Limitations of Stabilizing Effects of Fundamentalists Facing Positive Feedback Traders*

Michael Heinrich Baumann† Michaela Baumann

Alexander Erler

Universität Bayreuth

Universitätsstraße 30, D-95447 Bayreuth, Germany

michael.baumann (at) uni-bayreuth.de

Abstract

We analyze financial interactions between fundamentalists and chartists within a heterogeneous agent model, focusing on discovering whether the presence

*This paper was presented at the XVIII. Workshop on Quantitative Finance 2017 (QFW2017), University of Milano-Bicocca, Milano, Italy, and at the XVI. Annual European Economics and Finance Society Conference 2017 (EEFS2017) at University of Ljubljana, Ljubljana, Slovenia, and is accepted for presentation at the 49th Money, Macro and Finance Research Group Annual Conference 2017 (MMF2017), King's Business School, King's College London, Strand, London, UK.

†The work of Michael H. Baumann is supported by a scholarship from Hanns-Seidel-Stiftung e.V. (HSS), funded by Bundesministerium für Bildung und Forschung (BMBF).

of fundamentalists is enough to stabilize prices. In contrast to related work, which is based on simulations, we analytically prove that the presence of fundamentalists is not sufficient to avoid asset price bubbles. The behavior of trend followers can result in exploding prices irrespective of fundamentalists' investment decisions. We derive the upper boundaries for positive feedback traders' investments necessary to avoid exploding prices. In this situation, intervention measures might be necessary in order to stabilize stock/asset markets.

Keywords

Heterogeneous Agent; Feedback Trading; Fundamentalists; Chartists; Trend Followers; Financial Bubbles

JEL classification codes

D84, G01, G11

1 Motivation

Financial market bubbles have repeatedly caused major macroeconomic problems, a very prominent example of which was the subprime havoc of 2007/2008 (Reinhart and Rogoff, 2009). While misguided macroeconomic policies are chief among the usual suspects in trying to understand such aberrations, an important strand of the literature focuses on the question of whether specific behavior of market participants is responsible for price bubbles. In particular, heterogeneous agent models (HAMs) analyze how both chartists and fundamentalists are able to determine asset price movements (Hommes, 2006a).

Chartists, for example trend followers, trade based only on information about the price process, that is, they assume that all important information is present in the asset price (Graham et al., 1934). In contrast, fundamentalists have some fundamental value in mind and trade based on perceived over- or undervaluation of the underlying asset. Trend followers magnify the current trend, either positively or negatively, because their trading is based on the philosophy that the greater the absolute value of the slope of the price process, the more that should be invested or disinvested (Covel, 2004). Fundamentalists, in contrast, invest or disinvest, that is, increase or decrease their investment, when the price is below or above the fundamental value, thereby pushing the asset price toward its fundamental value. Traders act out of self-interest with the intention of making a profit, and give little thought to how their actions will impact prices. As a consequence of the two different investment strategies, the presence of chartists can cause exploding prices (De Long et al., 1990b), whereas fundamentalists are associated with a stabilizing influence on assets. Thus, the following question arises:

> *Are the balancing effects of fundamentalists strong enough compensate for the destabilizing impacts of chartists?*

HAMs are increasingly employed in search of an answer to this question (Gaunersdorfer and Hommes, 2005; Hommes, 2002; Lux, 1995, 1998;

Lux and Marchesi, 1999, 2000).[1] The models typically use bounded rational agents, (imperfect) heuristics or rules of thumb, and nonlinear dynamics (which might be chaotic). Some studies find that the stabilizing effects of fundamentalists are not necessarily strong enough to stabilize markets (Hommes, 2006a). However, the results are usually obtained via simulations and are not analytically proven (Hommes, 2006a).[2] thus leading to a second question:

> *Is it possible to analytically prove that chartists' behavior can lead to exploding prices irrespective of fundamentalists' compensatory effects?*

The main contribution of our paper is a mathematically rigorous proof that chartists' behavior—specifically, the behavior of linear feedback traders without rational expectations and without information about the market

[1] These studies provide useful explanations for many stylized facts, including excess volatility, high trading volume, temporary bubbles, trend following, sudden crashes, mean reversion, clustered volatility and fat tailed distribution returns. For an excellent overview regarding HAM see the work of Hommes (2006a).

[2] An exception is the work of De Long et al. (1990b) which investigates the effect of positive feedback traders and informed speculators, who evaluate and consider the needs of the other market participants, especially the growing needs of the positive feedback traders, in a three-period market model facing fundamentalists. De Long et al. (1990b) show that the interaction of these two trader types pushes the price away from the fundamental value under specific assumptions and despite the fundamentalists' stabilizing behavior. The present work differs from the work of De Long et al. (1990b) in that we do not investigate how two types of traders—positive feedback traders and informed speculators—jointly push up the price but instead look only at trend followers, nor do we assume a predetermined end of the market.

(e.g., fundamental value, trading volume, or even prices)—can overcome the stabilizing effects of traders with rational expectations of the fundamental value. Put differently, prices explode because the stabilizing effects of fundamentalists are outweighed by linear feedback traders. Unstable price developments are the result, which in turn increase the likelihood of a financial bubble. As shown in the proof, thresholds for model-inherent values can be specified that make certain the occurrence of a bubble. Furthermore, there are certain values of external parameters that allow the thresholds of the inherent values to be met. The analysis reveals that even fundamentalists without any liquidity constraints and with perfect information about the price, the fundamental value, and the market's characteristics are not sufficient to stabilize a very simply constructed market based on (excess) demand if the feedback trader's initial investment is large enough.

The field of applied mathematics has many new results concerning technical trading strategies (Barmish and Primbs, 2011, 2015). For example, the performance properties of chartist strategies have been proven and explanations given for why it is reasonable to trade according to a feedback strategy. In contrast to the feedback trading literature, where the price taker property is usually presumed, we study the effects of trading strategies in an HAM that displays phenomena caused by (excess) demand (Baumann, 2015a).

The paper is organized as follows: Section 2 explains the price model as well as the investment strategies of feedback traders and fundamentalists. Section 3 answers the main question of the paper, that is, whether the presence of fundamentalists is sufficient to stabilize the market. Section 4 presents further results based on the calculations of Section 3 and Section 5 provides ideas for future work and concludes the paper.

2 Model Structure

The model consists of a one asset market and is populated with two types of heterogeneous agents—fundamentalists and chartists. Their interaction with the market maker is illustrated in Section 2.1. Section 2.2 presents the price process in the interactive market model. Sections 2.3, 2.4, and 2.5 introduce the traders and their expectations. For simplification of the analysis we assume that there is only one feedback trader, that is we treat all existing feedback traders as one average feedback trader.[3]

2.1 Timeline

At the beginning of every period $t \in \{0, 1, \dots, T\}$, each agent $\ell \in \{FT, FU\}$, where FT is the feedback trader (chartist) and FU the fundamentalist, decides how to invest based on his investment strategy, where T is unknown or even ∞. Each investment strategy I_t^ℓ is guided by a different heuristic (rule of thumb). Based on the strategy chosen, each agent then allocates his financial resources among the asset market. The trader is aware of past market data and of expectations of future fundamental values $\mathbb{E}[f_{t+1}]$. The resulting

[3]There is indeed no difference between one feedback trader with an initial investment I_0^{FT} and fixed K, see Equation (3), and n feedback traders with initial investments $\frac{I_0^{FT}}{n}$ and the same K. That is, for the feedback traders this summarization is without loss of generality (WLOG). Whether this assumption is WLOG for fundamentalists, too, is left to future work.

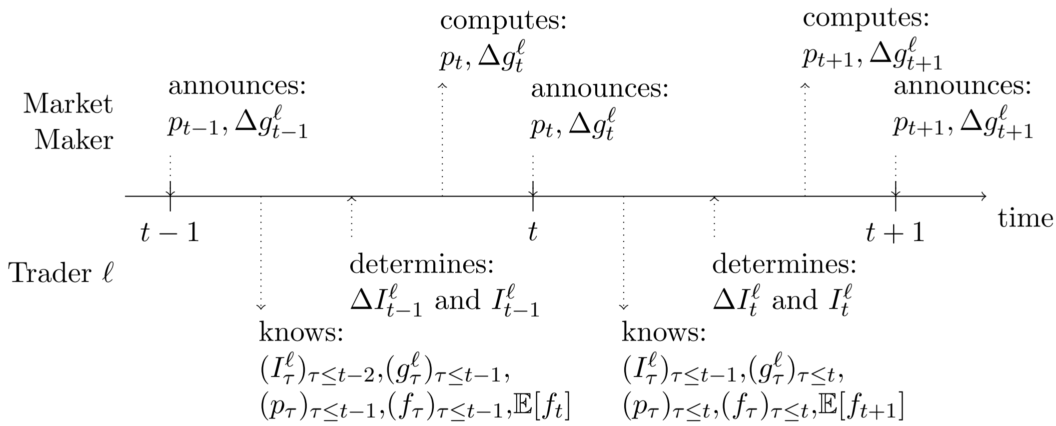


Figure 1: Timeline of the traders' and the market maker's decisions and interactions with $\Delta g^{\ell}_t = I^{\ell}_{t-1} \cdot \frac{\Delta p_t}{p_{t-1}}$.

changes in the investments, denoted by ΔI^{ℓ}_t, are cleared by a market maker who adjusts asset prices according to (excess) demand. After the traders have observed the price change Δp_t, and hence their own gains or losses Δg^{ℓ}_t in the recent period, they use this information in making their next investment decision.[4] It is assumed that at the end of each period the investment is capitalized and reinvested. Based on this understanding the price model is constructed. The timeline of the traders' and the market maker's decisions and interactions is shown in Figure 1.

2.2 Price Process for the Interactive Market Model

In feedback trading literature, price is usually determined through a certain price process, for example, geometric Brownian motion (GBM), which is exogenously given (Barmish and Primbs, 2015). This implies that the traders

[4]For all processes α_t we set $\Delta \alpha_t = \alpha_t - \alpha_{t-1}$ as the change of the underlying process, e.g., Δg^{ℓ}_t is the period profit while g^{ℓ}_t is the overall gain/loss of trader ℓ.

are not able to influence the price. To avoid this price taker property, which is a strong restriction of every market model, agent-based price models have evolved in the academic economics literature (Hommes, 2006a). According to these models, the price is a function of traders' investment decisions. We denote the sum of all traders' changes of investment at time t with $\Delta I_t = \sum_\ell \Delta I_t^\ell$. Based on the idea of interacting agents, Baumann (2015a) constructs a pricing model that fulfills the law of (excess) demand, namely

(I1) $p_{t+1} = p_t$, if $\Delta I_t = 0$

(I2) $p_{t+1} \to \infty$, if $\Delta I_t \to \infty$

(I3) $p_{t+1} \to 0$, if $\Delta I_t \to -\infty$

(I4) p_{t+1} strictly monotonous increasing in ΔI_t

For simplification, we assume an infinite supply,[5] and thus the law of supply and demand reduces to a law of (excess) demand.[6] This model, which

[5]Infinite supply is, for example, given for synthetic assests, betting slips, etc. These assets are produced by the market maker without any restriction. Thus, the market maker can clear the market for sure. It follows that the market maker sets the new asset price according to the changes in the asset demand only.

[6]Alternatively, one can define (I1)-(I4) by use of the buying/selling decision $B_t := I_t - \frac{p_t}{p_{t-1}} \cdot I_{t-1}$ instead of the change of investment $I_t - I_{t-1}$. Then, a change of investment caused by price increase would not affect the price. Based on simulations, use of the buying/selling decision instead of the change of investment affects the proposition of this paper only quantitatively, not qualitatively. However, a finite supply would make the analysis much more complicated. In the work of Baumann and Baumann (2015), both the HAM for stocks using B_t and the HAM for synthetic assets using ΔI_t are presented.

is in a sense a natural generalization of the GBM (proven in Baumann, 2015a), in its general form is given by

$$p_{t+1} = p_t \cdot e^{M^{-1}\Delta I_t} \quad (1)$$

$$= p_0 \cdot e^{M^{-1} I_t} \quad (2)$$

where $M > 0$ is a scaling factor expressing the trading volume of the underlying asset.[7] This pricing model is closed through a market maker (Drescher and Herz, 2012). As is common practice, the market maker acts as a privileged trader that sets prices according to (excess) demand (see Figure 2) and hence ensures market clearing (cf. the role of a broker in stock markets) (Hommes, 2006a).[8] Baumann and Baumann (2015) show that this market model meets several stylized facts formulated by Hommes (2006b).

2.3 Feedback Traders

Barmish (2011); Barmish and Primbs (2011, 2015); Baumann (2015a) outline a special class of trading strategies based on control techniques, namely, feedback trading. Traders engaged in this sort of strategy are called feedback traders and utilize neither fundamentals nor the absolute asset value in making their investments; they take into account only their own gains and losses. Their strategy thus depends on prices relative to their previous investments,

[7] The pricing rule of Equation (1) is similar to that one Batista et al. (2015) use. Unless otherwise stated, for simplicity M is set to $M = 1$.

[8] Possible profit making by and survival of the market maker will not be discussed in the work at hand but is an interesting topic for future work.

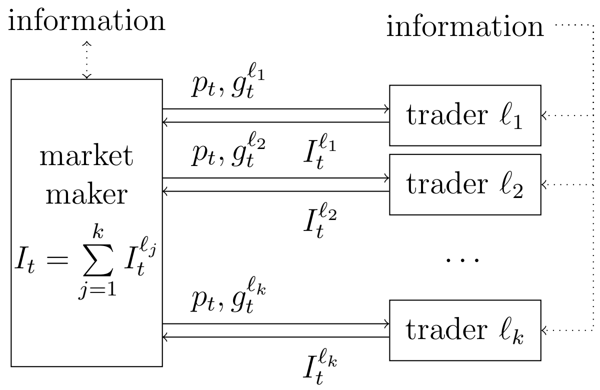


Figure 2: Schematic representation of the role of the market maker with k traders.

that is, feedback traders are chartists because gains or losses, respectively, are a function of the price but not of any fundamental value. From a control theoretic point of view, feedback traders treat the price like a disturbance variable and their strategy needs to be robust to this disturbing influence. In calculating a certain trader's gain, the market maker takes into account the trader's investment and the asset price.[9] Therefore, for feedback traders not only is it true that the investment affects the gain, but also that the gain determines the investment.[10]

One specific feedback strategy, discussed by Barmish and Primbs (2011, 2012); Baumann (2015b), is the (positive) linear feedback strategy

$$I_t^{FT} := I_0^{FT} + K \cdot g_t^{FT} \tag{3}$$

[9]The price is a function of all traders' investment; see Section 2.2 and, especially, Figure 2.

[10]In the literature, continuous time models are usually applied whereas this analysis uses a discrete time model because this is, as mentioned by Barmish and Primbs (2011), the weaker, more general assumption.

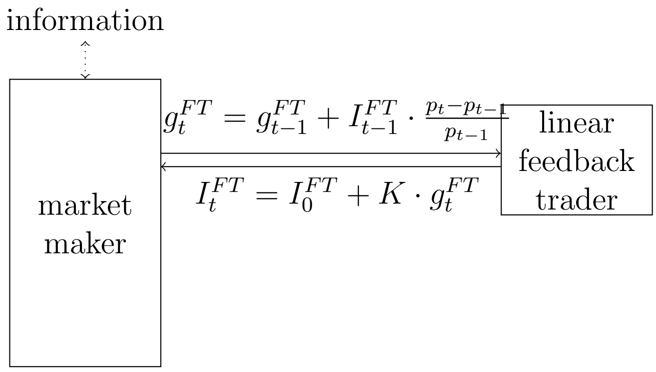


Figure 3: Schematic interaction between market maker and linear feedback trader.

where the linear feedback trader calculates his investment I_t^{FT} at time t as a linear function of his gain/loss function g_t^{FT} using the initial investment $I_0^{FT} > 0$ and a feedback parameter $K > 0$. Figure 3 shows a feedback loop between the gain or loss g^{FT} of a linear feedback trader and his investment I^{FT}. By calculating the gain or loss of a specific trader (or group of traders) ℓ via

$$g_t^{\ell} = \sum_{i=1}^{t} I_{i-1}^{\ell} \cdot \frac{p_i - p_{i-1}}{p_{i-1}} \tag{4}$$

where p_t denotes the price process[11] and I_t^{ℓ} the trader's investment at time t, it follows that linear feedback traders are trend followers given $I_t^{FT} > 0$ (see also Equation (7)). A trader is called a trend follower (cf. Covel, 2004) if his investment increases when prices are rising and decreases when prices are falling. Note that the particular investment amount at time $t \geq 1$ is given by

$$\Delta I_t^{FT} = I_t^{FT} - I_{t-1}^{FT} \tag{5}$$

[11]The relative price change $\frac{p_t - p_{t-1}}{p_{t-1}}$ is called return on investment (ROI).

$$= K \cdot (g_t^{FT} - g_{t-1}^{FT}) \tag{6}$$

$$= K \cdot I_{t-1}^{FT} \cdot \frac{p_i - p_{i-1}}{p_{i-1}} \tag{7}$$

whereas I_t^{FT} denotes the total investment at time t (all individual investment amounts up to time t) of feedback trader FT. Rising prices lead to increasing gain for the linear feedback trader if $I_t^{FT} > 0$ and thus his investment increas, too. Analogously, falling prices lower the gain and the trader disinvests. Baumann (2015a) shows that in the event only one feedback trader is acting on the market with the price process described by Equation (1), it holds that

$$I_t > 0 \ \forall t, \tag{8}$$

$$\Delta I_t > 0 \ \forall t, \text{ and} \tag{9}$$

$$\text{if } \exists t : \Delta I_t > \Delta I_{t-1} \tag{10}$$

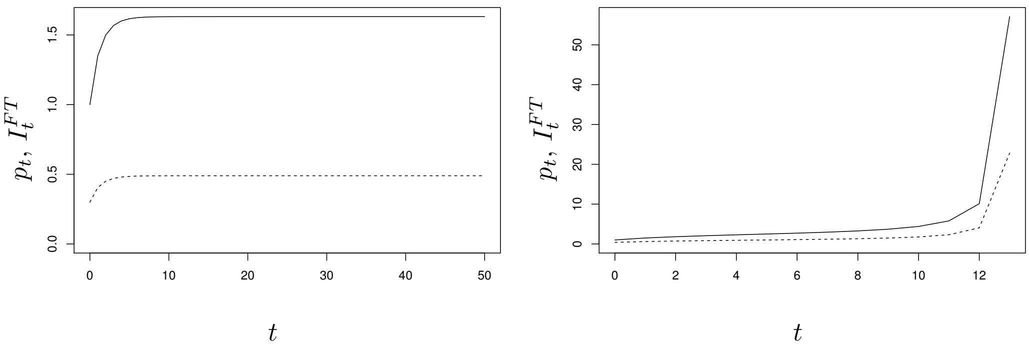
$$\Rightarrow \Delta I_{t+1} > \Delta I_t. \tag{11}$$

This is important as it will be shown that, together with the results of Section 3, the price explosion effects of feedback traders, that would have occurred in absence of fundamentalists can be compensated by fundamentalists—at least to a certain degree.

Two typical investment paths can be identified in the scenario where only one feedback-based trader is acting on the market. The two paths are shown in Figure 4a and Figure 4b where the asset price p_t is indicated with a solid line and the feedback trader's investment with a dashed one. If I_0^{FT} lies below a specific threshold, I_t^{FT} converges (Figure 4a), if it is above this threshold the investment explodes (Figure 4b). Baumann (2015a) provides a non-closed formula determining the threshold. Specific values can be derived only through a simulation like the one in Figure 4 and by algorithmically localizing the threshold.

Control-based trading strategies like the one presented by Barmish and Primbs (2014) are interesting to further analyze since the literature contains several notable results, for example, the guarantee of non-negative profit for the simultaneously long short (SLS) strategy that has initial investment zero and consists of two linear feedback strategies for continuously differentiable prices (arbitrage!). For prices following a GBM, for price processes allowing for jumps (Merton's jump diffusion model [MJDM]), and for all essentially linear prices one can expect positive profit for the SLS strategy, i.e., $\mathbb{E}[g_t^{SLS}] > 0$ while $I_0^{SLS} = 0$. [12] However, all of these settings assume the price taker property, as the price process is defined independently of the traders' investments. In contrast, here we abandon the price taker property

[12] Barmish and Primbs (2011) show that a SLS strategy that is the sum of two particular and opposed linear feedback strategies with initial investment zero always makes a positive profit under the assumption of continuously differentiable prices. Furthermore, if in a continuous time model prices follow a GBM the SLS strategy is expected to yield non-negative profit (as proven in Barmish and Primbs, 2011). Baumann (2015b) shows that even for a discontinuous price process characterized through MJDM, SLS trading is still profitable, independent of intensity, type, and height of jumps. Continuously differentiable prices, GBM, and MJDM all fulfill the price taker property that is usually assumed in literature about feedback trading (see, e.g., Barmish, 2011; Barmish and Primbs, 2011). Furthermore, Baumann and Grüne (2015) show that $I_0^{SLS} = 0$ and $\mathbb{E}[g_t^{SLS}] > 0$ under the price taker assumption hold even for all essentially linear prices.



(a) I_0^{FT} below a specific threshold: I_t^{FT} (dashed line) converges. $\{p_0 = 1, M = 1, T = 50, FT(I_0^{FT} = 0.3, K = 1)\}$.

(b) I_0^{FT} above a specific threshold: I_t^{FT} (dashed line) diverges. $\{p_0 = 1, M = 1, T = 13, FT(I_0^{FT} = 0.4, K = 1)\}$.

Figure 4: Investment of feedback traders is indicated with a dashed line, development of the asset price p_t is indicated with a solid line; note the different scaling of the vertical axes.

assumption and instead consider an interactive market model, as introduced by Baumann (2015a), as we want to examine the price's behavior under heterogeneous agents.

By transforming Equation (7), the feedback trader's investment rate, we see that linear feedback traders follow a strategy that can be written as

$$I_t^{FT} = I_{t-1}^{FT} + K \cdot I_{t-1}^{FT} \cdot \frac{p_i - p_{i-1}}{p_{i-1}} \tag{12}$$

$$= I_{t-1}^{FT} + K \cdot I_{t-1}^{FT} \cdot (e^{M^{-1}\Delta I_{t-1}} - 1) \tag{13}$$

which leads to an investment of

$$I_t^{FT} = I_{t-1}^{FT} + K \cdot I_{t-1}^{FT} \cdot (e^{M^{-1}\Delta I_{t-1}^{FT}} - 1), \tag{14}$$

when only one trader, the linear feedback trader, is acting on the market.

To sum up, the idea behind the linear feedback trading strategy is that money can be made by following the price trend.

2.4 Fundamentalists

As explained in Section 1, fundamentalists invest when the price is below the fundamental value $f_t > 0$ and disinvest when the price is above the fundamental value.[13] Thus, it is of particular interest how much fundamentalists invest or disinvest in the respective cases. For deterministic fundamental values f_t, i.e., the fundamental value is a function in t, one way of determining the investment rate is

$$\Delta I_t^{FU} = M \cdot \ln \frac{f_{t+1}}{p_t}. \tag{15}$$

In this case fundamentalists do not need to estimate the fundamental value because it is fixed and certain. Traders following the investment rule of Equation (15) could be called *strong fundamentalists* because their investment strategy could push the price back to its fundamental value at any time. If the strong fundamentalist is the only trader buying/selling at time t, then for any $p_t > 0$ and f_{t+1} it follows:

$$p_{t+1} = p_t \cdot e^{\ln \frac{f_{t+1}}{p_t}} \tag{16}$$

[13]If, for example, the fundamental value is below the asset price, fundamentalists conclude that the price will decrease in the long run, not directly in the next step. So they do not necessarily disinvest so much that their investment becomes negative, but they reduce their investment which is also a disinvestment.

$$= p_t \cdot \frac{f_{t+1}}{p_t} \tag{17}$$

$$= f_{t+1} \tag{18}$$

Section 2.5 presents the case of a fundamentalist trading based on a distorted fundamental value. It turns out, however, that this distortion does not affect the general behavior of the market model.

2.5 Expectations and Noise

Some types of traders, for example, informed speculators (De Long et al., 1990b), base their trading decisions on rational expectations. Is this the case for feedback traders and fundamentalists?

In general, for feedback traders and trend followers, the answer is "no," as they only assume the existence of a trend. For example, based on the current slope of asset price development ($p_t - p_{t-1}$) they forecast the future direction of the asset. However, fundamentalists are assumed to have rational expectations (see, e.g., Drescher and Herz, 2012). Generally, they pursue the strategy

$$\Delta I_t^{FU} = M \cdot \ln \frac{\mathbb{E}[f_{t+1}|f_t]}{p_t}. \tag{19}$$

Even a casual observation of real markets makes it clear that price fluctuations are seldom purely rational. There is always noise and uncertainty in the market, a factor considered essential by many economists (see, e.g., Black, 1986; De Long et al., 1990a). Some reasons for noise include that traders make mistakes, trade on unreliable (noisy) information, or simply enjoy trading and are not overly concerned with being rational about it.

Here, we do not assume that traders are making mistakes, as this would lead to completely unexpected, unsystematic behavior. Furthermore, both feedback traders and fundamentalists do follow a specified strategy. Thus the only way noise could enter the market is through noisy information. However, the traders' investments as well as the price, announced by the market maker (see Figure 1), are not distorted. The only information that could be noisy is that about the fundamental value. In this case, the fundamentalist has to estimate f_{t+1} at time t and trade according to Equation (19). Since it is unreasonable that $|f_{t+1} - \mathbb{E}[f_{t+1}]|$ becomes arbitrary large but exploding prices imply $|p_t - f_t| \to \infty$, the effects of noisy information do not play a decisive role.

Therefore, we a priori consider f_t a deterministic fundamental value in the presented work.

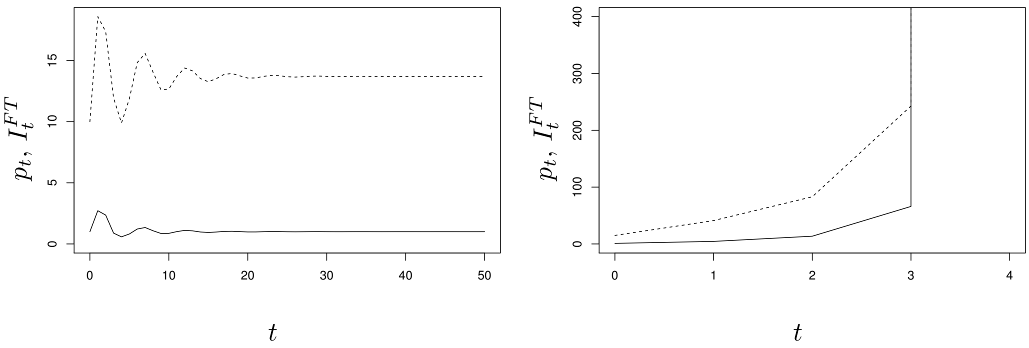
3 Proof of Limitations of Fundamentalists' Stabilizing Effects

In this section we demonstrate, analytically and mathematically rigorously, that fundamentalists are not always able to stabilize markets through their trading actions. We inductively prove, in contrast to simulations, that effects of linear feedback traders dominate those of fundamentalists and destabilize markets.

Since we habe already defined the pricing model (Equation (1)) and the traders, the next task is to check whether fundamentalists defined according to Equation (15) are able to stabilize the price when trading simultaneously on the market with linear feedback traders following Equations (3) and (4). To simplify the notation, we set $f_t \equiv 1$. This is one special case, but if we can show the destabilizing effects of feedback traders' investment strategy for this case, it will prove that fundamentalists do not always have market stabilizing effects. The proof proceeds without using technical trading restrictions, for example, limits on feedback traders' investment amount.

These two trader types are suitable for analyzing the problem because if it turns out that prices explode for appropriately chosen parameters I_0^{FT} and K of linear feedback traders even when acting on a market with fundamentalists who are employing an investment strategy that could bring prices close to the fundamental value at every point of time, it will be strong evidence that chartists' rules, in this case the linear feedback strategy are able to overcome the effects of strong fundamentalists. Why it is enough to consider only linear feedback traders and fundamentalists and no other type of traders, some of which are presented by Baumann and Baumann (2015), becomes obvious when taking into consideration that if feedback traders' investment goes to infinity which means prices explode, then also the absolute value of fundamentalists' investment goes to infinity. Thus, compared to the exploding investments of feedback traders and fundamentalists, the relatively small investment[14] of other possible traders may be neglected at least for our

[14]Trend followers invest a lot when prices rise strongly and fundamentalist disinvest a lot when price greatly exceeds the fundamental value, i.e., the investment of trend followers goes against infinity and that of fundamentalists goes against minus infinity. For traders who neither predicate their investment on the distance of fundamental value and price



(a) Price and feedback trader's investment converging, i.e., fundamentalists' effects predominate; parameters $\{p_0 = 1, M = 10, T = 50, FU(f_t \equiv 1), F(I_0^{FT} = 10, K = 0.5)\}$.

(b) Price and feedback trader's investment diverging, i.e., feedback traders' effects predominate; parameters $\{p_0 = 1, M = 10, T = 3, FU(f_t \equiv 1), F(I_0^{FT} = 15, K = 0.5)\}$.

Figure 5: Two typical situations in a market involving feedback traders and fundamentalists (notice: T differs in the two figures for purposes of readability).

analysis.[15]

Simulations reveal two typical price developments (see Figures 5a and 5b). In Figure 5a, fundamentalists' effects predominate and the price stabilizes around the fundamental value. In Section 4, this converging investment effect is shown when $K = 1$, where K is the feedback parameter from Equation (3),

---

nor on the slope of the price it is unreasonable that their investment goes against (minus) infinity.

[15]For moving average traders (MA) and noise traders (NO), both presented by Baumann and Baumann (2015) and needed for a valid market model as also shown by Baumann and Baumann (2015), usually $|\Delta I_t^{MA}| \leq \Delta I^*$ and $P(|\Delta_t^{NO}| > B) \to 0$ holds for $B \to \infty$. Thus, only $|\Delta I_t^{FT}|$ and $|\Delta I_t^{FU}|$ can become arbitrarily large which is the only interesting contribution for bubble analysis.

and a specific limit value for I_0^{FT} is computed. In Figure 5b, however, market development is not that obvious. At first glance, the figure might suggest that prices explode. But as the simulation software reaches its limits, it becomes unclear whether or not prices level out in these simulation scenarios. We therefore need an analytical examination. In cases like those shown in the simulated Figure 5b, the proposition of Theorem 1 determines with certainty whether the investment of feedback traders is in fact exploding, or whether this only looks to be the case due to simulation insufficiencies and the investment will eventually stabilize, but with a greater amplitude, for example, as in Figure 5a.

To simplify the expressions in the model, we assume that $f_t \equiv 1$ and $p_0 = 1$ in all upcoming equations. This choice is just one possible scaling but does not change the model's dynamics in general. It holds:[16]

$$\begin{aligned}
\Delta I_t^{FU} &= M \cdot \ln \frac{f_{t+1}}{p_t} && (20)\\
&= -M \cdot \ln e^{M^{-1} I_{t-1}} && (21)\\
&= -I_{t-1} && (22)\\
&= -I_{t-1}^{FT} - I_{t-1}^{FU} && (23)\\
\Rightarrow \; I_t^{FU} &= -I_{t-1}^{FT} && (24)\\
\Rightarrow \; \Delta I_t^{FU} &= -\Delta I_{t-1}^{FT} && (25)
\end{aligned}$$

[16]We define a process α_t as $(\alpha_t)_{t\in\mathbb{Z}} \subset \mathbb{R}$ with $\alpha_t = 0 \; \forall t < 0$. Furthermore, we define the Δ-operator as $\Delta^k \alpha_t := \Delta^{k-1}\alpha_t - \Delta^{k-1}\alpha_{t-1}$, $\Delta^1 \alpha_t := \Delta\alpha_t = \alpha_t - \alpha_{t-1}$, and $\Delta^0 \alpha_t := \alpha_t$. A price process p_t is strictly positive, i.e., $(p_t)_t > 0$ for all $t \geq 0$.

With this, we can specify Equation (13), which describes the investment of the feedback traders:

$$\Delta I_t^{FT} = K \cdot I_{t-1}^{FT}(e^{M^{-1}(\Delta I_{t-1}^{FT} + \Delta I_{t-1}^{FU})} - 1) \tag{26}$$

$$= K \cdot I_{t-1}^{FT}(e^{M^{-1}(\Delta I_{t-1}^{FT} - \Delta I_{t-2}^{FT})} - 1) \tag{27}$$

$$= K \cdot I_{t-1}^{FT}(e^{M^{-1}\Delta^2 I_{t-1}^{FT}} - 1) \tag{28}$$

Theorem 1 tells us conditions for the feedback trader's investment I^{FT} and its derivatives for which prices explode. Note that the following implication holds:

$$\Delta^k I_{t-1}^{FT} > a \ \wedge \ \Delta^{k+1} I_t^{FT} > b \ \Rightarrow \ \Delta^k I_t^{FT} > a + b. \tag{29}$$

We obtain this directly from the definition of the delta operator which is equivalent to

$$\Delta^k I_t^{FT} = \Delta^{k+1} I_t^{FT} + \Delta^k I_{t-1}^{FT}. \tag{30}$$

Theorem 1. For the investment of the positive linear feedback trader (Equation (3)) interacting with a strong fundamentalist (Equation (15)) on the market model (Equation (1)), under conditions

$$\Delta^3 I_t^{FT} > M \cdot \ln 2, \tag{31}$$

$$\Delta^2 I_t^{FT} > M \cdot \ln 2 \cdot \max\left\{1, K^{-1}\right\}, \tag{32}$$

$$\Delta I_{t-1}^{FT} > 0, \text{ and} \tag{33}$$

$$I_{t-2}^{FT} > 0 \tag{34}$$

it follows that

$$\Delta^k I_{t+1}^{FT} > M \cdot \ln 2 \;\forall k \in \{0,1,2,3\} \tag{35}$$

and

$$\Delta^2 I_{t+1}^{FT} > M \cdot \ln 2 \cdot K^{-1}. \tag{36}$$

This means, the feedback trader's investment, the slope of investment, the curvature of investment, and the increase of the curvature of the investment are strictly greater than $M \cdot \ln 2$ for all $t \geq t^*$ for some t^*. All in all, this is a fast exploding investment, which leads to an equally quickly exploding price.

$$p_{t+1} = p_t \cdot e^{M^{-1}\cdot(\Delta I_t^{FU} + \Delta I_t^{FT})} \tag{37}$$

$$= p_t \cdot e^{\ln \frac{f_{t+1}}{p_t}} \cdot e^{M^{-1}\cdot \Delta I_t^{FT}} \tag{38}$$

$$= f_{t+1} \cdot e^{M^{-1}\cdot \Delta I_t^{FT}} \tag{39}$$

As an interpretation, note that under Equation (24), fundamentalists always respond one period later with the inversed investment of feedback traders.

In other words, the feedback trader's investment increases, the rate of increase increases, and the rate of this growth increases. Furthermore, all of these growth rates are bounded from below. Since the fundamentalist's investment is minus the investment of the feedback trader from one period before the ratio of the (dis-)invested amounts is strictly increasing, that is the feedback trader's exploding effect predominates the fundamentalist's stabilizing one. Theorem 1 is proven by induction in the following.

Proof. It is enough to prove Equation (35) for $k = 3$ as all other inequalities can then be derived from Equation (29), respectively, Equation (30).

$$
\begin{aligned}
\frac{1}{K}\Delta^3 I^{FT}_{t+1} &= \frac{1}{K}(\Delta^2 I^{FT}_{t+1} - \Delta^2 I^{FT}_{t}) && (40)\\
&= \frac{1}{K}(\Delta I^{FT}_{t+1} - 2\Delta I^{FT}_{t} + \Delta I^{FT}_{t-1}) && (41)\\
&= I^{FT}_{t}(e^{M^{-1}\Delta^2 I^{FT}_{t}} - 1) && (42)\\
&\quad - 2I^{FT}_{t-1}(e^{M^{-1}\Delta^2 I^{FT}_{t-1}} - 1) && (43)\\
&\quad + I^{FT}_{t-2}(e^{M^{-1}\Delta^2 I^{FT}_{t-2}} - 1) && (44)\\
&= (I^{FT}_{t-2} + \Delta I^{FT}_{t-1} + \Delta I^{FT}_{t})(e^{M^{-1}\Delta^2 I^{FT}_{t}} - 1) && (45)\\
&\quad - 2(I^{FT}_{t-2} + \Delta I^{FT}_{t-1})(e^{M^{-1}\Delta^2 I^{FT}_{t-1}} - 1) && (46)\\
&\quad + I^{FT}_{t-2}(e^{M^{-1}\Delta^2 I^{FT}_{t-2}} - 1) && (47)\\
&= I^{FT}_{t-2}(e^{M^{-1}\Delta^2 I^{FT}_{t}} - 1) && (48)\\
&\quad + \Delta I^{FT}_{t-1}(e^{M^{-1}\Delta^2 I^{FT}_{t}} - 1) && (49)\\
&\quad + \Delta I^{FT}_{t}(e^{M^{-1}\Delta^2 I^{FT}_{t}} - 1) && (50)\\
&\quad - 2I^{FT}_{t-2}(e^{M^{-1}\Delta^2 I^{FT}_{t-1}} - 1) && (51)\\
&\quad - 2\Delta I^{FT}_{t-1}(e^{M^{-1}\Delta^2 I^{FT}_{t-1}} - 1) && (52)\\
&\quad + I^{FT}_{t-2}(e^{M^{-1}\Delta^2 I^{FT}_{t-2}} - 1) && (53)\\
&= I^{FT}_{t-2}(e^{M^{-1}\Delta^2 I^{FT}_{t}} - 2e^{M^{-1}\Delta^2 I^{FT}_{t-1}} + e^{M^{-1}\Delta^2 I^{FT}_{t-2}}) && (54)\\
&\quad + 2\Delta I^{FT}_{t-1}(e^{M^{-1}\Delta^2 I^{FT}_{t}} - e^{M^{-1}\Delta^2 I^{FT}_{t-1}}) && (55)\\
&\quad + \Delta I^{FT}_{t}(e^{M^{-1}\Delta^2 I^{FT}_{t}} - 1) && (56)
\end{aligned}
$$

We evaluate these summands separately:

$$(55) = 2\Delta I_{t-1}^{FT}(e^{M^{-1}\Delta^2 I_{t-1}^{FT} + M^{-1}\Delta^3 I_t^{FT}} - e^{M^{-1}\Delta^2 I_{t-1}^{FT}}) \quad (57)$$

$$= 2\Delta I_{t-1}^{FT} e^{M^{-1}\Delta^2 I_{t-1}^{FT}}(e^{M^{-1}\Delta^3 I_t^{FT}} - 1) \quad (58)$$

$$> 2\Delta I_{t-1}^{FT} e^{M^{-1}\Delta^2 I_{t-1}^{FT}}(2 - 1) \quad (59)$$

$$> 0 \quad (60)$$

$$(56) = (\Delta I_{t-1}^{FT} + \Delta^2 I_t^{FT})(e^{M^{-1}\Delta^2 I_t^{FT}} - 1) \quad (61)$$

$$> 0 + M \cdot \max\{\ln 2, \frac{\ln 2}{K}\} \quad (62)$$

$$> M \cdot \frac{\ln 2}{K} \quad (63)$$

$$(54) = I_{t-2}^{FT}(e^{M^{-1}\Delta^2 I_{t-2}^{FT} + M^{-1}\Delta^3 I_{t-1}^{FT} + M^{-1}\Delta^3 I_t^{FT}} \quad (64)$$

$$- 2e^{M^{-1}\Delta^2 I_{t-2}^{FT} + M^{-1}\Delta^3 I_{t-1}^{FT}} + e^{M^{-1}\Delta^2 I_{t-2}^{FT}}) \quad (65)$$

$$= I_{t-2}^{FT} e^{M^{-1}\Delta^2 I_{t-2}^{FT}}(e^{M^{-1}\Delta^3 I_{t-1}^{FT}}(e^{M^{-1}\Delta^3 I_t^{FT}} - 2) + 1) \quad (66)$$

$$> I_{t-2}^{FT} e^{M^{-1}\Delta^2 I_{t-2}^{FT}}(e^{M^{-1}\Delta^3 I_{t-1}^{FT}}(2 - 2) + 1) \quad (67)$$

$$= I_{t-2}^{FT} e^{M^{-1}\Delta^2 I_{t-2}^{FT}} \quad (68)$$

$$> 0 \quad (69)$$

As a result, we obtain

$$\Delta^3 I_{t+1}^{FT} > M \cdot \ln 2. \quad (70)$$

□

That the conditions for the endogenous variables $I_{t-2}^{FT}, \Delta I_{t-1}^{FT}, \Delta^2 I_t^{FT}, \Delta^3 I_t^{FT}$

may be fulfilled for some t (and some parameter assignment) is shown in Table 1 in which the investment development of the feedback trader and its derivatives are listed for $I_0^{FT} = 15$, $K = 0.5$, and $M = 10$. In short, there are exogenous variables that lead to price explosion. This demonstrates that feedback traders' effects are able to overcome fundamentalists' effects.

On the other hand, Table 2 sets out a situation where price would explode when only feedback traders are acting on the market. Equations (8)–(10) hold for the feedback traders, so, according to Baumann (2015a), their investment causes a bubble even in the absence of any other traders. However, if fundamentalists enter the market price explosion is prevented, as the investment rates tend to 0 at time $t = 80$ in Table 2. Clearly, the conditions of Theorem 1 for feedback traders are not satisfied.

In summary, even a strong fundamentalist investment rule, that is a strategy without any restrictions and involving a possibly infinitely large investment amount, is not able to stabilize the market when a trader using very simple linear feedback strategy with an adequate initial investment is acting on the market, too. Market failures can happen, prices may explode, and the investment behavior of strong fundamentalists cannot prevent this.

4 Further Results for $K = 1$

To this point, we have demonstrated that the pricing model described in Equation (1) together with a quite simple chartist rule can create a financial bubble even when fundamentalists are acrive in the market. We now discuss some further results and special features of the market model when the linear feedback trader is assumend to ba reasonable all-in feedback trader, that is, one who invests all of his gain but nothing more, i.e., $K = 1$. Formula simplifications for the feedback trader's investment are given for two specific cases. The first is a market with only a linear feedback trader; the second is a market with a linear feedback trader and a fundamentalist. Furthermore, as mentioned in Section 3, if existing, the limits of the investment are calculated. For simplicity we set $M = 1$.

4.1 Case 1: All-In Linear Feedback Trader

In the first case with only one linear feedback trader, Equation (14) simplifies to:

$$I_t^{FT} = I_{t-1}^{FT} + I_{t-1}^{FT} \cdot (e^{\Delta I_{t-1}^{FT}} - 1) \tag{71}$$

$$= I_{t-1}^{FT} \cdot e^{\Delta I_{t-1}^{FT}} \tag{72}$$

$$= I_0^{FT} \cdot e^{\Delta I_0^{FT}} \cdots e^{\Delta I_{t-1}^{FT}} \tag{73}$$

$$= I_0^{FT} \cdot e^{I_{t-1}^{FT}} \tag{74}$$

In the case that I_t^{FT} converges to some value it follows

$$\lim_{t \to \infty} I_t^{FT} = -lw(-I_0^{FT}), \tag{75}$$

where lw denotes the Lambert-W-function.[17]

[17]The Lambert-W-function is the inverse function of $f(x) = x \cdot e^x$.

4.2 Case 2: All-In Linear Feedback Trader and Fundamentalist

In the second case, to arrive at the simplified feedback trader's investment amount, we need to rewrite Equations (13) and (28) to take into consideration the fundamentalist's investment amount:

$$I_t^{FT} = I_{t-1}^{FT} + I_{t-1}^{FT}(e^{\Delta^2 I_{t-1}^{FT}} - 1) \tag{76}$$

$$= I_0^{FT} \cdot e^{\Delta^2 I_0^{FT}} \cdots e^{\Delta^2 I_{t-1}^{FT}} \tag{77}$$

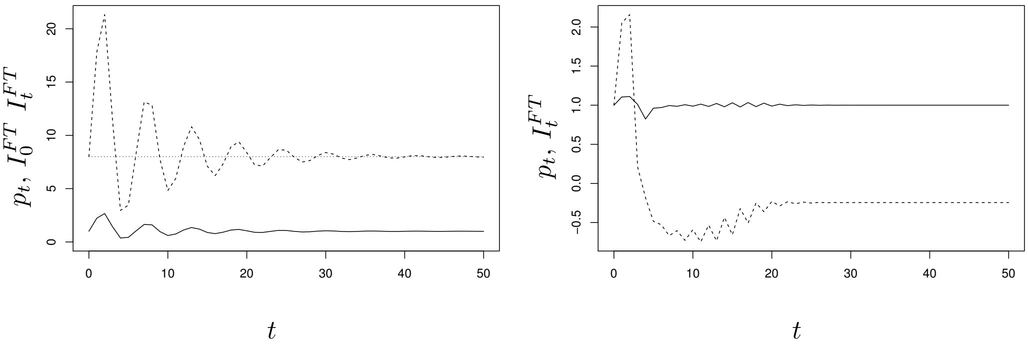
$$= I_0^{FT} \cdot e^{\Delta I_{t-1}^{FT}} \tag{78}$$

In the case where only linear feedback traders and fundamentalists are acting on the market—which is also the setting of the main result in Section 3—with $K = 1$ it is easy to calculate the limit of the feedback trader's investment, assuming that one exists (see, e.g., Figure 6): If $I_t^{FT} \to c \in \mathbb{R} \Rightarrow \Delta I_t^{FT} \to 0$ and by using Equation (78):

$$\lim_{t\to\infty} I_t^{FT} = I_0^{FT} \tag{79}$$

Figure 6a illustrates the outcome. Note that, due to Equation (28) (market with feedback trader and fundamentalist), for $K \in (0, 1]$ we have $I_t^{FT} \geq 0$, i.e., the linear feedback trader is a long trader. In contrast, for $K > 1$, negative investments I_t^{FT} may occur, as shown in Figure 6b.[18]

[18]When rewriting Equation (12) to $I_t^{FT} = I_{t-1}^{FT}\left(1 + K \cdot \frac{p_t - p_{t-1}}{p_{t-1}}\right)$ and noting that the return on investment is always bigger than minus one, it follows that for $K > 0$ it holds $I_t^{FT} > 0$, too.



(a) Feedback trader's investment converging to the initial investment I_0^{FT}; parameters $\{p_0 = 1, M = 10, T = 50, FU(f_t \equiv 1), F(I_0^{FT} = 8, K = 1)\}$.

(b) Negative investment of feedback trader; parameters $\{p_0 = 1, M = 10, T = 50, FU(f_t \equiv 1), F(I_0^{FT} = 1, K = 10)\}$.

Figure 6: Two specific situations in a market model involving feedback traders and fundamentalists.

5 Conclusion

Our analysis indicates that trend followers may cause price explosions regardless of fundamentalists' investment decisions. Specifically, Theorem 1 and its proof analytically show that a fundamentalist's investment strategy, that is a strategy that pushes prices toward their fundamental values, can be insufficient to dominate linear feedback trading strategies. However, the potential for feedback traders' to create a bubble appears to be lower (Equations (31)–(34)) when fundamentalists are active in the market (cf. Equations (8)–(10)). Although the results indicate that fundamentalists have a stabilizing effect, this effect is limited up to some threshold value (cf. Table 2).

The analysis also shows that for identical investment decisions price movements are more volatile at higher price levels compared to at lower price levels. Thus, even this simply constructed market model is able to capture certain market phenomena.

For future research, it might be interesting to allow for different proportions of fundamentalists and trend followers and therewith analyze the market behavior depending on this proportion. Also adding sentimentalists, always adapting the better working strategy of the other traders, might be an interesting generalization.

Given our results and the fact that financial bubbles are associated with high economic costs an important question arises: Seeing as fundamentalists do not appear to be an adequate market stabilizing force, is there another type of trader, perhaps the market maker, that would be able to stabilize prices in a market-appropriate way and, if so, what would such a trader look like?

Generally, our analysis supports the view that intervention measures or at least some kind of incentive system is necessary to stabilize asset markets and prevent financial bubbles. Such measures could for example be the direct intervention of some control authority, progressive transaction costs, or trading restrictions.

Acknowledgments

The authors thank Lars Grüne, Chair for Applied Mathematics, and Bernhard Herz, Chair of International Economics and Finance, both with University of Bayreuth, for discussing the results, supervising the work, and revising the paper.

The authors also thank Mrs. Deborah Willow, WillowHut Publishing Services, for revising the work.

Furthermore, the authors wish to thank the audience of the XVIII. Workshop on Quantitative Finance 2017, especially Mr. Adeyemi Adewale Sonubiand, with Università degli Studi di Milano-Bicocca, who was the respective discussant, and the audience of the XVI. Annual European Economics and Finance Society Conference 2017.

	I_t^{FT}	ΔI_t^{FT}	$\Delta^2 I_t^{FT}$	$\Delta^3 I_t^{FT}$
$t = 0$	15	15	15	15
$t = 1$	41.112668	26.112668	11.112668	-3.88733197
$t = 2$	83.0106859	41.8980179	15.7853499	4.67268186
$t = 3$	242.716956	159.70627	117.808252	102.022902
$t = 4$	15864296.3	15864053.3	15863893.8	15863776

Table 1: The boxed table entries fulfill the conditions of Theorem 1 for $t = 3$ for which prices explode; market parameters are as in Figure 5b.

	I_t^{FT}	ΔI_t^{FT}	$\Delta^2 I_t^{FT}$	$\Delta^3 I_t^{FT}$
$t = 0$	0.4	0.4	0.4	0.4
$t = 1$	0.59672988	0.19672988	-0.2032701	-0.6032701
...				
$t = 5$	0.39734101	0.02131807	0.05226206	-0.069616
...				
$t = 80$	≈0.4	≈0	≈0	≈0

Table 2: The table shows a situation where price would explode without fundamentalists but is stabilized by them. The investment parameters are the same as for Figure 4b where prices explode. The boxed cells fulfill the conditions required by Equations (8)-(10).

References

B. R. Barmish. On performance limits of feedback control-based stock trading strategies. In *IEEE American Control Conference (ACC)*, pages 3874–3879, 2011. doi: 10.1109/ACC.2011.5990879.

B. R. Barmish and J. A. Primbs. On arbitrage possibilities via linear feedback in an idealized Brownian motion stock market. In *IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, pages 2889–2894, 2011. doi: 10.1109/CDC.2011.6160731.

B. R. Barmish and J. A. Primbs. On market-neutral stock trading arbitrage via linear feedback. In *IEEE American Control Conference (ACC)*, pages 3693–3698, 2012. doi: 10.1109/ACC.2012.6315392.

B. R. Barmish and J. A. Primbs. Stock trading via feedback control. In *Encyclopedia of Systems and Control*, pages 1–10, 2014. doi: 10.1007/978-1-4471-5102-9_131-1.

B. R. Barmish and J. A. Primbs. On a new paradigm for stock trading via a model-free feedback controller. *IEEE Transactions on Automatic Control*, PP(99):1–15, 2015. ISSN 0018-9286. doi: 10.1109/TAC.2015.2444078.

J. d. G. Batista, D. Massaro, J.-P. Bouchaud, D. Challet, and C. Hommes. Do investors trade too much? a laboratory experiment. *arXiv preprint arXiv:1512.03743*, 2015.

M. H. Baumann. Effects of linear feedback trading in an interactive market model. In *American Control Conference (ACC)*, pages 3880–3885, 2015a. doi: 10.1109/ACC.2015.7171935.

M. H. Baumann. On stock trading via feedback control when underlying stock returns are discontinuous. University of Bayreuth, 2015b. URL `https://eref.uni-bayreuth.de/21282/`. Preprint; submitted to IEEE Transactions on Automatic Control (TAC).

M. H. Baumann and M. Baumann. Stylized facts in a phenomena emulating heterogeneous agent model. 2015. In preparation.

M. H. Baumann and L. Grüne. Positive expected feedback trading gain for all essentially linearly representable prices. 2015. URL `https://eref.uni-bayreuth.de/22592/`. Preprint.

F. Black. Noise. *Journal of Finance*, 41(3):529–543, 1986. doi: 10.1111/j.1540-6261.1986.tb04513.x.

M. Covel. *Trend Following: How Great Traders Make Millions in Up or Down Markets*. FT Press, 1st edition, 2004. ISBN 978-0-131-44603-7.

J. B. De Long, A. Shleifer, L. H. Summers, and R. J. Waldmann. Noise trader risk in financial markets. *Journal of Political Economy*, 98(4):703–738, 1990a. URL `http://www.jstor.org/stable/2937765`.

J. B. De Long, A. Shleifer, L. H. Summers, and R. J. Waldmann. Positive feedback investment strategies and destabilizing rational speculation. *Journal of Finance*, 45(2):379–395, 1990b. doi: 10.1111/j.1540-6261.1990.tb03695.x.

C. Drescher and B. Herz. Monetary shocks in bounded efficient financial markets with bounded rational agents. *Discussion Paper 09-12, Wirtschaftswissenschaftliche Diskussionspapiere*, 2012. ISSN 1611-3837.

A. Gaunersdorfer and C. H. Hommes. A nonlinear structural model for volatility clustering. CeNDEF Working Papers 05-02, Universiteit van Amsterdam, Center for Nonlinear Dynamics in Economics and Finance, 2005.

B. Graham, D. L. F. Dodd, and S. Cottle. *Security Analysis.* New York: McGraw-Hill, 1934.

C. H. Hommes. Modeling the stylized facts in finance through simple nonlinear adaptive systems. *Proceedings of the National Academy of Sciences of the United States of America*, 99(4):7221–7228, 2002.

C. H. Hommes. Heterogeneous agent models in economics and finance. In L. Tesfatsion and K. L. Judd, editors, *Handbook of Computational Economics, Volume 2*, pages 1109–1186. Elsevier, 2006a.

C. H. Hommes. Interacting agents in finance. *Tinbergen Institute Discussion Paper No. 06-029/1*, 2006b. doi: http://dx.doi.org/10.2139/ssrn.894221.

T. Lux. Herd behaviour, bubbles and crashes. *Economic Journal*, 105(431): 881–896, 1995.

T. Lux. The socio-economic dynamics of speculative markets: interacting agents, chaos, and the fat tails of return distributions. *Journal of Economic Behavior & Organization*, 33(2):143–165, 1998.

T. Lux and M. Marchesi. Scaling and criticality in a stochastic multi-agent model of a financial market. *Nature*, 397:498–500, 1999.

T. Lux and M. Marchesi. Volatility clustering in financial markets: A microsimulation of interacting agents. *International Journal of Theoretical and Applied Finance*, 3(4):675–702, 2000.

C. M. Reinhart and K. Rogoff. *This Time is Different: Eight Centuries of Financial Folly.* Princeton University Press, 2009.