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Optimal Control of the Two-Dimensional Vlasov-Maxwell-System

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0 Introduction

0.1 The system

The time evolution of a collisionless plasma is modeled by the Vlasov-Maxwell system. Collisions among the plasma particles can be neglected if the plasma is sufficiently rarefied or hot. The particles only interact through electromagnetic fields created collectively. We only consider plasmas consisting of just one particle species, for example, electrons. This work can immediately be adapted to the case of several particle species. For the sake of simplicity, we choose units such that physical constants like the speed of light, the charge and rest mass of an individual particle are normalized to unity.
Allowing the particles to move at relativistic speeds, the three-dimensional Vlasov-Maxwell system is given by

$$\begin{aligned} \partial_t f + \widehat{p} \cdot \partial_x f + (E + \widehat{p} \times B) \cdot \partial_p f =& 0, & (0.1)\\ \partial_t E - \mathrm{curl}_x B =& -j_f, & (0.2)\\ \partial_t B + \mathrm{curl}_x E =& 0, & (0.3)\\ \mathrm{div}_x E =& \rho, & (0.4)\\ \mathrm{div}_x B =& 0, & (0.5)\\ \rho_f =& 4\pi \int f \; dp, & (0.6)\\ j_f =& 4\pi \int \widehat{p} f \; dp. & (0.7) \end{aligned}$$

Here, the Vlasov equation is (0.1) and the Maxwell equations of electrodynamics are (0.2)-(0.5). Vlasov and Maxwell equations are coupled via (0.6) and (0.7). In particular, $f = f(t, x, p)$ denotes the density of the particles on phase space, and $E = E(t, x)$, $B = B(t, x)$ are the electromagnetic fields, whereby $t \in \mathbb{R}$, x, and $p \in \mathbb{R}^3$ stand for time, position in space, and momentum. The abbreviation

$$\widehat{p} = \frac{p}{\sqrt{1 + |p|^2}}$$

denotes the velocity of a particle with momentum p. Furthermore, some moments of f appear as source terms in the Maxwell equations, that is to say j_f and ρ_f which equal the current and charge density up to the constant 4π.
Considering the Cauchy problem for the above system, we moreover demand

$$f(0, x, p) = \mathring{f}(x, p), E(0, x) = \mathring{E}(x), B(0, x) = \mathring{B}(x),$$

where $\mathring{f} \geq 0$, $\mathring{E}$, and $\mathring{B}$ are some given initial data.
However, we have not readily explained the source term ρ in (0.4). If we would demand $\mathrm{div}_x E = \rho_f$ this would lead to a seeming contradiction: Formally integrating this equation with respect to x (and assuming $E \to 0$ rapidly enough at ∞) leads to $\int \rho_f \; dx = 0$ and hence $f = 0$ by $\mathring{f} \geq 0$. This problem is caused by our simplifying

restriction to one species of particles and is resolved by adding some terms to ρ_f, for example a neutralizing background density, so that we have a total charge density ρ with vanishing space integral.
Unfortunately, existence of global, classical solutions for general (smooth) data is an open problem. In fact, the results of Section 1.2 have not been verified yet in the case of three dimensions. It is only known that global weak solutions can be obtained. For a detailed insight concerning this matter see [14].
Therefore we only consider a 'two-dimensional' version of the problem, in the following sense: All functions shall be independent of the third variables x_3 and p_3. This describes a plasma where the particles only move in the (x_1, x_2)-plane, but the plasma extends in the x_3-direction infinitely. To ensure that these properties are preserved in time, we have to demand that the electric field lies in the plane and that the magnetic field is perpendicular to the plane so that $E = (E_1(t,x), E_2(t,x), 0)$ and $B = (0, 0, B(t,x))$. Here and in the following, let $x = (x_1, x_2)$ and $p = (p_1, p_2)$ be two-dimensional variables. Hence the magnetic field is always divergence free. Now the Vlasov-Maxwell system reads

$$\begin{aligned}
\partial_t f + \widehat{p} \cdot \partial_x f + (E + (\widehat{p}_2, -\widehat{p}_1) B) \cdot \partial_p f =& 0,\\
\partial_t E_1 - \partial_{x_2} B =& - j_{f,1},\\
\partial_t E_2 + \partial_{x_1} B =& - j_{f,2},\\
\partial_t B + \partial_{x_1} E_2 - \partial_{x_2} E_1 =& 0,\\
\mathrm{div}_x E =& \rho,\\
(f, E, B)|_{t=0} =& \left(\mathring{f}, \mathring{E}, \mathring{B}\right).
\end{aligned}$$

The goal is to control the plasma in a proper way. Thereto we add external currents U to the system, in applications generated by inductors. These currents, like the electric field and the current density of the plasma particles, have to lie in the plane and have to be independent of the third space coordinate. Of course, there will be an external charge density ρ_{ext} corresponding to the external current. It is natural to assume local conservation of the external charge,

$$\partial_t \rho_{\mathrm{ext}} + \mathrm{div}_x U = 0.$$

Hence we can eliminate ρ_{ext} via

$$\rho_{\mathrm{ext}} = \mathring{\rho}_{\mathrm{ext}} - \int_0^t \mathrm{div}_x U \; d\tau.$$

The initial value $\mathring{\rho}_{\mathrm{ext}}$ will be added to the background density. This total background density will be neglected throughout this work.

Therefore we consider the controlled Vlasov-Maxwell system

$$\begin{aligned}
\partial_t f + \widehat{p} \cdot \partial_x f + \left(E - \widehat{p}^{\perp} B\right) \cdot \partial_p f &= 0, \\
\partial_t E_1 - \partial_{x_2} B &= -\, j_{f,1} - U_1, \\
\partial_t E_2 + \partial_{x_1} B &= -\, j_{f,2} - U_2, \\
\partial_t B + \partial_{x_1} E_2 - \partial_{x_2} E_1 &= 0, \\
\operatorname{div}_x E &= \rho_f - \int_0^t \operatorname{div}_x U \; d\tau, \\
(f, E, B)|_{t=0} &= \left(\mathring{f}, \mathring{E}, \mathring{B}\right)
\end{aligned} \tag{CVM}$$

on a finite time interval $[0, T]$ with given $T > 0$; here we introduced the abbreviation

$$a^{\perp} = (-a_2, a_1)$$

for $a \in \mathbb{R}^2$.

It is well known that L^q-norms (with respect to (x, p), $1 \le q \le \infty$) of f are preserved in time by f solving the Vlasov equation since the vector field $\left(\widehat{p}, E - \widehat{p}^{\perp} B\right)$ is divergence free in (x, p). Therefore, especially, the L^1-norm (with respect to x) of the charge density ρ_f is constant in time.

The outline of our work is the following: In the first part, we have to prove unique solvability of (CVM). Of course, some regularity assumptions on the external current and the initial data have to be made in order to prove existence of classical solutions. In the second part, we consider an optimal control problem. On the one hand, we want the shape of the plasma to be close to some desired shape. On the other hand, the energy of the external currents shall be as small as possible. These two aims lead to minimizing some objective function. To analyze the optimal control problem, it is convenient to show differentiability of the control-to-state operator first. After that, we prove existence of a minimizer and deduce first order optimality conditions and the adjoint equation.

0.2 Some notation and simple computations

We denote by $B_r(x)$ the open ball with radius $r > 0$ and center $x \in X$ where X is a normed space. Furthermore, we abbreviate $B_r := B_r(0)$.

For a function

$$g\colon\ [0, T] \times \mathbb{R}^j \to \mathbb{R}^k$$

we abbreviate

$$g(t) := g(t, \cdot) : \mathbb{R}^j \to \mathbb{R}^k$$

for $0 \le t \le T$.

Sometimes, denoting certain function spaces, we omit the set where these functions are defined. Which set is meant should be obvious, in fact the largest possible set like

$[0,T]\times\mathbb{R}^j$ (including time) or $\mathbb{R}^j$ (not including time).
We use the abbreviations

$$\xi=\frac{y-x}{t-\tau},\ es=\frac{-2\left(\xi+\widehat{p}\right)}{1+\widehat{p}\cdot\xi},\ bs=\frac{-2\xi\cdot\widehat{p}^{\perp}}{1+\widehat{p}\cdot\xi},$$
$$et=\frac{-2\left(1-\left|\widehat{p}\right|^2\right)\left(\xi+\widehat{p}\right)}{\left(1+\widehat{p}\cdot\xi\right)^2},\ bt=\frac{-2\left(1-\left|\widehat{p}\right|^2\right)\xi\cdot\widehat{p}^{\perp}}{\left(1+\widehat{p}\cdot\xi\right)^2}.$$

We state some fundamental properties which will be used several times:

Remark 0.1. i) For $|p|\le r$ and $|\xi|\le 1$ we can estimate

$$\left|\partial_p\left(bs\right)\right|,\left|\partial_p\left(es\right)\right|,\left|\partial_p\partial_\xi\left(bs\right)\right|,\left|\partial_p\partial_\xi\left(es\right)\right|,\left|bt\right|,\left|et\right|,\left|\partial_{(\xi,p)}\left(bt\right)\right|,\left|\partial_{(\xi,p)}\left(et\right)\right|\le C\left(r\right)$$

where $C\left(r\right)>0$ is a constant only depending on r, since

$$\left|1+\widehat{p}\cdot\xi\right|\ge 1-\left|\widehat{p}\right|\left|\xi\right|\ge 1-\frac{r}{\sqrt{1+r^2}}>0.$$

ii) We compute

$$\int_{|x-y|<t-\tau}\frac{dy}{\sqrt{\left(t-\tau\right)^2-\left|x-y\right|^2}}=2\pi\int_0^{t-\tau}s\left(\left(t-\tau\right)^2-s^2\right)^{-\frac{1}{2}}ds=2\pi\left(t-\tau\right)$$

and

$$\begin{aligned}&\int_0^t\int_{|x-y|<t-\tau}\frac{dyd\tau}{\left(t-\tau\right)^{l+1}\sqrt{1-\left|\xi\right|^2}}\\=&\int_0^t\int_{|x-y|<t-\tau}\frac{dyd\tau}{\left(t-\tau\right)^{l}\sqrt{\left(t-\tau\right)^2-\left|x-y\right|^2}}\\=&2\pi\int_0^t\left(t-\tau\right)^{-l+1}d\tau\le\frac{2\pi}{2-l}T^{2-l}=C\left(T,l\right)<\infty\end{aligned}$$

for $l<2$.

iii) It holds that $\frac{\partial\xi}{\partial y_j}=\left(t-\tau\right)^{-1}e_j$ and $\frac{\partial\xi}{\partial\tau}=\xi\left(t-\tau\right)^{-1}$.

0.3 Maxwell equations

We will have to consider first order and second order Maxwell equations. In three dimensions, with general current and charge densities, they read

$$\begin{aligned}\partial_tE-\mathrm{curl}_xB=&-j,\\\partial_tB+\mathrm{curl}_xE=&0,\\\mathrm{div}_xE=&\rho,\\\mathrm{div}_xB=&0,\\\left(E,B\right)\left(0\right)=&\left(\mathring{E},\mathring{B}\right),\end{aligned}\tag{1stME3D}$$

and

$$\begin{aligned} \partial_t^2 E - \Delta E =& -\partial_t j - \partial_x \rho, \\ E(0) =& \mathring{E}, \\ \partial_t E(0) =& \mathrm{curl}_x \mathring{B} - j(0), \\ \partial_t^2 B - \Delta B =& \mathrm{curl}_x j, \\ B(0) =& \mathring{B}, \\ \partial_t B(0) =& -\mathrm{curl}_x \mathring{E}, \end{aligned} \tag{2ndME3D}$$

respectively. It is well known that both systems are equivalent for E, $B \in C^2$, ρ, $j \in C^1$ if the compatibility constraints

$$\begin{aligned} \mathrm{div}\mathring{E} =& \rho(0), \\ \mathrm{div}\mathring{B} =& 0 \end{aligned} \tag{CC3D}$$

are satisfied and local conservation of charge holds, i.e.

$$\partial_t \rho + \mathrm{div}_x j = 0. \tag{LC}$$

Therefore, under these assumptions we may switch between first order and second order Maxwell equations.
Moreover, the divergence equations of (1stME3D) are redundant if (CC3D) and (LC) hold, since then

$$\partial_t (\mathrm{div}_x E - \rho) = \mathrm{div}_x (\mathrm{curl}_x B - j) - \partial_t \rho = -\partial_t \rho - \mathrm{div}_x j = 0$$

and

$$\partial_t \mathrm{div}_x B = -\mathrm{div}_x \mathrm{curl}_x E = 0.$$

Applying these assertions to our 'two-dimensional' setting with fields $(E_1, E_2, 0)$ and $(0, 0, B)$ we conclude:

Lemma 0.2. *Let $\mathring{E}$ and $\mathring{B}$ be of class C^2 and E, $B \in C^2$, and ρ, $j \in C^1$. If the conditions*

$$\mathrm{div}\mathring{E} = \rho(0) \tag{CC}$$

and

$$\partial_t \rho + \mathrm{div}_x j = 0 \tag{LC}$$

are satisfied, then it holds that:

i) If

$$\partial_t E_1 - \partial_{x_2} B = -j_1,$$

$$\begin{aligned}\partial_t E_2 + \partial_{x_1} B =& - j_2,\\ \partial_t B + \partial_{x_1} E_2 - \partial_{x_2} E_1 =& 0,\\ (E, B)(0) =& \left(\mathring{E}, \mathring{B}\right),\end{aligned}$$

we have $\operatorname{div}_x E = \rho$ *globally in time.*

ii) The systems of first order Maxwell equations

$$\begin{aligned}\partial_t E_1 - \partial_{x_2} B =& - j_1,\\ \partial_t E_2 + \partial_{x_1} B =& - j_2,\\ \partial_t B + \partial_{x_1} E_2 - \partial_{x_2} E_1 =& 0,\\ (E, B)(0) =& \left(\mathring{E}, \mathring{B}\right),\end{aligned} \tag{1stME}$$

and second order Maxwell equations

$$\begin{aligned}\partial_t^2 E - \Delta E =& - \partial_t j - \partial_x \rho,\\ E(0) =& \mathring{E},\\ \partial_t E(0) =& \left(\partial_{x_2} \mathring{B}, -\partial_{x_1} \mathring{B}\right) - j(0),\\ \partial_t^2 B - \Delta B =& \partial_{x_1} j_2 - \partial_{x_2} j_1,\\ B(0) =& \mathring{B},\\ \partial_t B(0) =& - \partial_{x_1} \mathring{E}_2 + \partial_{x_2} \mathring{E}_1,\end{aligned} \tag{2ndME}$$

are equivalent.

We give a quite general condition that guarantees (LC).

Lemma 0.3. *Let* $g \in C$*, and* f*,* d*, and* K *of class* C^1 *with* $\operatorname{div}_p K = 0$ *and* $f(t, x, \cdot)$ *compactly supported for each* $t \in [0, T]$ *and* $x \in \mathbb{R}^2$*. Assume*

$$\partial_t f + \widehat{p} \cdot \partial_x f + K \cdot \partial_p f = g,$$

and that

$$\int g \; dp = 0$$

holds. Then $\rho = \rho_f - \int_0^t \operatorname{div}_x d \; d\tau$ *and* $j = j_f + d$ *satisfy* (LC)*.*

Proof. Firstly,

$$\partial_t \left(- \int_0^t \operatorname{div}_x d \; d\tau \right) + \operatorname{div}_x d = 0$$

is obvious. Furthermore, integrating the Vlasov equation with respect to p instantly yields

$$\partial_t \rho_f + \operatorname{div}_x j_f = 0.$$

□

Since (2ndME) consists of Cauchy problems for wave equations, we will need a solution formula for the 2D wave equation. In two dimensions, the (in C^2 unique) solution of the Cauchy problem

$$\begin{aligned}\partial_t^2 u - \Delta u =& f,\\ u(0) =& g,\\ \partial_t u(0) =& h,\end{aligned}$$

is given by the well known formula

$$\begin{aligned}u(t,x) =& \frac{1}{2\pi}\int_0^t \int_{|x-y|<t-\tau} \frac{f(\tau,y)}{\sqrt{(t-\tau)^2 - |x-y|^2}}\, dyd\tau\\ &+ \frac{1}{2\pi}\int_{B_1} \frac{g(x+ty) + t\nabla g(x+ty)\cdot y + th(x+ty)}{\sqrt{1-|y|^2}}\, dy.\end{aligned}$$

Unfortunately, for this to be a solution it is required that f, $h \in C^2$, and $g \in C^3$. Nevertheless, such a solution formula can be obtained if the data are less regular:

Lemma 0.4. *Let $M := [0,T] \times \mathbb{R}^2$ and $u \in C^2(M)$, $f \in C(M)$, $g \in C^1(\mathbb{R}^2)$, and $h \in C(\mathbb{R}^2)$ with*

$$\begin{aligned}\partial_t^2 u - \Delta u =& f,\\ u(0) =& g,\\ \partial_t u(0) =& h.\end{aligned}$$

Then u is given by

$$\begin{aligned}u(t,x) =& \frac{1}{2\pi}\int_0^t \int_{|x-y|<t-\tau} \frac{f(\tau,y)}{\sqrt{(t-\tau)^2 - |x-y|^2}}\, dyd\tau\\ &+ \frac{1}{2\pi}\int_{B_1} \frac{g(x+ty) + t\nabla g(x+ty)\cdot y + th(x+ty)}{\sqrt{1-|y|^2}}\, dy\end{aligned}$$

and we have

$$\|u\|_{L^\infty(M)} \le \frac{1}{2}T^2 \|f\|_\infty + T\|g\|_{W^{1,\infty}} + T\|h\|_\infty .$$

Proof. Let $(t,x) \in M$ and $K_{t,x} := \{(\tau,y) \in M \mid 0 \le \tau \le t, |x-y| \le t-\tau\}$, the closed wave cone corresponding to (t,x). Since $K_{t,x} \subset M$ is bounded we may choose $(u_k) \subset C^\infty$ with $u_k \to u$ in $C_b^2(K_{t,x})$ for $k \to \infty$. Then we have $(f_k) := (\partial_t^2 u_k - \Delta u_k)$, $(g_k) := (u_k(0))$, $(h_k) := (\partial_t u_k(0)) \subset C^\infty$ with $f_k \to f$ in $C_b(K_{t,x})$, $g_k \to g$ in $C_b^1(B_t(x))$, and $h_k \to h$ in $C_b(B_t(x))$ for $k \to \infty$. Applying the solution formula for u_k yields

$$u(t,x) = \lim_{k\to\infty} u_k(t,x)$$

$$= \lim_{k\to\infty} \frac{1}{2\pi} \int_0^t \int_{|x-y|<t-\tau} \frac{f_k(\tau, y)}{\sqrt{(t-\tau)^2 - |x-y|^2}} \, dy d\tau$$
$$+ \frac{1}{2\pi} \int_{B_1} \frac{g_k(x+ty) + t\nabla g_k(x+ty)\cdot y + th_k(x+ty)}{\sqrt{1-|y|^2}} \, dy$$
$$= \frac{1}{2\pi} \int_0^t \int_{|x-y|<t-\tau} \frac{f(\tau, y)}{\sqrt{(t-\tau)^2 - |x-y|^2}} \, dy d\tau$$
$$+ \frac{1}{2\pi} \int_{B_1} \frac{g(x+ty) + t\nabla g(x+ty)\cdot y + th(x+ty)}{\sqrt{1-|y|^2}} \, dy;$$

note that all kernels are integrable.
The estimate is derived straightforwardly. □

0.4 Control space for classical solutions

In the following let $L > 0$,

$$U \in V := \left\{ d \in W^{2,1}\left(0, T; C_b^4\left(\mathbb{R}^2; \mathbb{R}^2\right)\right) \mid d(t,x) = 0 \text{ for } |x| \geq L \right\},$$

and let V be equipped with the $W^{2,1}\left(0, T; C_b^4\left(\mathbb{R}^2; \mathbb{R}^2\right)\right)$-norm.

1 Existence results

1.1 Estimates on the fields

1.1.1 A generalized system

The most important instrument to get certain bounds is to have representations of the fields. One can use the solution formula for the wave equation and after some transformation of the integral expressions Gronwall-like estimates on the density and the fields can be derived. These bounds, for instance, will imply that the sequences constructed in Section 1.3 converge in a certain sense. Having that in mind it is useful not to work with the system (CVM) but with a somewhat generalized one with second order Maxwell equations:

$$\begin{aligned}
\partial_t f + \widehat{p}\cdot\partial_x f + \alpha(p) K \cdot \partial_p f &= g, \\
\partial_t^2 E - \Delta E &= -\partial_t j_f - \partial_t d - \partial_x \rho_f + \partial_x \int_0^t \mathrm{div}_x d \, d\tau, \\
\partial_t^2 B - \Delta B &= \partial_{x_1} j_{f,2} - \partial_{x_2} j_{f,1} + \partial_{x_1} d_2 - \partial_{x_2} d_1, \\
(f, E, B)(0) &= \left(\mathring{f}, \mathring{E}, \mathring{B}\right), \\
\partial_t E(0) &= \left(\partial_{x_2}\mathring{B}, -\partial_{x_1}\mathring{B}\right) - j_{\mathring{f}} - d(0), \\
\partial_t B(0) &= -\partial_{x_1}\mathring{E}_2 + \partial_{x_2}\mathring{E}_1,
\end{aligned} \tag{GVM}$$

with initial data $\mathring{f}$ of class C_c^1 and $\mathring{E}$, $\mathring{B}$ of class C_b^2.
Now we assume that we already have functions f, K of class C^1, E, B of class C^2, g of class C_b, d of class $C^1\left(0,T;C_b^2\right)$ and α of class C_b^1 satisfying (GVM). Furthermore we assume that $\mathrm{div}_p K = 0$ and that there is a $r > 0$ in such a way, that $f(t,x,p) = g(t,x,p) = 0$ if $|p| > r$.

1.1.2 Estimates on the density

Theorem 1.1. *The density f and its (x,p)-derivatives are estimated by*

i)

$$\|f(t)\|_\infty \leq \left\|\mathring{f}\right\|_\infty + \int_0^t \|g(\tau)\|_\infty \, d\tau$$

if $g \in C$ and

ii)

$$\|\partial_{x,p} f(t)\|_\infty \leq \left(\left\|\partial_{x,p}\mathring{f}\right\|_\infty + \int_0^t \|\partial_{x,p} g(\tau)\|_\infty \, d\tau\right) \cdot \exp\left(\int_0^t \|\partial_{x,p}(\alpha K)(\tau)\|_\infty \, d\tau\right)$$

if $g \in C^1$.

Proof. If $g \in C^1$ we have (cf. [13], p. 14)

$$\begin{aligned} f(t,x,p) =& \mathring{f}((X,P)(0,t,x,p)) + \int_0^t g(s,(X,P)(s,t,x,p))\, ds, \\ \partial_{x,p} f(t,x,p) =& \left(\partial_{x,p}\mathring{f}\right)((X,P)(0,t,x,p)) + \int_0^t (\partial_{x,p} g)(s,(X,P)(s,t,z))\, ds \\ & - \int_0^t (\partial_{x,p} f)(s,(X,P)(s,t,z))(\partial_{x,p}(\alpha K))(s,(X,P)(s,t,z))\, ds \end{aligned} \tag{1.1}$$

where the characteristics of the Vlasov equation in (GVM) are defined via

$$\dot{X} = \widehat{P},\ \dot{P} = \alpha(P) K(s,X,P)$$

with initial condition $(X,P)(t,t,x,p) = (x,p)$. Thus the first estimate is obvious and the second is a result of

$$\begin{aligned} \|\partial_{x,p} f(t)\|_\infty \leq & \left\|\partial_{x,p}\mathring{f}\right\|_\infty + \int_0^t \|\partial_{x,p} g(\tau)\|_\infty \, d\tau \\ & + \int_0^t \|\partial_{x,p} f(\tau)\|_\infty \|\partial_{x,p}(\alpha K)(\tau)\|_\infty \, d\tau \end{aligned}$$

and applying Gronwall's inequality.
If g is only continuous, let $(f_k) \subset C^\infty$ with $f_k \to f$ in C_b^1. This is possible since supp$f$ is compact as a consequence of f vanishing for $|p| > r$ and $\left|\dot{X}\right| \leq 1$. Therefore we have $\partial_t f_k + \widehat{p} \cdot \partial_x f_k + \alpha K \cdot \partial_p f_k \in C^1$. Applying (1.1) for f_k we conclude

$$\begin{aligned}
&f(t,x,p) = \lim_{k\to\infty} f_k(t,x,p) \\
&= \lim_{k\to\infty} (f_k(0))((X,P)(0,t,x,p)) \\
&\quad + \int_0^t (\partial_t f_k + \widehat{p} \cdot \partial_x f_k + \alpha K \cdot \partial_p f_k)(s,(X,P)(s,t,x,p))\,ds \\
&= \mathring{f}((X,P)(0,t,x,p)) + \int_0^t (\partial_t f + \widehat{p} \cdot \partial_x f + \alpha K \cdot \partial_p f)(s,(X,P)(s,t,x,p))\,ds \\
&= \mathring{f}((X,P)(0,t,x,p)) + \int_0^t g(s,(X,P)(s,t,x,p))\,ds
\end{aligned}$$

which implies i). □

The p-support condition on f is satisfied if supp$\alpha \subset B_R$ for some $R > 0$: Obviously for $|p| > \max\{R, r, r_0\}$ (where $\mathrm{supp}_p \mathring{f} \subset B_{r_0}$) we have $\dot{P}(s,t,x,p) = 0$, hence $P(s,t,x,p) = p$ and therefore $\mathring{f}((X,P)(0,t,x,p)) = g(s,(X,P)(s,t,x,p)) = 0$.
In the following we denote by $C > 0$ some generic constant that may change from line to line, but is only dependent on T, r, and α (i.e. its C_b^1-norm). All estimates for fixed p are made under the tacit assumption $|p| \leq r$.

1.1.3 Representation of the fields

We can derive integral expressions for the fields E and B proceeding similarly to [6]. Here and in the following we omit the dependence on the variables of integration if the functions to be integrated are evaluated at exactly these variables; for example, we shortly write $\int a\, db$ instead of $\int a(b)\, db$.

Theorem 1.2. *We have $E = E^0 + ES + ET + ED$ and $B = B^0 + BS + BT + BD$ where E^0, B^0 are functionals of the initial data and $d(0)$, and where*

$$ES_j = \int_0^t \int_{|x-y|<t-\tau} \int \frac{(\alpha\partial_p(es_j) + es_j \nabla\alpha) \cdot Kf + (es_j)\,g}{\sqrt{(t-\tau)^2 - |x-y|^2}}\, dpdyd\tau,$$

$$BS = \int_0^t \int_{|x-y|<t-\tau} \int \frac{(\alpha\partial_p(bs) + bs \nabla\alpha) \cdot Kf + (bs)\,g}{\sqrt{(t-\tau)^2 - |x-y|^2}}\, dpdyd\tau,$$

$$ET_j = \int_0^t \int_{|x-y|<t-\tau} \int \frac{et_j}{(t-\tau)\sqrt{(t-\tau)^2 - |x-y|^2}} f\, dpdyd\tau,$$

$$BT = \int_0^t \int_{|x-y|<t-\tau} \int \frac{bt}{(t-\tau)\sqrt{(t-\tau)^2 - |x-y|^2}} f\, dpdyd\tau,$$

$$ED_j = -\frac{1}{2\pi}\int_0^t \int_{|x-y|<t-\tau} \frac{\partial_t d_j - \int_0^\tau \partial_{x_j} \mathrm{div}_x d \; ds}{\sqrt{(t-\tau)^2 - |x-y|^2}} \; dy d\tau,$$

$$BD = \frac{1}{2\pi}\int_0^t \int_{|x-y|<t-\tau} \frac{\partial_{x_1} d_2 - \partial_{x_2} d_1}{\sqrt{(t-\tau)^2 - |x-y|^2}} \; dy d\tau.$$

Furthermore the estimate

$$\begin{aligned} \|E(t)\|_\infty + \|B(t)\|_\infty \leq & C\left(\left\|\mathring{f}\right\|_\infty + \left\|\mathring{E}\right\|_{C_b^1} + \left\|\mathring{B}\right\|_{C_b^1} + \|d\|_{W^{1,1}\left(0,T;C_b^2\right)}\right) \\ & + C\int_0^t \left(\left(1 + \|K(\tau)\|_\infty\right)\|f(\tau)\|_\infty + \|g(\tau)\|_\infty\right) d\tau \end{aligned}$$

holds.

If additionally $\mathring{E}$, $\mathring{B} \in C_c$, *and* d *is compactly supported in* x *uniformly in* t, *so are also the fields.*

Proof. Let

$$S := \partial_t + \widehat{p}\cdot\partial_x, \; T := \frac{\partial_x - \xi\partial_t}{\sqrt{1-|\xi|^2}}.$$

Confusion with the time T seems unlikely. The use of these differential operators will be helpful because S turns up in the Vlasov equation and the properties of T ensure that an integration by parts with the wave cone as the integration domain will be nice to handle. We can express t- and x-derivatives in terms of S and T:

$$\begin{aligned} \partial_t =& \frac{S - \sqrt{1-|\xi|^2}\,\widehat{p}\cdot T}{1+\xi\cdot\widehat{p}}, \\ \partial_{x_1} =& \frac{\xi_1 S + \sqrt{1-|\xi|^2}\left(\left(1+\xi_2\widehat{p}_2\right)T_1 - \xi_1\widehat{p}_2 T_2\right)}{1+\xi\cdot\widehat{p}}, \\ \partial_{x_2} =& \frac{\xi_2 S + \sqrt{1-|\xi|^2}\left(-\xi_2\widehat{p}_1 T_1 + \left(1+\xi_1\widehat{p}_1\right)T_2\right)}{1+\xi\cdot\widehat{p}}. \end{aligned} \tag{1.2}$$

This can easily be seen; simply invert

$$\begin{pmatrix} 1 & \widehat{p}_1 & \widehat{p}_2 \\ \frac{-\xi_1}{\sqrt{1-|\xi|^2}} & \frac{1}{\sqrt{1-|\xi|^2}} & 0 \\ \frac{-\xi_2}{\sqrt{1-|\xi|^2}} & 0 & \frac{1}{\sqrt{1-|\xi|^2}} \end{pmatrix}.$$

A crucial property of T is the following: For any $h = h(\tau, y)$ of class C^1 we have

$$\partial_{y_j}\left(\frac{h(\tau,y)}{\sqrt{1-|\xi|^2}}\right) + \partial_\tau\left(-\frac{\xi_j h(\tau,y)}{\sqrt{1-|\xi|^2}}\right)$$

$$
\begin{aligned}
=&\frac{\partial_{x_j} h(\tau,y) - \xi_j \partial_t h(\tau,y)}{\sqrt{1-|\xi|^2}} + h(\tau,y)\left(\partial_{y_j}\left(\frac{1}{\sqrt{1-|\xi|^2}}\right) - \partial_\tau\left(\frac{\xi_j}{\sqrt{1-|\xi|^2}}\right)\right) \\
=&T_j h(\tau,y) \qquad (1.3)
\end{aligned}
$$

since the bracket in the second line vanishes.
First we consider the magnetic field B. It satisfies an inhomogeneous wave equation with certain initial conditions:

$$
\begin{aligned}
\partial_t^2 B - \Delta B =& \partial_{x_1} j_{f,2} - \partial_{x_2} j_{f,1} + \partial_{x_1} d_2 - \partial_{x_2} d_1, \\
B(0) =& \mathring{B}, \\
\partial_t B(0) =& -\partial_{x_1} \mathring{E}_2 + \partial_{x_2} \mathring{E}_1.
\end{aligned}
$$

Lemma 0.4 yields

$$
\begin{aligned}
B =& \widetilde{B}^0 + \frac{1}{2\pi}\int_0^t \int_{|x-y|<t-\tau} \frac{\partial_{x_1} j_{f,2} - \partial_{x_2} j_{f,1} + \partial_{x_1} d_2 - \partial_{x_2} d_1}{\sqrt{(t-\tau)^2 - |x-y|^2}}\, dyd\tau \\
=& \widetilde{B}^0 + 2\int_0^t \int_{|x-y|<t-\tau} \int \frac{\widehat{p}_2 \partial_{x_1} f - \widehat{p}_1 \partial_{x_2} f}{(t-\tau)\sqrt{1-|\xi|^2}}\, dpdyd\tau + BD
\end{aligned}
$$

where $\widetilde{B}^0$ satisfies

$$
\begin{aligned}
\partial_t^2 \widetilde{B}^0 - \Delta \widetilde{B}^0 =& 0, \\
\widetilde{B}^0(0) =& \mathring{B}, \\
\partial_t \widetilde{B}^0(0) =& -\partial_{x_1} \mathring{E}_2 + \partial_{x_2} \mathring{E}_1
\end{aligned}
$$

and is a functional of the initial data with

$$
\left\|\widetilde{B}^0\right\|_\infty \le C\left(\left\|\mathring{E}\right\|_{C_b^1} + \left\|\mathring{B}\right\|_{C_b^1}\right).
$$

Applying (1.2) we have

$$
\begin{aligned}
B - \widetilde{B}^0 - BD =& 2\int_0^t \int_{|x-y|<t-\tau} \int \frac{dpdyd\tau}{(t-\tau)\sqrt{1-|\xi|^2}\,(1+\xi\cdot\widehat{p})} \\
&\cdot\left(\widehat{p}_2\left(\xi_1 S + \sqrt{1-|\xi|^2}\left((1+\xi_2\widehat{p}_2) T_1 - \xi_1 \widehat{p}_2 T_2\right)\right)\right. \\
&\left. - \widehat{p}_1\left(\xi_2 S + \sqrt{1-|\xi|^2}\left(-\xi_2 \widehat{p}_1 T_1 + (1+\xi_1\widehat{p}_1) T_2\right)\right)\right) f \\
=& \int_0^t \int_{|x-y|<t-\tau} \int \left(\frac{2(\xi_1\widehat{p}_2 - \xi_2\widehat{p}_1) S f}{(t-\tau)\sqrt{1-|\xi|^2}\,(1+\xi\cdot\widehat{p})}\right.
\end{aligned}
$$

$$+\frac{2\left(\widehat{p}_2+\xi_2\left|\widehat{p}\right|^2\right)T_1 f}{(t-\tau)\left(1+\xi\cdot\widehat{p}\right)}-\frac{2\left(\widehat{p}_1+\xi_1\left|\widehat{p}\right|^2\right)T_2 f}{(t-\tau)\left(1+\xi\cdot\widehat{p}\right)}\Bigg) dpdyd\tau$$

$$=: I_S + I_{T_1} + I_{T_2}.$$

Obviously because of the Vlasov equation in (GVM) we can write

$$Sf = -\alpha K\cdot\partial_p f + g = -\nabla_p\cdot(\alpha K f) + \nabla\alpha\cdot K f + g$$

where we used the assumption that K is divergence free with respect to p; hence $I_S = BS$ after an integration by parts in p.
Next we consider I_{T_1}. With $A := \frac{2\left(\widehat{p}_2+\xi_2|\widehat{p}|^2\right)}{(t-\tau)(1+\widehat{p}\cdot\xi)}$ and the use of (1.3) we get

$$I_{T_1} = \int\int_0^t\int_{|x-y|<t-\tau} A\nabla_{(\tau,y)}\cdot\left(\frac{(1,0,-\xi_1)\,f}{\sqrt{1-|\xi|^2}}\right)dyd\tau dp.$$

Now it would be nice to integrate by parts with respect to (τ, y). For this sake (note that the integrand is singular at $|x-y| = t-\tau$) let $0<\epsilon<1$ and compute for fixed p

$$\begin{aligned}
&\int_0^t\int_{|x-y|<(1-\epsilon)(t-\tau)} A\nabla_{(\tau,y)}\cdot\left(\frac{(1,0,-\xi_1)\,f}{\sqrt{1-|\xi|^2}}\right)dyd\tau\\
=&-\int_0^t\int_{|x-y|<(1-\epsilon)(t-\tau)} \nabla_{(\tau,y)}A\cdot\left(\frac{(1,0,-\xi_1)\,f}{\sqrt{1-|\xi|^2}}\right)dyd\tau\\
&+\int_{|x-y|<(1-\epsilon)t}\left(A\frac{(1,0,-\xi_1)\,f}{\sqrt{1-|\xi|^2}}\right)\Bigg|_{\tau=0}\cdot(0,0,-1)\,dy\\
&+\int_0^t\int_{|x-y|=(1-\epsilon)(t-\tau)} A\frac{(1,0,-\xi_1)\,f}{\sqrt{1-|\xi|^2}}\cdot\frac{\left(\frac{y-x}{|y-x|},1-\epsilon\right)}{\sqrt{1+(1-\epsilon)^2}}\,dyd\tau.
\end{aligned}\tag{1.4}$$

Here, the last term should vanish for $\epsilon\to 0$ (this is the reason why we introduced T). Indeed, because of $|\xi| = \frac{|y-x|}{t-\tau} = 1-\epsilon$ and $\sqrt{1+(1-\epsilon)^2}\geq 1$ we can estimate

$$\begin{aligned}
&\left|\frac{(1,0,-\xi_1)}{\sqrt{1-|\xi|^2}}\cdot\frac{\left(\frac{y-x}{|y-x|},1-\epsilon\right)}{\sqrt{1+(1-\epsilon)^2}}\right|\leq\left|\frac{\frac{y_1-x_1}{|y-x|}-(1-\epsilon)\,\xi_1}{\sqrt{1-(1-\epsilon)^2}}\right|\\
=&\frac{|y_1-x_1|}{\sqrt{1-(1-\epsilon)^2}}\left|\frac{1}{|y-x|}-\frac{1-\epsilon}{t-\tau}\right| = \frac{|y_1-x_1|}{(t-\tau)\sqrt{1-(1-\epsilon)^2}}\left|\frac{1}{1-\epsilon}-1+\epsilon\right|
\end{aligned}$$

$$\leq \frac{1-\epsilon}{\sqrt{1-(1-\epsilon)^2}} \left| \frac{1}{1-\epsilon} - 1 + \epsilon \right| = \sqrt{1-(1-\epsilon)^2}.$$

Hence the last term of (1.4) converges to 0 (note that $|A| \leq C(t-\tau)^{-1}$, $|f| \leq \left\| \mathring{f} \right\|_\infty + \int_0^T \|g(\tau)\|_\infty d\tau =: \widetilde{C} < \infty$):

$$\begin{aligned}
&\int_0^t \int_{|x-y|=(1-\epsilon)(t-\tau)} \left| A \frac{(1,0,-\xi_1) f}{\sqrt{1-|\xi|^2}} \cdot \frac{\left(\frac{y-x}{|y-x|}, 1-\epsilon\right)}{\sqrt{1+(1-\epsilon)^2}} \right| dyd\tau \\
\leq & C\widetilde{C}\sqrt{1-(1-\epsilon)^2} \int_0^t \int_{|x-y|=(1-\epsilon)(t-\tau)} (t-\tau)^{-1} dyd\tau \\
= & 2C\widetilde{C}\pi\sqrt{1-(1-\epsilon)^2}(1-\epsilon)t \to 0
\end{aligned} \tag{1.5}$$

for $\epsilon \to 0$. Now letting $\epsilon \to 0$ in (1.4) and integrating over p we conclude

$$I_{T_1} = data_1 - \int_0^t \int_{|x-y|<(t-\tau)} \int \nabla_{(\tau,y)} A \cdot \left(\frac{(1,0,-\xi_1) f}{\sqrt{1-|\xi|^2}} \right) dpdyd\tau,$$

where

$$\begin{aligned}
data_1 :=& \int_{|x-y|<t} \int \left(A \frac{(1,0,-\xi_1) f}{\sqrt{1-|\xi|^2}} \right) \Bigg|_{\tau=0} \cdot (0,0,-1) \, dpdy, \\
|data_1| \leq & C \left\| \mathring{f} \right\|_\infty \int_{|x-y|<t} \left(t^2 - |x-y|^2 \right)^{-\frac{1}{2}} dy \leq C \left\| \mathring{f} \right\|_\infty,
\end{aligned} \tag{1.6}$$

is a functional of the initial data. After the computation of

$$\begin{aligned}
&\frac{1}{2} \nabla_{(\tau,y)} A \cdot (1,0,-\xi_1) \\
=& \partial_{y_1} \left(\frac{\widehat{p}_2 + \xi_2 |\widehat{p}|^2}{t-\tau+\widehat{p}\cdot(y-x)} \right) - \xi_1 \partial_\tau \left(\frac{\widehat{p}_2 + \xi_2 |\widehat{p}|^2}{t-\tau+\widehat{p}\cdot(y-x)} \right) \\
=& \frac{-\widehat{p}_1 \left(\widehat{p}_2 + \xi_2 |\widehat{p}|^2 \right) - \xi_1 \left(\xi_2 (t-\tau)^{-1} |\widehat{p}|^2 (t-\tau+\widehat{p}\cdot(y-x)) + \widehat{p}_2 + \xi_2 |\widehat{p}|^2 \right)}{(t-\tau+\widehat{p}\cdot(y-x))^2} \\
=& - \frac{\xi_1 \xi_2 |\widehat{p}|^2}{(t-\tau)^2 (1+\widehat{p}\cdot\xi)} - \frac{(\widehat{p}_1 + \xi_1) \left(\widehat{p}_2 + \xi_2 |\widehat{p}|^2 \right)}{(t-\tau)^2 (1+\widehat{p}\cdot\xi)^2}
\end{aligned}$$

we finally get

$$I_{T_1} = data_1 + 2 \int_0^t \int_{|x-y|<t-\tau} \int \frac{\left(\frac{\xi_1\xi_2|\widehat{p}|^2}{1+\widehat{p}\cdot\xi} + \frac{(\widehat{p}_1+\xi_1)\left(\widehat{p}_2+\xi_2|\widehat{p}|^2\right)}{(1+\widehat{p}\cdot\xi)^2} \right) f}{(t-\tau)\sqrt{(t-\tau)^2 - |x-y|^2}} \, dpdyd\tau.$$

Similarly we proceed with I_{T_2} to derive

$$I_{T_2} = data_2 - 2\int_0^t \int_{|x-y|<t-\tau} \int \frac{\left(\frac{\xi_1\xi_2|\widehat{p}|^2}{1+\widehat{p}\cdot\xi} + \frac{(\widehat{p}_2+\xi_2)\left(\widehat{p}_1+\xi_1|\widehat{p}|^2\right)}{(1+\widehat{p}\cdot\xi)^2}\right) f}{(t-\tau)\sqrt{(t-\tau)^2 - |x-y|^2}}\, dpdyd\tau.$$

Therefore

$$I_{T_1} + I_{T_2} = data_1 + data_2 + BT,$$

and after defining

$$B^0 := \widetilde{B}^0 + data_1 + data_2$$

we finally get the desired representation

$$B = \widetilde{B}^0 + I_S + I_{T_1} + I_{T_2} + BD = B^0 + BS + BT + BD$$

of the magnetic field.
Of course, the representations for the electric field E can be derived in a very similar way. For example, one starts with

$$\begin{aligned}
\partial_t^2 E_1 - \Delta E_1 =& - \partial_t j_{f,1} - \partial_{x_1}\rho_f + \partial_{x_1}\int_0^t \mathrm{div}_x d\; d\tau - \partial_t d_1,\\
E_1(0) =& \mathring{E}_1,\\
\partial_t E_1(0) =& \partial_{x_2}\mathring{B} - j_{\mathring{f},1} - d_1(0).
\end{aligned}$$

Hence the solution $\widetilde{E}_1^0$ of the homogeneous wave equation with these initial data is estimated by

$$\left\|\widetilde{E}_1^0\right\|_\infty \le C\left(\left\|\mathring{E}\right\|_{C_b^1} + \left\|\mathring{B}\right\|_{C_b^1} + \|d\|_\infty\right).$$

For the inhomogeneous part one can proceed similarly as before (cf. [6], p. 338 ff.). The support assertion is an immediate consequence of the representation formula. Physically, this is a result of the fact that electromagnetic fields can not propagate faster than the speed of light. Furthermore, the remaining estimate is a consequence of Remark 0.1. □

Remark 1.3. If $f(t,x,\cdot)$ is compactly supported for every t, x, but not necessarily uniformly in t, x, nevertheless the fields are given by the formula above. For this, one does not need the uniformity. However, the estimates can not be obtained.

1.1.4 First derivatives of the fields

The next step is to differentiate these representation formulas and deriving certain estimates. The method is similar to the previous one.
The constant C may now only depend on T, r, the initial data (i.e. their C_b^2-norms), and $\|\alpha\|_{C_b^1}$.

Theorem 1.4. *If $g \in C^1$ and $d \in W^{2,1}\left(0,T;C_b^3\right)$, then the derivatives of the S-, T-, and D-terms are given by*

$$\partial_{x_i}BS = \int_0^t \int_{|x-y|<t-\tau} \int \frac{\left(\alpha\partial_p\left(bs\right) + bs\nabla\alpha\right)\cdot\left(f\partial_{x_i}K + K\partial_{x_i}f\right) + bs\partial_{x_i}g}{\sqrt{\left(t-\tau\right)^2 - |x-y|^2}}\ dpdyd\tau,$$

$$\partial_{x_i}BT = \int_0^t \int_{|x-y|<t-\tau} \int \frac{bt}{\left(t-\tau\right)\sqrt{\left(t-\tau\right)^2 - |x-y|^2}}\partial_{x_i}f\ dpdyd\tau,$$

$$\partial_{x_i}BD = \frac{1}{2\pi}\int_0^t \int_{|x-y|<t-\tau} \frac{\partial_{x_i}\partial_{x_1}d_2 - \partial_{x_i}\partial_{x_2}d_1}{\sqrt{\left(t-\tau\right)^2 - |x-y|^2}}\ dyd\tau,$$

$$\partial_{x_i}ES = \int_0^t \int_{|x-y|<t-\tau} \int \frac{\left(\alpha\partial_p\left(es\right) + es\nabla\alpha\right)\cdot\left(f\partial_{x_i}K + K\partial_{x_i}f\right) + es\partial_{x_i}g}{\sqrt{\left(t-\tau\right)^2 - |x-y|^2}}\ dpdyd\tau,$$

$$\partial_{x_i}ET = \int_0^t \int_{|x-y|<t-\tau} \int \frac{et}{\left(t-\tau\right)\sqrt{\left(t-\tau\right)^2 - |x-y|^2}}\partial_{x_i}f\ dpdyd\tau,$$

$$\partial_{x_i}ED = \frac{1}{2\pi}\int_0^t \int_{|x-y|<t-\tau} \frac{\partial_t\partial_{x_i}d - \int_0^\tau \partial_{x_i}\partial_x \mathrm{div}_x d\ ds}{\sqrt{\left(t-\tau\right)^2 - |x-y|^2}}\ dyd\tau,$$

$$\begin{aligned}\partial_tBS = &\int_0^t \int_{|x-y|<t-\tau} \int \frac{\left(\alpha\partial_p\left(bs\right) + bs\nabla\alpha\right)\cdot\left(f\partial_tK + K\partial_tf\right) + bs\partial_tg}{\sqrt{\left(t-\tau\right)^2 - |x-y|^2}}\ dpdyd\tau \\ &+ \int_{|x-y|<t} \int \frac{\left(\alpha\partial_p\left(bs\right) + bs\nabla\alpha\right)|_{\tau=0}\cdot K\left(0\right)\mathring{f} + bs|_{\tau=0}\,g\left(0\right)}{\sqrt{t^2 - |x-y|^2}}\ dpdy,\end{aligned}$$

$$\begin{aligned}\partial_tBT = &\int_0^t \int_{|x-y|<t-\tau} \int \frac{bt}{\left(t-\tau\right)\sqrt{\left(t-\tau\right)^2 - |x-y|^2}}\partial_tf\ dpdyd\tau \\ &+ \int_{|x-y|<t} \int \frac{bt|_{\tau=0}}{t\sqrt{t^2 - |x-y|^2}}\mathring{f}\ dpdy,\end{aligned}$$

$$\begin{aligned}\partial_tBD = &\frac{1}{2\pi}\int_0^t \int_{|x-y|<t-\tau} \frac{\partial_t\partial_{x_1}d_2 - \partial_t\partial_{x_2}d_1}{\sqrt{\left(t-\tau\right)^2 - |x-y|^2}}\ dyd\tau \\ &+ \frac{1}{2\pi}\int_{|x-y|<t} \frac{\partial_{x_1}d_2\left(0\right) - \partial_{x_2}d_1\left(0\right)}{\sqrt{t^2 - |x-y|^2}}\ dy,\end{aligned}$$

$$\partial_t ES = \int_0^t \int_{|x-y|<t-\tau} \int \frac{(\alpha\partial_p (es) + es\nabla\alpha) \cdot (f\partial_t K + K\partial_t f) + es\partial_t g}{\sqrt{(t-\tau)^2 - |x-y|^2}} \, dpdyd\tau$$
$$+ \int_{|x-y|<t} \int \frac{(\alpha\partial_p (es) + es\nabla\alpha)|_{\tau=0} \cdot K(0) \mathring{f} + es|_{\tau=0}\, g(0)}{\sqrt{t^2 - |x-y|^2}} \, dpdy,$$
$$\partial_t ET = \int_0^t \int_{|x-y|<t-\tau} \int \frac{et}{(t-\tau)\sqrt{(t-\tau)^2 - |x-y|^2}} \partial_t f \, dpdyd\tau$$
$$+ \int_{|x-y|<t} \int \frac{et|_{\tau=0}}{t\sqrt{t^2 - |x-y|^2}} \mathring{f} \, dpdy,$$
$$\partial_t ED = -\frac{1}{2\pi} \int_0^t \int_{|x-y|<t-\tau} \frac{\partial_t^2 d - \partial_x \mathrm{div}_x d}{\sqrt{(t-\tau)^2 - |x-y|^2}} \, dyd\tau$$
$$-\frac{1}{2\pi} \int_{|x-y|<t} \frac{\partial_t d_j(0)}{\sqrt{t^2 - |x-y|^2}} \, dy.$$

Furthermore the derivatives are estimated by

$$\begin{aligned}\|\partial_{t,x} E(t)\|_\infty + \|\partial_{t,x} B(t)\|_\infty \leq & C\left(1 + \|K\|_\infty + \|f\|_\infty + \|g\|_\infty\right)\left(1 + \|K\|_\infty\right)^2 \\ & \cdot \left(1 + \ln_+ \left(|||\partial_{x,p} f|||_{[0,t]}\right) + \int_0^t \|\partial_{t,x,p} K(\tau)\|_\infty \, d\tau\right) \\ & + C \int_0^t \|\partial_{t,x} g(\tau)\|_\infty \, d\tau + C\|d\|_{W^{2,1}\left(0,T;C_b^3\right)}\end{aligned}$$

if $\|K\|_\infty < \infty$. *Here* $|||a|||_{[0,t]} := \sup_{0\leq\tau\leq t} \|a(\tau)\|_\infty$.

Proof. For instance,

$$BT = \int_0^t \int_{|z|<s} \int \frac{bt\left(\frac{z}{s}, p\right)}{s\sqrt{s^2 - |z|^2}} f(t-s, x+z, p) \, dpdyds.$$

Thus we can differentiate under the integral sign as a consequence of Remark 0.1 which leads to the given formula.
Firstly, we want to bound $\partial_{x_i} BS$. The part with $\partial_{x_i} g$ is straightforwardly estimated by $C\int_0^t \|\partial_x g(\tau)\|_\infty d\tau$ and the part with $f\partial_{x_i} K$ by $C\|f\|_\infty \int_0^t \|\partial_x K(\tau)\|_\infty d\tau$. In the remaining part with $K\partial_{x_i} f$, again we write ∂_{x_i} in terms of S and T. For simplicity, we only consider $i = 1$; of course, one can proceed with $i = 2$ analogously. We split the integral into three terms:

$$\int_0^t \int_{|x-y|<t-\tau} \int \frac{(\alpha\partial_p (bs) + bs\nabla\alpha) \cdot K\partial_{x_1} f}{\sqrt{(t-\tau)^2 - |x-y|^2}} \, dpdyd\tau$$

$$
\begin{aligned}
&= \int_0^t \int_{|x-y|<t-\tau} \int \frac{(\alpha \partial_p (bs) + bs \nabla \alpha) \cdot K}{(1+\widehat{p}\cdot\xi)(t-\tau)\sqrt{1-|\xi|^2}} \\
&\quad \cdot \left(\xi_1 S f + \sqrt{1-|\xi|^2} \left((1+\xi_2 \widehat{p}_2) T_1 - \xi_1 \widehat{p}_2 T_2 \right) f \right) dp dy d\tau \\
&=: J_S + J_{T_1} + J_{T_2}.
\end{aligned}
$$

With $Sf = -\nabla_p \cdot (\alpha K f) + \nabla \alpha \cdot K f + g$ and after integrating by parts in p we conclude

$$
\begin{aligned}
J_S = &\int_0^t \int_{|x-y|<t-\tau} \int \frac{\xi_1 \; dp dy d\tau}{(t-\tau)\sqrt{1-|\xi|^2}} \\
&\cdot K \cdot \left(\alpha f \partial_p \left(\frac{K \cdot (\alpha \partial_p (bs) + bs \nabla \alpha)}{1+\widehat{p}\cdot\xi} \right) + \frac{\alpha \partial_p (bs) + bs \nabla \alpha}{1+\widehat{p}\cdot\xi} (\nabla \alpha \cdot K f + g) \right)
\end{aligned}
$$

and hence

$$
\begin{aligned}
|J_S| \leq &C \int_0^t \left(\|K(\tau)\|_\infty \|f(\tau)\|_\infty \left(1 + \|\partial_p K(\tau)\|_\infty\right) \right. \\
&\left. + \|K(\tau)\|_\infty \left(\|K(\tau)\|_\infty \|f(\tau)\|_\infty + \|g(\tau)\|_\infty \right) \right) d\tau \\
\leq &C \|K\|_\infty \|f\|_\infty \int_0^t \|\partial_p K(\tau)\|_\infty d\tau + C \|K\|_\infty \left(\|f\|_\infty (1 + \|K\|_\infty) + \|g\|_\infty \right).
\end{aligned}
$$

Next we consider J_{T_1}. Define $A := \frac{(1+\xi_2 \widehat{p}_2)\partial_p(bs)}{(t-\tau)(1+\widehat{p}\cdot\xi)}$ and use (1.3) to derive

$$
J_{T_1} = \int_0^t \int_{|x-y|<t-\tau} \int A \cdot K \nabla_{(\tau,y)} \cdot \left(\frac{(1,0,-\xi_1) f}{\sqrt{1-|\xi|^2}} \right) dp dy d\tau.
$$

Now J_{T_1} has the same form as I_{T_1} from the previous theorem. Hence we can proceed similarly as before. Note that $|AKf| \leq C \|K\|_\infty \|f\|_\infty (t-\tau)^{-1} = \widetilde{C} (t-\tau)^{-1}$, therefore the surface term with $|x-y| = (1-\epsilon)(t-\tau)$ will vanish as well for $\epsilon \to 0$. Hence

$$
\begin{aligned}
J_{T_1} = &- \int_0^t \int_{|x-y|<t-\tau} \int \nabla_{(\tau,y)} (A \cdot K) \cdot \left(\frac{(1,0,-\xi_1) f}{\sqrt{1-|\xi|^2}} \right) dp dy d\tau \\
&+ \int_{|x-y|<t} \int \left(A \cdot K \frac{(1,0,-\xi_1) f}{\sqrt{1-|\xi|^2}} \right) \Bigg|_{\tau=0} \cdot (0,0,-1) \, dp dy.
\end{aligned}
$$

The second term is estimated by $C \|K\|_\infty$ like $data_1$. For the first term we have the inequality (recall Remark 0.1)

$$
\left| \frac{\partial A}{\partial \tau} \right| = \left| \partial_\tau \left(\frac{(1+\xi_2 \widehat{p}_2) \partial_p (bs)}{t-\tau+\widehat{p}\cdot(y-x)} \right) \right|
$$

$$
\begin{aligned}
&= \left| \frac{\widehat{p}_2 \xi_2 \partial_p (bs) + (1 + \xi_2 \widehat{p}_2) \partial_p (\partial_\xi (bs)) \cdot \xi}{(t-\tau)(t-\tau+\widehat{p}\cdot(y-x))} + \frac{(1+\xi_2\widehat{p}_2)\partial_p (bs)}{(t-\tau+\widehat{p}\cdot(y-x))^2} \right| \\
&\leq C (t-\tau)^{-2}.
\end{aligned}
$$

Similarly, the same estimate holds for $\left|\frac{\partial A}{\partial y_j}\right|$. Therefore we conclude

$$
\begin{aligned}
|J_{T_1}| \leq & C + C \int_0^t \left((t-\tau)^{-2} \|K(\tau)\|_\infty + (t-\tau)^{-1} \|\partial_{t,x} K(\tau)\|_\infty \right) \|f(\tau)\|_\infty \\
& \cdot \int_{|x-y|<t-\tau} \left(1 - |\xi|^2\right)^{-\frac{1}{2}} dy d\tau \\
\leq & C + C \|f\|_\infty \left(\|K\|_\infty + \int_0^t \|\partial_{t,x} K(\tau)\|_\infty d\tau \right).
\end{aligned}
$$

In the same way one can easily establish the same estimate for J_{T_2} as well.
Secondly, we consider $\partial_{x_i} BT$, and again, without loss of generality, only $i = 1$. As before, write

$$
\begin{aligned}
\partial_{x_1} BT = & \int_0^t \int_{|x-y|<t-\tau} \int \frac{bt}{(t-\tau)\sqrt{(t-\tau)^2 - |x-y|^2}} \partial_{x_1} f \, dp dy d\tau \\
= & \int_0^t \int_{|x-y|<t-\tau} \int \frac{bt}{(1+\widehat{p}\cdot\xi)(t-\tau)^2 \sqrt{1-|\xi|^2}} \\
& \cdot \left(\xi_1 S f + \sqrt{1-|\xi|^2} \left((1+\xi_2 \widehat{p}_2) T_1 - \xi_1 \widehat{p}_2 T_2 \right) f \right) dp dy d\tau \\
=: & L_S + L_{T_1} + L_{T_2}.
\end{aligned}
$$

First the S-term is handled as always:

$$
L_S = \int_0^t \int_{|x-y|<t-\tau} \int \frac{\xi_1 \, dp dy d\tau}{(t-\tau)^2 \sqrt{1-|\xi|^2}} \left(\alpha K f \partial_p \left(\frac{bt}{1+\widehat{p}\cdot\xi} \right) + \frac{bt (\nabla\alpha \cdot K f + g)}{1+\widehat{p}\cdot\xi} \right)
$$

and therefore

$$
|L_S| \leq C \left(\|K\|_\infty \|f\|_\infty + \|g\|_\infty \right).
$$

Next we proceed with L_{T_1}. Here the kernel is $A := \frac{bt(1+\xi_2\widehat{p}_2)}{(t-\tau)^2(1+\widehat{p}\cdot\xi)}$. Now we have to be careful because we can only estimate $|A| \leq C(t-\tau)^{-2}$. This is too weak since in an estimate like (1.5) we would arrive at $\int_0^t (t-\tau)^{-1} d\tau$ which is not finite. Thus let $\delta \in]0,t]$ to be chosen later and only consider the integral expression of L_{T_1} for $\tau \in [0, t-\delta]$. Here we are allowed to integrate by parts as before, and the crucial surface term vanishes because now A is even bounded. Instead, we get an additional surface term at $\tau = t - \delta$. Altogether we derive

$$\begin{aligned}
&\int_0^{t-\delta}\int_{|x-y|<t-\tau}\int AT_1 f\ dpdyd\tau \\
=&-\int_0^{t-\delta}\int_{|x-y|<t-\tau}\int \nabla_{(\tau,y)}A\cdot\left(\frac{(1,0,-\xi_1)\,f}{\sqrt{1-|\xi|^2}}\right)dpdyd\tau \\
&+\int_{|x-y|<t}\int\left(A\frac{(1,0,-\xi_1)\,f}{\sqrt{1-|\xi|^2}}\right)\Bigg|_{\tau=0}\cdot(0,0,-1)\,dpdy \\
&+\int_{|x-y|<\delta}\int\left(A\frac{(1,0,-\xi_1)\,f}{\sqrt{1-|\xi|^2}}\right)\Bigg|_{\tau=t-\delta}\cdot(0,0,1)\,dpdy.
\end{aligned}$$

Now, the second term is easily estimated by C and the third term by

$$C\delta^{-2}\left\|f\right\|_\infty\int_{|x-y|<\delta}\left(1-\frac{|x-y|^2}{\delta^2}\right)^{-\frac{1}{2}}dy = C\delta^{-2}\delta^2\left\|f\right\|_\infty = C\left\|f\right\|_\infty$$

independently of δ. For the first term we estimate

$$\begin{aligned}
&\left|\frac{\partial A}{\partial\tau}\right| = \left|\partial_\tau\left(\frac{(1+\xi_2\widehat{p}_2)\,bt}{(t-\tau)\,(t-\tau+\widehat{p}\cdot(y-x))}\right)\right| \\
=&\left|\frac{\widehat{p}_2\xi_2 bt+(1+\xi_2\widehat{p}_2)\,\partial_\xi\,(bt)\cdot\xi}{(t-\tau)^2\,(t-\tau+\widehat{p}\cdot(y-x))}+\frac{(1+\xi_2\widehat{p}_2)\,bt}{(t-\tau)^2\,(t-\tau+\widehat{p}\cdot(y-x))}\right. \\
&\left.+\frac{(1+\xi_2\widehat{p}_2)\,bt}{(t-\tau)\,(t-\tau+\widehat{p}\cdot(y-x))^2}\right| \\
\leq& C\,(t-\tau)^{-3}
\end{aligned}$$

and similarly $\left|\frac{\partial A}{\partial y_j}\right| \leq C\,(t-\tau)^{-3}$. Hence

$$\begin{aligned}
&\left|\int_0^{t-\delta}\int_{|x-y|<(t-\tau)}\int\nabla_{(\tau,y)}A\cdot\left(\frac{(1,0,-\xi_1)\,f}{\sqrt{1-|\xi|^2}}\right)dpdyd\tau\right| \\
\leq& C\left\|f\right\|_\infty\int_0^{t-\delta}(t-\tau)^{-3}\int_{|x-y|<t-\tau}\left(1-|\xi|^2\right)^{-\frac{1}{2}}dyd\tau = C\left\|f\right\|_\infty\int_0^{t-\delta}(t-\tau)^{-1}\,d\tau \\
=& C\left\|f\right\|_\infty\ln\frac{t}{\delta}
\end{aligned}$$

and after collecting the bounds we have

$$\left|\int_0^{t-\delta}\int_{|x-y|<t-\tau}\int AT_1 f\ dpdyd\tau\right| \leq C + C\left\|f\right\|_\infty\left(1+\ln\frac{t}{\delta}\right).$$

There remains the part where $\tau \in [t-\delta, t]$. Using the Vlasov equation we can estimate for $\tau \leq t$

$$|\partial_t f(\tau, y, p)| \leq C(1 + \|K\|_\infty) |||\partial_{x,p} f|||_{[0,t]} + \|g\|_\infty$$

and thus

$$|T_1 f(\tau, y, p)| \leq \left(C(1 + \|K\|_\infty) |||\partial_{x,p} f|||_{[0,t]} + \|g\|_\infty \right) \left(1 - |\xi|^2\right)^{-\frac{1}{2}}.$$

Therefore we conclude

$$\begin{aligned}
&\left| \int_{t-\delta}^{t} \int_{|x-y|<t-\tau} \int A T_1 f \; dp dy d\tau \right| \\
&\leq \left(C(1 + \|K\|_\infty) |||\partial_{x,p} f|||_{[0,t]} + \|g\|_\infty \right) \int_{t-\delta}^{t} \int_{|x-y|<t-\tau} (t-\tau)^{-2} \left(1 - |\xi|^2\right)^{-\frac{1}{2}} dy d\tau \\
&= \left(C(1 + \|K\|_\infty) |||\partial_{x,p} f|||_{[0,t]} + \|g\|_\infty \right) \delta.
\end{aligned}$$

Collecting the respective estimates it holds that

$$|L_{T_1}| \leq C(1 + \|K\|_\infty + \|f\|_\infty + \|g\|_\infty) \left(1 + \ln \frac{t}{\delta} + |||\partial_{x,p} f|||_{[0,t]} \, \delta \right).$$

Now choose $\delta := \min \left\{ t, |||\partial_{x,p} f|||_{[0,t]}^{-1} \right\}$ to conclude in both cases the final estimate

$$\begin{aligned}
|L_{T_1}| &\leq C(1 + \|K\|_\infty + \|f\|_\infty + \|g\|_\infty) \left(1 + \ln_+ \left(t \, |||\partial_{x,p} f|||_{[0,t]}\right)\right) \\
&\leq C(1 + \|K\|_\infty + \|f\|_\infty + \|g\|_\infty) \left(1 + \ln_+ \left(|||\partial_{x,p} f|||_{[0,t]}\right)\right)
\end{aligned}$$

since $\ln_+(ta) \leq \ln_+ t + \ln_+ a \leq C + \ln_+ a$ for $a > 0$. Of course, the same estimate holds for L_{T_2} as well.
Next, $\partial_{x_i} BD$ is straightforwardly estimated by $C \|d\|_{W^{1,1}\left(0,T;C_b^2\right)}$.
Last but not least, we have $\left|\partial_{t,x} \widetilde{B}^0\right| \leq C$ because $\widetilde{B}^0$ satisfies a homogeneous wave equation with controlled initial data, and $|\partial_{t,x} data_j| \leq C$ because, for instance, we can compute

$$\partial_{x_i} data_1 = \int_{|x-y|<t} \int \left(A \frac{(1, 0, -\xi_1) \, \partial_{x_i} f}{\sqrt{1 - |\xi|^2}} \right) \Bigg|_{\tau=0} \cdot (0, 0, -1) \, dp dy$$

like $\partial_{x_i} BT$ above.
All these considerations can be done for the electric field and its representation in the same way. The only slight difference is that there appears $d(0)$ in the initial conditions for $\widetilde{E}^0$ which leads to $\left|\partial_{t,x} \widetilde{E}^0\right| \leq C \|\partial_{t,x} d\|_\infty$ which is no problem at all. Moreover,

$\partial_{x_i} ED$ is estimated by $C\,\|d\|_{W^{1,1}\left(0,T;C_b^3\right)}$.
Now there only remain the t-derivatives of the S-, T-, and D-terms. Each first term is handled as before; for the S- and T-parts split ∂_t in terms of S and T and proceed analogously. The latter terms of the S- and T-parts are easily estimated by $C\left(1+\|K\|_\infty+\|g\|_\infty\right)$. The latter terms of the D-parts are estimated by $C\,\|d\|_{C^1\left(0,T;C_b^1\right)} \le C\,\|d\|_{W^{2,1}\left(0,T;C_b^1\right)}$ and the first by $C\,\|d\|_{W^{2,1}\left(0,T;C_b^3\right)}$.
Thus, after collecting all bounds, we finally get the desired estimate. □

1.2 A-priori bounds on the support with respect to p

The most important property that is exploited later while showing global existence of a solution of (CVM), is to have a-priori bounds on the p-support of f. This means: If we have a solution (f,E,B) of (CVM) on $[0,T[$ with $f\in C^1$ and E, B of class C^2, we have to show that

$$P(t) := \inf\{a>0 \mid f(\tau,x,p)=0 \text{ for all } |p|\ge a,\ 0\le\tau\le t\}+3$$

is controlled, i.e. $P(t)\le Q$ for $0\le t<T$ where $Q>0$ is some constant only dependent on T, the initial data (i.e. their C_b^1-norms and $P(0)$), L, and $\|U\|_V$ (the '+3' in the definition of P makes no sense at first sight but will be convenient later to estimate $\ln P\ge 1$, for instance). In the following the constants C may also only depend on these numbers. Note that, per definition, P is monotonically increasing and that $|f|\le \left\|\mathring{f}\right\|_\infty$. Moreover, $P(t)<\infty$ for each $0\le t<T$ because we have an a priori estimate on the x-support of f via $\left|\dot{X}\right|\le 1$, so that $\operatorname{supp}_x f\subset B_s$, and on the compact set $[0,t]\times\overline{B_s}$ the electromagnetic fields are bounded; hence the force field $E-\widehat{p}^\perp B$ is bounded there. Furthermore, (LC) holds by Lemma 0.3. Therefore and with Remark 1.3 we have the representations of the fields as given in Theorem 1.2. Moreover, we can also demand that (f,E,B) solves

$$\begin{aligned}
\partial_t f+\widehat{p}\cdot\partial_x f+\left(E-\widehat{p}^\perp B\right)\cdot\partial_p f &=0,\\
\partial_t^2 E-\Delta E &=-\partial_t j_f-\partial_t U-\partial_x\rho_f+\partial_x\int_0^t \operatorname{div}_x U\,d\tau,\\
\partial_t^2 B-\Delta B &=\partial_{x_1} j_{f,2}-\partial_{x_2} j_{f,1}+\partial_{x_1}U_2-\partial_{x_2}U_1,\\
(f,E,B)(0) &=\left(\mathring{f},\mathring{E},\mathring{B}\right),\\
\partial_t E(0) &=\left(\partial_{x_2}\mathring{B},-\partial_{x_1}\mathring{B}\right)-j_{\mathring{f}}-U(0),\\
\partial_t B(0) &=-\partial_{x_1}\mathring{E}_2+\partial_{x_2}\mathring{E}_1
\end{aligned} \tag{CVM2nd}$$

instead of (CVM) since both systems are equivalent by Lemma 0.2.
We use the notation

$$\omega := \frac{y-x}{|y-x|},\ a\wedge b := a_1b_2-a_2b_1,\ K := E-\widehat{p}^\perp B$$

and follow [7].

1.2.1 Energy estimates

The consideration of certain energies corresponding to (CVM) will be an important tool. Before that we have to state a theorem of Kato concerning linear symmetric hyperbolic systems, cf. [12], Theorem I:

Lemma 1.5. *Consider the problem*

$$\begin{aligned} \partial_t u + \sum_{i=1}^{n} a_i \partial_{x_i} u =& g, \\ u(0) =& \mathring{u} \end{aligned} \tag{LSHS}$$

with $u, g\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$, $a_i\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^{m \times m}$.
Let $s \in \mathbb{N}$ *with* $s > \frac{n}{2} + 1$, $1 \leq s' \leq s$, *and let the following assumptions hold for all* $0 \leq t \leq T$, $x \in \mathbb{R}^n$, *and* $1 \leq i \leq n$:

i) $a_i \in C\left(0,T; H_l^0\left(\mathbb{R}^n; \mathbb{R}^{m\times m}\right)\right)$,

ii) $\|a_i(t)\|_{H_l^s} \leq K$,

iii) $a_i(t,x)$ *is symmetric,*

iv) $g \in L^1\left(0,T; H^{s'}\left(\mathbb{R}^n;\mathbb{R}^m\right)\right) \cap C\left(0,T; H^{s'-1}\left(\mathbb{R}^n;\mathbb{R}^m\right)\right)$,

v) $\mathring{u} \in H^{s'}\left(\mathbb{R}^n;\mathbb{R}^m\right)$.

Then (LSHS) *has a solution* $u \in C\left(0,T; H^{s'}\left(\mathbb{R}^n;\mathbb{R}^m\right)\right) \cap C^1\left(0,T; H^{s'-1}\left(\mathbb{R}^n;\mathbb{R}^m\right)\right)$ *which is unique in the bigger class* $C\left(0,T; H^1\left(\mathbb{R}^n;\mathbb{R}^m\right)\right) \cap C^1\left(0,T; L^2\left(\mathbb{R}^n;\mathbb{R}^m\right)\right)$. *Furthermore the estimate*

$$\|u(t)\|_{H^r} \leq \exp(CKT)\left(\|\mathring{u}\|_{H^r} + C\int_0^T \|g(\tau)\|_{H^r}\,d\tau\right)$$

with $C = C(n,s)$ *holds for* $0 \leq r \leq s'$.

The lemma above is not the full version of Kato's theorem, but enough for our purpose.
Note that the so-called 'Local Sobolev Spaces' are defined as

$$H_l^s\left(\mathbb{R}^n;\mathbb{R}^j\right) := \left\{ z\colon \mathbb{R}^n \to \mathbb{R}^j \mid \|z\|_{H_l^s} := \sup_{x\in\mathbb{R}^n} \|z\|_{H^s(B_1(x))} < \infty \right\}.$$

The following lemma is the key lemma of this section:

Lemma 1.6. *Let* $0 \leq R \leq T$. *The estimates* *i)*

$$\sup_{x\in\mathbb{R}^2}\int_{|y-x|<R}\left(\frac{1}{2}|E|^2+\frac{1}{2}B^2+4\pi\int f\sqrt{1+|p|^2}dp\right)dy\leq C,$$

ii)

$$\sup_{x\in\mathbb{R}^2}\int_0^t\int_{|y-x|=t-\tau+R}\left(\frac{1}{2}(E\cdot\omega)^2+\frac{1}{2}(B+\omega\wedge E)^2\right.$$
$$\left.+4\pi\int f\sqrt{1+|p|^2}\,(1+\widehat{p}\cdot\omega)\,dp\right)dS_y d\tau\leq C,$$

iii)

$$\sup_{x\in\mathbb{R}^2}\int_{|y-x|<R}\rho_f^{\frac{3}{2}}\;dy\leq C,$$

iv)

$$\sup_{x\in\mathbb{R}^2}\int_{|y-x|<R}\left(\int\frac{f}{\sqrt{1+|p|^2}}\;dp\right)^3dy\leq C$$

hold for all $t\in[0,T[$.

Proof. We split the electro-magnetic fields into internal and external fields; precisely, they are defined by

$$\begin{aligned}\partial_t E_{\text{int},1}-\partial_{x_2}B_{\text{int}}&=-j_{f,1},\\ \partial_t E_{\text{int},2}+\partial_{x_1}B_{\text{int}}&=-j_{f,2},\\ \partial_t B_{\text{int}}+\partial_{x_1}E_{\text{int},2}-\partial_{x_2}E_{\text{int},1}&=0,\\ (E_{\text{int}},B_{\text{int}})(0)&=\left(\mathring{E},\mathring{B}\right)\end{aligned}$$

and

$$\begin{aligned}\partial_t E_{\text{ext},1}-\partial_{x_2}B_{\text{ext}}&=-U_1,\\ \partial_t E_{\text{ext},2}+\partial_{x_1}B_{\text{ext}}&=-U_2,\\ \partial_t B_{\text{ext}}+\partial_{x_1}E_{\text{ext},2}-\partial_{x_2}E_{\text{ext},1}&=0,\\ (E_{\text{ext}},B_{\text{ext}})(0)&=0.\end{aligned}$$

Indeed, the existence of $(E_{\text{ext}},B_{\text{ext}})=:u$ is guaranteed by Kato's theorem ($n=2$, $s=s'=3$) since there is a solution of

$$\partial_t u+\sum_{i=1}^2 a_i\partial_{x_i}u=\begin{pmatrix}-U_1\\-U_2\\0\end{pmatrix},$$

$$u(0) = 0$$

with

$$a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ a_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Because of $U \in V$ we have $E_{\text{ext}}, B_{\text{ext}} \in C\left(0,T;H^3\right) \cap C^1\left(0,T;H^2\right) \subset C^1$; furthermore

$$\left\|\left(E_{\text{ext}}, B_{\text{ext}}\right)(t)\right\|_\infty \leq C \left\|\left(E_{\text{ext}}, B_{\text{ext}}\right)(t)\right\|_{H^2} \leq C \int_0^T \|U(\tau)\|_{H^2} d\tau \leq C \|U\|_V = C;$$

here we needed the support condition on U. Because of the linearity of the Maxwell equations it holds that $E_{\text{int}} := E - E_{\text{ext}}$ and $B_{\text{int}} := B - B_{\text{ext}}$ solve their equations mentioned earlier and are of class C^1. Now let

$$e_{\text{int}} := \frac{1}{2}\left|E_{\text{int}}\right|^2 + \frac{1}{2}B_{\text{int}}^2 + 4\pi \int f\sqrt{1+|p|^2}dp$$

which is physically the energy density of the internal system and

$$e := \frac{1}{2}|E|^2 + \frac{1}{2}B^2 + 4\pi \int f\sqrt{1+|p|^2}dp.$$

We have

$$\begin{aligned}
&\partial_t e_{\text{int}} + \text{div}_x\left(-B_{\text{int}}E_{\text{int}}^\perp + 4\pi \int fp\ dp\right) \\
=&E_{\text{int}} \cdot \partial_t E_{\text{int}} + B_{\text{int}}\partial_t B_{\text{int}} + 4\pi \int \partial_t f\sqrt{1+|p|^2}dp + E_{\text{int},2}\partial_{x_1}B_{\text{int}} + B_{\text{int}}\partial_{x_1}E_{\text{int},2} \\
&- E_{\text{int},1}\partial_{x_2}B_{\text{int}} - B_{\text{int}}\partial_{x_2}E_{\text{int},1} + 4\pi \int \partial_x f \cdot p\ dp \\
=&- E_{\text{int}} \cdot j_f - 4\pi \int K \cdot \partial_p f\sqrt{1+|p|^2}dp \\
=&- E_{\text{int}} \cdot j_f + 4\pi E \cdot \int f\partial_p\sqrt{1+|p|^2}dp + 4\pi B \int f\text{div}_p p^\perp dp \\
=&E_{\text{ext}} \cdot j_f
\end{aligned}$$

where we made use of the respective Vlasov-Maxwell equations, $\partial_p\sqrt{1+|p|^2} = \widehat{p}$, and $\text{div}_p p^\perp = 0$. We integrate this identity over a suitable set and arrive at

$$\begin{aligned}
&\int_0^t \int_{|y-x|<t-\tau+R} E_{\text{ext}} \cdot j_f\ dyd\tau \\
=&\int_0^t \int_{|y-x|<t-\tau+R} \left(\partial_\tau e_{\text{int}} + \text{div}_y\left(-B_{\text{int}}E_{\text{int}}^\perp + 4\pi \int fpdp\right)\right) dyd\tau
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{|y-x|<t+R} e_{\text{int}}(0,y)\,dy + \int_{|y-x|<R} e_{\text{int}}(t,y)\,dy \\
&\quad + \frac{1}{\sqrt{2}}\int_0^t \int_{|y-x|=t-\tau+R} \left(e_{\text{int}} + \omega\cdot\left(-B_{\text{int}}E_{\text{int}}^{\perp} + 4\pi\int fp dp\right)\right) dS_y d\tau
\end{aligned} \tag{1.7}
$$

after an integration by parts in (τ, y). The integrand of the last integral is non-negative because of (note that $1+\widehat{p}\cdot\omega \geq 1 - 1\cdot 1 = 0$ and $|\omega| = 1$)

$$
\begin{aligned}
0 \leq & d_{\text{int}} := \frac{1}{2}(E_{\text{int}}\cdot\omega)^2 + \frac{1}{2}(B_{\text{int}} + \omega\wedge E_{\text{int}})^2 + 4\pi\int f\sqrt{1+|p|^2}\,(1+\widehat{p}\cdot\omega)\,dp \\
=& \frac{1}{2}E_{\text{int},1}^2\omega_1^2 + \frac{1}{2}E_{\text{int},2}^2\omega_2^2 + \frac{1}{2}B_{\text{int}}^2 + B_{\text{int}}\omega_1 E_{\text{int},2} - B_{\text{int}}\omega_2 E_{\text{int},1} + \frac{1}{2}E_{\text{int},2}^2\omega_1^2 \\
& + \frac{1}{2}E_{\text{int},1}^2\omega_2^2 + 4\pi\int f\sqrt{1+|p|^2}dp + \omega\cdot 4\pi\int fp\ dp \\
=& \frac{1}{2}E_{\text{int},1}^2 + \frac{1}{2}E_{\text{int},2}^2 + \frac{1}{2}B_{\text{int}}^2 + 4\pi\int f\sqrt{1+|p|^2}dp + \omega_1 B_{\text{int}}E_{\text{int},2} - \omega_2 B_{\text{int}}E_{\text{int},1} \\
& + \omega\cdot 4\pi\int fp\ dp \\
=& e_{\text{int}} + \omega\cdot\left(-B_{\text{int}}E_{\text{int}}^{\perp} + 4\pi\int fp\ dp\right).
\end{aligned} \tag{1.8}
$$

The left hand side of (1.7) has to be investigated. The external fields are bounded by C, so

$$
\left|\int_0^t\int_{|y-x|<t-\tau+R} E_{\text{ext}}\cdot j_f\ dy d\tau\right| \leq C\int_0^t \|j_f(\tau)\|_{L^1}\,d\tau \leq C\int_0^t \|\rho_f(\tau)\|_{L^1}\,d\tau \leq C \tag{1.9}
$$

since the L^1-norm of ρ_f is constant in time.
Now we can prove the assertions using (1.7), (1.8), and (1.9):

i) We have

$$
\int_{|y-x|<R} e_{\text{int}}\ dy \leq \int_{|y-x|<t+R} e_{\text{int}}(0,y)\,dy + C \leq C(R+t)^2 + C \leq C
$$

since $t, R \leq T$. Together with

$$
e \leq 2e_{\text{int}} + |E_{\text{ext}}|^2 + |B_{\text{ext}}|^2 \leq 2e_{\text{int}} + C
$$

we conclude

$$
\int_{|y-x|<R} e\ dy \leq C + CR^2 \leq C.
$$

ii) Similarly,

$$\int_0^t \int_{|y-x|=t-\tau+R} d_{\text{int}}\ dS_y d\tau \leq \sqrt{2} \int_{|y-x|<t+R} e_{\text{int}}\left(0,y\right) dy + C \leq C$$

and

$$\begin{aligned} d :=& \frac{1}{2}\left(E\cdot\omega\right)^2 + \frac{1}{2}\left(B+\omega\wedge E\right)^2 + 4\pi \int f\sqrt{1+|p|^2}\left(1+\widehat{p}\cdot\omega\right) dp \\ \leq& 2d_{\text{int}} + 2\left|E_{\text{ext}}\right|^2 + \left|B_{\text{ext}}\right|^2 \leq 2d_{\text{int}} + C \end{aligned}$$

yield

$$\int_0^t \int_{|y-x|=t-\tau+R} d\ dS_y d\tau \leq C + Ct\left(t+R\right)^2 \leq C.$$

iii) For $r>0$ it holds that

$$\begin{aligned} \rho_f =& 4\pi \int f\ dp = 4\pi \int_{|p|<r} f\ dp + 4\pi \int_{|p|\geq r} f\ dp \\ \leq& Cr^2 + 4\pi r^{-1} \int_{|p|\geq r} f\sqrt{1+|p|^2} dp \leq C\left(r^2 + r^{-1}e\right). \end{aligned}$$

Now choose $r := e^{\frac{1}{3}} > 0$ to derive $\rho_f \leq Ce^{\frac{2}{3}}$ (if $e=0$ then also $\rho_f = 0$) and hence

$$\int_{|y-x|<R} \rho_f^{\frac{3}{2}}\ dy \leq C \int_{|y-x|<R} e\ dy \leq C.$$

iv) Similarly,

$$\begin{aligned} \int \frac{f}{\sqrt{1+|p|^2}}\ dp \leq& C \int_{|p|<r} \frac{1}{\sqrt{1+|p|^2}}\ dp + \frac{1}{1+r^2} \int_{|p|\geq r} f\sqrt{1+|p|^2} dp \\ \leq& C \int_0^r \frac{s}{\sqrt{1+s^2}}\ ds + \frac{1}{r^2} e \leq C\left(r + r^{-2}e\right) \leq Ce^{\frac{1}{3}} \end{aligned}$$

for again $r := e^{\frac{1}{3}}$ which yields

$$\int_{|y-x|<R} \left(\int \frac{f}{\sqrt{1+|p|^2}}\ dp\right)^3 dy \leq C \int_{|y-x|<R} e\ dy \leq C.$$

□

Next we have to establish an important inequality:

Lemma 1.7. *The inequality*

$$(\widehat{p}\wedge\omega)^2 \leq 2\,(1+\widehat{p}\cdot\xi)$$

holds (for $|\xi|\leq 1$).

Proof. On the one hand we have with $|\omega|=1$

$$\begin{aligned}1+|p|^2-(p\cdot\omega)^2 =&1+|p|^2-p_1^2\omega_1^2-2p_1p_2\omega_1\omega_2-p_2^2\omega_2^2\\ =&1+p_1^2\omega_2^2+p_2^2\omega_1^2-2p_1p_2\omega_1\omega_2=1+(p\wedge\omega)^2\geq(p\wedge\omega)^2\end{aligned}$$

and on the other hand

$$\sqrt{1+|p|^2}-p\cdot\omega\leq\sqrt{1+|p|^2}+|p|\leq 2\sqrt{1+|p|^2}.$$

Putting these two estimates together we conclude

$$\begin{aligned}\sqrt{1+|p|^2}\,(1+\widehat{p}\cdot\omega)=&\sqrt{1+|p|^2}+p\cdot\omega=\frac{1+|p|^2-(p\cdot\omega)^2}{\sqrt{1+|p|^2}-p\cdot\omega}\geq\frac{(p\wedge\omega)^2}{2\sqrt{1+|p|^2}}\\ =&\frac{1}{2}\,(\widehat{p}\wedge\omega)^2\,\sqrt{1+|p|^2}\end{aligned}$$

and therefore

$$(\widehat{p}\wedge\omega)^2\leq 2\,(1+\widehat{p}\cdot\omega)\,. \tag{1.10}$$

Since

$$|\xi|\,\widehat{p}\cdot\omega=\widehat{p}\cdot\xi,$$

$\widehat{p}\cdot\omega$ and $\widehat{p}\cdot\xi$ have the same sign. If they are negative, then $\widehat{p}\cdot\omega\leq\widehat{p}\cdot\xi$ because of $|\xi|\leq 1$ which, together with (1.10), implies the assertion. If they are ≥ 0, then

$$(\widehat{p}\wedge\omega)^2\leq|\widehat{p}|^2\,|\omega|^2\leq 1\leq 2\,(1+\widehat{p}\cdot\xi)$$

holds. ∎

1.2.2 Estimates on the S-terms

The crucial problem is to estimate the fields in a proper way. Unfortunately, the estimates of Section 1.1 can not be applied because, of course, we can not assume that $P(t)$ is controlled. The first step is to estimate the respective S-terms.

Lemma 1.8. *We have*

$$|ES_1|+|ES_2|+|BS|\leq C\int_0^t\int_{|x-y|<t-\tau}\int\frac{\left(|E|+|B|+\frac{|E\cdot\omega|+|B+\omega\wedge E|}{1+\widehat{p}\cdot\xi}\right)f}{\sqrt{1+|p|^2}\sqrt{(t-\tau)^2-|x-y|^2}}\,dpdyd\tau.$$

Proof. First it holds that

$$|\xi - \omega| = |y - x| \left| \frac{|y - x| - (t - \tau)}{(t - \tau)\,|y - x|} \right| = 1 - |\xi| \leq 1 - |\widehat{p}|\,|\xi| \leq 1 + \widehat{p} \cdot \xi. \tag{1.11}$$

For the estimate on BS compute

$$\begin{aligned}
\frac{1}{2} K \cdot \partial_p\,(bs) =& K \cdot \partial_p \left(\frac{\xi_1 p_2 - \xi_2 p_1}{\sqrt{1 + |p|^2} + \widehat{p} \cdot \xi} \right) \\
=& \frac{K \cdot \left(\left(\sqrt{1 + |p|^2} + p \cdot \xi \right) \xi^{\perp} - (\xi_1 p_2 - \xi_2 p_1)\,(\widehat{p} + \xi) \right)}{\left(\sqrt{1 + |p|^2} + p \cdot \xi \right)^2} \\
=& \frac{K \cdot \big((1 + \widehat{p} \cdot \xi)\,\xi^{\perp} - \xi \wedge \widehat{p}\,(\widehat{p} + \xi) \big)}{\sqrt{1 + |p|^2}\,(1 + p \cdot \xi)^2} \\
=& \frac{L\,(E, B)}{\sqrt{1 + |p|^2}\,(1 + p \cdot \xi)^2}
\end{aligned} \tag{1.12}$$

where the linear function $L \colon \mathbb{R}^3 \to \mathbb{R}$ (for fixed p and ξ) is defined as

$$L\,(E, B) := (1 + \widehat{p} \cdot \xi)\,(\xi \wedge E - B\widehat{p} \cdot \xi) - (\xi \wedge \widehat{p})\,(E \cdot \widehat{p} + E \cdot \xi + B\xi \wedge \widehat{p})\,.$$

Note that $\xi \wedge E = E \cdot \xi^{\perp}$ and $\xi \wedge \widehat{p} = -\xi \cdot \widehat{p}^{\perp}$. The set $\left\{ (\omega, 0)\,, \left(\omega^{\perp}, 1\right), \left(\omega^{\perp}, -1\right) \right\}$ is an orthogonal basis of $\mathbb{R}^3$ because of

$$(\omega \cdot z)\,\omega + (\omega \wedge z)\,\omega^{\perp} = \left(\omega_1^2 z_1 + \omega_2^2 z_1, \omega_2^2 z_2 + \omega_2^2 z_1\right) = z$$

for each $z \in \mathbb{R}^2$. Hence we can write

$$\begin{aligned}
L\,(E, B) =& \left(L\,(\omega, 0)\,(\omega, 0) + \frac{1}{2} L\left(\omega^{\perp}, 1\right)\left(\omega^{\perp}, 1\right) + \frac{1}{2} L\left(\omega^{\perp}, -1\right)\left(\omega^{\perp}, -1\right) \right) \cdot (E, B) \\
=:& \left(A_1\,(\omega, 0) + A_2\left(\omega^{\perp}, 1\right) + A_3\left(\omega^{\perp}, -1\right) \right) \cdot (E, B)\,.
\end{aligned}$$

The only thing remaining now is to estimate the coefficients A_i. With $\xi \wedge \omega = \omega^{\perp} \cdot \xi = 0$, $\omega\,|\xi| = \xi$, $\xi \cdot \omega = |\xi|$, and the use of (1.11) and Lemma 1.7 we have

$$\begin{aligned}
|A_1| =& |\xi \wedge \widehat{p}\,(\omega \cdot \widehat{p} + \omega \cdot \xi)| = |\xi|\,|\omega \wedge \widehat{p}|\,||\xi| + \omega \cdot \widehat{p}| \leq ||\xi| - 1 + 1 + \widehat{p} \cdot \xi + (\omega - \xi) \cdot \widehat{p}| \\
\leq& 1 - |\xi| + 1 + \widehat{p} \cdot \xi + 1 - |\xi| \leq 3\,(1 + \widehat{p} \cdot \xi)\,,
\end{aligned}$$

$$\begin{aligned}
2\,|A_2| =& \left| (1 + \widehat{p} \cdot \xi)\left(\xi \wedge \omega^{\perp} - \widehat{p} \cdot \xi\right) - \xi \wedge \widehat{p}\left(\omega^{\perp} \cdot \widehat{p} + \xi \wedge \widehat{p}\right) \right| \\
\leq& 2\,(1 + \widehat{p} \cdot \xi) + |\xi|\,(1 + |\xi|)\,(\omega \wedge \widehat{p})^2 \leq 6\,(1 + \widehat{p} \cdot \xi)\,,
\end{aligned}$$

and

$$\begin{aligned}2\left|A_3\right| &= \left|\left(1+\widehat{p}\cdot\xi\right)\left(\xi\wedge\omega^{\perp}+\widehat{p}\cdot\xi\right)-\xi\wedge\widehat{p}\left(\omega^{\perp}\cdot\widehat{p}-\xi\wedge\widehat{p}\right)\right| \\ &= \left|\left(1+\widehat{p}\cdot\xi\right)\left(\left|\xi\right|-1+1+\widehat{p}\cdot\xi\right)-\xi\wedge\widehat{p}\left(1-\left|\xi\right|\right)\omega\wedge\widehat{p}\right| \\ &\leq 2\left(1+\widehat{p}\cdot\xi\right)^2+\left(1-\left|\xi\right|\right)\left(\omega\wedge\widehat{p}\right)^2\leq 4\left(1+\widehat{p}\cdot\xi\right)^2.\end{aligned}$$

These estimates yield

$$\begin{aligned}\left|L\left(E,B\right)\right| \leq& 3\left(1+\widehat{p}\cdot\xi\right)\left|\left(\omega,0\right)\cdot\left(E,B\right)\right|+3\left(1+\widehat{p}\cdot\xi\right)\left|\left(\omega^{\perp},1\right)\cdot\left(E,B\right)\right| \\ &+2\left(1+\widehat{p}\cdot\xi\right)^2\left|\left(\omega^{\perp},-1\right)\cdot\left(E,B\right)\right| \\ \leq& 3\left(1+\widehat{p}\cdot\xi\right)\left(\left|E\cdot\omega\right|+\left|B+\omega\wedge E\right|\right)+2\left(1+\widehat{p}\cdot\xi\right)^2\left(\left|E\right|+\left|B\right|\right).\end{aligned}$$

Together with (1.12) this implies the desired estimate on BS. Similarly, one can proceed with ES_k. Details can be found in [7], p. 362 ff. □

Lemma 1.9. *Let*

$$\sigma_S\left(t,x,\xi\right):=\int\frac{f}{\sqrt{1+\left|p\right|^2}\left(1+\widehat{p}\cdot\xi\right)}\,dp$$

for $\left|\xi\right|<1$*. Then the estimate*

$$0\leq\sigma_S\leq CP\left(t\right)\min\left\{P\left(t\right),\left(1-\left|\xi\right|^2\right)^{-\frac{1}{2}}\right\}$$

holds.

Proof. We have

$$\begin{aligned}0\leq\sigma_S \leq& C\int_{\left|p\right|\leq P(t)}\frac{1}{\sqrt{1+\left|p\right|^2}\left(1+\widehat{p}\cdot\xi\right)}\,dp \\ =& C\int_0^{P(t)}\int_0^{\pi}\frac{u}{\sqrt{1+u^2}\left(1-\widehat{u}\left|\xi\right|\cos\varphi\right)}\,d\varphi du\end{aligned}$$

and

$$\int_0^{\pi}\frac{1}{1-\widehat{u}\left|\xi\right|\cos\varphi}\,d\varphi=\frac{\pi}{\sqrt{1-\widehat{u}^2\left|\xi\right|^2}}\leq\frac{\pi}{\sqrt{1-\widehat{u}\left|\xi\right|}} \tag{1.13}$$

since $\widehat{u}\left|\xi\right|\leq 1$. For an estimate on the last term compute

$$\frac{1}{1-\widehat{u}\left|\xi\right|}=\frac{\sqrt{1+u^2}}{\sqrt{1+u^2}-u\left|\xi\right|}=\frac{\sqrt{1+u^2}\left(\sqrt{1+u^2}+u\left|\xi\right|\right)}{1+u^2\left(1-\left|\xi\right|^2\right)}\leq\frac{2\left(1+u^2\right)}{1+u^2\left(1-\left|\xi\right|^2\right)}$$

$$\leq 2\min\left\{1+u^2, \frac{1+u^2}{1-|\xi|^2+u^2\left(1-|\xi|^2\right)}\right\} = 2\min\left\{1+u^2, \left(1-|\xi|^2\right)^{-1}\right\} \tag{1.14}$$

which leads to

$$\int_0^\pi \frac{1}{1-\widehat{u}\,|\xi|\cos\varphi}\,d\varphi \leq C\min\left\{\sqrt{1+u^2}, \left(1-|\xi|^2\right)^{-\frac{1}{2}}\right\}.$$

Hence, on the one hand we can estimate

$$\sigma_S \leq C\int_0^{P(t)} u\ du = CP(t)^2$$

and on the other hand

$$\sigma_S \leq C\left(1-|\xi|^2\right)^{-\frac{1}{2}}\int_0^{P(t)} \frac{u}{\sqrt{1+u^2}}\ du \leq CP(t)\left(1-|\xi|^2\right)^{-\frac{1}{2}}.$$

□

The next step is to further estimate the S-terms.

Lemma 1.10. *It holds that*

$$\int_0^t\int_{|x-y|<t-\tau} \frac{\sigma_S\left(|E\cdot\omega|+|B+\omega\wedge E|\right)}{\sqrt{(t-\tau)^2-|x-y|^2}}\ dyd\tau \leq CP(t)\ln P(t).$$

Proof. Denote

$$K_g(\tau,y,\omega) := |E(\tau,y)\cdot\omega| + |B(\tau,y)+\omega\wedge E(t,y)|$$

and have in mind that

$$K_g^2 \leq 2|E\cdot\omega|^2 + 2|B+\omega\wedge E|^2.$$

Now rewrite the integral above substituting $\psi := \frac{1}{2}(t-\tau-s)$ and later $r := t-\tau-2\psi$:

$$\begin{aligned}
&\int_0^t\int_{|x-y|<t-\tau} \frac{\sigma_S K_g}{\sqrt{(t-\tau)^2-|x-y|^2}}\ dyd\tau \\
=&\int_0^t\int_0^{t-\tau}\int_{|x-y|=s} \frac{\sigma_S K_g}{\sqrt{(t-\tau)^2-s^2}}\ dS_y ds d\tau \\
=&\int_0^t\int_0^{\frac{1}{2}(t-\tau)}\int_{|x-y|=t-\tau-2\psi} \frac{\sigma_S K_g}{\sqrt{\psi}\sqrt{t-\tau-\psi}}\ dS_y d\psi d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\frac{t}{2}} \int_0^{t-2\psi} \int_{|x-y|=t-\tau-2\psi} \frac{\sigma_S K_g}{\sqrt{\psi}\sqrt{t-\tau-\psi}} \, dS_y d\tau d\psi \\
&= \int_0^{\frac{t}{2}} \int_0^{t-2\psi} \int_{|x-y|=r} \frac{\sigma_S \left(t-r-2\psi, y, \frac{y-x}{r+2\psi}\right) K_g \left(t-r-2\psi, y, \omega\right)}{\sqrt{\psi}\sqrt{r+\psi}} \, dS_y dr d\psi.
\end{aligned}
$$

Let $0 \leq \epsilon \leq \frac{t}{2}$ to be chosen later and split the ψ-integral. Firstly, consider $\psi \in \left[\epsilon, \frac{t}{2}\right]$. Lemma 1.9 yields with $|y-x| = r$

$$
\begin{aligned}
\sigma_S \left(t-r-2\psi, y, \frac{y-x}{r+2\psi}\right) \leq& CP\left(t-r-2\psi\right) \left(1 - \frac{|y-x|^2}{\left(r+2\psi\right)^2}\right)^{-\frac{1}{2}} \\
\leq& \frac{CP\left(t\right)\left(r+2\psi\right)}{\sqrt{\left(r+2\psi\right)^2 - r^2}} = \frac{CP\left(t\right)\left(r+2\psi\right)}{2\sqrt{\psi}\sqrt{r+\psi}}. \qquad (1.15)
\end{aligned}
$$

Furthermore we have

$$
\begin{aligned}
&\int_0^{t-2\psi} \int_{|x-y|=r} K_g^2 \left(t-r-2\psi, y, \omega\right) dS_y dr \\
=& \int_0^{t-2\psi} \int_{|x-y|=t-\tau-2\psi} K_g^2 \left(\tau, y, \omega\right) dS_y d\tau \\
\leq& C \int_0^{t-2\psi} \int_{|x-y|=t-\tau-2\psi} dS_y d\tau \\
&+ C \int_0^{t-2\psi} \int_{|x-y|=t-\tau-2\psi} \left(\left(E\cdot\omega\right)^2 + \left(B + \omega \wedge E\right)^2\right) dS_y d\tau \\
\leq& C \qquad (1.16)
\end{aligned}
$$

where we used Lemma 1.6 ii) (with $R = 0$) for the term in the fourth line. The inequalities (1.15) and (1.16) together with Hölder's inequality imply

$$
\begin{aligned}
&\int_\epsilon^{\frac{t}{2}} \int_0^{t-2\psi} \int_{|x-y|=r} \frac{\sigma_S \left(t-r-2\psi, y, \frac{y-x}{r+2\psi}\right) K_g \left(t-r-2\psi, y, \omega\right)}{\sqrt{\psi}\sqrt{r+\psi}} \, dS_y dr d\psi \\
\leq& CP\left(t\right) \int_\epsilon^{\frac{t}{2}} \int_0^{t-2\psi} \int_{|y-x|=r} K_g \left(t-r-2\psi, y, \omega\right) \frac{r+2\psi}{\psi\left(r+\psi\right)} \, dS_y dr d\psi \\
\leq& CP\left(t\right) \int_\epsilon^{\frac{t}{2}} \left(\int_0^{t-2\psi} \int_{|y-x|=r} K_g^2 \left(t-r-2\psi, y, \omega\right) dS_y dr\right)^{\frac{1}{2}} \\
&\cdot \left(\int_0^{t-2\psi} \int_{|y-x|=r} \left(\frac{r+2\psi}{\psi\left(r+\psi\right)}\right)^2 dS_y dr\right)^{\frac{1}{2}} d\psi \\
\leq& CP\left(t\right) \int_\epsilon^{\frac{t}{2}} \left(\int_0^{t-2\psi} \frac{4r}{\psi^2} \, dr\right)^{\frac{1}{2}} d\psi
\end{aligned}
$$

$$\leq CP(t)\int_\epsilon^{\frac{t}{2}} \frac{1}{\psi}\, d\psi = CP(t)\ln\left(\frac{t}{2\epsilon}\right).$$

Secondly, consider $\psi \in [0,\epsilon]$ and estimate with Lemma 1.9

$$\sigma_S\left(t-r-2\psi, y, \frac{y-x}{r+2\psi}\right) \leq CP(t-r-2\psi)^2 \leq CP(t)^2.$$

An analogue Hölder estimate as before leads to

$$\int_0^\epsilon \int_0^{t-2\psi} \int_{|x-y|=r} \frac{\sigma_S\left(t-r-2\psi, y, \frac{y-x}{r+2\psi}\right) K_g(t-r-2\psi, y, \omega)}{\sqrt{\psi}\sqrt{r+\psi}}\, dS_y dr d\psi$$
$$\leq CP(t)^2 \int_0^\epsilon \left(\int_0^{t-2\psi}\int_{|y-x|=r} \frac{1}{\psi(r+\psi)}\, dS_y dr\right)^{\frac{1}{2}} d\psi$$
$$\leq CP(t)^2 \int_0^\epsilon \left(\frac{t}{\psi}\right)^{\frac{1}{2}} d\psi \leq CP(t)^2\sqrt{\epsilon}.$$

Collecting the bounds we arrive at

$$\int_0^t \int_{|x-y|<t-\tau} \frac{\sigma_S K_g}{\sqrt{(t-\tau)^2 - |x-y|^2}}\, dy d\tau \leq CP(t)\left(\ln\left(\frac{t}{2\epsilon}\right) + P(t)\sqrt{\epsilon}\right).$$

Now choose $\epsilon := \min\left\{P(t)^{-2}, \frac{t}{2}\right\}$ which completes the proof because on the one hand, if $\epsilon = \frac{t}{2} \leq P(t)^{-2}$ we have

$$CP(t)\left(\ln\left(\frac{t}{2\epsilon}\right) + P(t)\sqrt{\epsilon}\right) = CP(t)^2 P(t)^{-1} \leq CP(t)\ln P(t),$$

and on the other hand, if $\epsilon = P(t)^{-2} \leq \frac{t}{2}$ we have

$$CP(t)\left(\ln\left(\frac{t}{2\epsilon}\right) + P(t)\sqrt{\epsilon}\right) = CP(t)\left(\ln\left(\frac{t}{2}P(t)^2\right) + 1\right)$$
$$\leq CP(t)\left(\ln_+\left(\frac{t}{2}\right) + 2\ln P(t) + 1\right) \leq CP(t)\ln P(t).$$

Note that in both cases the definition of P is convenient since $\ln P \geq 1$. □

Collecting the previous lemmata we get the following Gronwall-like estimate on the S-terms:

Lemma 1.11. *We have*

$$|ES_1| + |ES_2| + |BS| \leq CP(t)\ln P(t) + C\int_0^t \left(\|E(\tau)\|_\infty + \|B(\tau)\|_\infty\right) d\tau.$$

Proof. Lemmata 1.8 and 1.10 imply

$$\begin{aligned}
&|ES_1| + |ES_2| + |BS| \\
\leq & C \int_0^t \left(\|E(\tau)\|_\infty + \|B(\tau)\|_\infty\right) \int_{|y-x|<t-\tau} \int \frac{f \; dpdy}{\sqrt{1+|p|^2}\sqrt{(t-\tau)^2 - |x-y|^2}} \, d\tau \\
&+ CP(t) \ln P(t).
\end{aligned}$$

Since by Hölder's inequality and Lemma 1.6 iv) (with $R = t - \tau$)

$$\begin{aligned}
&\int_{|y-x|<t-\tau} \int \frac{f(\tau,y,p)}{\sqrt{1+|p|^2}\sqrt{(t-\tau)^2 - |x-y|^2}} \, dpdy \\
\leq & \left(\int_{|y-x|<t-\tau} \left(\int \frac{f(\tau,y,p)}{\sqrt{1+|p|^2}} \, dp \right)^3 dy \right)^{\frac{1}{3}} \left(\int_{|x-y|<t-\tau} \left((t-\tau)^2 - |x-y|^2 \right)^{-\frac{3}{4}} dy \right)^{\frac{2}{3}} \\
\leq & C \left(\int_0^{t-\tau} r \left((t-\tau)^2 - r^2 \right)^{-\frac{3}{4}} dr \right)^{\frac{2}{3}} \leq C (t-\tau)^{\frac{1}{3}} \leq C,
\end{aligned}$$

the assertion is proved. □

1.2.3 Estimates on the T-terms

Next we have to take care of the T-terms.

Lemma 1.12. *Let*

$$\sigma_T(t,x,\xi) := \int f(t,x,p) \left(|et_1| + |et_2| + |bt|\right)(\xi,p) \, dp$$

for $|\xi| < 1$. *Then the estimate*

$$\sigma_T(t,x,\xi) \leq C \min \left\{ P(t)^2, P(t)^{\frac{1}{2}} e(t,x)^{\frac{1}{2}} (1-|\xi|)^{-\frac{1}{4}} \right\}$$

holds.

Proof. First, the inequality

$$|\xi + \widehat{p}|^2 = |\xi|^2 + |\widehat{p}|^2 + 2\widehat{p} \cdot \xi \leq 2 + 2\widehat{p} \cdot \xi$$

yields together with Lemma 1.7

$$\begin{aligned}
|et_1| + |et_2| + |bt| = & \frac{2\left(1 - |\widehat{p}|^2\right)}{(1 + \widehat{p} \cdot \xi)^2} \left(|\xi_1 + \widehat{p}_1| + |\xi_2 + \widehat{p}_2| + |\xi \wedge \widehat{p}|\right) \\
\leq & C \left(1 - |\widehat{p}|^2\right) (1 + \widehat{p} \cdot \xi)^{-\frac{3}{2}}. \qquad (1.17)
\end{aligned}$$

On the other hand we have with (1.13)

$$\int_0^\pi \frac{1}{(1-\widehat{u}\,|\xi|\cos\varphi)^{\frac{3}{2}}}\,d\varphi \leq \frac{1}{\sqrt{1-\widehat{u}\,|\xi|}}\int_0^\pi \frac{1}{1-\widehat{u}\,|\xi|\cos\varphi}\,d\varphi \leq \frac{C}{1-\widehat{u}\,|\xi|} \tag{1.18}$$

which implies for $0 \leq R \leq P(t)$ with the use of (1.14)

$$\begin{aligned}\int_{|p|<R}\left(1-|\widehat{p}|^2\right)(1+\widehat{p}\cdot\xi)^{-\frac{3}{2}}\,dp &= \int_0^R\int_0^\pi \frac{u\left(1-\widehat{u}^2\right)}{(1-\widehat{u}\,|\xi|\cos\varphi)^{\frac{3}{2}}}\,d\varphi du\\ \leq& C\int_0^R \frac{u}{(1+u^2)(1-\widehat{u}\,|\xi|)}\,du \leq C\int_0^R u\,du = CR^2.\end{aligned} \tag{1.19}$$

Hence, (1.17) and (1.19) lead to

$$\sigma_T \leq C\int_{|p|<P(t)}\left(1-|\widehat{p}|^2\right)(1+\widehat{p}\cdot\xi)^{-\frac{3}{2}}\,dp \leq CP(t)^2 \tag{1.20}$$

which is the first part of the assertion. For the second part, first note that

$$1-|\widehat{p}|^2 = (1+|\widehat{p}|)(1-|\widehat{p}|) \leq 2(1-|\widehat{p}|\,|\xi|) \leq 2(1+\widehat{p}\cdot\xi)\,.$$

This and Hölder's inequality yield together with (1.18) and (1.14)

$$\begin{aligned}&\int_{R<|p|<P(t)} f\left(1-|\widehat{p}|^2\right)(1+\widehat{p}\cdot\xi)^{-\frac{3}{2}}\,dp \leq 2\int_{R<|p|<P(t)} f\,(1+\widehat{p}\cdot\xi)^{-\frac{1}{2}}\,dp\\ \leq& 2\left(\int_{|p|>R} f^{\frac{3}{2}}\,dp\right)^{\frac{2}{3}}\left(\int_{|p|<P(t)}(1+\widehat{p}\cdot\xi)^{-\frac{3}{2}}\,dp\right)^{\frac{1}{3}}\\ \leq& C\left(\frac{1}{\sqrt{1+R^2}}\int f\sqrt{1+|p|^2}dp\right)^{\frac{2}{3}}\left(\int_0^{P(t)} u\int_0^\pi (1-\widehat{u}\,|\xi|\cos\varphi)^{-\frac{3}{2}}\,d\varphi dp\right)^{\frac{1}{3}}\\ \leq& Ce^{\frac{2}{3}}R^{-\frac{2}{3}}\left(\int_0^{P(t)}\frac{u}{1-\widehat{u}\,|\xi|}\,du\right)^{\frac{1}{3}} \leq Ce^{\frac{2}{3}}R^{-\frac{2}{3}}\left(1-|\xi|^2\right)^{-\frac{1}{3}}P(t)^{\frac{2}{3}}\,.\end{aligned} \tag{1.21}$$

Putting (1.17), (1.19), and (1.21) together, we conclude that for $0 \leq R \leq P(t)$ the estimate

$$\sigma_T \leq C\left(R^2 + e^{\frac{2}{3}}R^{-\frac{2}{3}}\left(1-|\xi|^2\right)^{-\frac{1}{3}}P(t)^{\frac{2}{3}}\right)$$

holds. If $e^{\frac{1}{4}}\left(1-|\xi|^2\right)^{-\frac{1}{8}}P(t)^{\frac{1}{4}} \leq P(t)$ choose R as the left hand side to get the desired estimate

$$\sigma_T \leq Ce^{\frac{1}{2}}\left(1-|\xi|^2\right)^{-\frac{1}{4}}P(t)^{\frac{1}{2}}\,.$$

If $e^{\frac{1}{4}}\left(1-|\xi|^2\right)^{-\frac{1}{8}} P(t)^{\frac{1}{4}} > P(t)$ simply use (1.20) to conclude

$$\sigma_T \leq CP(t)^2 \leq Ce^{\frac{1}{2}}\left(1-|\xi|^2\right)^{-\frac{1}{4}} P(t)^{\frac{1}{2}}.$$

□

Lemma 1.13. *The T-terms satisfy the inequality*

$$|ET_1| + |ET_2| + |BT| \leq CP(t)\ln P(t).$$

Proof. First note that

$$|ET_1| + |ET_2| + |BT| \leq \int_0^t \int_{|x-y|<t-\tau} \frac{\sigma_T(\tau,y,\xi)}{(t-\tau)\sqrt{(t-\tau)^2-|x-y|^2}}\, dyd\tau.$$

In the following let $0<\delta\leq t$ and $0<\epsilon<1$ to be chosen later and split the integral into several parts. Firstly, Lemma 1.12 yields

$$\begin{aligned}
&\int_0^{t-\delta}\int_{(1-\epsilon)(t-\tau)<|x-y|<t-\tau} \frac{\sigma_T(\tau,y,\xi)}{(t-\tau)\sqrt{(t-\tau)^2-|x-y|^2}}\, dyd\tau \\
\leq & CP(t)^2 \int_0^t (t-\tau)^{-1} \int_{(1-\epsilon)(t-\tau)}^{t-\tau} \frac{r}{\sqrt{(t-\tau)^2-r^2}}\, drd\tau = CtP(t)^2\sqrt{1-(1-\epsilon)^2} \\
\leq & CP(t)^2\sqrt{\epsilon}. \qquad (1.22)
\end{aligned}$$

Secondly, with Lemmata 1.12, 1.6 i) (with $R=(1-\epsilon)(t-\tau)$), and Hölder's inequality we estimate

$$\begin{aligned}
&\int_0^{t-\delta}\int_{|x-y|<(1-\epsilon)(t-\tau)} \frac{\sigma_T(\tau,y,\xi)}{(t-\tau)\sqrt{(t-\tau)^2-|x-y|^2}}\, dyd\tau \\
\leq & CP(t)^{\frac{1}{2}} \int_0^{t-\delta}\int_{|x-y|<(1-\epsilon)(t-\tau)} \frac{e^{\frac{1}{2}}\left(1-|\xi|^2\right)^{-\frac{1}{4}}}{(t-\tau)\sqrt{(t-\tau)^2-|x-y|^2}}\, dyd\tau \\
\leq & CP(t)^{\frac{1}{2}} \int_0^{t-\delta} (t-\tau)^{-1}\left(\int_{|x-y|<(1-\epsilon)(t-\tau)} e\ dy\right)^{\frac{1}{2}} \\
&\cdot\left(\int_{|x-y|<(1-\epsilon)(t-\tau)} \frac{\left(1-|\xi|^2\right)^{-\frac{1}{2}}}{(t-\tau)^2-|x-y|^2}\, dy\right)^{\frac{1}{2}} d\tau \\
\leq & CP(t)^{\frac{1}{2}} \int_0^{t-\delta} (t-\tau)^{-1}\left(\int_0^{(1-\epsilon)(t-\tau)} \frac{(t-\tau)r}{\left((t-\tau)^2-r^2\right)^{\frac{3}{2}}}\, dr\right)^{\frac{1}{2}} d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq CP(t)^{\frac{1}{2}}\left(\left(1-(1-\epsilon)^2\right)^{-\frac{1}{2}}-1\right)^{\frac{1}{2}}\int_0^{t-\delta}(t-\tau)^{-1}\,d\tau\\
&\leq CP(t)^{\frac{1}{2}}\,\epsilon^{-\frac{1}{4}}\ln\frac{t}{\delta}. \qquad (1.23)
\end{aligned}
$$

Thirdly, with Lemma 1.12 we have

$$
\begin{aligned}
&\int_{t-\delta}^{t}\int_{|x-y|<t-\tau}\frac{\sigma_T(\tau,y,\xi)}{(t-\tau)\sqrt{(t-\tau)^2-|x-y|^2}}\,dyd\tau\\
\leq &CP(t)^2\int_{t-\delta}^{t}(t-\tau)^{-1}\int_0^{t-\tau}\frac{r}{\sqrt{(t-\tau)^2-r^2}}\,drd\tau=CP(t)^2\,\delta. \qquad (1.24)
\end{aligned}
$$

Combining (1.22), (1.23), and (1.24) we arrive at

$$
|ET_1|+|ET_2|+|BT|\leq C\left(P(t)^2\sqrt{\epsilon}+P(t)^{\frac{1}{2}}\,\epsilon^{-\frac{1}{4}}\ln\frac{t}{\delta}+P(t)^2\,\delta\right).
$$

Now first choose $\delta:=\min\left\{P(t)^{-1},t\right\}$. If $\delta=P(t)^{-1}$ then the right hand side above equals

$$
C\left(P(t)^2\sqrt{\epsilon}+P(t)^{\frac{1}{2}}\,\epsilon^{-\frac{1}{4}}\ln(tP(t))+P(t)\right),
$$

if $\delta=t$ it equals

$$
C\left(P(t)^2\sqrt{\epsilon}+P(t)^2\,t\right)\leq C\left(P(t)^2\sqrt{\epsilon}+P(t)\right).
$$

Hence in both cases the estimate

$$
|ET_1|+|ET_2|+|BT|\leq C\left(P(t)^2\sqrt{\epsilon}+P(t)^{\frac{1}{2}}\,\epsilon^{-\frac{1}{4}}\ln((1+t)\,P(t))+P(t)\right)
$$

holds (again, here and in the following, have in mind that $P\geq 3$). Now choose $\epsilon:=P(t)^{-2}(\ln P(t))^{\frac{4}{3}}$ (note that $a^{-2}(\ln a)^{\frac{4}{3}}\leq a^{-\frac{2}{3}}<1$ for $a\geq 3$) which yields

$$
\begin{aligned}
&|ET_1|+|ET_2|+|BT|\\
\leq &C\left(P(t)(\ln P(t))^{\frac{2}{3}}+P(t)(\ln P(t))^{-\frac{1}{3}}\ln((1+t)\,P(t))+P(t)\right)\\
\leq &C(1+\ln(t+1)+1+1)\,P(t)(\ln P(t))^{\frac{2}{3}}\leq CP(t)\ln P(t).
\end{aligned}
$$

□

1.2.4 Conclusion

Altogether, we now have estimated the S- and T-terms of the fields. Having a look at the representation formula we notice that the terms E^0, B^0, ED, and BD still have to be considered. Fortunately, this is quite easy. By Lemma 0.4 we have $\left|\widetilde{B}^0\right| \leq C$. Furthermore we can also estimate $|data_i|$ by C; see (1.6) and note that the constant there only depends on $\left\|\mathring{f}\right\|_\infty$, $P(0)$, and T. Analogously, we proceed with E^0; again, a slight difference is that there is $\|U(0)\|_\infty$ additionally in the estimate, but this term is also controlled by C. Furthermore we straightforwardly estimate (recall Remark 0.1)

$$|ED|, |BD| \leq C \|U\|_{W^{1,1}\left(0,T;C_b^2\right)} \leq C.$$

Now we can finally prove:

Theorem 1.14. *The a-priori bound $P(t) \leq Q$ holds, where Q is only dependent on T, the C_b^1-norms of the initial data,* $\operatorname{supp}_p \mathring{f}$ *(which basically coincides with $P(0)$), L, and $\|U\|_V$.*

Proof. Collecting all bounds on the fields we arrive at

$$\|E(t)\|_\infty + \|B(t)\|_\infty \leq C + CP(t)\ln P(t) + C\int_0^t \left(\|E(\tau)\|_\infty + \|B(\tau)\|_\infty\right) d\tau.$$

This implies

$$\|E(t)\|_\infty + \|B(t)\|_\infty \leq C + CP\left(\widetilde{t}\right)\ln P\left(\widetilde{t}\right) + C\int_0^t \left(\|E(\tau)\|_\infty + \|B(\tau)\|_\infty\right) d\tau$$

for $0 \leq t \leq \widetilde{t} < T$. Now applying Gronwall's inequality for $t \in \left[0, \widetilde{t}\right]$ and then setting $t = \widetilde{t}$ yield

$$\|E(t)\|_\infty + \|B(t)\|_\infty \leq C + CP(t)\ln P(t). \tag{1.25}$$

Have a look at the characteristics

$$\dot{X} = \widehat{P},\ \dot{P} = \left(E - \widehat{P}^\perp B\right)(s, X),\ (X, P)(t, t, x, p) = (x, p)$$

which are well defined. We have

$$\begin{aligned} |P(0,t,x,p)| &\geq |p| - \int_0^t \left|\dot{P}(s,t,x,p)\right| ds \geq |p| - \int_0^t \left(\|E(s)\|_\infty + \|B(s)\|_\infty\right) ds \\ &\geq |p| - C - C\int_0^t P(s)\ln P(s)\, ds \end{aligned}$$

which yields with $f(t,x,p) = \mathring{f}((X,P)(0,t,x,p))$ that

$$P(t) \leq P(0) + C + C\int_0^t P(s)\ln P(s)\, ds = C + C\int_0^t P(s)\ln P(s)\, ds.$$

Hence we conclude

$$P(t) \leq \exp\left(\exp(Ct)\ln C\right)$$

because the right hand side solves the respective integral equation. Finally this leads to the desired estimate

$$P(t) \leq Q := \exp\left(\exp\left(CT\right)\ln C\right). \tag{1.26}$$

□

1.3 Existence of classical solutions

In the following we want to construct a solution of (CVM).

1.3.1 The iteration scheme

We now work with initial data $\mathring{f} \geq 0$ of class C_c^2, $\mathring{E}$, $\mathring{B}$ of class C_b^3, and control $U \in V$ that satisfy (CC), i.e. $\mathrm{div}\mathring{E} = \rho_{\mathring{f}}$. Unfortunately, we have to approximate these functions, so let $\mathring{f}_k \to \mathring{f}$ in C_b^2, $\mathring{E}_k \to \mathring{E}$ and $\mathring{B}_k \to \mathring{B}$ in C_b^3 with $\mathring{f}_k \in C_c^\infty$, $\mathring{E}_k$, $\mathring{B}_k \in C^\infty$, and furthermore $U_k \to U$ in V with $U_k \in C^\infty$ (note that C^∞ is dense in V). Without loss of generality we can assume that $\left\|\mathring{f}_k\right\|_{C_b^2} \leq 2\left\|\mathring{f}\right\|_{C_b^2}$ and likewise for $\mathring{E}_k$, $\mathring{B}_k$, and U_k, and that $\mathrm{supp}_p\mathring{f}_k \subset B_{2Q}$ with the Q obtained from the previous section. The strategy to obtain a solution of (CVM) is the following: By iteration we construct densities f_k and fields E_k, B_k in such a way that these functions will converge in a proper sense and that we may pass to the limit in (CVM). However, it is more convenient to work with a modified system. As the previous section suggests, it is crucial to control the p-support of f. For this reason we first consider a cut-off system on $[0, T]$ where we modify the original Vlasov equation and use the second order Maxwell equations ((CC) and (LC) need not hold for the iterates):

$$\begin{aligned}
\partial_t f + \widehat{p}\cdot\partial_x f + \alpha(p)\left(E - \widehat{p}^\perp B\right)\cdot\partial_p f &= 0,\\
\partial_t^2 E - \Delta E &= -\partial_t j_f - \partial_t U - \partial_x\rho_f + \partial_x\int_0^t \mathrm{div}_x U\, d\tau,\\
\partial_t^2 B - \Delta B &= \partial_{x_1} j_{f,2} - \partial_{x_2} j_{f,1} + \partial_{x_1} U_2 - \partial_{x_2} U_1,\\
(f, E, B)(0) &= \left(\mathring{f}, \mathring{E}, \mathring{B}\right),\\
\partial_t E(0) &= \left(\partial_{x_2}\mathring{B}, -\partial_{x_1}\mathring{B}\right) - j_{\mathring{f}} - U(0),\\
\partial_t B(0) &= -\partial_{x_1}\mathring{E}_2 + \partial_{x_2}\mathring{E}_1.
\end{aligned} \tag{αVM}$$

Here, let the cut-off function α be of class $C_c^\infty\left(\mathbb{R}^2\right)$ with $\alpha(p) = 1$ for $|p| \leq 2Q$. The property of the constant Q will imply that a solution of (αVM) is also a solution of (CVM). The generic constants $C > 0$ may now depend on T, $\left\|\mathring{f}\right\|_{C_b^2}$, $\left\|\mathring{E}\right\|_{C_b^3}$, $\left\|\mathring{B}\right\|_{C_b^3}$, $P(0)$, L, $\|U\|_V$, $\|\alpha\|_{C_b^1}$, and Q (in fact, on the one hand, α can be chosen in such a way that $\|\alpha\|_{C_b^1} \leq 1$, and on the other hand, Q only depends on the initial data, T, L,

and $\|U\|_V$; hence the dependence on $\|\alpha\|_{C_b^1}$ and Q can be neglected).
We start the iteration with $f_0(t,x,p) := \mathring{f}_0(x,p)$, $E_0(t,x) := \mathring{E}_0(x)$, and $B_0(t,x,p) := \mathring{B}_0(x)$. The induction hypothesis is that f_k, E_k, and B_k are of class C^∞ and that the fields are bounded. Given f_{k-1}, E_{k-1}, and B_{k-1}, we firstly define f_k as the solution of

$$\begin{aligned} \partial_t f_k + \widehat{p}\cdot\partial_x f_k + \alpha(p)\left(E_{k-1} - \widehat{p}^\perp B_{k-1}\right)\cdot\partial_p f_k =&0, \\ f_k(0) =&\mathring{f}_k, \end{aligned} \tag{1.27}$$

namely

$$f_k(t,x,p) = \mathring{f}_k\left(X_k(0,t,x,p), P_k(0,t,x,p)\right)$$

with the characteristics defined by

$$\begin{aligned} \dot{X}_k =&\widehat{P}_k, & X_k(t,t,x,p) =&x, \\ \dot{P}_k =&\alpha(P_k)\left(E_{k-1} - \widehat{P}_k^\perp B_{k-1}\right)(s,X_k), & P_k(t,t,x,p) =&p. \end{aligned}$$

We apply a result of Hartman, cf. [3], Corollary 4.1, which roughly says that the dependence on initial conditions of solutions of an ordinary differential equation is as regular as the right hand side of the differential equation. We conclude that X_k and P_k are of class C^∞ in all four variables by the induction hypothesis. This yields that even $f_k \in C^\infty$. Since α is compactly supported the p-support of f_k is controlled by a constant C. Hence, ρ_{f_k} and j_{f_k} are well defined as $C^\infty \cap C_b^1$-functions.
Secondly, we define E_k and B_k as the solution of

$$\begin{aligned} \partial_t^2 E_k - \Delta E_k =& -\partial_t j_{f_k} - \partial_t U_k - \partial_x \rho_{f_k} + \partial_x \int_0^t \mathrm{div}_x U_k \, d\tau, \\ \partial_t^2 B_k - \Delta B_k =&\partial_{x_1} j_{f_k,2} - \partial_{x_2} j_{f_k,1} + \partial_{x_1} U_{k,2} - \partial_{x_2} U_{k,1}, \\ (E_k, B_k)(0) =& \left(\mathring{E}_k, \mathring{B}_k\right), \\ \partial_t E_k(0) =& \left(\partial_{x_2}\mathring{B}_k, -\partial_{x_1}\mathring{B}_k\right) - j_{\mathring{f}_k} - U_k(0), \\ \partial_t B_k(0) =& -\partial_{x_1}\mathring{E}_{k,2} + \partial_{x_2}\mathring{E}_{k,1}. \end{aligned} \tag{1.28}$$

Indeed, we can solve these wave equations by applying the solution formula given before Lemma 0.4. Since the right hand sides of the above equations are of class C^∞ and bounded, so are also E_k and B_k.

1.3.2 Certain bounds

Now we apply Theorems 1.1, 1.2, and 1.4 to get nice estimates which will allow us to conclude that, indeed, there is a solution of (αVM). The first goal is to show that (f_k, E_k, B_k) is bounded in the $\|\cdot\|_{C_b^1}$-norm uniformly in k. In the following write $K_k := E_k - \widehat{p}^\perp B_k$ for simplicity. We have

$$\partial_t f_k + \widehat{p}\cdot\partial_x f_k + \alpha K_{k-1}\cdot\partial_p f_k =0,$$

$$\begin{aligned}
\partial_t^2 E_k - \Delta E_k =& - \partial_t j_{f_k} - \partial_t U_k - \partial_x \rho_{f_k} + \partial_x \int_0^t \mathrm{div}_x U_k \ d\tau, \\
\partial_t^2 B_k - \Delta B_k =& \partial_{x_1} j_{f_k,2} - \partial_{x_2} j_{f_k,1} + \partial_{x_1} U_{k,2} - \partial_{x_2} U_{k,1}, \\
(f_k, E_k, B_k)(0) =& \left(\mathring{f}_k, \mathring{E}_k, \mathring{B}_k \right), \\
\partial_t E_k (0) =& \left(\partial_{x_2} \mathring{B}_k, -\partial_{x_1} \mathring{B}_k \right) - j_{\mathring{f}_k} - U_k (0), \\
\partial_t B_k (0) =& - \partial_{x_1} \mathring{E}_{k,2} + \partial_{x_2} \mathring{E}_{k,1}.
\end{aligned}$$

Applying Theorems 1.1 and then 1.2 we get

$$\|f_k (t)\|_\infty \leq \left\| \mathring{f}_k \right\|_\infty \leq C$$

and

$$\begin{aligned}
& \|E_k (t)\|_\infty + \|B_k (t)\|_\infty \\
\leq & C \left(\left\| \mathring{f}_k \right\|_\infty + \left\| \mathring{E}_k \right\|_{C_b^1} + \left\| \mathring{B}_k \right\|_{C_b^1} + \|U_k\|_{W^{1,1}\left(0,T;C_b^2\right)} \right) \\
& + C \int_0^t \left(1 + \|E_{k-1} (\tau)\|_\infty + \|B_{k-1} (\tau)\|_\infty\right) \|f_k (\tau)\|_\infty \, d\tau \\
\leq & C + C \int_0^t \left(\|E_{k-1} (\tau)\|_\infty + \|B_{k-1} (\tau)\|_\infty\right) d\tau.
\end{aligned}$$

Hence

$$\|E_k (t)\|_\infty + \|B_k (t)\|_\infty \leq C$$

which is a consequence of the following lemma:

Lemma 1.15. *i) Let (a_k) satisfy*

$$\begin{aligned}
a_k (t) \leq & C_1 + C_2 \int_0^t a_{k-1} (\tau) \, d\tau \ \text{for} \ \ k \geq 1, \\
a_0 (t) \leq & C_1
\end{aligned}$$

for $t \in [0, T]$. Then we have

$$a_k (t) \leq C_1 e^{C_2 T}.$$

ii) Let $\left(b_k^l\right)$ be non-negative and satisfy

$$\begin{aligned}
b_k^l (t) \leq & C z_k^l + C \int_0^t b_{k-1}^{l-1} (\tau) \, d\tau \ \text{for} \ \ k, l \geq 1, \\
b_k^l (t) \leq & C
\end{aligned}$$

for $t \in [0,T]$ and assume $z_k^l \to 0$ for $k,l \to \infty$ (i.e. $\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall k,l \geq N$: $\left|z_k^l\right| < \epsilon$). Then we have

$$b_k^l(t) \to 0,\ \ k,l \to \infty$$

uniformly in t.

Proof. i) It is sufficient to show via induction that

$$a_k(t) \leq C_1 \sum_{i=0}^{k} \frac{C_2^i t^i}{i!}.$$

The estimate obviously holds for $k=0$. If it is true for $k-1$, then we also get

$$a_k(t) \leq C_1 + C_2 \int_0^t a_{k-1}(\tau)\,d\tau \leq C_1 + C_2 \int_0^t C_1 \sum_{i=0}^{k-1} \frac{C_2^i \tau^i}{i!}\ d\tau = C_1 \sum_{i=0}^{k} \frac{C_2^i t^i}{i!}.$$

ii) We show

$$b_k^l(t) \leq C(1+t)\, z_k^l \sum_{i=0}^{m-1} \frac{C^i t^i}{i!} + C^m \int_0^t \frac{(t-\tau)^{m-1}}{(m-1)!} b_{k-m}^{l-m}(\tau)\,d\tau$$

for $1 \leq m \leq \min\{k,l\}$ via induction. The case $m=1$ is part of the assumption and the iteration step works as well:

$$\begin{aligned}
b_k^l(t) \leq & C(1+t)\, z_k^l \sum_{i=0}^{m-1} \frac{C^i t^i}{i!} + C^m \int_0^t \frac{(t-\tau)^{m-1}}{(m-1)!} b_{k-m}^{l-m}(\tau)\,d\tau \\
\leq & C(1+t)\, z_k^l \sum_{i=0}^{m-1} \frac{C^i t^i}{i!} + C^{m+1} \int_0^t \frac{(t-\tau)^{m-1}}{(m-1)!} \int_0^\tau \left(z_k^l + b_{k-(m+1)}^{l-(m+1)}(s) \right) ds d\tau \\
\leq & C(1+t)\, z_k^l \sum_{i=0}^{m-1} \frac{C^i t^i}{i!} + C^{m+1}(1+t)\, z_k^l \int_0^t \frac{(t-\tau)^{m-1}}{(m-1)!}\ d\tau \\
& + C^{m+1} \int_0^t \int_s^t \frac{(t-\tau)^{m-1}}{(m-1)!}\ d\tau\ b_{k-(m+1)}^{l-(m+1)}(s)\,ds \\
= & C(1+t)\, z_k^l \sum_{i=0}^{m} \frac{C^i t^i}{i!} + C^{m+1} \int_0^t \frac{(t-s)^m}{m!} b_{k-(m+1)}^{l-(m+1)}(s)\,ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
b_k^l(t) \leq & C(1+T)\, e^{CT} z_k^l + C C^m \int_0^t \frac{(t-\tau)^{m-1}}{(m-1)!}\ d\tau \\
\leq & C(1+T)\, e^{CT} z_k^l + \frac{C^{m+1} T^m}{m!} \to 0
\end{aligned}$$

for $k,l \to \infty$ (with now $m := \min\{k,l\}$) uniformly in t.

$\square$

The second part of the lemma above will be needed not until later.
With the knowledge that (f_k, E_k, B_k) is bounded in the $\|\cdot\|_\infty$-norm we can now apply Theorem 1.4 to conclude

$$\begin{aligned}&\|\partial_{t,x}E_k(t)\|_\infty + \|\partial_{t,x}B_k(t)\|_\infty\\ \leq& C\left(1+\ln_+\left(|||\partial_{x,p}f_k|||_{[0,t]}\right)+\int_0^t\left(\|\partial_{t,x}E_{k-1}(\tau)\|_\infty+\|\partial_{t,x}B_{k-1}(\tau)\|_\infty\right)d\tau\right).\end{aligned}$$

Now Theorem 1.1 yields

$$\begin{aligned}&\ln_+\left(|||\partial_{x,p}f_k|||_{[0,t]}\right)\\ \leq&\ln_+\left(C\exp\left(C+C\int_0^t\left(\|\partial_x E_{k-1}(\tau)\|_\infty+\|\partial_x B_{k-1}(\tau)\|_\infty\right)d\tau\right)\right)\\ \leq&\ln_+ C+C+C\int_0^t\left(\|\partial_x E_{k-1}(\tau)\|_\infty+\|\partial_x B_{k-1}(\tau)\|_\infty\right)d\tau.\end{aligned}$$

Putting these last two estimates together we have

$$\|\partial_{t,x}E_k(t)\|_\infty + \|\partial_{t,x}B_k(t)\|_\infty \leq C + C\int_0^t\left(\|\partial_{t,x}E_{k-1}(\tau)\|_\infty+\|\partial_{t,x}B_{k-1}(\tau)\|_\infty\right)d\tau$$

which implies with Lemma 1.15 that the left hand side is bounded by C. Hence also $\|\partial_{x,p}f_k(t)\|_\infty$ is estimated by C, and so is the t-derivative of f_k as a consequence of the Vlasov equation.
Furthermore we would like to bound (f_k, E_k, B_k) in the $\|\cdot\|_{C_b^2}$-norm. For this sake we have to differentiate the system (αVM) with respect to x_i and then p_i. On the one hand we derive the system

$$\begin{aligned}\partial_t\partial_{x_i}f_k+\widehat{p}\cdot\partial_x\partial_{x_i}f_k+&\\ \alpha K_{k-1}\cdot\partial_p\partial_{x_i}f_k=&-\alpha\partial_{x_i}K_{k-1}\cdot\partial_p f_k,\\ \partial_t^2\partial_{x_i}E_k-\Delta\partial_{x_i}E_k=&-\partial_t j_{\partial_{x_i}f_k}-\partial_t\partial_{x_i}U_k-\partial_x\rho_{\partial_{x_i}f_k}+\partial_x\int_0^t \mathrm{div}_x\partial_{x_i}U_k\ d\tau,\\ \partial_t^2\partial_{x_i}B_k-\Delta\partial_{x_i}B_k=&\partial_{x_1}j_{\partial_{x_i}f_k,2}-\partial_{x_2}j_{\partial_{x_i}f_k,1}+\partial_{x_1}\partial_{x_i}U_{k,2}-\partial_{x_2}\partial_{x_i}U_{k,1},\\ (\partial_{x_i}f_k,\partial_{x_i}E_k,\partial_{x_i}B_k)(0)=&\left(\partial_{x_i}\mathring{f}_k,\partial_{x_i}\mathring{E}_k,\partial_{x_i}\mathring{B}_k\right),\\ \partial_t\partial_{x_i}E_k(0)=&\left(\partial_{x_2}\partial_{x_i}\mathring{B}_k,-\partial_{x_1}\partial_{x_i}\mathring{B}_k\right)-j_{\partial_{x_i}\mathring{f}_k}-\partial_{x_i}U_k(0),\\ \partial_t\partial_{x_i}B_k(0)=&-\partial_{x_1}\partial_{x_i}\mathring{E}_{k,2}+\partial_{x_2}\partial_{x_i}\mathring{E}_{k,1}\end{aligned}\tag{1.29}$$

which has the form of (GVM) with the unknowns $\partial_{x_i}f_k$, $\partial_{x_i}E_k$, and $\partial_{x_i}B_k$, and on the other hand

$$\begin{aligned}\partial_t\partial_{p_i}f_k+\widehat{p}\cdot\partial_x\partial_{p_i}f_k+\alpha K_{k-1}\cdot\partial_p\partial_{p_i}f_k=&-\partial_{p_i}\widehat{p}\cdot\partial_x f_k-K_{k-1}\partial_{p_i}\alpha\cdot\partial_p f_k\\ &-\alpha\partial_{p_i}\widehat{p}^\perp B_{k-1}\cdot\partial_p f_k,\\ \partial_{p_i}f_k(0)=&\partial_{p_i}\mathring{f}_k.\end{aligned}\tag{1.30}$$

Theorem 1.1 yields (note that the corresponding force field αK_{k-1} is already controlled in C_b^1 and use the already known bounds)

$$\begin{aligned}&\|\partial_{x,p}\partial_{x_i} f_k(t)\|_\infty \\ \leq& C + C\int_0^t \left(\|\partial_x\partial_{x_i}E_{k-1}(\tau)\|_\infty + \|\partial_x\partial_{x_i}B_{k-1}(\tau)\|_\infty + \|\partial_{x,p}\partial_p f_k(\tau)\|_\infty\right) d\tau\end{aligned}$$

and

$$\|\partial_{x,p}\partial_{p_i} f_k(t)\|_\infty \leq C + C\int_0^t \left\|\partial_{x,p}^2 f_k(\tau)\right\|_\infty d\tau;$$

together

$$\begin{aligned}&\left\|\partial_{x,p}^2 f_k(t)\right\|_\infty \\ \leq& C + C\int_0^t \left(\left\|\partial_x^2 E_{k-1}(\tau)\right\|_\infty + \left\|\partial_x^2 B_{k-1}(\tau)\right\|_\infty + \left\|\partial_{x,p}^2 f_k(\tau)\right\|_\infty\right) d\tau\end{aligned}$$

which yields via Gronwall's inequality

$$\left\|\partial_{x,p}^2 f_k(t)\right\|_\infty \leq C + C\int_0^t \left(\left\|\partial_x^2 E_{k-1}(\tau)\right\|_\infty + \left\|\partial_x^2 B_{k-1}(\tau)\right\|_\infty\right) d\tau$$

and hence

$$\|\partial_{t,x,p}\partial_{x,p} f_k(t)\|_\infty \leq C + C\int_0^t \left(\left\|\partial_x^2 E_{k-1}(\tau)\right\|_\infty + \left\|\partial_x^2 B_{k-1}(\tau)\right\|_\infty\right) d\tau$$

by the Vlasov equations of (1.29) and (1.30).
Furthermore with Theorem 1.4 applied to (1.29), we can estimate (note again that K_{k-1} is controlled and $\ln_+ a \leq a$)

$$\begin{aligned}&\|\partial_{t,x}\partial_x E_k(t)\|_\infty + \|\partial_{t,x}\partial_x B_k(t)\|_\infty \\ \leq& C + C\left|\left\|\partial_{x,p}^2 f_k\right\|\right|_{[0,t]} \\ &+ C\int_0^t \left(\|\partial_{t,x}\partial_x E_{k-1}(\tau)\|_\infty + \|\partial_{t,x}\partial_x B_{k-1}(\tau)\|_\infty + \|\partial_{t,x,p}\partial_{x,p} f_k(\tau)\|_\infty\right) d\tau \\ \leq& C + C\int_0^t \left(\|\partial_{t,x}\partial_x E_{k-1}(\tau)\|_\infty + \|\partial_{t,x}\partial_x B_{k-1}(\tau)\|_\infty\right) d\tau.\end{aligned}$$

In the estimate above, it is important that we have four space derivatives of U so that $\|\partial_x U_k\|_{W^{2,1}\left(0,T;C_b^3\right)}$ is controlled by C. Hence again, $\partial_{t,x}\partial_x E_k(t)$ and $\partial_{t,x}\partial_x B_k(t)$ are bounded by C uniformly in t and therefore also $\partial_{t,x,p}\partial_{x,p} f_k(t)$. Furthermore, on the one hand, recalling system (1.29), we find that also $\partial_t^2 E_k$ and $\partial_t^2 B_k$ are bounded by C. On the other hand, with this knowledge we conclude that the same holds for the ∂_t^2-derivative of f since

$$\partial_t^2 f_k + \widehat{p}\cdot\partial_t\partial_x f_k + \alpha\partial_t K_{k-1}\cdot\partial_p f_k + \alpha K_{k-1}\cdot\partial_t\partial_p f_k = 0.$$

Therefore, the iterates are bounded in $W^{2,\infty}$.

1.3.3 Convergence of the iteration scheme and regularity of the solution

Now, having all desired bounds together, we somehow have to show that our sequences converge in a suitable sense. For this sake, define $f_k^l := f_k - f_l$ and likewise for the fields and subtract the equations of the l-th step from those of the k-th step:

$$\begin{aligned}
\partial_t f_k^l + \widehat{p} \cdot \partial_x f_k^l + \alpha K_{k-1} \cdot \partial_p f_k^l =& - \alpha K_{k-1}^{l-1} \cdot \partial_p f_l, \\
\partial_t^2 E_k^l - \Delta E_k^l =& - \partial_t j_{f_k^l} - \partial_t U_k^l - \partial_x \rho_{f_k^l} + \partial_x \int_0^t \mathrm{div}_x U_k^l \, d\tau, \\
\partial_t^2 B_k^l - \Delta B_k^l =& \partial_{x_1} j_{f_k^l,2} - \partial_{x_2} j_{f_k^l,1} + \partial_{x_1} U_{k,2}^l - \partial_{x_2} U_{k,1}^l, \\
\left(f_k^l, E_k^l, B_k^l\right)(0) =& \left(\mathring{f}_k^l, \mathring{E}_k^l, \mathring{B}_k^l\right), \\
\partial_t E_k^l(0) =& \left(\partial_{x_2} \mathring{B}_k^l, -\partial_{x_1} \mathring{B}_k^l\right) - j_{\mathring{f}_k^l} - U_k^l(0), \\
\partial_t B_k^l(0) =& - \partial_{x_1} \mathring{E}_{k,2}^l + \partial_{x_2} \mathring{E}_{k,1}^l.
\end{aligned}$$

Denote $z_k^l := \left\|\mathring{f}_k^l\right\|_{C_b^2} + \left\|\mathring{E}_k^l\right\|_{C_b^3} + \left\|\mathring{B}_k^l\right\|_{C_b^3} + \left\|U_k^l\right\|_V$. Theorems 1.1 and 1.4 yield together with the known bounds

$$\left\|f_k^l(t)\right\|_\infty \leq z_k^l + C \int_0^t \left(\left\|E_{k-1}^{l-1}(\tau)\right\|_\infty + \left\|B_{k-1}^{l-1}(\tau)\right\|_\infty\right) d\tau$$

and then

$$\begin{aligned}
&\left\|E_k^l(t)\right\|_\infty + \left\|B_k^l(t)\right\|_\infty \\
\leq& C z_k^l + C \int_0^t \left(\left\|f_k^l(\tau)\right\|_\infty + \left\|E_{k-1}^{l-1}(\tau)\right\|_\infty + \left\|B_{k-1}^{l-1}(\tau)\right\|_\infty\right) d\tau \\
\leq& C z_k^l + C \int_0^t \left(\left\|E_{k-1}^{l-1}(\tau)\right\|_\infty + \left\|B_{k-1}^{l-1}(\tau)\right\|_\infty\right) d\tau.
\end{aligned}$$

Thus, as a consequence of Lemma 1.15 ii), (E_k) and (B_k) and hence also (f_k) are Cauchy sequences in the $\|\cdot\|_\infty$-norm. The respective limits, denoted by E, B, and f, are, of course, the candidates for the solution of (αVM).
It would be nice if we had Cauchy sequences also with respect to the C_b^1-norm. For this reason now subtract the systems (1.29) and (1.30) for step l from those of the step k so that we have

$$\begin{aligned}
&\partial_t \partial_{x_i} f_k^l + \widehat{p} \cdot \partial_x \partial_{x_i} f_k^l + \\
&\qquad \alpha K_{k-1} \cdot \partial_p \partial_{x_i} f_k^l = - \alpha K_{k-1}^{l-1} \cdot \partial_p \partial_{x_i} f_k - \alpha \partial_{x_i} K_{k-1}^{l-1} \cdot \partial_p f_k - \alpha \partial_{x_i} K_{l-1} \cdot \partial_p f_k^l, \\
&\partial_t^2 \partial_{x_i} E_k^l - \Delta \partial_{x_i} E_k^l = - \partial_t j_{\partial_{x_i} f_k^l} - \partial_t \partial_{x_i} U_k^l - \partial_x \rho_{\partial_{x_i} f_k^l} + \partial_x \int_0^t \mathrm{div}_x \partial_{x_i} U_k^l \, d\tau, \\
&\partial_t^2 \partial_{x_i} B_k^l - \Delta \partial_{x_i} B_k^l = \partial_{x_1} j_{\partial_{x_i} f_k^l,2} - \partial_{x_2} j_{\partial_{x_i} f_k^l,1} + \partial_{x_1} \partial_{x_i} U_{k,2}^l - \partial_{x_2} \partial_{x_i} U_{k,1}^l, \\
&\left(\partial_{x_i} f_k^l, \partial_{x_i} E_k^l, \partial_{x_i} B_k^l\right)(0) = \left(\partial_{x_i} \mathring{f}_k^l, \partial_{x_i} \mathring{E}_k^l, \partial_{x_i} \mathring{B}_k^l\right),
\end{aligned}$$

$$\begin{aligned}\partial_t\partial_{x_i}E_k^l(0) &= \left(\partial_{x_2}\partial_{x_i}\mathring{B}_k^l, -\partial_{x_1}\partial_{x_i}\mathring{B}_k^l\right) - j_{\partial_{x_i}\mathring{f}_k^l} - \partial_{x_i}U_k^l(0),\\ \partial_t\partial_{x_i}B_k^l(0) &= -\partial_{x_1}\partial_{x_i}\mathring{E}_{k,2}^l + \partial_{x_2}\partial_{x_i}\mathring{E}_{k,1}^l\end{aligned}$$

and

$$\begin{aligned}\partial_t\partial_{p_i}f_k^l + \widehat{p}\cdot\partial_x\partial_{p_i}f_k^l + \alpha K_{k-1}\cdot\partial_p\partial_{p_i}f_k^l =& -\alpha K_{k-1}^{l-1}\cdot\partial_p\partial_{p_i}f_l - \partial_{p_i}\widehat{p}\cdot\partial_x f_k^l\\ &- \left(K_{k-1}\partial_{p_i}\alpha - \alpha\partial_{p_i}\widehat{p}^\perp B_{k-1}\right)\cdot\partial_p f_k^l\\ &- \left(K_{k-1}^{l-1}\partial_{p_i}\alpha - \alpha\partial_{p_i}\widehat{p}^\perp B_{k-1}^{l-1}\right)\cdot\partial_p f_l,\\ \partial_{p_i}f_k^l(0) =& \partial_{p_i}\mathring{f}_k^l.\end{aligned}$$

As before, Theorem 1.1 (and the estimates above) imply

$$\left\|f_k^l(t)\right\|_{C_b^1} \le Cz_k^l + C\int_0^t\left(\left\|f_k^l(\tau)\right\|_{C_b^1} + \left\|E_{k-1}^{l-1}(\tau)\right\|_{C_b^1} + \left\|B_{k-1}^{l-1}(\tau)\right\|_{C_b^1}\right)d\tau,$$

so by Gronwall's inequality

$$\left\|f_k^l(t)\right\|_{C_b^1} \le Cz_k^l + C\int_0^t\left(\left\|E_{k-1}^{l-1}(\tau)\right\|_{C_b^1} + \left\|B_{k-1}^{l-1}(\tau)\right\|_{C_b^1}\right)d\tau.$$

On the other hand Theorem 1.2 yields

$$\begin{aligned}&\left\|E_k^l(t)\right\|_{C_b^1} + \left\|B_k^l(t)\right\|_{C_b^1}\\ \le& Cz_k^l + C\int_0^t\left(\left\|f_k^l(\tau)\right\|_{C_b^1} + \left\|E_{k-1}^{l-1}(\tau)\right\|_{C_b^1} + \left\|B_{k-1}^{l-1}(\tau)\right\|_{C_b^1}\right)d\tau\\ \le& Cz_k^l + C\int_0^t\left(\left\|E_{k-1}^{l-1}(\tau)\right\|_{C_b^1} + \left\|B_{k-1}^{l-1}(\tau)\right\|_{C_b^1}\right)d\tau.\end{aligned}$$

Applying Lemma 1.15 we conclude that $(f_k, E_k, B_k)(t)$ are Cauchy sequences in the C_b^1-norm uniformly in t. Exactly the same can be done for the t-derivatives starting with

$$\begin{aligned}&\partial_t^2 f_k^l + \widehat{p}\cdot\partial_x\partial_t f_k^l +\\ &\quad\alpha K_{k-1}\cdot\partial_p\partial_t f_k^l = -\alpha K_{k-1}^{l-1}\cdot\partial_p\partial_t f_k - \alpha\partial_t K_{k-1}^{l-1}\cdot\partial_p f_k - \alpha\partial_t K_{l-1}\cdot\partial_p f_k^l,\\ &\partial_t^2\partial_t E_k^l - \Delta\partial_t E_k^l = -\partial_t j_{\partial_t f_k^l} - \partial_t^2 U_k^l - \partial_x\rho_{\partial_t f_k^l} + \partial_x\int_0^t \mathrm{div}_x\partial_t U_k^l\,d\tau,\\ &\partial_t^2\partial_t B_k^l - \Delta\partial_t B_k^l = \partial_{x_1}j_{\partial_t f_k^l,2} - \partial_{x_2}j_{\partial_t f_k^l,1} + \partial_{x_1}\partial_t U_{k,2}^l - \partial_{x_2}\partial_t U_{k,1}^l,\\ &\partial_t f_k^l(0) = -\widehat{p}\cdot\partial_x\mathring{f}_k^l - \alpha\mathring{K}_{k-1}\cdot\partial_p\mathring{f}_k^l - \alpha\mathring{K}_{k-1}^{l-1}\cdot\partial_p\mathring{f}_l,\\ &\partial_t E_k^l(0) = \left(\partial_{x_2}\mathring{B}_k^l, -\partial_{x_1}\mathring{B}_k^l\right) - j_{\mathring{f}_k^l} - U_k^l(0),\\ &\partial_t B_k^l(0) = -\partial_{x_1}\mathring{E}_{k,2}^l + \partial_{x_2}\mathring{E}_{k,1}^l,\\ &\partial_t^2 E_k^l(0) = \Delta\mathring{E}_k^l - 4\pi\int\widehat{p}\left(\widehat{p}\cdot\partial_x\mathring{f}_k^l + \alpha\left(\mathring{K}_{k-1}\cdot\partial_p\mathring{f}_k^l + \mathring{K}_{k-1}^{l-1}\cdot\partial_p\mathring{f}_l\right)\right)dp\end{aligned}$$

$$\begin{aligned}
&-\partial_t U_k^l(0) - \partial_x \rho_{\mathring{f}_k^l},\\
\partial_t^2 B_k^l(0) =& \Delta \mathring{B}_k^l + \partial_{x_1} j_{\mathring{f}_k^l,2} - \partial_{x_2} j_{\mathring{f}_k^l,1} + \partial_{x_1} U_{k,2}^l(0) - \partial_{x_2} U_{k,1}^l(0).
\end{aligned}$$

The initial conditions for the ∂_t^2-terms are a result of (1.27) and (1.28).
Altogether, (f_k, E_k, B_k) is a Cauchy sequence in the whole C_b^1-norm.
For later considerations it will be convenient that the density and the fields are even C_b^2. Since all second derivatives are bounded in $L^\infty\left([0,T]\times\mathbb{R}^j\right)$ ($j = 4$ or 2 respectively) they converge, after extracting a suitable subsequence, in the weak-*-sense. Of course, these limits have to be the respective weak derivatives of f, E, and B. The remaining part is to show that the weak derivatives just obtained are in fact classical ones. For this sake, have a look at the representation formula for $\partial_{x_i}\partial_{x_j} B_k$; use system (1.29) and Theorem 1.4:

$$\begin{aligned}
&\partial_{x_i}\partial_{x_j} B_k - \partial_{x_i}\overline{B}_k^0\\
=&\int_0^t \int_{|x-y|<t-\tau} \int \frac{bt}{(t-\tau)\sqrt{(t-\tau)^2 - |x-y|^2}} \partial_{x_i}\partial_{x_j} f_k \; dpdyd\tau\\
&+\int_0^t \int_{|x-y|<t-\tau} \int \frac{(\alpha\partial_p(bs) + bs\nabla\alpha)\cdot \partial_{x_j} f_k \partial_{x_i} K_{k-1}}{\sqrt{(t-\tau)^2 - |x-y|^2}} \; dpdyd\tau\\
&+\int_0^t \int_{|x-y|<t-\tau} \int \frac{(\alpha\partial_p(bs) + bs\nabla\alpha)\cdot K_{k-1}\partial_{x_i}\partial_{x_j} f_k}{\sqrt{(t-\tau)^2 - |x-y|^2}} \; dpdyd\tau\\
&-\int_0^t \int_{|x-y|<t-\tau} \int \frac{(bs)\,\alpha\partial_{x_i}\partial_{x_j} K_{k-1}\cdot \partial_p f_k}{\sqrt{(t-\tau)^2 - |x-y|^2}} \; dpdyd\tau\\
&-\int_0^t \int_{|x-y|<t-\tau} \int \frac{(bs)\,\alpha\partial_{x_j} K_{k-1}\cdot \partial_{x_i}\partial_p f_k}{\sqrt{(t-\tau)^2 - |x-y|^2}} \; dpdyd\tau\\
&+\frac{1}{2\pi}\int_0^t \int_{|x-y|<t-\tau} \frac{\partial_{x_1}\partial_{x_i} U_{k,2} - \partial_{x_2}\partial_{x_i} U_{k,2}}{\sqrt{(t-\tau)^2 - |x-y|^2}} \; dyd\tau.
\end{aligned}$$

Here, $\overline{B}_k^0$ is the 'B^0' of system (1.29) and converges to the respective expression without indices (recall Lemma 0.4 and the definition of $data_i$).
We are allowed to pass to the limit in the integral expressions because all kernels are integrable, (f_k, E_k, B_k) converge in C_b^1, the second derivatives weak-* in L^∞, and U_k in V. Hence we can omit the indices in the equation above or equivalently

$$\begin{aligned}
&\partial_{x_i}\partial_{x_j} B - \partial_{x_i}\overline{B}^0\\
=&\int_0^t \int_{|x-y|<t-\tau} \int \frac{bt}{(t-\tau)\sqrt{(t-\tau)^2 - |x-y|^2}} \partial_{x_i}\partial_{x_j} f \; dpdyd\tau
\end{aligned}$$

$$+\int_0^t \int_{|x-y|<t-\tau} \int \frac{(\alpha\partial_p(bs) + bs\nabla\alpha)\cdot\partial_{x_i}\left(K\partial_{x_j}f\right)}{\sqrt{(t-\tau)^2 - |x-y|^2}}\, dpdyd\tau$$
$$-\int_0^t \int_{|x-y|<t-\tau} \int \frac{(bs)\,\alpha\partial_{x_i}\left(\partial_{x_j}K\cdot\partial_p f\right)}{\sqrt{(t-\tau)^2 - |x-y|^2}}\, dpdyd\tau$$
$$+\frac{1}{2\pi}\int_0^t \int_{|x-y|<t-\tau} \frac{\partial_{x_1}U_2 - \partial_{x_2}U_1}{\sqrt{(t-\tau)^2 - |x-y|^2}}\, dyd\tau$$

and conclude that $\partial_{x_i}\partial_{x_j}B$ is continuous which is an immediate consequence of $U \in V$ and the following lemma:

Lemma 1.16. *Denote $M := \{(s,z) \in [0,T]\times\mathbb{R}^n \mid 0 \le s \le T,\ |z| < s\}$ and let $h \in C\left([0,T]\times\mathbb{R}^{n+m}\right)$ with uniform support in $p \in \mathbb{R}^m$, i.e. $\mathrm{supp}_p h \subset B_r$, and let $w \in C^1(M\times B_r)$ and $\gamma \in \{t, x_1, \dots x_n\}$. Furthermore let one of the following options hold:*

i) $h \in W^{1,\infty}\left([0,T]\times\mathbb{R}^{n+m}\right)$ and $w \in L^1(M\times B_r)$,

ii) $h \in W^{1,1}\left(0,T;L^\infty\left(\mathbb{R}^{n+m}\right)\right)$ if $\gamma = t$ or $h \in L^\infty\left(0,T;W^{1,\infty}\left(\mathbb{R}^{n+m}\right)\right)$ if $\gamma = x_i$ respectively, and

$$\int_{s-d<|z|<s}\int_{B_r} |w(s,z,p)|\, dpdz \to 0$$

for $d \to 0$ uniformly in $s \in [0,T]$.

Then

$$H(t,x) := \int_0^t \int_{|x-y|<t-\tau} \int (\partial_\gamma h)(\tau,y,p)\, w(t-\tau, y-x, p)\, dpdyd\tau$$
$$= \int_0^t \int_{|z|<s} \int (\partial_\gamma h)(t-s, x+z, p)\, w(s,z,p)\, dpdzds$$

is continuous in $(t,x) \in [0,T]\times\mathbb{R}^n$.

Proof. Let $\gamma = x_i$ and $\epsilon > 0$ be given. For $(t,x) \in [0,T]\times\mathbb{R}^n$ and $d > 0$ define

$$I_d(t,x) := \int_0^t \int_{s-d<|z|<s} \int (\partial_{x_i}h)(t-s, x+z, p)\, w(s,z,p)\, dpdzds$$

and estimate in case i)

$$|I_d(t,x)| \le \|\partial_{x_i}h\|_\infty \int_0^T \int_{s-d<|z|<s}\int_{B_r} |w(s,z,p)|\, dpdzds \to 0$$

and in case ii)

$$|I_d(t,x)| \le \int_0^T \|\partial_{x_i}h(s)\|_\infty\, ds \left\| s \mapsto \int_{s-d<|z|<s}\int_{B_r} |w(s,z,p)|\, dpdz \right\|_\infty \to 0$$

for $d \to 0$ uniformly in (t,x). Thus we can choose d so that $|I_d(t,x)| < \frac{\epsilon}{4}$ for all (t,x). For now fixed d consider the remaining integral and integrate by parts

$$\begin{aligned}
J_d(t,x) :=& \int_0^t \int_{|z|<s-d} \int (\partial_{x_i} h)(t-s, x+z, p)\, w(s,z,p)\, dpdzds \\
=& \int_0^t \int_{|z|<s-d} \int (\partial_{z_i} h)(t-s, x+z, p)\, w(s,z,p)\, dpdzds \\
=& -\int_0^t \int_{|z|<s-d} \int h(t-s, x+z, p)\, \partial_{z_i} w(s,z,p)\, dpdzds \\
&+ \int_0^t \int_{|z|=s-d} \int h(t-s, x+z, p)\, w(s,z,p)\, \frac{1}{\sqrt{2}}\, dpdS_z ds \\
&+ \int_{|z|<t-d} \int h(0, x+z, p)\, w(t,z,p)\, dpdz.
\end{aligned}$$

This is allowed because the integration domain is away from the possibly singular set $|z| = s$. For that very reason J_d is obviously continuous by the standard theorem for parameter integrals, so if $(\delta t, \delta x)$ is small enough (with $t + \delta t \in [0,T]$) we have

$$|J_d(t+\delta t, x+\delta x) - J_d(t,x)| < \frac{\epsilon}{2}.$$

Finally with $H = I_d + J_d$ we conclude

$$\begin{aligned}
&|H(t+\delta t, x+\delta x) - H(t,x)| \\
\leq& |I_d(t+\delta t, x+\delta x)| + |I_d(t,x)| + |J_d(t+\delta t, x+\delta x) - J_d(t,x)| < \epsilon.
\end{aligned}$$

Analogously, one proves the assertion for $\gamma = t$. □

This lemma is applicable since f has uniform support in p, $\partial_x f$, $\partial_p f$, and $\partial_x K$ are of class $W^{1,\infty}$, $|bs|$, $|bt| \leq C(r)$, and by Remark 0.1.
Next, we have a representation formula for $\partial_t \partial_{x_j} B_k$ according to Theorem 1.4. Analogously we conclude that $\partial_t \partial_{x_j} B$ is continuous. For this, note that the terms without an $\int_0^t$-integral are easy to handle since there only initial values appear.
The procedure for E is nearly the same. The only critical point is to ensure that

$$\int_0^t \int_{|x-y|<t-\tau} \frac{\partial_t^2 \partial_{x_j} U}{\sqrt{(t-\tau)^2 - |x-y|^2}}\, dyd\tau$$

is continuous for $U \in V$. To this end, we can apply Lemma 1.16 with $h = \partial_t \partial_{x_j} U \chi$ where $\chi = \chi(p) \in C_c^\infty(\mathbb{R}^2)$ with $\int \chi\, dp = 1$. Note that $\partial_t \partial_{x_j} U$ is continuous and of class $W^{1,1}(0,T; L^\infty)$ by $U \in V$, and that

$$\int_{s-d<|z|<s} \frac{1}{\sqrt{s^2 - |z|^2}}\, dz = 2\pi \sqrt{2sd - d^2}\, 1_{s \geq d} \leq 2\pi \sqrt{T}\sqrt{d}.$$

So there only remain the ∂_t^2-derivatives of E and B. By the known convergence, we can pass to the limit in (αVM) so that the Vlasov equation holds everywhere and the Maxwell equations almost everywhere. With this knowledge and the just proven fact that the second space derivatives of the fields are continuous, we conclude that also the ∂_t^2-derivatives are continuous. Now the fact that the weak derivatives are continuous instantly implies that they are classical ones. Therefore the fields are of class C^2. Thus the characteristics

$$\dot{X} = \widehat{P},\ \dot{P} = \alpha(P)\left(E - \widehat{P}^{\perp}B\right)(s, X),\ (X, P)(t, t, x, p) = (x, p)$$

are well defined and of class C^2 in (t, x, p) (see [3] again). Hence

$$f(t, x, p) = \mathring{f}((X, P)(0, t, x, p))$$

is also of class C^2.

We have solved (αVM), but actually (CVM) is to be solved: Obviously, (αVM) coincides with (CVM2nd) as long as f vanishes for $|p| \geq Q$. But this property is guaranteed by Section 1.2. Therefore (f, E, B) is a solution of (CVM2nd) and hence of (CVM) by equivalence.

In the following we also may neglect α in the equations for the iterates because of $\alpha = 1$ on B_{2Q} and $f_k \to f$ uniformly; thus f_k vanishes for $|p| \geq 2Q$ if k is large enough.

We collect some properties of (f, E, B):

Theorem 1.17. *There is a solution (f, E, B) of* (CVM) *with:*

i) *f, E, and B are of class C^2,*

ii) *f vanishes for $|p| \geq Q$ or $|x| \geq R + T$ (where Q only depends on T, the initial data (their C_b^1-norms and $P(0)$), and $\|U\|_V$, and where* $\operatorname{supp}_x \mathring{f} \subset B_R$*),*

iii) *E, B vanish for $|x| \geq \widetilde{R} + L + R + T$ if their initial data are compactly supported, i.e.* supp $\mathring{E}$, supp $\mathring{B} \subset B_{\widetilde{R}}$,

iv) *the C_b^2-norms of the solution are estimated by a constant only depending on T, the initial data (their C_b^2-norms and $P(0)$), L, and $\|U\|_V$.*

Proof. For ii) note that $\left|\dot{X}\right| \leq 1$, for iii) recall the representation formula of the fields, and iv) holds because it holds for all iterates, they converge in C_b^1 and their second derivatives weakly-* in L^∞. □

1.3.4 Uniqueness

We prove uniqueness of the solution.

Theorem 1.18. *The obtained solution (f, E, B) of* (CVM) *is unique in $C^1 \times \left(C^2\right)^2$.*

Proof. Let $\left(\widetilde{f}, \widetilde{E}, \widetilde{B}\right)$ (with the above regularity) solve (CVM) too and define $\overline{f} := \widetilde{f} - f$ and so on. Then we have the system

$$
\begin{aligned}
\partial_t \overline{f} + \widehat{p} \cdot \partial_x \overline{f} + \left(\widetilde{E} - \widehat{p}^{\perp} \widetilde{B}\right) \cdot \partial_p \overline{f} =& - \left(\overline{E} - \widehat{p}^{\perp} \overline{B}\right) \cdot \partial_p f, \\
\partial_t \overline{E}_1 - \partial_{x_2} \overline{B} =& - j_{\overline{f},1}, \\
\partial_t \overline{E}_2 + \partial_{x_1} \overline{B} =& - j_{\overline{f},2}, \\
\partial_t \overline{B} + \partial_{x_1} \overline{E}_2 - \partial_{x_2} \overline{E}_1 =& 0, \\
\left(\overline{f}, \overline{E}, \overline{B}\right)(0) =& 0.
\end{aligned}
$$

Theorem 1.1 yields

$$
\left\|\overline{f}(t)\right\|_\infty \leq \|\partial_p f\|_\infty \int_0^t \left(\left\|\overline{E}(\tau)\right\|_\infty + \left\|\overline{B}(\tau)\right\|_\infty\right) d\tau
$$

and Theorem 1.2 implies

$$
\begin{aligned}
\left\|\overline{E}(t)\right\|_\infty + \left\|\overline{B}(t)\right\|_\infty \leq& C(T, Q) \left(1 + \left\|\widetilde{E}\right\|_\infty + \left\|\widetilde{B}\right\|_\infty + \|\partial_p f\|_\infty\right) \\
& \cdot \int_0^t \left(\left\|\overline{f}(\tau)\right\|_\infty + \left\|\overline{E}(\tau)\right\|_\infty + \left\|\overline{B}(\tau)\right\|_\infty\right) d\tau
\end{aligned}
$$

since the assertions of Section 1.2, especially Theorem 1.14, hold both for f and $\widetilde{f}$, and since the p-integral of the right hand side of the Vlasov equation vanishes so that (LC) holds for the system above by Lemma 0.3.
These two estimates and Gronwall's inequality instantly yield $\left(\overline{f}, \overline{E}, \overline{B}\right) = 0$; note that $f \in C_b^1$ and that $\left\|\widetilde{E}\right\|_\infty$ and $\left\|\widetilde{B}\right\|_\infty$ are finite because of (1.25) and (1.26). $\square$

Moreover, it is possible to show that the solution is unique in an even larger class. Here, the constructed solution satisfies the conditions if $\mathring{E}$ and $\mathring{B}$ are compactly supported.

Theorem 1.19. *A solution* (f, E, B) *of* (CVM) *with the properties*

i) f, E, *and* B *are of class* $W^{1,\infty} \cap H^1 \cap H^1\left(0, T; L^2\right)$,

ii) $\operatorname{supp} f \subset [0, T] \times B_r^2$ *for some* $r > 0$,

is unique (here, 'solution' means that (CVM) *holds pointwise almost everywhere).*

Proof. Let $\left(\widetilde{f}, \widetilde{E}, \widetilde{B}\right)$ (with the above properties) solve (CVM) too and define $\overline{f} := \widetilde{f} - f$ and so on. Then we have the system

$$
\begin{aligned}
\partial_t \overline{f} + \widehat{p} \cdot \partial_x \overline{f} + \left(\widetilde{E} - \widehat{p}^{\perp} \widetilde{B}\right) \cdot \partial_p \overline{f} =& - \left(\overline{E} - \widehat{p}^{\perp} \overline{B}\right) \cdot \partial_p f, \\
\partial_t \overline{E}_1 - \partial_{x_2} \overline{B} =& - j_{\overline{f},1},
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \overline{E}_2 + \partial_{x_1} \overline{B} =& - j_{\overline{f},2},\\
\partial_t \overline{B} + \partial_{x_1} \overline{E}_2 - \partial_{x_2} \overline{E}_1 =& 0,\\
\left(\overline{f}, \overline{E}, \overline{B}\right)(0) =& 0.
\end{aligned}
$$

Note that initial values make sense because of $H^1\left(0,T;L^2\right) \hookrightarrow C\left(0,T;L^2\right)$. We have

$$
\begin{aligned}
&\frac{1}{2}\left\|\overline{f}(t)\right\|_{L^2}^2\\
=&\int_0^t \int\int \overline{f}\partial_t \overline{f}\ dpdxd\tau\\
=&\int_0^t \int\int \overline{f}\left(-\widehat{p}\cdot\partial_x \overline{f} - \left(\widetilde{E} - \widehat{p}^{\perp}\widetilde{B}\right)\cdot\partial_p \overline{f} - \left(\overline{E} - \widehat{p}^{\perp}\overline{B}\right)\cdot\partial_p f\right) dpdxd\tau\\
=&\int_0^t \int\int \left(-\frac{1}{2}\mathrm{div}_x\left(\widehat{p}\overline{f}^2\right) - \frac{1}{2}\mathrm{div}_p\left(\left(\widetilde{E} - \widehat{p}^{\perp}\widetilde{B}\right)\overline{f}^2\right) - \overline{f}\left(\overline{E} - \widehat{p}^{\perp}\overline{B}\right)\cdot\partial_p f\right)\\
&\qquad dpdxd\tau\\
=&-\int_0^t \int\int \overline{f}\left(\overline{E} - \widehat{p}^{\perp}\overline{B}\right)\cdot\partial_p f\ dpdxd\tau\\
\leq& \|f\|_{W^{1,\infty}}\int_0^t \left\|\overline{f}(\tau)\right\|_{L^2}\left(\left\|\overline{E}(\tau)\right\|_{L^2} + \left\|\overline{B}(\tau)\right\|_{L^2}\right) d\tau,
\end{aligned}
$$

which implies

$$
\left\|\overline{f}(t)\right\|_{L^2} \leq \|f\|_{W^{1,\infty}}\int_0^t \left(\left\|\overline{E}(\tau)\right\|_{L^2} + \left\|\overline{B}(\tau)\right\|_{L^2}\right) d\tau
$$

via the quadratic version of Gronwall's inequality, cf. [2], Theorem 5. Similarly,

$$
\begin{aligned}
\frac{1}{2}\left\|\overline{B}(t)\right\|_{L^2}^2 =& \int_0^t\int \overline{B}\partial_t\overline{B}\ dxd\tau = \int_0^t\int \overline{B}\left(-\partial_{x_1}\overline{E}_2 + \partial_{x_2}\overline{E}_1\right) dxd\tau\\
=& \int_0^t\int \left(\overline{E}_2\partial_{x_1}\overline{B} - \overline{E}_1\partial_{x_2}\overline{B}\right) dxd\tau\\
=& \int_0^t\int \left(-\overline{E}\cdot\partial_t\overline{E} - \overline{E}\cdot j_{\overline{f}}\right) dxd\tau.
\end{aligned}
$$

Note that in the integration by parts no surface terms appear because of E, $B \in H^1$. This computation leads to

$$
\begin{aligned}
&\frac{1}{2}\left(\left\|\overline{E}(t)\right\|_{L^2}^2 + \left\|\overline{B}(t)\right\|_{L^2}^2\right) = \int_0^t\int -\overline{E}\cdot j_{\overline{f}}\ dxd\tau\\
\leq& \int_0^t \left\|\overline{E}(\tau)\right\|_{L^2}\left\|j_{\overline{f}}(\tau)\right\|_{L^2} d\tau \leq C(r)\int_0^t \left(\left\|\overline{E}(\tau)\right\|_{L^2} + \left\|\overline{B}(\tau)\right\|_{L^2}\right)\left\|\overline{f}(\tau)\right\|_{L^2} d\tau.
\end{aligned}
$$

Here, the last inequality holds because $\overline{f}$ vanishes as soon as $|p| > r$. Now again, the quadratic Gronwall lemma implies

$$\begin{aligned}\left\|\overline{E}(t)\right\|_{L^2} + \left\|\overline{B}(t)\right\|_{L^2} \leq & C(r)\int_0^t \left\|\overline{f}(\tau)\right\|_{L^2} d\tau \\ \leq & C(r,T)\|f\|_{W^{1,\infty}} \int_0^t \left(\left\|\overline{E}(\tau)\right\|_{L^2} + \left\|\overline{B}(\tau)\right\|_{L^2}\right) d\tau.\end{aligned}$$

This yields $\left(\overline{E}, \overline{B}\right) = 0$ and hence also $\overline{f} = 0$. □

2 The control-to-state operator

From now on the initial data always stay fixed with $0 \leq \mathring{f} \in C_c^2$ and $\mathring{E}$, $\mathring{B} \in C_c^3$, and $\mathrm{div}\mathring{E} = \rho_{\mathring{f}}$. As a result of the last section we may define the control-to-state operator via

$$\begin{aligned} S\colon V \to & C_b^2\left([0,T] \times \mathbb{R}^4\right) \times C_b^2\left([0,T] \times \mathbb{R}^2; \mathbb{R}^2\right) \times C_b^2\left([0,T] \times \mathbb{R}^2\right), \\ U \mapsto & (f, E, B).\end{aligned}$$

The goal is to show that S is differentiable with respect to suitable norms.

2.1 Lipschitz continuity

First we have to investigate whether S is Lipschitz continuous; to be more precise, locally Lipschitz continuous. Let $U, \delta U \in V$ and denote $(f, E, B) = S(U)$, $\left(\overline{f}, \overline{E}, \overline{B}\right) = S(U + \delta U)$, and $\left(\widetilde{f}, \widetilde{E}, \widetilde{B}\right) = S(U + \delta U) - S(U)$. We arrive at the system

$$\begin{aligned} \partial_t \widetilde{f} + \widehat{p} \cdot \partial_x \widetilde{f} + \left(E - \widehat{p}^\perp B\right) \cdot \partial_p \widetilde{f} = & -\left(\widetilde{E} - \widehat{p}^\perp \widetilde{B}\right) \cdot \partial_p \overline{f}, \\ \partial_t \widetilde{E}_1 - \partial_{x_2} \widetilde{B} = & -j_{\widetilde{f},1} - \delta U_1, \\ \partial_t \widetilde{E}_2 + \partial_{x_1} \widetilde{B} = & -j_{\widetilde{f},2} - \delta U_2, \\ \partial_t \widetilde{B} + \partial_{x_1} \widetilde{E}_2 - \partial_{x_2} \widetilde{E}_1 = & 0, \\ \left(\widetilde{f}, \widetilde{E}, \widetilde{B}\right)(0) = & 0, \end{aligned} \tag{2.1}$$

which is equivalent to the system with second order Maxwell equations because of Lemmata 0.2 and 0.3.

Note that the x- and p-support of the density and the C_b^1-norm of the solution is controlled by a constant dependent on T, the initial data, L, and the V-norm of the control, see Theorem 1.17. Therefore we can perform the same estimates also on the $\overline{\cdot}$-solution with a constant dependent on T, the initial data, L, and $\|U\|_V$ because, for instance, for $\|\delta U\|_V \leq 1$ we have $\|U + \delta U\|_V \leq \|U\|_V + 1$. Hence we will only show the locally Lipschitz continuity of S. Now we apply the results of Section 1.1:

$$\left\|\widetilde{f}(t)\right\|_\infty \leq C\int_0^t \left(\left\|\widetilde{E}(\tau)\right\|_\infty + \left\|\widetilde{B}(\tau)\right\|_\infty\right) d\tau$$

and

$$\begin{aligned}
&\left\|\widetilde{E}\,(t)\right\|_{\infty}+\left\|\widetilde{B}\,(t)\right\|_{\infty}\\
\leq &C\int_0^t\left(\left\|\widetilde{f}\,(\tau)\right\|_{\infty}+\left\|\widetilde{E}\,(\tau)\right\|_{\infty}+\left\|\widetilde{B}\,(\tau)\right\|_{\infty}\right)d\tau+C\left\|\delta U\right\|_{W^{1,1}\left(0,T;C_b^2\right)}\\
\leq &C\int_0^t\left(\left\|\widetilde{E}\,(\tau)\right\|_{\infty}+\left\|\widetilde{B}\,(\tau)\right\|_{\infty}\right)d\tau+C\left\|\delta U\right\|_{V}
\end{aligned}$$

which yields

$$\left\|\widetilde{f}\,(t)\right\|_{\infty}+\left\|\widetilde{E}\,(t)\right\|_{\infty}+\left\|\widetilde{B}\,(t)\right\|_{\infty}\leq C\left\|\delta U\right\|_{V}.$$

Differentiating (2.1) with respect to x and then to p implies

$$\begin{aligned}
\partial_t\partial_{x_i}\widetilde{f}+\widehat{p}\cdot\partial_x\partial_{x_i}\widetilde{f}+\left(E-\widehat{p}^{\perp}B\right)\cdot\partial_p\partial_{x_i}\widetilde{f}=&-\left(\partial_{x_i}E-\widehat{p}^{\perp}\partial_{x_i}B\right)\cdot\partial_p\widetilde{f}\\
&-\left(\partial_{x_i}\widetilde{E}-\widehat{p}^{\perp}\partial_{x_i}\widetilde{B}\right)\cdot\partial_p\overline{f}\\
&-\left(\widetilde{E}-\widehat{p}^{\perp}\widetilde{B}\right)\cdot\partial_p\partial_{x_i}\overline{f},\\
\partial_t\partial_{x_i}\widetilde{E}_1-\partial_{x_2}\partial_{x_i}\widetilde{B}=&-j_{\partial_{x_i}\widetilde{f},1}-\partial_{x_i}\delta U_1,\\
\partial_t\partial_{x_i}\widetilde{E}_2+\partial_{x_1}\partial_{x_i}\widetilde{B}=&-j_{\partial_{x_i}\widetilde{f},2}-\partial_{x_i}\delta U_2,\\
\partial_t\partial_{x_i}\widetilde{B}+\partial_{x_1}\partial_{x_i}\widetilde{E}_2-\partial_{x_2}\partial_{x_i}\widetilde{E}_1=&0,\\
\left(\partial_{x_i}\widetilde{f},\partial_{x_i}\widetilde{E},\partial_{x_i}\widetilde{B}\right)(0)=&0,
\end{aligned}$$

and

$$\begin{aligned}
\partial_t\partial_{p_i}\widetilde{f}+\widehat{p}\cdot\partial_x\partial_{p_i}\widetilde{f}+\left(E-\widehat{p}^{\perp}B\right)\cdot\partial_p\partial_{p_i}\widetilde{f}=&-\partial_{p_i}\widehat{p}\cdot\partial_x\widetilde{f}+\partial_{p_i}\widehat{p}^{\perp}B\cdot\partial_p\widetilde{f}\\
&-\left(\widetilde{E}-\widehat{p}^{\perp}\widetilde{B}\right)\cdot\partial_p\partial_{p_i}\overline{f}+\partial_{p_i}\widehat{p}^{\perp}\widetilde{B}\cdot\partial_p\overline{f},\\
\partial_{p_i}\widetilde{f}\,(0)=&0.
\end{aligned}$$

The estimate of Theorem 1.2 is not directly applicable because some terms need not be regular enough. But these two systems somehow also hold for the iterates ($(U+\delta U)_k$ denotes the smooth approximation of $U+\delta U$):

$$\begin{aligned}
&\partial_t\partial_{x_i}\widetilde{f}_k+\widehat{p}\cdot\partial_x\partial_{x_i}\widetilde{f}_k+\\
&\left(E_{k-1}-\widehat{p}^{\perp}B_{k-1}\right)\cdot\partial_p\partial_{x_i}\widetilde{f}_k=-\left(\partial_{x_i}E_{k-1}-\widehat{p}^{\perp}\partial_{x_i}B_{k-1}\right)\cdot\partial_p\widetilde{f}_k\\
&\qquad-\left(\partial_{x_i}\widetilde{E}_{k-1}-\widehat{p}^{\perp}\partial_{x_i}\widetilde{B}_{k-1}\right)\cdot\partial_p\overline{f}_k\\
&\qquad-\left(\widetilde{E}_{k-1}-\widehat{p}^{\perp}\widetilde{B}_{k-1}\right)\cdot\partial_p\partial_{x_i}\overline{f}_k,\\
&\partial_t^2\partial_{x_i}\widetilde{E}_k-\Delta\partial_{x_i}\widetilde{E}_k=-\partial_t j_{\partial_{x_i}\widetilde{f}_k}-\partial_t\partial_{x_i}\left(\left(U+\delta U\right)_k-U_k\right)-\partial_x\rho_{\partial_{x_i}\widetilde{f}_k}
\end{aligned}$$

$$
\begin{aligned}
&+ \partial_x \int_0^t \operatorname{div}_x \partial_{x_i} \left((U+\delta U)_k - U_k \right) \, d\tau, \\
\partial_t^2 \partial_{x_i} \widetilde{B}_k - \Delta \partial_{x_i} \widetilde{B}_k =& \partial_{x_1} j_{\partial_{x_i} \widetilde{f}_k, 2} - \partial_{x_2} j_{\partial_{x_i} \widetilde{f}_k, 1} + \partial_{x_1} \partial_{x_i} \left((U+\delta U)_k - U_k \right)_2 \\
&- \partial_{x_2} \partial_{x_i} \left((U+\delta U)_k - U_k \right)_1, \\
\left(\partial_{x_i} E_k, \partial_{x_i} B_k \right)(0) =& 0, \\
\partial_t \partial_{x_i} E_k(0) =& - \partial_{x_i} \left((U+\delta U)_k - U_k \right)(0), \\
\partial_t \partial_{x_i} B_k(0) =& 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_t \partial_{p_i} \widetilde{f}_k + \widehat{p} \cdot \partial_x \partial_{p_i} \widetilde{f}_k + \\
&\left(E_{k-1} - \widehat{p}^{\perp} B_{k-1} \right) \cdot \partial_p \partial_{p_i} \widetilde{f}_k = - \partial_{p_i} \widehat{p} \cdot \partial_x \widetilde{f}_k + \partial_{p_i} \widehat{p}^{\perp} B_{k-1} \cdot \partial_p \widetilde{f}_k \\
&\qquad - \left(\widetilde{E}_{k-1} - \widehat{p}^{\perp} \widetilde{B}_{k-1} \right) \cdot \partial_p \partial_{p_i} \overline{f}_k + \partial_{p_i} \widehat{p}^{\perp} \widetilde{B}_{k-1} \cdot \partial_p \overline{f}_k, \\
&\partial_{p_i} \widetilde{f}(0) = 0.
\end{aligned}
$$

Therefore we get the estimates

$$
\left\| \widetilde{f}_k(t) \right\|_{C_b^1} \le C \int_0^t \left(\left\| \widetilde{f}_k(\tau) \right\|_{C_b^1} + \left\| \widetilde{E}_{k-1}(\tau) \right\|_{C_b^1} + \left\| \widetilde{B}_{k-1}(\tau) \right\|_{C_b^1} \right) d\tau
$$

which yields

$$
\left\| \widetilde{f}_k(t) \right\|_{C_b^1} \le C \int_0^t \left(\left\| \widetilde{E}_{k-1}(\tau) \right\|_{C_b^1} + \left\| \widetilde{B}_{k-1}(\tau) \right\|_{C_b^1} \right) d\tau,
$$

and

$$
\begin{aligned}
&\left\| \partial_{t,x} \widetilde{E}_k(t) \right\|_\infty + \left\| \partial_{t,x} \widetilde{B}_k(t) \right\|_\infty \\
\le& C \int_0^t \left(\left\| \widetilde{f}_k(\tau) \right\|_{C_b^1} + \left\| \partial_{t,x} \widetilde{E}_{k-1}(\tau) \right\|_\infty + \left\| \partial_{t,x} \widetilde{B}_{k-1}(\tau) \right\|_\infty \right) d\tau \\
&+ C \left\| (U+\delta U)_k - U_k \right\|_{W^{1,1}\left(0,T;C_b^3\right)} \\
\le& C \int_0^t \left(\left\| \partial_{t,x} \widetilde{E}_{k-1}(\tau) \right\|_\infty + \left\| \partial_{t,x} \widetilde{B}_{k-1}(\tau) \right\|_\infty \right) d\tau + C \left\| \delta U \right\|_V,
\end{aligned}
$$

hence with Lemma 1.15 i)

$$
\left\| \widetilde{f}_k(t) \right\|_{C_b^1} + \left\| \widetilde{E}_k \right\|_{C_b^1} + \left\| \widetilde{B}_k \right\|_{C_b^1} \le C \left\| \delta U \right\|_V.
$$

Since the iterates converge in C_b^1 to the respective functions, the inequality above also holds without indices. With this knowledge and the Vlasov equation of (2.1) we even conclude

$$
\left\| \left(\widetilde{f}, \widetilde{E}, \widetilde{B} \right) \right\|_{C_b^1} \le C \left\| \delta U \right\|_V.
$$

Thus we have proved:

Theorem 2.1. *$S\colon V \to C_b^1\left([0,T]\times\mathbb{R}^4\right)\times C_b^1\left([0,T]\times\mathbb{R}^2\right)^3$ is locally Lipschitz continuous.*

2.2 Solvability of a linearized system

To show even differentiability of S we will have to analyze a linearized system of the form

$$\begin{aligned}\partial_t f+\widehat{p}\cdot\partial_x f+G\cdot\partial_p f=&\left(E-\widehat{p}^{\perp}B\right)\cdot g+a,\\ \partial_t E_1-\partial_{x_2}B=&-j_{f,1}+h_1,\\ \partial_t E_2+\partial_{x_1}B=&-j_{f,2}+h_2,\\ \partial_t B+\partial_{x_1}E_2-\partial_{x_2}E_1=&0,\\ (f,E,B)(0)=&0\end{aligned}\tag{LVM}$$

with already given functions $a\in L^1\left(0,T;L^2\right)$, $G\in C_b^2$ with $\mathrm{div}_p G=0$, $g\in C_b^1$ with $g=\partial_p\widetilde{g}$ for some $\widetilde{g}\in C_b^2$ and $g(t,x,p)=0$ for $|x|\geq r$ or $|p|\geq r$ for some $r>0$, and $h\in V$. We call (f,E,B) a solution of (LVM) if f, E, and B are of class $C\cap H^1$, the equalities hold pointwise almost everywhere, and f vanishes for $|p|\geq R$ for some $R>0$.
A crucial estimate is the following:

Theorem 2.2. *Let (f,E,B) be a solution of* (LVM)*. Then*

$$\|f(t)\|_{L^2}+\|E(t)\|_{L^2}+\|B(t)\|_{L^2}\leq C\left(R,\|g\|_\infty,T\right)\int_0^t\left(\|a(\tau)\|_{L^2}+\|h(\tau)\|_{L^2}\right)d\tau.$$

Proof. The proof is very similar to that of Theorem 1.19. First we have

$$\begin{aligned}&\frac{1}{2}\|f(t)\|_{L^2}^2\\ =&\int_0^t\int\int f\partial_t f\ dpdxd\tau\\ =&\int_0^t\int\int f\left(-\widehat{p}\cdot\partial_x f-G\cdot\partial_p f+\left(E-\widehat{p}^{\perp}B\right)\cdot g+a\right)dpdxd\tau\\ =&\int_0^t\int\int\left(-\frac{1}{2}\mathrm{div}_x\left(\widehat{p}f^2\right)-\frac{1}{2}\mathrm{div}_p\left(Gf^2\right)+f\left(E-\widehat{p}^{\perp}B\right)\cdot g+fa\right)dpdxd\tau\\ =&\int_0^t\int\int\left(f\left(E-\widehat{p}^{\perp}B\right)\cdot g+fa\right)dpdxd\tau\\ \leq&C(R)\left(1+\|g\|_\infty\right)\int_0^t\|f(\tau)\|_{L^2}\left(\|E(\tau)\|_{L^2}+\|B(\tau)\|_{L^2}+\|a(\tau)\|_{L^2}\right)d\tau,\end{aligned}$$

which implies

$$\|f(t)\|_{L^2}\leq C(R)\left(1+\|g\|_\infty\right)\int_0^t\left(\|E(\tau)\|_{L^2}+\|B(\tau)\|_{L^2}+\|a(\tau)\|_{L^2}\right)d\tau$$

via the quadratic version of Gronwall's inequality. Similarly we have

$$\begin{aligned}\frac{1}{2}\left\|B(t)\right\|_{L^2}^2 &= \int_0^t\int B\partial_t B\ dxd\tau = \int_0^t\int B\left(-\partial_{x_1}E_2+\partial_{x_2}E_1\right)dxd\tau\\ &= \int_0^t\int \left(E_2\partial_{x_1}B - E_1\partial_{x_2}B\right)dxd\tau\\ &= \int_0^t\int \left(-E\cdot\partial_t E - E\cdot j_f - E\cdot h\right)dxd\tau\end{aligned}$$

Note that in the integration by parts no surface terms appear because of E, $B\in H^1$. This computation leads to

$$\begin{aligned}&\frac{1}{2}\left(\left\|E(t)\right\|_{L^2}^2+\left\|B(t)\right\|_{L^2}^2\right) = \int_0^t\int -E\cdot j_f - E\cdot h\ dxd\tau\\ \le&\int_0^t \left\|E(\tau)\right\|_{L^2}\left(\left\|j_f(\tau)\right\|_{L^2}+\left\|h(\tau)\right\|_{L^2}\right)d\tau\\ \le& C(R)\int_0^t\left(\left\|E(\tau)\right\|_{L^2}+\left\|B(\tau)\right\|_{L^2}\right)\left(\left\|f(\tau)\right\|_{L^2}+\left\|h(\tau)\right\|_{L^2}\right)d\tau.\end{aligned}$$

Here, the last inequality holds because f vanishes as soon as $|p|$ is big enough. Now again, the quadratic Gronwall lemma implies

$$\begin{aligned}&\left\|E(t)\right\|_{L^2}+\left\|B(t)\right\|_{L^2}\\ \le& C(R)\int_0^t\left(\left\|f(\tau)\right\|_{L^2}+\left\|h(\tau)\right\|_{L^2}\right)d\tau\\ \le& C\left(R,\left\|g\right\|_\infty,T\right)\int_0^t\left(\left\|E(\tau)\right\|_{L^2}+\left\|B(\tau)\right\|_{L^2}+\left\|a(\tau)\right\|_{L^2}+\left\|h(\tau)\right\|_{L^2}\right)d\tau.\end{aligned}$$

This yields

$$\left\|E(t)\right\|_{L^2}+\left\|B(t)\right\|_{L^2}\le C\left(R,\left\|g\right\|_\infty,T\right)\int_0^t\left(\left\|a(\tau)\right\|_{L^2}+\left\|h(\tau)\right\|_{L^2}\right)d\tau$$

and then also

$$\left\|f(t)\right\|_{L^2}\le C\left(R,\left\|g\right\|_\infty,T\right)\int_0^t\left(\left\|a(\tau)\right\|_{L^2}+\left\|h(\tau)\right\|_{L^2}\right)d\tau.$$

□

We approximate G, $\widetilde{g}$, and h with smooth functions G_k, $\widetilde{g}_k$, and h_k which are converging to G, $\widetilde{g}$, and h in C_b^2 and V respectively, and define $g_k := \partial_p\widetilde{g}_k$. Without loss of generality we may assume that $\left\|G_k\right\|_{C_b^2}\le 2\left\|G\right\|_{C_b^2}$, $\left\|g_k\right\|_{C_b^1}\le 2\left\|g\right\|_{C_b^1}$, $\left\|h_k\right\|_V\le 2\left\|h\right\|_V$ and that $h_k(t,x)=0$ for $|x|\ge 2L$ and $g_k(t,x,p)=0$ for $|x|\ge 2r$ or $|p|\ge 2r$. These properties will often be used in the following. The constant C may now depend on T, L, r, $\|G\|_{C_b^2}$, $\|g\|_{C_b^1}$, and $\|h\|_V$. To show solvability of (LVM) for $a = 0$ we proceed similarly as before. Define $f_0 = E_{0,1} = E_{0,2} = B_0 = 0$ and solve in the k-th step

$$\begin{aligned}\partial_t f_k + \widehat{p} \cdot \partial_x f_k + G_k \cdot \partial_p f_k =& \left(E_{k-1} - \widehat{p}^\perp B_{k-1}\right) \cdot g_k,\\ f_k(0) =& 0\end{aligned}$$

by defining

$$f_k(t,x,p) = \int_0^t \left(\left(E_{k-1} - \widehat{p}^\perp B_{k-1}\right) \cdot g_k\right)(X_k(0,t,x,p), P_k(0,t,x,p))\, d\tau$$

with the characteristics

$$\begin{aligned}\dot{X}_k =& \widehat{P}_k, & X_k(t,t,x,p) =& x,\\ \dot{P}_k =& G_k(s, X_k, P_k), & P_k(t,t,x,p) =& p,\end{aligned}$$

and then solving

$$\begin{aligned}\partial_t^2 E_k - \Delta E_k =& -\partial_t j_{f_k} - \partial_t h_k - \partial_x \rho_{f_k} + \partial_x \int_0^t \mathrm{div}_x h_k \; d\tau,\\ \partial_t^2 B_k - \Delta B_k =& \partial_{x_1} j_{f_k,2} - \partial_{x_2} j_{f_k,1} + \partial_{x_1} h_{k,2} - \partial_{x_2} h_{k,1},\\ (E_k, B_k)(0) =& 0,\\ \partial_t E_k(0) =& -U_k(0),\\ \partial_t B_k(0) =& 0.\end{aligned}$$

All iterates are of class C^∞ as in Section 1.3.1. Furthermore, the characteristics are independent of the solution sequence (f_k, E_k, B_k). Thus we instantly have $\left|\dot{P}_k\right| \le C$, so $|P_k - p| \le CT$. Having a look at the formula for f_k we conclude that f_k vanishes as soon as

$$|p| \ge 2r + CT =: Q \tag{2.2}$$

since then the integrand vanishes as a result of

$$|P_k(s,t,x,p)| \ge |p| - |P_k - p| \ge 2r + CT - CT = 2r.$$

The same can be done for the x-coordinate starting with $\left|\dot{X}_k\right| \le 1$; hence $f_k(t,x,p) = 0$ for $|x| \ge 2r + T$. The assertions of Section 1.1 are directly applicable. We do not have to insert some α because of the already known bound on the p-support of f_k. Therefore (LC) holds for the iterated system and we can thus switch between first order and second order Maxwell equations; note that $\left(E_{k-1} - \widehat{p}^\perp B_{k-1}\right) \cdot g_k = \mathrm{div}_p \left(\left(E_{k-1} - \widehat{p}^\perp B_{k-1}\right) \widetilde{g}_k\right)$.

Now we proceed like in Section 1.3.1 and first want to bound C_b^1-norms. We find

$$\|f_k(t)\|_\infty \le C \int_0^t \left(\|E_{k-1}(\tau)\|_\infty + \|B_{k-1}(\tau)\|_\infty\right) d\tau$$

and

$$\begin{aligned}\|E_k(t)\|_\infty+\|B_k(t)\|_\infty\le&C+C\int_0^t\left(\|f_k(\tau)\|_\infty+\|E_{k-1}(\tau)\|_\infty+\|B_{k-1}(\tau)\|_\infty\right)d\tau\\\le&C+C\int_0^t\left(\|E_{k-1}(\tau)\|_\infty+\|B_{k-1}(\tau)\|_\infty\right)d\tau\end{aligned}\tag{2.3}$$

which implies that E_k, B_k, and hence f_k are bounded in $\|\cdot\|_\infty$. For the first derivatives we arrive at

$$\|\partial_{x,p}f_k(t)\|_\infty\le C+C\int_0^t\left(\|\partial_x E_{k-1}(\tau)\|_\infty+\|\partial_x B_{k-1}(\tau)\|_\infty\right)d\tau$$

and

$$\begin{aligned}&\|\partial_{t,x}E_k(t)\|_\infty+\|\partial_{t,x}B_k(t)\|_\infty\\\le&C+C\ln_+\left(|||\partial_{x,p}f_k|||_{[0,t]}\right)+C\int_0^t\left(\|\partial_{t,x}E_{k-1}(\tau)\|_\infty+\|\partial_{t,x}B_{k-1}(\tau)\|_\infty\right)d\tau\\\le&C+C\int_0^t\left(\|\partial_{t,x}E_{k-1}(\tau)\|_\infty+\|\partial_{t,x}B_{k-1}(\tau)\|_\infty\right)d\tau.\end{aligned}$$

Thereby and by the Vlasov equation all first derivatives of f_k, E_k, and B_k are bounded by C.
Note that the assumptions on G, h, and g were exploited. On the one hand we could estimate $\|G_k\|_{C_b^2}$, $\|h_k\|_V$, and $\|g_k\|_{C_b^1}$ by C, on the other hand the source terms of the Vlasov equations vanish for $|p|\ge 2r$ uniformly in k.
Next we have to show some Cauchy properties of the sequences. Use the notation like in the previous section, but now denote $K_k:=E_k-\widehat{p}^\perp B_k$ and $z_k^l:=\left\|G_k^l\right\|_{C_b^2}+\left\|h_k^l\right\|_V+\left\|g_k^l\right\|_{C_b^1}$. Then we have the system

$$\begin{aligned}\partial_t f_k^l+\widehat{p}\cdot\partial_x f_k^l+G_k\cdot\partial_p f_k^l=&-G_k^l\cdot\partial_p f_l+K_{k-1}^{l-1}\cdot g_k+K_{l-1}\cdot g_k^l,\\\partial_t E_{k,1}^l-\partial_{x_2}B_k^l=&-j_{f_k^l,1}-h_{k,1}^l,\\\partial_t E_{k,2}+\partial_{x_1}B_k^l=&-j_{f_k^l,2}-h_{k,2}^l,\\\partial_t B_k^l+\partial_{x_1}E_{k,2}-\partial_{x_2}E_{k,1}=&0,\\\left(f_k^l,E_k^l,B_k^l\right)(0)=&0\end{aligned}$$

which yields the estimates

$$\left\|f_k^l(t)\right\|_\infty\le Cz_k^l+C\int_0^t\left(\left\|E_{k-1}^{l-1}(\tau)\right\|_\infty+\left\|B_{k-1}^{l-1}(\tau)\right\|_\infty\right)d\tau,$$

and on the other hand

$$\left\|E_k^l(t)\right\|_\infty+\left\|B_k^l(t)\right\|_\infty$$

$$\begin{aligned}
&\leq C z_k^l + C \int_0^t \left(\left\| f_k^l(\tau) \right\|_\infty + \left\| E_{k-1}^{l-1}(\tau) \right\|_\infty + \left\| B_{k-1}^{l-1}(\tau) \right\|_\infty \right) d\tau \\
&\leq C z_k^l + C \int_0^t \left(\left\| E_{k-1}^{l-1}(\tau) \right\|_\infty + \left\| B_{k-1}^{l-1}(\tau) \right\|_\infty \right) d\tau.
\end{aligned}$$

These are the same estimates as in Section 1.3.1, so we conclude again that (f_k, E_k, B_k) is a Cauchy sequence in the C_b-norm.
Unfortunately, we can not show the Cauchy property with respect to the C_b^1-norm. For this we would first have to bound second derivatives of f_k which would require control of second derivatives of g_k. This, on the other hand, would require a smoother g. But for the later application we will not have more regularity of g than C_b^1.
Thus we have to proceed differently: Since f_k, E_k, and B_k are bounded in the C_b^1-norm, their first derivatives converge, after extracting a suitable subsequence, to the respective derivatives of f, E, and B in L^∞ in the weak-*-sense. Because of

$$\begin{aligned}
&\left| \int_0^T \int \int \left(G_k \cdot \partial_p f_k \varphi - G \cdot \partial_p f \varphi \right) dp dx d\tau \right| \\
\leq & \int_0^T \int \int \left| G_k - G \right| \left| \partial_p f_k \right| |\varphi| \, dp dx d\tau + \left| \int_0^T \int \int G \left(\partial_p f_k - \partial_p f \right) \varphi \; dp dx d\tau \right| \\
\leq & C \left\| G_k - G \right\|_\infty \|\varphi\|_{L^1} + \left| \int_0^T \int \int G \left(\partial_p f_k - \partial_p f \right) \varphi \; dp dx d\tau \right| \to 0
\end{aligned}$$

for $k \to \infty$ for any test function φ, (f, E, B) satisfies (LVM) pointwise almost everywhere; the other terms are obviously easier to handle. Altogether we have found a solution of (LVM) of class $C \cap W^{1,\infty}$. Furthermore it is also of class H^1 because all sequence elements have compact support with respect to x, p or x respectively uniformly in t and k; for the fields recall the representation formula.
For uniqueness, let (f_1, E_1, B_1) be a solution of (LVM) too and define $f_2 := f - f_1$ and so on which yields

$$\begin{aligned}
\partial_t f_2 + \widehat{p} \cdot \partial_x f_2 + G \cdot \partial_p f_2 =& \left(E_2 - \widehat{p}^\perp B_2 \right) \cdot g, \\
\partial_t E_{2,1} - \partial_{x_2} B_2 =& - j_{f_2,1}, \\
\partial_t E_{2,2} + \partial_{x_1} B_2 =& - j_{f_2,2}, \\
\partial_t B_2 + \partial_{x_1} E_{2,2} - \partial_{x_2} E_{2,1} =& 0, \\
(f_2, E_2, B_2)(0) =& 0.
\end{aligned}$$

Applying Theorem 2.2 this instantly implies that f, E, and B vanish.

2.3 Differentiability

We want to study the differentiability of $S \colon V \to C\left(0, T; L^2\left(\mathbb{R}^4\right)\right) \times C\left(0, T; L^2\left(\mathbb{R}^2\right)\right)^3$.
Let $U \in V$ and let $\delta U \in V$ be some perturbation. In the following denote $(f, E, B) =$

$S(U)$ and $\left(\overline{f},\overline{E},\overline{B}\right) = S(U+\delta U)$. The candidate for the linearization is $S'(U)\,\delta U = (\delta f, \delta E, \delta B)$ where the right hand side satisfies

$$\begin{aligned}
\partial_t \delta f + \widehat{p}\cdot\partial_x \delta f + \left(E - \widehat{p}^{\perp} B\right)\cdot \partial_p \delta f =& - \left(\delta E - \widehat{p}^{\perp}\delta B\right)\cdot\partial_p f,\\
\partial_t \delta E_1 - \partial_{x_2}\delta B =& - j_{\delta f,1} - \delta U_1,\\
\partial_t \delta E_2 + \partial_{x_1}\delta B =& - j_{\delta f,2} - \delta U_2,\\
\partial_t \delta B + \partial_{x_1}\delta E_2 - \partial_{x_2}\delta E_1 =& 0,\\
(\delta f, \delta E, \delta B)(0) =& 0.
\end{aligned}$$

Indeed, this system can be solved because of $G := E - \widehat{p}^{\perp} B \in C_b^2$ (note that $\mathrm{div}_p G = 0$), $g := -\partial_p f \in C_b^1$, and $h := \delta U \in V$. First we note that $S'(U)$ is linear and that by Theorem 2.2

$$\|(\delta f, \delta E, \delta B)\|_{C(0,T;L^2)} \leq C \int_0^T \|\delta U(t)\|_{L^2}\, dt \leq C\,\|\delta U\|_V \tag{2.4}$$

which says that $S'(U)$ is bounded. The last inequality holds because of supp $\delta U(t) \subset B_L$.
The next step is to show that $S(U+\delta U) - S(U) - S'(U)\,\delta U$ is 'small'. Defining $\widetilde{f} := \overline{f} - f - \delta f$ and so on and subtracting the respective equations yield

$$\begin{aligned}
\partial_t \widetilde{f} + \widehat{p}\cdot\partial_x \widetilde{f} + \left(E - \widehat{p}^{\perp} B\right)\cdot\partial_p \widetilde{f} =& - \left(\widetilde{E} - \widehat{p}^{\perp}\widetilde{B}\right)\cdot\partial_p f\\
& - \left(\overline{E} - E - \widehat{p}^{\perp}\left(\overline{B} - B\right)\right)\cdot\partial_p\left(\overline{f} - f\right),\\
\partial_t \widetilde{E}_1 - \partial_{x_2}\widetilde{B} =& - j_{\widetilde{f},1},\\
\partial_t \widetilde{E}_2 + \partial_{x_1}\widetilde{B} =& - j_{\widetilde{f},2},\\
\partial_t \widetilde{B} + \partial_{x_1}\widetilde{E}_2 - \partial_{x_2}\widetilde{E}_1 =& 0,\\
\left(\widetilde{f},\widetilde{E},\widetilde{B}\right)(0) =& 0.
\end{aligned}$$

Applying Theorem 2.2 we conclude

$$\left\|\left(\widetilde{f},\widetilde{E},\widetilde{B}\right)\right\|_{C(0,T;L^2)} \leq C \int_0^T \|a(t)\|_{L^2}\, dt$$

where

$$a := -\left(\overline{E} - E - \widehat{p}^{\perp}\left(\overline{B} - B\right)\right)\cdot\partial_p\left(\overline{f} - f\right).$$

Here we have to exploit the Lipschitz property of S. Theorem 2.1 yields

$$\|a(t)\|_{L^2} \leq C\left(\left\|\overline{E} - E\right\|_\infty + \left\|\overline{B} - B\right\|_\infty\right)\left\|\overline{f} - f\right\|_{C_b^1} \leq C\,\|\delta U\|_V^2.$$

Note that for the first inequality the fact was used that $\overline{f}$ and f have compact support in x and p uniformly in t and independent of $\|\delta U\|_V$ for, for instance, $\|\delta U\|_V \leq 1$

(recall Theorem 1.17 and the reasoning in Section 2.1).
Finally we arrive at

$$\left\|\left(\widetilde{f},\widetilde{E},\widetilde{B}\right)\right\|_{C(0,T;L^2)}\leq C\left\|\delta U\right\|_V^2 \tag{2.5}$$

which proves part of i) of the following theorem:

Theorem 2.3. *i)* $S\colon V\to W:=C\left(0,T;L^2\left(\mathbb{R}^4\right)\right)\times C\left(0,T;L^2\left(\mathbb{R}^2\right)\right)^3$ *is continuously Fréchet-differentiable.*

ii) $\Phi:=\rho\circ S_1\colon V\to C\left(0,T;L^2\left(\mathbb{R}^2\right)\right)$, $U\mapsto\rho_f$ *is continuously Fréchet-differentiable.*

iii) $\overline{\Phi}:=\rho\circ S_1\colon V\to C\left(0,T;L^1\left(\mathbb{R}^2\right)\right)$, $U\mapsto\rho_f$ *is continuously Fréchet-differentiable.*

Proof. For part ii) define

$$\Phi'\left(U\right)\delta U:=\rho_{\delta f}. \tag{2.6}$$

Now it is crucial to bound the p-support of $\overline{f}$, f, and δf by a constant $C>0$ only depending on T, the initial data, L, and $\left\|U\right\|_V$. We first consider δf. The control of the p-support in (2.2) holds for all iterates and hence for δf. The constant there only depends on T, $\left\|G\right\|_\infty=\left\|E-\widehat{p}^\perp B\right\|_\infty$, the p-support of $\partial_p f$, and L. Because of Theorem 1.17 the absolute values of the fields E and B and the p-support of f are controlled by some constant only depending on T, the initial data, L, and $\left\|U\right\|_V$. Hence we have together with (2.4)

$$\left\|\rho_{\delta f}\left(t\right)\right\|_{L^2}=\left(\int\left|\int\delta f\;dp\right|^2dx\right)^{\frac{1}{2}}\leq C\left(\int\int\left|\delta f\right|^2dpdx\right)^{\frac{1}{2}}\leq C\left\|\delta U\right\|_V$$

which implies that $\Phi'\left(U\right)$ is bounded. Furthermore the p-supports of $\overline{f}$ and f only depend on T, the initial data, L, and $\left\|U\right\|_V$ (for again $\left\|\delta U\right\|_V\leq 1$ for example). Hence the same assertion holds for $\widetilde{f}=\overline{f}-f-\delta f$ and therefore with (2.5)

$$\left\|\rho_{\widetilde{f}}\left(t\right)\right\|_{L^2}=\left(\int\left|\int\widetilde{f}\;dp\right|^2dx\right)^{\frac{1}{2}}\leq C\left(\int\int\left|\widetilde{f}\right|^2dpdx\right)^{\frac{1}{2}}\leq C\left\|\delta U\right\|_V^2.$$

Together with the equality

$$\Phi\left(U+\delta U\right)-\Phi\left(U\right)-\Phi'\left(U\right)\delta U=\rho_{\overline{f}}-\rho_f-\rho_{\delta f}=\rho_{\widetilde{f}}$$

this instantly yields that $\Phi'\left(U\right)$ is indeed the Fréchet-derivative of Φ in U.
Part iii) is an instant consequence of ii) and the support assertions discussed above. The derivative of $\overline{\Phi}$ is given by (2.6) as before.
To show continuity of S', let $\delta V\in V$ with $\left\|\delta V\right\|_V\leq 1$. We have to investigate

$$\left(\check{f},\check{E},\check{B}\right):=\left(f^1,E^1,B^1\right)-\left(f^0,E^0,B^0\right):=S'\left(U+\delta U\right)\delta V-S'\left(U\right)\delta V.$$

Applying the previously given formula for S' we arrive at

$$\begin{aligned}
\partial_t \check{f} + \widehat{p} \cdot \partial_x \check{f} + \left(\overline{E} - \widehat{p}^{\perp} \overline{B}\right) \cdot \partial_p \check{f} =& - \left(\check{E} - \widehat{p}^{\perp} \check{B}\right) \cdot \partial_p \overline{f} - \left(E^0 - \widehat{p}^{\perp} B^0\right) \cdot \partial_p \left(\overline{f} - f\right) \\
&- \left(\overline{E} - E - \widehat{p}^{\perp} \left(\overline{B} - B\right)\right) \cdot \partial_p f^0, \\
\partial_t \check{E}_1 - \partial_{x_2} \check{B} =& - j_{\check{f},1}, \\
\partial_t \check{E}_2 + \partial_{x_1} \check{B} =& - j_{\check{f},2}, \\
\partial_t \check{B} + \partial_{x_1} \check{E}_2 - \partial_{x_2} \check{E}_1 =& 0, \\
\left(\check{f}, \check{E}, \check{B}\right)(0) =& 0.
\end{aligned}$$

By (2.2) and the conclusion after (2.3) we know that the p-support of f^0 and the absolute values of E^0 and B^0 are controlled by a constant only depending on T, the initial data, L, $\|U\|_V$, and $\|\delta V\|_V$ (the latter can be neglected, of course). The dependence on some terms in f, E, and B can be eliminated like in the beginning of this proof. Hence, proceeding as before and using Theorem 2.2 and the locally Lipschitz continuity of S, we conclude

$$\left\|\left(\check{f}, \check{E}, \check{B}\right)\right\|_W \leq C \|\delta U\|_V$$

where C only depends on T, the initial data, L, and $\|U\|_V$. This leads to

$$\|S'(U + \delta U) - S'(U)\|_{L(V,W)} \leq C \|\delta U\|_V$$

which says that S' is even locally Lipschitz continuous.
Using the assertions for the p-support of f^0 and f^1 (controlled by a constant only depending on T, the initial data, L, and $\|U\|_V$ if $\|\delta U\|_V \leq 1$) we conclude

$$\left\|\rho_{\check{f}}\right\|_{C(0,T;L^2)}, \left\|\rho_{\check{f}}\right\|_{C(0,T;L^1)} \leq C \left\|\check{f}\right\|_{C(0,T;L^2)} \leq C \|\delta U\|_V$$

as before. This implies that Φ' and $\overline{\Phi}'$ are locally Lipschitz continuous. □

3 Optimal control problem

Now we consider some optimal control problems. We want to minimize some objective function that depends on the external control U and the state (f, E, B). The control and the state are coupled via (CVM) so that (CVM) appears as a constraint.
We first give thought to a problem with general controls and a general objective function. Then we proceed with optimizing problems where the objective function is explicitly given and where the control set is restricted to such controls that are realizable in applications concerning the control of a plasma.

3.1 General problem

3.1.1 Control space

Until now we have worked with the control space

$$V = \left\{U \in W^{2,1}\left(0, T; C_b^4\left(\mathbb{R}^2; \mathbb{R}^2\right)\right) \mid U(t,x) = 0 \text{ for } |x| \geq L\right\}.$$

To apply standard optimization techniques it is necessary that the control space is reflexive. Hence we choose ($\gamma > 2$)

$$\mathcal{U} := \left\{U \in H^2\left(0,T;W^{5,\gamma}\left(\mathbb{R}^2;\mathbb{R}^2\right)\right) \mid U(t,x) = 0 \text{ for } |x| \geq L\right\}$$

equipped with the $H^2\left(0,T;W^{5,\gamma}\right)$-norm. By Sobolev's embedding theorems, $\mathcal{U}$ is continuously embedded in V.
In accordance with Theorems 1.17 and 2.3, we have already proved that there is a continuously differentiable control-to-state operator

$$\begin{aligned} S\colon V &\to \left(C_b^2\left([0,T]\times\mathbb{R}^4\right)\times C_b^2\left([0,T]\times\mathbb{R}^2;\mathbb{R}^2\right)\times C_b^2\left([0,T]\times\mathbb{R}^2\right), \|\cdot\|_{C(0,T;L^2)}\right), \\ U &\mapsto (f,E,B), \end{aligned}$$

such that (CVM) holds for (f,E,B) and control U. Furthermore, the map

$$U \mapsto \rho_f$$

is continuously differentiable with respect to the $C\left(0,T;L^2\right)$- and $C\left(0,T;L^1\right)$-norm in the image space. Moreover, the C_b^2-norm and the x- and p-support of (f,E,B) are controlled by a constant only depending on T, L, the initial data, and $\|U\|_V$.
By $\mathcal{U} \hookrightarrow V$, these assertions also hold with $\mathcal{U}$ instead of V.

3.1.2 Existence of minimizers

We consider the general problem

$$\begin{aligned} \min_{(f,E,B)\in\left(C^2\cap H^1\right)^3,U\in\mathcal{U}} \quad & \phi(f,E,B,U) \\ \text{s.t.} \quad & (f,E,B) = S(U). \end{aligned} \tag{GP}$$

We have to specify some assumptions on ϕ:

Condition 3.1. i) $\phi\colon \left(C^2\cap H^1\right)^3\times\mathcal{U}\to\mathbb{R}\cup\{\infty\}$ and $\phi\not\equiv\infty$,

ii) ϕ is coercive in $U\in\mathcal{U}$, i.e. in general: Let X, Y be normed spaces; $\psi\colon X\times Y\to\mathbb{R}$ is said to be coercive in $y\in Y$ iff for all sequences $(y_k)\subset Y$ with $\|y_k\|_Y\to\infty$, $k\to\infty$, then also $\psi(x_k,y_k)\to\infty$, $k\to\infty$, for any sequence $(x_k)\subset X$,

iii) ϕ is weakly lower semicontinuous in the following sense: if $(f_k,E_k,B_k)\rightharpoonup(f,E,B)$ in H^1 and $U_k\rightharpoonup U$ in $\mathcal{U}$, then $\phi(f,E,B,U)\leq\liminf_{k\to\infty}\phi(f_k,E_k,B_k,U_k)$.

These assumptions allow us to prove existence of a (not necessarily unique) minimizer. We will first prove a lemma that will be useful later:

Lemma 3.2. *Let $(U_k)\subset V$ be bounded and $(f_k,E_k,B_k)=S(U_k)$. Then, after extracting a suitable subsequence, it holds that:*

i) The sequences (f_k), (E_k), and (B_k) converge weakly in H^1 and $H^1\left(0,T;L^2\right)$, weakly- in $W^{1,\infty}$, and strongly in L^2 to some f, E, and B.*

ii) There is $r > 0$ so that f, E, B, and, for all $k \in \mathbb{N}$, f_k, E_k, and B_k vanish if $|x| \geq r$ or $|p| \geq r$.

iii) If additionally $U_k \to U$ in the sense of distributions for some $U \in V$ for $k \to \infty$, then $(f, E, B) = S(U)$ and f, E, and B are of class C_b^2.

Proof. By Theorem 1.17, on the one hand, (f_k, E_k, B_k) is bounded in the C_b^1-norm. On the other hand, f_k vanishes as soon as $|p|$ is big enough uniformly in k. Moreover, f_k, E_k, and B_k vanish as soon as $|x|$ is big enough. Hence (f_k, E_k, B_k) is also bounded in H^1 and in $H^1\left(0, T; L^2\right)$. Together with the boundedness in C_b^1, (f_k, E_k, B_k) converge, after extracting a suitable subsequence, to some (f, E, B), namely weakly in H^1 and $H^1\left(0, T; L^2\right)$, and weakly-* in $W^{1,\infty}$. This proves ii) and part of i).
For the remaining part of i) (strong convergence in L^2) we have to exploit some compactness. This compactness is guaranteed by the theorem of Rellich-Kondrachov. By the reasoning above, (f_k, E_k, B_k) are bounded in H^1 and in fact, only a bounded subset of the x- and p-space matters. Hence (a subsequence of) (f_k, E_k, B_k) converges strongly in L^2 to the limit (f, E, B).
For iii), we have to pass to the limit in (CVM). First, the initial conditions are preserved in the limit since $H^1\left(0, T; L^2\right) \hookrightarrow C\left(0, T; L^2\right)$. Furthermore the Vlasov and Maxwell equations hold pointwise almost everywhere for the limit functions: The only difficult part is the nonlinear term in the Vlasov equation. To handle this, we have to make use of the strong convergence in L^2 obtained above. We find for each $\varphi \in C_c^\infty\left(]0, T[\times \mathbb{R}^4\right)$ that

$$\begin{aligned}
&\left|\int_0^T \int\int \left(\left(E_k - \widehat{p}^\perp B_k\right) \cdot \partial_p f_k - \left(E - \widehat{p}^\perp B\right) \cdot \partial_p f\right) \varphi \, dpdxdt\right| \\
\leq &\left|\int_0^T \int\int \left(E - \widehat{p}^\perp B\right) \cdot \left(\partial_p f_k - \partial_p f\right) \varphi \, dpdxdt\right| \\
&+ \|\partial_p f_k\|_\infty \int_0^T \int\int \left(|E_k - E| + |B_k - B|\right) |\varphi| \, dpdxdt.
\end{aligned}$$

Both terms converge to 0 for $k \to \infty$ since $f_k \rightharpoonup f$ in H^1, $E_k \to E$, $B_k \to B$ in L^2, and f_k is bounded in C_b^1. Therefore, altogether, (CVM) holds pointwise almost everywhere. Now we can apply Theorem 1.19 to conclude (f, E, B) equals $S(U)$ and is hence of class C_b^2. □

Theorem 3.3. *Let ϕ satisfy Condition 3.1. Then there is a minimizer of* (GP).

Proof. We consider a minimizing sequence (f_k, E_k, B_k, U_k) with $(f_k, E_k, B_k) = S(U_k)$ and

$$\lim_{k\to\infty} \phi\left(f_k, E_k, B_k, U_k\right) = m := \inf_{U\in\mathcal{U}, (f,E,B)=S(U)} \phi\left(f, E, B, U\right) \in \mathbb{R} \cup \{-\infty\}.$$

By coercivity in U, cf. Condition 3.1 ii), (U_k) is bounded in $\mathcal{U}$ and therefore in V. Hence we may extract a weakly convergent subsequence (also denoted by U_k)

since $H^2\left(0,T;W^{5,\gamma}\right)$ is reflexive. The weak limit U is the candidate for being an optimal control. Of course, by weak convergence, U vanishes for $|x| \geq L$; hence $U \in \mathcal{U}$. Because of $\mathcal{U} \hookrightarrow L^1$ we also get $U_k \rightharpoonup U$ in L^1 and hence $U_k \to U$ in the sense of distributions. Lemma 3.2 yields $(f_k, E_k, B_k) \rightharpoonup (f, E, B)$ in H^1 (after extracting a suitable subsequence) and $(f, E, B) = S(U)$. Together with the weak lower semicontinuity of ϕ, see Condition 3.1 iii), we instantly get $\phi(f, E, B, U) = m$ which proves optimality. □

3.1.3 Occurring problems

In order to be able of examining some problem that is somehow application-oriented, we first have to think about possible problems concerning the conditions on the objective function ϕ. Especially the coercivity in U will make some trouble since the $\mathcal{U}$-norm is pretty strong. One can try to guarantee these conditions in various ways. We give two examples and comment their disadvantages:

- The objective function contains some cost term of the control in the full $\mathcal{U}$-norm (or even a stronger norm), so that for example $\phi(f, E, B, U) = \psi(f, E, B) + \|U\|_{\mathcal{U}}^2$. Then ϕ is obviously coercive in $U \in \mathcal{U}$. But typically in applications, such a strong cost term makes no sense. For instance, the energy in the external current U can be measured by its L^2-norm (with respect to x). Therefore, a cost term containing derivatives of even fifth order in space has no physical motivation. Even if we ignore physical backgrounds and establish first order optimality conditions for such a ϕ we would arrive at an equality containing all derivatives controlled in $\mathcal{U}$. This means, we would have to solve a nonlinear partial differential equation including x-derivatives of the optimal control up to tenth order and additionally mixed with t-derivatives. A numerical approach would hardly be successful.

- We add another constraint $\|U\|_{\mathcal{U}} \leq K$ for some $K > 0$. Then $\widetilde{\phi}(f, E, B, U) := \phi(f, E, B, U) + \chi_{B_K}(U)$ (where $\chi_A(a) = \begin{cases} 0, & a \in A, \\ \infty, & a \notin A \end{cases}$ for some set A) is coercive in $U \in \mathcal{U}$ if for example $\phi \geq 0$ (typically ϕ is indeed non-negative). Ignoring the physical reasonableness of that constraint and rather concentrating on mathematical aspects we note that this approach would lead to first order optimality conditions in which a Lagrange multiplier with respect to the new constraint will occur. This Lagrange multiplier will be an element of the dual space of $\mathcal{U}$ which is very irregular since $\mathcal{U}$ is very regular. Again, from a numerical point of view, these conditions will be hard to handle.

On the other hand, we can not simply use a less regular control space. Firstly, we need $\mathcal{U} \hookrightarrow V$ to ensure that the control-to-state operator is differentiable; this will be useful later. Secondly, $\mathcal{U}$ needs to be reflexive to extract (in some sense) converging subsequences from a minimizing sequence. Here we should remark that we also could demand $W^{2,p}$-regularity in time for $p > 1$ instead of H^2-regularity which would allow more controls if $1 < p < 2$. However, working in a H^2-setting (at least in time) is more convenient.

3.2 An optimization problem with realizable external currents

3.2.1 Motivation

As the previous considerations suggest it would be nice if we somehow eliminated the variability of the control with respect to the space coordinate. This can be achieved by only considering controls of the form

$$U(t,x) = \sum_{j=1}^{N} u_j(t)\, z_j\left(R(\alpha_j)\, x + b_j\right)$$

where the functions $0 \not\equiv z_j \in C_b^6\left(\mathbb{R}^2;\mathbb{R}^2\right)$ with z_j vanishing for $|x| \geq r_j > 0$ are fixed and we only vary the functions $u_j \in H^2([0,T])$, the angles of the rotation matrices $R(\alpha_j) = \begin{pmatrix} \cos\alpha_j & -\sin\alpha_j \\ \sin\alpha_j & \cos\alpha_j \end{pmatrix}$, and the translation vectors $b_j \in \mathbb{R}^2$.
From a physical point of view, this model describes an ensemble of N inductors with 'size' r_j that can suitably be placed in space initially by changing α_j and b_j but that stay fixed in time. Since the size of a laboratory is limited, we assume $|b_j| \leq r$ for some $r > 0$. Therefore, U is an element of V if we set $L = r + \max\{r_j \mid j = 1, \ldots N\}$. Each inductor generates a current z_j at full capacity that is tangential to the plane and that extends infinitely in the third space dimension. We control the system by turning these inductors on whereby the capacity u_j is suitably adjusted as a function of time. Hence we will have to consider an additional constraint $|u_j| \leq 1$. Physically, the consideration only of controls of the above form is no substantial restriction at all because only such control fields are realizable in applications.

3.2.2 Formulation

The problem to be considered is the following:

$$\begin{aligned} \min_{\substack{(f,E,B)\in\left(C^2\cap H^1\right)^3, u\in H^2([0,T])^N,\\ \alpha\in\mathbb{R}^N, b\in\left(\mathbb{R}^2\right)^N}} \quad & \frac{1}{2}\left\|\rho_f - \rho_d\right\|^2_{L^2([0,T]\times\mathbb{R}^2)} + \frac{\beta}{2}\sum_{j=1}^{N} c_j \Big(\left\|u_j\right\|^2_{L^2([0,T])} \\ & + \beta_1 \left\|\partial_t u_j\right\|^2_{L^2([0,T])} + \beta_2 \left\|\partial_t^2 u_j\right\|^2_{L^2([0,T])}\Big) \\ \text{s.t. } & (f,E,B) = S\left(\sum_{j=1}^{N} u_j z_j\left(R(\alpha_j)\cdot + b_j\right)\right), \\ & |u_j| \leq 1, |b_j| \leq r, j = 1,\ldots,N \end{aligned} \tag{P1}$$

where $c_j := \left\|z_j\right\|_{L^2(\mathbb{R}^2;\mathbb{R}^2)}$. We give some comments on the objective function:

- The charge density shall be as close as possible to some given desired density $\rho_d = \rho_d(t,x) \in L^2\left([0,T]\times\mathbb{R}^2\right)$. One could consider the L^2-norm of some $f - f_d$ instead but the space coordinates of the particles are of actual interest rather than their momenta.

- Furthermore, the cost term containing the control shall be as small as possible. We have to use the full H^2-norm of the u_j in the regularization term because thereby the objective function is coercive in $u \in H^2([0,T])^N$. On the other hand, from a practical point of view, one would like to have the u_j nearly constant rather than oscillating and therefore having to turn the inductors on and off quite often. However, the L^2-norms of the u_j itself are more interesting than the ones of their derivatives. Hence it is suitable to choose $0 < \beta_1, \beta_2 \ll 1$.

- The parameter $\beta > 0$ indicates which of the two aims mentioned above shall rather be achieved.

3.2.3 Existence of minimizers

Section 3.1.2 is useful for showing existence of minimizers of (P1).

Theorem 3.4. *There is a minimizer of* (P1).

Proof. The objective function, abbreviated by $\phi = \phi(f,E,B,u,\alpha,b) = \phi_1(f) + \phi_2(u)$ (let ϕ_1 be the term with $\rho_f - \rho_d$ and ϕ_2 be the sum), is coercive in $u \in H^2([0,T])^N$ because of

$$\phi(f,E,B,u,\alpha,b) \geq \frac{\beta}{2}\min\{1,\beta_1,\beta_2\}\min\{c_j \mid j = 1,\ldots,N\}\|u\|^2_{(H^2)^N}$$

where $\|u\|_{(H^2)^N} = \sum_{j=1}^N \|u_j\|_{H^2([0,T])}$. Hence, considering a minimizing sequence $\left(f_k, E_k, B_k, u^k, \alpha^k, b^k\right)$ (we use upper indices for u^k, α^k, and b^k to avoid confusion with their components) with $(f_k, E_k, B_k) = S\left(\sum_{j=1}^N u_j^k z_j\left(R\left(\alpha_j^k\right)\cdot + b_j^k\right)\right)$, $\left|u_j^k\right| \leq 1$, and $\left|b_j^k\right| \leq r$, we conclude that:

- $\left(u^k\right)$ is bounded in $\left(H^2\right)^N$; hence $u^k \rightharpoonup u$ in $\left(H^2\right)^N$ for some $u \in \left(H^2\right)^N$ for $k \to \infty$,

- $\left(\alpha^k\right)$ is bounded in $\mathbb{R}^N$ since we may assume $\alpha_j^k \in [0, 2\pi]$ without loss of generality; hence $\alpha^k \to \alpha$ in $\mathbb{R}^N$ for some $\alpha \in \mathbb{R}^N$ for $k \to \infty$,

- $\left(b^k\right)$ is bounded in $\left(\mathbb{R}^2\right)^N$; hence $b^k \to b$ in $\left(\mathbb{R}^2\right)^N$ for some $b \in \left(\mathbb{R}^2\right)^N$ for $k \to \infty$,

possibly after extracting a suitable subsequence. The constraints $|u_j| \leq 1$ and $|b_j| \leq r$ are obviously preserved by weak and strong convergence, respectively, and the fact that closed convex subsets (of normed spaces) are weakly closed.

Furthermore, the sequence $(U_k) := \left(\sum_{j=1}^N u_j^k z_j\left(R\left(\alpha_j^k\right)\cdot + b_j^k\right)\right)$ is bounded in V because of $H^2([0,T]) \hookrightarrow W^{2,1}([0,T])$ and $\left\|z_j\left(R\left(\alpha_j^k\right)\cdot + b_j^k\right)\right\|_{C_b^4} = \|z_j\|_{C_b^4}$.
Now we would like to apply Lemma 3.2. To show $U_k \to U := \sum_{j=1}^N u_j z_j\left(R\left(\alpha_j\right)\cdot + b_j\right)$ in the sense of distributions, let $\varphi \in C_c^\infty\left(]0,T[\times\mathbb{R}^2\right)$. We have by Hölder's inequality

$$\begin{aligned}&\left|\int_0^T\int\left(U_k - U\right)\varphi\, dxdt\right|\\ \leq&\sum_{j=1}^N\left(\left|\int_0^T\left(u_j^k - u_j\right)\int z_j\left(R\left(\alpha_j\right)x + b_j\right)\varphi\, dxdt\right|\right.\\ &\left.\quad+\|\varphi\|_\infty\left\|u_j^k\right\|_{L^2}\sqrt{T}\int\left|z_j^k\left(R\left(\alpha_j^k\right)x + b_j^k\right) - z_j\left(R\left(\alpha_j\right)x + b_j\right)\right| dx\right).\end{aligned}$$

Here, the first term converges to 0 for $k\to\infty$ because of $u^k \rightharpoonup u$ in $\left(H^2\right)^N$. The same holds for the second term since $\left(u_j^k\right)$ is bounded in L^2 and $z_j\left(R\left(\alpha_j^k\right)\cdot + b_j^k\right) \to z_j\left(R\left(\alpha_j\right)\cdot + b_j\right)$ in L^1 for each $j = 1,\ldots,N$; the last property is a consequence of $\alpha^k \to \alpha$ and $b^k \to b$ since we have by the mean value theorem

$$\begin{aligned}&\int\left|z_j\left(R\left(\alpha_j^k\right)x + b_j^k\right) - z_j\left(R\left(\alpha_j\right)x + b_j\right)\right| dx\\ \leq&\|Dz_j\|_\infty\int_{|x|\leq L}\left(\left|R\left(\alpha_j^k\right) - R\left(\alpha_j\right)\right||x| + \left|b_j^k - b_j\right|\right)dx\\ \leq&\|Dz_j\|_\infty\pi L^2\left(\|DR\|_\infty L + 1\right)\left(\left|\alpha_j^k - \alpha_j\right| + \left|b_j^k - b_j\right|\right).\end{aligned}$$

Therefore, Lemma 3.2 is applicable and delivers some f, E, and B so that (CVM) is preserved in the limit. The remaining part is to show that U is indeed an optimal control. Firstly, $u^k \rightharpoonup u$ in $\left(H^2\right)^N$ instantly implies $\phi_2(u) \leq \liminf_{k\to\infty}\phi_2\left(u^k\right)$. Secondly, by Lemma 3.2, all f_k and f have compact support with respect to p uniformly in k, and $f_k \to f$ in L^2. These properties yield $\rho_{f_k} \to \rho_f$ in L^2 by Hölder's inequality and therefore $\phi_1(f) = \lim_{k\to\infty}\phi_1(f_k)$. This finally proves the desired optimality. $\square$

3.2.4 Differentiability of the objective function

Next we study the differentiability of the objective function. To this end we like to exploit Theorem 2.3. First we have to prove some technical lemmata:

Lemma 3.5. *i) Let $K \subset \mathbb{R}^n$ be open and bounded. Then the map*

$$\begin{aligned}H\colon \mathbb{R}^{n\times n}\times\mathbb{R}^n \to& C_b^l\left(K;\mathbb{R}^m\right),\\ (A,c)\mapsto& A\cdot + c\end{aligned}$$

is continuously Fréchet-differentiable for each $l \in \mathbb{N}_0$.

ii) Let $g \in C_b^{k+2}(\mathbb{R}^n; \mathbb{R}^m)$ and $K \subset \mathbb{R}^n$ be open. Then the superposition map

$$\begin{aligned} M_g \colon C_b^l(K; \mathbb{R}^n) &\to C_b^k(K; \mathbb{R}^m), \\ h &\mapsto g \circ h \end{aligned}$$

is continuously Fréchet-differentiable for each $l \geq k$.

iii) Let $g \in C_b^{k+2}(\mathbb{R}^n; \mathbb{R}^m)$ and $K \subset \mathbb{R}^n$ be open and bounded. Then the map

$$\begin{aligned} G_g \colon \mathbb{R}^{n \times n} \times \mathbb{R}^n &\to C_b^k(K; \mathbb{R}^m), \\ (A, c) &\mapsto g(A \cdot + c) \end{aligned}$$

is continuously Fréchet-differentiable with derivative

$$G_g'(A, c)(\delta A, \delta c) = Dg(A \cdot + c)(\delta A \cdot + \delta c).$$

Proof. Note that H is well defined by K being bounded. For this very reason H is a bounded linear operator and hence of class C^1 with derivative

$$H'(A, c)(\delta A, \delta c) = \delta A \cdot + \delta c.$$

For part ii), let

$$M_g'(h)\,\delta h = Dg \circ h \cdot \delta h$$

be the candidate for the linearization. $M_g'(h)$ is bounded because of

$$\|Dg \circ h \cdot \delta h\|_{C_b^k} \leq C\left(\|g\|_{C_b^{k+1}}, \|h\|_{C_b^k}\right) \|\delta h\|_{C_b^k}.$$

Furthermore, we have by mean value theorem

$$M_g(h + \delta h) - M_g(h) - M_g'(h)\,\delta h = \int_0^1 \left(Dg(h(\cdot) + t\delta h(\cdot)) - Dg(h(\cdot))\right) dt \cdot \delta h(\cdot).$$

The C_b^k-norm of this expression can be estimated by

$$C\left(\|g\|_{C_b^{k+2}}, \|h\|_{C_b^k}\right) \|\delta h\|_{C_b^k}^2$$

since $D^j g$ is Lipschitz continuous for $0 \leq j \leq k$. Hence M_g is differentiable. To prove continuity of M_g' we examine

$$M_g'\left(\overline{h}\right)\delta h - M_g'(h)\,\delta h = \left(Dg \circ \overline{h} - Dg \circ h\right) \cdot \delta h.$$

Again, by Lipschitz continuity of $D^j g$ we conclude

$$\left\|M_g'\left(\overline{h}\right)\delta h - M_g'(h)\,\delta h\right\|_{C_b^k} \leq C\left(\|g\|_{C_b^{k+2}}\right) \left\|\overline{h} - h\right\|_{C_b^k} \|\delta h\|_{C_b^k}$$

which proves even Lipschitz continuity of M_g'.
Part iii) follows from i) and ii) by a simple application of the chain rule. $\square$

Lemma 3.6. *The map*

$$\begin{aligned} Q\colon W^{2,1}([0,T]) \times C_b^4\left(\mathbb{R}^2\right) \to & V, \\ (h,q) \mapsto & hq, \end{aligned}$$

is continuously Fréchet-differentiable with derivative

$$Q'(h,q)(\delta h, \delta q) = q\delta h + h\delta q.$$

Proof. Q is bilinear and bounded. $\square$

Now we can apply the chain rule to conclude:

Theorem 3.7. *i) The solution map*

$$\begin{aligned} \Xi\colon \left(H^2([0,T])\right)^N \times \mathbb{R}^N \times \left(\mathbb{R}^2\right)^N \to & C\left(0,T;L^2\left(\mathbb{R}^4\right)\right) \times C\left(0,T;L^2\left(\mathbb{R}^2\right)\right)^3, \\ (u,\alpha,b) \mapsto & (f,E,B) = S\left(\sum_{j=1}^N u_j z_j\left(R(\alpha_j)\cdot + b_j\right)\right) \end{aligned}$$

is continuously Fréchet-differentiable and $\Xi'(u,\alpha,b)(\delta u, \delta\alpha, \delta b) = (\delta f, \delta E, \delta B)$ *satisfies*

$$\begin{aligned} \partial_t \delta f + \widehat{p}\cdot\partial_x \delta f + \left(E - \widehat{p}^\perp B\right)\cdot\partial_p \delta f =& -\left(\delta E - \widehat{p}^\perp \delta B\right)\cdot \partial_p f, \\ \partial_t \delta E_1 - \partial_{x_2}\delta B =& -j_{\delta f,1} - \delta U_1, \\ \partial_t \delta E_2 + \partial_{x_1}\delta B =& -j_{\delta f,2} - \delta U_2, \\ \partial_t \delta B + \partial_{x_1}\delta E_2 - \partial_{x_2}\delta E_1 =& 0, \\ (\delta f, \delta E, \delta B)(0) =& 0 \end{aligned}$$

where

$$\delta U = \sum_{j=1}^N \left(u_j D z_j\left(R(\alpha_j)\cdot + b_j\right)\left(R'(\alpha_j)\delta\alpha_j \cdot + \delta b_j\right) + z_j\left(R(\alpha_j)\cdot + b_j\right)\delta u_j\right).$$

ii) The maps

$$\begin{aligned} \Psi\colon \left(H^2([0,T])\right)^N \times \mathbb{R}^N \times \left(\mathbb{R}^2\right)^N \to & C\left(0,T;L^2\left(\mathbb{R}^2\right)\right), \\ (u,\alpha,b) \mapsto & \rho_f \end{aligned}$$

and

$$\begin{aligned} \overline{\Psi}\colon \left(H^2([0,T])\right)^N \times \mathbb{R}^N \times \left(\mathbb{R}^2\right)^N \to & C\left(0,T;L^1\left(\mathbb{R}^2\right)\right), \\ (u,\alpha,b) \mapsto & \rho_f \end{aligned}$$

are continuously Fréchet-differentiable and $\Psi'(u,\alpha,b)(\delta u, \delta\alpha, \delta b) = \rho_{\delta f}$ *with the* δf *from above.*

iii) The objective function

$$
\begin{aligned}
\overline{\phi}\colon \left(H^2\left([0,T]\right)\right)^N \times \mathbb{R}^N \times \left(\mathbb{R}^2\right)^N \to & \mathbb{R}, \\
(u,\alpha,b) \mapsto & \frac{1}{2}\left\|\rho_f - \rho_d\right\|_{L^2}^2 + \frac{\beta}{2}\sum_{j=1}^N c_j \Big(\left\|u_j\right\|_{L^2}^2 \\
& + \beta_1 \left\|\partial_t u_j\right\|_{L^2}^2 + \beta_2 \left\|\partial_t^2 u_j\right\|_{L^2}^2\Big)
\end{aligned}
$$

is continuously Fréchet-differentiable and

$$
\begin{aligned}
\overline{\phi}'\left(u,\alpha,b\right)\left(\delta u,\delta\alpha,\delta b\right) = & \left\langle \rho_f - \rho_d, \rho_{\delta f}\right\rangle_{L^2} + \beta\sum_{j=1}^N c_j \left(\left\langle u_j, \delta u_j\right\rangle_{L^2}\right. \\
& \left. + \beta_1 \left\langle \partial_t u_j, \partial_t \delta u_j\right\rangle_{L^2} + \beta_2 \left\langle \partial_t^2 u_j, \partial_t^2 \delta u_j\right\rangle_{L^2}\right)
\end{aligned}
$$

with the δf from above.

Proof. For i), using the chain rule, Theorem 2.3, Lemmata 3.5 (with $K = B_L$; recall that all possible controls vanish for $|x| \geq L$) and 3.6, and the formula for the respective derivatives given there, and the obvious fact that $\beta \mapsto R(\beta)$ is differentiable leads to

$$\Xi'\left(u,\alpha,b\right)\left(\delta u,\delta\alpha,\delta b\right) = S'\left(U\right)\delta U$$

where

$$U = \sum_{j=1}^N u_j z_j \left(R\left(\alpha_j\right)\cdot + b_j\right)$$

and

$$\delta U = \sum_{j=1}^N Q'\left(u_j, z_j\left(R\left(\alpha_j\right)\cdot + b_j\right)\right)\Big(\delta u_j, G'_{z_j}\left(R\left(\alpha_j\right), b_j\right)\left(R'\left(\alpha_j\right)\delta\alpha_j, \delta b_j\right)\Big).$$

The formulas of Φ' and $\overline{\Phi}'$ in Theorem 2.3 imply ii).
Part iii) is simple computation; note that $C\left(0,T;L^2\left(\mathbb{R}^2\right)\right) \hookrightarrow L^2\left([0,T]\times\mathbb{R}^2\right)$. □

3.2.5 Optimality conditions

Now we want to deduce first order optimality conditions for a minimizer of (P1). First we write (P1) in the equivalent form

$$
\begin{aligned}
\min_{\substack{u\in H^2([0,T])^N,\\ \alpha\in\mathbb{R}^N, b\in\left(\mathbb{R}^2\right)^N}} \; & \frac{1}{2}\left\|\Psi\left(u,\alpha,b\right) - \rho_d\right\|_{L^2([0,T]\times\mathbb{R}^2)}^2 + \frac{\beta}{2}\sum_{j=1}^N c_j \Big(\left\|u_j\right\|_{L^2([0,T])}^2 \\
& + \beta_1 \left\|\partial_t u_j\right\|_{L^2([0,T])}^2 + \beta_2 \left\|\partial_t^2 u_j\right\|_{L^2([0,T])}^2\Big) \\
\text{s.t.} \; & -u_j + 1 \geq 0, u_j + 1 \geq 0, r^2 - \left|b_j\right|^2 \geq 0, j = 1,\ldots,N.
\end{aligned}
\tag{P1'}
$$

Here, the objective function $\overline{\phi} = \overline{\phi}(u, \alpha, b) = \phi(\Xi(u, \alpha, b), u, \alpha, b)$ is a function of only the control.
The constraints will lead to corresponding Lagrange multipliers. In general, to prove their existence, some condition on the constraints is necessary. On this account we state a theorem of Zowe and Kurcyusz, see [18], which is based on a fundamental work of Robinson, [15]:

Lemma 3.8. *Let X, Y be Banach spaces, $C \subset X$ non-empty, closed, and convex, $K \subset Y$ a closed convex cone (K is a 'cone' means $0 \in K$, $x \in K \Rightarrow \lambda x \in K\ \forall \lambda > 0$), $\overline{\phi}\colon X \to \mathbb{R}$ Fréchet-differentiable, and $g\colon X \to Y$ continuously Fréchet-differentiable. Denote for $A \subset X$ (and similarly for $A \subset Y$)*

$$A^{+} = \{x^* \in X^* \mid x^* a \geq 0\ \forall a \in A\}$$

and denote for $x \in X$ and $y \in Y$

$$\begin{aligned} C(x) &= \{\lambda(c - x) \mid c \in C, \lambda \geq 0\}, \\ K(y) &= \{k - \lambda y \mid k \in K, \lambda \geq 0\}. \end{aligned}$$

Let $\overline{x} \in X$ be a local minimizer (i.e., a local minimizer of the objective function restricted to all feasible points) of the problem

$$\begin{aligned} &\min_{x \in X}\ \overline{\phi}(x) \\ &\text{s.t. } x \in C, g(x) \in K, \end{aligned}$$

and let the constraint qualification

$$g'(\overline{x})\, C(\overline{x}) - K(g(\overline{x})) = Y \tag{CQ}$$

hold.
Then there is a Lagrange multiplier $y^* \in Y^*$ for the problem above at $\overline{x}$, i.e.

i) $y^* \in K^{+}$,

ii) $y^* g(\overline{x}) = 0$,

iii) $\overline{\phi}'(\overline{x}) - y^* \circ g'(\overline{x}) \in C(\overline{x})^{+}$.

This lemma is applicable to our problem (P1'): By Theorem 3.7, our objective function is differentiable from

$$X = \left(H^2([0, T])\right)^N \times \mathbb{R}^N \times \left(\mathbb{R}^2\right)^N$$

to $\mathbb{R}$. Writing

$$\begin{aligned} x &= (u, \alpha, b), \\ C &= X, \end{aligned}$$

$$\begin{aligned} Y &= (C([0,T]))^{2N} \times \mathbb{R}^N, \\ K &= \{(w,c) \in Y \mid w_i \geq 0, 1 \leq i \leq 2N; c_i \geq 0, 1 \leq i \leq N\}, \\ g(x) &= \left(-u_1+1,\dots,-u_N+1,u_1+1,\dots,u_N+1,r^2-|b_1|^2,\dots,r^2-|b_N|^2\right), \end{aligned}$$

our constraints read

$$x \in C, g(x) \in K.$$

Of course, K is a closed convex cone and g is continuously Fréchet-differentiable. The crucial condition on the constraints can be verified for each feasible $x = (u, \alpha, b) \in X$ (not necessarily a minimizer). In fact, this is quite easy since the constraints are harmless. It holds that

$$g'(x)\,\delta x = (-\delta u, \delta u, -2b_1 \cdot \delta b_1, \dots, -2b_N \cdot \delta b_N); \tag{3.1}$$

note that $H^2([0,T]) \hookrightarrow C([0,T])$.
To show (CQ) we have to find $\delta x = (\delta u, \delta\alpha, \delta b) \in X$, $\lambda \in \mathbb{R}_{\geq 0}$, and $(\theta^+, \theta^-, \eta) \in \left((C([0,T]))^N\right)^2 \times \mathbb{R}^N$ with θ_j^+, $\theta_j^- \geq 0$, and $\eta_j \geq 0$, satisfying

$$\begin{aligned} &(-\delta u, \delta u, -2b_1 \cdot \delta b_1, \dots, -2b_N \cdot \delta b_N) - \left(\theta^+, \theta^-, \eta\right) \\ &\qquad +\lambda\left(-u+1, u+1, r^2-|b_1|^2,\dots,r^2-|b_N|^2\right) = \left(w^+, w^-, c\right) \end{aligned} \tag{3.2}$$

where $(w^+, w^-, c) \in \left((C([0,T]))^N\right)^2 \times \mathbb{R}^N$ is given. First note that the choice of $\delta\alpha$ is arbitrary since it does not appear in the equation above. Next, we abbreviate

$$\begin{aligned} \vartheta^+ &:= \max_{i=1,\dots,N} \left\|w_i^+\right\|_\infty, \\ \vartheta^- &:= \max_{i=1,\dots,N} \left\|w_i^-\right\|_\infty. \end{aligned}$$

Now let

$$\begin{aligned} \lambda &:= \left(\frac{1}{2}\left(\vartheta^+ + \vartheta^-\right) + 1\right)\sigma, \\ \theta_j^+ &:= \left(\vartheta^+ - u_j + 1\right)\sigma - w_j^+, \\ \theta_j^- &:= \left(\vartheta^- + u_j + 1\right)\sigma - w_j^-, \\ \delta u_j &:= -\frac{1}{2}\left(\vartheta^+(u_j+1) + \vartheta^-(u_j-1)\right)\sigma \end{aligned}$$

with $\sigma \geq 1$ to be chosen later. Obviously, $\lambda \geq 0$ and δu_j is of class H^2. Furthermore, θ_j^+, $\theta_j^- \in C([0,T])$ and are ≥ 0 by choice of ϑ^+, ϑ^-, and feasibility of x. Thereby, the equation

$$(-\delta u, \delta u) - \left(\theta^+, \theta^-\right) + \lambda(-u+1, u+1) = \left(w^+, w^-\right)$$

can easily be verified and implies part of (3.2). For the remaining part, let

$$\sigma \geq \frac{1}{r^2} \max_{j=1,\ldots,N} |c_j| + 1$$

and consider two cases for each $j = 1, \ldots, N$. If $b_j = 0$ choose δb_j arbitrarily and $\eta_j := \lambda r^2 - c_j$. If $b_j \neq 0$ choose $\delta b_j := -\frac{c_j}{2} \frac{b_j}{|b_j|^2}$ and $\eta_j := \lambda \left(r^2 - |b_j|^2 \right)$. In both cases, $\eta_j \geq 0$ either by choice of σ or by feasibility of x, and moreover

$$-2b_j \cdot \delta b_j - \eta_j + \lambda \left(r^2 - |b_j|^2 \right) = c_j.$$

Altogether we conclude that, indeed, (3.2) holds and hence (CQ) is satisfied for any feasible $x \in X$.
We should remark that for convex constraints, commonly, the less general Slater's condition is verified. For this, there has to be an interior point of the cone K; to be more precise, Slater's condition demands the existence of $x \in X$ with $g(x) \in \mathrm{int} K$. In our problem, the existence of such a point is guaranteed by the fact that the cone of positive functions has a non-empty interior in $C([0,T])$.
Furthermore, it will be more convenient that we have $C([0,T])$ in the first components of Y instead of $H^2([0,T])$ because under this condition a corresponding Lagrange multiplier is an element of the dual space of $C([0,T])$ which is more regular than that of $H^2([0,T])$. It would be even better if we could use $L^2([0,T])$ instead of $C([0,T])$, but in such a setting the construction above does not work since $\|w\|_\infty$ is not finite in general for a L^2-function w. Also Slater's condition is not satisfied since the cone of positive functions has empty interior in L^2.
With Lemma 3.8 we can deduce the following KKT-conditions for a minimizer of (P1').
We denote by $M([0,T]) \cong C([0,T])^*$ the set of regular Borel measures on $[0,T]$.

Theorem 3.9. *Let $\overline{x} = \left(\overline{u}, \overline{\alpha}, \overline{b}\right)$ be a minimizer of* (P1'). *Then there are Lagrange multipliers λ_j^+ (corresponding to the constraint $u_j \leq 1$), $\lambda_j^- \in M([0,T])$ (corresponding to $u_j \geq -1$), $\mu_j \in \mathbb{R}$ (corresponding to $|b_j| \leq r$), $j = 1, \ldots, N$, satisfying:*

i) (Primal feasibility):

$$|\overline{u}_j| \leq 1, \left|\overline{b}_j\right| \leq r.$$

ii) (Dual feasibility):

$$\lambda_j^+, \lambda_j^- \geq 0,$$

i.e.,

$$\lambda_j^+ v, \lambda_j^- v \geq 0$$

for all $v \in C([0,T])$ with $v \geq 0$, and

$$\mu_j \geq 0.$$

iii) (Complementary slackness):

$$\begin{aligned}\lambda_j^+ \left(\overline{u}_j - 1\right) =&0,\\ \lambda_j^- \left(\overline{u}_j + 1\right) =&0,\\ \mu_j \left(r^2 - \left|\overline{b}_j\right|^2\right) =&0.\end{aligned}$$

iv) (Stationarity): For all $(\delta u, \delta\alpha, \delta b) \in \left(H^2\left([0,T]\right)\right)^N \times \mathbb{R}^N \times \left(\mathbb{R}^2\right)^N$ *it holds that*

$$\begin{aligned}&\left\langle \rho_{\overline{f}} - \rho_d, \rho_{\delta f}\right\rangle_{L^2}\\ &+\beta \sum_{j=1}^N c_j \left(\left\langle \overline{u}_j, \delta u_j\right\rangle_{L^2} + \beta_1 \left\langle \partial_t \overline{u}_j, \partial_t \delta u_j\right\rangle_{L^2} + \beta_2 \left\langle \partial_t^2 \overline{u}_j, \partial_t^2 \delta u_j\right\rangle_{L^2}\right)\\ =&\sum_{j=1}^N \left(\left(\lambda_j^- - \lambda_j^+\right)\delta u_j - 2\mu_j \overline{b}_j \cdot \delta b_j\right)\end{aligned}$$

where δf *is obtained by solving*

$$\begin{aligned}\partial_t \delta f + \widehat{p}\cdot\partial_x \delta f + \left(\overline{E} - \widehat{p}^\perp \overline{B}\right)\cdot\partial_p \delta f =& -\left(\delta E - \widehat{p}^\perp \delta B\right)\cdot \partial_p \overline{f},\\ \partial_t \delta E_1 - \partial_{x_2}\delta B =& -j_{\delta f,1} - \delta U_1,\\ \partial_t \delta E_2 + \partial_{x_1}\delta B =& -j_{\delta f,2} - \delta U_2,\\ \partial_t \delta B + \partial_{x_1}\delta E_2 - \partial_{x_2}\delta E_1 =&0,\\ \left(\delta f, \delta E, \delta B\right)(0) =&0\end{aligned}$$

with

$$\delta U = \sum_{j=1}^N \left(\overline{u}_j D z_j \left(R\left(\overline{\alpha}_j\right)\cdot + \overline{b}_j\right)\left(R'\left(\overline{\alpha}_j\right)\delta\alpha_j \cdot + \delta b_j\right) + z_j \left(R\left(\overline{\alpha}_j\right)\cdot + \overline{b}_j\right)\delta u_j\right)$$

and $\left(\overline{f}, \overline{E}, \overline{B}\right)$ *satisfying*

$$\begin{aligned}\partial_t \overline{f} + \widehat{p}\cdot\partial_x \overline{f} + \left(\overline{E} - \widehat{p}^\perp \overline{B}\right)\cdot\partial_p \overline{f} =&0,\\ \partial_t \overline{E}_1 - \partial_{x_2}\overline{B} =& -j_{\overline{f},1} - \overline{U}_1,\\ \partial_t \overline{E}_2 + \partial_{x_1}\overline{B} =& -j_{\overline{f},2} - \overline{U}_2,\\ \partial_t \overline{B} + \partial_{x_1}\overline{E}_2 - \partial_{x_2}\overline{E}_1 =&0,\\ \left(\overline{f}, \overline{E}, \overline{B}\right)(0) =& \left(\mathring{f}, \mathring{E}, \mathring{B}\right)\end{aligned}$$

with

$$\overline{U} = \sum_{j=1}^N \overline{u}_j z_j \left(R\left(\overline{\alpha}_j\right)\cdot + \overline{b}_j\right).$$

Proof. First, i) simply states the feasibility of the minimizer.
The assertions ii)-iv) correspond to i)-iii) of Lemma 3.8. The Lagrange multiplier $y^* \in Y^*$ obtained there can be written as

$$y^* = \left(\lambda_1^+, \dots, \lambda_N^+, \lambda_1^-, \dots, \lambda_N^-, \mu_1, \dots, \mu_N\right)$$

with λ_j^+, $\lambda_j^- \in \left(C\left([0,T]\right)\right)^* \cong M\left([0,T]\right)$, and $\mu_j \in \mathbb{R}$. The acting on some $y = \left(w_1^+, \dots, w_N^+, w_1^-, \dots, w_N^-, c_1, \dots, c_N\right) \in Y$ is given by

$$y^* y = \sum_{j=1}^{N} \left(\lambda_j^+ w_j^+ + \lambda_j^- w_j^- + \mu_j c_j\right).$$

Therefore $y^* \in K^+$ implies ii) by only considering some $\left(0, \dots, 0, w_j^+, 0, \dots, 0\right)$, or $\left(0, \dots, 0, w_j^-, 0, \dots, 0\right)$, or $(0, \dots, 0, c_j, 0, \dots, 0)$.
Next, $y^* g\left(\overline{x}\right) = 0$ leads to

$$\sum_{j=1}^{N} \left(\lambda_j^+ \left(1 - \overline{u}_j\right) + \lambda_j^- \left(\overline{u}_j + 1\right) + \mu_j \left(r^2 - \left|\overline{b}_j\right|^2\right)\right) = 0.$$

By i) and ii), all summands are non-negative and therefore have to vanish which implies iii).
Finally, $\overline{\phi}'\left(\overline{x}\right) - y^* \circ g'\left(\overline{x}\right) \in C\left(\overline{x}\right)^+$ leads to iv) because of

$$C\left(\overline{x}\right)^+ = X^+ = \{0\},$$

Theorem 3.7 iii), and (3.1). □

Conversely, we can ignore the constraint $g(x) \in K$ and define the feasible set

$$C := \{x = (u, \alpha, b) \in X \mid |u_j| \leq 1, |b_j| \leq r, j = 1, \dots, N\}$$

which is a convex and closed subset of X. Now we can instantly apply Lemma 3.8 (to be precise, put $K = Y = \{0\}$ and $g = 0$ there). Only part iii) delivers a non-trivial assertion:

Theorem 3.10. *Let* $\overline{x} = \left(\overline{u}, \overline{\alpha}, \overline{b}\right)$ *be a minimizer of* (P1'). *Then it holds that:*

i) (Feasibility):

$$\left|\overline{u}_j\right| \leq 1, \left|\overline{b}_j\right| \leq r.$$

ii) (Stationarity): For all $(u, \alpha, b) \in C$, *and defining*

$$(\delta u, \delta\alpha, \delta b) := \left(u - \overline{u}, \alpha - \overline{\alpha}, b - \overline{b}\right),$$

we have

$$0 \leq \left\langle \rho_{\overline{f}} - \rho_d, \rho_{\delta f} \right\rangle_{L^2}$$

$$+\beta\sum_{j=1}^{N}c_j\left(\langle\overline{u}_j,\delta u_j\rangle_{L^2}+\beta_1\langle\partial_t\overline{u}_j,\partial_t\delta u_j\rangle_{L^2}+\beta_2\left\langle\partial_t^2\overline{u}_j,\partial_t^2\delta u_j\right\rangle_{L^2}\right)$$

where δf is obtained by solving

$$\begin{aligned}\partial_t\delta f+\widehat{p}\cdot\partial_x\delta f+\left(\overline{E}-\widehat{p}^{\perp}\overline{B}\right)\cdot\partial_p\delta f=&-\left(\delta E-\widehat{p}^{\perp}\delta B\right)\cdot\partial_p\overline{f},\\ \partial_t\delta E_1-\partial_{x_2}\delta B=&-j_{\delta f,1}-\delta U_1,\\ \partial_t\delta E_2+\partial_{x_1}\delta B=&-j_{\delta f,2}-\delta U_2,\\ \partial_t\delta B+\partial_{x_1}\delta E_2-\partial_{x_2}\delta E_1=&0,\\ (\delta f,\delta E,\delta B)(0)=&0\end{aligned}$$

with

$$\delta U=\sum_{j=1}^{N}\left(\overline{u}_jDz_j\left(R\left(\overline{\alpha}_j\right)\cdot+\overline{b}_j\right)\left(R'\left(\overline{\alpha}_j\right)\delta\alpha_j\cdot+\delta b_j\right)+z_j\left(R\left(\overline{\alpha}_j\right)\cdot+\overline{b}_j\right)\delta u_j\right)$$

and $\left(\overline{f},\overline{E},\overline{B}\right)$ satisfying

$$\begin{aligned}\partial_t\overline{f}+\widehat{p}\cdot\partial_x\overline{f}+\left(\overline{E}-\widehat{p}^{\perp}\overline{B}\right)\cdot\partial_p\overline{f}=&0,\\ \partial_t\overline{E}_1-\partial_{x_2}\overline{B}=&-j_{\overline{f},1}-\overline{U}_1,\\ \partial_t\overline{E}_2+\partial_{x_1}\overline{B}=&-j_{\overline{f},2}-\overline{U}_2,\\ \partial_t\overline{B}+\partial_{x_1}\overline{E}_2-\partial_{x_2}\overline{E}_1=&0,\\ \left(\overline{f},\overline{E},\overline{B}\right)(0)=&\left(\mathring{f},\mathring{E},\mathring{B}\right)\end{aligned}$$

with

$$\overline{U}=\sum_{j=1}^{N}\overline{u}_jz_j\left(R\left(\overline{\alpha}_j\right)\cdot+\overline{b}_j\right).$$

3.2.6 Adjoint equation

Considering the optimality conditions above, we note that we have to compute $\overline{\phi}'$ and thus the whole derivative Ξ' at an optimal point $\overline{x}$. However, there is a more efficient way, the adjoint approach.

Considering a general setting, we let y be the state and x the control. Furthermore write the differential equation in the form $F(y,x)=0\in Z$ where Z is a Banach space. Moreover assume that there is a differentiable solution operator $y(\cdot)$ so that $F(y(x),x)=0$. Finally, let the objective function be given in the form $\overline{\phi}(x)=\phi(y(x),x)$.

Assuming in the following that all derivatives exist in a proper sense, we first compute

$$\overline{\phi}'(x)=\partial_y\phi(y(x),x)\,y'(x)+\partial_x\phi(y(x),x)=y'(x)^*\,\partial_y\phi(y(x),x)+\partial_x\phi(y(x),x).$$

Therefore, we only need $y'(x)^* \partial_y \phi(y(x), x)$ and not the whole derivative $y'(x)$. On the other hand, differentiating $F(y(x), x) = 0$ yields

$$0 = \partial_y F(y(x), x)\, y'(x) + \partial_x F(y(x), x) = y'(x)^* \partial_y F(y(x), x) + \partial_x F(y(x), x).$$

Now we define the adjoint state $q \in Z^*$ as the solution of the adjoint equation

$$\partial_y F(y(x), x)^* q = -\partial_y \phi(y(x), x).$$

Of course, the solvability of this linear, inhomogeneous differential equation (system) has to be examined.
Thus we conclude

$$\begin{aligned}\partial_x F(y(x), x)^* q =& q\partial_x F(y(x), x) = -q\left(y'(x)^* \partial_y F(y(x), x)\right)\\ =& -y'(x)^* \left(\partial_y F(y(x), x)^* q\right)\\ =& y'(x)^* \partial_y \phi(y(x), x).\end{aligned}$$

Therefore, following the adjoint approach, we firstly solve the adjoint equation

$$\partial_y F(y(x), x)^* q = -\partial_y \phi(y(x), x)$$

and secondly compute

$$\overline{\phi}'(x) = \partial_x F(y(x), x)^* q + \partial_x \phi(y(x), x). \tag{3.3}$$

In order to apply these considerations to our problem we have to define F suitably. Here, 'suitably' means that the differentiability of F and the differentiability of the control-to-state operator Ξ have to fit together. In other words, $F(y, x)$ should be differentiable with respect to the $C\left(0, T; L^2\right)$-norm in the state variable $y = (f, E, B)$. In the following let

$$\begin{aligned}M_R := \big\{&(f, E, B) \in C_c^2\left([0,T] \times \mathbb{R}^4\right) \times C_c^2\left([0,T] \times \mathbb{R}^2; \mathbb{R}^2\right) \times C_c^2\left([0,T] \times \mathbb{R}^2\right) \mid\\ &f(t, x, p) = 0 \text{ for all } |p| \geq R\big\}\end{aligned}$$

for some $R > 0$, and let M_R be equipped with the $C\left(0, T; L^2\right)$-norm. Here, the index 'c' means 'compactly supported with respect to x and p' (or x respectively).
Furthermore let X be as in the previous section and

$$Z := H^1\left([0,T] \times \mathbb{R}^4\right)^* \times \left(H^1\left([0,T] \times \mathbb{R}^2\right)^*\right)^3 \times L^2\left(\mathbb{R}^4\right)^* \times \left(L^2\left(\mathbb{R}^2\right)^*\right)^3.$$

Now define

$$F_R \colon M_R \times X \to Z$$

via

$$F_R\left((f, E, B), (u, \alpha, b)\right)(g, h_1, h_2, h_3, a_1, a_2, a_3, a_4)$$

$$
\begin{aligned}
= \Bigg(& -\int_0^T \int \int \left(\partial_t g + \widehat{p} \cdot \partial_x g + \left(E - \widehat{p}^{\perp} B\right) \cdot \partial_p g\right) f \; dpdxdt \\
& + \langle g(T), f(T) \rangle_{L^2} - \langle g(0), f(0) \rangle_{L^2}, \\
& \int_0^T \int \left(-E_1 \partial_t h_1 + B \partial_{x_2} h_1 + j_{f,1} h_1 + U_1 h_1\right) dxdt \\
& + \langle h_1(T), E_1(T) \rangle_{L^2} - \langle h_1(0), E_1(0) \rangle_{L^2}, \\
& \int_0^T \int \left(-E_2 \partial_t h_2 - B \partial_{x_1} h_2 + j_{f,2} h_2 + U_2 h_2\right) dxdt \\
& + \langle h_2(T), E_2(T) \rangle_{L^2} - \langle h_2(0), E_2(0) \rangle_{L^2}, \\
& \int_0^T \int \left(-B \partial_t h_3 - E_2 \partial_{x_1} h_3 + E_1 \partial_{x_2} h_3\right) dxdt \\
& + \langle h_3(T), B(T) \rangle_{L^2} - \langle h_3(0), B(0) \rangle_{L^2}, \\
& \int \int \left(f(0) - \mathring{f}\right) a_1 \; dpdx, \int \left(E_1(0) - \mathring{E}_1\right) a_2 \; dx, \int \left(E_2(0) - \mathring{E}\right) a_3 \; dx, \\
& \int \left(B(0) - \mathring{B}\right) a_4 \; dx \Bigg)
\end{aligned}
$$

where

$$
U = \sum_{j=1}^{N} u_j z_j \left(R(\alpha_j) \cdot + b_j\right).
$$

After several integrations by parts, it is obvious that (f, E, B) solves (CVM) with control U iff $F_R((f, E, B), (u, \alpha, b)) = 0$ for any $R > 0$ with $\mathrm{supp}_p f \subset B_R$.
Since no derivatives of the state $y = (f, E, B)$ appear above and the state is of class C_b, $\partial_y F_R$ exists and is given by

$$
\begin{aligned}
& \partial_y F_R((f, E, B), (u, \alpha, b)) (\delta f, \delta E, \delta B) (g, h_1, h_2, h_3, a_1, a_2, a_3, a_4) \\
= \Bigg(& -\int_0^T \int \int \left(\left(\partial_t g + \widehat{p} \cdot \partial_x g + \left(E - \widehat{p}^{\perp} B\right) \cdot \partial_p g\right) \delta f + \left(\delta E - \widehat{p}^{\perp} \delta B\right) f \cdot \partial_p g\right) dpdxdt \\
& + \langle g(T), \delta f(T) \rangle_{L^2} - \langle g(0), \delta f(0) \rangle_{L^2}, \\
& \int_0^T \int \left(-\delta E_1 \partial_t h_1 + \delta B \partial_{x_2} h_1 + j_{\delta f,1} h_1\right) dxdt \\
& + \langle h_1(T), \delta E_1(T) \rangle_{L^2} - \langle h_1(0), \delta E_1(0) \rangle_{L^2}, \\
& \int_0^T \int \left(-\delta E_2 \partial_t h_2 - \delta B \partial_{x_1} h_2 + j_{\delta f,2} h_2\right) dxdt \\
& + \langle h_2(T), \delta E_2(T) \rangle_{L^2} - \langle h_2(0), \delta E_2(0) \rangle_{L^2}, \\
& \int_0^T \int \left(-\delta B \partial_t h_3 - \delta E_2 \partial_{x_1} h_3 + \delta E_1 \partial_{x_2} h_3\right) dxdt
\end{aligned}
$$

$$+ \langle h_3(T), \delta B(T)\rangle_{L^2} - \langle h_3(0), \delta B(0)\rangle_{L^2},$$
$$\left.\int\int \delta f(0)\, a_1 \; dpdx, \int \delta E_1(0)\, a_2 \; dx, \int \delta E_2(0)\, a_3 \; dx, \int \delta B(0)\, a_4 \; dx\right)$$

for $(\delta f, \delta E, \delta B) \in M_R$. Note that it is crucial that f vanishes for $|p| \geq R$ so that for $i = 1, 2$ the linear map

$$(f, E, B) \mapsto \int_0^T \int j_{f,i} \cdot dxdt \in H^1\left([0,T] \times \mathbb{R}^2\right)^*$$

is bounded by

$$\left|\int_0^T \int j_{f,i} h_i \; dxdt\right| \leq C(T,R)\, \|f\|_{C(0,T;L^2)} \|h_i\|_{H^1}$$

and hence differentiable.
On the other hand we have

$$\partial_y \phi\left((f,E,B), (u,\alpha,b)\right)(\delta f, \delta E, \delta B) = \langle \rho_f - \rho_d, \rho_{\delta f}\rangle_{L^2}.$$

Here again, the support condition given in the definition of M_R is important to estimate

$$\left|\int_0^T \int (\rho_f - \rho_d)\, \rho_{\delta f} \; dxdt\right| \leq C(T,R)\, \|\rho_f - \rho_d\|_{L^2} \|\delta f\|_{C(0,T;L^2)}$$

and

$$\int_0^T \int \rho_{\delta f}^2 \; dxdt \leq C(T,R)\, \|\delta f\|^2_{C(0,T;L^2)}.$$

Now we search for an adjoint state

$$\begin{aligned} q =& (g, h_1, h_2, h_3, a_1, a_2, a_3, a_4) \\ \in& Z^* \cong H^1\left([0,T]\times\mathbb{R}^4\right) \times \left(H^1\left([0,T]\times\mathbb{R}^2\right)\right)^3 \times L^2\left(\mathbb{R}^4\right) \times \left(L^2\left(\mathbb{R}^2\right)\right)^3 \end{aligned}$$

satisfying

$$\partial_y F_R(y(x), x)^* q = -\partial_y \phi(y(x), x).$$

In other words, after integrating by parts once,

$$\begin{aligned} &-\int_0^T \int\int \left(\partial_t g + \widehat{p} \cdot \partial_x g + \left(E - \widehat{p}^\perp B\right) \cdot \partial_p g - 4\pi\left(\widehat{p}_1 h_1 + \widehat{p}_2 h_2\right)\right) \delta f \; dpdxdt \\ &+ \int_0^T \int \left(-\partial_t h_1 + \partial_{x_2} h_3 + \int g \partial_{p_1} f \; dp\right) \delta E_1 \; dxdt \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T\int\left(-\partial_t h_2-\partial_{x_1}h_3+\int g\partial_{p_2}f\ dp\right)\delta E_2\ dxdt\\
&+\int_0^T\int\left(-\partial_t h_3+\partial_{x_2}h_1-\partial_{x_1}h_2-\int g\widehat{p}^{\perp}\cdot\partial_p f\ dp\right)\delta B\ dxdt\\
&+\langle g(T),\delta f(T)\rangle_{L^2}-\langle g(0)-a_1,\delta f(0)\rangle_{L^2}+\langle h_1(T),\delta E_1(T)\rangle_{L^2}\\
&-\langle h_1(0)-a_2,\delta E_1(0)\rangle_{L^2}+\langle h_2(T),\delta E_2(T)\rangle_{L^2}-\langle h_2(0)-a_3,\delta E_2(0)\rangle_{L^2}\\
&+\langle h_3(T),\delta B(T)\rangle_{L^2}-\langle h_3(0)-a_4,\delta B(0)\rangle_{L^2}\\
=&-\int_0^T\int\int 4\pi\left(\rho_f-\rho_d\right)\delta f\ dpdxdt
\end{aligned}
\tag{3.4}
$$

for all $(\delta f,\delta E,\delta B)\in M_R$. Therefore the adjoint state solves the adjoint system

$$
\begin{aligned}
\partial_t g+\widehat{p}\cdot\partial_x g+\left(E-\widehat{p}^{\perp}B\right)\cdot\partial_p g=&4\pi\left(\widehat{p}_1h_1+\widehat{p}_2h_2\right)+4\pi\left(\rho_f-\rho_d\right),\\
\partial_t h_1-\partial_{x_2}h_3=&\int g\partial_{p_1}f\ dp',\\
\partial_t h_2+\partial_{x_1}h_3=&\int g\partial_{p_2}f\ dp',\\
\partial_t h_3-\partial_{x_2}h_1+\partial_{x_1}h_2=&-\int g\widehat{p'}^{\perp}\cdot\partial_p f\ dp',\\
\left(g,h_1,h_2,h_3\right)(T)=&0
\end{aligned}
\tag{Ad}
$$

for $|p|<R$. Since $R>0$ (with $\mathrm{supp}_p f\subset B_R$) is arbitrary, it is natural to demand (Ad) holds globally on $[0,T]\times\mathbb{R}^4$. Conversely, if (Ad) holds for all p, then (3.4) holds for for all $(\delta f,\delta E,\delta B)\in M_R$ for any $R>0$ if we simply set $a_1=g(0)$, $(a_2,a_3,a_4)=(h_1,h_2,h_3)(0)$. The latter equations are unsubstantial and will be ignored.
In accordance with (3.3), we compute the derivative of $\overline{\phi}$ via

$$
\begin{aligned}
\overline{\phi}'(u,\alpha,b)(\delta u,\delta\alpha,\delta b)=\int_0^T\int\left(\delta U_1h_1+\delta U_2h_2\right)dxdt+\beta\sum_{j=1}^N c_j\big(&\langle u_j,\delta u_j\rangle_{L^2}\\
&+\beta_1\langle\partial_t u_j,\partial_t\delta u_j\rangle_{L^2}+\beta_2\left\langle\partial_t^2u_j,\partial_t^2\delta u_j\right\rangle_{L^2}\big)
\end{aligned}
$$

where

$$
\delta U=\sum_{j=1}^N\left(u_jDz_j\left(R\left(\alpha_j\right)\cdot+b_j\right)\left(R'\left(\alpha_j\right)\delta\alpha_j\cdot+\delta b_j\right)+z_j\left(R\left(\alpha_j\right)\cdot+b_j\right)\delta u_j\right).
$$

System (Ad) has to be investigated. It is a final value problem which can easily be turned into an initial value problem via $\widetilde{g}(t,x,p)=g(T-t,-x,-p)$ and $\widetilde{h}(t,x)=h(T-t,-x)$, so that the left hand sides of the differential equations in (Ad) do not change. In other words, the hyperbolic system (Ad) is time reversible.
To show unique solvability of (Ad), one can proceed similar to the dealing with (LVM). Yet there are some differences. Firstly, the source terms in the Maxwell equations are

not the current densities induced by g but some other moments of g. Additionally, even in the fourth equation of (Ad) a source term appears. Hence we have to prove analogues of Theorems 1.2 and 1.4 with more general source terms. Secondly, the right hand side of the Vlasov equation (and hence a solution g) does not have compact support with respect to p. But this will not cause any problems since in a representation formula for h there will appear a factor $\partial_p f$ (or first derivatives of $\partial_p f$). Because of the known fact that f is compactly supported with respect to p uniformly in t, x, we do not have to demand that g has this property. In Section 1.1 we had to assume this property for the density since the integral defining the current density induced by this density contains the factor $\widehat{p}$ which is obviously not compactly supported in p.
Instead of examining the solvability of (Ad) further, we consider another optimization problem.

3.3 The problem of keeping the plasma in a certain container

3.3.1 Formulation

In real applications, like controlling a plasma in a fusion reactor, the actual goal is to keep the plasma in a certain container, for instance a torus. Furthermore, one can hardly determine a best appropriate shape ρ_d. Thus, it is suitable to impose a constraint on the charge density ρ_f which only allows densities that are zero or, better, nearly zero, on a forbidden (measurable) set $A \subset \mathbb{R}^2$. For example, A could be the complement of the torus. The new constraint to be considered is

$$\int_0^T \int_A \rho_f \, dxdt \leq \epsilon.$$

Simultaneously, we abolish the term with ρ_f in the objective function.
Now, the question arises how to choose ϵ. In order to guarantee existence of feasible points, we define (recall the notation $\rho_f = \overline{\Psi}(u, \alpha, b) \in C\left(0, T; L^1\right)$)

$$\epsilon := \inf \left\{ \int_0^T \int_A \overline{\Psi}(u, \alpha, b) \, dxdt \mid |u_j| \leq 1, |b_j| \leq r, j = 1, \ldots, N \right\} + \widetilde{\epsilon}$$

where $\widetilde{\epsilon} > 0$ is chosen small. In fact, one will hope that the infimum above equals zero, but this will not further be investigated.
Therefore, our new problem reads

$$\begin{aligned} \min_{\substack{u \in H^2([0,T])^N, \\ \alpha \in \mathbb{R}^N, b \in \left(\mathbb{R}^2\right)^N}} \quad & \sum_{j=1}^N c_j \left(\|u_j\|^2_{L^2([0,T])} + \beta_1 \|\partial_t u_j\|^2_{L^2([0,T])} + \beta_2 \left\|\partial_t^2 u_j\right\|^2_{L^2([0,T])} \right) \\ \text{s.t.} \quad & |u_j| \leq 1, |b_j| \leq r, j = 1, \ldots, N, \\ & \int_0^T \int_A \overline{\Psi}(u, \alpha, b) \, dxdt \leq \epsilon. \end{aligned} \tag{P2}$$

3.3.2 Existence of minimizers

Theorem 3.11. *There is a minimizer of* (P2).

Proof. The proof is very similar to the proof of Theorem 3.4. Note that feasible points exist by definition of ϵ. Additionally, we only have to show that the new constraint

$$\int_0^T \int_A \overline{\Psi}(u,\alpha,b)\,dxdt \le \epsilon \tag{3.5}$$

is preserved in the limit. Because of $f_k \to f$ in L^2 and f_k having compact support with respect to x and p uniformly in k, we conclude that $f_k \to f$ also in L^1. Hence $\rho_{f_k} \to \rho_f$ in L^1 which instantly implies that (3.5) is preserved in the limit. □

3.3.3 Optimality conditions

The new nonlinear and non-convex constraint in (P2) including the state is harder to handle than the constraints on the control. First we note that

$$\begin{aligned} H\colon X \to& \mathbb{R}, \\ (u,\alpha,b) \mapsto& \int_0^T \int_A \overline{\Psi}(u,\alpha,b)\,dxdt \end{aligned}$$

is continuously Fréchet-differentiable by Theorem 3.7 ii) and $C\left(0,T;L^1\left(\mathbb{R}^2\right)\right) \hookrightarrow L^1\left([0,T]\times\mathbb{R}^2\right)$ with derivative

$$H'(u,\alpha,b)(\delta u,\delta\alpha,\delta b) = \int_0^T \int_A \overline{\Psi}'(u,\alpha,b)(\delta u,\delta\alpha,\delta b)\,dxdt.$$

We proceed similarly as before. Writing

$$C := \{x = (u,\alpha,b) \in X \mid |u_j| \le 1, |b_j| \le r, j = 1,\dots,N\}$$

our constraints are

$$x \in C,\ \ g(x) = \epsilon - H(x) \in \mathbb{R}_{\ge 0}.$$

Note that g is of class C^1.
Again, we want (CQ) to be satisfied at an optimal point $\overline{x} = \left(\overline{u},\overline{\alpha},\overline{b}\right)$. However, a problem arises if $H(\overline{x}) = \epsilon$ and $H'(\overline{x}) = 0$ since then

$$g'(\overline{x})\,X - \left(\mathbb{R}_{\ge 0} - \mathbb{R}_{\ge 0}\, g(\overline{x})\right) = \mathbb{R}_{\le 0} \ne \mathbb{R}.$$

Hence (CQ) can not be satisfied.
In all other cases we can show (CQ). Let $d \in \mathbb{R}$ be given. Firstly, if $g(\overline{x}) > 0$ define

$$\delta x := 0,$$

$$\lambda := \frac{|d|}{g(\overline{x})},$$
$$k := |d| - d \geq 0.$$

Secondly, if $H'(\overline{x}) \neq 0$ let $\delta x \in X$ with $H'(\overline{x})\,\delta x = -d$ and set $k = \lambda = 0$.
In both cases we arrive at

$$g'(\overline{x})\,\delta x - (k - \lambda g(\overline{x})) = d.$$

Thus (CQ) is satisfied.
This leads to:

Theorem 3.12. *Let $\overline{x} = \left(\overline{u}, \overline{\alpha}, \overline{b}\right)$ be a minimizer of* (P2). *Then 1) or 2) holds:*

1) There is $\mu \in \mathbb{R}$ such that the following assertions hold:

i) (Primal feasibility):

$$|\overline{u}_j| \leq 1, \left|\overline{b}_j\right| \leq r, \int_0^T \int_A \overline{\Psi}\left(\overline{u}, \overline{\alpha}, \overline{b}\right) dxdt \leq \epsilon.$$

ii) (Dual feasibility):

$$\mu \geq 0.$$

iii) (Complementary slackness):

$$\mu = 0$$

or

$$\int_0^T \int_A \overline{\Psi}\left(u, \alpha, b\right) dxdt = \epsilon.$$

iv) (Stationarity): For all $(u, \alpha, b) \in C$, and defining

$$(\delta u, \delta\alpha, \delta b) := \left(u - \overline{u}, \alpha - \overline{\alpha}, b - \overline{b}\right),$$

we have

$$\begin{aligned}&\sum_{j=1}^N c_j \left(\langle \overline{u}_j, \delta u_j\rangle_{L^2} + \beta_1 \langle \partial_t \overline{u}_j, \partial_t \delta u_j\rangle_{L^2} + \beta_2 \left\langle \partial_t^2 \overline{u}_j, \partial_t^2 \delta u_j\right\rangle_{L^2}\right)\\ \geq& \mu \int_0^T \int_A \rho_{\delta f}\, dxdt\end{aligned}$$

where δf is obtained by solving

$$\partial_t \delta f + \widehat{p} \cdot \partial_x \delta f + \left(\overline{E} - \widehat{p}^{\perp}\overline{B}\right) \cdot \partial_p \delta f = -\left(\delta E - \widehat{p}^{\perp}\delta B\right) \cdot \partial_p \overline{f},$$

$$
\begin{aligned}
\partial_t \delta E_1 - \partial_{x_2} \delta B =& - j_{\delta f,1} - \delta U_1, \\
\partial_t \delta E_2 + \partial_{x_1} \delta B =& - j_{\delta f,2} - \delta U_2, \\
\partial_t \delta B + \partial_{x_1} \delta E_2 - \partial_{x_2} \delta E_1 =& 0, \\
(\delta f, \delta E, \delta B)(0) =& 0
\end{aligned}
$$

with

$$
\delta U = \sum_{j=1}^{N} \left(\overline{u}_j D z_j \left(R(\overline{\alpha}_j) \cdot + \bar{b}_j \right) \left(R'(\overline{\alpha}_j) \delta\alpha_j \cdot + \delta b_j \right) + z_j \left(R(\overline{\alpha}_j) \cdot + \bar{b}_j \right) \delta u_j \right)
$$

and $\left(\overline{f}, \overline{E}, \overline{B}\right)$ *satisfying*

$$
\begin{aligned}
\partial_t \overline{f} + \widehat{p} \cdot \partial_x \overline{f} + \left(\overline{E} - \widehat{p}^{\perp} \overline{B} \right) \cdot \partial_p \overline{f} =& 0, \\
\partial_t \overline{E}_1 - \partial_{x_2} \overline{B} =& - j_{\overline{f},1} - \overline{U}_1, \\
\partial_t \overline{E}_2 + \partial_{x_1} \overline{B} =& - j_{\overline{f},2} - \overline{U}_2, \\
\partial_t \overline{B} + \partial_{x_1} \overline{E}_2 - \partial_{x_2} \overline{E}_1 =& 0, \\
\left(\overline{f}, \overline{E}, \overline{B}\right)(0) =& \left(\mathring{f}, \mathring{E}, \mathring{B} \right)
\end{aligned}
$$

with

$$
\overline{U} = \sum_{j=1}^{N} \overline{u}_j z_j \left(R(\overline{\alpha}_j) \cdot + \bar{b}_j \right).
$$

2) It holds that:

i)

$$
\int_0^T \int_A \overline{\Psi}(u, \alpha, b) \, dx dt = \epsilon.
$$

ii) For all $(\delta u, \delta\alpha, \delta b) \in X$ *we have*

$$
\int_0^T \int_A \rho_{\delta f} dx dt = 0
$$

where δf *is obtained by solving*

$$
\begin{aligned}
\partial_t \delta f + \widehat{p} \cdot \partial_x \delta f + \left(\overline{E} - \widehat{p}^{\perp} \overline{B} \right) \cdot \partial_p \delta f =& - \left(\delta E - \widehat{p}^{\perp} \delta B \right) \cdot \partial_p \overline{f}, \\
\partial_t \delta E_1 - \partial_{x_2} \delta B =& - j_{\delta f,1} - \delta U_1, \\
\partial_t \delta E_2 + \partial_{x_1} \delta B =& - j_{\delta f,2} - \delta U_2, \\
\partial_t \delta B + \partial_{x_1} \delta E_2 - \partial_{x_2} \delta E_1 =& 0, \\
(\delta f, \delta E, \delta B)(0) =& 0
\end{aligned}
$$

with

$$\delta U = \sum_{j=1}^{N} \left(\overline{u}_j Dz_j \left(R\left(\overline{\alpha}_j\right) \cdot + \overline{b}_j \right) \left(R'\left(\overline{\alpha}_j\right) \delta\alpha_j \cdot + \delta b_j \right) + z_j \left(R\left(\overline{\alpha}_j\right) \cdot + \overline{b}_j \right) \delta u_j \right)$$

and $\left(\overline{f}, \overline{E}, \overline{B}\right)$ *satisfying*

$$\begin{aligned} \partial_t \overline{f} + \widehat{p} \cdot \partial_x \overline{f} + \left(\overline{E} - \widehat{p}^{\perp} \overline{B}\right) \cdot \partial_p \overline{f} =& 0, \\ \partial_t \overline{E}_1 - \partial_{x_2} \overline{B} =& -j_{\overline{f},1} - \overline{U}_1, \\ \partial_t \overline{E}_2 + \partial_{x_1} \overline{B} =& -j_{\overline{f},2} - \overline{U}_2, \\ \partial_t \overline{B} + \partial_{x_1} \overline{E}_2 - \partial_{x_2} \overline{E}_1 =& 0, \\ \left(\overline{f}, \overline{E}, \overline{B}\right)(0) =& \left(\mathring{f}, \mathring{E}, \mathring{B}\right) \end{aligned}$$

with

$$\overline{U} = \sum_{j=1}^{N} \overline{u}_j z_j \left(R\left(\overline{\alpha}_j\right) \cdot + \overline{b}_j \right).$$

Proof. If 2) does not hold, then (CQ) is satisfied and Lemma 3.8 implies 1). □

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