient markets are found. However, there are studies indicating that there are (with constant fundamentals) long term trends (possibly sinusodial) (Granger and Morgenstern, 1962; Saad et al., 1998). Another problem, related to the data-dredging issue, is the overfitting problem when constructing trading strategies. Bailey et al. (2014) state that often trading rules are defined by too much use of past data s.t. there is no meaning for future developments anymore.

Second, there is the joint hypothesis problem, which states that market efficiency can (nearly) always only be tested when simultaneously using a market model. A consequence is that if a test fails, no one can say whether the market efficiency hypothesis is wrong or whether the used market model is insufficient. With this argument, all critics to the efficient market hypothesis in all versions that rely not only on general market assumptions but on specific market models can be defended.

An exception of this joint hypothesis problem are so-called event studies (Fama et al., 1969). Event studies analyze how fast and to which extent stock prices adjust to announcements, i.e. to new public information. So, event studies lie in the field of the semi-strong form of the market efficiency hypothesis and not in the field of the weak one. There are studies that indicate that stock prices do not adjust instantaneously to new events, but there are also papers stating that too early reaction and too late reaction as well as overreaction and underreaction are averaging out (Stickel, 1985; Fama, 1995).

And last, there is the momentum effect, which states that assets that performed well over the last few months will do so over the next few months (approximately up to twelve months) and similar for bad assets (Jegadeesh and Titman, 1993, 2001; Fama and French, 1996, 2008). According to Moskowitz (2010) this effect can be explained by behavioral economics: Traders tend to underreact at first and overreact with a delay, both causing momentum. Related to the momentum effect is the disposition effect, which states that traders tend to sell rising assets too early to lock in the winnings while sell falling assets too late since they hope the charts become better again. Both behaviors make a delay in the assets' reaction to news in the fundamentals: After good news prices do not rise adequately since some traders sell for locking in winnings and after bad news prices do not fall adequately because some traders hold their shares too long because they hope for good news. For a trader to exploit the momentum effect there is the problem that this effect eventually stops and the traders can hardly anticipate when this will happen. Nonetheless, the momentum effect can be exploited. The only way to explain this in an efficient market is to take risk into account: Investing in well performing assets has to be riskier than investing in poor performing assets. This seems to be quite counterintuitive, but from another perspective, maybe the asset is only performing better because the underlying firm is accepting more risk in its work. Other explanations for this increase of risk are that for well performing firms it is harder to hold the performance level and that the investment opportunities might have changed due to the better performance (cf. Moskowitz, 2010).

To sum up, there are various versions (strong, semi-strong, weak) and definitions (with or without risk, with or without trading and information cost, comparison to random strategies or not) of the efficient market hypothesis, but in all versions chartists are not allowed to make money on average. The attacks to market efficiency are always considered to be empirical, while the defense is empirical and theoretical.

In the work at hand, we provide some theoretical critics without running into the joint hypothesis problem or into the overfitting problem because we construct our trading rules model-free, i.e., in the definition of these rules no market model is assumed, and in the analyses of our strategies, we only start with specific market models but generalize the performance properties to markets where just a few general assumptions have to be fulfilled. For the construction and the analysis of our technical trading rules we need some basics from the field of stochastic analysis, which are given next.

Chapter 3

Stochastic Analysis in Continuous Time

In this chapter, we give a very brief overview of stochastic analysis, which is the field of mathematics dealing with stochastic processes, stochastic integration, and stochastic differential equations. This is important to understand the analysis of the trading strategies we are interested in. We need to know how to calculate gains as stochastic integrals, what trading strategies are meaningful and allowed, e.g., not looking into the future, and what are stochastic differential equations, because by means of these equations often market models are defined.

We restrict this overview to the more general case of continuous time and we skip the proofs, except for the cases where they are important for the understanding. The proofs can be found in the literature of stochastic analysis, especially in the work of Protter (2005); Øksendal (2003); Applebaum (2009); Kühn (2016); McKean jr. (1969). This chapter is following Kühn (2016) and also Protter (2005) very closely. Some of the definitions and theorems in this chapter are not directly needed for this work, but they are needed indirectly or for the general understanding of stochastic analysis or for those readers who want to read in the cited references.

3.1 Basics of Stochastic Analysis

Before starting with stochastic integrals, which is the most important concept of this chapter, we have to define and discuss a few sets, functions, sequences, etc. All these concepts building upon stochastics and probability theory.

Definition 1 (Basic Setting)**.** *Let Ω be a non-empty set of outcomes. We call $\mathcal{F} \subset \wp(\Omega)$ a σ-algebra if it contains Ω, is closed under complement, and is closed under countable unions. The pair $(\Omega, \mathcal{F})$ is called a measurable space. All elements of $\mathcal{F}$ are called measurable. A function*

$$\mathbb{P} : \mathcal{F} \to [0, 1]$$

with $\mathbb{P}(\Omega) = 1$ *and* $\mathbb{P}\left(\dot{\bigcup}_{n\in\mathbb{N}} A_i\right) = \sum_{n\in\mathbb{N}} \mathbb{P}(A_i)$ *for all countable sequences of disjoint sets* $A_i \in \mathcal{F}$ *is called probability measure. The triple* $(\Omega, \mathcal{F}, \mathbb{P})$ *is called a probability space. A set* N *is called a null set if* $N \in \mathcal{F}$ *and* $\mathbb{P}(N) = 0$. *A probability space is called complete if all subsets of all null sets are measurable. A filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ *is a family of* σ*-algebras with* $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ *for all* $s \leq t$. *We call* $T \in \mathbb{R}^+$ *the horizon of the model and* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *a filtered probability space.*

The σ-algebra $\mathcal{F}_t$ can be interpreted as the information that is known at time t, i.e., if $A \in \mathcal{F}_t$ then at time t we know whether A happened or not. Note that every probability space can be transformed into a complete one: Let's define

$$\tilde{\mathcal{F}} = \{\tilde{A} \subset \Omega \mid \exists A_1, A_2 \in \mathcal{F} \text{ s.t. } A_1 \subset \tilde{A} \subset A_2 \text{ and } \mathbb{P}(A_2 \setminus A_1) = 0\}$$

and $\tilde{\mathbb{P}} : \tilde{\mathcal{F}} \to [0,1]$, $\tilde{A} \mapsto \mathbb{P}(A_1) = \mathbb{P}(A_2)$. One can prove that $\tilde{\mathcal{F}}$ is a σ-algebra, that $\tilde{\mathbb{P}}$ is well defined, that $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is complete, and that $\tilde{\mathcal{F}}$ is the smallest complete σ-algebra that contains $\mathcal{F}$.

Definition 2 (Usual Conditions)**.** *A filtered and complete probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *fulfills the usual conditions if* $\mathcal{F}_0$ *contains all null sets of* $\mathcal{F}$ *and if* $\mathbb{F}$ *is right-continuous, i.e., if*

$$\mathcal{F}_t = \mathcal{F}_{t+}$$

for all $t \in [0, T)$ *with* $\mathcal{F}_{t+} := \bigcap_{u\in(t,T]} \mathcal{F}_u$.

That means the information that is available at time t is exactly the same as directly after t or—the other way around—it cannot happen that an information is not available at time t but directly after t. Note that $\mathbb{F}$ does not need to be left-continuous, i.e., knowing all information up to t does not mean that the information at time t is already known.

Definition 3 (Random Variables and Stochastic Processes)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $(\Omega', \mathcal{F}')$ *be a measurable space. A function* $Z : \Omega \to \Omega'$ *is called random variable (and also measurable) if* $Z^{-1}(A') \in \mathcal{F}$ *for all* $A' \in \mathcal{F}'$. *If* $(\Omega', \mathcal{F}') = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, Z *is called a real-valued random variable,* $\mathcal{B}(\mathbb{R})$ *is the Borel* σ*-algebra on* $\mathbb{R}$, *which is the smallest* σ*-algebra that contains all open sets of* $\mathbb{R}$. *For any* $\mathcal{M} \subset \wp(\Omega)$ *we denote with* $\sigma(\mathcal{M})$ *the smallest* σ*-algebra that contains* $\mathcal{M}$. *For a real-valued random variable* Z *we define* $\sigma(Z) = \{Z^{-1}(A) \mid A \in \mathcal{B}(\mathbb{R})\}$, *which is a* σ*-algebra. A mapping* $X : \Omega \times [0, T] \to \mathbb{R}$ *is called a (real-valued) stochastic process if* $X_t : \Omega \to \mathbb{R}$, $\omega \mapsto X(\omega, t)$ *is a random variable for all* t *(i.e., if* X_t *is* $\mathcal{F}$*-measurable). If* X_t *is additionally* $\mathcal{F}_t$*-measurable for all* t, *we call* X *an adapted stochastic process. For a stochastic process* X *for all* t *we call* $(\mathcal{F}_t^0(X))_{t\in[0,T]}$ *with* $\mathcal{F}_t^0(X) = \sigma(\{X_s^{-1}(A) \mid s \leq t, A \in \mathcal{B}(\mathbb{R})\})$ *the natural filtration of* X.

Sometimes, we use $\mathbb{R}_0^+$ instead of $[0, T]$, which is just a little bit more general. Note that any stochastic process is adapted to its natural filtration. The mapping $t \mapsto X_t(\omega)$ is called a sample path of X. The natural filtration of a right-continuous stochastic process (i.e. a stochastic process that has only right-continuous paths ($\lim_{s\to t, s>t} X_s(\omega) =$

$X_t\ \forall\omega$)) does not necessarily have to be right-continuous: Let's have a look at the process X with $X_t = (t - t_0)^+ Z$ for any $t_0 \in (0, T)$ and any non-constant real-valued random variable Z. It holds that $F_t^0(X) = \{0, \Omega\}$ if $t \leq t_0$ and $F_t^0(X) = \sigma(Z)$ if $t > t_0$. It follows that $\mathcal{F}_{t_0}^0 \neq \mathcal{F}_{t_0+}^0$.

For any filtration $\mathbb{F}$ we can define $\tilde{\mathbb{F}}$ via $\tilde{\mathcal{F}}_t = \bigcap_{u \in (t,T]} \sigma(\mathcal{F}_u, \mathcal{N})$ if $t \in [0, T)$ and $\tilde{\mathcal{F}}_t = \sigma(\mathcal{F}_T, \mathcal{N})$ if $t = T$, with $\mathcal{N} = \{A \in \mathcal{F} \mid \mathbb{P}(A) = 0\}$, which fulfills the usual conditions. When adding the null sets to any right-continuous filtration, this new filtration is automatically right-continuous: It holds $\bigcap_{u \in (t,T]} \sigma(\mathcal{F}_u, \mathcal{N}) = \sigma\left(\bigcap_{u \in (t,T]} \mathcal{F}_u, \mathcal{N}\right)$ for any filtration $\mathbb{F}$. We always assume the usual conditions.

Definition 4 (Stopping Time)**.** *A random variable $\tau : \Omega \to [0, 1]$ is called stopping time if for all t it holds $\{\tau \leq t\} := \{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{F}_t$. The set $\mathcal{F}_\tau = \{A \in \mathcal{F} \mid A \cap \{\tau \leq t\} \in \mathcal{F}_t\ \forall t \in [0, T]\}$ is the stopping time σ-algebra of τ.*

That means, for a stopping time τ at time t we know whether τ happened yet. When τ is a stopping time it holds $\{\tau < t\} \in \mathcal{F}_t$. Since $\mathbb{F}$ is right-continuous it additionally holds that if $\{\tau < t\} \in \mathcal{F}_t$ for all t it follows that τ is a stopping time.

Theorem 5. Let X be a counting process, that is $X_t = \sum_{n \in \mathbb{N}} \mathbb{I}_{[\tau_n, T]}(t)$ with stopping times τ_n. The filtration $\mathcal{F}_t^0(X) = \sigma(X_s, s \leq t)$ is right-continuous.

Proof in Protter (2005, Thm. I.25).

Definition 6 (Equivalence of Stochastic Processes)**.** *Let X and Y be two stochastic processes. We call X a version of Y and vice versa if $\mathbb{P}(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) = \mathbb{P}(Y_{t_1} \in A_1, \ldots, Y_{t_n} \in A_n)$ for all $n \in \mathbb{N}$, $t_1, \ldots, t_n \in [0, T]$, and $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$. The process X is a modification of Y and vice versa when $\mathbb{P}(X_t = Y_t) = 1$ for all t. The processes X and Y are indistinguishable if $\mathbb{P}(X_t = Y_t\ \forall t \in [0, T]) = 1$.*

If two processes are indistinguishable, they are modifications. If they are modifications, they are versions. Two versions do not need to be defined on the same space. If X and Y are indistinguishable, there exists $N \in \mathcal{F}_0$ s.t. $t \mapsto X_t(\omega)$ equals $t \mapsto Y_t(\omega)$ for all $\omega \in \Omega \setminus N$. If X and Y are modifications of each other, for all t there exists $N_t \in \mathcal{F}_t$ s.t. $X_t(\omega) = Y_t(\omega)$ for all $\omega \in \Omega \setminus N_t$. But $\bigcup_{t \in [0,T]} N_t$ does not need to be measurable.

For the difference of indistinguishable processes and modifications we give a small example: Let $X \equiv 0$ and $Y_t(\omega) = \mathbb{I}_{t = U(\omega)}$ with U being uniformly distributed on $[0, T]$. It holds $\mathbb{P}(X_t = Y_t) = \mathbb{P}(U \neq t) = 1$ for all t, but $\mathbb{P}(X_t = Y_t\ \forall t) = 0$.

If X and Y are modifications of each other and the paths of X and Y are right-continuous, X and Y are indistinguishable. The proof is based on the set $\bigcup_{t \in \mathbb{Q} \cap [0,T)} N_t \cup N_T$, which is a null set. For details see Kühn (2016, pp. 7f) and Protter (2005, Thm. I.2).

Definition 7 (Càdlàg). *A stochastic process X is called càdlàg ("continue à droite, limite à gauche") if all paths of X are right-continuous and $\lim_{s\to t, s<t} X_s(\omega) = X_{t-}(\omega)$ exists in $\mathbb{R}$ for all $t \in (0, T]$.*

We set $X_{0-}(\omega) = X_0(\omega)$, $\Delta X_t = X_t - X_{t-}$, $X_- = (X_{t-})_{t\in[0,T]}$, and $\Delta X = (\Delta X_t)_{t\in[0,T]}$. When X and Y are indistinguishable, then X_- and Y_- are indistinguishable, too, as well as ΔX and ΔY (cf. Kühn, 2016, Def. 2.13).

Theorem 8. Let $f : [0, T] \to \mathbb{R}$ be a càdlàg function. For all $n \in \mathbb{N}$ there is only a finite number of jumps with absolute value bigger than $\frac{1}{n}$ and $\{t \mid \Delta f_t \neq 0\}$ is countable.

Proof in Kühn (2016, p. 8). Usually, asset prices are modeled via càdlàg processes. This class allows for price jumps, e.g., caused by shocks, but does not allow for double jumps. Double jump means that neither the right limit nor the left limit equals the value of the process at the jump time.

For a stochastic process X and a set $B \in \mathcal{B}(\mathbb{R})$ we call $\tau(\omega) = \inf\{t > 0 \mid X_t(\omega) \in B\}$ the hitting time of B for X. If X is an adapted process with right-continuous or left-continuous paths and B an open set, then the hitting time of B is a stopping time (Kühn, 2016, p. 8).

Definition 9 (Conditional Expectation). *Let Z be a real-valued, $\mathcal{F}$-measurable random variable and let $\mathcal{G} \subset \mathcal{F}$ be a sub σ-algebra. If $Z \geq 0$ or $\mathbb{E}[Z] < \infty$ there exists an a.s. unique $\mathcal{G}$-measurable random variable Z' s.t. $\mathbb{E}[\mathbb{I}_A Z] = \mathbb{E}[\mathbb{I}_A Z']$ $\forall A \in \mathcal{G}$. With $Z' =: \mathbb{E}[Z|\mathcal{G}]$ we denote one version of this conditional expectation of Z under information $\mathcal{G}$.*

If neither $Z \geq 0$ nor $\mathbb{E}[Z] < \infty$ holds, but instead $\mathbb{E}[|Z||\mathcal{G}] < \infty$, we can set $\mathbb{E}[Z|\mathcal{G}] = \mathbb{E}[Z^+|\mathcal{G}] - \mathbb{E}[Z^-|\mathcal{G}]$. Note that in general a conditional expectation is not deterministic.

Definition 10 (Martingale). *An adapted, càdlàg process $(X_t)_{t\in[0,T]}$ with $\mathbb{E}[|X_t|] < \infty$ for all t is called a martingale if $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ a.s. for all $s \leq t$. It is called a submartingale if $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ a.s. for all $s \leq t$ and a supermartingale if $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$ a.s. for all $s \leq t$.*

The definition of martingales is equivalent to $\mathbb{E}[\mathbb{I}_A(X_t - X_s)] = 0$ $\forall s \leq t$ and $A \in \mathcal{F}_s$. Note that the definition of martingale depends on $\mathbb{P}$ and $\mathbb{F}$.

Theorem 11. If H is a random variable with $\mathbb{E}[|H|] < \infty$, there exists exactly one $\mathbb{P}$-martingale X with $X_T = H$ *a.s.* (modulo indistinguishability). This is a càdlàg transformation of the process $t \mapsto \mathbb{E}[H|\mathcal{F}_t]$, i.e. a process where all paths are made right-continuous.

Proof in Dellacherie and Meyer (1978).

Definition 12 (One Dimensional Brownian Motion). *An adapted process $(B_t)_{t\geq 0}$ with $B_0 = 0$ is a Brownian motion if $B_t - B_s$ is independent of $\mathcal{F}_s$ for all $0 \leq s \leq t < \infty$, if $B_t - B_s \sim \mathcal{N}(0, t - s)$ for all $0 \leq s \leq t < \infty$, and if all paths $t \mapsto B_t(\omega)$ are continuous.*

Theorem 13. A Brownian motion exists.

Proof in Klenke (2006). Note that this definition is over-determined. One could either drop the assumption of Gaussian distributed increments and instead use any distribution with mean 0 and variance $t-s$. By use of the central limit theorem the Gaussian distribution follows. Or, one could drop the assumption of continuous paths as this property would follow by use of the Kolmogorov Chentsov continuity theorem, i.e., there would exist a continuous modification of the process. Without the usual conditions the definition had to be modified to *a.a.* paths. But by use of the usual conditions we could set all non-continuous paths to zero and would have a Brownian motion where all paths are continuous (cf. Kühn, 2016, Bem. 2.27).

Let $(B_t)_t$ be a Brownian motion and $s \leq t$. Since it holds $\mathbb{E}[B_t|\mathcal{F}_s] = B_s + \mathbb{E}[B_t - B_s|\mathcal{F}_s] = B_s + \mathbb{E}[B_t - B_s] = B_s$, the Brownian motion is a martingale. In a similar way it can be shown that $(B_t^2 - t)_t$ is a martingale, too, and exploiting $Z \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow \mathbb{E}\left[e^Z\right] = e^{\mu + \frac{\sigma^2}{2}}$ and $\mathbb{V}\left[e^Z\right] = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$ that also $\left(e^{aB_t - \frac{a^2 t}{2}}\right)_t$ is a martingale for all real a (Kühn, 2016, Thm. 2.28).

Definition 14 (Grid and Mesh)**.** *For $k \in \mathbb{N}$ we call $\pi = (t_0, \ldots, t_k)$ a grid on $[a, b]$ if $a = t_0 < t_1 < \ldots < t_k = b$. We define the mesh size $mesh(\pi) = \max\limits_{j=1,\ldots,k}(|t_j - t_{j-1}|)$.*

Theorem 15. Let B be a Brownian motion, $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of grids on $[a, a + t]$ (i.e., $\pi_n = (t_0^n, \ldots, t_{k_n}^n)$, $k_n \in \mathbb{N}$, and $a = t_0^n < t_1^n < \ldots < t_{k_n}^n = a + t$) with $\lim\limits_{n\to\infty} mesh(\pi_n) = 0$. Define $\pi_n^{(2)}(B) = \sum\limits_{j=1,\ldots,k_n} (B_{t_j^n} - B_{t_{j-1}^n})^2$.

Then it holds $\lim\limits_{n\to\infty} \mathbb{E}\left[\left(\pi_n^{(2)}(B) - t\right)^2\right] = 0$ (convergence in L^2 (mean square), which implies convergence in probability ($\forall \varepsilon > 0 : \lim\limits_{n\to\infty} \mathbb{P}\left(\left|\pi_n^{(2)}(B) - t\right| > \varepsilon\right) = 0$)).

Proof in Kühn (2016, Thm. 2.30).

Theorem 16. If the sequence of grids is refining, the convergence is *a.s.* (i.e., $\mathbb{P}\left(\lim\limits_{n\to\infty} \pi_n^{(2)}(B) = t\right) = 1$).

Proven in Protter (2005, Thm. I.28 (and Thm. I.14)). By use of grids with mesh size to zero we defined $\pi_n^{(2)}$, which is something like a quadratic variation (with $n \to \infty$). Later, we formally define the quadratic variation and see that this quadratic variation with grids equals the formally defined quadratic variation for Brownian motions (and, actually, for all semimartingales). Note that Thm. 15 tells us a property of the Brownian motion in general and not of its single paths, like the next theorem.

Theorem 17. For $0 < a < b$ and $r \in \overline{\mathbb{R}^+}$ and for *a.a.* sample paths $t \mapsto B_t(\omega)$ there exists a sequence of grids π_n on $[a, b]$ (depending on ω) with $\lim\limits_{n\to\infty} mesh(\pi_n) = 0$ and $\pi_n^{(2)}(B(\omega)) = \sum\limits_{j=1,\ldots,k_n} \left(B_{t_j^n}(\omega) - B_{t_{j-1}^n}(\omega)\right)^2 = r$.

(Cf. Kühn, 2016, Thm. 2.31).

Definition 18 (Variation)**.** *Let $(X_t)_{t\geq 0}$ be a càdlàg stochastic process. With $V(X)_t = \sup_{n\in\mathbb{N}} \sum_{k=1,\dots,2^n} \left| X_{\frac{k}{2^n}t} - X_{\frac{k-1}{2^n}t} \right|$ we denote the variation of X. The process $V(X)_t$ is $[0,\infty]$-valued. We say that X has finite variation if $V(X)_t < \infty$ a.s. for all $t < \infty$.*

It holds that $V(X)_t = \lim_{n\to\infty} \sum_{k=1,\dots,2^n} \left| X_{\frac{k}{2^n}t} - X_{\frac{k-1}{2^n}t} \right|$. If X is right-continuous (or even left-continuous) it holds $V(X)_t = \sup_{\pi=(t_0,\dots,t_n)} \sum_{k=1,\dots,n} |X_{t_k} - X_{t_{k-1}}|$, where all grids π are allowed. The advantage of the dyadic definition is that $V(X)_t$ is $\mathcal{F}_t$-measurable. If X is càdlàg and has finite variation, the mapping $t \mapsto V(X)_t$ is non-decreasing and càdlàg (cf. Kühn, 2016, Thm. 2.34).

Theorem 19. The Brownian motion has *a.s.* an unbounded variation.

Proof in Kühn (2016, Thm. 2.35). This property makes the definition of stochastic integrals somehow tricky as we see later on.

The filtration $\mathcal{F}_t(B) = \sigma(B_s, s \leq t, \mathcal{N})$ with $\mathcal{N}$ being the set of null sets of $\sigma(B_s, s \leq T)$ is an example for a right-continuous filtration. Proven in Karatzas and Shreve (1991, Chap. 2, Prop. 7.7, and Thm. 7.9).

So far, we learned about the definition and some properties of stochastic processes, as they are needed for constructing price movements. However, we need trading strategies, which are also processes, and trading gains to understand how trading in stochastic finance works.

3.2 Stochastic Integration—Simple Predictable Processes

In this section, we define trading gains as stochastic integrals. For this, we start with trading strategies that are constructed similarly to discrete time trading, i.e., at first we allow the trader to reallocate the portfolio only finitely often.

Definition 20 (Stochastic Intervals)**.** *For stopping times τ and σ, we define the stochastic intervals*

$$\rrbracket \tau, \sigma \rrbracket = \{(\omega, t) \in \Omega \times [0,T] \mid \tau(\omega) < t \leq \sigma(\omega)\},$$

$$\rrbracket \tau, \sigma \llbracket = \{(\omega, t) \in \Omega \times [0,T] \mid \tau(\omega) < t < \sigma(\omega)\},$$

$$\llbracket \tau, \sigma \rrbracket = \{(\omega, t) \in \Omega \times [0,T] \mid \tau(\omega) \leq t \leq \sigma(\omega)\},$$

$$\llbracket \tau, \sigma \llbracket = \{(\omega, t) \in \Omega \times [0,T] \mid \tau(\omega) \leq t < \sigma(\omega)\},$$

and

$$\llbracket \tau \rrbracket = \llbracket \tau, \tau \rrbracket.$$

Definition 21 (Simple Predictable Process)**.** *A process H is called simple predictable if it can be written as*

$$H_t(\omega) = \sum_{i=1,\dots,n} Z_{i-1}(\omega) \mathbb{I}_{\rrbracket T_{i-1}, T_i \rrbracket}(\omega, t)$$

with $n \in \mathbb{N}$, $(T_i)_{i=1,\ldots,n}$ stopping times with $0 = T_0 \leq T_1 \leq \ldots \leq T_n = T$, and Z_i $\mathcal{F}_{T_i}$-measurable random variables ($i = 0, \ldots, n$) with $|Z_i| < \infty$. With $\mathcal{S}$ we denote the set of all simple predictable processes.

Definition 22 (Stochastic Integral)**.** *Let H be a simple predictable process and X be a càdlàg process. We define the stochastic integral $I_X(H)$ via*

$$I_X(H) = \sum_{i=1,\ldots,n} Z_{i-1}(X_{T_i} - X_{T_{i-1}}) \in L^0(\Omega, \mathcal{F}, \mathbb{P}).$$

With $L^0(\Omega, \mathcal{F}, \mathbb{P})$ we denote the set of equivalence classes of measurable functions where we identify functions that are *almost* everywhere identical. We call H the integrand and X the integrator. This definition is path-by-path and thus does not depend on the representation of H.

Stochastic integrals are very important in financial mathematics. In a discrete time setting $\{0, \ldots, N\}$, a strategy is a stochastic process $(H_i)_{i=1,\ldots,N}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_i)_{i=0,\ldots,n}$ that is (in a discrete sense) predictable, i.e., H_i is $\mathcal{F}_{i-1}$-measurable. A price process in discrete time is given by an adapted stochastic process X, i.e., X_i is $\mathcal{F}_i$-measurable and the trading gain is given by $\sum\limits_{i=1,\ldots,n} H_i(X_i - X_{i-1})$. Our integral $I_X(H)$ (in continuous time) actually is a generalization of the discrete time gain (Kühn, 2016, p. 17). Thus, stochastic integrals can be used for the calculation of trading gains. The restriction of the integrand to simple predictable processes means that the trading portfolio is reallocated only finitely often. Our aim is to generalize the definition of stochastic integrals even more.

As mentioned above, the use of the Brownian motion as an integrator is somehow tricky because it has an unbounded variation. However, since Brownian motions are an often used process in the definition of complex market models, it is our aim to define integrals even for Brownian motions as integrators. But before going to Brownian motions, we restrict our analysis to processes with finite variation. The set of all adapted, càdlàg processes X with $V(X)_T < \infty$ is denoted by $\mathcal{V}$. A process is called increasing if all paths are non-decreasing. The subset of $\mathcal{V}$ of all adapted, càdlàg, increasing processes X with $V(X)_T < \infty$ is denoted by $\mathcal{V}^+$.

For all $X \in \mathcal{V}$ there exists exactly one pair $(A, B) \in \mathcal{V}^+ \times \mathcal{V}^+$ s.t. $X = X_0 + A - B$ and $V(X) = A + B$ and $A_0 = B_0 = 0$. If there is another pair $(A', B') \in \mathcal{V}^+ \times \mathcal{V}^+$ with $X = X_0 + A' - B'$ and $A_0 = B_0 = 0$ it holds $V(X)_t \geq A_t + B_t$ $\forall t$. The proof is based on the construction of $A = \frac{X - X_0 + V(X)}{2}$ and $B = \frac{V(X) - X + X_0}{2}$ and can be found in Kühn (2016, p. 18).

Via $A \in \mathcal{V}^+$ we can define random measures (depending on ω) on $\mathcal{B}([0,T])$ by $\mu_A((s,t], \omega) = A_t(\omega) - A_s(\omega)$ with $s \leq t$. Actually, μ_A is a pre-measure on the ring of all finite unions of intervals $(s,t]$ ($s \leq t$), which can be extended to a measure on the corresponding σ-algebra (Brokate and Kersting, 2011, Satz XI.2 and the following example).

If H is bounded and all paths $t \mapsto H_t(\omega)$ are Borel measurable, we can define pathwise, i.e. ω-by-ω, the Lebesgue-Stieltjes integral via

$$\begin{aligned}\int_0^T H_s(\omega)dX_s(\omega) &= \int_0^T H_s(\omega)dA_s(\omega) - \int_0^T H_s(\omega)dB_s(\omega)\\ &= \int_0^T H_s(\omega)d\mu_A(ds,\omega) - \int_0^T H_s(\omega)d\mu_B(ds,\omega).\end{aligned}$$

If H is simple predictable, this is exactly $I_X(H)$. If $t \mapsto H_t(\omega)$ is continuous (or just left-continuous) this is also a Riemann-Stieltjes integral, which can be approximated via step functions (simple predictable integrands).

Next, we investigate why it is in general not possible to define a meaningful stochastic integral ω-by-ω. As mentioned above, this definition of stochastic integrals does not work in general when X is of unbounded variation. For any function $x : [0,1] \to \mathbb{R}$ with $\sup\limits_{t\in[0,1]} |x(t)| < \infty$ (which is true if x is continuous or càdlàg) but $V(x)_1 = \infty$ (e.g., x is a path of a Brownian motion), we can find a sequence $(h_n)_{n\in\mathbb{N}}$ of functions $h_n : [0,1] \to \mathbb{R}$ with $\sup\limits_{t\in[0,1]} |h_n(t)| \leq 1$ $\forall n$ but $I_x(h_n) \to \infty$ for $n \to \infty$ (Kühn, 2016, p. 20). With the Banach-Steinhaus theorem it follows that there exists a continuous (and thus bounded) function $h : [0,1] \to \mathbb{R}$ with $\limsup\limits_{n\to\infty} \sum\limits_{i=1,\dots,2^n} h\left(\frac{i-1}{2^n}\right)\left[x\left(\frac{i}{2^n}\right) - x\left(\frac{i-1}{2^n}\right)\right] = \infty$, i.e., there exists a continuous function s.t. $I_x(h) \to \infty$ if the grid (which is a dyadic grid) is refined with $mesh \to 0$. That means, for each path of a Brownian motion (or even other processes with unbounded variation) we find a continuous function as integrand so that the integral goes to infinity. This is not how we want to model financial markets because Brownian motions should be possible as integrators. However, these functions h_n use at time t_{i-1} information of time t_i. That means, in this example the traders can look into the future when using the strategies h_n. Thus, next we restrict our integrand to strategies where the trader does not look into the future and define stochastic integrals for this set of integrands. Note that our stochastic integral shall still allow the interpretation as trading gain.

Definition 23 (Predictable Process)**.** *The smallest σ-algebra that contains $A\times\{0\}$ $\forall A \in \mathcal{F}_0$ and $A \times (s,t]$ $\forall 0 \leq s < t \leq T, A \in \mathcal{F}_s$ is called the predictable σ-algebra $\mathcal{P}$. With*

$$\mathcal{E} = \{A \times \{0\}\ \forall A \in \mathcal{F}_0\} \cup \{A \times (s,t]\ \forall 0 \leq s < t \leq T, A \in \mathcal{F}_s\}$$

it holds $\mathcal{P} = \sigma(\mathcal{E}) = \{M \subset \Omega \times [0,T] \mid M \in \mathcal{A}\ \forall \mathcal{A}\ \sigma\text{-algebra on } \Omega \times [0,T] \text{ with } \mathcal{E} \subset \mathcal{A}\}$. A process $H : \Omega \times [0,T] \to \mathbb{R}$ is called predictable if it is $\mathcal{P}$-measurable, i.e., $\mathcal{P}$-$\mathcal{B}(\mathbb{R})$-measurable.

We define $\mathcal{F}_{t-} = \sigma\left(\bigcup\limits_{s<t} \mathcal{F}_s\right)$. Let H be a predictable process. Then it holds for all $t \in (0,T]$ that H_t is $\mathcal{F}_{t-}$-measurable and for all ω that the path $t \mapsto H_t(\omega)$ is $\mathcal{B}([0,T])$-measurable. A proof can be found in Kühn (2016, p. 22), which uses the fact that for a non-empty set $\tilde{\Omega}$ and $\mathcal{E} \subset \wp(\tilde{\Omega}) \ni B$ it holds that $B \cap \sigma(\mathcal{E}) = \sigma_B(B \cap \mathcal{E})$.

The intersection of a set M and a set of sets $\mathcal{M}$ with $\mathcal{M} \subset \tilde{\Omega} \ni M$ is defined via $M \cap \mathcal{M} := \{C \subset \tilde{\Omega} \mid \exists A \in \mathcal{M} \text{ with } C = A \cap M\}$. With $\sigma(\mathcal{E})$ we denote the smallest σ-algebra that contains $\mathcal{E}$ on the space $\tilde{\Omega}$ and with $\sigma_B(\mathcal{E})$ we analogously denote the smallest σ-algebra that contains $\mathcal{E}$ on the space B. To not allow all adapted processes as integrands is reasonable because there are adapted processes that would lead to arbitrage possibilities if they were allowed as trading strategies. An example is given in Kühn (2016, p. 23).

Note that a process which is $\mathcal{F}_{t-}$-measurable for all t is not necessarily predictable. Every predictable process is adapted, but not every adapted process is predictable.

It would also be possible to define the predictable σ-algebra through the set of all left-continuous, adapted processes or even through all sets $A \times \{0\}, A \in \mathcal{F}_0$ and $[\![0, \tau]\!]$ with τ being a stopping time. That means, with

$$\mathcal{E}' = \{X^{-1}([a, b)) \mid a, b \in \mathbb{R}, X \text{ adapted and left-continuous}\}$$

and

$$\mathcal{E}'' = \{A \times \{0\} \mid A \in \mathcal{F}_0\} \cup \{[\![0, \tau]\!] \mid \tau \text{ stopping time}\}$$

it holds

$$\sigma(\mathcal{E}) = \sigma(\mathcal{E}') = \sigma(\mathcal{E}'').$$

In short we can write $\sigma(\mathcal{E}') = \sigma(\{X : \Omega \times [0, T] \to \mathbb{R} \mid X \text{ adapted and left-continuous}\})$. A proof is given in Kühn (2016, pp. 23ff) and uses that for all $\mathcal{M}_1, \mathcal{M}_2 \subset \wp(\tilde{\Omega})$ (with $\tilde{\Omega}$ being a non-empty set) it holds $\mathcal{M}_1 \subset \sigma(\mathcal{M}_2) \Rightarrow \sigma(\mathcal{M}_1) \subset \sigma(\mathcal{M}_2)$. It was also possible to replace "left-continuous" by càglàd ("continue à gauche, limite à droite"), i.e., by the property that all paths of X are left-continuous and $\lim_{s \to t, s > t} X_s(\omega) =: X_{t+}(\omega)$ exists in $\mathbb{R}$ for all $t \in [0, T)$. It follows that all left-continuous, adapted processes are predictable.

Simply spoken, when all information that is known in the interval $[0, t)$ is used, the value of a predictable process at time t is already known. Thus, it is reasonable that an adapted, left-continuous process is predictable. Note again that not every process which is $\mathcal{F}_{t-}$-measurable is predictable. That the integrator has to be right-continuous is caused by the applicability of this theory to financial markets. When a price process jumps at time t (due to new information at time t) we want the value of the price process at time t to be the new value. Also other definitions for the integrator would be possible from a mathematicians point of view (cf. Kühn, 2016, Bem. 3.17). The combination of right-continuous integrators and left-continuous (or in general: predictable) integrands fits exactly to financial markets. Note that while the integrator has to fulfill some regularity constraints, e.g., càdlàg, the integrand just has to be predictable. In Kühn (2016, Bsp. 3.18) the following example of a somehow irregular but predictable process is given. The process $H_t = \mathbb{I}_{\{B_t \geq 1\}}$ with B a standard Brownian motion is predictable because it holds $\{(\omega, t) \in \Omega \times [0, T] \mid H_t(\omega) = 1\} = \{(\omega, t) \in \Omega \times [0, T] \mid B_t(\omega) \in [1, \infty)\}$ and the right side is a part of the predictable σ-algebra $\mathcal{P}$ since $\mathcal{P}$ can be defined by means of all adapted, left-continuous processes (and the Brownian motion is adapted and left-continuous). However, H_t is neither in all points of time left-continuous nor in all points of time there exists the limit from the right.

Now, we further analyze the integral $I_X : \mathcal{S} \to L^0$, which is linear. We say that a sequence of stochastic processes $(H_n)_{n\in\mathbb{N}}$ converges uniformly to H on $\Omega \times [0,T]$ if $\lim_{n\to\infty} \sup_{(\omega,t)\in\Omega\times[0,T]} |H_n(\omega,t)) - H(\omega,t)| = 0$.

Definition 24 (Good Integrator)**.** *Let X be an adapted, càdlàg process. We call X a good integrator if I_X is continuous in the following sense: For all $(H_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ and $H \in \mathcal{S}$ it holds $H_n \to H$ uniformly on $\Omega \times [0,T] \Rightarrow I_X(H_n) \to I_X(H)$ in probability.*

Note that the definition would be equivalent if we replaced H by 0. Next, we define the stochastic integral for good integrators X and adapted, càglàd integrands H (which could be extended to bounded, predictable integrands). In all cases, the integrand has to be predictable (Kühn, 2016, Bem. 3.22).

Definition 25 (Absolutely Continuous Measures)**.** *Let $\mathbb{P}$ and $\mathbb{O}$ be two measures on the same probability space. We say that $\mathbb{O}$ is absolutely continuous with respect to $\mathbb{P}$ if $\mathbb{P}(A) = 0 \Rightarrow \mathbb{O}(A) = 0 \ \forall A \in \mathcal{F}$. In this case we write $\mathbb{O} \ll \mathbb{P}$.*

Theorem 26. If $\mathbb{O} \ll \mathbb{P}$ and $(Z_n)_{n\in\mathbb{N}}$ is a sequence of real-valued random variables that converges to Z in $\mathbb{P}$-probability, then it converges to Z in $\mathbb{O}$-probability, too.

A proof can be found in Kühn (2016, Lemma 3.25). With this, for $\mathbb{O} \ll \mathbb{P}$ it directly follows that every good integrator in $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a good integrator in $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{O})$, too. This property is very important in stochastic finance, especially when considering hedging problems.

Theorem 27. If $X \in \mathcal{V}$, it follows that X is a good integrator.

Proof in Kühn (2016, Thm. 3.27).

Definition 28 (Square Integrable Process)**.** *A stochastic process X is called square integrable if $\mathbb{E}\left[X_t^2\right] < \infty \ \forall t \in [0,T]$.*

Doob's optional sampling theorem (not to be confused with Doob's optional stopping theorem) says that for a martingale X and $\tau_1 \leq \tau_2$ stopping times on $[0,T]$ it holds that $X_{\tau_1} = \mathbb{E}\left[X_{\tau_2} | \mathcal{F}_{\tau_1}\right]$ *a.s.* This theorem can be used, as done in Kühn (2016, Thm. 3.29), to show that all square integrable martingales are good integrators.

Definition 29 (Localization)**.** *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of stopping times on $[0,T]$ with $T_1 \leq T_2 \leq \ldots$. We say that $(T_n)_{n\in\mathbb{N}}$ is localizing if $\mathbb{P}(T_n = T) \to 1$ for $n \to \infty$. The process $X_t^{T_n} := X_{t\wedge T_n}$ is called a stopped process.*

For a process X, it holds $X^T = X_t \mathbb{I}_{t<T} + X_T \mathbb{I}_{t\geq T}$. Furthermore, we define $X^{T-} = X_t \mathbb{I}_{t<T} + X_{T-} \mathbb{I}_{t\geq T}$ with $X_{0-} = 0$ the pre-stopped process.

Note that for *a.a.* ω T has to be reached. For example, $T_n = T - \frac{1}{n}$ $(T > 1)$ is not localizing. (If we had $\mathbb{R}_0^+$ instead of $[0,T]$, the condition had to be $\mathbb{P}(T_n \geq t) \to 1$ for $n \to \infty \ \forall t \in \mathbb{R}_0^+$.)

Definition 30 (Local)**.** *A stochastic process X is a local martingale if there exists a localizing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ s.t. the stopped process X^{T_n} is a martingale for all $n \in \mathbb{N}$.*

A stochastic process X is called locally bounded if there exists a localizing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ s.t. the stopped process X^{T_n} is bounded for all $n \in \mathbb{N}$.

In general, if $\mathcal{C}$ is a class of stochastic processes, we can define the corresponding local class $\mathcal{C}_{loc}$ via $X \in \mathcal{C}_{loc}$ if there exists a localizing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ s.t. the stopped process $X^{T_n} \in \mathcal{C}$ for all $n \in \mathbb{N}$.

It holds $\mathcal{C} \subset \mathcal{C}_{loc}$. An example for a local martingale that is not a martingale is the trading gain of a doubling-up strategy. Every bounded, local martingale is a martingale (Kühn, 2016, Prop. 3.35). If $(T_n)_{n\in\mathbb{N}}$ is a localizing sequence of stopping times and the stopped processes X^{T_n} are good integrators for all $n \in \mathbb{N}$, X is a good integrator, too (Kühn, 2016, Thm. 3.36). Note that the set of all good integrators is a vector space.

Definition 31 (Semimartingale)**.** *An adapted, càdlàg process X is called a semimartingale if there exists a local martingale M and an adapted process with finite variation A s.t.*

$$X = M + A.$$

Theorem 32 (Fundamental Theorem of Local Martingales)**.** For all $c > 0$ a local martingale M can be decomposed into $M = N + A$ where A is a local martingale with finite variation and N is a local martingale with jumps with absolute values bounded by c.

This theorem is proven in Protter (2005, pp. 102ff). It follows that every local martingale can be additively decomposed into a square integrable, local martingale and a process with finite variation (Kühn, 2016, Korollar 3.41).

Theorem 33 (Bichteler-Dellacherie)**.** The set of good integrators is exactly the set of semimartingales.

A proof can be found in Protter (2005, Chap. III) (cf. Kühn, 2016, Thm. 3.39).

The stochastic integral $I_X : \mathcal{S} \to L^0$ is a function that maps simple predictable stochastic processes to random variables, i.e., the integral is evaluated at time T. Since, e.g., the trading gain develops over the time, we now want the stochastic integral to map to a space of stochastic processes.

3.3 Stochastic Integration—Adapted, Càglàd Processes

We denote the space of all adapted, right-continuous processes with $\tilde{\mathbb{D}}$, the space of all adapted, càdlàg processes with $\mathbb{D} \subset \tilde{\mathbb{D}}$, the space of all adapted, left-continuous processes with $\tilde{\mathbb{L}}$, and the space of all adapted, càglàd processes with $\mathbb{L} \subset \tilde{\mathbb{L}}$.

Definition 34 (Stochastic Integral as a Stochastic Process). *Let*

$$H = \sum_{i=1,\dots,n} Z_{i-1}\mathbb{I}_{]\!]T_{i-1},T_i]\!]} \in \mathcal{S}$$

with Z_i $\mathcal{F}_{T_i}$-measurable $(i = 1,\dots,n-1)$ and $0 = T_0 \leq T_1 \leq \dots \leq T_n = T$ stopping times and X be an adapted, càdlàg process. We call the linear function $J_X : \mathcal{S} \to \mathbb{D}$ with

$$J_X(H)_t = \sum_{i=1,\dots,n} Z_{i-1}(X_{T_i\wedge t} - X_{T_{i-1}\wedge t})$$

the stochastic integral of H over X. We also denote this process by $H \bullet X$.

Thus, $H \bullet X_t$ is the value of the process at time $t \in [0,T]$ and it holds $J_X(H)_T = I_X(H)$.

Definition 35 (*up* Convergence). *Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of stochastic processes and H be a stochastic process. We say that H_n converges uniformly in probability (up) to H if $\sup_{t\in[0,T]} |H_{n,t} - H_t| \to 0$ for $n \to \infty$ in probability.*

That means, in $t \in [0,T]$ it has to converge uniformly but in $\omega \in \Omega$ the convergence has just to be in probability. It holds that we can construct a metric $d : \mathbb{D}\times\mathbb{D} \to \mathbb{R}_0^+$ by use of the *up* convergence, i.e., for this metric it has to hold for all $(Y_n)_{n\in\mathbb{N}} \subset \mathbb{D}$ and $Y \in \mathbb{D}$ that $Y_n \to Y$ *up*$, n \to \infty \Leftrightarrow d(Y_n, Y) \to 0, n \to \infty$. One possibility of such a metric is

$$d_{up}(X,Y) = \mathbb{E}\left[\sup_{t\in[0,T]} |X_t - Y_t| \wedge 1\right].$$

Actually, this is a metric on the equivalence classes of $\mathbb{D}$ where we identify indistinguishable processes because $d_{up}(X,Y) = 0$ if and only if X and Y are indistinguishable. This metric d_{up} is also a metric on $\mathcal{S}\times\mathcal{S}$ and we denote it in both cases by d_{up}.

On a metric space (M,d) it holds for all $(x_n)_{n\in\mathbb{N}} \subset M$ and $x \in M$ that

$$d(x_n,x) \to 0, n\to\infty \Leftrightarrow \forall (n_k)_{k\in\mathbb{N}}\ \exists\, (n_{k_l})_{l\in\mathbb{N}} \text{ s.t. } d\left(x_{n_{k_l}}, x\right) \to 0, l \to \infty.$$

Note that there exists no metric $\tilde{d}$ s.t. $Y_n \to Y$ *a.s.*$, n \to \infty \Leftrightarrow \tilde{d}(Y_n,Y) \to 0, n \to \infty$ (Kühn, 2016, Bem. 3.44).

Definition 36 (*up* Metric Spaces). *We denote $\mathcal{S}$ together with the metric d_{up} by $\mathcal{S}_{up} = (\mathcal{S}, d_{up})$ and $\mathbb{D}$ together with the metric d_{up} by $\mathbb{D}_{up} = (\mathbb{D}, d_{up})$.*

Theorem 37. If X is a good integrator, $J_X : \mathcal{S}_{up} \to \mathbb{D}_{up}$ is continuous, i.e., for all $(H_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ and $H \in \mathcal{S}$ it holds that $H_n \to H$ *up* $\Rightarrow J_X(H_n) \to J_X(H)$ *up*.

Proven in Kühn (2016, Thm. 3.46). Let Y be a stochastic process. We define $Y_t^* = \sup_{0\leq s\leq t} |Y_s|$ and $Y^* = \sup_{0\leq s\leq T} |Y_s|$.

Definition 38 (*up* Cauchy Sequence)**.** *A sequence $(X_n)_{n\in N}$ of stochastic processes is called an up Cauchy sequence if $\forall\varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N : d_{up}(X_n, X_m) \leq \varepsilon$.*

A sequence $(X_n)_{n\in N}$ of stochastic processes is an *up* Cauchy sequence if and only if $\forall\varepsilon > 0\ \mathbb{P}\left(\sup\limits_{t\in[0,T]} |X_{n,t} - X_{m,t}| > \varepsilon\right) \to 0,\ n, m \to \infty.$

Theorem 39. If X is a good integrator, for all $(H_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ it holds that if $(H_n)_{n\in\mathbb{N}}$ is an *up* Cauchy sequence, $(J_X(H_n))_{n\in\mathbb{N}}$ is an *up* Cauchy sequence, too.

Proven in Kühn (2016, Korollar 3.48). Theorem 39 allows us to define the stochastic integral in a meaningful sense (even for Brownian motions as integrators) as a limit.

Theorem 40. Every *up* Cauchy sequence in $\mathbb{D}_{up}$ has a limit in $\mathbb{D}_{up}$, i.e., $\mathbb{D}_{up}$ is complete.

A proof is in Kühn (2016, Thm. 3.49).

Definition 41 (the closure of $\mathcal{S}$)**.** *With $\overline{\mathcal{S}_{up}}$ we denote the closure of $\mathcal{S}$ under d_{up}, i.e., $\overline{\mathcal{S}_{up}} = \{H : \Omega \times [0, T] \to \mathbb{R} \mid \exists (H_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ s.t. $H_n \to H$ up$\}$.*

Note that for all $H \in \overline{\mathcal{S}_{up}}$ $\sup\limits_{t\in[0,T]} |H_{n,t} - H_t|$ has to be $\mathcal{F}$-measurable. All $H \in \overline{\mathcal{S}_{up}}$ have to be adapted and left-continuous (Kühn, 2016, Bem. 3.51 and Thm. 3.49).

Definition 42 (Stochastic Integral on $\overline{\mathcal{S}_{up}}$)**.** *Let X be a good integrator, $H \in \overline{\mathcal{S}_{up}}$, and $(H_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ be a sequence with $H_n \to H$, up. The stochastic integral of H over X is $\lim\limits_{n\to\infty} J_X(H_n) =: J_X(H) =: H \bullet X$.*

Theorem 43. The definition for $H \bullet X$ with X being a good integrator and $H \in \overline{\mathcal{S}_{up}}$ is well defined, i.e., it does not depend on which $(H_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ is chosen.

Proof in Kühn (2016, Bem. 3.53).

Theorem 44. The mapping $J_X : \overline{\mathcal{S}_{up}} \to \mathbb{D}$ is the unique, continuous function on $\overline{\mathcal{S}_{up}}$ that equals $J_X : \mathcal{S} \to \mathbb{D}$ on $\mathcal{S}$ (when identifying indistinguishable processes).

Proven in Kühn (2016, Thm. 3.54). Let

$$\mathbb{L}_0 = \{H \in \mathbb{L} \mid H_0 = 0\}.$$

Theorem 45. It holds that $\mathcal{S} \subset \mathbb{L}_0$ dense with respect to d_{up} and that $\overline{\mathcal{S}_{up}} = \mathbb{L}_0$.

Proof in Kühn (2016, Thm. 3.55). Thus, $H \bullet X$ is defined for all $H \in \mathbb{L}_0$ and all semimartingales X.

Now, we calculate the integral $B \bullet B_t$ (B is a standard Brownian motion). The Brownian motion is in $\mathbb{L}_0$ and is a martingale, thus the integral is defined. We set

$B_{n,u} = \sum\limits_{k=1,\dots,2^n} B_{\frac{k-1}{2^n}T} \mathbb{I}_{\left(\frac{k-1}{2^n}T, \frac{k}{2^n}T\right]}(u), u \in [0,T]$. It holds $B_n \in \mathcal{S}\ \forall n \in \mathbb{N}$. Since the paths of B are uniformly continuous, it holds $B_n \to B$ *up*, $n \to \infty$. We calculate

$$\begin{aligned} J_B(B_n)_t &= \sum_{k=1,\dots,2^n \text{ s.t. } \frac{k}{2^n}T \le t} B_{\frac{k-1}{2^n}T}\left(B_{\frac{k}{2^n}T} - B_{\frac{k-1}{2^n}T}\right) \\ &= \sum_{k=1,\dots,2^n \text{ s.t. } \frac{k}{2^n}T \le t} \frac{1}{2}\left(B^2_{\frac{k}{2^n}T} - B^2_{\frac{k-1}{2^n}T}\right) \\ &\quad - \sum_{k=1,\dots,2^n \text{ s.t. } \frac{k}{2^n}T \le t} \frac{1}{2}\left(B_{\frac{k}{2^n}T} - B_{\frac{k-1}{2^n}T}\right)^2. \end{aligned}$$

The first sum converges to $\frac{1}{2}B_t^2$ and the second sum to $\frac{1}{2}t$ (Thm. 16). Since the stochastic integral is continuous, the grid is dense in $[0,T]$, and $t \mapsto \frac{1}{2}B_t^2 - \frac{1}{2}t$ as well as $t \mapsto J_B(B)_t$ are right-continuous, it follows $\mathbb{P}\left(J_B(B)_t = \frac{1}{2}B_t^2 - \frac{1}{2}t\ \forall t \in [0,T]\right) = 1$.

Let A be a continuous process with finite variation and $A_0 = 0$. It follows $A \bullet A_t = \frac{1}{2}A_t^2$ (Kühn, 2016, p. 41). When we have $J_X(X)$ for any continuous, good integrator X and X_n defined analogously to B_n converging *up* to X, it follows that $J_X(X_n)$ is an *up* Cauchy sequence. The limit of $\sum\limits_{k=1,\dots,2^n}\left(X_{\frac{k}{2^n}T} - X_{\frac{k-1}{2^n}T}\right)^2$ $(n \to \infty)$ exists, which means that for a continuous, good integrator the so-called quadratic variation on a grid with mesh size to zero exists (cf. Thm. 15). Actually, the quadratic variation (on a grid with mesh size to zero) exists even for non-continuous, good integrators, which is not clear at all since X_n does not need to converge to X *up* but pointwise. It is important to note that a pointwise convergence does not necessarily imply a convergence in the *up* sense. However, for some results, the pointwise convergence is enough, like for the quadratic variation.

Now the question arises why this effort for the definition of the stochastic integral is needed. Another way would be to shrink the set of possible integrators to (continuous) processes with finite variation. However, as the next example shows, this would not be a good choice for financial mathematics.

In a financial market with a riskless bond $S^0 \equiv 0$ and a risky asset modeled via a continuous, non-constant stochastic process with finite variation, which is a good integrator, we can define the left-continuous, adapted integrand H via $H_t = \left(S_t^1 - S_0^1\right)\mathbb{I}_{t \le t_0}$ where $t_0 \in (0,T]$ is a point of time where $\mathbb{P}(S_{t_0}^1 \neq S_0^1) > 1$, which exists. Applying the result for $A \bullet A_t$ for continuous processes with finite variation, the trading gain is given via $H \bullet S_T^1 = \left(S^1 - S_0^1\right) \bullet \left(S^1 - S_0^1\right)_{t_0} = \frac{1}{2}\left(S_{t_0}^1 - S_0^1\right)^2$, which is non-negative with probability 1 and positive with probability > 0. That means, this would be an arbitrage possibility (cf. Kühn, 2016, Prop. 3.58).

In the following, we give a few more results concerning the stochastic integral. If τ is a $[0,T]$-valued stopping time, it holds $(H \bullet X)^\tau = \left(H\mathbb{I}_{[\![0,\tau]\!]}\right) \bullet X = H \bullet (X^\tau)$ (Kühn, 2016, pp. 42f). The jump process of the integral $s \mapsto \Delta(H \bullet X)_s$ and the process $s \mapsto H_s(\Delta X_s)$ are indistinguishable (Kühn, 2016, p. 43).

So far, we have seen that our stochastic integral maps a good integrator (i.e. a semimartingale) and an integrand ($\in \mathbb{L}_0 = \overline{\mathcal{S}_{up}}$) into the set $\mathbb{D}$. However, is the integral itself a semimartingale again?

Theorem 46. The process $H \bullet X$ is a good integrator and it holds for $G \in \overline{\mathcal{S}_{up}}$ that $G \bullet (H \bullet X) = (GH) \bullet X$.

Proven in (Kühn, 2016, p. 43). This property is called associativity.

If H is bounded and X is a square integrable martingale, $H \bullet X$ is a square integrable martingale, too (Kühn, 2016, pp. 43f). When $H \in \overline{\mathcal{S}_{up}}$, then H is locally bounded. If X is a local square integrable martingale, i.e., if there exists a localizing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ s.t. $\mathbb{E}\left[\left(X_T^{T_n}\right)^2\right] < \infty$ and X^{T_n} is a martingale for all $n \in \mathbb{N}$, then $H \bullet X$ is a local square integrable martingale, too (Kühn, 2016, p. 45).

It is possible to relax the restriction of the integrands ($H \in \mathbb{L}_0$) to locally bounded, predictable processes H, which requires much more effort (Protter, 2005, Chap. IV.). It also holds that for locally bounded integrands and local martingales as integrators, the integral is a local martingale. If the integrand is bounded and the integrator is a square integrable martingale, the integral is a square integrable martingale, too. But for a bounded integrand and a martingale as integrator, the integral does not need to be a martingale (Kühn, 2016, Bem. 3.60).

Definition 47 (Uniformly Integrable)**.** *A sequence of real-valued random variables $(Z_n)_{n\in\mathbb{N}}$ is called uniformly integrable if $\mathbb{E}[|Z_n|] < \infty \ \forall n \geq 1$ and*

$$\lim_{z\to\infty}\left(\sup_{n\geq 1}\int\limits_{|Z_n|>z} |Z_n| d\mathbb{P}\right) = 0.$$

Theorem 48. A sequence of real-valued random variables $(Z_n)_{n\in\mathbb{N}}$ is uniformly integrable if and only if there exists a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\frac{\varphi(x)}{x} \to \infty, x \to \infty$ s.t. $\sup\limits_{n\in\mathbb{N}} \mathbb{E}[\varphi(|Z_n|)] < \infty$.

(Kühn, 2016, p. 44).

Definition 49 (Random Grid Tends to Identity)**.** *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of random grids given through $\sigma_n = (T_0^n, T_1^n, \ldots, T_{k_n}^n)$ with $0 = T_0^n \leq T_1^n \leq \ldots \leq T_{k_n}^n = T$ stopping times, $k_n \in \mathbb{N}$. We say that $(\sigma_n)_{n\in\mathbb{N}}$ tends to identity if*

$$\|\sigma_n\| = \max_{i=1,\ldots,k_n} |T_i^n - T_{i-1}^n| \to 0 \ a.s., \ n \to \infty.$$

Let Y be a stochastic process and $\sigma = (T_0, T_1, \ldots, T_k)$ be a random grid. Via

$$Y^{(\sigma)}(\omega, t) = \sum_{i=1,\ldots,k} Y_{T_{i-1}}(\omega, t)\mathbb{I}_{]\!]T_{i-1},T_i]\!]}(\omega, t)$$

we define the simple predictable process $Y^{(\sigma)}$. And for an integrator X we define the integral

$$Y^{(\sigma)} \bullet X = \sum_{i=1,\ldots,k} Y_{T_{i-1}}(X^{T_{i-1}} - X^{T_i}).$$

Theorem 50. If X is a semimartingale, Y is an element of $\mathbb{D}_0(= \{Y \in \mathbb{D} \mid Y_0 = 0\})$ or $\mathbb{L}_0$, and $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of random grids tending to identity, it holds $Y^{(\sigma_n)} \bullet X \to Y_- \bullet X$ *up*, $n \to \infty$.

Proven in Protter (2005, Thm. II.21). Note that in general $Y^{(\sigma_n)} \to Y_-, n \to \infty$ is true only pointwise but not *up* (if Y is continuous, it is true also *up*). Like for the quadratic variation on a grid with mesh size to zero, the step functions do not need to converge *up* to the integrand, however, the sequence of integrals converges *up*.

Definition 51 (Quadratic Variation of a Semimartingale)**.** *Let X and Y be two semimartingales, which especially means that X and Y are elements of $\mathbb{D}$. We define the quadratic variation process of X through $[X,X] = X^2 - X_0^2 - 2X_- \bullet X$. The quadratic covariation of X and Y is defined through $[X,Y] = XY - X_0Y_0 - X_- \bullet Y - Y_- \bullet X$.*

It holds $[X,Y] = \frac{1}{2}([X+Y, X+Y] - [X,X] - [Y,Y])$, which is called the polarization identity. If B is a standard Brownian motion, it holds $[B,B]_t = B_t^2 - 2B \bullet B_t = B_t^2 - (B_t^2 - t) = t \; \forall t \geq 0$. That means, for a Brownian motion the formally defined quadratic variation equals the limit of the quadratic variation for grids with mesh size to zero (cf. Thm. 15). This is true for all semimartingales.

Let X be a semimartingale. The quadratic variation of X is an increasing, adapted, càdlàg process.

Theorem 52. It holds $[X,X]_0 = 0$ and $\Delta[X,X]_t = [X,X]_t - [X,X]_{t_-} = (\Delta X_t)^2$. If $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of random grids tending to identity (with $0 = T_0^n \leq T_1^n \leq \ldots \leq T_{k_n}^n = T$ stopping times), then $\sum_{i=1,\ldots,k_n} \left(X^{T_i^n} - X^{T_{i-1}^n}\right)^2 \to [X,X]$ *up*, $n \to \infty$. If τ is a stopping time, it holds $[X^\tau, X] = [X, X^\tau] = [X^\tau, X^\tau] = [X,X]^\tau$.

A proof can be found in Kühn (2016, Thm. 3.65).

Theorem 53. The process $[X,Y]$ has finite variation and, thus, is a semimartingale.

This theorem is proven in Kühn (2016, Korollar 3.66).

Theorem 54. Let X and Y be semimartingales. The process XY is a semimartingale and, thus, the vector space of all semimartingales equipped with that product is an algebra.

A proof is given in Kühn (2016, Korollar 3.67). The process $[X,X]$ is càdlàg and increasing and it holds $\Delta[X,X]_t = (\Delta X_t)^2$, $t \geq 0$, thus, we can path-by-path decompose the process into a continuous part and a jump part via $[X,X]_t = [X,X]_t^c +$

$\sum_{0<s\leq t}(\Delta X_s)^2,\ t\geq 0$, with $[X,X]^c_t = \lim_{n\to\infty}\left([X,X]_t - \sum_{0<s\leq t,\ |\Delta X_s|>\frac{1}{n}}(\Delta X_s)^2\right)$ (Protter, 2005, Thm. II.22). Note that there is in general no canonical numbering of the jumps since it is possible that there are infinite many jumps in any neighborhood of zero. Since there is only a finite number of jumps bigger than $\frac{1}{n}$, the last sum is well defined. Since $[X,X]_t$ is increasing and $\Delta[X,X]_t = (\Delta X_t)^2,\ t\geq 0$, the difference is non-negative. Because the difference is decreasing in n, the limit exists. Note that in general it is not possible to decompose a process X (with unbounded variation) into a continuous part and a jump part. For all increasing processes A it is possible to define the continuous part via $A^c_t = \lim_{n\to\infty}\left(A_t - \sum_{0\leq s<t,\ A_{s+}-A_{s-}>\frac{1}{n}}(A_{s+}-A_{s-})\right) - (A_t - A_{t-})$ (Kühn, 2016, p. 49).

Let X and Y be semimartingales. It holds $[X,Y]_0 = 0$ and $\Delta[X,Y] = \Delta X\Delta Y$. Thus, again, we can path-by-path decompose the process into a continuous part and a jump part via $[X,Y]_t = [X,Y]^c_t + \sum_{0<s\leq t}(\Delta X_s)(\Delta Y_s), t\geq 0$ (Protter, 2005, Thm. II.23).

Theorem 55. *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of random grids tending to identity with $0 = T^n_0 \leq T^n_1 \leq \ldots \leq T^n_{k_n} = T$ stopping times. It holds $\sum_{i=1,\ldots,k_n}(X^{T^n_i} - X^{T^n_{i-1}})(Y^{T^n_i} - Y^{T^n_{i-1}}) \to [X,Y]$ up, $n\to\infty$. If τ is a stopping time, it holds $[X^\tau,Y] = [X,Y^\tau] = [X^\tau,Y^\tau] = [X,Y]^\tau$.*

The proof is in Kühn (2016, Thm. 3.69).

Let X be an adapted, càdlàg process with bounded variation, then it holds $[X,X]^c = 0$. If X is additionally continuous, it holds that $[X,X] = X_0^2$ is constant (Protter, 2005, Thm. II.26). Let X be a semimartingale with $[X,X]^c = 0$ and Y be any semimartingale, it holds $[X,Y]_t = X_0Y_0 + \sum_{0<s\leq t}\Delta X_s\Delta Y_s$ (Protter, 2005, Thm. II.28). Especially, then it holds $[X,Y]^c = 0$ and $[X,Y] = 0$ if X or Y has continuous paths.

Theorem 56. *Let X be a local martingale with continuous paths and $[X,X]_T = 0$. Then X is constant, i.e., $X_t = x_0 \in \mathbb{R}\ \forall t\in[0,T]$.*

Proof in Protter (2005, Thm. II.27). Note that the Brownian motion has infinite variation (cf. Protter, 2005, Thm. I.29).

Theorem 57. *If X and Y are semimartingales and $H,K\in\mathbb{L}_0$, it holds $[H\bullet X, K\bullet Y] = (HK)\bullet[X,Y]$.*

Proven in Kühn (2016, Thm. 3.71).

Theorem 58. *Let M be a local martingale. The process M is a square integrable martingale if and only if $\mathbb{E}[[M,M]_T] < \infty$. In this case, it holds for the variance $\mathbb{V}[M_t] = \mathbb{E}[M_t^2] - M_0^2 = \mathbb{E}[[M,M]_t]\ \forall t\in[0,T]$.*

Proof in Kühn (2016, Thm. 3.72). It follows for a one-dimensional standard Brownian motion B and $H \in \mathbb{L}_0$ with $\mathbb{E}\left[\int_0^t H_s^2 ds\right] < \infty$ that $\mathbb{E}[H \bullet B_t] = 0$ and $\mathbb{E}[(H \bullet B_t)^2] = \mathbb{E}\left[\int_0^t H_s^2 ds\right]$ $\forall t \in [0, T]$ (Kühn, 2016, Korollar 3.74).

Doob's martingale inequality states that for a martingale or a non-negative submartingale $(M_t)_{t\geq 0}$ which has to be right-continuous if $p > 1$ and $T > 0$, it holds $\left\|\sup_{t\leq T} |M_t|\right\|_p \leq \frac{p}{p-1}\|M_T\|_p$.

Theorem 59. If H is an adapted, càdlàg process, X and Y are semimartingales, and $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of random grids tending to identity, then it holds

$$\sum_{i=1,\dots,k_n} H_{T_{i-1}^n}(X^{T_i^n} - X^{T_{i-1}^n})(Y^{T_i^n} - Y^{T_{i-1}^n}) \to H_- \bullet [X, Y] \; up, \; n \to \infty.$$

Proof in Kühn (2016, Thm. 3.75).

3.4 Lemma of Itō

After having constructed the stochastic integral, another aim of this chapter is to define stochastic differential equations (actually: stochastic integral equations). For this, we need something like a stochastic version of integration by parts. This is what the Lemma of Itō tells us. Note that the $\int$-sign is also used for stochastic integrals. (Note that we write $X^{i\cdot}$ to distinguish X^2 from $X^{2\cdot}$.)

Theorem 60 (Lemma of Itō)**.** Let $X = (X^{1\cdot}, \dots, X^{d\cdot})$ be a d-tuple of semimartingales and $f : \mathbb{R}^d \to \mathbb{R}$ be a twice continuously differentiable function. The Lemma of Itō (Itō's formula) states that $f(X)$ again is a semimartingale, which is indistinguishable to

$$\begin{aligned} f(X_t) =& f(X_0) + \sum_{i=1,\dots,d} \left(\frac{\partial f}{\partial x_i}(X_-)\right) \bullet X_t^i \\ &+ \frac{1}{2} \sum_{1\leq i,j\leq d} \left(\frac{\partial^2 f}{\partial x_i x_j}\right) \bullet \left[X^{i\cdot}, X^{j\cdot}\right]_t^c \\ &+ \sum_{0<s\leq t} \left[f(X_s) - f(X_{s-}) - \sum_{i=1,\dots,d} \frac{\partial f}{\partial x_i}(X_{s-})\Delta X_s^i\right] \end{aligned}$$

(Protter, 2005, Thm. II.32 and Thm. II.33). Especially, if $d = 1$, X is a semimartingale, and f is a $\mathcal{C}^2$ real function, $f(X)$ is a semimartingale, too, and it holds

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t f''(X_{s-})d[X, X]_s^c$$

$$+ \sum_{0<s\leq t} [f(X_s) - f(X_{s-}) - f'(X_{s-}\Delta X_s)].$$

If $d = 1$ and A is an adapted, càdlàg process with finite variation, Itō's formula reduces to

$$f(A_t) = f(A_0) + \int\limits_0^t f(A_s) dA_s^c + \sum_{0<s\leq t} (f(A_s) - f(A_{s-}))$$

(Kühn, 2016, p. 54).

If $X_t^{1\cdot} = t$ it holds $[X^{1\cdot}, X^{j\cdot}] = 0 \ \forall j$ and $\Delta X^{1\cdot} = 0$. If furthermore $d = 2$ and $X^{2\cdot}$ is a Brownian motion B, it holds (for f once differentiable in the first argument and twice in the second),

$$f(t, B_t) = f(0,0) + \int\limits_0^t \partial_1 f(s, B_s) ds + \int\limits_0^t \partial_2 f(s, B_s) dB_s + \frac{1}{2} \int\limits_0^t \partial_{22} f(s, B_s) ds$$

(Kühn, 2016, Bem. 3.78).

Theorem 61. If X is a continuous semimartingale and $f \in \mathcal{C}^2(\mathbb{R})$, $f(X)$ is a semimartingale again and it holds

$$f(X_t) = f(X_0) + f'(X) \bullet X_t + \frac{1}{2} f''(X) \bullet [X, X]_t.$$

(Kühn, 2016, Korollar 3.79).

If $d = 2$, $X_t^{1\cdot} = t$, and $X_t^{2\cdot} = \tilde{X}_t$, which itself is given via a so-called stochastic differential equation $\tilde{X}_t = \tilde{X}_0 + \int\limits_0^t a_s ds + \int\limits_0^t b_s dB_s$ with existing integrals (this type of equation is defined in the next section), we can use Itō's formula to obtain

$$f(t, \tilde{X}_t) = \int_0^t \left(\partial_2 f(s, \tilde{X}_s) a_s + \partial_1 f(s, \tilde{X}_s) + \frac{1}{2} \partial_{22} f(s, \tilde{X}_s) b_s^2 \right) ds + \int_0^t \partial_2 f(s, \tilde{X}_s) b_s dB_s$$

(Kuo, 2006, Thm. 7.4.3, p. 103).

Theorem 62. Let X be a semimartingale with $X_0 = 0$. There exists exactly one semimartingale Y that fulfills

$$Y_t = 1 + Y_- \bullet X_t \ \forall t \geq 0.$$

The process Y is given by

$$\begin{aligned} Y_t &= e^{X_t - \frac{1}{2}[X,X]_t^c} \prod_{0<s\leq t} \left((1 + \Delta X_s) e^{-\Delta X_s} \right) \\ &= e^{X_t - \frac{1}{2}[X,X]_t} \prod_{0<s\leq t} \left((1 + \Delta X_s) e^{-\Delta X_s + \frac{1}{2}(\Delta X_s)^2} \right) \end{aligned}$$

where the product converges.

Proven in Protter (2005, Thm. II.37) (cf. Kühn, 2016, Thm. 3.81).

Definition 63 (Stochastic Exponential). *Let X be a semimartingale with $X_0 = 0$. The unique solution of $Z_t = 1 + Z_- \bullet X_t$ $\forall t \geq 0$ is called the stochastic exponential of X (or the Doléans-Dade exponential of X). We write $\mathcal{E}(X) = Z$, i.e., $\mathcal{E}(X)_t = 1 + \mathcal{E}(X)_- \bullet X_t$.*

Definition 64 (Geometric Brownian Motion without Drift). *Let $X_t = \sigma B_t,\ t \geq 0,\ \sigma \in \mathbb{R}$ and B a Brownian motion. It holds $\mathcal{E}(\sigma B)_t = e^{\sigma B_t - \frac{1}{2}[\sigma B, \sigma B]_t} = e^{\sigma B_t - \frac{\sigma^2}{2} t}$, which is called a geometric Brownian motion (without drift).*

Theorem 65. The stochastic exponential is with probability one positive if and only if $\Delta X > -1$. If X is a local martingale, $\mathcal{E}(X)$ is a local martingale, too.

(Kühn, 2016, Bem. 3.84).

Definition 66 (Lévy Process). *Let $(X_t)_{t \in I}$ (with an index set $I = [0, T]$ or $I = \mathbb{R}^+$). We say that X has independent increments if for any $n \in \mathbb{N}_{\geq 3}$ and $0 \leq t_1 < t_2 < \ldots < t_n$ $(\leq T$ or $< \infty)$, $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}}$ are stochastically independent.*

We say that X has stationary increments if for all $h > 0$ and all $t_1, t_2 \in I$ s.t. $t_1 + h, t_2 + h \in I$ it holds $X_{t_1+h} - X_{t_1} \sim X_{t_2+h} - X_{t_2}$.

A process X is called stochastically continuous if for all $\varepsilon > 0$ and all $t_0 > 0$ it holds $\lim\limits_{t \to t_0} \mathbb{P}(|X_t - X_{t_0}| > \varepsilon) = 0$.

All stochastically continuous processes X with independent and stationary increments with $X_0 = 0$ a.s. are called Lévy processes.

Theorem 67. If X is a Lévy process and a martingale, $\mathcal{E}(X)$ is a martingale, too.

(Kühn, 2016, Bem. 3.84).

Theorem 68 (Lévy's Theorem). The process X is a Brownian motion if and only if X is a local martingale with $X_0 = 0$ and $[X, X]_t = t$ $\forall t \geq 0$

(Kühn, 2016, Thm. 3.86).

Definition 69 (Poisson Process). *The process $N_t = \sum\limits_{k=1}^{\infty} \mathbb{I}_{S_k \leq t}$ with $S_k = \sum\limits_{i=1}^{k} T_i$ and $(T_i)_{i \in \mathbb{N}}$ a sequence of independently and identically exponentially distributed random variables with parameter $\lambda > 0$ (the jump rate) is called Poisson process.*

It holds that N_t is Poisson distributed with parameter λt.

Definition 70 (Compound Poisson Process). *Let additionally to Def. 69 $(U_k)_{k \in \mathbb{N}}$ be an i.i.d. sequence of random variables that is independent of $(T_i)_{i \in \mathbb{N}}$. The process $Y_t = \sum\limits_{k=1}^{\infty} U_k \mathbb{I}_{S_k \leq t}$ is called compound Poisson process, the U_k's are called the jump heights.*

Compound poisson processes are also called Poisson-driven processes. Important examples of Lévy processes are the standard Brownian motion, Poisson processes, and compound Poisson processes. Note that Lévy processes do not need to have continuous paths. An example of a stochastically continuous process with potentially discontinuous paths is the (compound) Lévy process: It holds

$$\lim_{t \to t_0} \mathbb{P}(|Y_t - Y_{t_0}| > \varepsilon) \leq \lim_{t \to t_0} \mathbb{P}(|Y_t - Y_{t_0}| > 0) \leq \lim_{t \to t_0} \left(1 - e^{-\lambda |t - t_0|}\right) = 0.$$

3.5 Stochastic Differential Equations

With the stochastic exponential, we have seen an example for a stochastic differential equation (SDE). Note that SDE is only a name for a type of equation that is actually a stochastic integral equation. There is a broad literature on SDEs to which we refer the interested reader, for example Protter (2005, Chap. V.) and Applebaum (2009); Øksendal (2003). Here, we just give a few basic results on SDEs. As known from ordinary differential equations (ODEs), the Lipschitz property is important for solution results. Thus, it is not surprising that we have to define Lipschitz continuity for stochastic processes first. But, there is more than one possibility for this definition.

Definition 71 (Lipschitz)**.** *A function $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is called Lipschitz if there exists $k \in \mathbb{R}$ s.t. $|f(t,x) - f(t,y)| \leq k\|x - y\|_2$ and $t \mapsto f(t,x)$ is càdlàg [sic!] (cf. Protter, 2005, pp. 255ff) for all x. We call f autonomous if $f(t,x) = f(x), t \geq 0$ for all x.*

A function $f : \mathbb{R}^+ \times \Omega \times \mathbb{R}^n \to \mathbb{R}$ is called random Lipschitz if $(t,\omega) \mapsto f(t,\omega,x)$ is in $\mathbb{L}$ for all fixed x and if there exists a finite random variable K s.t. for each (t,ω) $|f(t,\omega,x) - f(t,\omega,y)| \leq K(\omega)\|x - y\|_2$.

An operator $F : \mathbb{D}^n \to \mathbb{D}$ is called process Lipschitz if for all $X, Y \in \mathbb{D}^n$ and for any stopping time T it holds $X^{T-} = Y^{T-} \Rightarrow F(X)^{T-} = F(Y)^{T-}$ (where X^{T-} is meant component-by-component) and if there exists an adapted process $K \in \mathbb{L}$ s.t. $|F(X)_t - F(Y)_t| \leq K_t\|X_t - Y_t\|_2$ $\forall\omega$.

An operator $F : \mathbb{D}^n \to \mathbb{D}$ is called functional Lipschitz if for all $X, Y \in \mathbb{D}^n$ and for any stopping time T it holds $X^{T-} = Y^{T-} \Rightarrow F(X)^{T-} = F(Y)^{T-}$ and if there exists an increasing finite process $K \in \mathbb{L}$ s.t. $|F(X)_t - F(Y)_t| \leq K_t \sup_{s \leq t}\|X_s - Y_s\|_2$ $\forall\omega$.

Theorem 72 (Stochastic Differential Equation I.)**.** Let Z be a semimartingale with $Z_0 = 0$ and $f : \mathbb{R}^+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ random Lipschitz. Let X_0 be a finite and $\mathcal{F}_0$-measurable random variable. Then the equation $X_t = X_0 + \int_0^t f(s,\omega,X_{s-})dZ_s(\omega)$ has a unique solution, which is a semimartingale.

Proven in Protter (2005, Thm. V.6).

Theorem 73 (Stochastic Differential Equation II.)**.** Let Z be a d-dimensional vector of semimartingales with $Z_0^i = 0$ $\forall i = 1, \ldots, d$. Let $J \in \mathbb{D}^n$ be a vector of processes and let $F_j^i : \mathbb{D}^n \to \mathbb{D}$ be functional Lipschitz operators ($i = 1, \ldots, n,\ j = 1, \ldots, d$). The system $X_t^i = J_t^i + \sum_{j=1,\dots,d} \int_0^t F_j^i(X)_{s-} dZ_s^j$ has a unique solution in $\mathbb{D}^n$. If J is a vector of semimartingales, X is a vector of semimartingales, too.

Proof in Protter (2005, Thm. V.7).

The probably most important SDE is that one of a geometric Brownian motion (GBM) with drift, which is given through

$$X_t = X_0 + \int_0^t \mu X_s d_s + \int_0^t \sigma X_s dB_s,$$

where B is a standard Brownian motion, $X_0 \in \mathbb{R}^+$ *a.s.* is the initial value, $\mu > -1$ is the drift, and $\sigma > 0$ is the volatility. Note that the X is continuous and, thus, $X_s = X_{s-}$. By use of Itō's formula, it can be obtained that

$$X_t = X_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right),$$

which is a supermartingale for $\mu \in (-1, 0)$, a martingale for $\mu = 0$, and a submartingale for $\mu > 0$. In all cases, the GBM is a semimartingale. The GBM is often used as the price process in financial mathematics, e.g., in the Black-Scholes model, which is used for option pricing (Black and Scholes, 1973). The GBM (as a semimartingale) can be used as integrator and a predictable trading strategy as integrand to calculate the trading gain via an integral.

Note that every càdlàg supermartingale and every submartingale is a semimartingale (Protter, 2005, Thm. III.32 and Corollary). The GBM has the disadvantage that it is a continuous process. However, on "real" stock markets, jumps happen, due to, e.g., new information or no trading on the weekend. Thus, Merton (1976) developed another important price process given by the stochastic differential equation

$$X_t = X_0 + \int_0^t (\mu - \kappa\lambda) X_s d_s + \int_0^t \sigma X_s dB_s + \int_0^t X_s dN_s,$$

which is called Merton's jump diffusion model (MJDM). This is a generalization to the GBM, where N_t is a compound Poisson process with jump rate $\lambda > 0$ and an i.i.d. sequence of jump heights $(Y_k - 1)_{k \in \mathbb{N}}$ with $Y_k > 0$ *a.s.*, $Y_k \geq 0$ and expected jump height $\kappa > 0$. The solution of this SDE is given by Merton (1976) through

$$X_t = X_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right) \prod_{k=1}^{N} Y_i.$$

Here, N is a Poisson distributed random variable with parameter λt, which denotes the number of jumps up to time t. The term $-\lambda\kappa$ compensates the expected movement of the price due to the jumps s.t. μ is the drift of the price process again. A very detailed solution of MJDM with explanations to jump processes can be found in the work of Privault (2017).

Two other important price process models in stochastic finance are the Ornstein-Uhlenbeck process, which is sometimes also called Vašíček model, and the Cox-Ingersoll-Ross model, which is an extension of the previous one. Both processes are mean reverting, i.e., if the price is above the respective mean reversion level, the drift is negative and vice versa. The Ornstein-Uhlenbeck process is given through the SDE

$$X_t = X_0 + \int_0^t \theta(\mu - X_s)ds + \int_0^t \sigma dB_s,$$

where $X_0 \in \mathbb{R}$ is the initial price, $\theta > 0$ is the so-called mean reversion speed, $\mu \in \mathbb{R}$ is the mean reversion level, $\sigma > 0$ is the volatility or diffusion level, and B_t is a standard Brownian motion. Note that the diffusion term is independent of the price level. Thus, the Ornstein-Uhlenbeck process has the disadvantage that it can become negative. However, it has the advantage that one can compute the solution analytically (but not without an integral), which is

$$X_t = X_0 \cdot \exp(-\theta t) + \mu(1 - \exp(-\theta t)) + \int_0^t \sigma \cdot \exp(\theta(s-t))dB_s.$$

The Cox-Ingersoll-Ross model is given through the SDE

$$X_t = X_0 + \int_0^t \theta(\mu - X_s)ds + \int_0^t \sigma\sqrt{X_s}dB_s,$$

with the same kinds of parameters as above. If additionally $\mu > 0$ is satisfied, the process avoids negative values. However, there is no analytical solution.

Now, we have finished the overview on stochastic calculus in finance. The most important things we learned are the conditions under which a stochastic integral is well-defined: The integrand has to be locally bounded and predictable, e.g., adapted and càglàd, and the integrator has to be a semimartingale, which is càdlàg. Additionally, we have learned about Itō's formula, which allows us to solve (some) stochastic differential equations, and we have seen some examples for stochastic processes, which are often used as price models. At the very end of this chapter, we want to mention that the stochastic integral used in the work at hand is called Itō integral and that there is another possibility for defining stochastic integrals developed by R. L. Stratonowitsch, which is not common in stochastic finance.

Chapter 4

New Findings in Stochastic Calculus

In this chapter, we present a few new findings in stochastic calculus, which were inspired by the analysis of feedback trading rules. Besides the application to the investigation of feedback trading strategies, these findings contribute to the theory of stochastic processes or random variables.

4.1 Another Stochastic Fubini-type Theorem

As seen in the Cox-Ingersoll-Ross model, price processes are sometimes modeled via SDEs that cannot be solved analytically. However, often an analytical solution of an SDE is not needed at all since the expected value of the solution (as a function of time) is enough for applications.

The aim of this section is to show that it is allowed to switch the expectation operator and the Itō integral (under specific conditions). With this, we then apply the expectation operator on both sides of an SDE to get a deterministic ODE for the expectation of the solution. The main contribution of this section is the next theorem: It states that it is allowed to switch the two integrals of interest, namely the Itō integral and $\mathbb{E}$ if the integrator is linear in expectation.

Theorem 74. Let Z be a semimartingale with $\mathbb{E}[Z_t - Z_s] = \zeta(t-s)\ \forall 0 \leq s \leq t \leq T$. Further let $X \in \mathbb{D}_0$ be integrable (i.e., $\mathbb{E}[|X_t|] < \infty\ \forall t$) and $Z_t - Z_s$ independent of X_s for all $0 \leq s \leq t \leq T$. Let $Y_t = X \bullet Z_t = \int_0^t X_{s-} dZ_s$ be integrable, too, and $\mathbb{E}[X_t]$ continuous. Then it holds that

$$\mathbb{E}[Y_t] = \int_0^t \mathbb{E}[X_s]\zeta ds.$$

Proof. If $X \in \mathbb{D}_0$ it follows that $X_- \in \mathbb{L}_0$. We define a sequence of random grids through $\sigma_n = (0, \frac{T}{n}, \frac{2T}{n}, \ldots, T)$, cf. Def. 49. Note that $(\sigma_n)_n$ tends to identity and that all σ_n are deterministic. We define the sequence of simple predictable processes X^n via

$$X^n(\omega,t) = \sum_{i=1,\dots,2^n} X_{\frac{(i-1)T}{2^n}}(\omega,t)\mathbb{I}_{\left]\frac{(i-1)T}{2^n},\frac{iT}{2^n}\right]}(t).$$

With Thm. 50 it follows that $X^n \bullet Z \to X \bullet Z$, *up*.

We choose a subsequence of X^n s.t. the convergence $X^n \bullet Z \to X \bullet Z$ is uniformly in time and *a.s.* in ω (which is possible since the limit is in probability). Additionally, we set all $X^n(\omega) \equiv 0$ where either the convergence does not hold (because it is just *a.s.*) or where the distance (as the supremum over t) between $(X^n \bullet Z)(\omega)$ and $(X \bullet Z)(\omega)$ is ≥ 1. We rename this new sequence to X^n and note that nothing changes concerning the convergence, besides that the convergence is dominated by the integrable function $|Y_t| + 1$ (a).

The convergence $X^n \to X$ is pointwise in t for *a.a.* ω. For each t (and *a.a.* ω), we can find an n^* so that $|X^n_t(\omega) - X_t(\omega)| < 1$ for all $n \geq n^*$. That means, we can treat the convergence like it was bounded (with boundary $|X_t| + 1$) (e).

Furthermore, we define for all X^n a sequence $X^{n,m}$ $(m \geq n)$ of representations via

$$X^{n,m}(\omega,t) = \sum_{i=1,\dots,2^n} \sum_{j=1,\dots,2^{m-n}} X_{\frac{(i-1)T}{2^n}}(\omega,t)\mathbb{I}_{\left]\frac{(i-1)T}{2^n}+\frac{(j-1)T}{2^m},\frac{(i-1)T}{2^n}+\frac{jT}{2^m}\right]}(t).$$

Note that $X^{n,n} = X^n$ and that all $X^{n,m}$ are just representations of X^n for all $m \geq n$ (i.e., all $X^{n,m}$ and X^n are exactly the same function; convergences are monotonous) (b). It holds that X_u and X^n_u are independent of $Z_w - Z_v$ for all $0 \leq u \leq v \leq w \leq T$ (c). For shortening the notation, we insert a subscript t at the end of the formulae instead of subscript $\wedge t$ in each random variable. Further, note that $\mathbb{E}[X_t]$ is bounded on $[0,T]$, thus, for a sequence that converges to $\mathbb{E}[X_t]$ this convergence can assumed to be bounded (with the same argument as above) (d).

This leads to:

$$\begin{aligned}
\mathbb{E}[Y_t] =& \mathbb{E}[X \bullet Z_t] \\
=& \mathbb{E}\left[\lim_{n\to\infty} X^n \bullet Z_t\right] \\
\overset{(a)}{=}& \lim_{n\to\infty} \mathbb{E}[X^n \bullet Z_t] \\
=& \lim_{n\to\infty} \mathbb{E}\left[\lim_{m\geq n,\ m\to\infty} X^{n,m} \bullet Z_t\right] \\
\overset{(b)}{=}& \lim_{n\to\infty} \lim_{m\geq n,\ m\to\infty} \mathbb{E}[X^{n,m} \bullet Z_t] \\
=& \lim_{n\to\infty} \lim_{m\geq n,\ m\to\infty} \mathbb{E}\left[\sum_{j=1,\dots,2^m} X^n_{\frac{(j-1)T}{2^m}}\left(Z_{\frac{jT}{2^m}} - Z_{\frac{(j-1)T}{2^m}}\right)\right]_t
\end{aligned}$$

$$\stackrel{(c)}{=} \lim_{n\to\infty} \lim_{m\geq n,\ m\to\infty} \left(\sum_{j=1,\dots,2^m} \mathbb{E}\left[X^n_{\frac{(j-1)T}{2^m}}\right] \mathbb{E}\left[Z_{\frac{jT}{2^m}} - Z_{\frac{(j-1)T}{2^m}}\right] \right)_t$$
$$= \lim_{n\to\infty} \left(\lim_{m\geq n,\ m\to\infty} \sum_{j=1,\dots,2^m} \mathbb{E}\left[X^n_{\frac{(j-1)T}{2^m}}\right] \cdot \frac{\zeta}{2^m} \right)_t$$
$$= \lim_{n\to\infty} \int_0^t \mathbb{E}[X^n_s]\zeta ds$$
$$\stackrel{(d)}{=} \int_0^t \lim_{n\to\infty} \mathbb{E}[X^n_s]\zeta ds$$
$$\stackrel{(e)}{=} \int_0^t \mathbb{E}\left[\lim_{n\to\infty} X^n_s\right]\zeta ds$$
$$= \int_0^t \mathbb{E}[X_s]\zeta ds$$

□

Next, we present an alternative proof, which is more constructive and does not use Thm. 50.

Proof. Since X_{t-} is predictable we find a sequence of simple predictable processes $(H^n_t)_t$ so that $H^n \to X,\ n \to \infty,\ up$ and $H^n \bullet Z \to X \bullet Z, n \to \infty,\ up$. In the next step we choose a subsequence so that $H_n \to X$ is uniformly in time and $a.s.$ in ω (which is possible since the limit is in probability). Note that the limit for the integral is still up. For shortening the notation, we rename it to H^n again.

Now, H^n_t is of the form $H^n_t(\omega) = \sum_{i=1,\dots,k_n} M^n_{i-1}(\omega)\mathbb{I}_{]\!]T^n_{i-1},T^n_i]\!]}(\omega,t)$ with $k_n \in \mathbb{N}$, $(T^n_i)_{i=1,\dots,k_n}$ stopping times with $0 = T^n_0 \leq T^n_1 \leq \ldots \leq T^n_{k_n} = T$ and M^n_i $\mathcal{F}_{T^n_i}$-measurable random variables $(i = 0,\dots,k_n - 1)$ with $|M^n_i| < \infty$.

We note that we can replace M^n_{i-1} by $X_{T^n_{i-1}}$. This is true since $M^n_{i-1} = H^n_{T^n_{i-1}+}$ and on the interval $]\!]T^n_{i-1}, T^n_i]\!]$ the distance between H^n and X goes to zero (for $a.a.$ ω and $n \to \infty$) and the distance between $X_{T^n_{i-1}}$ and $H^n_{T^n_{i-1}+}$ goes to zero (for $a.a.$ ω and $n \to \infty$). For clear, $X_{T^n_i}$ are $\mathcal{F}_{T^n_i}$-measurable random variables $(i = 0,\dots,k_n - 1)$ with $\left|X_{T^n_i}\right| < \infty$.

As a consequence, the new sequence of processes with $X_{T^n_{i-1}}$ instead of M^n_{i-1} is a sequence of simple predictable processes and still converges uniformly in time and $a.s.$ in ω to X_{t-} and still $H^n \bullet Z \to X \bullet Z,\ n \to \infty,\ up$ holds (with another rename for H^n).

Now we choose another subsequence (of that subsequence) so that the latter limit is $a.s.$ in ω, too. And we set all $H^n(\omega) \equiv 0$ if the distances (as the supremum over t) between $H^n(\omega)$ and $X(\omega)$ or between $(H^n \bullet Z)(\omega)$ and $(X \bullet Z)(\omega)$ is ≥ 1. And we set $H^n \equiv 0$ for all ω where H^n does not converge to X or where $H^n \bullet Z$ does not converge to $X \bullet Z$. These definitions do not change anything on the convergences, save that the convergences are dominated now.

Since $H^n \bullet Z$ converges uniformly in t to Y $a.s.$ and Y is integrable, we can use the dominated convergence theorem (e.g., with boundary $|Y_t|+1$) to obtain $\lim_{n\to\infty} \mathbb{E}[(H^n \bullet Z)_t] = \mathbb{E}[Y_t]$. Now we have to calculate $\mathbb{E}[(H^n \bullet Z)_t]$. For each H^n_t, which is a simple predictable process, we define a sequence $(H^{n,m})_m$ of simple predictable processes via $H^{n,m} = \sum_{j=1,\ldots,2^m} H^n_{\frac{(j-1)T}{2^m}} \mathbb{I}_{\left(\frac{(j-1)T}{2^m}, \frac{jT}{2^m}\right]}$ $(m \geq 0)$.

With $\vartheta_i^{n,m-}(\omega)$ we denote the largest point of the grid $\{\frac{0}{2^m}, \frac{T}{2^m}, \ldots, T\}$ with $\vartheta_i^{n,m-}(\omega) \leq T_i^n(\omega)$ and with $\vartheta_i^{n,m+}(\omega)$ we denote the smallest point of the grid $\{\frac{0}{2^m}, \frac{T}{2^m}, \ldots, T\}$, with $\vartheta_i^{n,m+}(\omega) \geq T_i^n(\omega)$. Without loss of generality, we choose m big enough s.t. all jumps (that are not at the same point of time) of $H^n(\omega)$ are separated by the dyadic grid (ω-by-ω). It holds, since Z is càdlàg:

$$
\begin{aligned}
&\sup_{t\in[0,T]} |(H^{n,m} \bullet Z)(\omega)_t - (H^n \bullet Z)(\omega)_t)| \\
&\leq \sup_{t\in[0,T]} \left(\sum_{i=1,\ldots,n_k} \left| \left(H^n_{\vartheta_i^{n,m+}}(\omega) - H^n_{\vartheta_i^{n,m-}}(\omega) \right) \left(Z_{\vartheta_i^{n,m+}} - Z_{T_i^n} \right) \right| \right)_t \\
&\to 0, \ m \to \infty
\end{aligned}
$$

For all ω, $H^{n,m} \bullet Z$ converges to $H^n \bullet Z$ $(m \to \infty)$, especially there exists an m^* so that the distance is smaller than 1 for all $m \geq m^*$ (for all t). That means, again, we can use the dominated convergence theorem (with boundary $|H^n \bullet Z_t| + 1$) to get

$$
\begin{aligned}
\mathbb{E}\left[(H^n \bullet Z)_t\right] =& \mathbb{E}\left[\left(\lim_{m\to\infty} H^{n,m} \right) \bullet Z_t \right] \\
=& \mathbb{E}\left[\lim_{m\to\infty} \left(H^{n,m} \bullet Z\right)_t \right] \\
=& \lim_{m\to\infty} \mathbb{E}\left[H^{n,m} \bullet Z_t\right] \\
=& \lim_{m\to\infty} \mathbb{E}\left[\left(\sum_{j=1,\ldots,2^m} H^n_{\frac{(j-1)T}{2^m}} \left(Z_{\frac{(j-1)T}{2^m}} - Z_{\frac{jT}{2^m}} \right) \right)_t \right] \\
=& \lim_{m\to\infty} \left(\sum_{j=1,\ldots,2^m} \mathbb{E}\left[H^n_{\frac{(j-1)T}{2^m}} \right] \mathbb{E}\left[Z_{\frac{(j-1)T}{2^m}} - Z_{\frac{jT}{2^m}} \right] \right)_t \\
=& \lim_{m\to\infty} \left(\sum_{j=1,\ldots,2^m} \mathbb{E}\left[H^n_{\frac{(j-1)T}{2^m}} \right] \zeta \frac{1}{2^m} \right)_t \\
=& \int_0^t \mathbb{E}\left[H^n_s\right] \zeta ds.
\end{aligned}
$$

Here, we used that X is independent of the increments of Z and thus H^n are also independent. Putting these results together and using a third and a fourth time the dominated convergence theorem (but these times for X with boundary $|X_t|+1$ and for $\mathbb{E}[X]$ which is bounded on $[0,T]$) completes the proof: $\mathbb{E}[Y_t] = \lim_{n\to\infty}\mathbb{E}[(H^n \bullet Z)_t] = \lim_{n\to\infty}\int_0^t \mathbb{E}[H^n_s]\zeta ds = \int_0^t \lim_{n\to\infty}\mathbb{E}[H^n_s]\zeta ds = \int_0^t \mathbb{E}\left[\lim_{n\to\infty} H^n_s\right]\zeta ds = \int_0^t \mathbb{E}[X_s]\zeta ds$ (since X is càglàd).

□

Now, we apply this theorem to SDEs.

Theorem 75. Let Z be a d-dimensional vector of semimartingales with stochastically independent and stationary increments, which implies that there are $\zeta^j \in \mathbb{R}$ s.t. $\mathbb{E}[Z^j_t - Z^j_s] = \zeta^j(t-s)$, and $Z^i_0 = 0 \;\forall i = 1,\dots,d$. Let $F^i_j : \mathbb{D}^n \to \mathbb{D}$ be linear operators $(i = 1,\dots,n,\; j = 1,\dots,d)$. Let $J \in \mathbb{D}^n$ be a vector of processes with $\mathbb{E}[J_t] = \delta_t \in \mathbb{R}^n$. We define

$$X^i_t = J^i_t + \sum_{j=1,\dots,d} \int_0^t F^i_j(X)_{s-} dZ^j_s.$$

Let all Z^j be independent of each other. If X_t is an integrable process, it holds with $\mathbb{E}[X^i_t] = \xi^i_t$ that

$$\xi^i_t = \delta^i_t + \sum_{j=1,\dots,d} \int_0^t F^i_j(\xi_s)\zeta^j dt,$$

which is an ordinary differential equation (ODE).

Proof. First, note that if $F^i_j : \mathbb{D}^n \to \mathbb{D}$ are linear operators $(i = 1,\dots,n,\; j = 1,\dots,d)$, F^i_j are functional Lipschitz. Note that $\mathbb{E}[Z^j_t - Z^j_s] = \zeta^j(t-s)$ is justified since Z has stationary increments. We note that X^i_t is independent of the increments $Z^j_{t+h} - Z^j_t$ (due to the SDE and the independent increments of Z). Further, note that ξ^i is continuous since Z has stationary increments and due to the construction of the SDE (otherwise $sup_{t\in[0,T]}|\xi^i_t| = \infty$). So we can apply the expectation operator on both sides of the SDE and use Thm. 74.

□

Theorem 75 is very helpful in the case of SDEs (which fulfill the conditions of the theorem) that cannot be solved analytically (or only with very high effort). When we are interested only in the expectation of the solution, we do not need to solve the SDE, instead we can apply the theorem.

Before coming to the next section, we mention that there exist several stochastic Fubini theorems or Fubini-type theorems in the literature, e.g., the works of Hille (2014); Van Neerven and Veraar (2005); Veraar (2012); Kailath et al. (1978); Berger and Mizel

(1979) and Protter (2005, Thm. IV.64, Thm. IV.65). To the best of the authors knowledge these settings are different to our assumptions. For example, they use different probability spaces for the expectation and for the stochastic process or the measure is time dependent and the integration is performed in time over that measure.

4.2 An Extension of Wald's Lemma

Generally, the expectation of a sum equals the sum of the expected summands. However, sometimes there is the problem that the expectation of a sum shall be calculated where the number of summands is random (and also the summands). This case is analyzed by the Lemma of Wald (1944).

Theorem 76 (Wald's Lemma)**.** Let $X = (X_n)_{n\in\mathbb{N}}$ be a sequence of identically distributed and integrable random variables and let N be an $\mathbb{N}$-valued, integrable random variable, independent of X. It holds:

$$\mathbb{E}\left[\sum_{n=1,\ldots,N} X_n\right] = \mathbb{E}[N]\mathbb{E}[X_1]$$

This is true since, if N is independent of X, we can calculate the conditional expected value

$$\mathbb{E}\left[\sum_{n=1,\ldots,N} X_n \middle| N = k\right] = \mathbb{E}\left[\sum_{n=1,\ldots,k} X_n\right] = k\mathbb{E}[X_1].$$

Thus, $\mathbb{E}\left[\sum_{n=1,\ldots,N} X_n \middle| N\right] = N\mathbb{E}[X_1]$ and so

$$\mathbb{E}\left[\sum_{n=1,\ldots,N} X_n\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{n=1,\ldots,N} X_n \middle| N\right]\right] = \mathbb{E}[N]\mathbb{E}[X_1].$$

Now, we modify this result to the product case. That means, we calculate the expectation of a product where both the number of factors and the factors are random.

Theorem 77. Let $X = (X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. and integrable random variables with $\mathbb{E}[X_1] = x$ and let N be an $\mathbb{N}$-valued, integrable random variable, independent of X. It holds:

$$\mathbb{E}\left[\prod_{n=1,\ldots,N} X_n\right] = \mathbb{E}\left[x^N\right]$$

Proof. If N is independent of X, we can calculate the conditional expected value

$$\mathbb{E}\left[\prod_{n=1,\ldots,N} X_n \middle| N = k\right] = \mathbb{E}\left[\prod_{n=1,\ldots,k} X_n\right] = x^k.$$

Thus, $\mathbb{E}\left[\prod_{n=1,\ldots,N} X_n \middle| N\right] = x^N$ and so

$$\mathbb{E}\left[\prod_{n=1,\ldots,N} X_n\right] = \mathbb{E}\left[\mathbb{E}\left[\prod_{n=1,\ldots,N} X_n \middle| N\right]\right] = \mathbb{E}\left[x^N\right].$$

□

Note that in general this is not equal to $x^{\mathbb{E}[N]}$. In general, the computation of $\mathbb{E}\left[x^N\right]$ is not an easy task. However, during this work we use this theorem for the case when N is Poisson distributed and we see that in this specific case, the expected product can be calculated.

Chapter 5

Motivation of Feedback-based Trading

Now, we have learned about market efficiency and about stochastic analysis. That means, we know the basics to construct and analyze trading strategies and to discuss the results. Next, we construct and analyze (technical) trading strategies. A literature review giving the most important results concerning technical trading rules is given in Chap. 6.

Like mentioned in Chap. 1, we want our strategies to be model-free. The idea of model-free strategies is that a trader does neither need to assume a market model nor to estimate parameters like the sign of the trend. Thus, a model-free strategy is constructed without any assumptions on the market model and shall be robust against disturbing influences and unknown parameters. Note that we use specific price models at the beginning of Chap. 9 for analyzing the performance of our strategies, not for constructing them. For this aim, we rely on control theoretic techniques to construct trading strategies, which are called feedback trading strategies or, simply, feedback rules.

In general, any trader is buying and selling assets and tries to make a profit when trading. For this, the trader determines the investment I, which generates the gain g. That means, for all traders, the gain is a function of the investment (and of the price). A feedback trader ℓ treats financial markets or, indeed, the portfolio (custody account) like a machine. The trader controls the output of the machine (of the portfolio) by using input variables. The input variable is the investment I^ℓ, which generates the overall gain g^ℓ.

In the feedback case, the output determines the input, i.e. the investment. That means, the investment is calculated as a function of the output, i.e., $I^\ell = h^\ell\left(g^\ell\right)$ for some function h^ℓ. The price process $p > 0$ can be seen as a disturbance variable and is used indirectly for determining I^ℓ through

$$g_t^\ell = \int_0^t \left(\frac{I^\ell}{p}\right)_{\tau-} dp_\tau.$$

Thus, feedback traders are chartists. Note that there are other types of chartist strategies that use much more data that can be obtained by analyzing the chart, i.e. the price history, and not only the own gain. Nonetheless, feedback traders are just one special class of chartists that use g_t (or in a wider setting g_τ ($\tau \in [0,t]$)) instead of p_τ ($\tau \in [0,t]$). In Fig. 5.1, the schematic interaction of a trader and the broker, who is the manager of the portfolio, is depicted. The broker is an intermediate person between the trader and the market, which produces the price. As mentioned above, for all traders, the gain g is a function of the investment I (and the price p). If the trader is a feedback trader, the investment is a function of the gain as well: $I = h(g)$. It is because of this feedback loop, that such trading strategies are called feedback strategies.

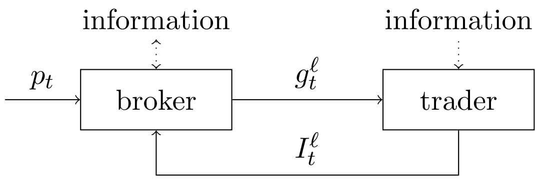

Figure 5.1: Schematic interaction between broker and trader

There is still the question how the function h should look like. One basic feedback trading strategy is the linear feedback long trader L with investment rule

$$I_t^L = I_0^* + K g_t^L,$$

where $I_0^* > 0$ is the initial investment and $K > 0$ is the feedback parameter. When following this strategy, the trader starts with the initial investment I_0^* and then adds K times the trader's own gain (the gain of this strategy, the so-called long side) to the initial investment. Note that $g_0^L = 0$. That means, the feedback parameter K specifies how much of the earned money is reinvested in the asset. If $K = 1$, the trader is a buy-and-hold trader, if $K > 1$, the trader is investing more money than earned, which is called leverage. If the price process and the time scale are continuous, this trader is a long trader, i.e., the investment fulfills $I_t > 0$ for all $t \geq 0$, and a trend follower (since $K > 1$ and $p_t > 0$ the trader is buying if the price rises and vice versa). An investment I_t is called long if $I_t > 0$ (or sometimes also $I_t \geq 0$) and short if $I_t < 0$ (or $I_t \leq 0$). If $K \in (0,1)$ the trader is selling if the price rises though the investment is still rising due to the value change.

Analogously to the linear long strategy, one can construct a linear feedback short trader S with investment strategy

$$I_t^S = -I_0^* - K g_t^S,$$

with g_t^S being the short side's gain. This trader is (when time and price are continuous) an anti trend following (depending on K only the investment rises or the trader is really buying if the price falls) short investor who loses money when prices rise and earns money when prices fall. These two strategies are analyzed extensively in the literature (e.g. Barmish and Primbs, 2011, 2016) for two reasons. On the one hand, the investment formulae are easy to handle (because these strategies are mathematically closed formulae), on the other hand, one can construct more complex strategies using these two linear strategies. In the following, for reasons of shortening the notation we also use the purely formal abbreviation "d" instead of writing stochastic integrals. A detailed literature review containing the most important results concerning linear feedback rules is given in Chap. 6.

A long trader makes money when the price rises and loses money when the price falls. For a short trader this holds vice versa. As mentioned above, the trading strategy we want to construct should be model free, that means especially, we do not want to need an estimation whether the price rises or falls on average, i.e., we do not want to estimate the trend. An idea for solving the problem that our strategies shall be model-free, i.e., especially they shall be constructed without an estimator for the trend, but that the trend can be positive or negative (and depending on the sign of the trend the long or the short rule would perform well—or not), is to take the superposition of the linear long controller and the linear short controller: the simultaneously long short trading rule

$$I_t = I_t^{SLS} = I_t^L + I_t^S,$$

which is schematically depicted in Fig. 5.2. Since the trader does not know whether the price rises or falls it seems to be reasonable to invest simultaneously long and short. However, one might guess that gains and losses may average out. Note that it is important that the gains of the long and the short side are calculated separately. That means, we use two independently calculated feedback rules and analyze the gain of the superposition, which is $g_t^{SLS} = g_t^L + g_t^S$.

So far, the SLS controller is just an idea. We do not know if it works. That it is really reasonable to use the SLS rule, is shown in the following chapters, where we show in an example that the gains and losses do not average out and where we give several performance results on SLS trading (from the literature and new ones). Note that this controller is model-free because there is no specific price model assumed for constructing it. To enhance readability we often write $I(t)$ and $g(t)$ instead of $I^{SLS}(t)$ and $g^{SLS}(t)$, respectively. When there are different strategies used (like in Sec. 9.4), we again write SLS. In the SLS rule, the long side's gain and the short side's gain are calculated separately and the same parameter values $K > 0$ and $I_0^* > 0$ are used for the long and the short side. A flow diagram for the SLS rule is given in Fig. 5.3.

It is important to note that (e.g., for the long side) $dg^L(t) = I^L(t) \cdot \frac{dp(t)}{p(t)}$ is short for $g_t^L = \int_0^t \left(\frac{I^L}{p}\right)_{\tau-} dp_\tau$. That leads to a strategy that is predictable. This is—in discrete time—reached through $\Delta g_t^L = \frac{I_{t-1}^L}{p_{t-1}} \cdot (p_t - p_{t-1})$ and $I_{t-1}^L = I_0^* + K g_{t-1}^L$. When calculating the limits for continuous time trading, this is a well defined integral if the price p is a good integrator.

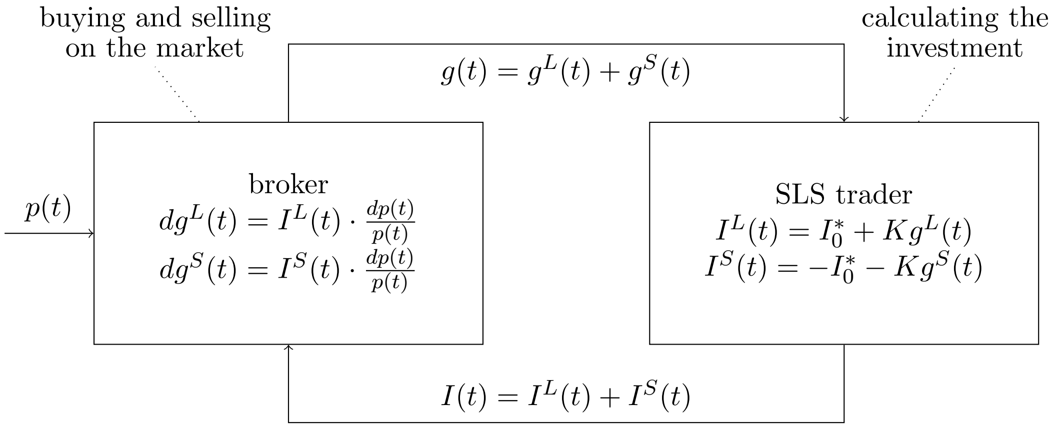

Figure 5.2: Schematic interaction between broker and SLS trader

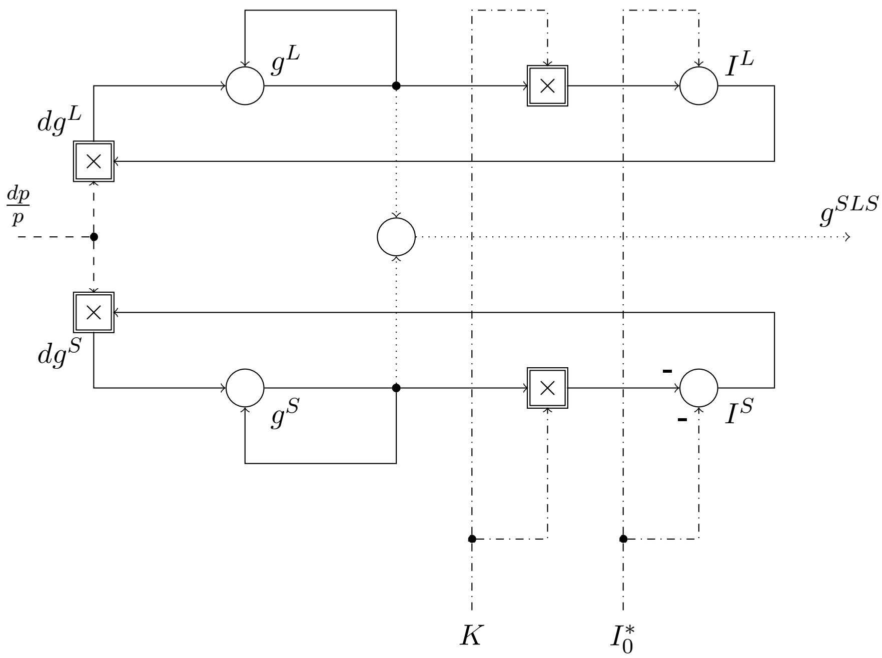

Figure 5.3: Flow diagram for the SLS controller with disturbance variable return on investment $\frac{dp}{p}$, i.e. price, and output variable gain g^{SLS}. The SLS trader's parameters are $K > 0$ and $I_0^* > 0$

Chapter 6

Literature Review for Feedback Trading

In Chap. 5, we have seen the construction of the SLS trading rule. However, it is not clear why a trader should rely on this strategy and whether this strategy works well (e.g., has positive gains on average). In this chapter, we present the most important results from the related literature to SLS trading and give a short overview on the feedback trading literature in general. Empirical papers have found that technical trading (i.e. chartist rules) can work well (like Avramov et al., 2017). However, these strategies are still deemed as reading tea leafs, e.g., by p-hacking arguments. The aim of the feedback trading literature is to put technical trading on a mathematical basis.

In this chapter, we consider different sets of underlying price models on which the SLS rule is analyzed. Note that none of these models was originally used for constructing the SLS strategy.

6.1 Continuously Differentiable Prices

The first set of prices considered in the literature and, thus, in this work is the set of continuously differentiable prices, i.e., $p \in \mathcal{C}^1([0,T],\mathbb{R}^+)$. The analysis of this section is following Barmish (2008, 2011); Barmish and Primbs (2012). We provide the analysis of such market models although we know that continuously differentiable prices are rather unrealistic. Still, there are at least two reasons for proceeding that way: First, the result in this setting is much stronger than in other settings, second, the analysis is deterministic and, thus, relatively easy. In continuous time settings we use (t) and in discrete time settings a subscript t for distinguishing the different time scales.

We start with the SLS rule as introduced in Chap. 5. Note that in this setting

$$I(t) = I^L(t) + I^S(t),$$

$$I^L(t) = I_0^* + Kg^L(t),$$

$$I^S(t) = -I_0^* - Kg^L(t),$$
$$g^L(t) = \int_0^t I^L(\tau) \cdot \frac{\dot{p}(\tau)}{p(\tau)} d\tau,$$

and

$$g^S(t) = \int_0^t I^S(\tau) \cdot \frac{\dot{p}(\tau)}{p(\tau)} d\tau.$$

Next, we analyze performance properties of the SLS rule—in this setting, we show that the gain of the SLS rule is non-negative for all possible market developments and zero only for a special case. We start this analysis with the long side. It holds:

$$\dot{I}^L(t) = K\dot{g}^L(t)$$

and

$$\dot{g}^L(t) = I^L(t) \cdot \frac{\dot{p}(t)}{p(t)}$$

Putting these two formulae together leads to

$$\frac{\dot{I}^L(t)}{I^L(t)} = K \cdot \frac{\dot{p}(t)}{p(t)}.$$

Now we can integrate both sides, leading to:

$$\begin{aligned} \int_0^t \frac{\dot{I}^L(\tau)}{I^L(\tau)} d\tau &= K \int_0^t \frac{\dot{p}(\tau)}{p(\tau)} d\tau \\ \Leftrightarrow \quad \ln I^L(t) - \ln I^L(0) &= K(\ln p(t) - \ln p(0)) \\ \Leftrightarrow \quad \ln \frac{I^L(t)}{I^L(0)} &= \ln \left(\frac{p(t)}{p(0)}\right)^K \end{aligned}$$

and by use of the exponential operator

$$\frac{I^L(t)}{I^L(0)} = \left(\left(\frac{p(t)}{p(0)}\right)^K\right)$$

Putting this into the definition of $I^L(t) = I_0^* + Kg^L(t)$ and recalling that $I^L(0) = I_0^*$ gives us

$$I_0^* \left(\frac{p(t)}{p(0)}\right)^K = I_0^* + Kg^L(t)$$

and the long side's gain

$$g^L(t) = \frac{I_0^*}{K} \left(\left(\frac{p(t)}{p(0)}\right)^K - 1\right).$$

Substituting I_0^* by $-I_0^*$ and K by $-K$ directly leads to the short side's gain

$$g^S(t) = \frac{I_0^*}{K}\left(\left(\frac{p(t)}{p(0)}\right)^{-K} - 1\right).$$

Knowing that $g(t) = g^L(t) + g^S(t)$ lets us conduct

$$g(t) = \frac{I_0^*}{K}\left(\left(\frac{p(t)}{p(0)}\right)^{K} + \left(\frac{p(t)}{p(0)}\right)^{-K} - 2\right).$$

It is easy to see that $g(t) = 0$ if and only if $p(t) = p(0)$ and that $g(t) > 0$ for all other $p(t) > 0$. This is an arbitrage opportunity.

Theorem 78. If the price process is continuously differentiable, the SLS rule leads to the gain/loss function

$$g(t) = \frac{I_0^*}{K}\left(\left(\frac{p(t)}{p(0)}\right)^{K} + \left(\frac{p(t)}{p(0)}\right)^{-K} - 2\right),$$

which is zero if and only if $p(t) = p(0)$ and positive for all $0 < p(t) \neq p(0)$.

Next, we briefly state some further results of the respective SLS literature. Barmish (2008) analyzes the SLS rule in the $p \in \mathcal{C}^1$-setting, introduces an SLS rule with saturation (i.e. with a boundary for the investment I_{max}), and makes simulations with real stock prices. Barmish (2011) discusses trading controllers with and without derivatives (the latter ones are called differentiator-free), analyzes the SLS rule in the $p \in \mathcal{C}^1$-setting, analyzes an SLS rule with reset (i.e., when either $I^L < I_{min}$ or $I^S > -I_{min}$ the SLS rule is set back to I_0^* or $-I_0^*$, resp. That means, if the investment is close to zero, the controller starts with its initial values. Since the expected gain is positive for all t (under a GBM assumption), it is positive for all intervals until the restarts), and performs simulations on real world stock data. Barmish and Primbs (2012) analyze the SLS rule in the $p \in \mathcal{C}^1$-setting as well as in the CAPM, e.g., concerning the gain, the expected gain, and the worst case loss, and also perform backtests on real stock data.

6.2 Geometric Brownian Motion

As mentioned above, continuously differentiable prices are a very hard assumption. This is the reason why in the literature another market model is analyzed extensively, the geometric Brownian motion (with drift). The GBM is a very common market model and, e.g., used in the Black-Scholes model for option pricing (Black and Scholes, 1973). In this section, we follow the analysis of the GBM market of Barmish and Primbs (2011, 2016). All the market models discussed in the feedback trading literature have to be understood as so-called proving grounds: Using these theoretical models, performance properties have to be proven before testing the strategy on real data.

In a purely formal way the GBM can be formulated as

$$\frac{dp}{p}(t) = \mu dt + \sigma dW(t),$$

with $\mu > -1$ being the drift, $\sigma > 0$ the volatility, and $W(t)$ a standard Brownian motion (Wiener process). For the remainder of this work, if there is any trading strategy, we assume that this strategy is predictable and locally bounded and all integrators, e.g., prices, are semimartingales. That means, we do not write $t-$ in the subscript of the integrands all the time, even if this is meant. To sum up, we choose all strategies and prices so that all integrals are well defined. For the discrete time models analyzed in the work at hand, we will see that all discrete time stochastic integrals are well defined (especially that the strategy is predictable) and that the limits fit exactly to our definitions of continuous time stochastic integrals.

Again, we first analyze the long side of the SLS rule. It holds

$$dg^L(t) = I^L(t) \cdot \frac{dp}{p}(t) = I^L(t)(\mu dt + \sigma dW(t))$$

and with $I^L(t) = I_0^* + Kg^L(t)$ it follows

$$dI^L(t) = KI^L(t)(\mu dt + \sigma dW(t)),$$

i.e.,

$$\frac{dI^L}{I^L}(t) = K\mu dt + K\sigma dW(t),$$

which is a GBM again.

The solution of this GBM is

$$I^L(t) = I_0^* e^{\left(\mu K - \frac{\sigma^2 K^2}{2}\right)t + \sigma K W(t)}.$$

We know that $g^L(t) = \frac{I^L(t) - I_0^*}{K}$, $g^L(0) = 0$, and that the solution of the GBM of the price process is

$$\frac{p(t)}{p(0)} = e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)},$$

which together leads to a gain/loss function of

$$g^L(t) = \frac{I_0^*}{K}\left(\left(\frac{p(t)}{p(0)}\right)^K e^{\frac{\sigma^2}{2}(K-K^2)t} - 1\right).$$

Substituting I_0^* by $-I_0^*$ and K by $-K$ leads to

$$g^S(t) = \frac{I_0^*}{K}\left(\left(\frac{p(t)}{p(0)}\right)^{-K} e^{-\frac{\sigma^2}{2}(K+K^2)t} - 1\right)$$

for the short side's gain and finally to the next theorem.

Theorem 79. The gain/loss function of the SLS rule for GBMs is given through

$$g(t) = \frac{I_0^*}{K}\left(\left(\frac{p(t)}{p(0)}\right)^{K} e^{\frac{\sigma^2}{2}(K-K^2)t} + \left(\frac{p(t)}{p(0)}\right)^{-K} e^{-\frac{\sigma^2}{2}(K+K^2)t} - 2\right).$$

Note that this formula is meant path-by-path, i.e., if we know the development of p, we know the development of g^L. However, the formula for g^L is not path dependent. In the formula for g^L we only have to know $p(t)$ and the fixed initial price $p(0)$. The path on which the price develops to $p(t)$ is not important and instead of a stochastic integral in this gain/loss formula there is only one random variable, namely $p(t)$.

In the case of a GBM, the SLS rule does not provide arbitrage. When, e.g., setting $p(t)/p(0) = 1$ and $K = 1$, we get $g(t) = I_0^*(e^{-\sigma^2 t} - 1)$, which is non-negative if and only if $-\sigma^2 t \geq 0$. That means, for all $t > 0$ the SLS rule made a loss.

However, recognizing that $\frac{p(t)}{p(0)}$ is log-normally distributed, i.e., $\ln\left(\frac{p(t)}{p(0)}\right) \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$, and that $\ln\left(\left(\frac{p(t)}{p(0)}\right)^K\right) \sim \mathcal{N}\left(\left(K\mu - \frac{K\sigma^2}{2}\right)t, K^2\sigma^2 t\right)$ allows us to compute the expected gain. It holds:

$$\mathbb{E}\left[\left(\frac{p(t)}{p(0)}\right)^{K}\right] = e^{\left(K\mu - \frac{K\sigma^2}{2} + \frac{K^2\sigma^2}{2}\right)t}$$

and

$$\mathbb{E}\left[\left(\frac{p(t)}{p(0)}\right)^{-K}\right] = e^{\left(-K\mu + \frac{K\sigma^2}{2} + \frac{K^2\sigma^2}{2}\right)t}$$

This leads to:

$$\begin{aligned}\mathbb{E}[g(t)] =& \frac{I_0^*}{K}\left(e^{\left(K\mu - \frac{K\sigma^2}{2} + \frac{K^2\sigma^2}{2}\right)t} e^{\frac{\sigma^2}{2}(K-K^2)t} + e^{\left(-K\mu + \frac{K\sigma^2}{2} + \frac{K^2\sigma^2}{2}\right)t} e^{-\frac{\sigma^2}{2}(K+K^2)t} - 2\right)\\ =& \frac{I_0^*}{K}\left(e^{K\mu t} + e^{-K\mu t} - 2\right)\end{aligned}$$

In particular, the expected gain is zero if and only if $\mu = 0$ and positive for all other trends $\mu > -1$ ($\forall t > 0$).

Theorem 80. For the expected gain of the SLS rule when underlying prices follow a GBM, it holds

$$\mathbb{E}[g(t)] = \frac{I_0^*}{K}\left(e^{K\mu} + e^{-K\mu} - 2\right),$$

which is zero if and only if the trend is zero and positive for all other trends ($\forall t > 0$).

The property $\mathbb{E}[g(t)] \geq 0$ and $\mathbb{E}[g(t)] > 0$ whenever $\mu \neq 0$ is called the robust positive expectation property (RPEP). We emphasize that these results do not imply an arbitrage possibility in the classical sense, because an arbitrage strategy is a strategy π with $\pi_0 \leq 0$ and $\pi(t) \geq 0$ and $\mathbb{P}(\pi(t) > 0) > 0$. However, it holds that the discounted net gain is positive in expectation: $e^{-\mu t}\mathbb{E}[g(t)] > 0$ ($\mu \neq 0, t > 0$) (with the discounting factor $e^{-\mu t}$).

Before coming to a simulation of GBMs and corresponding SLS trading, we briefly summarize the related literature to this section. Barmish and Primbs (2011) analyze the SLS rule in the GBM model, conduct formulae for the gain/loss function, the expected gain, the variance of the gain, and the density function of the gain, and provide some simulations. Barmish and Primbs (2016) analyze the SLS rule in the setting of continuously differentiable prices and for GBM prices and deal with formulae for the gain, the expected gain, the variance of the gain, the density function of the gain, and the probability of loosing. Additionally, SLS rules with a saturation boundary are analyzed and the leverage problem is investigated, i.e., the amount of money an SLS trader needs for investment related to how much money is in the trader's account V. Further work on SLS trading in GBM markets was done by Dokuchaev and Savkin (1998a,b, 2002, 2004); Dokuchaev (2012). Especially, Dokuchaev (2012, Thm. 4.3, Equations (4.16) and (4.17)) provides a structurally similar formula for the expected trading gain of the SLS rule within the GBM model as Barmish and Primbs (2011), just expressed differently:

$$\mathbb{E}[g(t)] = \frac{2I_0^*}{K}(\cosh(K\mu t) - 1) > 0 \; \forall \mu \neq 0, \; t > 0$$

Barmish and Primbs (2014) analyze the standard SLS rule as well as the SLS rule with saturation and especially discuss practical and basic topics like order filling, gain/loss accounting, broker margin rate, simulation, short selling, and feedback loops. Malekpour and Barmish (2013) analyze the risk of SLS trading in a GBM setting using drawdown, i.e. the maximal loss in the account V (either absolute: $D_{max}(V) = \max_{0 \leq s \leq t \leq T}(V(s) - V(t))$; or relative: $d_{max}(V) = \max_{0 \leq s \leq t \leq T} \frac{V(s)-V(t)}{V(s)}$). The account value of a trader is the initial account plus the gain $V(t) = V_0^* + g(t)$. That means, the drawdown of a trading path is a number that states how much money a trader lost maximally during the path (in several versions). This number is used for measuring how risky a trading rule is. In that paper, another representation of SLS is used where the linear rules are $I^L = KV^L$ and $I^S = -KV^S$ and formulae for the expected absolute drawdown and for an upper boundary of the relative drawdown are derived. Malekpour and Barmish (2014b) motivate some problems in feedback trading like controllers with delay, i.e., controllers that do not take into account all past period gains, or moving averages (which are used in many classical technical trading strategies (Ivanova et al., 2014)) and how the crossing points of prices and moving averages can be calculated and how moving average strategies and linear feedback strategies can be combined. Additionally, the importance of discrete time models and of backtests is stated and drawdown is discussed. Barmish et al. (2013) explain how to simulate SLS trading paths, Iwarere and Barmish (2014) analyze the SLS strategy when prices are governed by a binomial or quadrinomial tree (Cox-Ross-Rubinstein model), Malekpour and Barmish (2012) discuss how to optimally choose K in the SLS strategy, and Malekpour and Barmish (2014a) deal with the problem of how to measure risk for feedback trading strategies. Iwarere and Barmish (2010); Primbs and Barmish (2012) analyze other feedback trading strategies and connect classical technical trading rules to feedback trading. Malekpour et al. (2013) use a controller with an integral part and Primbs and Barmish (2011) connect feedback trading to option pricing.

At the end of this section, we present two illustrating pictures on SLS trading in a GBM market. For these pictures, we can only use a discrete time scale and, thus, we have to discretize the price path and the strategy. In Fig. 6.1 we can see a path of a Wiener process, i.e. of a Brownian motion, and the corresponding path of the geometric Brownian motion with parameters $\mu = 0.1$ and $\sigma = 0.2$. In Fig. 6.2 the GBM is depicted together with the (discrete time) trading gain of an SLS strategy with parameters $I_0^* = 1$ and $K = 4$ and with the investment levels of the short side and the long side. Additionally, there is a circle for the theoretical trading gain (in a continuous time model) from Thm. 79 and a triangle for the expected trading gain (also in a continuous time model) from Thm. 80.

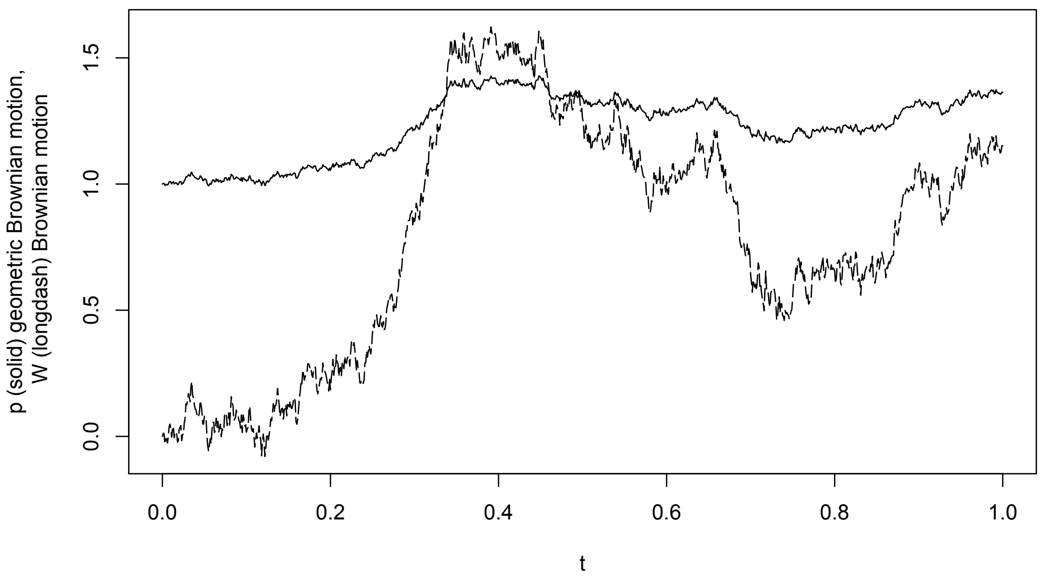

Figure 6.1: A path of a Brownian motion (Wiener process) with the corresponding path of the geometric Brownian motion (GBM) with parameters $\mu = 0.1$ and $\sigma = 0.2$

6.3 Time-Varying Geometric Brownian Motion

Primbs and Barmish (2013, 2017) show that the robust positive expectation property (RPEP; the property that the expected gain is non-negative and zero only for a null set in the parameter space) also holds when the trend $\mu(t)$ as well as the volatility $\sigma(t)$ of the GBM are time dependent. A time-varying GBM (tvGBM) is described via

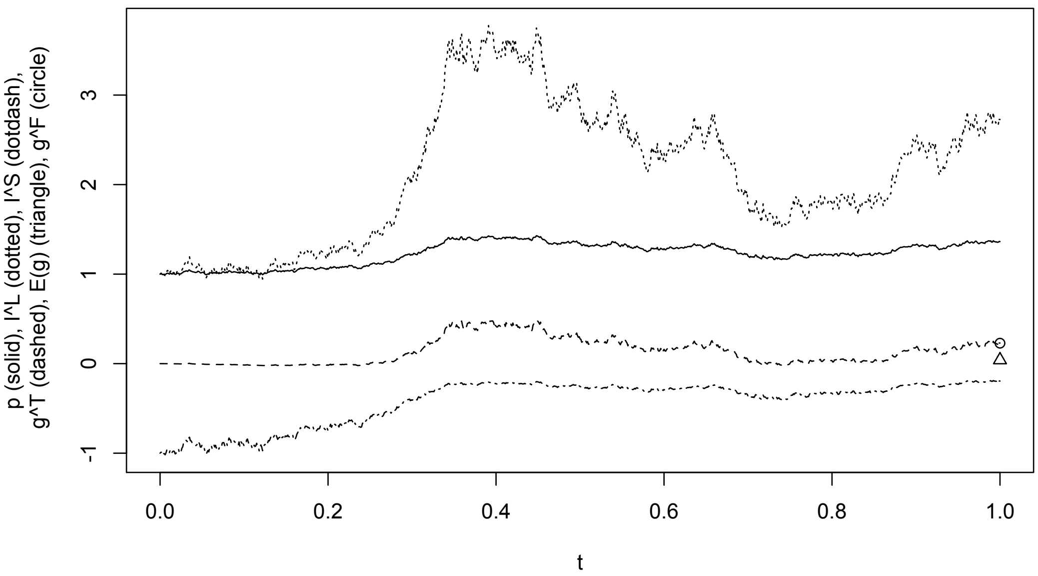

Figure 6.2: The path of a geometric Brownian motion (GBM) from Fig. 6.1 together with the trading gain of an SLS rule with parameters $I_0^* = 1$ and $K = 4$ and the investment paths of the long and the short side

$\frac{dp}{p}(t) = \mu(t)dt + \sigma(t)dW(t)$ with integrable functions μ and σ.

Theorem 81. For a time-varying GBM with trend $\mu(t)$ and volatility $\sigma(t)$ and the SLS trading rule it holds:

$$\mathbb{E}[g(t)] = \frac{I_0^*}{K}\left(exp\left(K\int_0^t \mu(s)ds\right) + exp\left(-K\int_0^t \mu(s)ds\right) - 2\right).$$

It further holds $\mathbb{E}[g(t)] \geq 0$ and, whenever $\int_0^t \mu(s)ds \neq 0$, it holds $\mathbb{E}[g(t)] > 0$, too.

6.4 Controller with Delay and Constant Trend

Malekpour and Barmish (2016) note an interesting and especially practical problem of the SLS rule. Since the SLS strategy is calculated by use of the overall gain, price behaviors that happened a long time ago have the same impact on the investment decision of the trader as if they happened a few days ago. Consider a price development where in the phase after the trader entered the market, the price rose a lot and then stayed nearly constant for a long time. The trader's long (short) side would have made (lost) a lot of money in the first period and then stayed approximately constant. As a consequence of the feedback loop, the investment of the trader is still very high and long, which seems to be questionable since prices stayed nearly constant for a long time. Malekpour and

Barmish (2016) introduce a new strategy called Initially Long-Short (ILS) with delay as the superposition of a linear long rule with delay $I_t^{Ld} = I_0^* + K(g_t^{Ld} - g_{t-m}^{Ld})$ and a linear short rule with delay $I_t^{Sd} = -I_0^* - K(g_t^{Sd} - g_{t-m}^{Sd})$. The strategy is defined and analyzed in a discrete time setting with a time grid $\{0, 1, 2, \ldots\}$ with fixed time steps, e.g., days. The word *initially* denotes the fact that (due to the discrete time grid) only at the initial time one can be sure that the long (short) side is truly long (short). Among other market requirements, the main assumption by Malekpour and Barmish (2016) is $\mathbb{E}\left[\frac{p_t - p_{t-1}}{p_{t-1}}\right] = \mu \neq 0$, which is needed to show that the positive robust expectation property still holds. In the ILS strategy only the period gains of the last m days are taken into account. While on the one hand the idea of not taking into account too old price and gain developments makes the ILS rule of Malekpour and Barmish (2016) favorable to the standard SLS rule, on the other hand the hard delay definition seems to be a little bit problematic. Consider a price history where m days ago an important event happened at the market, for example a sudden crash, which made the short side much more important. Today, this event will be taken into account, tomorrow, this will not be the case. That means, the strategy will change substantially only because an important event happened exactly m days ago, where the number m is idiosyncratically chosen by the trader. A point to think about that is not discussed in detail by Malekpour and Barmish (2016) is that the trader is assumed to be a price taker. But the trader decides to trade, e.g., daily and the expected return on investment on a daily basis is assumed to equal μ. That means, the trader indirectly influences the expected return on investment by choosing a trading frequency, which at a first glance may be contradiction to the price taker property. However, this is not a problem, as shown in the work at hand (in Sec. 9.3).

At the end of this chapter, we mention that there is much more work on technical trading, e.g., the work of Calafiore and Monastero (2010, 2012); Calafiore (2008, 2009); Cover (2011); Cover and Ordentlich (1996); Mudchanatongsuk et al. (2008); Taylor and Allen (1992); Primbs and Sung (2009). However, since these papers do not use the same market assumptions as in the work at hand and this chapter shall not be too lengthy, we end the literature review here. In the next chapter, the market assumptions of this work are explained in detail.

Chapter 7

Market Requirements

Before starting the analytical work on the performance of the SLS rule, we discuss some market requirements, which are in line with the related literature (Barmish and Primbs, 2011, 2016) and which have to be fulfilled in our work:

- *Short Selling,* i.e. the possibility for short selling: By construction of the SLS strategy, the short side's strategy requires for sure (and maybe also the long side) the possibility for short selling, i.e. for a negative investment. If the stock under trade is big enough, usually the trader can short the stock.
- *Costless Trading:* There are no additional costs associated with buying or selling an asset, i.e., there are no trading costs on the market. The assumption of costless trading was in the past a strong argument of the defenders of the efficient market hypothesis to show that chartist strategies cannot work in practice (cf. Fama, 1991). However, in times of flat-rate stock trading offers, this assumption might be less problematic. We can say that it is approximately true for big trading companies and highly liquid stocks.
- *Adequate Resources:* We assume that the trader has adequate resources, i.e., the trader has always enough money for trading. In other words, the trader's financial resources are big enough so that all desired transactions can be executed and there are no financial constraints, which could prohibit any desired transaction. Again, this is plausible for big trading companies if the investment is not too high, however, the maximum amount of the investment and the needed size of the trading company depend on the particular case and cannot be specified in general. That means, the adequate resources assumption is justified if the trader is big enough, e.g., a mutual fund, and if the trader is not trading too much of the single asset under trade.
- *Price Taker Property:* In the performance part of this work, Chap. 9, we assume the price taker property. That means, the trader is not able to influence the asset's price, neither directly nor through buying or selling decisions, i.e., the trader's actions do not have any influence on the market, especially, they do not affect the

stock price. This is approximately true if, again, the investment is not too high compared to the size of the stock's underlying firm. In the effects part of this work, Chap. 10, we relax this assumption.

- *Perfect Liquidity:* There is neither a gap between bid and ask price nor any waiting time for transaction execution, i.e., the trader can arbitrarily buy and sell stocks. Stock prices do not change during one transaction. This is no strong assumption for large companies' stocks under trade.

To sum up, the market requirements are more or less true for a big, rich trader trading small amounts of stocks of a big underlying firm. We need a few additional technical assumptions for the analysis of the SLS rule.

- *Time Scale:* For some results we assume the continuous time scale $[0, T]$ and for some the discrete one $\{0, 1, 2, \ldots, T\}$ or $\{0, h, 2h, \ldots, Nh\}$ with a variable mesh size h. Other results are for sampled-data systems, i.e., we assume a continuous time price process, but a discrete time trading strategy, which is likely the most realistic assumption for the time scale.

- *Bond:* We assume that there is a riskless bond available and, which is a much harder assumption, that the interest rate equals the margin rate. Additionally and without loss of generality, we assume the margin and interest rate of the bond to be zero. This can be done since we can use the bond (with a rate unequal to zero) as numéraire. That means, all rates and trends are somehow relative to the numéraire. If a stock is assumed to be risk neutral, i.e., if the stock's trend equals the rate of the riskless bond, in our setting, the trend of this stock was zero. In a market model, which is completely risk neutral, the trends of all stocks would equal zero. However, according to efficient market definitions involving risk, it is plausible that a risky asset has a higher trend than a riskless one. Thus, excluding the zero trend case from some results is justified.

- *Real Number of Shares:* We allow the number of shares a trader holds to be real. If a trader buys stocks directly from the firm issuing it, the number of bought shares is an integer. When buying via a broker (which is also needed for short selling), the number of shares can be assumed to be rational—depending on the broker. Hence, the assumption of a real number of shares is not too hard and commonly accepted in stochastic finance.

Chapter 8

Small Example

In the next chapter, we analyze the performance of the SLS rule in detail. To give some feeling how SLS trading works we first provide a few very small examples. For that reason, we assume a market on the time scale $\{0, 1, 2\}$ and a (not necessarily recombining) binomial tree model, as shown in Fig. 8.1. The price process starts with the initial price p_0. In the time points $i = 1, 2$ the price goes up by the factor p_i^u with probability $0 < q_i < 1$ and down by the factor p_i^d with probability $1 - q_i$. For non-arbitrage reasons it holds $0 < p_i^d < 1 < p_i^u$, since the rate is $r = 1$ throughout the whole work. We define the trend at time $i = 0, 1$ as $\mu_i = \mathbb{E}\left[\frac{p_{i+1} - p_i}{p_i}\middle|\mathcal{F}_i\right] = p_{i+1}^u q_{i+1} + p_{i+1}^d (1 - q_{i+1}) - 1$.

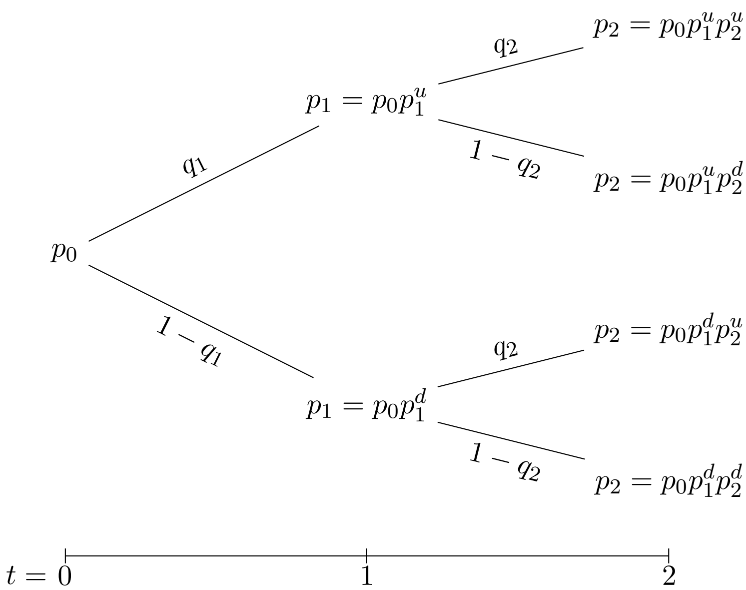

Figure 8.1: General binomial tree model with two periods

In discrete time settings we calculate the gain of a strategy ℓ according to

$$g_i^\ell = g_{i-1}^\ell + I_{i-1}^\ell \frac{p_i - p_{i-1}}{p_{i-1}}.$$

That means, the amount of stocks held at time i is $\frac{I_{i-1}^\ell}{p_{i-1}}$, which is known at time $i-1$, i.e., the strategy is predictable in a discrete time sense.

Note that even though the initial investment of the SLS rule is always zero (since $I_0 = I_0^L + I_0^S = I_0^* - I_0^* = 0$) this does not mean that the bond value of an SLS trader is always zero: Depending on the price behavior, the trader has to borrow money from the bank or put money on the bank. Especially, the account value of a trader is $V_t = V_0 + g_t$ where we assume $V_0 = g_0^L = g_0^S = 0$. The difference of the account value and the investment level $I_t = I_t^L + I_t^S$ has to be cleared by the bond value B.

In the following, we calculate the trading behavior for SLS traders in a few exemplary market models. In these examples market parameters or trader parameters vary. Thus, we can get a feeling for the underlying mechanism of SLS trading and see the effects of the parameters.

In the examples in Tabs. 8.1, 8.2, and 8.3 we set a constant trend with $p_1^u = p_2^u = 1.1$, $p_1^d = p_2^d = 0.8$, and $q_1 = q_2 = 0.5$. The trader's parameters are the fixed initial investment $I_0^* = 10$ and varying values for the feedback parameter $K = 2$, $K = 1$, and $K = 0.5$. Thus, we can see the effects of K. In Tab. 8.1 it holds $K = 2$, $\mu_0 = \mu_1 = -5\%$, and $\mathbb{E}[g_2] = 0.1$. In Tab. 8.2 it holds $K = 1$, $\mu_0 = \mu_1 = -5\%$, and $\mathbb{E}[g_2] = 0.05$. And in Tab. 8.3 it holds $K = 0.5$, $\mu_0 = \mu_1 = -5\%$, and $\mathbb{E}[g_2] = 0.025$. So far, we could see that reducing K reduces the expected gain but also the worst-case loss (from 0.8 via 0.4 to 0.2).

path	t	p	I^L	g^L	I^S	g^S	V	B
p_0	0	1	10	0	-10	0	0	0
$p_0p_1^u$	1	1.1	12	1	-8	-1	0	-4
$p_0p_1^d$	1	0.8	6	-2	-14	2	0	8
$p_0p_1^up_2^u$	2	1.21	14.4	2.2	-6.4	-1.8	0.4	-7.6
$p_0p_1^up_2^d$	2	0.88	7.2	-1.4	-11.2	0.6	-0.8	3.2
$p_0p_1^dp_2^u$	2	0.88	7.2	-1.4	-11.2	0.6	-0.8	3.2
$p_0p_1^dp_2^d$	2	0.64	3.6	-3.2	-19.6	4.8	1.6	17.6

Table 8.1: Example for SLS trading on a binomial tree with $p_1^u = p_2^u = 1.1$, $p_1^d = p_2^d = 0.8$, $q_1 = q_2 = 0.5$, $I_0^* = 10$, and $K = 2$

In the examples in Tabs. 8.1, 8.4, and 8.5 we fix $p_1^u = p_2^u = 1.1$, $p_1^d = 0.8$, $q_1 = q_2 = 0.5$, $I_0^* = 10$, and $K = 2$. Now, we vary the trend through varying $p_2^d = 0.8, 0.65, 0.95$. In Tab. 8.4 it holds $p_2^d = 0.65$, $\mu_0 = -5\%$, $\mu_1 = -12.5\%$, and $\mathbb{E}[g_2] = 0.25$. In Tab. 8.5 it holds $p_2^d = 0.65$, $\mu_0 = -5\%$, $\mu_1 = 2.5\%$, and $\mathbb{E}[g_2] = -0.05$. Note that in Tab. 8.1 the trend is constant (and negative), thus, the sign of the trend does not change. Also in Tab. 8.4 the sign of the trend does not change, though the trend varies. However in

path	t	p	I^L	g^L	I^S	g^S	V	B
p_0	0	1	10	0	-10	0	0	0
$p_0p_1^u$	1	1.1	11	1	-9	-1	0	-2
$p_0p_1^d$	1	0.8	8	-2	-12	2	0	4
$p_0p_1^up_2^u$	2	1.21	12.1	2.1	-8.1	-1.9	0.2	-3.8
$p_0p_1^up_2^d$	2	0.88	8.8	-1.2	-10.8	0.8	-0.4	1.6
$p_0p_1^dp_2^u$	2	0.88	8.8	-1.2	-10.8	0.8	-0.4	1.6
$p_0p_1^dp_2^d$	2	0.64	6.4	-3.6	-14.4	4.4	0.8	8.8

Table 8.2: Example for SLS trading on a binomial tree with $p_1^u = p_2^u = 1.1$, $p_1^d = p_2^d = 0.8$, $q_1 = q_2 = 0.5$, $I_0^* = 10$, and $K = 1$

path	t	p	I^L	g^L	I^S	g^S	V	B
p_0	0	1	10	0	-10	0	0	0
$p_0p_1^u$	1	1.1	10.5	1	-9.5	-1	0	-1
$p_0p_1^d$	1	0.8	9	-2	-11	2	0	2
$p_0p_1^up_2^u$	2	1.21	11.025	2.05	-9.025	-1.95	0.1	-1.9
$p_0p_1^up_2^d$	2	0.88	9.45	-1.1	-10.45	0.9	-0.2	0.8
$p_0p_1^dp_2^u$	2	0.88	9.45	-1.1	-10.45	0.9	-0.2	0.8
$p_0p_1^dp_2^d$	2	0.64	8.1	-3.8	-12.1	4.2	0.4	4.4

Table 8.3: Example for SLS trading on a binomial tree with $p_1^u = p_2^u = 1.1$, $p_1^d = p_2^d = 0.8$, $q_1 = q_2 = 0.5$, $I_0^* = 10$, and $K = 0.5$

Tab. 8.5 the sign of the trend changes. In our examples, we can only expect a negative gain in the case where the sign of the trend changes. This is generalized in the next chapter, when the SLS rule is analyzed mathematically.

At the end of this small example chapter, we illustrate that for the SLS rule (indeed: for an expected positive gain of the SLS rule) it is not important whether the trend is positive or negative. For this reason, we compare Tab. 8.1 and Tab. 8.6 with $p_1^u = p_2^u = 1.2$, $p_1^d = p_2^d = 0.9$, $q_1 = q_2 = 0.5$, $I_0^* = 10$, and $K = 2$. In Tab. 8.6 it holds $\mu_0 = \mu_1 = 5\%$ and also $\mathbb{E}[g_2] = 0.1$. Even this property, that it does not matter whether the trend is, e.g., $+5\%$ or -5% is generalized in the next chapter.

Before analyzing the performance of the SLS rule in the next chapter, we give an intuitive (heuristic) explanation why this strategy works well: The SLS strategy starts with both investing equally long and short. When prices rise, the long side makes money and the short side looses it, when prices fall, it is the other way round. But due to compound interests, returns on investment of, e.g., three times 10%, generate a higher gain on the long side than the loss on the short side is. And for returns of investment of, e.g., three times -10%, the short side's gain surpasses the long side's loss, also due to compound interests. Note that this is no proof but important for the intuition of SLS trading.

path	t	p	I^L	g^L	I^S	g^S	V	B
p_0	0	1	10	0	-10	0	0	0
$p_0p_1^u$	1	1.1	12	1	-8	-1	0	-4
$p_0p_1^d$	1	0.8	6	-2	-14	2	0	8
$p_0p_1^up_2^u$	2	1.21	14.4	2.2	-6.4	-1.8	0.4	-7.6
$p_0p_1^up_2^d$	2	0.715	3.6	-3.2	-13.6	1.8	-1.4	8.6
$p_0p_1^dp_2^u$	2	0.88	7.2	-1.4	-11.2	0.6	-0.8	3.2
$p_0p_1^dp_2^d$	2	0.52	1.8	-4.1	-23.8	6.9	2.8	24.8

Table 8.4: Example for SLS trading on a binomial tree with $p_1^u = p_2^u = 1.1$, $p_1^d = 0.8$, $p_2^d = 0.65$,$q_1 = q_2 = 0.5$, $I_0^* = 10$, and $K = 2$

path	t	p	I^L	g^L	I^S	g^S	V	B
p_0	0	1	10	0	-10	0	0	0
$p_0p_1^u$	1	1.1	12	1	-8	-1	0	-4
$p_0p_1^d$	1	0.8	6	-2	-14	2	0	8
$p_0p_1^up_2^u$	2	1.21	14.4	2.2	-6.4	-1.8	0.4	-7.6
$p_0p_1^up_2^d$	2	1.045	10.8	0.4	-8.8	-0.6	-0.2	-2.2
$p_0p_1^dp_2^u$	2	0.88	7.2	-1.4	-11.2	0.6	-0.8	3.2
$p_0p_1^dp_2^d$	2	0.76	5.4	-2.3	-15.4	2.7	0.4	10.4

Table 8.5: Example for SLS trading on a binomial tree with $p_1^u = p_2^u = 1.1$, $p_1^d = 0.8$, $p_2^d = 0.95$,$q_1 = q_2 = 0.5$, $I_0^* = 10$, and $K = 2$

path	t	p	I^L	g^L	I^S	g^S	V	B
p_0	0	1	10	0	-10	0	0	0
$p_0p_1^u$	1	1.2	14	2	-6	-2	0	-8
$p_0p_1^d$	1	0.9	8	-1	-12	1	0	4
$p_0p_1^up_2^u$	2	1.44	19.6	4.8	-3.6	-3.2	1.6	-14.4
$p_0p_1^up_2^d$	2	1.08	11.2	0.6	-7.2	-1.4	-0.8	-4.8
$p_0p_1^dp_2^u$	2	1.08	11.2	0.6	-7.2	-1.4	-0.8	-4.8
$p_0p_1^dp_2^d$	2	0.81	6.4	-1.8	-14.4	2.2	0.4	8.4

Table 8.6: Example for SLS trading on a binomial tree with $p_1^u = p_2^u = 1.2$, $p_1^d = p_2^d = 0.9$, $q_1 = q_2 = 0.5$, $I_0^* = 10$, and $K = 2$

Chapter 9

Performance of Feedback Trading Strategies

In this chapter, we analyze the performance of the simultaneously long short (SLS) feedback rule in more general market models than the models in the related literature. It is commonly accepted that continuously differentiable prices are rather unrealistic. However, there are also some critics concerning the use of the GBM. A first drawback of the GBM is that the trend is fixed. As seen in the works of Primbs and Barmish (2013, 2017) this can be generalized to time-varying GBMs. However, there are even more points of critics. Despite of the continuous time scale, which seems to be a little bit unrealistic, the biggest issue is that the paths of the GBM and even of the time-varying GBM are *a.s.* continuous. But on real markets there are jumps, e.g., caused by new information or by no trading times during the night or during the weekend. Thus, in a first step we use a (continuous time) model that allows for jumps, Merton's jump diffusion model (MJDM).

The whole chapter is closely following the work of the author of this thesis on the performance of feedback trading and especially of SLS trading (Baumann, 2017a,b; Baumann and Grüne, 2016, 2017). The proofs are provided here since they are originally done by the author of this work. However, the proofs can also be found in the respective papers.

9.1 Merton's Jump Diffusion Model

Neither continuously differentiable prices nor the GBM allow for price jumps. The literature shows that in the context of option pricing jumps have a high influence on hedging, namely, with jumps markets become incomplete (cf. Merton, 1973, 1976; Black and Scholes, 1973). Hence, we can ask the question what happens to the SLS strategy's robust positive expectation property (RPEP) if there are jumps into the model, or more precisely what happens to the SLS strategy's RPEP in MJDM. This section follows Baumann (2017b) and considers a stock price governed by Merton's jump diffusion model.

The relative price change in Merton's model is given through

$$\frac{dp(t)}{p(t)} = \tilde{\mu}dt + \sigma dW(t) + dN(t),$$

where $W(t)$ is a Wiener process, $N(t)$ is a Poisson-driven process with jump intensity $\lambda > 0$ and jumps $(Y_i - 1)$ i.i.d. with existing first moment (this is equivalent to $\mathbb{E}[Y_i - 1] < \infty$). Note that a Poisson process is a special Poisson-driven process with $Y_i - 1 \equiv 1$ and that generally $\int_0^t \mathrm{d}N_\tau$ is discontinuous. We assume $Y_i > 0$ for the reason of positive prices and define $\kappa := \mathbb{E}[Y_i - 1]$. The parameter $\sigma > 0$ denotes the volatility and $\tilde{\mu}$ denotes the trend. Note that we do not make any assumptions on the distribution of the random variables $Y_i > 0$. One possibility for the distribution of the Y_is, which we use only for the computer simulations, is $\ln Y_i \sim \mathcal{N}(\mu_{Y_i}, \sigma_{Y_i})$.

Summarizing, MJDM is somehow a generalization of the GBM. In this section, we use p for MJDM and b for the GBM

$$\frac{db(t)}{b(t)} = \tilde{\mu}dt + \sigma dW(t).$$

The distinction between b and p is made for simplicity of the formulae. We set

$$\tilde{\mu} := \mu - \lambda\kappa$$

where parameter $\mu > -1$ denotes the jumpless trend, i.e. the trend of MJDM without any jumps, which is also a GBM. That means, if there were no jumps, the trend $\tilde{\mu}$ is exactly the jumpless trend μ. If the jumps are positive in expectation ($\mathbb{E}[N(t) - N(s)] = \lambda\kappa(t - s)$, $t \geq s$; sometimes denoted by: $\mathbb{E}[dN(t)] = \lambda\kappa dt$) the trend has to be adjusted downwards ($-\lambda\kappa$) and vice versa (to ensure that μ is the trend). Furthermore, we set $p(0) = p_0 = b_0 = b(0)$ which implies $\mathbb{E}[p(t)] = \mathbb{E}[b(t)]$ for all t if $\mu = \tilde{\mu}$.

It can be shown, using Itō's lemma (in the general form containing jump processes, e.g., Poisson-driven processes) that the solution of MJDM is

$$p(t) = p_0 e^{\left(\tilde{\mu} - \frac{\sigma^2}{2}\right)t + \sigma W(t)} \prod_{i=1}^{N} Y_i = b(t) \prod_{i=1}^{N} Y_i,$$

where $\tilde{\mu} = \mu - \lambda\kappa$, $p_0 > 0$, $N \sim \mathcal{P}oi(\lambda t)$ and $W(t)$, Y_i, and N all are independently distributed (cf. Merton, 1971, 1976; Kushner, 1972; McKean jr., 1969). Merton's jump diffusion model is a stochastic process with a countably infinite number of jumps for $t \to \infty$. The time between two jumps is independently and identically exponentially distributed with parameter $\lambda > 0$. The number of jumps which occurred up to time t is Poisson distributed with parameter λt. Between every two jumps the process follows a GBM with jump-adjusted trend $\tilde{\mu}$.

This model is in continuous time and we allow for continuous time trading. At every point of time $t > 0$ the trader knows the own gain, the price, and is able to buy and sell stocks to adjust the investment. Although in real markets traders actually cannot trade continuously, in times of high-frequency-trading this assumption can be considered approximately satisfied. Note that there is indeed another approach of how to bring jumps in the model. In a discrete time model every price movement is a jump. However, there is a well-known theory on SDEs and we firstly rely on a widely accepted jump model in continuous time. Furthermore, the statistical characteristics of a discretized GBM in general differ strongly from those of a discretized Merton's jump diffusion model.

We deduce formulae for the gain/loss function and examine the expected gain and the standard deviation of the gain for MJDM analytically. This analysis starts analogously to the calculations of Barmish and Primbs (2016). However, due to the possibility of jumps our calculations are more lengthy and we have to use more results from the mathematical literature like the theorem of Fubini-Tonelli.

Now we derive a formula for the gain $g(t)$ of an SLS trader in Merton's jump diffusion model. Therefore, we set

$$a := \exp\left(\tfrac{(K-K^2)\sigma^2 t}{2}\right) \text{ and } c := \exp\left(\tfrac{-(K+K^2)\sigma^2 t}{2}\right).$$

Theorem 82. For the SLS trading strategy and a stock price following MJDM, it holds that

$$\begin{aligned} g(t) &= \tfrac{I_0^*}{K}\Bigg(\left(\tfrac{b(t)}{b_0}\right)^K a\prod_{i=1}^N(1+K(Y_i-1)) \\ &\quad+\left(\tfrac{b(t)}{b_0}\right)^{-K} c\prod_{i=1}^N(1-K(Y_i-1))-2\Bigg) \\ &= \tfrac{I_0^*}{K}\Bigg(\left(\tfrac{p(t)}{p_0}\right)^K a\prod_{i=1}^N(Y_i^{-K}+KY_i^{-K}(Y_i-1)) \\ &\quad+\left(\tfrac{p(t)}{p_0}\right)^{-K} c\prod_{i=1}^N(Y_i^{K}-KY_i^{K}(Y_i-1))-2\Bigg). \end{aligned}$$

Proof. We decompose $g(t) = g^L(t) + g^S(t)$ with $g^L(t)$ and $g^S(t)$ given by $g^\ell(t) = \int_0^t I^\ell(\tau-)\cdot\frac{dp(\tau)}{p(\tau-)}$ (and skipping the minus in "$(\tau-)$"). At first, the long side's gain is considered. The change of the gain is described by the SDE

$$\begin{aligned} dg^L(t) &= I^L(t)\cdot\tfrac{dp(t)}{p(t)} \\ &= \left(I_0^* + Kg^L(t)\right)((\mu-\lambda\kappa)dt+\sigma dW(t)+dN(t)). \end{aligned}$$

With $I^L(t) = I_0^* + Kg^L(t)$ it follows

$$dI_t^L = K((\mu-\lambda\kappa)dt+\sigma dW_t+dN_t)I_t^L$$

and

$$\tfrac{dI^L(t)}{I^L(t)} = (K\mu-\lambda K\kappa)dt + K\sigma dW(t) + KdN(t)$$

$$= (K\mu - \lambda K\kappa)dt + K\sigma dW(t) + d\tilde{N}(t).$$

We remark that $\tilde{N}(t)$ again is a Poisson-driven process with jump intensity $\lambda > 0$ but with jumps $(X_i^L - 1)$ with $X_i^L := 1 + K(Y_i - 1)$. It holds $\mathbb{E}[X_i^L - 1] = K\kappa$ and $I^L(0) = I_0^*$. Thus, it follows (cf. Merton, 1976)

$$I^L(t) = I^L(0)e^{\left(K\mu - \lambda K\kappa - \frac{K^2\sigma^2}{2}\right)t + K\sigma W(t)} \prod_{i=1}^N X_i^L.$$

The resubstitution of $I^L(t) = I_0^* + Kg^L(t) \;\Leftrightarrow\; g^L(t) = \frac{1}{K}(I^L(t) - I_0^*)$ leads to

$$\begin{aligned} g^L(t) &= \tfrac{I_0^*}{K}\left(e^{\left(K\tilde{\mu} - \frac{K\sigma^2}{2}\right)t + K\sigma W(t)} a\prod_{i=1}^N X_i^L - 1\right) \\ &= \tfrac{I_0^*}{K}\left(\left(\tfrac{b(t)}{b_0}\right)^K a\prod_{i=1}^N (1 + K(Y_i - 1)) - 1\right) \\ &= \tfrac{I_0^*}{K}\left(\left(\tfrac{p(t)}{p_0}\right)^K a\prod_{i=1}^N (Y_i^{-K} + KY_i^{-K}(Y_i - 1)) - 1\right). \end{aligned}$$

Now, let us consider $X_i^S := 1 - K(Y_i - 1)$ and note $\mathbb{E}\left[X_i^S - 1\right] = -K\kappa$. Substituting K and I_0^* by $-K$ and $-I_0^*$, respectively, leads to the short side's gain

$$\begin{aligned} g^S(t) &= \tfrac{I_0^*}{K}\left(\left(\tfrac{b(t)}{b_0}\right)^{-K} c\prod_{i=1}^N X_i^S - 1\right) \\ &= \tfrac{I_0^*}{K}\left(\left(\tfrac{p(t)}{p_0}\right)^{-K} c\prod_{i=1}^N (Y_i^K - KY_i^K(Y_i - 1)) - 1\right). \end{aligned}$$

Together, this finishes the proof. □

This is a first theoretical result concerning the performance of SLS trading in MJDM. The formula tells us that the gain does not depend on the diffusion part (the GBM part) of the price process. Only $b(t)$ at time t and the jumps $(Y_i)_{i=1,\dots,N}$ up to time t are of importance. Moreover, only a countable number of random variables is present since $(b(t))_t$ is not used but just $b(t)$. Next, we analyze what we can expect for the gain at arbitrary time t.

9.1.1 Expected Gain in MJDM

In this section, we focus on a result concerning the expected gain. We obtain:

Theorem 83. The expected gain of the SLS trading strategy with a stock price following MJDM is

$$\mathbb{E}[g(t)] = \tfrac{I_0^*}{K}\left(e^{K\mu t} + e^{-K\mu t} - 2\right).$$

In particular, for all $t > 0$, $\mu \neq 0$, the expected gain of the SLS trading strategy is positive, i.e.,

$$\mathbb{E}[g(t)] > 0,$$

for all $\lambda > 0$, $(Y_i)_{i\in\mathbb{N}} > 0$.

Proof. In order to calculate the expected gain $\mathbb{E}[g(t)]$ we consider Thm. 82. Remembering that $b(t)$, N, and $(Y_i)_i$ are all independent and $\ln b(t) \sim \mathcal{N}\left(\tilde{\mu} - \frac{1}{2}\sigma^2, \sigma^2\right)$ we can transform

$$\begin{aligned}
\mathbb{E}[g(t)] = \tfrac{I_0^*}{K}\Bigg(&\mathbb{E}\left[\left(\tfrac{b(t)}{b_0}\right)^K\right] a\mathbb{E}\left[\textstyle\prod_{i=1}^N (1 + K(Y_i - 1))\right] \\
&+ \mathbb{E}\left[\left(\tfrac{b(t)}{b_0}\right)^{-K}\right] c\mathbb{E}\left[\textstyle\prod_{i=1}^N (1 - K(Y_i - 1))\right] - 2\Bigg) \\
= \tfrac{I_0^*}{K}\Bigg(&e^{K\tilde{\mu}t}\mathbb{E}\left[\textstyle\prod_{i=1}^N (1 + K(Y_i - 1))\right] \\
&+ e^{-K\tilde{\mu}t}\mathbb{E}\left[\textstyle\prod_{i=1}^N (1 - K(Y_i - 1))\right] - 2\Bigg).
\end{aligned}$$

Here, we used that if Z is a random variable with $\ln Z \sim \mathcal{N}(\mu_Z, \sigma_Z^2)$ the identity $\mathbb{E}\left[Z^K\right] = e^{K\mu_Z + \frac{1}{2}K^2\sigma_Z^2}$ holds.

The next step makes use of the theorem of Fubini-Tonelli. We assume that Y_i is defined on Ω_{Y_i}, N is defined on Ω_N, $\Omega_Y := \Omega_{Y_1}\times\Omega_{Y_2}\times\Omega_{Y_3}\times\ldots$, and $Y := Y_1\otimes Y_2\otimes Y_3\otimes\ldots$. Since

$$\begin{aligned}
&\int_{\Omega_N}\int_{\Omega_Y} \textstyle\prod_{i=1}^N (1 + 2K + K(Y_i - 1))dP_Y dP_N \\
&= \textstyle\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \int_{\Omega_Y} \prod_{i=1}^n (1 + 2K + K(Y_i - 1))dP_Y \\
&= \textstyle\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t}\mathbb{E}[1 + 2K + K(Y_1 - 1)]^n \\
&= e^{-\lambda t}\textstyle\sum_{n=0}^{\infty} \frac{(\lambda t(1+2K+K\kappa))^n}{n!} = e^{\lambda K t(\kappa+2)} < \infty
\end{aligned}$$

we can apply Fubini-Tonelli for calculating the expected values, because it holds
$1 + 2K + K(Y_i - 1) \geq \max\{|1 + K(Y_i - 1)|, |1 - K(Y_i - 1)|\}$:

$$\begin{aligned}
&\int_{\Omega_N\times\Omega_Y} \textstyle\prod_{i=1}^N (1 + K(Y_i - 1))d(P_N \otimes P_Y) \\
&= \int_{\Omega_N}\int_{\Omega_Y} \textstyle\prod_{i=1}^N (1 + K(Y_i - 1))dP_Y dP_N \\
&= \textstyle\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \int_{\Omega_Y} \prod_{i=1}^n (1 + K(Y_i - 1))dP_Y = e^{\lambda t K\kappa}
\end{aligned}$$

and analogously

$$\int_{\Omega_N\times\Omega_Y} \textstyle\prod_{i=1}^N (1 - K(Y_i - 1))d(P_N \otimes P_Y) = e^{-\lambda t K\kappa}.$$

It follows that

$$\begin{aligned}\mathbb{E}[g(t)] &= \tfrac{I_0^*}{K}\left(e^{K\tilde{\mu}t}e^{\lambda tK\kappa} + e^{-K\tilde{\mu}t}e^{-\lambda tK\kappa} - 2\right)\\ &= \tfrac{I_0^*}{K}(e^{K\mu t} + e^{-K\mu t} - 2)\end{aligned}$$

for all $\lambda > 0, (Y_i)_{i\in\mathbb{N}} > 0$.

To show that the expected gain is positive if $\mu \neq 0$ and $t > 0$, we note $e^x + e^{-x} - 2 > 0$ for all $\mathbb{R} \ni x \neq 0$. □

The formulae in Thm. 83 do not depend on the jumps' specifications, i.e., they neither depend on the jumps' intensity λ nor on the jumps' distribution Y_i ($\mathbb{E}[Y_i - 1] < \infty$).

Note that $g(t)$ is the total profit while $dg(t)$ is the periodical profit. The probability that exactly k jumps occurred up to time t is $\mathcal{P}oi_{\lambda t}(k) = \frac{(\lambda t)^k}{k!}e^{-\lambda t} > 0$. Nonetheless, $\mathbb{E}[g(t)]$ does not depend on the jump statistics. This can be explained as follows: In Thm. 83 all jump parameters disappear since the term $-\lambda\kappa$ was added in MJDM for fixing the trend, which is defined through the expectation operator. The expected value of the price depends only on the time and on the trend. Thus, it is plausible that this trend adjustment also makes the expected trading gain independent of the jumps.

Another way of proving the theorem uses Thm. 77 as follows:

Proof. The proof is nearly the same as above except for the calculation of

$$\mathbb{E}\left[\prod_{i=1}^{N}(1 \pm K(Y_i - 1))\right].$$

It holds with Thm. 77 since all random variables are stochastically independent and $\mathbb{E}[1 + K(Y_i - 1)] = 1 + \kappa K$ that $\mathbb{E}\left[\prod_{i=1}^{N}(1 + K(Y_i - 1))\right] = \mathbb{E}\left[(1 + \kappa K)^N\right]$.

In general, this is not easy to compute. However, if X is Poisson distributed with parameter λ and r is a real number, we know that

$$\mathbb{E}\left[r^X\right] = \sum_{i=0}^{\infty} r^i \cdot \frac{\lambda^i}{i!}e^{-\lambda} = e^{\lambda(r-1)}.$$

This leads to $\mathbb{E}\left[\prod_{i=1}^{N}(1 + K(Y_i - 1))\right] = e^{\lambda t\kappa K}$ and analogously to
$\mathbb{E}\left[\prod_{i=1}^{N}(1 - K(Y_i - 1))\right] = e^{-\lambda t\kappa K}$. □

9.1.2 Variance in MJDM

After having studied the expected value we now look at the variance of the SLS trader's gain $\mathbb{V}[g(t)]$ or, equivalently, at the standard deviation $\mathbb{S}[g(t)]$.

Theorem 84. The standard deviation of the gain of the SLS trading strategy with a stock price governed by MJDM is

$$\begin{aligned}\mathbb{S}[g(t)] = \tfrac{I_0^*}{K}((&e^{2Kt\mu} + e^{-2Kt\mu})(e^{K^2t(\sigma^2+\lambda\zeta)} - 1)\\ &+ 2(e^{-K^2t(\sigma^2+\lambda\zeta)} - 1))^{\frac{1}{2}}\end{aligned}$$

if the second moment of the jumps' distribution exists. Here we abbreviated $\zeta := \mathbb{E}[(Y_i - 1)^2] < \infty$.

Proof. Analogous to the calculation of the expected value, it can be shown that

$$\int_{\Omega_N}\int_{\Omega_Y}\prod_{i=1}^{N}((1+4K+4K^2)+(2K+4K^2)(Y_i-1)+K^2(Y_i-1)^2)\mathrm{d}P_Y\mathrm{d}P_N < \infty.$$

This allows for using Fubini-Tonelli. Note:

$$\begin{aligned}&(1+4K+4K^2)+(2K+4K^2)(Y_i-1)+K^2(Y_i-1)^2\\ \geq \max\{&|1+2K(Y_i-1)+K^2(Y_i-1)^2|,\\ &|1-2K(Y_i-1)+K^2(Y_i-1)^2|,\\ &|1-K^2(Y_i-1)^2|\}\end{aligned}$$

It holds

$$\int_{\Omega_N\times\Omega_Y}\prod_{i=1}^{N}((1+2K(Y_i-1)+K^2(Y_i-1)^2))\mathrm{d}(P_N\otimes P_Y) = e^{\lambda t(2K\kappa+K^2\zeta)},$$

$$\int_{\Omega_N\times\Omega_Y}\prod_{i=1}^{N}((1-2K(Y_i-1)+K^2(Y_i-1)^2))\mathrm{d}(P_N\otimes P_Y) = e^{\lambda t(-2K\kappa+K^2\zeta)},$$

and

$$\int_{\Omega_N\times\Omega_Y}\prod_{i=1}^{N}((1-K^2(Y_i-1)^2))\mathrm{d}(P_N\otimes P_Y) = e^{-\lambda tK^2\zeta}.$$

A well-known transformation for the variance is $\mathbb{V}[g(t)] = \mathbb{E}[g(t)^2]-(\mathbb{E}[g(t)])^2$. According to this alternative representation, we first calculate the second moment of the gain:

$$\begin{aligned}\mathbb{E}[g(t)^2] &= \tfrac{I_0^{*2}}{K^2}\mathbb{E}\Bigg[\Bigg(\left(\tfrac{b(t)}{b_0}\right)^{K} a\textstyle\prod_{i=1}^{N}(1+K(Y_i-1))\\ &+\left(\tfrac{b(t)}{b_0}\right)^{-K} c\textstyle\prod_{i=1}^{N}(1-K(Y_i-1))-2\Bigg)^2\Bigg]\\ &= \tfrac{I_0^{*2}}{K^2}\Bigg(e^{2\tilde{\mu}tK+\sigma^2K^2t}\mathbb{E}\left[\textstyle\prod_{i=1}^{N}(1+2K(Y_i-1)+K^2(Y_i-1)^2)\right]\\ &+e^{-2\tilde{\mu}Kt+\sigma^2K^2t}\mathbb{E}\left[\textstyle\prod_{i=1}^{N}(1-2K(Y_i-1)+K^2(Y_i-1)^2)\right]\end{aligned}$$

$$
\begin{aligned}
&+ 4 - 4e^{K\mu t} - 4e^{-K\mu t} \\
&+ 2e^{-K^2\sigma^2 t}\mathbb{E}\left[\prod_{i=1}^{N}(1 - K^2(Y_i - 1)^2)\right]\Bigg) \\
= &\tfrac{I_0^{*2}}{K^2}\Bigg(e^{K^2t\left(\sigma^2+\lambda\zeta\right)}\left(e^{2Kt\mu} + e^{-2Kt\mu}\right) \\
&+ 2e^{-K^2t(\sigma^2+\lambda\zeta)} - 4\left(e^{K\mu t} + e^{-K\mu t} - 1\right)\Bigg)
\end{aligned}
$$

With this and the formula for the expected gain, it follows

$$
\begin{aligned}
\mathbb{V}[g(t)] = \tfrac{I_0^{*2}}{K^2}\Bigg(&e^{K^2t(\sigma^2+\lambda\zeta)}\left(e^{2Kt\mu} + e^{-2Kt\mu}\right) \\
&+ 2e^{-K^2t(\sigma^2+\lambda\zeta)} - \left(e^{2K\mu t} + e^{-2K\mu t} + 2\right)\Bigg)
\end{aligned}
$$

which implies the claimed formula for $\mathbb{S}[g(t)]$. □

Note that for using Fubini-Tonelli not only the first but also the second moment of Y_i must exist. One interesting observation is that the variance of $g(t)$ depends on the jump intensity and the second moment of the jumps, but not on their first moment, i.e. on the expected height of the jumps. We see that the variance is strictly increasing in λ and in ζ, i.e., if the jumps' intensity or the jumps' variance grows, the gain's variance grows, too. It seems plausible that the more jumps occur, the more volatile the gain is, as well as the higher the variance in the jumps height is, the higher the volatility in the gain becomes.

We end this section with a discussion of the variance of the gain for the jumpless case (i.e. geometric Brownian motion). There are two possibilities to obtain the jumpless geometric Brownian motion from Merton's jump diffusion model. First, we can set $\lambda = 0$, i.e., the probability that a jump occurs is zero, or, second, we can define $Y_i \sim \delta_1$ where δ_d is the Dirac distribution (degenerate distribution) with parameter d, i.e., jumps do not affect the price process (that means, the jump height is zero). In the second case it follows that $\zeta = 0$. Since in the equation for $\mathbb{V}[g(t)]$, λ and ζ only occur as product $\lambda\zeta$, we can set $\lambda\zeta = 0$ to derive the variance formula for the jumpless case.

9.1.3 Simulations and Plots

In Figs. 9.1 and 9.2, the dependencies of the expected gain and the standard deviation on the feedback parameter K and the jumpless trend μ are illustrated. In the graphs, one of the parameters K or μ, resp., varies while all other parameters are fixed.

We can see that $\mathbb{E}[g(t)]$ is increasing in K and in $|\mu|$. The standard deviation is increasing faster in K and $|\mu|$. From these findings we can conclude two facts. First, if a trader is searching for an optimal K there are two opposed arguments for choosing

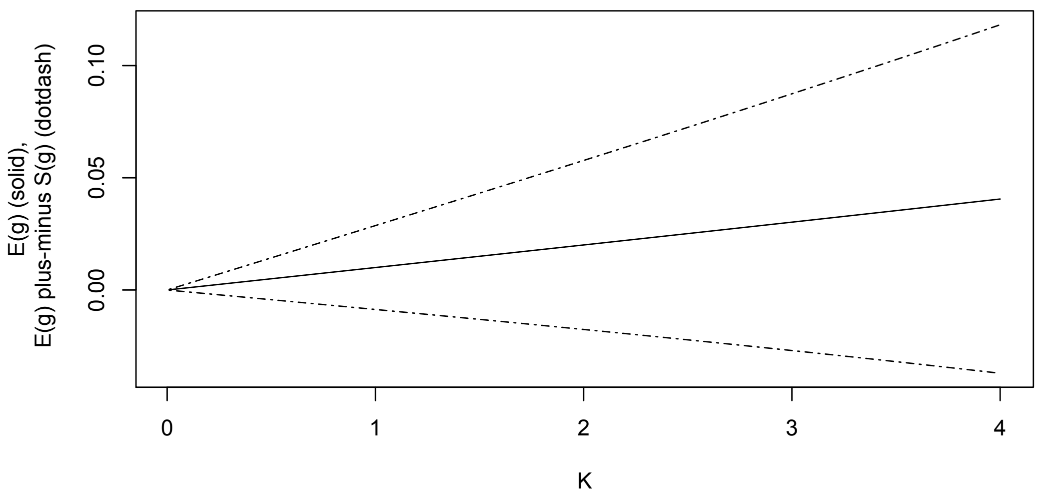

Figure 9.1: Dependency of the expected gain and of the standard deviation on K ($\in (0, 4]$). All other parameters respectively: $I_0^* = 1$, $\mu = 0.1$, $\sigma = 0.02$, $t = 1$, $\lambda = 2$, $\mu_{Y_i} = 0.02$, and $\sigma_{Y_i} = 0.05$

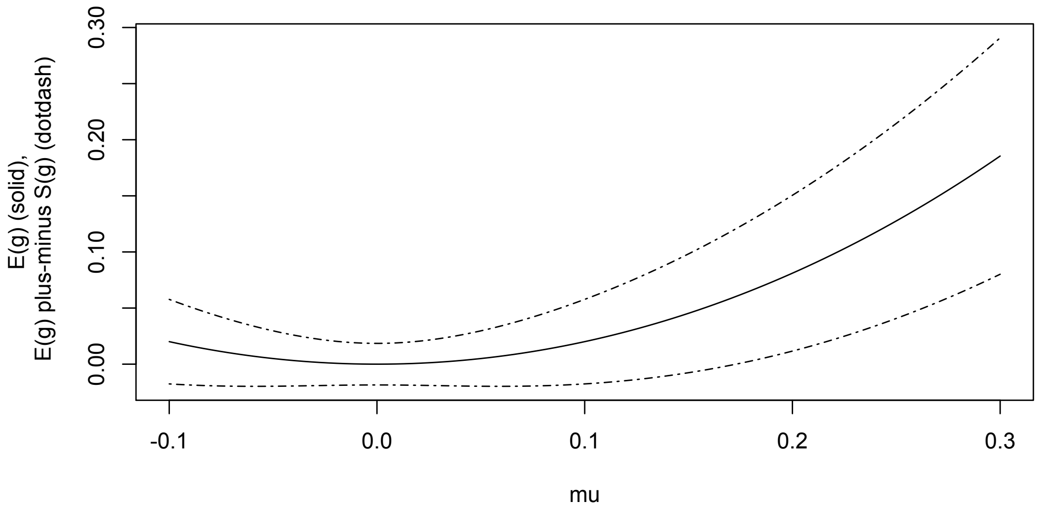

Figure 9.2: Dependency of the expected gain and of the standard deviation on μ ($\in [-10\%, 30\%]$). All other parameters: $I_0^* = 1$, $K = 2$, $\sigma = 0.02$, $t = 1$, $\lambda = 2$, $\mu_{Y_i} = 0.02$, and $\sigma_{Y_i} = 0.05$

K. A large K leads to higher expected gains but also to a higher variance of the gain and, thus, to higher risk. We conclude that the more risk-averse a trader is, the smaller the chosen K should be and vice versa (see Fig. 9.1). Second, we see that it does not matter whether $-1 < \mu < 0$ or $\mu > 0$, i.e., it does not matter if the trend is positive or negative. Hence, a trader does neither need to estimate the trend nor its sign since in all cases the SLS strategy leads to a positive expected gain (see Fig. 9.2). Furthermore, we notice that the standard deviation is increasing in λ and σ.

Note that for creating the graphs so far no stochastic process was simulated and no (pseudo) random number was generated. Thus, let us finally have a look at the histograms obtained from simulating the gains for 1,000 realizations of the underlying stochastic price process, shown in Figs. 9.3 and 9.4. These two histograms show two interesting facts. On the one hand, one can see that the trading results obtained by simulated real trading on stochastic processes with discrete time (g^T; Fig. 9.3) and the results calculated via the formula with the continuous time assumption (g^F; Fig. 9.4) do not differ very much. A slight difference is caused by the fact that the simulation is performed using a discretization in time while the formula assumes continuous trading. On the other hand, the histograms show that the gains are highly skewed (which is in line with Malekpour and Barmish (2012) and Malekpour and Barmish (2014a)), especially, the gains have a so-called fat tail to the positive side. That is, a negative gain with a small loss is more likely than a positive gain, but a high gain is more likely than a high loss. All in all, this leads to a positive mean. Note that $p(t)$, N, and Y_i are the same for both graphs. For all simulations the jumps are lognormally distributed, like it is recommended by Merton (1976) with $\ln Y_i \sim \mathcal{N}\left(\mu_{Y_i}, \sigma^2_{Y_i}\right)$.

At the end of this section, we show a path of a price process governed by MJDM and of SLS trading on this path. In Fig. 9.5 a price path is depicted and the jumps are marked with $\times$-signs. In Fig. 9.6 we see the price path of Fig. 9.5 and the investment paths of a linear long controller and of a linear short controller (with $I_0^* = 1$ and $K = 4$) as well as the SLS trading gain of these two linear controllers. The theoretical gain—calculated via Thm. 82—for the continuous time case is depicted with a circle and the expected gain with a triangle (Thm. 83).

9.2 Essentially Linearly Representable Prices

This section generalizes the RPEP of the SLS rule and follows Baumann and Grüne (2016) closely. For the trend of a GBM p it holds

$$\mathbb{E}\left[\frac{dp(t)}{p(t)}\right] = \mu dt$$

with $\mu > -1$. Note that this is again a purely formal notation. Furthermore, Barmish and Primbs (2011) show for this case that

$$\mathbb{E}[g(t)] = \frac{I_0^*}{K}\left(e^{K\mu t} + e^{-K\mu t} - 2\right),$$

which is positive for all $t > 0$ and $\mu \neq 0$, holds for SLS trading.

In Sec. 9.1, we have seen that the RPEP does not only hold for the GBM but also

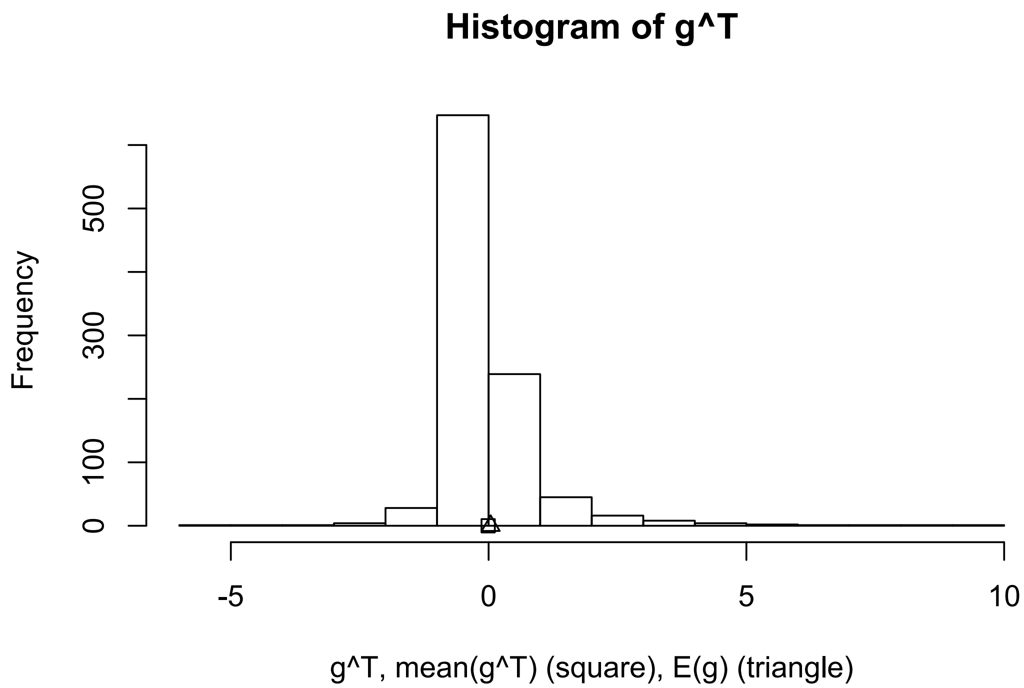

Figure 9.3: Histogram of the trading gain of 1,000 runs of SLS with a discretized MJDM with grid $\{0, 0.001, 0.002, \dots, 1\}$. The square marks the mean of the gains and the triangle the expected gain

for a discontinuous price model. More precisely, in this discontinuous price model prices follow MJDM, defined by the SDE

$$dp(t) = (\mu - \lambda\kappa)p(t)dt + \sigma p(t)dW(t) + p(t)dN(t),$$

where $N(t)$ is a Possion-driven process with jumps $Y_i - 1 > -1$, jump intensity λ, and an expected jump height κ. In MJDM, μ has the same functionality as the trend in the GBM ($\mathbb{E}\left[\frac{dp(t)}{p(t)}\right] = \mu dt$). It is shown that

$$\mathbb{E}[g(t)] = \frac{I_0^*}{K}\left(e^{K\mu t} + e^{-K\mu t} - 2\right) > 0$$

holds for all prices governed by MJDM, too. That means that the expected SLS trading gain neither depends on the jumps' intensity nor on their height or kind. Moreover, the expected gain is positive for all $\mu \neq 0$ and $t > 0$.

That means, the results for the GBM and for MJDM are the very same and depend

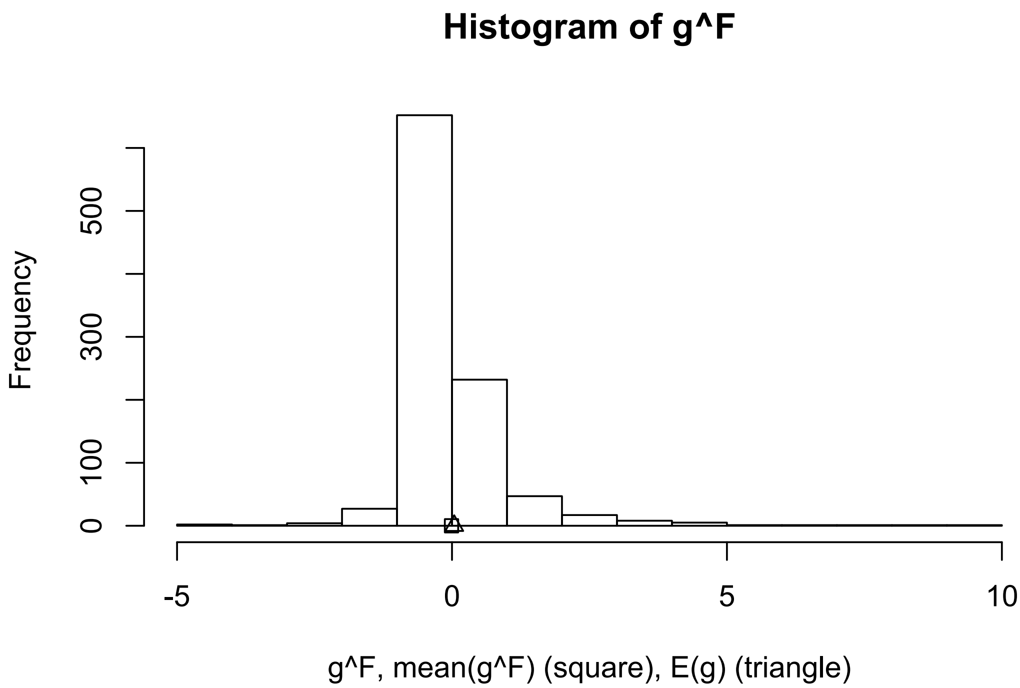

Figure 9.4: Histogram of the trading gain of 1,000 runs of SLS with MJDM in continuous time $[0, 1]$. The square marks the mean of the gains and the triangle the expected gain

only on I_0^* and K, which are the strategy's parameters, as well as on μ, which has the same functionality in both models. This leads us to the conjecture that a specific price model is not important to show the RPEP as long as the regularity for the trend holds.

Knowing this, we are now going to verify the robust positive expectation property for a rather large class of models including GBM and MJDM. We would like to mention that while this set contains the GBM and MJDM as special cases, our result does not make the literature addressing SLS trading for these price models obsolete because in these special cases there more (other) results are given. The models used in this section are still in continuous time, i.e., at every point of time $t \in [0, T]$, the trader has all information available up to t and adjusts the investment level $I(t)$.

To verify that the property of positive expected gain can be generalized from GBM and MJDM to a larger set of price models, a proper candidate set needs to be defined.

Definition 85. *We define the set of essentially linearly representable prices*

$$\mathcal{P} := \Big\{ p \,\Big|\, p \text{ is a solution of an SDE of the form:}$$

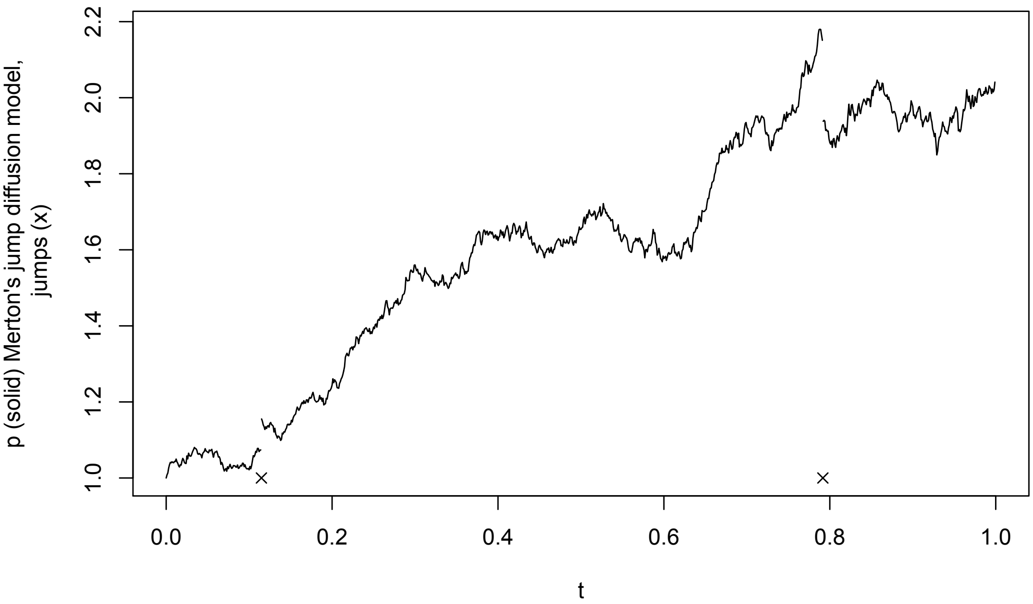

Figure 9.5: A simulation of a price process following Merton's jump diffusion model. The $\times$-signs mark the position of the jumps and, like for all simulations, the jumps are lognormally distributed. Parameters: $T = 1$, increment $\tau = 0.001$, $p_0 = 1$, $\mu = 0.1$, $\sigma = 0.2$, $\lambda = 5$, $\mu_{Y_i} = -0.1$, $\sigma_{Y_i} = 0.2$

$$dp(t) = \sum_{i=1}^{m} a_i p(t) dS^i(t) + \sum_{j=1}^{n} b_j(t, p(t)) dZ^j(t) \Big\},$$

with $a_i \in \mathbb{R}$, $S^i(t)$ stochastic processes (that are good integrators) with $\mathbb{E}[dS^i(t)] \equiv s_i dt$, with $s_i \in \mathbb{R}$, b_j $\mathcal{L}^1$-functions, and $Z^j(t)$ stochastic processes (that are good integrators) with $\mathbb{E}[dZ^j(t)] \equiv 0$. That means $\mathbb{E}[S^i(t) - S^i(u)] = s_i(t-u)$ and $\mathbb{E}[Z^j(t) - Z^j(u)] = 0$ for all $t \geq u$. All the processes $(S^i)_i, (Z^j)_j$ are assumed to be stochastically independent and $S^i(t)$ resp. $Z^j(t)$ are assumed to be stochastically independent of $S^i(t) - S^i(u)$ resp. $Z^j(t) - Z^j(u)$ for all $t > u \geq 0$. Moreover, we require that the parameters are chosen such that $(p(t))_t > 0$ a.s. and $(p(t))_t \geq 0$. The integrators and integrands have to be in a form s.t. the integrals exist (since p is the solution of an SDE it is obvious that a solution of the SDE representing p exists). Furthermore, we assume that this solution is unique. With $p \in \mathcal{P}$ we denote a specific price model, i.e. the prices given by one of the SDEs in $\mathcal{P}$ with fixed parameters.

It is important that the parameters are chosen in a way that $(p(t))_t > 0$ *a.s.* is garanteed. For instance, if $S^2(t)$ is a Poisson-driven process with lognormal minus one jumps, parameter a_2 has to be in $(0, 1]$. The set $\mathcal{P}$ is called the set of *essentially*

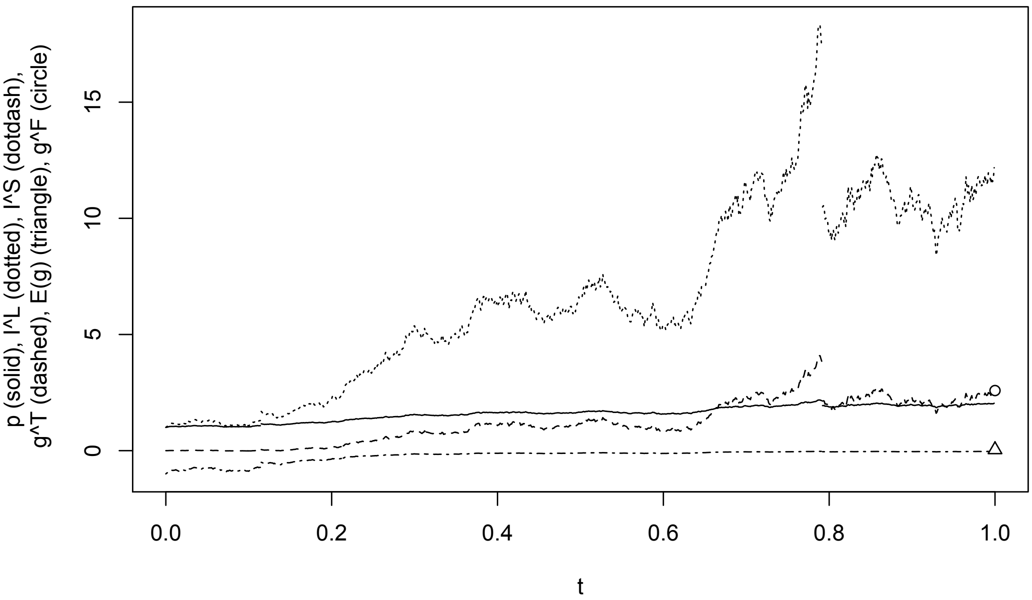

Figure 9.6: A simulation of SLS trading on MJDM. Price, investments of the long and the short side, and the SLS trading gain. The price path is exactly that one of Fig. 9.5. The trading parameters are $I_0^* = 1$ and $K = 4$

linearly representable prices because in the SDE representing p, all terms corresponding to processes with non-zero expectation, i.e. the essential ones, are linear in $p(t)$ and $\mathbb{E}[dS^i(t)] = const.dt$, i.e., one could call $S^i(t)$ expectedly linear. Note that $p(t)$ resp. $b^j(t, p(t))$ is stochastically independent of $S^i(t) - S^i(u)$ resp. $Z^j(t) - Z^j(u)$ for all $t > u \geq 0$ (and the very same is true for the SLS investment I instead of p).

For obtaining a GBM we set $m = 1$, $a_1 = \mu$, $S^1(t) = t$, $n = 1$, $b_1(t, p(t)) = \sigma p(t)$, and $Z^1(t) = W(t)$ (with W being a Wiener process). For the MJDM we have to set additionally resp. change $m = 2$, $a_1 = \mu - \lambda\kappa$, $a_2 = 1$, and $S(t)^2 = N(t)$. We also have $\mathbb{E}[dt] = 1dt$, $\mathbb{E}[dW(t)] = 0$, and $\mathbb{E}[dN(t)] = \lambda\kappa dt$.

Calculating the expected value $\mathbb{E}[p_t]$ for a price model $p \in \mathcal{P}$ is rather uncomplicated when using the SDE representing p and the stochastic independencies assumed above. We apply the expectation operator on both sides of the SDE, use Thm. 75, and get

$$d\mathbb{E}[p_t] = \sum_{i=1}^{m} a_i \mathbb{E}[p_t] s_i dt.$$

It follows that

$$\mathbb{E}[p_t] = p_0 e^{t \sum_{i=1}^{m} a_i s_i}.$$

We define
$$\sum_{i=1}^{m} a_i s_i =: \mu,$$
which again is the trend of the price process.

We remark that from this identity and the fact that $\mathbb{E}[dN_t] = \lambda\kappa dt$ in MJDM it becomes obvious why the term $-\lambda\kappa$ in the specification $a_1 = \mu - \lambda\kappa$ in MJDM is needed. Next, we derive a formula for the expected gain that holds for all $p \in \mathcal{P}$ and see that this expectation value is non-negative and moreover positive for a non-zero trend. In the following, it is shown that the expected gain of an SLS trader does not depend on a specific price model out of set $\mathcal{P}$.

Theorem 86. Given $\mu > -1$, for all price models $p \in \mathcal{P}$ satisfying $\sum_{i=1}^{m} a_i s_i = \mu$ the formula
$$\mathbb{E}[g_t] = \frac{I_0^*}{K}\left(e^{K\mu t} + e^{-K\mu t} - 2\right)$$
holds, implying $\mathbb{E}[g_t] > 0$ if $\mu \neq 0$ and $t > 0$.

Proof. Let $p \in \mathcal{P}$. It follows
$$dp_t = \sum_{i=1}^{m} a_i p_t dS_t^i + \sum_{j=1}^{n} b_j(t, p_t) dZ_t^j$$
and by means of $dI_t^L = K \cdot \frac{I_t^L}{p_t} dp_t$ it holds
$$dI_t^L = \sum_{i=1}^{m} K a_i I_t^L dS_t^i + \sum_{j=1}^{n} K \cdot \frac{I_t^L}{p_t} b_j(t, p_t) dZ_t^j.$$
Using the expectation operator together with the assumptions on the coefficients of the SDEs in $\mathcal{P}$ and Thm. 75 leads to
$$d\mathbb{E}[I_t^L] = \sum_{i=1}^{m} K a_i \mathbb{E}[I_t^L] s_i dt$$
and, thus,
$$\mathbb{E}[I_t^L] = I_0^* e^{Kt\sum_{i=1}^{m} a_i s_i} = I_0^* e^{K\mu t}.$$

Analogously, by substituting $K \mapsto -K$ and $I_0^* \mapsto -I_0^*$ we obtain for the linear short trader:
$$\mathbb{E}[I_t^S] = -I_0^* e^{-K\mu t}$$
It follows that
$$\mathbb{E}[g_t^L] = \frac{I_0^*}{K}(e^{K\mu t} - 1)$$
and
$$\mathbb{E}[g_t^S] = \frac{I_0^*}{K}(e^{-K\mu t} - 1).$$

Combining this leads to:

$$\mathbb{E}[g_t] = \frac{I_0^*}{K}(e^{K\mu t} + e^{-K\mu t} - 2)$$

Note that this formula for the SLS trader's gain holds for all $p \in \mathcal{P}$ satisfying $\sum_{i=1}^{m} a_i s_i = \mu$ and does not depend on the specific price model. The inequality

$$\mathbb{E}[g_t] > 0$$

directly follows for all $\mu \neq 0$ and $t > 0$. □

To illustrate the statement of Thm. 86 we consider an arbitrary $p \in \mathcal{P}$, e.g.,

$$dp_t = (\mu + \zeta)p_t dt + a p_t dN_t + \sigma\sqrt{p_t} dW_t$$

with μ being the risk-free interest rate, ζ a parameter making p satisfying the condition on μ, $a \in (0, 1], \sigma > 0$, W_t a Wiener processes, and N_t a Poisson-driven process with intensity λ and jumps $(Y_i - 1)$. For the jump distribution we assume $Y_i \sim Exp(\lambda_{Y_i})$. We define $\kappa := \mathbb{E}[Y - 1] = \frac{1}{\lambda_Y} - 1$. It holds that $\mathbb{E}[p_t] = p_0 e^{(\mu+\zeta+a\lambda\kappa)t}$. All parameters have to satisfy $p(t) > 0$ $a.s.$ Thus, we set $\zeta := -a\lambda\kappa$. The expected SLS trading gain is given through Thm. 86. Although our example market model is an unusual extension of the MJDM with a square root in the diffusion part and an a in the jump part, it falls into the class $\mathcal{P}$ for which Thm. 86 is valid. Thus, we do not have to solve the SDE to derive the expected value of the SLS trading strategy. Instead, we can apply Thm. 86.

9.3 Constant Trend Model in Discrete Time

So far, we analyzed the SLS rule in price models that were given through SDEs, either specific ones like the GBM or MJDM or for a whole set of SDEs. Now, we go a step further and do no longer rely on specific price models but instead define general requirements for models. We see that this is enough to show the robust positive expectation property. Furthermore, we see that our results do not depend on any model but only on the trend. At first, this analysis is done in discrete time, then we move on to sampled-data systems and continuous time. The main assumption in this section is that we have a constant trend. In Sec. 9.4, we relax this assumption. The GBM and MJDM are special cases for the general requirements of this section when calculating limits for continuous time. Before analyzing the SLS strategy in this setting, we have to specify the price processes of interest and the time grid on which we define the price processes.

We assume discrete time trading, i.e., at every point of time $t \in \mathcal{T} = \{0, h, 2h, \dots, T\}$ with $T = Nh$ and $h > 0$, the trader has all information available up to t and adjusts the investment I_t.

Definition 87. *Given $h > 0$ and $\mathcal{T}$ from above, the price processes of interest have the following properties:*

- *Stochastic Prices: the price process $(p_t)_{t\in\mathcal{T}}$ is a stochastic process*

- *Positive Prices: the price p_t is positive for all $t \in \mathcal{T}$*
- *Fixed Start Price: The start price $p_0 \in \mathbb{R}^+$ is deterministic*
- *Independent Multiplicative Growth: for all $k \in \mathbb{N}$ and all $t_0 < t_1 < \ldots < t_k \in \mathcal{T}$ it holds:*
$$p_{t_0}, \frac{p_{t_1}}{p_{t_0}}, \frac{p_{t_2}}{p_{t_1}}, \ldots, \frac{p_{t_k}}{p_{t_{k-1}}} \text{ are stochastically independent}$$
- *Constant Trend: the expected relative return is constant, i.e., there is $\mu_h > -1$ such that for all $t \in \mathcal{T} \setminus \{0\}$ it holds:*
$$\mathbb{E}\left[\frac{1}{p_{t-h}} \cdot \frac{p_t - p_{t-h}}{h}\right] = \mu_h$$
Note that this assumption is inspired by "$\mathbb{E}\left[\frac{dp(t)}{p(t)}\right] = \mu dt$" and that it is equivalent to:
$$\mathbb{E}\left[\frac{p_t}{p_{t-h}}\right] = \mu_h h + 1$$

If $h > 0$ is not fixed but considered to be a parameter of the trader (determined by the trading frequency), this appears to be a contradiction to the definition of μ_h since the relative return may then depend on the trading frequency. Section 9.3.2 shows why this is not a contradiction.

Theorem 88. For $t = nh$, a price process fulfilling Def. 87 has the expected value
$$\mathbb{E}[p_t] = p_0 \cdot z\left(\mu_h, \tfrac{1}{h}\right)^t,$$
with $z(x, m) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ given by $\quad z(x, m) \mapsto \left(1 + \dfrac{x}{m}\right)^m.$

Proof. This can be proven by calculation using Def. 87:
$$\begin{aligned}
\mathbb{E}[p_t] &= \mathbb{E}\left[p_0 \cdot \frac{p_h}{p_0} \cdot \frac{p_{2h}}{p_h} \cdots \frac{p_t}{p_{(n-1)h}}\right] \\
&= p_0 \cdot \prod_{i=1}^{n} \mathbb{E}\left[\frac{p_{ih}}{p_{(i-1)h}}\right] \\
&= p_0 \cdot (\mu_h h + 1)^n \;=\; p_0 \cdot \left((\mu_h h + 1)^{\frac{1}{h}}\right)^t
\end{aligned}$$
Now the definition of the function z proves the theorem. □

When defining $(\mathcal{F}_t)_{t \in \mathcal{T}}$ as the family of σ algebras containing the available information, with a very similar proof one can show that it holds:
$$\mathbb{E}\left[p_{t_2} | \mathcal{F}_{t_1}\right] = p_{t_1} \cdot \left((\mu_h h + 1)^{\frac{1}{h}}\right)^{t_2 - t_1} = p_{t_1} \cdot z\left(\mu_h, \tfrac{1}{h}\right)^{t_2 - t_1}$$

The next question that may arise is which processes fulfill Def. 87. Theorem 89 gives us one possibility to construct such processes.

Theorem 89. Let $(X_t)_{t\in\mathcal{T}} \subset \mathbb{R}$ be a Lévy process. Especially, this stochastic process has the following properties:

- Independent Growth: for all $k \in \mathbb{N}$ and all $t_0 < t_1 < \ldots < t_k \in \mathcal{T}$ it holds:

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}} \text{ are stochastically independent}$$

- Identically Distributed Growth: for all $t_1, t_2, t_3, t_4 \in \mathcal{T}$ with $t_2 - t_1 = t_4 - t_3$ it holds:

$$X_{t_2} - X_{t_1} \sim X_{t_4} - X_{t_3}$$

- Start at zero: $X_0 = 0$

Then for every $p_0 \in \mathbb{R}^+$ it holds that

$$p_t := p_0 \cdot e^{X_t} \; \forall t \in \mathcal{T}$$

fulfills Def. 87.

Proof. Obviously, p_t is a stochastic process which is positive and has a fixed start price. The independent multiplicative growth of p_t follows from the independent growth of X_t and of $X_0 = 0$. It remains to prove the constant trend: From the identically distributed growth it follows $X_{t_1} - X_{t_1-h} \sim X_{t_2} - X_{t_2-h}$ and thus $\frac{e^{X_{t_1}}}{e^{X_{t_1-h}}} \sim \frac{e^{X_{t_2}}}{e^{X_{t_2-h}}}$. Particularly, $\mathbb{E}\left[\frac{e^{X_{t_1}}}{e^{X_{t_1-h}}}\right] = \mathbb{E}\left[\frac{e^{X_{t_2}}}{e^{X_{t_2-h}}}\right]$ holds for all $t_1, t_2 \in \mathcal{T}$. This shows that $\mu_h := \left(\mathbb{E}\left[\frac{e^{X_{t_1}}}{e^{X_{t_1-h}}}\right] - 1\right) h^{-1}$ is well-defined. □

9.3.1 The Robust Positive Expectation Property

Now, after having understood the price dynamics we analyze the SLS trading strategy's performance in such a market. At first, we have a look at the linear long trader:

$$I_t^L = I_0^* + K g_t^L$$

We recall that

$$g_t^L = \sum_{\tau\in\{h,2h,\ldots,nh\}} I_{\tau-h}^L \cdot \frac{p_\tau - p_{\tau-h}}{p_{\tau-h}}.$$

So it holds:

$$I_t^L - I_{t-h}^L = K \cdot (g_t^L - g_{t-h}^L) = K \cdot I_{t-h}^L \cdot \frac{p_t - p_{t-h}}{p_{t-h}},$$

$$\frac{I_t^L - I_{t-h}^L}{h \cdot I_{t-h}^L} = K \cdot \frac{p_t - p_{t-h}}{h \cdot p_{t-h}},$$

and

$$\frac{I_t}{I_{t-h}} = K \cdot \left(\frac{p_t}{p_{t-h}} - 1\right) + 1$$

This directly leads to

$$\mathbb{E}\left[\frac{I_t^L - I_{t-h}^L}{h \cdot I_{t-h}^L}\right] = K\mu_h$$

and with an analogous proof to that one of Thm. 88 to Thm. 90.

Theorem 90. For the investment of a linear long trader it holds:

$$\mathbb{E}\left[I_t^L\right] = I_0^* \cdot z\left(K\mu_h, \tfrac{1}{h}\right)^t$$

□

From the closed form formula for the expected investment of the linear long trader we derive a similar formula for the expected gain of the linear long trader when using the definition of the linear long feedback rule:

$$\mathbb{E}\left[g_t^L\right] = \frac{I_0^*}{K} \cdot \left(z\left(K\mu_h, \tfrac{1}{h}\right)^t - 1\right)$$

By substituting $I_0^* \mapsto -I_0^*$ and $K \mapsto -K$ we get for the short side's investment and gain:

$$\mathbb{E}\left[I_t^S\right] = -I_0^* \cdot z\left(-K\mu_h, \tfrac{1}{h}\right)^t$$

and

$$\mathbb{E}\left[g_t^S\right] = \frac{I_0^*}{K} \cdot \left(z\left(-K\mu_h, \tfrac{1}{h}\right)^t - 1\right)$$

Recalling that $g_t = g_t^L + g_t^S$, we obtain Thm. 91.

Theorem 91. The expected gain of the SLS feedback trading strategy for prices defined by Def. 87 is:

$$\mathbb{E}[g_t] = \frac{I_0^*}{K} \cdot \left(z\left(K\mu_h, \tfrac{1}{h}\right)^t + z\left(-K\mu_h, \tfrac{1}{h}\right)^t - 2\right)$$

□

Next, we show that the expected gain is positive for all $\mathcal{T} \ni t > h$.

Theorem 92. The expected gain of the SLS feedback trading strategy is non-negative and is zero if and only if $t = 0$ or $t = h$.

Proof. We calculate:

$$\mathbb{E}[g_0] = 0$$

and

$$\mathbb{E}[g_h] = \frac{I_0^*}{K} \cdot ((1 + K\mu_h h) + (1 - K\mu_h h) - 2) = 0$$

For $t = nh$ with $n \geq 2$ the proof becomes a little more tricky:

$$\begin{aligned}
\mathbb{E}[g_t] &= \frac{I_0^*}{K} ((1 + K\mu_h h)^n + (1 - K\mu_h h)^n - 2) \\
&= \frac{I_0^*}{K} \left(\left(\sum_{i=0}^{n} \binom{n}{i} \cdot (K\mu_h h)^i \right) + \left(\sum_{i=0}^{n} \binom{n}{i} \cdot (-K\mu_h h)^i \right) - 2 \right) \\
&= \frac{2I_0^*}{K} \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} \cdot \left((K\mu_h h)^i\right)^2 > 0,
\end{aligned}$$

which shows the claim. □

9.3.2 A Sampled-Data System: Discrete Time Trading of Continuous Time Price Processes

In practice, the price of a stock is not only defined at the discrete trading times $t \in \mathcal{T}$, which are chosen by the trader. Ideally, one would model $p(t)$ as a continuous time price process, which is defined for all $t \in \mathbb{R}_0^+$. We recall that in order to distinguish the continuous time from the discrete time case, we write the time argument in brackets for continuous time processes, i.e. $p(t)$ instead of p_t. In a control theoretic notion, the discrete time controller derived in the last section is implemented as a sampled-data controller with sampling time $h > 0$. Hence, the sampling time $h > 0$ becomes a parameter of the trader and that appears to be a conflict between the fact that the trend μ_h depends on the trading frequency via h while on the other hand Chap. 7 demands the price taker property, i.e., that the trader is not able to influence the price.

In the following analysis we show that this contradiction can be resolved by assuming the price taker property for the continuous time returns rather than for the discrete time returns. To this end, we show that Def. 87 and the price taker property (see Chap. 7) can be met if we consider a constant trend μ for the continuous price model that is not influenced by the trader and a trader who trades on a discrete time grid with parameter $h > 0$, where h and μ are independent. For all $t_2 > t_1 \geq 0$ we assume:

$$\mathbb{E}\left[p(t_2)|\mathcal{F}_{t_1}\right] = p(t_1) \cdot e^{\mu(t_2 - t_1)},$$

which corresponds to Def. 87. This property is true, e.g., for the geometric Brownian motion and for Merton's jump diffusion model. It implies:

$$\mathbb{E}\left[p(t)\right] = p_0 \cdot e^{\mu t}$$

and

$$\mathbb{E}\left[\frac{p(t)}{p(t-h)}\middle|\mathcal{F}_{t-h}\right] = e^{\mu h} \ \ \forall h > 0,\, t \geq h$$

Since $e^{\mu h}$ is deterministic and thus independent of the realization of $p(t-h)$ it follows:

$$\mathbb{E}\left[\frac{p(t)}{p(t-h)}\right] = e^{\mu h} \ \ \forall h > 0,\, t \geq h$$

and thus

$$\mathbb{E}\left[\frac{p(t)-p(t-h)}{h \cdot p(t-h)}\right] = \frac{e^{\mu h}-1}{h} =: \mu_h$$

Hence, for all $h > 0$ we can appropriately choose μ_h. We note that with L'Hôspital's rule it is easily verified that $\mu_h \to \mu$ for $h \to 0$. Moreover, we can see that $0 < h$ and $\mu > -1$ implies $\mu_h > -1$.

From Thm. 92 it thus follows that for a continuous time process satisfying the first four properties of Def. 87 and $\mathbb{E}\left[p(t_2)|\mathcal{F}_{t_1}\right] = p(t_1) \cdot e^{\mu(t_2-t_1)}$ with $\mu > -1$, the discrete time SLS trading strategy with $0 < h$ yields positive expected gain $\mathbb{E}[g_t] > 0$ whenever $t \geq 2h$. We emphasize that this means that the important qualitative property, i.e. positive expected gain for *a.a.* parameters, holds independently of the length $h > 0$ of the sampling interval. This is in contrast to, e.g., stabilizing controllers, for which it is known that asymptotic stability of the closed loop may be lost if the sampling time is chosen too large (Nešić et al., 1999, 2009; Chen and Francis, 1991).

9.3.3 Limits for Continuous Time Trading

We end Sec. 9.3 by analyzing what happens if the trading frequency tends to infinity, i.e., if the time $h > 0$ between two trading times tends to 0. Clearly, this question only makes sense if $p(t)$ is a continuous time process, as in the previous section. As mentioned before, in order to obtain a meaningful limit we have to make sure that the stochastic Itō integral with respect to $dp(t)$ exists.

As in the previous section we assume

$$\mathbb{E}\left[p(t_2)|\mathcal{F}_{t_1}\right] = p_0 \cdot e^{\mu(t_2-t_1)}.$$

It directly follows:

$$\mathbb{E}\left[p(t)\right] = p_0 \cdot e^{\mu t}$$

Now, Thm. 92 can be applied. All results and definitions obtained so far can be transformed into similar results for continuous time trading when using

$$\lim_{m\to\infty} z(x,m) = e^x.$$

Considering the formula for the gain with $t_i = ih$, $n = t/h$ and letting $h \to 0$ we obtain:

$$g^\ell(t) = \int_0^t \frac{I^\ell(\tau)}{p(\tau)} dp(\tau)$$

$$\mathbb{E}\left[I^L(t)\right] = I_0^* \cdot e^{K\mu t},$$

$$\mathbb{E}\left[I^S(t)\right] = -I_0^* \cdot e^{-K\mu t},$$

$$\mathbb{E}\left[g^L(t)\right] = \frac{I_0^*}{K}\left(e^{K\mu t} - 1\right),$$

$$\mathbb{E}\left[g^S(t)\right] = \frac{I_0^*}{K}\left(e^{-K\mu t} - 1\right),$$

and last but not least

$$\mathbb{E}\left[g(t)\right] = \frac{I_0^*}{K}\left(e^{K\mu t} + e^{-K\mu t} - 2\right) > 0,$$

which is the desired formula for the expected gain $\mathbb{E}[g(t)]$.

When using the common and purely formal notation of stochastic differential equations, it holds

$$\mathbb{E}\left[\frac{dp(t)}{p(t)}\right] = \mu dt,$$

$$\frac{dI^L(t)}{I^L(t)} = K \cdot \frac{dp(t)}{p(t)},$$

$$\mathbb{E}\left[\frac{dI^L(t)}{I^L(t)}\right] = K\mu dt,$$

and

$$\mathbb{E}\left[\frac{dI^S(t)}{I^S(t)}\right] = -K\mu dt.$$

The conditions used here are exactly the same as in the continuous time setting in Barmish and Primbs (2011, 2016), which ensure the robust positive expectation property. Hence, in the limit for $h \to 0$, we recover the known results from the continuous time literature, but for a much more general class of price processes.

Actually, the continuous time limit of the expected gain is the same as given by Barmish and Primbs (2011, 2016). Thus, in the limit, our results are consistent with those for the continuous time SLS strategy analyzed in these references. The formula for the expected gain (Equation (4.16)) in Thm. 4.3 in the work of Dokuchaev (2012) is structurally similar to our expected gain's formula, which can be written as

$$\mathbb{E}[g(t)] = \frac{2I_0^*}{K}(\cosh(K\mu t) - 1) > 0.$$

The main difference of our results to these references is that our market model is much more general because the analysis in Barmish and Primbs (2011, 2016); Dokuchaev and Savkin (1998a,b, 2002, 2004); Dokuchaev (2012) are limited to stock prices governed by geometric Brownian motions.

9.4 Time-varying Trend Model in Discrete Time

The main feature of control-based trading strategies is that although market parameters like the expected return on investment are used when analyzing the strategies, the trader does neither have to know nor to estimate them for trading. Properties of the strategies hold for *a.a.* settings of the parameter values (given a measure on the parameter space that is absolutely continuous to the Lebesgue measure on this space). The subsequent analysis follows Baumann and Grüne (2017) but takes into account the ideas of Malekpour and Barmish (2016), who state that investment decisions should not rely (too much) on market behavior long ago, Primbs and Barmish (2013, 2017), who consider time-varying trends and volatilities, and Barmish and Primbs (2011, 2016); Baumann (2017b), who calculate expected gains and variances.

After having discussed market efficiency and control-based trading strategies, especially SLS trading, we construct a new, more general type of an SLS rule, which combines all the ideas above: the discounted SLS rule. The discounted SLS strategy is a class of trading rules that contains the standard SLS rule. The construction process as well as the analysis is based on refinements of the underlying time grids: Starting with discrete time price processes and thus discrete time trading, we end with continuous prices and continuous trading. The standard SLS rule is generalized by a discounting factor δ, the price process allows for time-varying parameters, and the analysis takes risk-adjusted returns into account. The mathematically proven results—either concerning all discounted SLS rules (including the standard rule) or only the standard SLS strategy—build a puzzle to market efficiency. Note that the construction of the discounted SLS rule is just the implementation of an idea, probably relevant for practical considerations. The generalization to time-varying trends (and volatilities) is the main feature of this section because there are new theoretical findings.

9.4.1 The Robust Positive Expectation Property

A controller with delay as presented by Malekpour and Barmish (2016) has the feature that too old (older than m days) events do not have any influence on the strategy, but it has a questionable feature, too: An event that is m days old is taken fully into account today but vanishes from the calculations tomorrow (i.e. after $m+1$ days). As an alternative controller type, we introduce the discounted SLS controller with discounting factor $\delta \in (0,1]$ (SLS$_\delta$). The main, and indeed the only, difference of a discounted rule to the standard rule $\ell \in L, S$ is that instead of the gain g_t^ℓ a discounted gain

$$f_t^{\ell\delta} = \sum_{i=1}^{n} I_{(i-1)h}^{\ell\delta} \cdot \frac{p_{ih} - p_{(i-1)h}}{p_{(i-1)h}} \cdot \delta^{-(i-1)h}$$

on a discrete time grid $\{0, h, 2h, \ldots\}$ with $h > 0$ and $t = nh$ is used. That means, the discounted SLS rule is

$$I_t^{SLS_\delta} = I_t^{L_\delta} + I_t^{S_\delta}$$

with

$$I_t^{L_\delta} = I_0^* + K f_t^{L_\delta}$$

and

$$I_t^{S_\delta} = -I_0^* - K f_t^{S_\delta}.$$

A flow diagram for the discounted SLS rule is given in Fig. 9.7. Note that for $\delta = 1$ this strategy is exactly the standard SLS strategy. The discounting factor δ specifies to which extent past information is used for calculating the current investment (cf. other economic discounting factors like, e.g., the game theoretic discounting factor in repeated games). Actually, in contrast to other examples we do not rate old information lower, but we rate new information higher. The higher δ is, the more influence past information has; for $\delta = 1$ all available information is equally weighted, for δ close to zero only the last available information is important. The discounted SLS strategy has, similar to the SLS strategy with delay, the advantage that (if $\delta < 1$) old information is not as important as new one. However, in contrast to the delay strategy the old information loses its weight gradually and not instantaneously. Note that if $\delta = 1$, the discounted SLS strategy is exactly the standard SLS rule and, thus, all results for SLS_δ also hold for the standard SLS strategy, which is SLS_1.

After having introduced the discounted SLS rule, we come to the basic novelty of this section: Different to the work in Sec. 9.3 is that we allow for a time-varying trend now:

$$\mathbb{E}\left[\frac{p_t - p_{t-h}}{h \cdot p_{t-h}}\right] =: \mu_{h;t-h}$$

(For the reason of non-negative prices, $\frac{p_t - p_{t-h}}{h \cdot p_{t-h}} \geq -1$ and $\mu_{h;t-h} > -1$ has to hold for all t and h.) This generalization is similar to that one done by Primbs and Barmish (2013, 2017) when extending the results for standard GBMs. Analogously to Baumann and Grüne (2017) and Sec. 9.3, we also assume positive, stochastic prices $(p_t)_{t \in \mathcal{T}} > 0$ ($\mathcal{T} = \{0, h, 2h, \ldots, T\}$, $T = Nh$, $t = nh$), $p_0 \in \mathbb{R}^+$, and independent multiplicative growth, i.e., for all $k \in \mathbb{N}$ and all $t_0 < t_1 < \ldots < t_k \in \mathcal{T}$ it holds that

$$p_{t_0}, \frac{p_{t_1}}{p_{t_0}}, \ldots, \frac{p_{t_k}}{p_{t_{k-1}}}$$

are stochastically independent. This is the *weak form of the market efficiency hypothesis* (cf. Chap. 2). Note that this stochastic independence holds when applying any measurable function on the growth rates, too. Again, there seems to be a contradiction to the price taker property: While on the one side h is chosen by the trader, on the other side the trend $\mu_{h;t}$ depends on h. But, as shown in Sec. 9.3, this problem can easily be solved either by use of so-called sampled-data systems or by calculating the limits for $h \to 0$.

In the following, we show that the positive robust expectation property does not hold

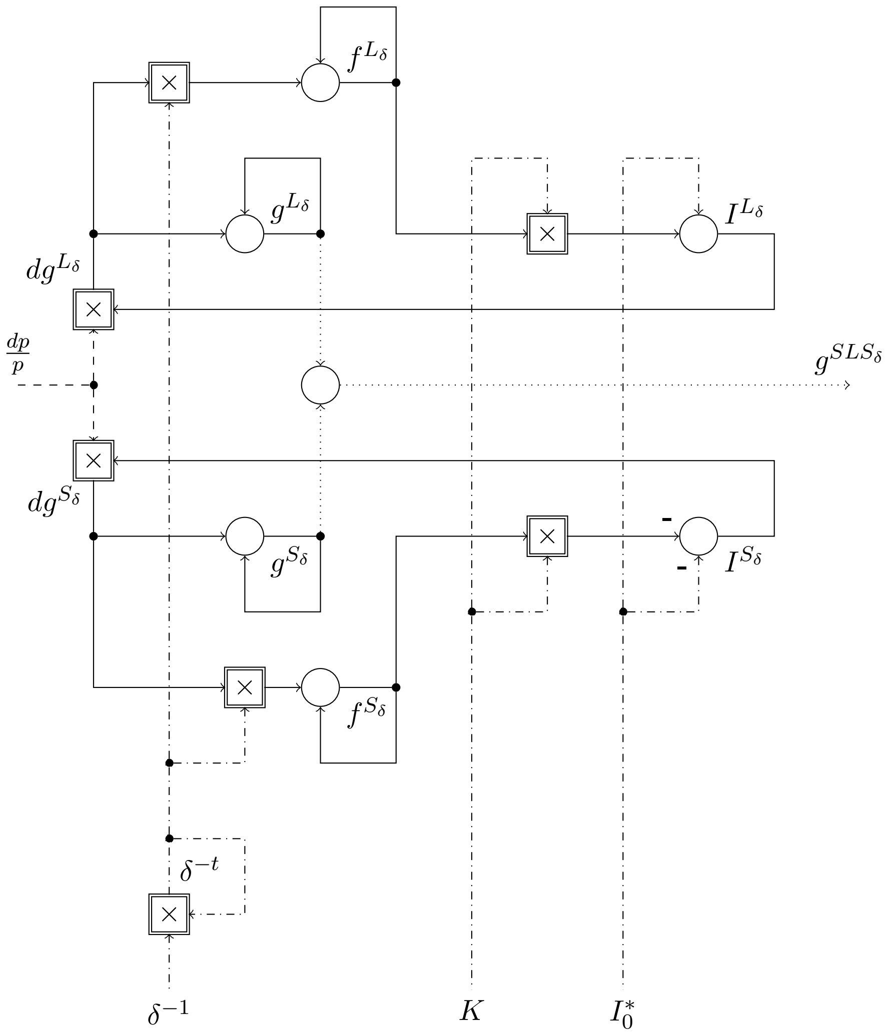

Figure 9.7: Flow diagram for the discounted SLS controller with input (or disturbance) variable return on investment $\frac{dp}{p}$ (i.e. price) and output variable gain g^{SLS_δ}. The SLS$_\delta$ trader's parameters are $K > 0$, $I_0^* > 0$, and $\delta \in (0, 1]$ (or $\delta^{-1} \in [1, \infty)$)

in general anymore (an example is given later in this section), but at least in two special cases. Note that in this time-varying setting, the robust positive expectation property does not hold in general even for $\delta = 1$. That means, not the novelty of discounting the strategy is the problem, but the time-varying trend.

First, we note that for the expected price it holds

$$\mathbb{E}\left[p_t\right] = \mathbb{E}\left[p_0 \cdot \prod_{i=1}^{n} \frac{p_{ih}}{p_{(i-1)h}}\right] = p_0 \cdot \prod_{i=1}^{n} \left(\mu_{h;(i-1)h} h + 1\right)$$

and

$$\mathbb{E}\left[p_{t_2}|\mathcal{F}_{t_1}\right] = p_{t_1} \cdot \prod_{i=n_1+1}^{n_2} (\mu_{h;(i-1)h}h + 1).$$

We start the analysis of the discounted SLS strategy with its long side. By the definition of $I_t^{L_\delta}$ and $f_t^{L_\delta}$ it follows

$$\frac{I_t^{L_\delta} - I_{t-h}^{L_\delta}}{h \cdot I_{t-h}^{L_\delta}} = K\delta^{-(t-h)} \cdot \frac{p_t - p_{t-h}}{h \cdot p_{t-h}}$$

and so

$$\mathbb{E}\left[\frac{I_t^{L_\delta} - I_{t-h}^{L_\delta}}{h \cdot I_{t-h}^{L_\delta}}\right] = K\delta^{-(t-h)}\mu_{h;t-h}.$$

It follows

$$\mathbb{E}\left[I_t^{L_\delta}\right] = I_0^* \cdot \prod_{i=1}^{n} \left(K\delta^{-(i-1)h}\mu_{h;(i-1)h}h + 1\right).$$

Again by the definition of $I_t^{L_\delta}$ it follows:

$$\mathbb{E}\left[f_t^{L_\delta}\right] = \frac{I_0^*}{K}\left(\prod_{i=1}^{n}\left(K\delta^{-(i-1)h}\mu_{h;(i-1)h}h + 1\right) - 1\right)$$

By substituting $I_0^* \mapsto -I_0^*$ and $K \mapsto -K$ the formula for $\mathbb{E}\left[f_t^{S_\delta}\right]$ follows.

Next, we investigate whether $\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right]$ is positive or not. The reader may ask why we are interested in the expected sum of the discounted gain of the short and the long side of the discounted SLS strategy. We can rewrite the undiscounted gain in the following way:

$$\begin{aligned} g_t^{\ell_\delta} &= \left(f_t^{\ell_\delta} - f_{t-h}^{\ell_\delta}\right)\delta^{t-h} + \left(f_{t-h}^{\ell_\delta} - f_{t-2h}^{\ell_\delta}\right)\delta^{t-2h} + \ldots + \left(f_h^{\ell_\delta} + 0\right) \cdot 1 \\ &= f_t^{\ell_\delta}\delta^{t-h} + f_{t-h}^{\ell_\delta}\delta^{t-2h}\left(1 - \delta^h\right) + \ldots + f_h^{\ell_\delta}\left(1 - \delta^h\right) \end{aligned}$$

Since it holds

$$\mathbb{E}\left[g_t^{SLS_\delta}\right] = \mathbb{E}\left[g_t^{L_\delta} + g_t^{S_\delta}\right]$$

and the expectation operator is linear, we conclude: When $\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right] > 0$ for all t, then $\mathbb{E}\left[g_t^{SLS_\delta}\right] > 0$, too. And this is what is really of interest: It is the positive robust expectation property. That means, we want to know whether $\mathbb{E}\left[g_t^{SLS_\delta}\right] > 0$. For this, we have to calculate $\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right] > 0$. In the case $\delta = 1$ it holds $\mathbb{E}\left[f_t^{\ell_1}\right] = \mathbb{E}\left[g_t^{\ell_1}\right]$.

Unfortunately, $\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right] > 0$ is not true for all t, all $\delta \in (0,1]$, and all $(\mu_{h;t})_t$. This can be seen by rewriting

$$\begin{aligned}
\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right] &= \frac{I_0^*}{K}\Bigg(\prod_{i=1}^{n}\left(K\delta^{-(i-1)h}\mu_{h;(i-1)h}h + 1\right) \\
&\quad + \prod_{i=1}^{n}\left(-K\delta^{-(i-1)h}\mu_{h;(i-1)h}h + 1\right) - 2\Bigg) \\
&= \frac{2I_0^*}{K}\sum_{\substack{\alpha\subset\{1,\dots,n\}\\ |\alpha|\ even\\ |\alpha|\neq 0}}\prod_{j\in\alpha} K\delta^{-(j-1)h}\mu_{h;(j-1)h}h.
\end{aligned}$$

When assuming a time-varying trend in discrete time, it is easy to find an example where this sum is negative. This is not a problem of discounting the SLS strategy, it is a problem of the time-varying trend if the time axis is non-continuous even in the standard SLS case, i.e., when $\delta = 1$. When setting $n = 2$, i.e., $\mathcal{T} = \{0, h, 2h\}$, with $\mu_{h;0} > 0$ and $\mu_{h;h} < 0$, which is the time-varying trend, and $\delta = 1$, i.e. (even) in the standard SLS case, it holds $\mathbb{E}\left[f_{2h}^{L_\delta} + f_{2h}^{S_\delta}\right] = 2KI_0^*h^2\delta^{-h}\mu_{h;0}\mu_{h;h} < 0$ and $\mathbb{E}\left[f_h^{L_\delta} + f_h^{S_\delta}\right] = 0$. It follows that $\mathbb{E}\left[g_{2h}^{SLS_\delta}\right] = \mathbb{E}\left[\left(f_{2h}^{L_\delta} + f_{2h}^{S_\delta}\right)\delta^h\right] + 0 < 0$.

However, there are (at least) two special cases where $\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right] > 0$ holds:

(i) First, when $n > 1$ and $\mu_{h;t} \geq 0$ for all t and $\mu_{h;t} > 0$ for at least two points of time t or when $\mu_{h;t} \leq 0$ for all t and $\mu_{h;t} < 0$ for at least two points of time t (since $|\alpha|$ is even). That means, whenever $\left(\mu_{h;(n-1)h}\right)_{n\in\{1,\dots,N\}}$ is non-negative (non-positive), $\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right]$ is non-negative. When there additionally exists $\nu \subset \{1,\dots,N\}$ with $|\nu| \geq 2$ so that $\left(\mu_{h;(j-1)h}\right)_{j\in\nu}$ is positive (negative), it holds that $\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right]$ is positive. The settings of Baumann and Grüne (2017) and Malekpour and Barmish (2016), i.e. μ or μ_h *const.* and non-zero, are a special case of case (i).

(ii) Second, when letting $h \to 0$ (i.e. $n \to \infty$) one can use the continuously compounded interest rate formula, which is a Vito Volterra style product integral, to see that

$$\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right] = \frac{I_0^*}{K}\left(exp\left(\int_0^t K\delta^{-s}\mu(s)ds\right) + exp\left(\int_0^t -K\delta^{-s}\mu(s)ds\right) - 2\right),$$

which is non-negative and additionally positive whenever $\bar{\mu}_\delta := \int_0^t \delta^{-s}\mu(s)ds \neq 0$.

In the continuous time case we proved that the robust positive expectation property still holds (since only for specific values of $\bar{\mu}_\delta$ the expected discounted gain and, thus, the expected gain is zero), cf. Figs. 9.8 and 9.9 for contour plots of the expected discounted SLS$_\delta$ trading gains as a function of $K > 0$ and $\bar{\mu}_\delta$. In Figs. 9.10 and 9.11 the expected discounted gains for different SLS$_\delta$ rules are depicted as functions of $\bar{\mu}_\delta$. Note that

$exp(x) + exp(-x) - 2 \geq 0 \; \forall x$ and equals zero if and only if $x = 0$.

The setting of Primbs and Barmish (2013, 2017) is a special case of case (ii) ($\delta = 1$) and all other results using GBMs or MJDM are also special cases of the case (i) (in the limit; $\delta = 1$). In the case (ii), $\mu(t)$ has to be a Riemann integrable function.

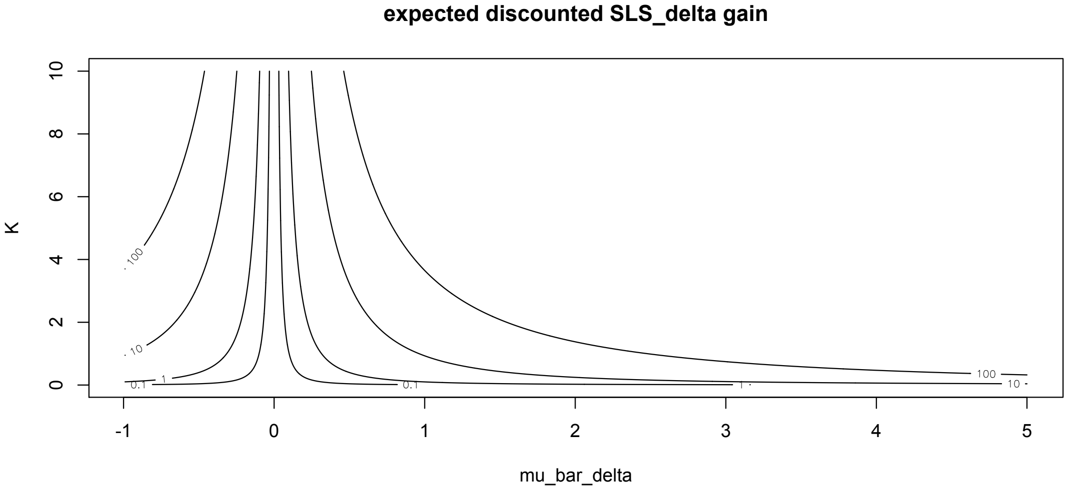

Figure 9.8: Contour plot of the expected discounted gain of the SLS$_\delta$ strategy for $K \in (0, 10]$ and $\bar{\mu}_\delta \in (-1, 5]$. The expected discounted gain is positive for all $(K, \bar{\mu}_\delta)$ with $\bar{\mu}_\delta \neq 0$

During every time interval with positive expected returns or negative expected returns, a trader using the discounted SLS rule has expected positive gains. Only when the expected return μ switches from rising to falling prices or vice versa the trader has to expect a loss. When increasing the trading frequency to continuous trading—which is nearly a realistic assumption in times of high frequency trading—and $\mu(t)$ is Riemann integrable, the measure of time points when μ is switching its direction goes to zero.

Mostly, in market efficiency literature, it is assumed that the price process is a random walk around its fundamental value. When allowing the fundamental value to be non-constant and assuming it to be not too wild, i.e., $\delta^{-t}\mu(t)$ ($\delta \in (0, 1]$) has to be Riemann integrable and $\bar{\mu}_\delta = \int_0^t \delta^{-s}\mu(s)ds \neq 0$, the SLS$_\delta$ trader can—when trading fast enough—expect a positive gain for all t and all discounting factors $\delta \in (0, 1]$. This should not be true in an efficient market.

9.4.2 Comparison to a Buy-and-Hold Rule

In this section, we compare the expected gain of the SLS rule to the expected gain of a buy-and-hold strategy. To keep the notation simple we rely on the standard SLS rule only, i.e., $\delta = 1$.

When comparing the expected gain of the SLS rule with that one of a buy-and-hold

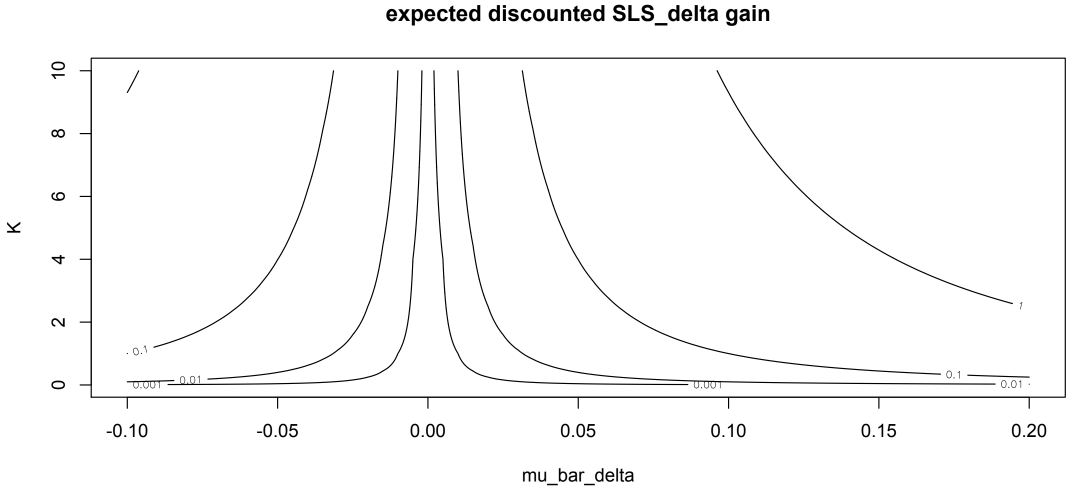

Figure 9.9: Contour plot of the expected discounted gain of the SLS_δ strategy for $K \in (0, 10]$ and $\bar{\mu}_\delta \in [-0.1, 0.2]$. The expected discounted gain is positive for all $(K, \bar{\mu}_\delta)$ with $\bar{\mu}_\delta \neq 0$

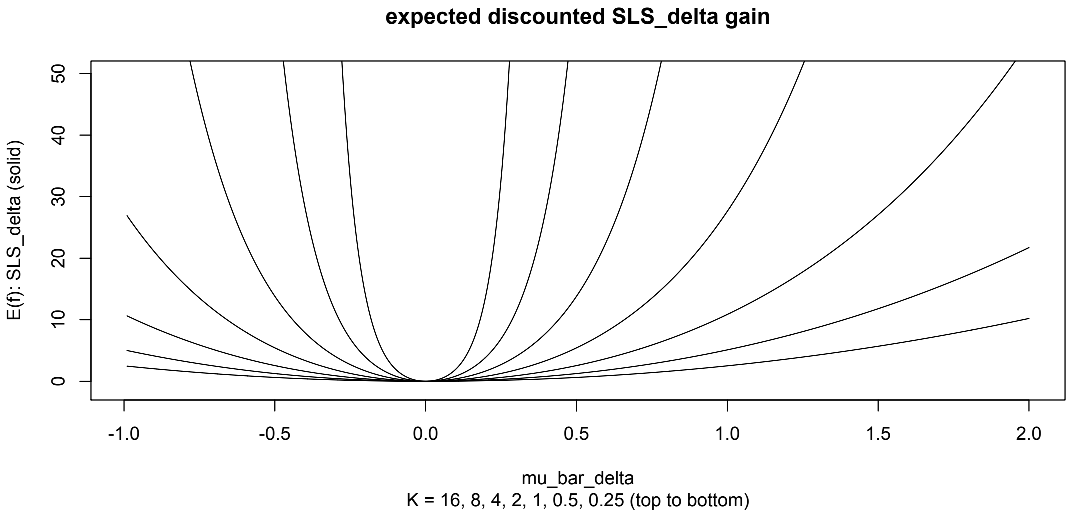

Figure 9.10: Expected discounted gain of different SLS_δ strategies with $I_0^* = 10$ and $K = 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}$ (from top to bottom). The average trend is in $\bar{\mu}_\delta \in (-1, 2]$

strategy (bnh), which is exactly the trader L_δ with $\delta = 1$, $K = 1$, and $I_0^* > 0$, it turns out that if $K > 1$ for all t with $\bar{\mu}(t) \in (-1, 0) \cup (B_{eg}(K, \bar{\mu}), \infty)$ the SLS rule is the dominant one and if $K \leq 1$ it still holds that for all t with $\bar{\mu}(t) \in (-1, 0)$ the SLS rule is dominant to the bnh rule (see Figs. 9.12, 9.13, 9.14, and 9.15 for graphs

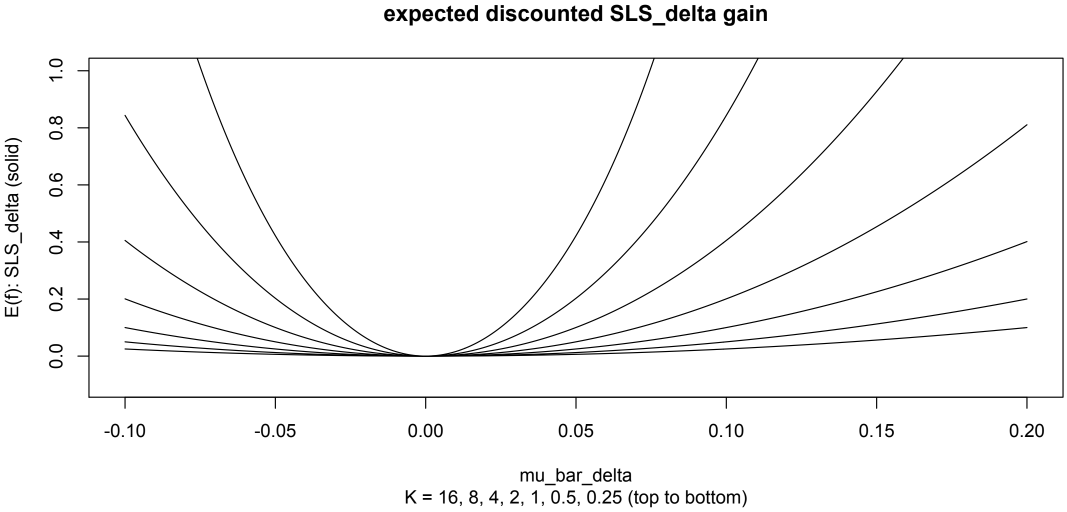

Figure 9.11: Expected discounted gain of different SLS_δ strategies with $I_0^* = 10$ and $K = 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}$ (from top to bottom). The average trend is in $\bar{\mu}_\delta \in [-0.1, 0.2]$

of the expected SLS gain and the expected bnh gain and contour plots of the expected difference of these strategies). The value $B_{eg}(K, \bar{\mu})$ depends on K and $\bar{\mu}$ and it holds: $B_{eg}(K, \bar{\mu}) \to 0$ for $K \to \infty$. Note that $\bar{\mu}(t) \notin [0, B_{eg}(K, \bar{\mu})]$ does not mean that the SLS is only dominant for special price paths, which would not be a result deserving attention. Since $\bar{\mu}(t) := \int_0^t \mu(s)ds$ with $\mu(t)dt = \mathbb{E}\left[\frac{dp(t)}{p(t)}\right]$ is the expected return of the price path that depends on changes in the fundamentals and all results so far concern expectations, the price paths are allowed to be random walks around the fundamental value when $\bar{\mu}(t)$ satisfies the condition $\bar{\mu}(t) \notin [0, B_{eg}(K, \bar{\mu})]$.

9.4.3 Risk-Adjusted Expected Return of SLS and Buy-and-Hold

For sure, there are some points to think about concerning the result about the robust positive expectation property in Secs. 9.4.1 and 9.4.2. The assumption that there are short time trends in expected returns (that can be caused by changes in fundamentals) is reasonable. The argument that the trader in practice has to achieve a positive gain on average when there are trading costs, is not really a solution of the puzzle (of positive expected gains in an efficient market) in times of flat-rate trading offers, especially when noticing that the difference of the expected gains of the SLS rule and of the bnh rule goes to infinity if $K \to \infty$ when $\bar{\mu}_\delta \neq 0$ or $\bar{\mu}_\delta \to \infty$ when $K > 1$ and trading costs in a highly liquid market can be assumed to be bounded. The same is true for the continuous trading assumption when considering high frequency trading. However, there is one argument against the robust positive expectation property of the SLS rule that puzzles us: the risk-adjustment.

Classically, the risk argument is given by the defenders of the market efficiency

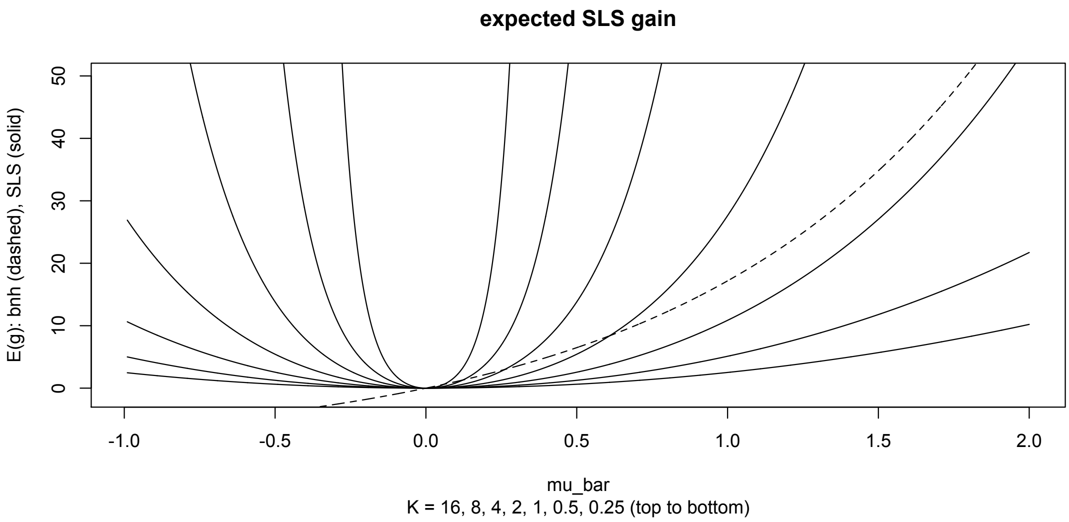

Figure 9.12: Expected gain of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}$ (from top to bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. The average trend is in $\bar{\mu} \in (-1, 2]$

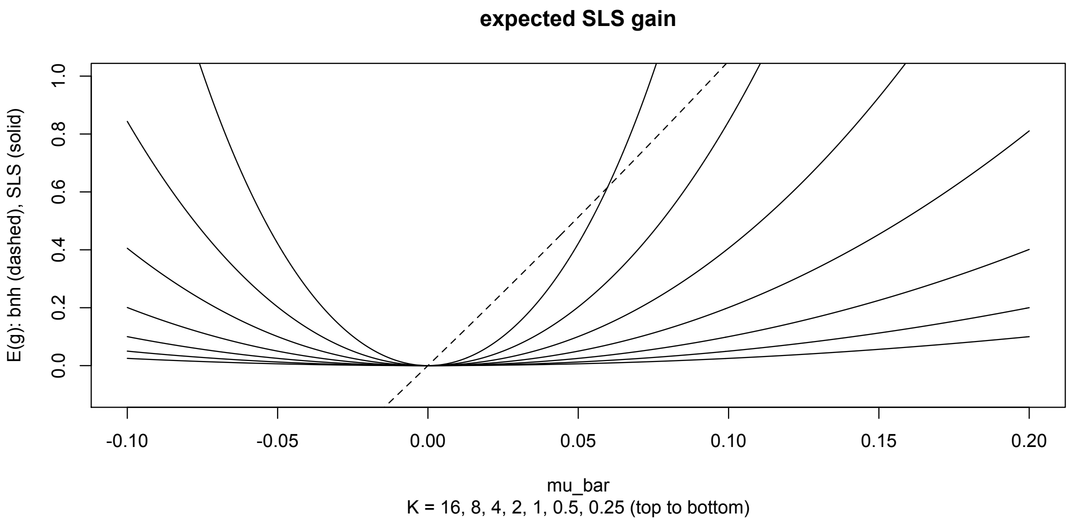

Figure 9.13: Expected gain of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}$ (from top to bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. The average trend is in $\bar{\mu} \in [-0.1, 0.2]$

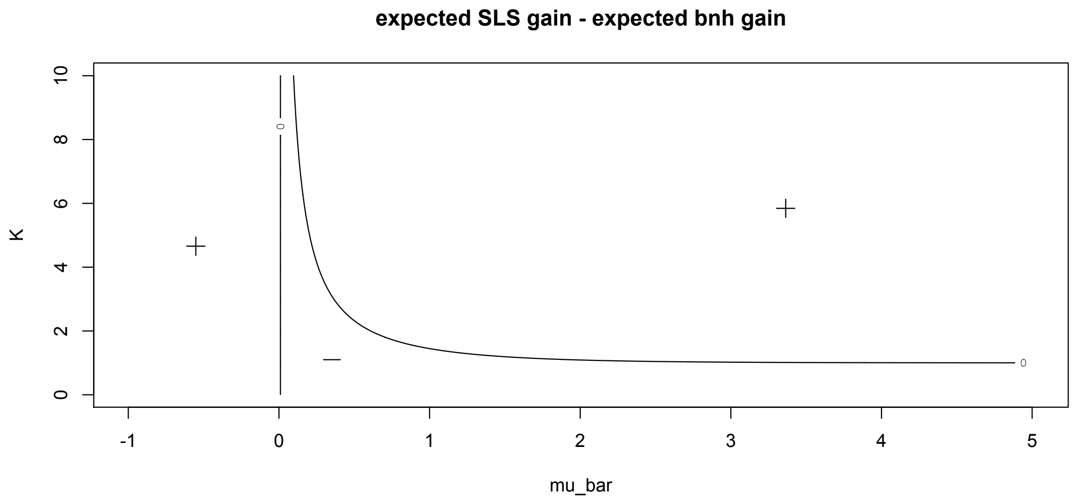

Figure 9.14: Contour plot of the expected difference of the gain of the SLS strategy and the bnh rule for $K \in (0, 10]$ and $\bar{\mu} \in (-1, 5]$. The expected difference is positive for all $(K, \bar{\mu})$ in the left as well as in the upper-right area

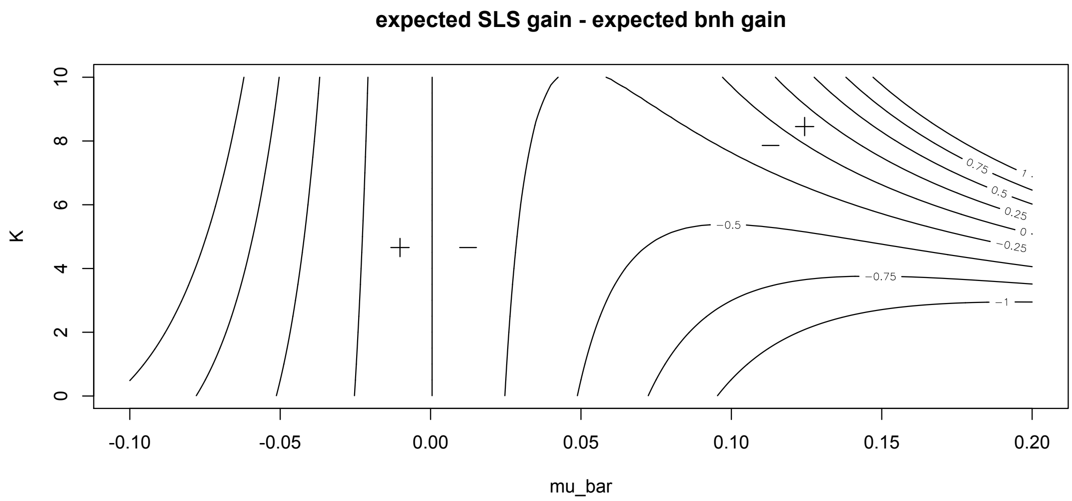

Figure 9.15: Contour plot of the expected difference of the gain of the SLS strategy and the bnh rule for $K \in (0, 10]$ and $\bar{\mu} \in [-0.1, 0.2]$. The expected difference is positive for all $(K, \bar{\mu})$ in the left as well as in the upper-right area

hypothesis when someone finds an external variable that allows for estimating higher expected returns of an asset. Then it is said that this external variable is just a better proxy for measuring risk and so it is concluded that the asset under investigation is more risky, which allows the asset to be more profitable (on average) without being a counter example to market efficiency. In the setting of this chapter, this is not applicable since there is only one asset under analysis. Even the discussion about the momentum effect (Moskowitz, 2010), i.e., higher momentum is related to higher risk, is not applicable to our setting because we do not have assumptions on the stock under trade. Here, only different trading strategies are considered. The only way to apply the risk adjustment argument to the SLS_δ rule is to use volatility, which we do next. As a proxy for volatility we use the standard deviation, which is not a risk measure in the sense of mathematical finance. At the end of this chapter and at the end of this thesis, the risk of the SLS rule and other definitions of it (cf. skewness) are discussed again. But for now, we use the most common choice.

For the remainder of this section, the analysis is restricted to the standard SLS rule, i.e., we set $\delta = 1$. For calculating the standard deviation of the SLS strategy, an assumption on the volatility of the underlying price process is needed. Analogous to the definition of the trend, it is set

$$\mathbb{E}\left[\frac{1}{h}\left(\frac{p_t - p_{t-h}}{p_{t-h}}\right)^2\right] =: \sigma^2_{h;t-h} > 0.$$

Note that also here a market parameter, namely $\sigma^2_{h;t}$, depends on h, which is chosen by the trader. However, the same argument as for $\mu_{h;t}$ holds (cf. Baumann and Grüne, 2017).

With this assumption it follows

$$\mathbb{E}\left[p_t^2\right] = p_0^2 \cdot \prod_{i=1}^{n}\left(\left(\sigma^2_{h;(i-1)h} + 2\mu_{h;(i-1)h}\right)h + 1\right)$$

and

$$\mathbb{E}\left[p_{t_2}^2 | \mathcal{F}_{t_1}\right] = p_{t_1}^2 \cdot \prod_{i=n_1+1}^{n_2}\left(\left(\sigma^2_{h;(i-1)h} + 2\mu_{h;(i-1)h}\right)h + 1\right).$$

Again, we start the analysis of the SLS strategy with its long side. Using the definition of I_t^L and g_t^L leads to:

$$\frac{1}{h}\left(\frac{I_t^L - I_{t-h}^L}{I_{t-h}^L}\right)^2 = \frac{K^2}{h}\left(\frac{p_t - p_{t-h}}{p_{t-h}}\right)^2$$

and

$$\mathbb{E}\left[\frac{1}{h}\left(\frac{I_t^L - I_{t-h}^L}{I_{t-h}^L}\right)^2\right] = K^2\sigma^2_{h;t-h}$$

It holds

$$\mathbb{E}\left[\left(I_t^L\right)^2\right] = I_0^{*2} \cdot \prod_{i=1}^{n} \left(\left(K^2\sigma^2_{h;(i-1)h} + 2K\mu_{h;(i-1)h}\right)h + 1\right).$$

Again by the definition of I_t^L it follows:

$$\mathbb{E}\left[\left(g_t^L\right)^2\right] = \frac{I_0^{*2}}{K^2}\left(\prod_{i=1}^{n}\left(\left(K^2\sigma^2_{h;(i-1)h} + 2K\mu_{h;(i-1)h}\right)h + 1\right) - 2\prod_{i=1}^{n}\left(K\mu_{h;(i-1)h}h + 1\right) + 1\right)$$

By substituting $I_0^* \mapsto -I_0^*$ and $K \mapsto -K$, the formula for $\mathbb{E}\left[\left(g_t^S\right)^2\right]$ follows. For calculating the standard deviation of the SLS strategy's gain, the mixed expectation of the long and the short side $\mathbb{E}\left[g_t^L g_t^S\right]$ are needed, too. It holds:

$$\frac{1}{h}\left(\frac{I_t^L - I_{t-h}^L}{I_{t-h}^L}\right)\left(\frac{I_t^S - I_{t-h}^S}{I_{t-h}^S}\right) = -\frac{K^2}{h}\left(\frac{p_t - p_{t-h}}{p_{t-h}}\right)^2$$

and

$$\mathbb{E}\left[\frac{1}{h}\left(\frac{I_t^L - I_{t-h}^L}{I_{t-h}^L}\right)\left(\frac{I_t^S - I_{t-h}^S}{I_{t-h}^S}\right)\right] = -K^2\sigma^2_{h;t-h}$$

With that it follows

$$\mathbb{E}\left[I_t^L I_t^S\right] = -I_0^{*2} \cdot \prod_{i=1}^{n}\left(-K^2\sigma^2_{h;(i-1)h}h + 1\right).$$

By the definitions of I_t^L and I_t^S it follows:

$$\mathbb{E}\left[g_t^L g_t^S\right] = \frac{I_0^{*2}}{K^2}\left(\prod_{i=1}^{n}\left(-K^2\sigma^2_{h;(i-1)h}h + 1\right) - \prod_{i=1}^{n}\left(K\mu_{h;(i-1)h}h + 1\right) - \prod_{i=1}^{n}\left(-K\mu_{h;(i-1)h}h + 1\right) + 1\right)$$

Now, all components needed for the calculation of

$$\mathbb{E}\left[\left(g^{SLS}(t)\right)^2\right] = \mathbb{E}\left[\left(g^L(t)\right)^2\right] + 2\mathbb{E}\left[g^L(t)g^S(t)\right] + \mathbb{E}\left[\left(g^S(t)\right)^2\right]$$

and

$$\mathbb{V}\left[g^{SLS}(t)\right] = \mathbb{E}\left[\left(g^{SLS}(t)\right)^2\right] - \left(\mathbb{E}\left[g^{SLS}(t)\right]\right)^2$$

are known. To keep the computation simple, we calculate the limit for continuous time trading $h \to 0$ and recall that $\bar{\mu}(t) = \bar{\mu}_1(t) = \int_0^t \mu(s)ds$ and define $\overline{\sigma^2}(t) := \int_0^t \sigma^2(s)ds$ (of course, $\sigma^2(t)$ has to be Riemann integrable as well). By use of the Vito Volterra style product integral, it follows:

$$\mathbb{E}\left[\left(g^{SLS}(t)\right)^2\right] = \mathbb{E}\left[\left(g^L(t)\right)^2 + \left(g^S(t)\right)^2 + 2g^L(t)g^S(t)\right]$$

$$
\begin{aligned}
&= \frac{I_0^{*2}}{K^2}\Bigg(exp\left(K^2\overline{\sigma^2}(t) + 2K\bar{\mu}(t)\right) - 2exp(K\bar{\mu}(t)) + 1 \\
&\quad + exp\left(K^2\overline{\sigma^2}(t) - 2K\bar{\mu}(t)\right) - 2exp(-K\bar{\mu}(t)) + 1 \\
&\quad + 2\left(exp\left(-K^2\overline{\sigma^2}(t)\right) - exp(K\bar{\mu}(t)) - exp(-K\bar{\mu}(t)) + 1\right)\Bigg)
\end{aligned}
$$

Combining the results for $\mathbb{E}\left[g^{SLS}(t)\right] = \frac{I_0^*}{K}(exp(K\bar{\mu}(t)) + exp(-K\bar{\mu}(t)) - 2)$ and $\mathbb{E}\left[\left(g^{SLS}(t)\right)^2\right]$ leads to the formula for the SLS rule's variance:

$$
\begin{aligned}
\mathbb{V}\left[g^{SLS}(t)\right] = \frac{I_0^{*2}}{K^2}\Bigg(&\left(exp\left(K^2\overline{\sigma^2}(t)\right) - 1\right)(exp(2K\bar{\mu}(t)) + exp(-2K\bar{\mu}(t))) \\
&+ 2\left(exp\left(-K^2\overline{\sigma^2}(t)\right) - 1\right)\Bigg)
\end{aligned}
$$

This expression fits exactly the results obtained in Sec. 9.1 and by Baumann (2017b) for MJDM (and the GBM).

After having derived the formulae for the expected gain and the variance of the SLS rule, next, we compare the results with the corresponding formulae for a simple buy-and-hold trader. It is easy to see that for the expected gain of a simple buy-and-hold strategy with initial investment I_0^* it holds

$$
\mathbb{E}\left[g^{bnh}(t)\right] = I_0^*(exp(\bar{\mu}(t)) - 1)
$$

and for the respective variance

$$
\mathbb{V}\left[g^{bnh}(t)\right] = I_0^{*2} exp(2\bar{\mu}(t))\left(exp\left(\overline{\sigma^2}(t)\right) - 1\right),
$$

for example by using the results for $g^L(t)$ and setting $K = 1$ (and $\delta = 1$).

For any strategy ℓ let

$$
rar(\ell; t) := \frac{\mathbb{E}\left[g^\ell(t)\right]}{\sqrt{\mathbb{V}\left[g^\ell(t)\right]}} = \frac{\mathbb{E}\left[g^\ell(t)\right]}{\mathbb{S}\left[g^\ell(t)\right]}
$$

be the risk-adjusted return of this strategy at time t. It is clear that $rar(SLS; t) > 0\ \forall t > 0, \bar{\mu}(t) \neq 0$, cf. Figs. 9.16 and 9.17 for contour plots of the risk-adjusted returns of the SLS strategy.

As suggested similarly by Malkiel (1973), we compare the risk-adjusted returns of the SLS rule with that of a buy-and-hold strategy. For all t with $\bar{\mu}(t) \in (-1, 0)$ the SLS rule is the dominant one. If $K \geq 1$ the bnh rule is dominant if $\bar{\mu}(t) > 0$. If $K < 1$ and $\bar{\mu}(t) > 0$ for some pairs $(K, \bar{\mu}(t))$ the SLS rule is dominant and for some the bnh rule,

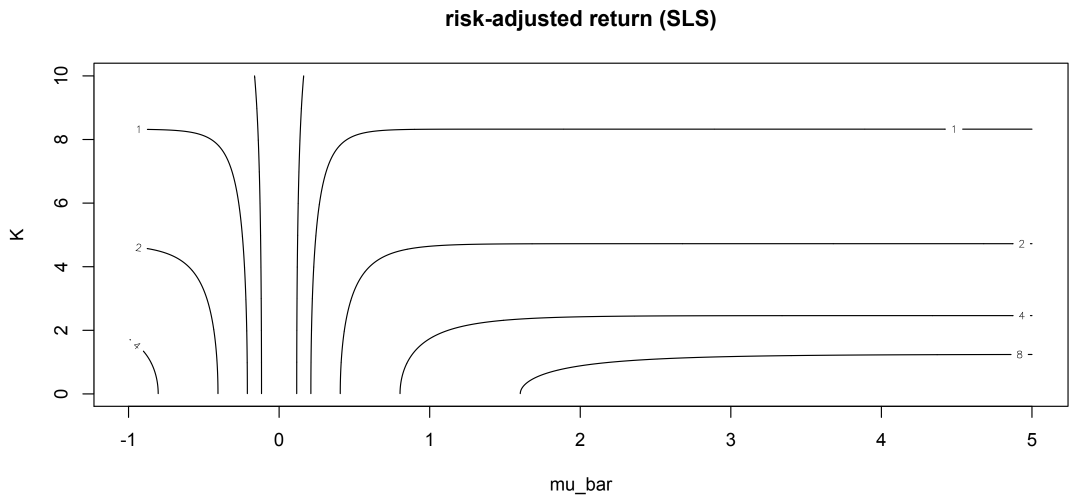

Figure 9.16: Contour plot of the risk-adjusted return of the SLS strategy for $K \in (0, 10]$ and $\bar{\mu}_\delta \in (-1, 5]$. For risk-adjustment we use the standard deviation. The risk-adjusted return is positive for all $(K, \bar{\mu}_\delta)$ with $\bar{\mu}_\delta \neq 0$

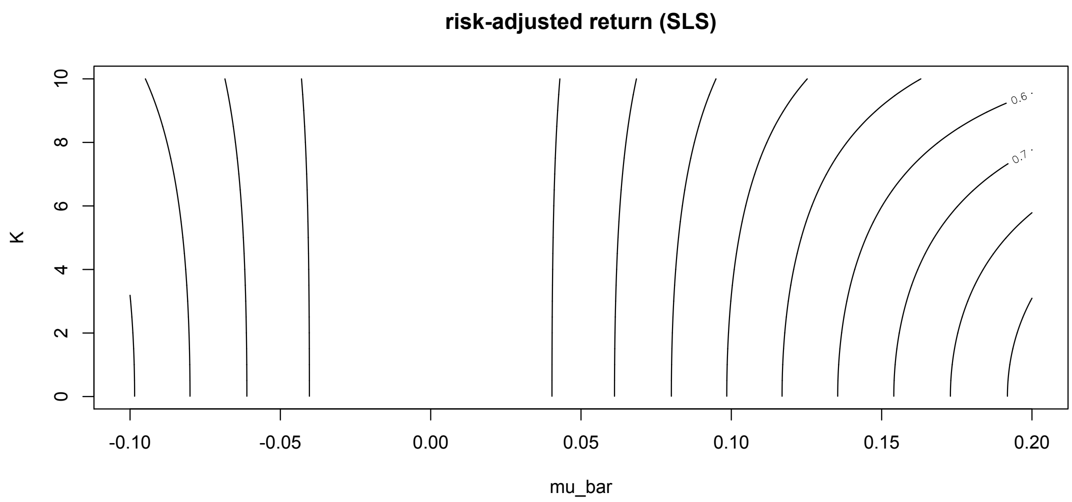

Figure 9.17: Contour plot of the risk-adjusted return of the SLS strategy for $K \in (0, 10]$ and $\bar{\mu}_\delta \in [-0.1, 0.2]$. For risk-adjustment we use the standard deviation. The risk-adjusted return is positive for all $(K, \bar{\mu}_\delta)$ with $\bar{\mu}_\delta \neq 0$

see Figs. 9.18, 9.19, 9.20, and 9.21 for graphs of the risk-adjusted returns of the SLS rule and the bnh rule (for varying $\overline{\sigma^2}$) and Figs. 9.22 and 9.23 for contour plots of the difference of the risk-adjusted returns of the SLS rule and the bnh strategy.

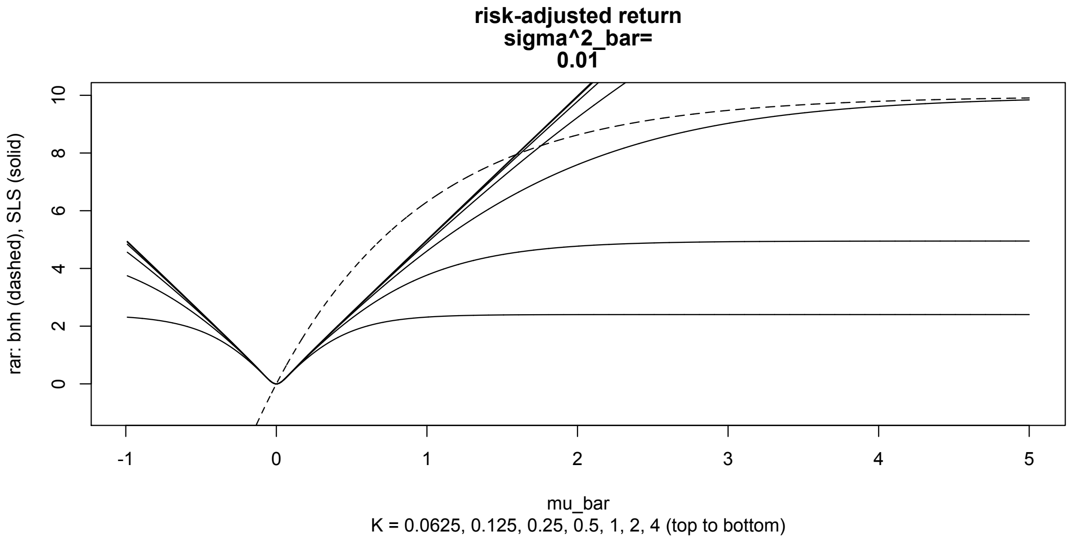

Figure 9.18: Risk-adjusted return of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ (from top to bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. All returns are adjusted with the respective standard deviation. The average trend is in $\bar{\mu} \in (-1, 5]$ and the average volatility is $\overline{\sigma^2} = 1\%$

Now, the question is whether the risk-adjustment and (at the same time) the comparison to the bnh rule is the solution to the conflict of the robust positive expectation property of the SLS rule to market efficiency. However, it is not. If a market as a whole (i.e. on average) is risk-neutral but not every single stock, a trader investing in a randomly selected portfolio (and this is what Malkiel (1973) actually suggests) can expect zero gain and therefore the risk-adjusted return is zero, too. If the trader uses the SLS rule stock-by-stock and there is only one single stock that is not risk-neutral (and it does not matter if the stock's expected return is too high or too low) the expected trading gain as well as the risk-adjusted return are positive.

For a practical application, there still remains the question how to choose K. If $\bar{\mu} < 0$, it does not matter whether $K > 1$ or $K < 1$ (in a qualitative manner) because expected gains and risk-adjusted returns are positive and even when compared to the bnh rule, for both the expected gains and risk-adjusted returns the SLS rule is dominant. When $\bar{\mu} > 0$, it also does not matter how to choose K when relying on expected gain or risk-adjusted return. However, when compared to the bnh rule it might be better to choose $K > 1$ when expected gain is the target function and to choose $K < 1$ when it is the risk-adjusted return. Note again that the comparison to the bnh strategy is questionable because the bnh rule is only better in specific cases for a single asset: A randomly selected portfolio should have a trend (and also a risk-adjusted trend) of exactly the bond's rate, i.e. of zero. A bnh trader faces the risk of a negative trend—an SLS trader does not.

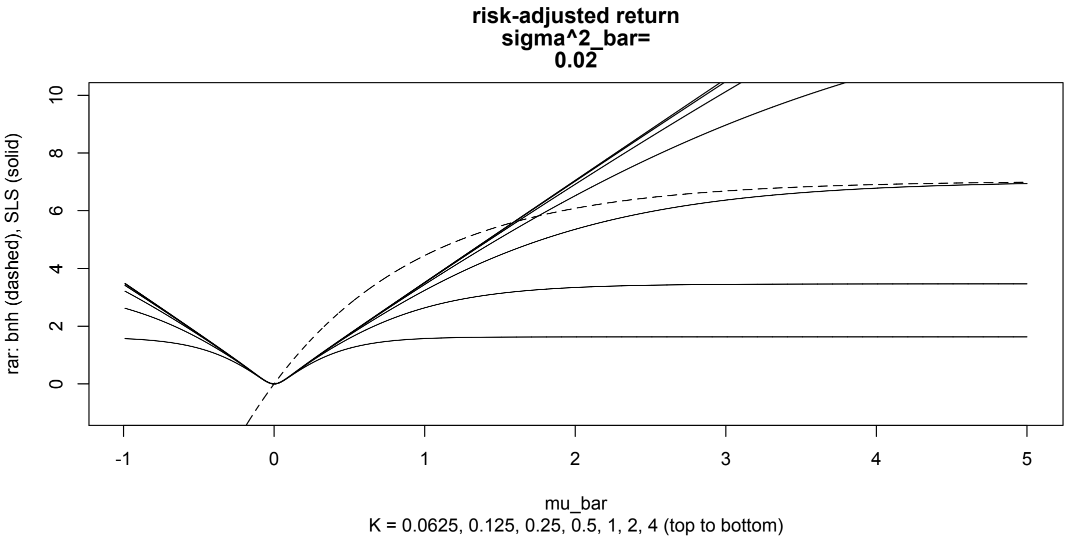

Figure 9.19: Risk-adjusted return of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ (from top to bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. All returns are adjusted with the respective standard deviation. The average trend is in $\bar{\mu} \in (-1, 5]$ and the average volatility is $\overline{\sigma^2} = 2\%$

9.5 Discussion of the Performance of SLS Trading

In the past, most puzzles for market efficiency came from empirical data and statistical methods. The puzzle presented in this chapter is a purely theoretical, mathematical one. We proved that for all price processes, which are random walks, it holds:

- If the price process is governed by Merton's jump diffusion model, the SLS rule lets the trader *a.s.* expect a positive gain.
- If the price process is in the set of essentially linearly representable prices, the robust positive expectation property holds.
- In discrete time, when the trend is constant and non-zero, the expected gain is positive.
- For a continuous time process with constant trend and discrete time trading, which is a sampled-data system, the expected gain is *a.s.* positive and does not depend on the trading interval, i.e. the sampling time.
- In discrete time, the expected gain of the discounted SLS_δ strategy for all discounting factors $\delta \in (0, 1]$, which includes the standard SLS rule ($\delta = 1$), is positive when $(\bar{\mu}_{h;t})_t \geq 0$ and $(\bar{\mu}_{h;t})_t > 0$ for at least two points of time or when $(\bar{\mu}_{h;t})_t \leq 0$ and $(\bar{\mu}_{h;t})_t < 0$ for at least two points of time.

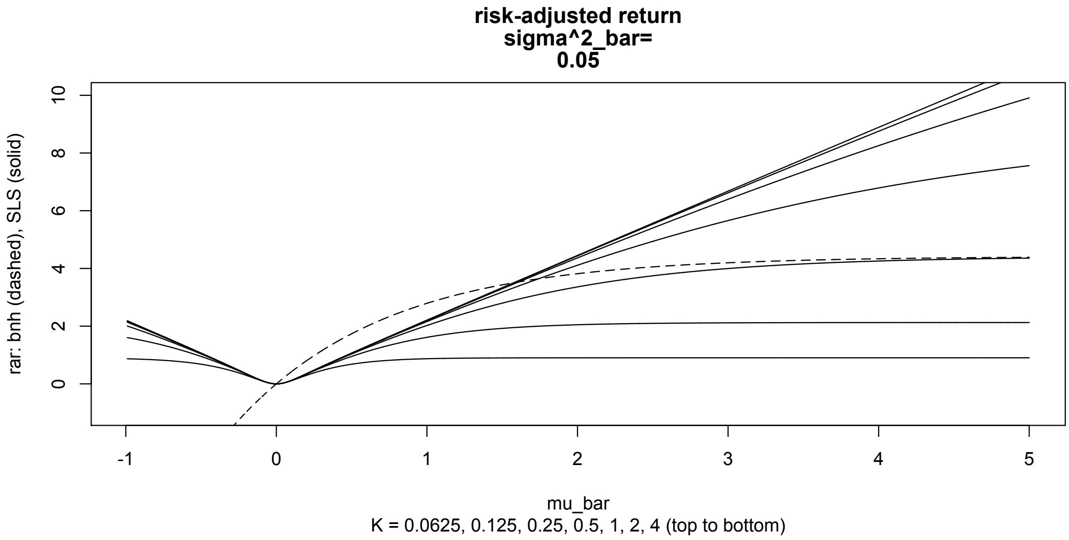

Figure 9.20: Risk-adjusted return of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ (from top to bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. All returns are adjusted with the respective standard deviation. The average trend is in $\bar{\mu} \in (-1, 5]$ and the average volatility is $\overline{\sigma^2} = 5\%$

- In continuous time, the expected gain of the discounted SLS_δ strategy for all discounting factors $\delta \in (0, 1]$, which includes the standard SLS rule ($\delta = 1$), is positive when $\bar{\mu}_\delta(t) \neq 0$.
- The expected gain of the standard SLS rule surpasses the expected gain of a simple buy-and-hold strategy for all $t > 0$ with $\bar{\mu}_\delta(t) \notin [0, B_{eg}(K, \bar{\mu})]$ if $K > 1$, with $B_{eg}(K) \to 0$ for $K \to \infty$, and all $\bar{\mu}_\delta(t) \notin \mathbb{R}_0^+$ if $K \leq 1$ (in continuous time).
- The risk-adjusted return of the standard SLS rule is positive for all $K > 0$, $-1 < \bar{\mu} \neq 0$, and $\overline{\sigma^2} > 0$.
- The risk-adjusted return of the standard SLS rule exceeds the risk-adjusted return of a simple buy-and-hold strategy for all $-1 < \bar{\mu} < 0$ and if $K \leq 1$ for some $0 < \bar{\mu}$.

That means, an SLS trader can expect positive gain (even in discrete time) on all arbitrary small intervals where the trend is not changing its sign. Only for that points of time where the trend changes its sign, the SLS trader is facing negative expected gains. Note that the price path itself can change its slope arbitrarily often. When the trend path is to some extent smooth and trading frequency is increased, the points of time where the trend changes its sign do carry less (or, when going to continuous time, even no) weight.

Clearly, there are some assumptions to discuss. Trading costs, for example, would decrease the expected gain of the SLS rule. However, as can be seen in Figs. 9.12 and 9.13,

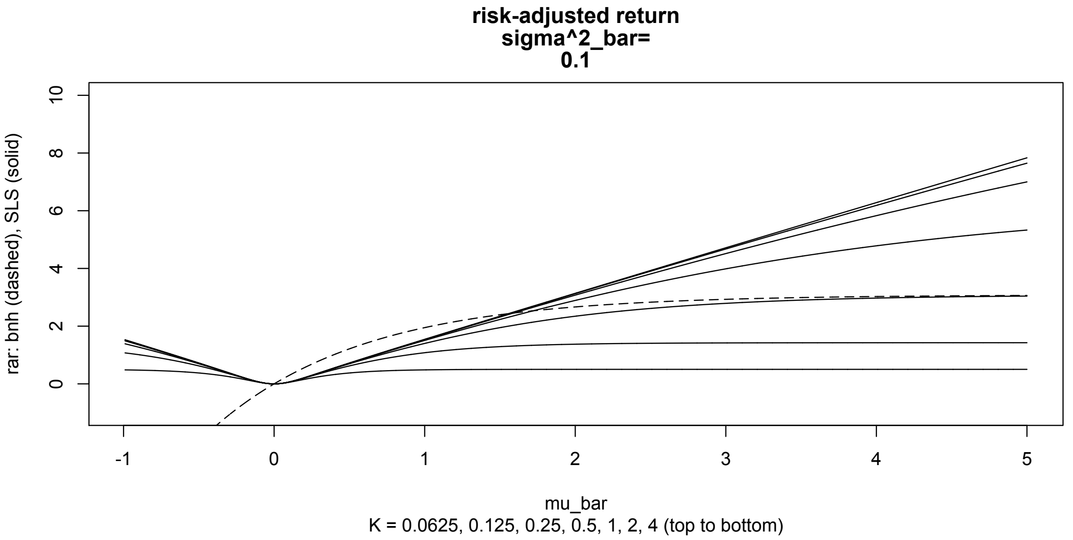

Figure 9.21: Risk-adjusted return of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ (from top to bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. All returns are adjusted with the respective standard deviation. The average trend is in $\bar{\mu} \in (-1, 5]$ and the average volatility is $\overline{\sigma^2} = 10\%$

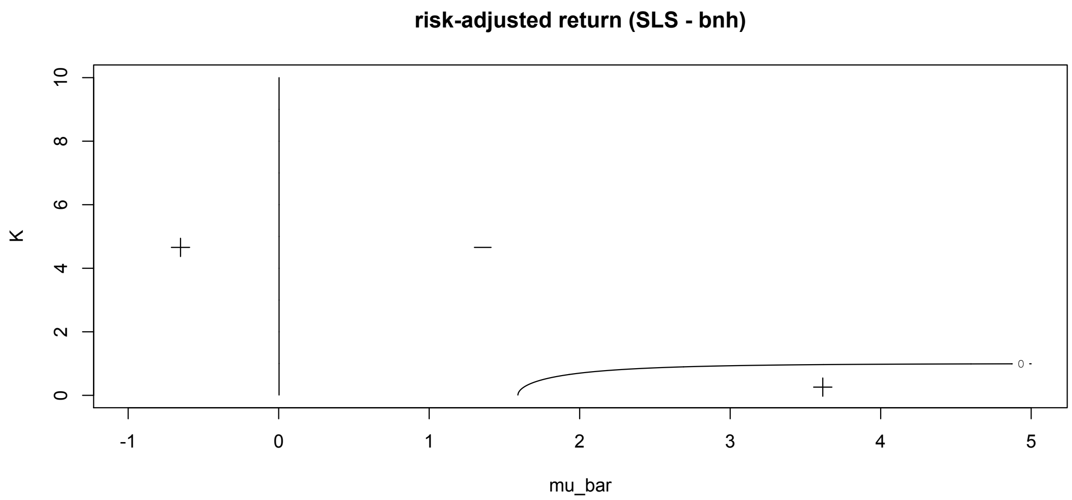

Figure 9.22: Contour plot of the difference of the risk-adjusted returns of the SLS rule and of a bnh rule. The average volatility is 1%. The SLS rule is dominant in the left and in the lower-right area

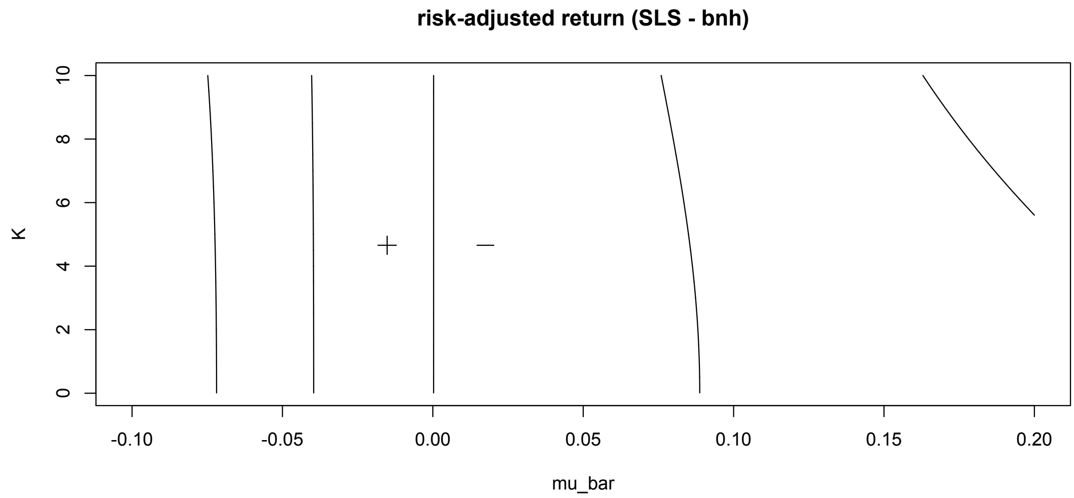

Figure 9.23: Contour plot of the difference of the risk-adjusted returns of the SLS rule and of a bnh rule. The average volatility is 1%. The SLS rule is dominant on the left side

the gap between the expected returns of the SLS and the buy-and-hold rule is widening heavily when increasing K (and the interval of $\bar{\mu}$ where the SLS rule is dominated tends to zero if $K \to \infty$). Thus, trading costs are not that important. Continuous time trading is a hard assumption. But since the results of this work do not rely on any price path but only on the trend process and there are high frequency trading possibilities, only a very hard non-trending assumption could invalidate these results. For example, one had to assume that for every point of time with a positive (negative) price trend, for every arbitrary small interval after that point of time, there has to be another point of time where the price trend is negative (positive). However, this would also imply that there are absolutely no identifiable trends in fundamental values. Adequate resources, perfect liquidity, short selling, and the price taker property can be seen as justified on modern stock exchanges when both the trader and the traded asset are big enough and I_0^* and K are chosen small enough (cf. Chap. 7).

If one asked us to solve the puzzle, the only—more or less—satisfying answer we could give is that the risk measure is inappropriate (maybe *skewness* would be better). But there are two problems: First, this idea only works when market efficiency is defined via risk-adjusted returns only (and not when it is defined via expected gain). And second, we would run in a problem very similar to the joint hypotheses problem: We conjecture that for nearly every trading strategy one could find a risk measure so that the risk-adjusted return is high and one so that it is low. And the other way around, we also conjecture that for nearly all risk measures one can find a trading strategy that beats the market and one that is beaten by it. No one could say *whether the risk measure* *or the market efficiency hypothesis is wrong.* Thus, we rely on a standard definition of risk-adjustment.

To sum up, there are three possibilities how to solve the problem whether the SLS rule is beating the market (for a big, rich trader that trades small amounts of highly liquid stocks of big underlying firms) or not. First, if we assumed that all assets are risk-neutral—and not only the market as a whole—the results would not hold. However, that would mean that in every point of time the trend of every single stock is exactly the trend of the bond, no matter how volatile the stock is. (Note that it is reasonable to assume that a high-volatile stock is riskier and, hence, should have a higher trend.) Second, and a little bit weaker than the first argument, if the trends (the trends and not only the price paths) of the stocks jump in every (infinitesimal small) interval from positive to negative or vice versa, again the results would not hold. And finally, third, that we cannot adequately measure risk. This leads to a *risk including joint hypotheses problem*, because there is not one risk measure everyone relies on. No one can say whether the used risk measure or the efficient market hypothesis is wrong. The last point is the most satisfying answer we can give. At the very end of this chapter, we mention that the robust positive expectation property is not an arbitrage possibility. The gain is not sure, it is only in expectation. And it needs potentially a very high number of experiments, i.e. of trading processes, to realize a positive expected gain on average.

Chapter 10

Effects of Technical Trading Rules

In Chap. 9, we analyzed the performance of technical trading rules, actually of SLS trading. At this point, we answer the question wether SLS trading or especially the long side of SLS trading is weakening financial stability. In the sections above, this did not matter since there we assumed the price taker property. However, when relaxing the price taker property and assuming a market model were buying decisions lead to increasing prices, positive trend following seems to produce bubbles: A trader who is investing long makes money if the price rises. If the trader is a trend follower, the trader buys, which leads to even more rising prices (and so on). We answer if in such a non-price-taker model, positive linear feedback trading for sure leads to a financial crisis, i.e. to a bubble, and if there are market internal mechanisms that prevent a bubble. Note that we are interested in the long side only because the short side of the feedback rule would push a price to zero (if it made a lot of money), which is not as dramatic as a bubble (from a macroeconomic point of view). If the long side made a lot of money, the short side's investment would be close to zero and the long side would (maybe) cause a bubble.

This chapter provides a market model that does not only allow prices to be influenced by the traders but also that the price is completely determined by the traders' investment decisions. We rely on a so-called heterogeneous agent model, i.e., all traders influence the price process and possibly indirectly each other, too. The model is discussed and different trading strategies are established. This section is based on Baumann (2015); Baumann et al. (2017).

10.1 Motivation for Our Heterogeneous Agent Model

Financial market bubbles have repeatedly caused major economic problems, a very prominent example for this was the crisis of 2007/2008 (Reinhart and Rogoff, 2009). An important strand of the financial crises literature focuses on the question of whether specific behavior of market participants is responsible for price bubbles. In particular, heterogeneous agent models (HAMs) analyze how heterogeneous traders, esp. chartists and fundamentalists, are able to determine asset price movements (Hommes, 2006a).

Chartists, for example trend followers, trade based only on information about the price process, that is, they assume that all important information is present in the asset price. Here, the first question arises. When we assume that there is only one active trader in the market, who trades with a market maker, and this trader is a long investing trend follower, does the combination of a market model where prices rise when traders buy and a single trader who buys even more when prices rise for sure lead to a bubble? Note that the market maker is a special trader (and not counted as active trader), who satisfies all buying and selling decisions, i.e., who clears the market, and sets new prices. In the literature it is often stated that trend followers magnify the current trend, either positively or negatively, because their trading is based on the philosophy that the greater the absolute value of the slope of the price process is, the more should be bought or sold (Covel, 2004).

In contrast to chartists, fundamentalists have some fundamental value in mind and trade based on perceived over- or undervaluation of the underlying asset. In particular, fundamentalists buy (sell) when the price is below (above) the fundamental value, thereby pushing the asset price toward its fundamental value.

All traders act out of self-interest with the intention of making a profit, and give little thought to how their actions impact prices. As a consequence of the two different investment strategies, the presence of chartists can cause exploding prices (De Long et al., 1990b), whereas fundamentalists are associated with a stabilizing influence on assets. Thus, the following question arises: Are the balancing effects of fundamentalists strong enough to compensate the destabilizing impacts of chartists? Heterogeneous agent models are increasingly employed in search of an answer to this question (Gaunersdorfer and Hommes, 2005; Hommes, 2002; Lux, 1995, 1998; Lux and Marchesi, 1999, 2000). These studies provide useful explanations for many stylized facts, including excess volatility, high trading volume, temporary bubbles, trend following, sudden crashes, mean reversion, clustered volatility, and fat tailed distribution returns. For an excellent overview regarding HAM see the work of Hommes (2006a). The models typically use bounded rational agents, (imperfect) heuristics or rules of thumb, and nonlinear dynamics (which might be chaotic). Some studies find that the stabilizing effects of fundamentalists are not necessarily strong enough to stabilize markets (Hommes, 2006a). However, the results are usually obtained via simulations and are not analytically proven (Hommes, 2006a). An exception is the work of De Long et al. (1990b) which investigates the effect of positive feedback traders and informed speculators, who evaluate and consider the needs of the other market participants, especially the growing needs of the positive feedback traders, in a three-period market model facing fundamentalists. De Long et al. (1990b) show that the interaction of these two trader types pushes the price away from the fundamental value under specific assumptions and despite the fundamentalists' stabilizing behavior. This chapter differs from the work of De Long et al. (1990b): We do not investigate how two types of traders, positive feedback traders and informed speculators, jointly push up the price but instead look only at trend followers, nor do we assume a predetermined end of the market. This leads to the question: Is it possible to analytically prove that chartists' behavior can lead to exploding prices irrespective of fundamentalists' compensatory effects?

The main contribution of this chapter is a mathematically rigorous proof that the behavior of chartists, specifically the behavior of linear feedback traders without rational expectations and without information about the market (e.g., fundamental value, trading volume, or even prices), can overcome the stabilizing effects of traders with rational expectations of the fundamental value. Put differently, prices explode because the stabilizing effects of fundamentalists are outweighed by linear feedback traders. As shown in the proof, thresholds for model-inherent values can be computed that make the occurrence of a bubble certain. Furthermore, there are specific values of external parameters that allow the thresholds of the inherent values to be met. The analysis reveals that even fundamentalists without any liquidity constraints and with perfect information about the price, the fundamental value, and the market's characteristics are not sufficient to stabilize a simply constructed market based on (excess) demand if the feedback trader's initial investment is large enough.

10.2 Model Structure

The model consists of a one asset market and is populated with different types of heterogeneous agents, e.g. fundamentalists (F), chartists (C), and noise traders (N). For simplification of the analysis, we assume that there is only one feedback trader, that is we treat all existing feedback traders as one big, average feedback trader. There is indeed no difference between one feedback trader with an initial investment I_0^C and fixed K and n feedback traders with initial investments $\frac{I_0^C}{n}$ and the same K. In the same manner, we identify all fundamentalists with the fundamentalist and all noise traders with the noise trader. We analyze four cases: A market with either only a noise trader, only a feedback trader, only a fundamentalist, or the most interesting case where a fundamentalist and a feedback trader are trading simultaneously. The case of a feedback trader and a noise trader acting simultaneously is analyzed by Baumann (2015). Note that the chartist used in this section is exactly the linear long feedback trader as introduced in Chap. 5. However, since the price dynamics in this section is totally different to the previous sections (especially, there is no price taker property anymore) also the trading dynamics strongly differs. This is the reason, why we use C instead of L in this section. Even if the chartist's/linear feedback trader's strategy is the same, the results obtained for trader C differ from those for trader L, due to the usage of HAMs.

At the beginning of every period $t \in \{0, 1, \ldots, T\}$, each active agent $\ell \in \{C, F, N\}$ decides how to invest based on the respective investment strategy, where T is unknown or even ∞. Each investment strategy I_t^ℓ is guided by a different heuristic (rule of thumb). Based on the strategy chosen, each agent then allocates the own financial resources among the asset market, which consists of one asset and one zero rate bond. The trader is aware of past market data and of expectations of future fundamental values $\mathbb{E}[f_{t+1}]$. The resulting changes in the investments, denoted by ΔI_t^ℓ or, better, the buying or selling decisions D_t^ℓ (demand function), are cleared by a market maker who adjusts asset prices according to (excess) demand. After the traders have observed the price change Δp_{t+1}, and hence their own gains or losses Δg^ℓ_{t+1} in the recent period, they use this information in making their next investment decision. For all processes α_t we set $\Delta\alpha_t = \alpha_t - \alpha_{t-1}$ as the change of the underlying process, e.g., Δg^ℓ_t is the period profit while g^ℓ_t is the overall gain/loss of trader ℓ. The timeline of the traders' and the market maker's decisions and interactions is shown in Fig. 10.1.

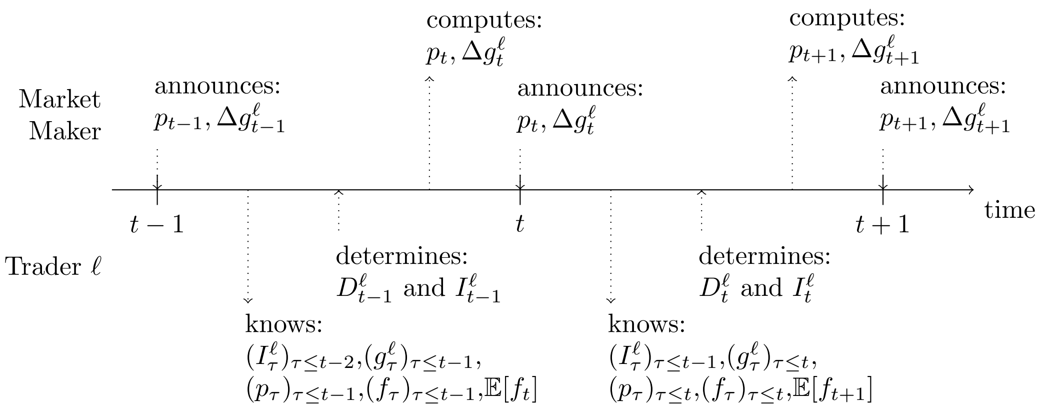

Figure 10.1: Timeline of the traders' and the market maker's decisions and interactions with $\Delta g^\ell_t = I^\ell_{t-1} \cdot \frac{\Delta p_t}{p_{t-1}}$

In the feedback trading literature, the price is usually determined through a certain price process, for example, a geometric Brownian motion (GBM), which is exogenously given (Barmish and Primbs, 2016). This implies that the traders are not able to influence the price. As explained in Sec. 10.1, agent-based price models have evolved in the academic economics literature (Hommes, 2006a) to avoid this price taker property. Note that the price taker property is a strong restriction. According to agent-based models, the price is a function of traders' investment decisions. Note that noise traders are important in the HAM literature to explain, e.g., stylized facts. However, in this chapter we only use them to show that our HAM is a generalization of the GBM.

We denote the sum of all traders' buying and selling decisions at time t with $D_t = \sum_\ell D^\ell_t$. Based on the idea of interacting agents, Baumann (2015) constructs a pricing model that fulfills the law of (excess) demand, similarly to

- $p_{t+1} = p_t$, if $D_t = 0$,
- $p_{t+1} \to \infty$, if $D_t \to \infty$,
- $p_{t+1} \to 0$, if $D_t \to -\infty$, and
- p_{t+1} strictly increasing in D_t.

Actually, Baumann (2015) uses ΔI instead of D, which does not change the dynamics substantially. The choice of Baumann (2015) is a little bit more unrealistic because value change caused by price changes in the previous period—and not by buying/selling decisions—cause price changes in the current period. However, the computation of the investment formulae is much shorter for ΔI. Here, we rely on the more realistic model, which uses D.

For simplification, we assume an infinite supply. Infinite supply is, for example, given for synthetic assests, betting slips, etc. These assets are produced by the market maker without any restriction. Thus, the market maker can clear the market for sure. It follows that the market maker sets the new asset price according to the asset demand only (without use of any external information). The buying/selling decision of trader ℓ is given trough

$$D_t^\ell := I_t^\ell - \frac{p_t}{p_{t-1}} \cdot I_{t-1}^\ell$$

for $t \geq 1$ and $D_0^\ell = I_0^\ell$. The second term $\frac{p_t}{p_{t-1}} \cdot I_{t-1}^\ell$ is the investment of trader ℓ at time t before the trader bought or sold anything, because I_{t-1}^ℓ is the investment at time $t-1$ and $\frac{p_t}{p_{t-1}}$ denotes how much this old investment has in- or decreased through price changes. The first term I_t^ℓ denotes the investment at time t according to the trader's strategy. As a consequence, the difference has to be bought or sold.

The model requirements are, e.g., fulfilled by the exponential function:

$$\begin{aligned} p_{t+1} &= p_t \cdot e^{M^{-1} D_t} \\ &= p_0 \cdot e^{M^{-1} B_t} \end{aligned}$$

where $M > 0$ is a scaling factor expressing the trading volume of the underlying asset and $B_t = \sum_{i=0}^t D_t$ is the amount of bought or sold assets up to time t (measured in currency units).

This pricing rule is similar to that one Batista et al. (2017) use. The works of Batista et al. (2017) and Baumann (2015) were developed independently. Unless otherwise stated, for simplicity of the notation M is set to $M = 1$. As mentioned above, our pricing model is cleared through a market maker (Drescher and Herz, 2012), cf. Fig. 10.2. We recall that the market maker acts as a privileged trader who sets prices according to (excess) demand (see Fig. 10.2) and hence ensures market clearing (cf. the role of a broker in stock markets) (Hommes, 2006a,b). Possible profit making by and survival of the market maker is not of interest for the work at hand.

This market model is a generalization of the GBM. This is easy to see when we define a noise trader (N). Noise is, according to Black (1986), essential for the function of markets. A noise trader's strategy (we model all noise traders as one big noise trader), is random but should follow a certain distribution, i.e., is not arbitrary. We set

$$D_t^N := M \cdot \left(\left(\mu - \tfrac{\sigma^2}{2} \right) + \sigma \cdot \Delta W_t \right),$$

where $M > 0$ is the scaling parameter for volumes of trade, $\mu \in \mathbb{R}$ is interpreted as a saving deposit per time step, $\sigma > 0$ specifies the volatility of the market, $\Delta W_t \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$ is a random walk that brings noise into the market and $-\frac{\sigma^2}{2}$ can be interpreted as risk aversion (note that $\Delta t = 1$). Such a demand function can be interpreted as a noise trader who has the right idea for the trend of the market in mind, but who's behavior is noisy and who is risk-averse: The noisier the behavior the less the trader is investing. The term μ is a target amount of money the noise trader wants to save. This construction of noise traders leads to the result that the presented market model is a natural generalization of the geometric Brownian motion (GBM).

Theorem 93. If the noise trader is the only trader on the market, for all $M > 0$ the paths p_t of the price process follow the paths of the geometric Brownian motion.

Proof. If there is only one trader that is a noise trader it holds $D_t = D_t^N$ and

$$p_{t+1} = p_t \cdot e^{M^{-1} D_t^N} = p_t \cdot e^{\left(\left(\mu - \frac{\sigma^2}{2}\right) + \sigma \cdot \Delta W_t\right)},$$

with $\Delta W_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$. This is exactly the formula of a discretized GBM with trend. $\square$

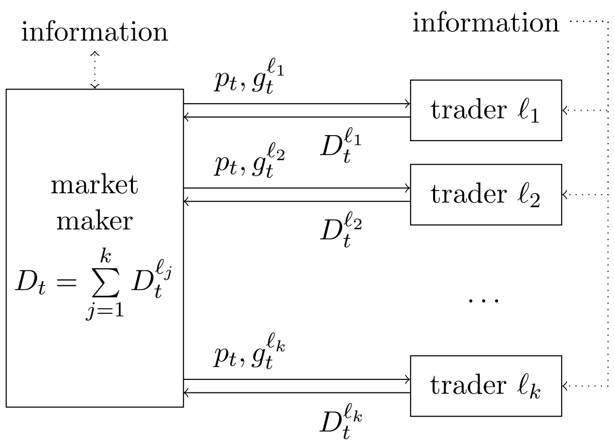

Figure 10.2: Schematic representation of the role of the market maker with k traders

So far, we constructed an HAM and analyzed the case when there is only a noise trader. Next, we investigate what happens if there is only a chartist (a linear long feedback trader) on the market.

10.3 Feedback Traders

When specifying their strategy, feedback traders take into account only their own gains and losses. The strategy, thus, depends on price changes and on their previous investments, that is, feedback traders are chartists because gains or losses, respectively, are a function of the price but not of any fundamental value. In calculating a certain trader's gain, the market maker takes into account the trader's investment and the asset price. Therefore, for feedback traders not only is it true that the investment affects the gain, but also that the gain determines the investment.

We rely on the (positive) linear feedback strategy

$$I_t^C := I_0^C + K \cdot g_t^C$$

where the linear feedback trader calculates the own investment I_t^C at time t as a linear function of the gain/loss function g_t^C using the initial investment $I_0^C > 0$ and a feedback parameter $K > 0$. This is the long side of the SLS rule. We rely on the long side only because this is the side possibly causing financial bubbles. In Fig. 10.3 a feedback loop between the gain or loss g^C of a linear feedback trader and the respective investment I^C is shown. By calculating the gain or loss of a specific trader (or group of traders) ℓ via

$$g_t^\ell = \sum_{i=1}^{t} I_{i-1}^\ell \cdot \frac{p_i - p_{i-1}}{p_{i-1}},$$

where p_t denotes the price process and I_t^ℓ the trader's investment at time t, it follows that linear feedback traders are trend followers given $I_t^C > 0$. The relative price change $\frac{p_t - p_{t-1}}{p_{t-1}}$ is called return on investment (ROI). A trader is called a trend follower (cf. Covel, 2004) if the trader is buying when prices are rising and selling when prices are falling. Note that the particular demand at time $t \geq 1$ is given by

$$\begin{aligned} D_t^C &= I_t^C - \frac{p_t}{p_{t-1}} I_{t-1}^C \\ &= I_{t-1}^C + K \cdot I_{t-1}^C \cdot \frac{p_t - p_{t-1}}{p_{t-1}} - \frac{p_t}{p_{t-1}} I_{t-1}^C \\ &= (K-1) \cdot I_{t-1}^C \cdot \frac{p_t - p_{t-1}}{p_{t-1}}, \end{aligned}$$

whereas I_t^C denotes the total investment at time t of feedback trader C.

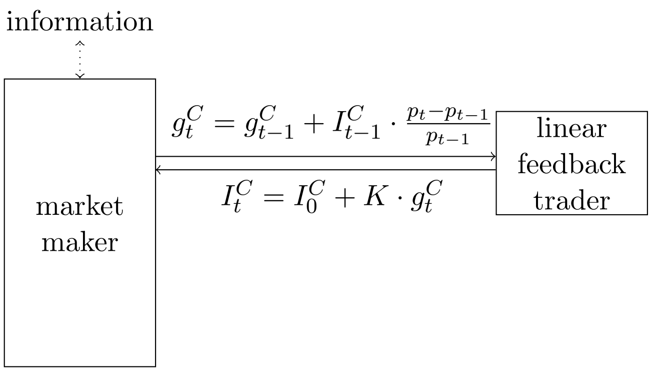

Figure 10.3: Schematic interaction between market maker and linear feedback trader

Note that

$$\Delta I_t^C = I_t^C - I_{t-1}^C$$

$$
\begin{aligned}
&= I_{t-1}^C + K \cdot I_{t-1}^C \cdot \frac{p_t - p_{t-1}}{p_{t-1}} - I_{t-1}^C \\
&= K \cdot I_{t-1}^C \cdot \frac{p_t - p_{t-1}}{p_{t-1}}.
\end{aligned}
$$

This means that $D_t^C = \frac{K-1}{K} \Delta I_t^C$ and thus $B_t^C = \frac{K-1}{K} I_t^C$. If $K \neq 1$, the trader is not only a buy-and-hold trader, but is really buying and selling. We can rewrite

$$D_t^C = K B_t^C \cdot \frac{p_t - p_{t-1}}{p_{t-1}}.$$

Now, we always assume $K > 1$, i.e., a trader who is buying more and more when making profit (because this is the interesting case for bubble investigation).

Rising prices lead to increasing gain for the linear feedback trader if $I_t^C > 0$ and, thus, the trader buys. Analogously, falling prices lower the gain and the trader sells.

Here, markets with purely linear feedback traders are studied, that means $I_t^\ell \equiv 0$ for all $\ell \neq C$. In this case, the feedback-based investment strategy is given by

$$
\begin{aligned}
&D_0^C = I_0^C > 0, \\
&D_1^C = (K-1) \cdot I_0^C \cdot \left(e^{M^{-1} I_0^C} - 1 \right), \text{ and} \\
&D_t^C = (K-1) \cdot I_{t-1}^C \cdot \left(e^{M^{-1} D_{t-1}^C} - 1 \right), \; t \geq 2.
\end{aligned}
$$

This leads us to the following theorem.

Theorem 94. If in our market maker model there is only one trader, a linear feedback trader C, trading with the market maker, the price dynamics (for $t \geq 2$) is:

$$\Delta B_t^C = K B_{t-1}^C \cdot \left(e^{M^{-1} \Delta B_{t-1}^C} - 1 \right)$$

Baumann (2015) shows in a very similar model that in the event only one feedback trader is acting on the market with the price process described in Sec. 10.2, it holds that

$$
\begin{aligned}
&I_t^C > 0 \;\forall t, \\
&D_t^C > 0 \;\forall t, \text{ and} \\
&D_t^C > D_{t-1}^C \Rightarrow \; D_{t+1}^C > D_t^C.
\end{aligned}
$$

We prove this in the remainder of this section.

Theorem 95. If the investments of all other traders are zero, the investment I_t^C and the demand function D_t^C of the linear feedback trader are positive.

Proof. The theorem is proven by induction. Because of $I_0^C > 0$ and $e^{M^{-1} I_0^C} > 1$, the initial inequality $D_1^C > 0$ is true. It follows $I_1^C = I_0^C (e^{M^{-1} I_0^C} - 1) + D_1^C > 0$. The induction step follows, as $e^{M^{-1} D_{t-1}^C} > 1$ and $I_t^C = I_{t-1}^C (e^{M^{-1} D_{t-1}^C} - 1) + D_t^C > 0$. □

It holds that $D_t^C > 0$ because of $I_0^C > 0$. This means that feedback traders' investment increases prices and thus also their gain, leading again to positive buying decisions and so on. But this does not necessarily have to end in a bubble. We say that a bubble occurs if $\exists t^* : \ \Delta p_{t+1} > \Delta p_t \ \forall t \geq t^*$. Note that if there are only chartists it holds that $p_t = p_{t-1} e^{M^{-1} D_{t-1}^C}$, i.e., $\Delta p_t = p_{t-1}\left(e^{M^{-1} D_{t-1}^C} - 1\right)$. Two typical demand paths can be identified in the scenario where only one feedback-based trader is acting on the market. The two paths are shown in Fig. 10.4 and Fig. 10.5 where the asset price p_t is indicated with a solid line and the feedback trader's investment with a dashed one. If I_0^C lies below a specific threshold, I_t^C converges (Fig. 10.4). If it is above this threshold, the investment explodes (Fig. 10.5). Baumann (2015) provides a non-closed formula determining the threshold. Specific values can be derived through a simulation like that one in Figs. 10.4 and 10.5 and by algorithmically localizing the threshold. That means, the demand function and, thus, the price can converge to some value.

Theorem 96. If the investments of all other traders are zero and $\exists t^* \in \mathcal{T} : \Delta D_{t^*}^C > 0$ then

$$\Delta D_t^C > 0$$

holds for all $t \geq t^*$. That means, the bought amount of stocks D_t^C of the feedback trader is strictly increasing for all $t \geq t^*$.

Proof. The induction step

$$D_t^C > D_{t-1}^C \Rightarrow D_{t+1}^C > D_t^C, \ t \geq 1,$$

has to be shown. This is true because of

$$D_{t+1}^C > D_t^C \Leftrightarrow I_t^C \cdot \left(e^{M^{-1} \cdot D_t^C} - 1\right) > I_{t-1}^C \cdot \left(e^{M^{-1} \cdot D_{t-1}^C} - 1\right),$$

$D_t^C > 0$ from which it follows $\Delta I_t^C > 0$, and the induction hypothesis. □

This is important as it is shown that, together with the results of Sec. 10.6, the price explosion effects of feedback traders that would possibly occur in absence of fundamentalists can be compensated by fundamentalists at least to a certain degree.

10.4 Fundamentalists

As explained in the introduction, Chap. 1, fundamentalists buy when the price is below the fundamental value $f_t > 0$ and sell when the price is above the fundamental value. If, for example, the fundamental value is below the asset price, fundamentalists conclude that the price decreases in the long run, not necessarily in the next step. So they possibly do not sell so much that their investment becomes negative, but they reduce their investment. Thus, it is of particular interest how much fundamentalists buy or sell in the respective cases. For deterministic fundamental values f_t, i.e., the fundamental

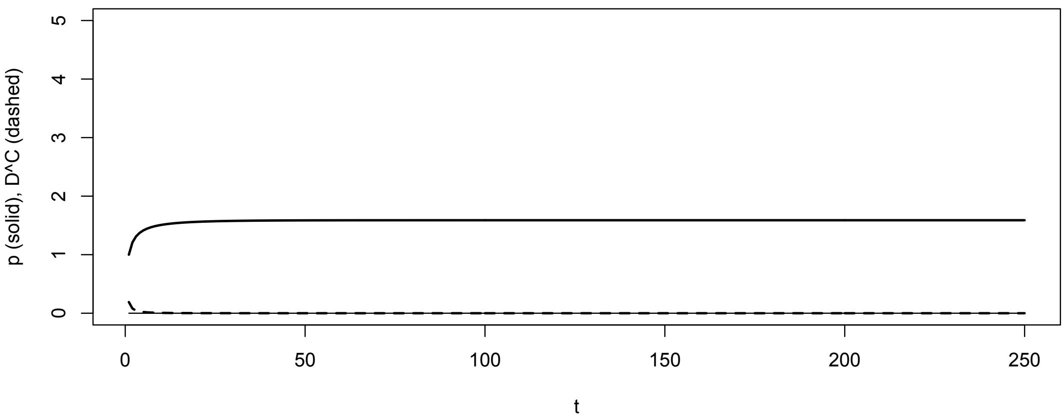

Figure 10.4: Demand of feedback trader is indicated with a dashed line, development of the asset price p_t is indicated with a solid line. The initial investment I_0^C is below a specific threshold: D_t^C (dashed line) converges $\{p_0 = 1, M = 1, T = 250, I_0^C = 0.191, K = 2\}$

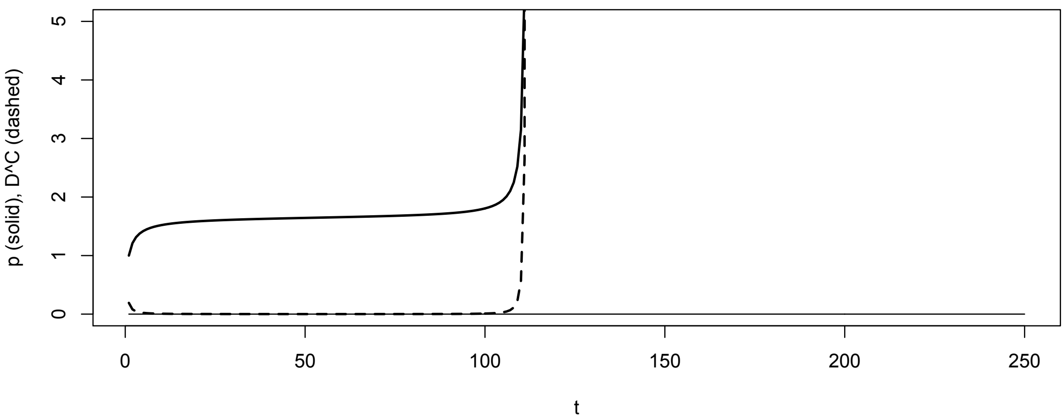

Figure 10.5: Demand of feedback trader is indicated with a dashed line, development of the asset price p_t is indicated with a solid line. The initial investment I_0^C is above a specific threshold: I_t^C (dashed line) diverges $\{p_0 = 1, M = 1, T = 250, I_0^C = 0.192, K = 2\}$

value is a function in t, one way of determining the demand rate is

$$D_t^F = M \cdot \ln \frac{f_{t+1}}{p_t}$$

(cf. Drescher and Herz, 2012). In this case, fundamentalists do not need to estimate the fundamental value because it is fixed and certain for the future period. Traders following this demand rule could be called *strong fundamentalists* because their investment strategy could push the price back to its fundamental value at any time.

Theorem 97. If the strong fundamentalist is the only trader buying/selling at time t, then for any $p_t > 0$ and f_{t+1} it follows:

$$p_{t+1} = f_{t+1}$$

Proof.

$$\begin{aligned} p_{t+1} &= p_t \cdot e^{\ln \frac{f_{t+1}}{p_t}} \\ &= p_t \cdot \frac{f_{t+1}}{p_t} \\ &= f_{t+1} \end{aligned}$$

□

Section 10.5 presents the case of a fundamentalist trading based on a distorted fundamental value. It turns out, however, that this distortion does not affect the general behavior of the market model.

10.5 Expectations and Noise

Some types of traders, for example informed speculators (De Long et al., 1990b), base their trading decisions on rational expectations. Is this the case for feedback traders and fundamentalists?

In general, for feedback traders and trend followers, the answer is "no," as they only assume the existence of a trend. For example, based on the current slope of asset price development ($p_t - p_{t-1}$) they forecast the future direction of the asset. However, fundamentalists are assumed to have rational expectations (see, e.g., Drescher and Herz, 2012). Generally, they pursue the strategy

$$D_t^F = M \cdot \ln \frac{\mathbb{E}[f_{t+1}|\mathcal{F}_t]}{p_t}.$$

Even a casual observation of real markets makes clear that price fluctuations are not always purely rational. There is always noise and uncertainty in the market, a factor considered essential by many economists (see, e.g., Black, 1986; De Long et al., 1990a). Some reasons for noise include that traders make mistakes, trade on unreliable (noisy) information, or simply enjoy trading and are not overly concerned with being rational about it.

Here, we do not assume that traders are making mistakes, as this could lead to unsystematic behavior, i.e., we do not want to take noise traders into account (a market with a linear feedback trader and a noise trader is analyzed by Baumann (2015)). Furthermore, both feedback traders and fundamentalists do follow a specified strategy. Thus, the only way noise could enter the market is through noisy information. However, the traders' investments as well as the price, announced by the market maker (see Fig. 10.1), are not distorted. The only information that could be noisy is that about the fundamental value. In this case, the fundamentalist has to estimate f_{t+1} at time t and trade according to $\mathbb{E}[f_{t+1}|\mathcal{F}_t]$. Since it is unreasonable that $|f_{t+1} - \mathbb{E}[f_{t+1}|\mathcal{F}_t]|$ becomes arbitrary large but exploding prices imply $|p_t - f_t| \to \infty$, the effects of noisy information do not play a decisive role. Therefore, we a priori consider f_t a deterministic fundamental value in this chapter of the presented work.

10.6 Proof of Limitations of Fundamentalists' Stabilizing Effects

In this section we demonstrate analytically and mathematically rigorously that fundamentalists are not always able to stabilize markets through their trading actions. Instead of using simulations, we inductively prove that effects of linear feedback traders dominate those of fundamentalists and destabilize markets.

Since we have already defined the pricing model and the traders, the next task is to check whether fundamentalists defined according to Sec. 10.4 are able to stabilize the price when trading simultaneously on the market with linear feedback traders following Sec. 10.3. To simplify the notation, we set $f_t \equiv 1$. This is one special case, but if we can show the destabilizing effects of feedback traders' investment strategy for this case, it proves that fundamentalists do not always have market stabilizing effects. The proof proceeds without using technical trading restrictions, for example, limits on feedback traders' investment amount.

The two trader types linear feedback trader C and fundamentalist F are suitable for analyzing the question of destabilizing effects of linear feedback traders because if it turns out that prices explode for appropriately chosen parameters I_0^C and K of linear feedback traders even when acting on a market with fundamentalists, who are employing an investment strategy that could bring prices close to the fundamental value at every point of time, it is strong evidence that chartists' rules, in this case the linear feedback strategy, are able to overcome the effects of strong fundamentalists in various market situations. Why it is enough to consider only linear feedback traders and fundamentalists and no other type of traders, some of which are presented by Ivanova et al. (2014), becomes obvious when taking into consideration that if feedback traders' investment goes to infinity which means prices explode, then also the absolute value of fundamentalists' investment goes to infinity. Thus, compared to the exploding investments of feedback traders and fundamentalists, the relatively small investment of other possible traders may be neglected at least for our analysis.

Trend followers invest a lot when prices rise strongly and fundamentalists disinvest a lot when the price greatly exceeds the fundamental value, i.e., the investment of trend followers goes to infinity and that of fundamentalists goes to minus infinity. For traders

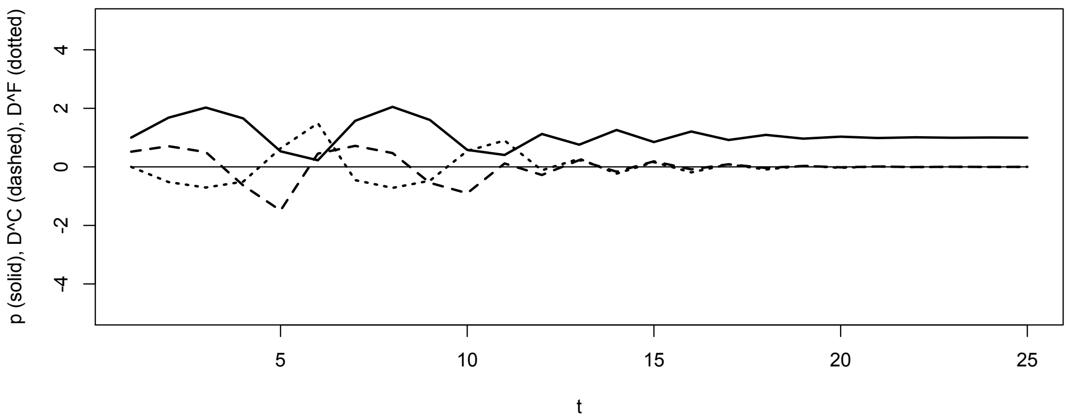

Figure 10.6: A typical situation in a market involving feedback traders and fundamentalists. Price and feedback trader's demand converge, i.e., fundamentalists' effects predominate; parameters $\{p_0 = 1, M = 1, T = 25, f_t \equiv 1, I_0^C = 5.19, K = 2\}$

who neither predicate their investment on the distance of fundamental value and price nor on the slope of the price it is unreasonable that their investment goes to (minus) infinity. Simulations reveal two typical price developments (see Figs. 10.6 and 10.7).

In Fig. 10.6, fundamentalists' effects predominate and the price stabilizes around the fundamental value. In Fig. 10.7, however, market development is not that obvious. At a first glance, the figure might suggest that prices explode. But as the simulation software reaches its limits, it becomes unclear whether or not prices level out in these simulation scenarios. We therefore need an analytical examination. In cases like those shown in the simulated Fig. 10.7, the proposition of Thm. 99 determines with certainty whether the bought amount of assets of the feedback traders is in fact exploding, or whether this only seems to be the case due to simulation insufficiencies and the portfolio eventually stabilizes, but with a greater amplitude as, for example, in Fig. 10.6.

To simplify the expressions in the model, we assume in addition to $f_t \equiv 1$ that $p_0 = 1$ in all upcoming equations. This choice is just one possible scaling but does not change the model's dynamics in general. We define a process α_t as $(\alpha_t)_{t\in\mathbb{Z}} \subset \mathbb{R}$ with $\alpha_t = 0 \ \forall t < 0$. Furthermore, we define the Δ-operator as $\Delta^k \alpha_t := \Delta^{k-1}\alpha_t - \Delta^{k-1}\alpha_{t-1}$, $\Delta^1 \alpha_t := \Delta\alpha_t = \alpha_t - \alpha_{t-1}$, and $\Delta^0\alpha_t := \alpha_t$. A price process p_t is strictly positive, i.e., $(p_t)_t > 0$ for all $t \geq 0$. It holds:

$$\begin{aligned} D_t^F &= M \cdot \ln \frac{f_{t+1}}{p_t} \\ &= -M \cdot \ln e^{M^{-1} B_{t-1}} \\ &= -B_{t-1} \end{aligned}$$

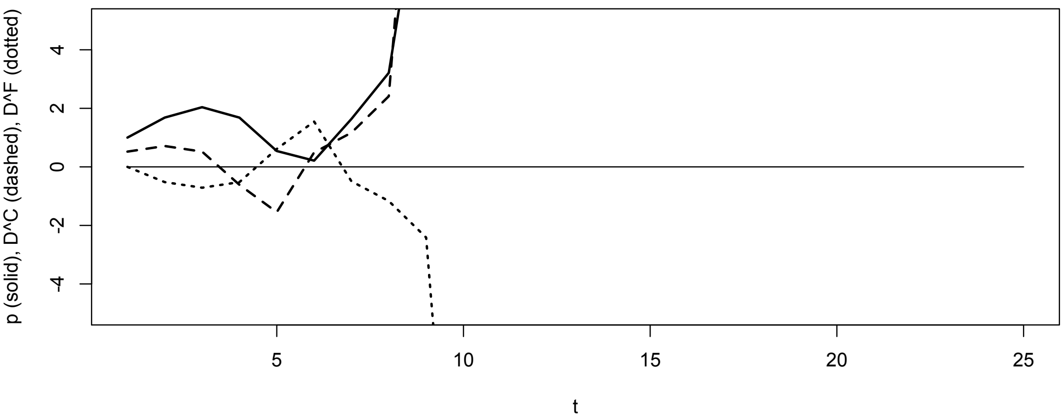

Figure 10.7: Another typical situation in a market involving feedback traders and fundamentalists. Price and feedback trader's demand diverge, i.e., feedback traders' effects predominate; parameters $\{p_0 = 1, M = 1, T = 25, f_t \equiv 1, I_0^C = 0.521, K = 2\}$

$$\begin{aligned} &= -B_{t-1}^C - B_{t-1}^F \\ \Rightarrow\ B_t^F &= -B_{t-1}^C \\ \Rightarrow\ D_t^F &= -D_{t-1}^C \end{aligned}$$

With this, we can specify the demand of the feedback traders:

$$\begin{aligned} D_t^C &= K \cdot B_{t-1}^C \left(e^{M^{-1}\left(D_{t-1}^C + D_{t-1}^F\right)} - 1 \right) \\ &= K \cdot B_{t-1}^C \left(e^{M^{-1}\left(D_{t-1}^C - D_{t-2}^C\right)} - 1 \right) \\ &= K \cdot B_{t-1}^C \left(e^{M^{-1}\Delta D_{t-1}^C} - 1 \right) \end{aligned}$$

Theorem 98. If there are exactly one linear feedback trader C and one fundamentalist F trading with the market maker, it holds:

$$\Delta B_t^C = K \cdot B_{t-1}^C \left(e^{M^{-1}\Delta^2 B_{t-1}^C} - 1 \right)$$

Theorem 99 tells us conditions for the feedback trader's cumulated demand B^C for which prices explode. Note that the following implication holds:

$$\Delta^k \alpha_{t-1}^C > a \ \wedge\ \Delta^{k+1} \alpha_t^C > b \ \Rightarrow\ \Delta^k \alpha_t^C > a + b.$$

We obtain this directly from the definition of the delta operator which is equivalent to

$$\Delta^k \alpha_t^C = \Delta^{k+1}\alpha_t^C + \Delta^k \alpha_{t-1}^C.$$

Note that $D_t^C = \Delta B_t^C$ and analogously for the derivatives.

Theorem 99. For the demand function resp. for the bought and sold assets of the positive linear feedback trader interacting with a strong fundamentalist on our market model, under conditions

$$\Delta^3 B_t^C > M \cdot \ln 2,$$
$$\Delta^2 B_t^C > M \cdot \ln 2$$
$$\Delta B_{t-1}^C > 0, \text{ and}$$
$$B_{t-2}^C > 0$$

for some $t \geq 2$, it follows that

$$\Delta^k B_{t+1}^C > M \cdot \ln 2 \ \forall k \in \{0, 1, 2, 3\}.$$

Theorem 99 is proven by induction in the following.

Proof. It is enough to prove the proposition for $k = 3$ as all other inequalities can then be derived from the definition of the Δ-operator.

$$\begin{aligned}
\frac{1}{K}\Delta^3 B_{t+1}^C &= \frac{1}{K}\left(\Delta^2 B_{t+1}^C - \Delta^2 B_t^C\right) \\
&= \frac{1}{K}\left(\Delta B_{t+1}^C - 2\Delta B_t^C + \Delta B_{t-1}^C\right) \\
&= B_t^C \left(e^{M^{-1}\Delta^2 B_t^C} - 1\right) \\
&\quad - 2B_{t-1}^C \left(e^{M^{-1}\Delta^2 B_{t-1}^C} - 1\right) \\
&\quad + B_{t-2}^C \left(e^{M^{-1}\Delta^2 B_{t-2}^C} - 1\right) \\
&= \left(B_{t-2}^C + \Delta B_{t-1}^C + \Delta B_t^C\right)\left(e^{M^{-1}\Delta^2 B_t^C} - 1\right) \\
&\quad - 2\left(B_{t-2}^C + \Delta B_{t-1}^C\right)\left(e^{M^{-1}\Delta^2 B_{t-1}^C} - 1\right) \\
&\quad + B_{t-2}^C \left(e^{M^{-1}\Delta^2 B_{t-2}^C} - 1\right) \\
&= B_{t-2}^C \left(e^{M^{-1}\Delta^2 B_t^C} - 1\right) \\
&\quad + \Delta B_{t-1}^C \left(e^{M^{-1}\Delta^2 B_t^C} - 1\right) \\
&\quad + \Delta B_t^C \left(e^{M^{-1}\Delta^2 B_t^C} - 1\right) \\
&\quad - 2B_{t-2}^C \left(e^{M^{-1}\Delta^2 B_{t-1}^C} - 1\right)
\end{aligned}$$

$$
\begin{aligned}
&- 2\Delta B^C_{t-1}\left(e^{M^{-1}\Delta^2 B^C_{t-1}} - 1\right)\\
&+ B^C_{t-2}\left(e^{M^{-1}\Delta^2 B^C_{t-2}} - 1\right)\\
=\ & B^C_{t-2}\left(e^{M^{-1}\Delta^2 B^C_{t}} - 2e^{M^{-1}\Delta^2 B^C_{t-1}} + e^{M^{-1}\Delta^2 B^C_{t-2}}\right) \quad (*)\\
&+ \Delta B^C_{t-1}\left(e^{M^{-1}\Delta^2 B^C_{t}} - 2e^{M^{-1}\Delta^2 B^C_{t-1}} + 1\right) \quad (**)\\
&+ \Delta B^C_{t}\left(e^{M^{-1}\Delta^2 B^C_{t}} - 1\right) \quad (***)
\end{aligned}
$$

We evaluate these summands separately:

$$
\begin{aligned}
(**) &= \Delta B^C_{t-1}\left(e^{M^{-1}\Delta^2 B^C_{t-1} + M^{-1}\Delta^3 B^C_{t}} - 2e^{M^{-1}\Delta^2 B^C_{t-1}} + 1\right)\\
&= \Delta B^C_{t-1}\left(e^{M^{-1}\Delta^2 B^C_{t-1}}\left(e^{M^{-1}\Delta^3 B^C_{t}} - 2\right) + 1\right)\\
&> \Delta B^C_{t-1}\left(e^{M^{-1}\Delta^2 B^C_{t-1}}(2-2) + 1\right)\\
&> 0\\
(***) &= \left(\Delta B^C_{t-1} + \Delta^2 B^C_{t}\right)\left(e^{M^{-1}\Delta^2 B^C_{t}} - 1\right)\\
&> 0 + M \cdot \ln 2\\
(*) &= B^C_{t-2}\Big(e^{M^{-1}\Delta^2 B^C_{t-2} + M^{-1}\Delta^3 B^C_{t-1} + M^{-1}\Delta^3 B^C_{t}}\\
&\quad - 2e^{M^{-1}\Delta^2 B^C_{t-2} + M^{-1}\Delta^3 B^C_{t-1}} + e^{M^{-1}\Delta^2 B^C_{t-2}}\Big)\\
&= B^C_{t-2} e^{M^{-1}\Delta^2 B^C_{t-2}}\left(e^{M^{-1}\Delta^3 B^C_{t-1}}\left(e^{M^{-1}\Delta^3 B^C_{t}} - 2\right) + 1\right)\\
&> B^C_{t-2} e^{M^{-1}\Delta^2 B^C_{t-2}}\left(e^{M^{-1}\Delta^3 B^C_{t-1}}(2-2) + 1\right)\\
&= B^C_{t-2} e^{M^{-1}\Delta^2 B^C_{t-2}}\\
&> 0
\end{aligned}
$$

As a result, we obtain

$$K^{-1}\Delta^3 B^C_{t+1} > M \cdot \ln 2$$

and since $K > 1$

$$\Delta^3 B^C_{t+1} > M \cdot \ln 2.$$

□

This means, the feedback trader's bought and sold assets, the demand, the slope of the demand, and the curvature of the demand are strictly greater than $M \cdot \ln 2$ for all $t \geq t^*$ for some t^*. All in all, this is a fast exploding demand, which leads to an equally quickly exploding price.

$$\begin{aligned} p_{t+1} &= p_t \cdot e^{M^{-1} \cdot \left(D_t^F + D_t^C\right)} \\ &= p_t \cdot e^{\ln \frac{f_{t+1}}{p_t}} \cdot e^{M^{-1} \cdot D_t^C} \\ &= f_{t+1} \cdot e^{M^{-1} \cdot D_t^C} \end{aligned}$$

Theorem 100. If there are exactly one fundamentalist F and one chartist C (a linear long feedback trader), the price dynamics satisfies for $t > 0$:

$$p_t = f_t e^{M^{-1} D_{t-1}^C}$$

Recall that $D_t^\ell = \Delta B_t^\ell$. As an interpretation, note that since $D_t^F = -D_{t-1}^C$, fundamentalists always respond one period later with minus the demand of the feedback traders. Theorem 99 tells us that the feedback trader's cumulated demand increases, the demand itself increases, and the first and second derivative increase, too. Furthermore, all of these growth rates are bounded from below. Since the fundamentalist's demand is minus the demand of the feedback trader from one period before, the ratio of the bought and sold amounts is strictly increasing, that is the feedback trader's exploding effect predominates the fundamentalist's stabilizing one.

That the conditions of Theorem 99 for the endogenous variables B_{t-2}^C, ΔB_{t-1}^C, $\Delta^2 B_t^C$, $\Delta^3 B_t^C$ may be fulfilled for some t (and some parameter assignment) is shown in Table 10.1 in which the demand development of the feedback trader is listed for $I_0^C = 0.521$, $K = 2$, and $M = 1$. In short, there are exogenous variables that lead to a price explosion. This demonstrates that feedback traders' effects are able to overcome fundamentalists' effects.

	$B_t^{FT} \approx$	$\Delta B_t^{FT} = D_t^{FT} \approx$	$\Delta D_t^{FT} \approx$	$\Delta^2 D_t^{FT} \approx$
$t=0$	$5.210000 \cdot 10^{-1}$	$5.210000 \cdot 10^{-1}$	$0.000000 \cdot 10^{0}$	$0.000000 \cdot 10^{0}$
$t=1$	$1.233426 \cdot 10^{0}$	$7.124264 \cdot 10^{-1}$	$1.914264 \cdot 10^{-1}$	$0.000000 \cdot 10^{0}$
$t=2$	$1.753872 \cdot 10^{0}$	$5.204459 \cdot 10^{-1}$	$-1.919805 \cdot 10^{-1}$	$-3.834069 \cdot 10^{-1}$
$t=3$	$1.141150 \cdot 10^{0}$	$-6.127224 \cdot 10^{-1}$	$-1.133168 \cdot 10^{0}$	$-9.411878 \cdot 10^{-1}$
$t=4$	$-4.062233 \cdot 10^{-1}$	$-1.547373 \cdot 10^{0}$	$-9.346507 \cdot 10^{-1}$	$1.985175 \cdot 10^{-1}$
$t=5$	$8.715681 \cdot 10^{-2}$	$4.933801 \cdot 10^{-1}$	$2.040753 \cdot 10^{0}$	$2.975404 \cdot 10^{0}$
$t=6$	$\boxed{1.254431 \cdot 10^{0}}$	$1.167274 \cdot 10^{0}$	$6.738944 \cdot 10^{-1}$	$-1.366859 \cdot 10^{0}$
$t=7$	$3.667613 \cdot 10^{0}$	$\boxed{2.413181 \cdot 10^{0}}$	$1.245907 \cdot 10^{0}$	$5.720125 \cdot 10^{-1}$
$t=8$	$2.183026 \cdot 10^{1}$	$1.816265 \cdot 10^{1}$	$\boxed{1.574946 \cdot 10^{1}}$	$\boxed{1.450356 \cdot 10^{1}}$
$t=9$	$3.019914 \cdot 10^{8}$	$3.019913 \cdot 10^{8}$	$3.019913 \cdot 10^{8}$	$3.019913 \cdot 10^{8}$

Table 10.1: The boxed table entries fulfill the conditions of Thm. 99 for $t = 8$ for which prices explode; market parameters are as in Figure 10.7 (Note: $\ln 2 \approx 0.6931472$) $\{p_0 = 1, M = 1, T = 25, f_t \equiv 1, I_0^C = 0.521, K = 2\}$

On the other hand, Table 10.2 sets out a situation where the price would explode when only feedback traders are acting on the market. The conditions of Thm. 96 hold for the feedback traders, so, according to Baumann (2015) resp. Thm. 96, their demand causes a bubble in the absence of any other traders. However, if fundamentalists enter the market, price explosion is prevented, as the demand rates tend to 0 at time $t = 73$ in Table 10.2. Clearly, the conditions of Thm. 99 for feedback traders are not satisfied.

	$B_t^{FT} \approx$	$\Delta B_t^{FT} = D_t^{FT} \approx$	$\Delta D_t^{FT} \approx$	$\Delta^2 D_t^{FT} \approx$
$t = 0$	0.1920000	$1.920000 \cdot 10^{-1}$	$0.000000 \cdot 10^{0}$	$0.000000 \cdot 10^{0}$
$t = 1$	0.2732815	$8.128148 \cdot 10^{-2}$	$-1.107185 \cdot 10^{-1}$	$0.000000 \cdot 10^{0}$
$t = 2$	0.2159966	$-5.728489 \cdot 10^{-2}$	$-1.385664 \cdot 10^{-1}$	$-2.784784 \cdot 10^{-2}$
$t = 3$	0.1600990	$-5.589755 \cdot 10^{-2}$	$1.387332 \cdot 10^{-3}$	$1.399537 \cdot 10^{-1}$
$t = 4$	$\boxed{0.1605436}$	$\boxed{4.445293 \cdot 10^{-4}}$	$\boxed{5.634208 \cdot 10^{-2}}$	$5.495475 \cdot 10^{-2}$
...	...	...	...	...
$t = 73$	0.1788845	0	0	0

Table 10.2: The table shows a situation where the price would explode without fundamentalists but is stabilized by them. The investment parameters are the same as for Figure 10.5 where prices explode. The boxed cells fulfill the conditions required by Thm. 96 $\{p_0 = 1, M = 1, T = 250, f_t \equiv 1, I_0^C = 0.192, K = 2\}$

In summary, even a strong fundamentalistic demand rule, that is a strategy without any restrictions and involving a possibly infinitely large demand, is not able to stabilize the market when a trader using a very simple linear feedback strategy with an adequate initial investment is acting on the market, too. Market failures can happen, prices may explode, and the trading behavior of strong fundamentalists cannot prevent this.

10.7 Discussion of Effects of Linear Feedback Trading

Our analysis indicates that trend followers may cause price explosions regardless of fundamentalists' investment decisions. Specifically, Thm. 99 and its proof analytically show that a fundamentalist's investment strategy, that is a strategy that pushes prices toward their fundamental values, can be insufficient to dominate linear feedback trading strategies. However, the potential for feedback traders' to create a bubble appears to be lower (Thm. 99) when fundamentalists are active in the market (cf. Thm. 96). Although the results indicate that fundamentalists have a stabilizing effect, this effect is limited up to some threshold value (cf. Table 10.2).

Given our results and the fact that financial bubbles are associated with high economic costs, an important question arises: Seeing that fundamentalists do not appear to be an adequate market stabilizing force, is there another type of trader that would be able to stabilize prices in a market-appropriate way and, if so, what would such a trader look like? Generally, our analysis supports the view that intervention measures or at least some kind of incentive system is necessary to stabilize asset markets and prevent financial bubbles. Such measures could, for example, be the direct intervention of some control authority, progressive transaction costs, or trading restrictions.

Chapter 11

Conclusion

In this work we learned about and discussed the hypothesis of efficient markets. An introduction into the basics of stochastic processes and integrals was given and another stochastic Fubini theorem was proven. Feedback trading, especially simultaneously long short (SLS) trading, was motivated and introduced. An overview of the most important works on feedback trading was given, some examples were calculated, and the corresponding market assumptions were discussed. Finally, in Chap. 9, the so-called robust positive expectation property (RPEP), which is the property that the expected gain for *a.a.* parameters is greater than zero (given a measure on the parameter space that is absolutely continuous to the Lebesgue measure on this space), has been generalized to Merton's jump diffusion model—which is interesting because jumps cause great problems in many fields of financial mathematics (cf. option pricing and hedging). Via essentially linearly representable prices the RPEP was further extended to discrete time models and sampled-data systems with constant trend and to discrete models with variable trend that is greater or lower zero and finally to continuous time markets in which a Riemann integrable trend must exist. In Chap. 10 we have seen that linear long feedback traders (as a part of the SLS strategy) can cause financial crises (i.e. bubbles) and this cannot necessarily be prevented by very powerful fundamentalists (who are commonly said to have a stabilizing effect).

The detailed discussions of the results on the performance and the impact of our linear feedback strategies are given in Sec. 9.5 and Sec. 10.7. In this chapter we debate on the questions whether and why strategies can exist in efficient markets that have the RPEP. One possibility to resolve this puzzle is to assume that all assets' prices are risk neutral. Then the expected profit would always be zero. It is reasonable to assume that traders do not know the expected chart of a price. However, it would be a very strong assumption that really all assets have the same trend as the bond (in expectation). This may be subjectively true for the trader, but not necessarily for the unknown real market measure. Another way to solve the dilemma is to consider a comparison with a buy-and-hold strategy. As we have already seen in Sec. 9.5, there are trends s.t. the expected profit of the SLS trader is worse than that of a buy-and-hold trader. However, the expected profit is still positive for *a.a.* trends, but the expected profit of the buy-and-hold strategy is only positive for positive trends. It is not a satisfactory answer that there is another strategy (the buy-and-hold rule) that is a little better for some trends (although for these trends the SLS rule still has expected positive gains), but much worse for other trends. And a comparison with a randomly selected buy-and-hold portfolio provides even less answers, since the market (on average) should always have a trend equal to the bond (which we have assumed to be zero) and, thus, any randomly selected portfolio should also have an expected trend of zero (otherwise the bond would be incorrect). Even an examination of risks and risk-adjusted returns does not provide any real answers. All risk-adjusted returns are still positive. The question about the right measure for risk arises. Maybe, skewness would be better than standard deviation, as the simulations and histograms suggest. However, one would run into a problem very similar to the joint hypothesis problem, namely that one does not know whether the risk measure or the hypothesis of efficient markets is wrong (we call this the "risk including joint hypotheses problem").

It only remains to rethink the assumptions and link them to the classic arguments of the defenders of the market efficiency hypothesis, actually, to the argument that if this strategy really works well, every trader would use it, which should destroy the good performance. In theory, this is also true for the long side of SLS trading, since the liquidity assumption would be destroyed: If considering that all traders were trend followers (because we assumed that it worked in the past) and now the price is rising, everyone would want to buy at the same time but nobody would sell. Thus, the stock would not be liquid and the strategy does no longer work. Hence, the same explanation holds for the (complete) SLS rule. The question arises why this does not happen in reality. The answer we suspect is the risk. As we have seen, the distribution of SLS profits is highly skewed, which means that the trader loses money with very high probability (like in the lottery, but with a positive expected gain—unlike to the lottery). That means that a trader would possibly need a very high number of experiments to realize the expected positive gain, which is connected to a high risk because maybe the trader's resources end before making the positive gains. This risk prevents many traders from using the SLS rule, which means that for those who use it, the liquidity assumption remains fulfilled at least for a short time. In the long term, even a few traders can self-destroy the market assumptions, for example, when the prices rise and they want to buy more and more, the liquidity and possibly also the adequate resources assumption will fail. On short horizons the SLS strategy works well (in expectation, i.e. on average) only if many other traders do not use it. This leads us to an interesting question for future research: Who makes the losses? The fundamentalists? The noise traders? Or the market maker or somebody else ...

There are two other approaches to resolve the problem—both rather non-mathematical. On the one hand, one could think about whether it is possible to construct a strategy that systematically exploits SLS traders. Then the performance of the SLS rule would be destroyed. When many traders use the SLS rule, many other traders would use this exploiting strategy, which would again decrease the number of SLS traders. Another explanation, rather from the field of behavioral finance, is that SLS traders or generally feedback traders may (too soon) want to lock in their profits. We think of a trend follower who makes profits and a still rising trend. However, at some point of time the trader gets afraid of possibly falling prices and sells to lock in the profits, which leads to less rising or even falling prices. Other traders would follow and sell, too, cf. herd behavior. Note that this only happens if humans decide and not computers, cf. the growing popularity of robo-advisors in the financial industry. It would also be interesting to model this "fear" in heterogeneous agent models. We think that the answer lies somewhere in between these explanations.

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List of Abbreviations

B	bought (and sold) amount of assets (in currency units)
bnh	buy-and-hold (strategy)
C	chartist (i.e. trend follower / feedback trader)
CAPM	Capital Asset Pricing Model
D^ℓ	demand
δ	discounting factor
Δ	difference (operator)
D/P	dividend yields
E/P	earnings per price
f^{ℓ_δ}	discounted gain
F	fundamentalist
g^ℓ	gain (i.e. gain/loss)
GBM	geometric Brownian motion
HAM	heterogeneous agent model
I^ℓ	investment
ILS	initially long short (strategy)
K	feedback parameter
ℓ	dummy for trader
L	long (i.e. linear long strategy / long side)
MJDM	Merton's jump diffusion model
N	noise trader
NA	no-arbitrage
ODE	(deterministic) ordinary differential equation
RPEP	robust positive expectation property (i.e. positive expected gains)
S	short (i.e. linear short strategy / short side)
SDE	stochastic differential equation (i.e. stochastic integral equation)
SLS	simultaneously long short (strategy)
SLS_δ	discounted simultaneously long short (strategy)
tvGBM	time-varying geometric Brownian motion

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