Beating the Market?*

Michael Heinrich Baumann†

Universität Bayreuth, Germany
michael.baumann@uni-bayreuth.de

August 2018

Abstract The efficient market hypothesis is highly discussed—supported and criticized—in economic literature. In its weakest form, it states that there are no price trends. When weakening the non-trending assumption only a little to arbitrary short, small, and fully unknown trends, I mathematically prove, for a specific class of control-based trading strategies, positive expected gains. Adjustments for risk and comparisons with buy-and-hold strategies do not satisfactorily solve the problem. In addition, in an exemplary backtesting study, when transaction costs and bid-ask-spreads are taken into account, I still observe, on average, positive gains. These strategies are model-free, i.e., a trader neither has to estimate market parameters as the trend's sign nor has to think about predictable patterns, etc. In this work, I bring together the economists' view on efficient markets and the engineers' view on feedback trading.

*This paper was presented at the Graduiertenseminar der VWL-Lehrstühle der Universität Bayreuth 2018, Schney, Lichtenfels, Germany, at the XIX Workshop on Quantitative Finance (QFW; Università degli Studi Roma Tre) 2018, Rome, Italy, at the 22^{nd} Annual International Conference on Macroeconomic Analysis and International Finance (ICMAIF; University of Crete) 2018, Rethymno, Crete, Greece, at the Seventeenth Annual European Economics and Finance Society Conference (EEFS; City, University of London) 2018, London, England, and at the 6th International PhD Meeting in Economics (University of Macedonia) 2018, Thessaloniki, Greece.

†The work of Michael H. Baumann was supported by Hanns-Seidel-Stiftung e.V. (HSS), funded by Bundesministerium für Bildung und Forschung (BMBF).

Keywords Technical Analysis; Efficient Market Hypothesis; Robust Positive Expectation Property; Simultaneously Long Short Trading; Control-based Trading Strategies

JEL codes C02, G11, G14

MSC 91G10, 91G99, 91B70

1 Introduction

While in the 1970s, the market efficiency hypothesis, i.e., the hypothesis that (specific) traders cannot expect excess returns, was highly accepted (Fama, 1965, 1970). Later on, it was highly criticized—yet also defended (Malkiel, 1989, 2005). Much of the criticism concerned so-called predictable patterns: for example, the January effect, i.e., high positive returns in the first two weeks of January. Defenders of the market efficiency hypothesis have several arguments against this, e.g., that patterns will self-destroy once published, or that small possible gains will vanish when trading costs have to be paid.

Additionally, the so-called joint hypotheses problem states that market efficiency and the used market model have to be tested, nearly always, simultaneously. That means when the test fails, no one knows whether the market is inefficient or whether the model used is insufficient. A second point of criticism of the criticism is the distinction between statistical inefficiency and economical inefficiency. The first means that one can construct a test for showing that there are, for example, predictable patterns. The second means that a trader has to be able to exploit this. And the last point used to defend the market efficiency hypothesis is that, even when one can construct a strategy with too high returns, e.g., by taking into account some external variables, it may be that these variables are better ratios for measuring risk. When introducing risk-adjusted returns, excess returns are no contradiction when they go hand in hand with excess risk.

In the next section, I will give a very short review of market efficiency, its criticism, and its defenses, i.e., the criticism of the criticism. Much of the discussion on market efficiency, technical trading, and beating the market follows the idea that a trader (i) has to find a predictable pattern, (ii) has to construct a trading strategy to exploit this pattern, and, (iii) has to test this new strategy against randomly selected broad index buy-and-hold strategies (Malkiel, 1973). However, a new strand of research—mainly in engineering sciences and mathematics—goes another way. In the view of the respective authors, task (i) can be skipped, allowing trading strategies to be constructed directly. These strategies usually are model-free and use neither predictions of patterns nor estimations of parameters. In short, and using the terminology of the control community: they are constructed to be robust against the price. Instead of task (iii), which relies on real market data, (performance) properties are proven mathematically. This way, the overfitting problem (cf. Bailey et al., 2014) is avoided.

While in the control literature, results on control-based trading strategies attract high attention, in the economical literature they are widely unknown. An aim of the work at hand is to review known results on a particular control-based strategy, the so-called Simultaneously Long Short (SLS) Strategy, to extend the results in different directions, and, finally, to bring them into the context of market efficiency, which is not adequately discussed in the control literature.

The paper is organized as follows: In Section 2, I discuss the literature on efficient markets as well as on feedback trading, and I explain SLS trading and the used market requirements. In Section 3, new results concerning SLS trading in a general market model with time-varying trends and volatilities are obtained, and risk as well as buy-and-hold strategies are considered. To account for trading costs and bid-ask-spreads—which are not considered in the analytical part of the work at hand—Section 4 is provided, in which I perform backtests on historic market data using bid-and-ask prices. After that, in Section 5, the standard SLS rule is generalized to the so-called discounted SLS rule, in which old data has less influence on the strategy. Finally, in Section 6, I discuss the results—especially in view of the efficient market hypothesis—and conclude the paper.

Since an aim of this work is to bring together economic ideas like market efficiency and control theoretic ideas like feedback trading, the one or the other might be uncommon to the reader. I will explain both views, that of the engineers and that of the economists, to both communities and discuss the differences that occur. After discussing the efficient market hypothesis, reviewing the feedback trading literature, and proving new properties of SLS strategies, all parts will be discussed together. At the end, since some of the performance properties (proven in the work at hand) do not fit with efficient markets, I discuss this puzzle.

2 Literature Review

In this section, I briefly discuss market efficiency, its criticism, and its defense. After that, I introduce the SLS rule, as it is known from the control literature, and state the most important results of this literature.

2.1 Review of Market Efficiency

In this section, I give a very brief overview of market efficiency. Because there is a very broad literature on this topic, and there are also a lot of very good and famous overviews, I refer the interested reader to these overviews (e.g., Fama, 1991; Malkiel, 2003). At the end of this work, in addition to the definition and discussion of market efficiency, I discuss some topics where definitions are not clear—focused on the discussion of the SLS strategy.

In its *strong* version, market efficiency states that all information is reflected in the price. That means no sophisticated trader—even no insider who has private information—performs on average better than a simple buy-and-hold trader. Insiders have access to all private and all public information. Public information means fundamentals and past returns. The strong version means that when there is no change in the fundamental value, all price movements are fully random with no trend. Mathematically speaking, the price process is a random walk around its fundamental value. A little bit weaker and maybe closer to markets is the assumption that only almost all information is incorporated in the price. But the costs for getting the missing information and for trading the asset are higher than the possible gain of exploiting this information (Fama, 1991).

The *semi-strong* version of the market efficiency hypothesis states that all public information is reflected in the price. That means insider trading may be profitable, which is widely accepted. For example, the findings on the effects of Value Line rank changes are a sign that insider trading may be profitable (Stickel, 1985) as summarized by Fama (1991). However, all public information is immediately incorporated in the asset price. The word *immediately* has to be understood in an averaged sense, i.e., markets may overreact or underreact to new information, and markets may reflect information too early or too late, but on average all these effects balance out (Fama, 1995). In other words, fundamental value analysis, i.e., trying to calculate the fundamental or intrinsic value (the real value), is on average not profitable at all because an asset's actual price is at any point in time the best estimate for the fundamental value (based on public information). Fundamentalists can make a profit when they find relevant information faster and better rate the effects to the fundamental values under analysis. Thus all fundamentalists do their best to be as fast and as accurate as possible—thereby adjusting prices instantaneously to the intrinsic values. Since no one knows who is the fastest and the best, on average fundamentalists cannot expect excess gains. Note that fundamentalists have access only to public information, i.e., to fundamentals and past returns.

Last, the *weak* version of market efficiency states that insider trading as well as fundamental analysis may be profitable, but technical analysis is not. Technical traders, who are also called technical analysts or chartists, have access only to past returns. However, the weak version states that no one can use past returns to predict future ones. Also, in this version, chartists, on average, cannot make money, markets have no memory, and patterns do not exist. Or, even a little bit weaker, when there exists a dependence of past and future returns, these anomalies are too small to be exploitable.

To sum up, in all versions of the market efficiency hypothesis, it is not possible, on average, for chartists to make money. Because there is a lot of literature on the profitability of technical trading, and there are numerous fund managers who rely on such strategies (Covel, 2004; Avramov et al., 2018), the task is always considered empirical. This means that chartist fund managers are challenged to provide statistics showing that their strategies outperform randomly selected buy-and-hold portfolios.

Hereafter, I summarize a selection of common criticism of the market efficiency hypothesis and state some arguments by the defenders of the hypothesis against these criticisms. One strand of criticism of the market efficiency hypothesis—actually, of its weak version—relies on predictable patterns. With statistical or data science methods, patterns were found, i.e., an on the average recurring behavior of stock market prices: the Monday effect (lower returns on Mondays, Cross (1973); French (1980)); the month effect (higher returns the last day of the month, Ariel (1987)); the holiday effect (higher returns on the day before a holiday, Ariel (1990)); and the most famous, the January effect (higher returns in January and even higher returns in the first five days of January, Keim (1983); Roll (1983)).

However—following Malkiel (2003)—predictable patterns will self-destroy once published. If the January effect existed, traders would buy on the last days of December and sell at the very beginning of January. That means the pattern would move a few days. Observing this, traders would buy and sell again a few days earlier. And so on. In the end, the January effect would be destroyed. A second attack on this strand of criticism is that the effects of (predictable) patterns are too small to exploit (Lakonishok and Smidt, 1988), especially when trading costs are considered. This last argument can be generalized: Just because there is a statistical inefficiency (i.e., predictability in returns shown by the use of data science methods) does not mean that a trader can profit from it—when the effect and the power of the statistic is small relative to additional costs. That means economical inefficiency must be shown by trading performance statistics.

Another strand of criticism of market efficiency is that stock returns may be predictable using some external variables, for example: dividend yields (D/P, Rozeff (1984); Shiller (1984)), earning per price ratios (E/P, Campbell and Shiller (1988)), or the firm's size (Banz, 1981). This is an attack on the semi-strong version of the efficient market hypothesis. But, as summarized by Fama (1991), these dependencies are either too small to exploit (especially when trading costs are taken into account) or—like in the case of the size effect—they have another reason: Taking into account some external variables with predictive power may just mean that these variables are better ratios for measuring risk. As mentioned above, the definition of market efficiency is not clear at all. Despite the statistical inefficiency vs. economical inefficiency problem, one can find statements in the literature, like *traders cannot expect excess returns* and *traders can only expect excess returns when they accept excess risk.* Often, the term *risk-adjusted gains* is used. Here, the next problem arises: How does one measure risk? Sometimes the Capital Asset Pricing Model's (CAPM's) β—or the standard deviation is used. I will come back to this problem in Section 6.

Another problem is that all empirical findings concerning market efficiency might be the result of *data-dredging* (also known as p-hacking), i.e., the results might be found by data-mining techniques—searching for significant p-values without causality or hypothesis. That means when one has enough data and is doing many, many tests, the probability of finding an anomaly is high, which does not prove inefficiency. The problem of p-hacking concerns the efficient market hypothesis in all its versions. Some studies indicate that there are (with constant fundamentals) long-term trends (possibly sinusodial) (Granger and Morgenstern, 1962; Saad et al., 1998) that do not face the problem of p-hacking. However, these studies have to deal with the issue that the trends may be not exploitable.

Additionally, there is the *joint hypotheses problem,* which states that market efficiency can (almost) always be tested only when simultaneously using a market model. A consequence is that when a test fails, no one can say whether the market efficiency hypothesis is wrong, or whether the used market model is insufficient. Since the joint hypotheses problem is a very strong argument, one that also concerns all versions of the efficient market hypothesis, the task for this work is to use no market model or at least a model as general as possible.

Exceptions to the joint hypotheses problem are the so-called *event studies* (Fama et al., 1969). Event studies analyze how fast and to what extent stock prices adjust to announcements, i.e., to new public information. So, event studies lie in the field of the semi-strong form of the market efficiency hypothesis. It is shown that prices may overreact or underreact to new information, and that reactions may be too early or too late. However, the defenders argue that, on average, all these anomalies vanish.

And last, there is the *momentum effect*: assets that performed well over the last few months will do so over the next few months, and similar for bad assets (Jegadeesh and Titman, 1993, 2001; Fama and French, 1996, 2008). Criticizing the weak form of market efficiency—based on empirical and statistical methods—this effect is explainable only by behavioral economics. Moskowitz (2010) explains why it is reasonable that assets with high momentum also have high risk. Another point against the momentum effect is that it eventually vanishes and thus is hard to exploit.

As shown above, there is a broad variety of criticism and defense of the efficient market hypothesis. Most past criticism was empirical and, thus, had the p-hacking problem. Theoretical critics often use a specific market model that leads to the joint hypotheses problem. To overcome the joint hypotheses problem, the data-dredging problem, and the overfitting problem (Bailey et al., 2014)—i.e., the problem that technical strategies might use too much past information to have any power for predicting the future—in the analytic part of the work at hand I present some criticism of the efficient market hypothesis, some that is purely theoretical, i.e., mathematical, and I use neither past data nor any market model, except some very basic market requirements. Only in the exemplary backtesting in Section 4 do I use historic market data.

2.2 Simultaneously Long Short (SLS) Trading

There is a strand of research in the control literature that seemingly does not care about market efficiency. There, by use of feedback techniques sourced in engineering sciences and analyzed in applied mathematics, trading strategies that are robust against noisy prices p_t are created. This control theoretic way of thinking is different from classical finance: Neither fundamentals f_t are calculated nor price patterns are searched for estimating future returns $\mathbb{E}\left[\frac{p_{t+h}-p_t}{p_t}\right]$ because the strategies do not use estimations of future returns.

Traders relying on control-based trading strategies are called feedback traders. They calculate their investment, i.e., their net asset position, which is an *input* variable to the system, i.e., to the financial market or actually to the trader's portfolio, at every point in time, as a function of an *output* variable of the system, usually the gain. Next, I define the Simultaneous Long Short (SLS) strategy as used in the control literature and present the most important results concerning this strategy.

As mentioned above, a feedback trader ℓ computes at time t the investment $I^\ell(t)$ as a function of the trader's own gain $g^\ell(t)$ and—some would call it naive—of nothing else:

$$I^\ell(t) = F\left(g^\ell(t)\right)$$

Since the results from the literature to be presented next are obtained in different market models, some in discrete time (indicated by subscript t), some in continuous time (indicated by t in brackets), I will give the definition of the strategy for a stochastic model in continuous time, which can easily be rewritten to other settings. The trader's gain is calculated by use of the investment and the return on investment:

$$g^\ell(t) := \int_0^t I^\ell(\tau) \cdot \frac{dp(\tau)}{p(\tau)} := \int_0^t \left(\frac{I^\ell}{p}\right)(\tau-)dp(\tau)$$

The integral is an Itō-Integral. The price process has to be a càdlàg semimartingale, which by construction makes the trading strategy $\left(\frac{I^\ell}{p}\right)(\tau-)$ a locally bounded, predictable process. The big question that has to be answered is how to choose function F. One possibility for F is the so-called *linear long* feedback trading rule:

$$I^L(t) = I_0^L + K^L g^L(t),$$

where $I_0^L > 0$ is the initial investment in the linear long rule and $K^L > 0$ is the so-called feedback parameter. It is easy to see that the linear long feedback trader is a long trader in continuous time when the price process is continuous, too. This means that this trader type makes money when prices rise and loses money when prices fall. When $K > 1$, this strategy is a trend-following rule, i.e., the trader buys when prices rise and vice versa. Since the required trading rule will be robust against variations in price, i.e., trend-following is a non-desired property, the linear long rule has to be modified. For this, the *linear short* feedback rule is defined first:

$$I^S(t) = -I_0^S - K^S g^S(t)$$

When time and price are continuous, this trader is a short investor who loses money when prices rise and earns money when prices fall ($I_0^S, K^S > 0$). When $K > 1$, this rule is an anti-trend-following strategy.

The simultaneously long short (SLS) rule is now simply defined as the superposition of the linear long and the linear short rule with the same parameters, i.e., $I_0^L = I_0^S := I_0^* > 0$ and $K^L = K^S := K > 0$:

$$I^{SLS}(t) = I^L(t) + I^S(t)$$

Note that the long side's gain g^L and the short side's gain g^S have to be calculated separately, i.e., the trader actually performs two feedback rules simultaneously. A flow diagram for the SLS rule is given in Fig. 1.

Fig. 1 about here.

2.3 Market Requirements

As can be seen, the short side's strategy requires for sure a possibility for

- *short selling*, i.e., a negative investment.

Besides this market requirement, a few more assumptions are needed in the analytic parts of the work at hand:

- *costless trading*, i.e., no additional costs related to buying or selling assets,
- *adequate resources*, i.e., no financial constraints that could prohibit any desired transaction,
- *perfect liquidity*, i.e., no bid-ask-spread and no waiting time, and
- the so-called *price-taker* property, i.e., no impact of the investment decisions on the price process.

In a discussion of the results of this paper, these assumptions are debated. But for now, I briefly justify these market requirements: short-selling and perfect liquidity should not be strong assumptions for large companies' stocks under trade. Costless trading, in the past a strong argument of the defenders of the efficient market hypothesis to show that chartist strategies cannot work in practice (cf. Fama, 1991), might be less controversial in times of flat-rate stock trading offers or when only trading over the counter. The adequate resources assumption is justified when the trader is big and rich enough, e.g., a mutual fund, and is not trading too much of the single asset under trade. The latter assumption also justifies the price-taker property.

The key assumption is high liquidity, because when trading a highly liquid stock in small amounts, it can be done over the counter. In this case, the price path is always the middle of the bid-and-ask price, which allows me to assume no bid-ask-spread. In the case of trading over the counter, the trader is really a price-taker and the trading costs are negligible.

In sum, for a big, rich trader who trades small amounts wirh an asset with a big underlying firm, the assumptions above can be accepted. Rich does not mean that the trader has an infinite amount of money all the time, like in the St. Petersburg paradox or when referring to doubling strategies. There are papers analyzing the leverage of the SLS rule, i.e., how many times the account value a trader needs to invest. Primbs and Barmish (2013, 2017) show that this leverage can be bounded. Additionally, it is easy to see that in a discrete time model with a given time horizon ($T < \infty$) and bounded stock returns—like in a binomial tree model (e.g., Cox-Ross-Rubinstein; CRR)—the maximal investment is bounded, too.

Furthermore, I assume

- a *risk-free bond* with
- *one interest rate for debit and credit,* and
- that the *interest rate equals zero.*

The existence of a zero-rate risk-free bond is easily achievable: When there is any bond, I can use this bond as numéraire. That means all stock returns are somehow relative to this bond. The assumption that the interest rate is the same for debit and credit is a harder assumption than the existence of a risk-free bond. Also, however, this assumption is actually not too hard for big traders. That means that the returns of all risk-neutral assets are zero. But I need another assumption, the existence of a

- *non-risk-neutral* asset.

The results obtained in this work (and in the related literature) only hold when the asset's trend is non-zero. Since the zero-trend case is always an exception in the results of this work, it is important that there be an asset with a trend unequal to zero. This assumption is justifiable because I can assume that a stock market as a whole is risk-neutral (on average), but not so each single stock. Additionally, it is reasonable in the sense of efficient markets that a volatile and, thus, more risky asset expects higher returns, i.e., they are positive relative to the numéraire. I do not need a stock with a higher expected return (compared to the bond/numéraire), only one with expected returns unequal to zero compared to the numéraire.

Note that the market requirements stated in this section concern the analytic parts of the work at hand. In Section 4, I provide a backtesting study on historic market data, including bid-ask-spread (i.e., imperfect liquidity), transaction costs, and interest rates. Thus, the results in Section 4 are empirical, not theoretical.

After having discussed the market requirements, in the next section I present the related work in the field of SLS trading. The authors of the respective papers assume more or less the same market requirements as I did above.

2.4 Literature Review on SLS Trading

The following literature review will give an idea about why the SLS strategy is an interesting one. Barmish (2011) shows that for continuously differentiable prices $p \in \mathcal{C}^1$

it holds

$$g^{SLS}_{\mathcal{C}^1}(t) = \frac{I_0^*}{K}\left(\left(\frac{p(t)}{p(0)}\right)^K + \left(\frac{p(t)}{p(0)}\right)^{-K} - 2\right)$$

from which follows that $g^{SLS}_{\mathcal{C}^1}(t) > 0$ for all price processes with $p(t) \in (-1, \infty) \setminus \{p(0)\}$. In other words, this is an *arbitrage* strategy. Note that this means that the gain at time t is independent of the process and only depends on the value of $p(t)$ at time t. Since $\mathcal{C}^1$ prices are a rather hard assumption, Barmish and Primbs (2011, 2016) show that when the underlying price process is governed by a geometric Brownian motion (GBM) (this is the price model used in the Black-Scholes model)

$$p_{GBM}(t) = p_0 \cdot exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right)$$

(with trend $\mu > -1$, volatility $\sigma > 0$, and a Wiener process (i.e., Brownian motion) $W(t)$), the SLS strategy is not an arbitrage strategy anymore. However, for the expected gain it holds:

$$\mathbb{E}\left[g^{SLS}_{GBM}(t)\right] = \frac{I_0^*}{K}(exp(K\mu t) + exp(-K\mu t) - 2)$$

Especially it follows $\mathbb{E}\left[g^{SLS}_{GBM}(t)\right] > 0$ whenever $\mu \neq 0$ holds. This is called the *robust positive expectation property.* Similar results are provided by Dokuchaev and Savkin (1998a,b, 2002, 2004); Dokuchaev (2012).

Primbs and Barmish (2013, 2017) show that the robust positive expectation property also holds when the trend $\mu(t)$ as well as the volatility $\sigma(t)$ of the GBM are time dependent. In fact, for a time-varying GBM (tvGBM) with trend $\mu(t)$ and volatility $\sigma(t)$ and the SLS trading rule, it holds:

$$\mathbb{E}\left[g^{SLS}_{tvGBM}(t)\right] = \frac{I_0^*}{K}\left(exp\left(K\int_0^t \mu(s)ds\right) + exp\left(-K\int_0^t \mu(s)ds\right) - 2\right).$$

For clear, whenever $\int_0^t \mu(s)ds \neq 0$ it holds $\mathbb{E}\left[g^{SLS}_{tvGBM}(t)\right] > 0$, too.

Iwarere and Barmish (2014) analyze the SLS strategy when prices are governed by tree models, and Barmish and Primbs (2012) use a market model motivated by the CAPM. Barmish (2008) and Malekpour et al. (2013) analyze strategies related to the SLS rule.

Baumann (2016) generalizes the results for SLS trading to prices governed by Merton's jump diffusion model (MJDM), which is given through:

$$p_{MJDM}(t) = p_0 \cdot exp\left(\left(\mu - \lambda\kappa - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right)\prod_{i=1}^{N} Y_i$$

Hereby, the GBM is extended by i.i.d. jumps $(Y_i - 1) > -1$ with jump intensity $\lambda > 0$, expected jump height $\mathbb{E}[Y_i - 1] = \kappa > 0$, and a number $N \sim \mathcal{P}oi(\lambda t)$ of jumps up to time t. Jumps are interesting in this context since they are known—in the fields of option pricing and hedging—for making markets incomplete (Merton, 1976). However, Baumann (2016) shows that the expected gain of the SLS strategy is:

$$\mathbb{E}\left[g_{MJDM}^{SLS}(t)\right] = \frac{I_0^*}{K}(exp(K\mu t) + exp(-K\mu t) - 2),$$

which is exactly the same as for the GBM. Baumann and Grüne (2016) further generalize this result to a set of price processes defined by stochastic differential equations called essentially linearly representable prices. Barmish and Primbs (2011) give a closed formula for the variance of the SLS trading rule when prices are governed by a GBM, and Baumann (2016) does this for MJDM prices as well.

Here, I want to mention again that the so-called linear long (short) trader is not necessarily long (short) when there are discontinuities—for example, when the price model allows for jumps, like MJDM, or when the model is in discrete time, as in the two papers discussed next.

Malekpour and Barmish (2016) note an interesting and especially practical problem of the SLS rule. Since the SLS strategy is calculated by use of the overall gain, price behaviors that happened a long time ago have the same impact on the investment decision of the trader as if they happened a few days ago. Imagine a price development where in the phase after the trader entered the market, the price rose a lot, then stayed nearly constant for a long time, and then decreased slowly. The trader's long (short) side would have made (lost) a lot of money in the first period, then stayed approximately constant. The slow decrease in later time does not level out the increase from a long time before. As a consequence of the feedback loop, the investment of the trader is still very high—and long—which seems to be questionable since prices stayed constant for so long, then decreased. Malekpour and Barmish (2016) introduce a new strategy called Initially Long-Short (ILS) with delay as the superposition of a linear long rule with delay $I_t^{Ld} = I_0^* + K\left(g_t^{Ld} - g_{t-m}^{Ld}\right)$ and a linear short rule with delay $I_t^{Sd} = -I_0^* - K\left(g_t^{Sd} - g_{t-m}^{Sd}\right)$. The strategy is defined and analyzed in a discrete time setting with a time grid $\{0, 1, 2, \ldots\}$ with fixed time steps, e.g., days. The word initially denotes the fact that only at the initial time can one be sure that the long (short) side is truly long (short). Among other market requirements, similar to that presumed in the work at hand, the main assumption by Malekpour and Barmish (2016) is $-1 < \mathbb{E}\left[\frac{p_t - p_{t-1}}{p_{t-1}}\right] = \mu \neq 0$, i.e., that the expected return on investment is non-zero and constant, which is needed to show that the robust positive expectation property still holds.

In the ILS strategy, only the period gains of the last m days are taken into account. On the one hand, while the idea of not taking into account a too-old price (and so gain) is an advantage of the ILS rule of Malekpour and Barmish (2016) compared to the standard SLS rule. On the other, the hard-delay definition seems to be a little bit problematic. Imagine a price history where $m-1$ days ago an important event happened at the market, for example a sudden crash, which made the short side much more important. Today, this event will be taken into account; tomorrow, this will not be the case. This means that the strategy will change substantially only because an important event happened exactly $m - 1$ days ago, where the number m was an idiosyncratic choice of the trader. A point to think about that is not discussed in detail by Malekpour and Barmish (2016) is that the trader is assumed to be a price-taker. However, the trader decides to trade, e.g., daily, and the expected return on investment on a daily basis is assumed to equal μ. That means the trader indirectly influences the expected return on investment by choosing a trading frequency, which at first glance seems to contradict the price-taker property. However, this is not a problem, as shown in the next paper reviewed below. I will come back to the idea of not taking into account too-old information in Section 5.

Also Baumann and Grüne (2017) use a discrete time setting at first, but with adjustable time steps $h > 0$: $\{0, h, 2h, \ldots\}$. Here, it is assumed that:

$$\mathbb{E}\left[\frac{p_t - p_{t-h}}{h \cdot p_{t-h}}\right] = \mu_h \neq 0,$$

which is the expected return on investment (eroi). For the standard SLS strategy, the expected gain is

$$\mathbb{E}\left[g_{eroi;t}^{SLS}\right] = \frac{I_0^*}{K}\left[(1 + K\mu_h h)^{\frac{t}{h}} + (1 - K\mu_h h)^{\frac{t}{h}} - 2\right],$$

positive whenever $\mu_h \neq 0$ and $t > h$. Even in this setting, the conjectural contradiction to the price-taker property is given: On the one hand, the trader chooses the trading frequency h; on the other hand, the expected return on investment μ_h has to be independent of the trader. To solve this, a (maybe more realistic) setting with an underlying continuous time price process but discrete time trades is introduced. Engineers call this a *sampled-data system.* The continuous time price process has to satisfy

$$\mathbb{E}\left[p(t_2)|\mathcal{F}_{t_1}\right] = p(t_1) \cdot exp\left(\mu\left(t_2 - t_1\right)\right)$$

for all $t_2 > t_1 \geq 0$. This μ is now independent of the trader's decision on trading time and with

$$\mu_h := \frac{exp(\mu h) - 1}{h}$$

the above theory is applicable. Finally, it shows that when calculating the limits for $h \to 0$, the results are fully in line with the known results for the GBM and MJDM. In sum, without assuming any fixed market model but only some core properties, like the expected return on investment and independent multiplicative growth of the price process, it shows that the SLS rule satisfies the robust positive expectation property, i.e., a positive expected gain.

2.5 Illustrative Example

Before going to the analytic part of this work, I present a small illustrative example of how SLS trading works in a non-recombinable binomial tree with time-varying parameters on the time grid $\{0, 1, 2\}$. Note that for the analytical findings in Section 3, no values of the parameters have to be known (or estimated), though for calculating this example, parameters were chosen. The parameters are: The initial price is $p_0 = 10$; the probability for a rising price $p_1 = p_0 + 20\% = 12$ in period one is 0.5; and for a falling price $p_1 = p_0 - 10\% = 9$ is also 0.5. In the second period, the price rises with probability 0.8 by 10% and drops with probability 0.2 by 10% (i.e., -10% increase). That means in period two, the price is $p_2 = 12 + 10\% = 13.2$; $p_2 = 12 - 10\% = 10.8$; $p_2 = 9 + 10\% = 9.9$; or $p_2 = 9 - 10\% = 8.1$. This leads to a trend of $\mu_1 = 5\%$ in period one and of $\mu_2 = 6\%$ in period two. Together with the trading parameters $I_0^* = 100$ and $K = 2$, this leads to the investments and gains depicted in Fig. 2. The variable B denotes the bond, i.e., the bank account. A positive B tells how much money the trader puts in the bank and a negative B how much money the trader has to borrow from the bank. The expected gain is $\mathbb{E}\left[g_2\right] = 0.4 \cdot 8 + 0.1 \cdot (-8) + 0.4 \cdot (-4) + 0.1 \cdot 4 = 1.2 > 0$.

Fig. 2 about here.

In the next section, I show that the expected gain was also positive when both μ_1 and

μ_2 were negative. The example above gives an idea how come SLS trading works. When prices go up (down) several times in a row, the trader profits from compound effects of the long (short) side, ones that exceed the losses on the other side (in expectation). Another point learned by this example is that the trader does not need an infinite amount of money: here it is bounded by 112. The system of SLS trading is not explainable with the St. Petersburg paradox because here the positive expected gain is achieved in round two and not only for $t \to \infty$. And even were I to add transaction costs of, e.g., 0.5 per trade, the expected gain would still be $0.2 > 0$, which shows that the classical transaction costs argument of the defenders of the efficient market hypothesis is not that strong in terms of SLS trading. I will come back to the problem of transaction costs and of bid-ask-spreads in Section 4.

3 Analysis of the Simultaneously Long Short Strategy with Time-varying Trends

The main feature of control-based trading strategies is that, although market parameters like the expected return on investment are used when analyzing the strategies, the trader neither needs to know nor to estimate them. Properties of the strategies hold for almost all settings of the parameter values. The following analysis generalizes the work of Baumann and Grüne (2017) but takes into account the ideas of Primbs and Barmish (2013, 2017) who consider time-varying trends and volatilities.

After having discussed market efficiency and control-based trading strategies, especially SLS trading, I present the analysis of the SLS rule in a general time-varying setting. This analysis is based on refinements of the underlying time grids: Starting with discrete time-price processes and thus discrete time-trading, I end with continuous prices and continuous trading. The price process allows for time-varying parameters, and in Section 3.2 the analysis takes risk-adjusted returns into account. The mathematically proven results build the already mentioned puzzle of market efficiency, which remains the aim of this work.

3.1 The Robust Positive Expectation Property

The basic novelty of this work, different from the work of Baumann and Grüne (2017), is that I allow for a time-varying trend:

$$\mathbb{E}\left[\frac{p_t - p_{t-h}}{h \cdot p_{t-h}}\right] =: \mu_{h;t-h}$$

(For the reason of non-negative prices, $\frac{p_t - p_{t-h}}{h \cdot p_{t-h}} \geq -1$ and $\mu_{h;t-h} > -1$ has to hold for all t and h.) This procedure is also a generalization of the work done by Primbs and Barmish (2013, 2017), which extends the results for standard GBMs to time-varying GBMs. Analogous to Baumann and Grüne (2017), I also assume positive, stochastic prices $(p_t)_{t \in \mathcal{T}} > 0$ ($\mathcal{T} = \{0, h, 2h, \ldots, T\}$, $T = Nh$, $t = nh$), $p_0 \in \mathbb{R}^+$, and independent multiplicative growth, i.e., for all $k \in \mathbb{N}$ and all $t_0 < t_1 < \ldots < t_k \in \mathcal{T}$ it holds that

$$p_{t_0}, \frac{p_{t_1}}{p_{t_0}}, \ldots, \frac{p_{t_k}}{p_{t_{k-1}}}$$

are stochastically independent. In other words, the returns of investment must be independent. This is the *weak form of the market efficiency hypothesis.* Note that this stochastic independence also holds when applying any measurable function on the growth rates. Again, there seems to be a contradiction in the price-taker property: While on one side, h is chosen by the trader, on the other side, the trend $\mu_{h;t}$ depends on h. But, as shown by Baumann and Grüne (2017), this problem can easily be solved—either by use of so-called sampled-data systems or by calculating the limits for $h \to 0$.

Here I will show that the robust positive expectation property does not, in general, hold anymore (an example is given later in this section). However, at least in two special cases, the robust positive expectation property is still valid. First, I note that for the expected price it holds

$$\mathbb{E}[p_t] = \mathbb{E}\left[p_0 \cdot \prod_{i=1}^{n} \frac{p_{ih}}{p_{(i-1)h}}\right] = p_0 \cdot \prod_{i=1}^{n} (\mu_{h;ih} h + 1)$$

and

$$\mathbb{E}[p_{t_2} | \mathcal{F}_{t_1}] = p_{t_1} \cdot \prod_{i=n_1+1}^{n_2} (\mu_{h;ih} h + 1).$$

I start the analysis of the SLS strategy with its long side. By the definition of I_t^L and $g_t^L = \sum_{i=1}^{n} I_{(i-1)h}^L \cdot \frac{p_{ih} - p_{(i-1)h}}{p_{(i-1)h}}$ it follows:

$$\frac{I_t^L - I_{t-h}^L}{h \cdot I_{t-h}^L} = K \cdot \frac{p_t - p_{t-h}}{h \cdot p_{t-h}}$$

and so

$$\mathbb{E}\left[\frac{I_t^L - I_{t-h}^L}{h \cdot I_{t-h}^L}\right] = K \mu_{h;t-h}.$$

It holds

$$\mathbb{E}\left[I_t^L\right] = I_0^* \cdot \prod_{i=1}^{n} \left(K\mu_{h;ih}h + 1\right).$$

Again by the definition of I_t^L it follows:

$$\mathbb{E}\left[g_t^L\right] = \frac{I_0^*}{K}\left(\prod_{i=1}^{n}\left(K\mu_{h;ih}h+1\right) - 1\right)$$

By substituting $I_0^* \mapsto -I_0^*$ and $K \mapsto -K$, the formula for $\mathbb{E}\left[g_t^S\right]$ is derived.

Next, I investigate whether $\mathbb{E}\left[g_t^{SLS}\right] = \mathbb{E}\left[g_t^L + g_t^S\right]$ is positive or not. Unfortunately, $\mathbb{E}\left[g_t^L + g_t^S\right] > 0$ is not true for all t and all $\left(\mu_{h;t}\right)_t$. This can be seen by rewriting

$$\begin{aligned}\mathbb{E}\left[g_t^{SLS}\right] &= \frac{I_0^*}{K}\left(\prod_{i=1}^{n}\left(K\mu_{h;ih}h+1\right) + \prod_{i=1}^{n}\left(-K\mu_{h;ih}h+1\right) - 2\right)\\ &= \frac{2I_0^*}{K}\sum_{\substack{\alpha\subset\{1,\ldots,n\}\\ |\alpha|\ even\\ |\alpha|\neq 0}}\prod_{j\in\alpha}K\mu_{h;jh}h.\end{aligned}$$

When assuming a time-varying trend in discrete time, it is easy to find an example where this sum is negative. When setting $n = 2$, i.e., $\mathcal{T} = \{0, h, 2h\}$, with $\mu_{h;h} > 0$ and $\mu_{h;2h} < 0$, which is a time-varying trend, it holds that $\mathbb{E}\left[g_{2h}^{SLS}\right] = 2KI_0^*h^2\mu_{h;h}\mu_{h;2h} < 0$.

However, there are (at least) two special cases where $\mathbb{E}\left[g_t^{SLS}\right] > 0$ holds. (i) One, when $n > 1$ and $\mu_{h;t} \geq 0$ for all t, and $\mu_{h;t} > 0$ for at least two points in time t or when $\mu_{h;t} \leq 0$ for all t and $\mu_{h;t} < 0$ for at least two points in time t (since $|\alpha|$ is *even*). That means, whenever $\left(\mu_{h;nh}\right)_{n\in\{1,\ldots,N\}}$ is non-negative (non-positive), $\mathbb{E}\left[g_t^{SLS}\right]$ is non-negative. When additionally there exists $\nu \subset \{1,\ldots,N\}$ with $|\nu| \geq 2$ so that $\left(\mu_{h;jh}\right)_{j\in\nu}$ is positive (negative), it holds that $\mathbb{E}\left[g_t^{SLS}\right]$ is positive. The settings of Baumann and Grüne (2017) and Malekpour and Barmish (2016), i.e., μ or μ_h *const.* and non-zero, are a special case of case (i).

(ii) Two, when letting $h \to 0$ (i.e., $n \to \infty$), one can use the continuously compounded interest rate formula, which is a Vito-Volterra-style product integral, to see

$$\mathbb{E}\left[g_t^{SLS}\right] = \frac{I_0^*}{K}\left(exp\left(\int_0^t K\mu(s)ds\right) + exp\left(\int_0^t -K\mu(s)ds\right) - 2\right),$$

which is positive whenever $\bar{\mu} := \int_0^t \mu(s)ds \neq 0$. That means, in the continuous time case, I proved that the robust positive expectation property still holds. Compare Figs. 3

and 4 for graphs of the expected SLS trading gain as functions of $\bar{\mu}$ for varying K, and Figs. 5 and 6 for contour plots of the expected SLS trading gains as a function of $K > 0$ and $\bar{\mu}$. Note that $exp(x) + exp(-x) - 2 \geq 0 \; \forall x$ and equals zero, if and only if $x = 0$.

The setting of Primbs and Barmish (2013, 2017) is a special case of case (ii), and all the results using GBMs or MJDM are special cases of the cases (i) (just on another time scale) and (ii). In case (ii), $\mu(t)$ has to be a Riemann integrable function.

Figs. 3, 4, 5, and 6 about here.

During every time interval with positive expected returns or negative expected returns, a trader using the SLS rule can expect positive gains. Only when the expected return μ switches from positive to negative or vice versa can the trader expect a loss. When increasing the trading frequency to continuous trading—which is nearly a realistic assumption in times of high-frequency trading—and $\mu(t)$ is Riemann integrable, the measure of points in time when μ is switching its sign goes to zero (given any measure that is absolutely continuous with the Lebesgue measure on the parameter space).

Mostly, in market efficiency literature, it is assumed that the price process is a random walk around its fundamental value. When allowing the fundamental value to be non-constant, and assuming it to be not too wild, i.e., $\mu(t)$ has to be Riemann integrable, i.e., $\bar{\mu} = \int_0^t \mu(s)ds \neq 0$ exists, the SLS trader can—when trading fast enough—expect a positive gain for all t. This should not be true in an efficient market.

3.2 Risk-Adjusted Expected Return

For sure, there are some points to think about concerning this result. The assumption that there are short time trends in expected returns (that can be caused by changes in fundamentals) is reasonable. The argument that the trader in practice has to achieve a positive gain on average when there are trading costs, in times of over-the-counter and flat-rate trading offers, it is not really a solution to the puzzle, and trading costs in a highly liquid market can be assumed to be bounded. (In Section 4, when performing a backtesting study on past price data, and when bid-ask-spreads and trading costs are taken into account, it turns out that, nevertheless, on average, positive gains can be observed.) The same is true for the continuous trading assumption when considering high frequency trading. However, there is one argument against the discounted SLS rule that puzzles me: the risk adjustment.

Classically, the risk argument is given by the defenders of the market efficiency hypothesis when someone finds an external variable that allows for estimating higher expected returns of an asset. Then it is said that this external variable is just a better proxy for measuring risk, so one concludes that the asset under investigation is more risky, which allows the asset to be more profitable (on average) without being a counterexample to market efficiency. In the setting of this paper, this is not applicable since there is only one asset under analysis, and there are no external variables. Even the discussion of the momentum effect (Moskowitz, 2010), i.e., higher momentum is related to higher risk, is not applicable to my setting because I do not need assumptions on the stock under trade. Here, only different trading strategies are considered. The only way to apply the risk adjustment argument to the SLS rule is to use volatility (standard deviation, which actually is not a risk measure in the sense of mathematical finance), which I will do next. At the end of the paper, the risk of the SLS rule and other definitions of it (cf. skewness) are discussed again. But for now, I use the most common choice.

For calculating the standard deviation of the SLS strategy, an assumption on the volatility of the underlying price process is needed. Analogous to the definition of the trend, it is set:

$$\mathbb{E}\left[\frac{1}{h}\left(\frac{p_t - p_{t-h}}{p_{t-h}}\right)^2\right] =: \sigma^2_{h;t-h} > 0$$

Note that also here there is a market parameter, namely $\sigma^2_{h;t}$ seemingly set by the trader via h. However, the same argument as for $\mu_{h;t}$ holds (cf. Baumann and Grüne, 2017).

With this assumption it follows that:

$$\mathbb{E}\left[p_t^2\right] = p_0^2 \cdot \prod_{i=1}^{n} \left(\left(\sigma^2_{h;ih} + 2\mu_{h;ih}\right) h + 1\right)$$

and

$$\mathbb{E}\left[p_{t_2}^2 | \mathcal{F}_{t_1}\right] = p_{t_1}^2 \cdot \prod_{i=n_1+1}^{n_2} \left(\left(\sigma^2_{h;ih} + 2\mu_{h;ih}\right) h + 1\right).$$

Again, I start the analysis of the SLS strategy with its long side. Using the definition of I_t^L and g_t^L leads to

$$\frac{1}{h}\left(\frac{I_t^L - I_{t-h}^L}{I_{t-h}^L}\right)^2 = \frac{K^2}{h}\left(\frac{p_t - p_{t-h}}{p_{t-h}}\right)^2$$

and so

$$\mathbb{E}\left[\frac{1}{h}\left(\frac{I_t^L - I_{t-h}^L}{I_{t-h}^L}\right)^2\right] = K^2\sigma^2_{h;t-h}.$$

It holds

$$\mathbb{E}\left[\left(I_t^L\right)^2\right] = I_0^{*2} \cdot \prod_{i=1}^{n} \left(\left(K^2\sigma^2_{h;ih} + 2K\mu_{h;ih}\right) h + 1\right).$$

Again by the definition of I_t^L it follows:

$$\mathbb{E}\left[\left(g_t^L\right)^2\right] = \frac{I_0^{*2}}{K^2}\left(\prod_{i=1}^{n}\left(\left(K^2\sigma_{h;ih}^2 + 2K\mu_{h;ih}\right)h + 1\right) - 2\prod_{i=1}^{n}\left(K\mu_{h;ih}h + 1\right) + 1\right)$$

Once more, by substituting $I_0^* \mapsto -I_0^*$ and $K \mapsto -K$, the formula for $\mathbb{E}\left[(g_t^S)^2\right]$ follows. For calculating the standard deviation of the SLS strategy's gain, the mixed expectation of the long and the short side $\mathbb{E}\left[g_t^L g_t^S\right]$ is needed, too.

It holds:

$$\frac{1}{h}\left(\frac{I_t^L - I_{t-h}^L}{I_{t-h}^L}\right)\left(\frac{I_t^S - I_{t-h}^S}{I_{t-h}^S}\right) = -\frac{K^2}{h}\left(\frac{p_t - p_{t-h}}{p_{t-h}}\right)^2$$

and

$$\mathbb{E}\left[\frac{1}{h}\left(\frac{I_t^L - I_{t-h}^L}{I_{t-h}^L}\right)\left(\frac{I_t^S - I_{t-h}^S}{I_{t-h}^S}\right)\right] = -K^2\sigma_{h;t-h}^2$$

With that it follows:

$$\mathbb{E}\left[I_t^L I_t^S\right] = -I_0^{*2}\cdot\prod_{i=1}^{n}\left(-K^2\sigma_{h;ih}^2 h + 1\right).$$

Now, by the definitions of I_t^L and I_t^S, it follows:

$$\mathbb{E}\left[g_t^L g_t^S\right] = \frac{I_0^{*2}}{K^2}\left(\prod_{i=1}^{n}\left(-K^2\sigma_{h;ih}^2 h + 1\right) - \prod_{i=1}^{n}\left(K\mu_{h;ih}h + 1\right) - \prod_{i=1}^{n}\left(-K\mu_{h;ih}h + 1\right) + 1\right)$$

Now, all components needed for the calculation of $\mathbb{E}\left[\left(g^{SLS}(t)\right)^2\right] = \mathbb{E}\left[\left(g^L(t)\right)^2\right] + 2\mathbb{E}\left[g^L(t)g^S(t)\right] + \mathbb{E}\left[\left(g^S(t)\right)^2\right]$ and $Var\left(g^{SLS}(t)\right) = \mathbb{E}\left[\left(g^{SLS}(t)\right)^2\right] - \left(\mathbb{E}\left[g^{SLS}(t)\right]\right)^2$ are known. To keep the computation simple, I calculate the limit for continuous time trading $h \to 0$ and define $\overline{\sigma^2}(t) := \int_0^t \sigma^2(s)ds$ (of course, $\sigma^2(t)$ has to be Riemann integrable as well). By use of the Vito-Volterra-style product integral, it follows:

$$\begin{aligned}
\mathbb{E}\left[\left(g^{SLS}(t)\right)^2\right] &= \mathbb{E}\left[\left(g^L(t)\right)^2 + \left(g^S(t)\right)^2 + 2g^L(t)g^S(t)\right] \\
&= \frac{I_0^{*2}}{K^2}\Bigg(exp\left(K^2\overline{\sigma^2}(t) + 2K\bar{\mu}(t)\right) - 2exp(K\bar{\mu}(t)) + 1 \\
&\quad + exp\left(K^2\overline{\sigma^2}(t) - 2K\bar{\mu}(t)\right) - 2exp(-K\bar{\mu}(t)) + 1 \\
&\quad + 2\left(exp\left(-K^2\overline{\sigma^2}(t)\right) - exp(K\bar{\mu}(t)) - exp(-K\bar{\mu}(t)) + 1\right)\Bigg)
\end{aligned}$$

Combining the results for $\mathbb{E}\left[g^{SLS}(t)\right] = \frac{I_0^*}{K}(exp(K\bar{\mu}(t)) + exp(-K\bar{\mu}(t)) - 2)$ and $\mathbb{E}\left[(g^{SLS}(t))^2\right]$ leads to the formula for the SLS rule variance:

$$Var\left(g^{SLS}(t)\right) = \frac{I_0^{*2}}{K^2}\Bigg(\left(exp\left(K^2\overline{\sigma^2}(t)\right) - 1\right)(exp(2K\bar{\mu}(t)) + exp(-2K\bar{\mu}(t))) \\ + 2\left(exp\left(-K^2\overline{\sigma^2}(t)\right) - 1\right)\Bigg)$$

This expression fits exactly the results obtained by Baumann (2016) for MJDM (and the GBM).

For any strategy ℓ, let

$$rar(\ell;t) := \frac{\mathbb{E}\left[g^{\ell}(t)\right]}{\sqrt{Var\left(g^{\ell}(t)\right)}}$$

be the risk-adjusted return of this strategy at time t. It is clear that $rar(SLS;t) > 0\ \forall t > 0, \bar{\mu}(t) \neq 0$, cf. Figs. 7, 8, 9, and 10 for graphs of the risk-adjusted return of the SLS rule as functions of $\bar{\mu}$ for varying K and $\overline{\sigma^2}$ as well as Figs. 11 and 12 for contour plots of the risk-adjusted returns of the SLS strategy.

Figs. 7, 8, 9, 10, 11, and 12 about here.

3.3 Comparison to Buy-and-Hold

Malkiel (1973) suggests the comparison of a trading strategy to a randomly selected buy-and-hold (bnh) portfolio for showing whether or not the strategy has excess returns. When assuming that the market has on average the same trend as the bond—(i.e., a risk-neutral market), which I assumed without loss of generality to be zero—all randomly selected bnh portfolios have an expected gain of zero, too. This means that the SLS rule is strictly better than any randomly selected bnh portfolio.

It is possible to compare the expected SLS gain stock-by-stock with the corresponding expected bnh gain (which I do next). However, this is not a solution to the puzzle, as I show in the remainder of this section.

Even when comparing stock-by-stock the expected gain of the SLS rule with a bnh strategy, which is exactly the trader L with $K = 1$ and $I_0^* > 0$, it turns out that when $K > 1$ for all t with $\bar{\mu}(t) \in (-1, 0) \cup (B_{eg}(K, \bar{\mu}), \infty)$, the SLS rule is the dominant one, and when $K \leq 1$, it still holds that for all t with $\bar{\mu}(t) \in (-1, 0)$, the SLS rule is dominant over the bnh rule (see Figs. 13, 14, 15, and 16 for graphs of the expected SLS gain, the expected bnh gain, and the contour plots of the expected difference of these strategies). It is easy to see that for the expected gain of a buy-and-hold strategy with initial investment I_0^*, it holds

$$\mathbb{E}\left[g^{bnh}(t)\right] = I_0^*(exp(\bar{\mu}(t)) - 1).$$

The value $B_{eg}(K, \bar{\mu})$ depends on K and $\bar{\mu}$ and it holds: $B_{eg}(K, \bar{\mu}) \to 0$ for $K \to \infty$. Note that $\bar{\mu}(t) \notin [0, B_{eg}(K, \bar{\mu})]$ does not mean that the SLS is only dominant for special price paths, which would not be a result deserving attention. Since $\bar{\mu}(t) = \int_0^t \mu(s)ds$ with $\mu(t)dt = \mathbb{E}\left[\frac{dp(t)}{p(t)}\right]$ is the expected trend of the price path that depends on changes in the fundamentals, and all results so far concern expectations, the price paths are allowed to be random walks around the fundamental value when $\bar{\mu}(t)$ satisfies the condition.

Figs. 13, 14, 15, and 16 about here.

It is clear that a buy-and-hold strategy has a positive expected gain when $\bar{\mu} > 0$ and a negative one when $\bar{\mu} < 0$. For some parameter settings, the bnh rule is dominant to the SLS rule. However, the expected SLS trading gain is positive for almost all parameters—the expected bnh gain is not. That means a bnh trader must know or estimate the average trend. An SLS trader has a positive expected trend with no estimation.

3.4 Buy-and-Hold and Risk

For the expected gain of a buy-and-hold strategy with initial investment I_0^*, it holds

$$\mathbb{E}\left[g^{bnh}(t)\right] = I_0^*(exp(\bar{\mu}(t)) - 1)$$

and for the respective variance

$$Var\left(g^{bnh}(t)\right) = I_0^{*2} exp(2\bar{\mu}(t))\left(exp\left(\overline{\sigma^2}(t)\right) - 1\right),$$

for example, by using the results for $g^L(t)$ and setting $K = 1$.

Next, I compare the risk-adjusted returns of the SLS rule and the buy-and-hold strategy. For all t with $\bar{\mu}(t) \in (-1, 0)$, the SLS rule is the dominant one, too. When $K \geq 1$, the bnh rule is dominant when $\bar{\mu}(t) > 0$. When $K < 1$ and $\bar{\mu}(t) > 0$, for some pairs $(K, \bar{\mu}(t))$, the SLS rule is dominant, and for some the bnh rule, see Figs. 17, 18, 19, and 20 for graphs of the risk-adjusted returns of the SLS rule and the bnh rule (for varying $\overline{\sigma^2}$ and varying K), and see Figs. 21 and 22 for contour plots of the difference between the risk-adjusted returns of the SLS rule and the bnh stratgy.

Figs. 17, 18, 19, 20, 21, and 22 about here.

Now, the question is whether the risk-adjustment and (at the same time) the comparison to the bnh rule is the solution to the robust positive expectation property of the SLS rule in an efficient market. However, it is not. When a market as a whole (i.e., on average) is risk-neutral but not in terms of every single stock, a trader investing in a randomly selected portfolio (and this is what Malkiel (1973) suggested) can expect zero gain; therefore the risk-adjusted return too is zero. When the trader uses the SLS rule stock-by-stock, and there is only one single stock that is not risk-neutral (and it does not matter whether the stock's expected return is too high or too low), the expected trading gain as well as the risk-adjusted return is positive. Indeed, it is reasonable that there are more and less volatile stocks that should have higher or lower trends, respectively.

3.5 The Choice of K

For a practical application, there remains the question of how to choose K. When $\bar{\mu} < 0$, it does not matter whether $K > 1$ or $K < 1$ (in a qualitative manner) because expected gains and risk-adjusted returns are positive, and even when compared to the bnh rule, for both expected gains and risk-adjusted returns, the SLS rule is dominant. When $\bar{\mu} > 0$, it also does not matter how to choose K when relying on expected gain or risk-adjusted return. However, when compared to the bnh rule, it might be better to choose $K > 1$ when expected gain is the target function, and to choose $K < 1$ when it is the risk-adjusted return. Please note again that the comparison to the bnh strategy is questionable because the bnh rule is only better in specific cases for a single asset: A randomly selected portfolio should have a gain (and a risk-adjusted return) of exactly the bond's rate, i.e., of zero. A bnh trader faces the risk of a negative trend—an SLS trader does not.

4 Backtesting with Trading Fees and Bid-Ask-Spreads

In Section 3, I proved mathematically that the SLS strategy has positive expected returns under specific assumptions. I also discussed some of these assumptions and investigated risk adjustments and comparisons to buy-and-hold rules. That means I already presented the theoretical puzzle of market efficiency and SLS trading. This section has two targets: First, I present backtest studies of the SLS rule on real historic market data. Second, in the simulations I allow for bid-ask-spreads, trading costs, and different interest rates for debit and credit.

4.1 Backtesting Trading Dynamics

Before simulating SLS trading for different parameters on 60 DAX charts, I have to modify the strategy in a few ways to make it applicable to real world data. Bid-and-ask prices have to be used, the number of stocks held should be an integer, trading fees lower trading gains, and a bank account with interest rates is added. That means, in detail, I define stock-by-stock on the discrete time grid $\mathcal{T} = \{0, 1, \ldots, T\}$ (with $T = 255$ for 2016, and $T = 252$ for 2017) with pa being the ask price and pb being the bid price:

- the price $p_t = \frac{pa_t + pb_t}{2}$
- the bid-ask-spread $spread_t = pa_t - pb_t$

Now, the SLS controller uses internally the known rules (with *round*-operators) but transmits to the broker only the total number of stocks to be held. That means, when, for example, the long side sends a buying signal and the short side a selling signal, only the difference is transmitted to the broker. When there is a buy or sell signal transmitted to the broker, the side causing the signal has to pay the trading costs. For example: One side gives a buy signal of 5 stocks and the other side a sell signal of 3 stocks; 2 stocks are bought and the side giving the signal 5 has to pay the fees. When both sides give signals in the same direction, each side has to pay for its own transaction.

I calculate the target investments

$$I_t^L = I_0^* + K g_t^L$$

and

$$I_t^S = -I_0^* - K g_t^S$$

which leads to a virtual number of stocks of

$$\#stock_t^L = round(I_t^L / p_t \ , \ 0)$$

and

$$\#stock_t^S = round(I_t^S / p_t \ , \ 0)$$

and virtual buy and sell signals of

$$buy_t^L = \#stock_t^L - \#stock_{t-1}^L$$

and

$$buy_t^S = \#stock_t^S - \#stock_{t-1}^S.$$

This leads to a number of stocks:

$$\#stock_t = \#stock_t^L + \#stock_t^S$$

The buy or sell signal transmitted to the broker is:

$$buy_t = buy_t^L + buy_t^S = \#stock_t - \#stock_{t-1}$$

With fee being the relative broker fee and m being the minimal broker fee per transaction, this leads to transaction fees of:

- When $buy_t > 0$ and $abs(buy_t^L) > abs(buy_t^S)$ and $buy_t^L \cdot buy_t^S < 0$:

$$costs_t^L = buy_t \cdot spread_t/2 + max(buy_t \cdot pa_t \cdot fee\ ,\ m)$$

- When $buy_t > 0$ and $abs(buy_t^L) < abs(buy_t^S)$ and $buy_t^L \cdot buy_t^S < 0$:

$$costs_t^S = buy_t \cdot spread_t/2 + max(buy_t \cdot pa_t \cdot fee\ ,\ m)$$

- When $buy_t < 0$ and $abs(buy_t^L) < abs(buy_t^S)$ and $buy_t^L \cdot buy_t^S < 0$:

$$costs_t^S = -buy_t \cdot spread_t/2 + max(-buy_t \cdot pb_t \cdot fee\ ,\ m)$$

- When $buy_t < 0$ and $abs(buy_t^L) > abs(buy_t^S)$ and $buy_t^L \cdot buy_t^S < 0$:

$$costs_t^L = -buy_t \cdot spread_t/2 + max(-buy_t \cdot pb_t \cdot fee\ ,\ m)$$

- When $buy_t > 0$ and $buy_t^L \cdot buy_t^S \geq 0$:

$$costs_t^L = buy_t^L \cdot spread_t/2 + max(buy_t \cdot pa_t \cdot fee\ ,\ m) \cdot buy_t^L/buy_t$$

  and

$$costs_t^S = buy_t^S \cdot spread_t/2 + max(buy_t \cdot pa_t \cdot fee\ ,\ m) \cdot buy_t^S/buy_t$$

- When $buy_t < 0$ and $buy_t^L \cdot buy_t^S \geq 0$:

$$costs_t^L = -buy_t^L \cdot spread_t/2 + max(-buy_t \cdot pb_t \cdot fee\ ,\ m) \cdot buy_t^L/buy_t$$

and

$$costs_t^S = -buy_t^S \cdot spread_t/2 + max(-buy_t \cdot pb_t \cdot fee\ ,\ m) \cdot buy_t^S/buy_t$$

These costs are used for lowering the gains. However, before calculating the gain, I have to calculate the bank account B with the interest rate $r1$ for credit (i.e., for money put in the bank) and the interest rate $-r2$ for debit (i.e., for money borrowed from the bank). The bank account as well as the gain/loss function start with zero. The dynamics are as follows:

- When $B_{t-1} \geq 0$ and $buy_t \geq 0$:

$$B_t = B_{t-1} \cdot (1 + r1) - buy_t \cdot pa_t$$

- When $B_{t-1} < 0$ and $buy_t \geq 0$:

$$B_t = B_{t-1} \cdot (1 - r2) - buy_t \cdot pa_t$$

- When $B_{t-1} \geq 0$ and $buy_t < 0$:

$$B_t = B_{t-1} \cdot (1 + r1) - buy_t \cdot pb_t$$

- When $B_{t-1} < 0$ and $buy_t < 0$:

$$B_t = B_{t-1} \cdot (1 - r2) - buy_t \cdot pb_t$$

This leads to the virtual trading gains of

$$g_t^L = g_{t-1}^L + \#stock_t^L \cdot (p_t - p_{t-1}) - costs_t^L$$

of the long side and

$$g_t^S = g_{t-1}^S + \#stock_t^S \cdot (p_t - p_{t-1}) - costs_t^S$$

of the short side, as well as a total gain (including interest rates) of

$$g_t = g_{t-1} + \#stock_t \cdot (p_t - p_{t-1}) - costs_t^L - costs_t^S$$

$$+ 1_{\{B_{t-1}>0\}} B_{t-1} \cdot r1 - 1_{\{B_{t-1}<0\}} B_{t-1} \cdot r2.$$

All in all, this leads stock-by-stock to a trading gain of g_T (minus the annual brokerage fee divided by the number of assets traded, e.g., 30 in case of the DAX).

4.2 Data, Results, and Criticism

The data set I use for backtesting contains 60 one-year charts with daily prices, namely the 30 stock charts indexed in the German stock index *DAX* for the years 2016 and 2017 (each with bid-and-ask prices) as provided by *THOMSON REUTERS DATASTREAM*. From the same source, I use the index data *DAX 30 PERFORMANCE-PRICE-INDEX* and the bond rate *BUBA-YIELD-LISTD-FEDRL-SEC 3-5Y MIDDLE RATE*. I chose the years 2016 and 2017 because in these years no firms were incorporated into or removed from the index. The trading fees were taken from *www.boerse-frankfurt.de/inhalt/handeln-handelskosten*, which leads to variable brokerage fees of

$$fee = 5.04\ BP,$$

but a minimal fee per trade of

$$m = 2,52 EUR.$$

The harmonic mean of the bond rate is $r = -0.5201459\%$. To make the results robust against different bond rates, I use

$$r1 = -1\ \%\text{ and } r2 = 15\ \%,$$

i.e., in all cases, the bond rates chosen are bad for the trader. Even when the influence is only marginal, I choose an annual fee of $25EUR$. In Table 1, the 30 assets listed in the DAX are given. These assets are used for backtesting because the requirements for SLS trading state that the traded stocks should be highly liquid (and the underlying firm should be big enough)—which is fulfilled for the stocks listed in the German stock index.

Table 1 about here.

In Table 2, I present the backtesting simulation results of the SLS rule on the 60 DAX charts (with bid-ask-spread) for 2016 and 2017. The trading gains for all stocks at the end of the respective years are given, as well as the maximum amount of money the trader has to borrow from the bank in these years for trading the respective asset (in brackets). At the end, the average trading gain when SLS trading stock-by-stock is calculated, and the maximum amount of money the trader has to borrow from the bank (in brackets) is given (which is not simply the sum of the maximal amounts for all stocks, but potentially less). All these values are simulated for $K = 2$ and $I_0^* = 5,000$. A histogram of the achieved trading gains is given in Fig. 28, and a graph for all 60 assets with the trading gains and the maximal amount of money borrowed from the bank can be found in Fig. 29.

The trading gains are between -397.80 EUR for Deutsche Telekom in 2017 (with maximal 2,050.96 EUR to be lent) and +10,500.83 for Deutsche Lufthansa in 2017 (with maximal 15,663.91 EUR to be lent). In total, a trader following this strategy in the years under analysis does not have to borrow more than 102,604.40 EUR from the bank. Thus, as mentioned in Section 2.3, the trader does not need an infinite amount of money.

To realize an excess return, the SLS rule needs a price path with clear trends (and it does not matter whether this is an upwards or a downwards trend). In case of an asset with positive and negative trends—as shown in Section 3—the SLS trader can expect a loss. Hence, the D:DTE stock seems to have a trend with a switching sign in 2017, while the D:LHA stock seems to have a clear trend in 2017. Having a look at Fig. 30 indicates that the Lufthansa chart goes clearly in one direction (up) and the Telekom chart is wobbling around its start price.

Table 2 about here.

Figs. 28, 29, and 30 about here.

In Fig. 31, a contour plot is depicted for the average trading gains of SLS trading for varying trader parameters for the DAX data. For small values of K as well as for small values of I_0^*, the trader, on average, has a loss. When the parameters are chosen large enough, the average gain is positive. This can be explained by the minimal transaction fee: in case of small investment amounts, the trader must pay the minimal fee for each trade, which is much higher than the relative fee. For large investments, the trader has to pay the relative fee (which is relatively smaller). In Table 3, the averaged trading gains over all stocks and years and the maximum amount of money to be borrowed are given for varying parameters: $K = 0.5,\ 1,\ 1.5,\ 2$, and $I_0^* = 500,\ 1,000,\ 1,500$, and $2,000$. These parameters are chosen because the border between positive and negative gain lies in these ranges.

Fig. 31 about here.

Table 3 about here.

To sum up, these results are a hint that the robust positive expectation property also holds for real world data with transaction costs. In the Histogram, Fig. 28, for the trading gains of the SLS rule for all 60 charts, the gains are highly skewed (which

I already mentioned and which is fully in line with the corresponding literature). That means, in order to realize the expected positive gain, a trader must perform SLS trading on many assets fulfilling the requirements.

However, there is also criticism of these results. When buying or selling a large amounts of stocks, even a backtest simulation with bid-and-ask prices is not realistic since the price can change during one trade. Thus, I_0^* and K should not be too large. Another point to consider is that the DAX rose from the beginning of 2016 to the end of 2017 by a factor of 1.256. When having a look at the maximum needed money and the gains, it might have been more profitable to invest in index mutual funds or index ETFs for the DAX (as a buy-and-hold trader). However, this only works when the DAX goes up, while SLS trading (in theory) also works when charts or the whole index goes down.

5 Extension: The Discounted Simultaneously Long Short Strategy

As mentioned in Section 2.4, Malekpour and Barmish (2016) state that investment decisions should not rely (too much) on past market behavior. A controller with delay as presented by Malekpour and Barmish (2016) has the favorable feature that too old (older than $m-1$ days) events have no influence on strategy, but it has a questionable feature, too: An event that is $m-1$ days old is taken fully into account today but vanishes from the calculations after m days. As an alternative controller type, I introduce the discounted SLS controller with discounting factor $\delta \in (0,1]$ (SLS$_\delta$). In this section, the standard SLS rule is generalized by discounting factor δ, and the price process allows for time-varying parameters. Again, the analysis is based on a refinement of time grinds. The main, and indeed the only, difference between a discounted rule ℓ_δ and a standard rule ℓ is that, instead of the gain g_t^ℓ, a discounted gain

$$f_t^{\ell_\delta} = \sum_{i=1}^{n} I_{(i-1)h}^{\ell_\delta} \cdot \frac{p_{ih} - p_{(i-1)h}}{p_{(i-1)h}} \cdot \delta^{-(i-1)h}$$

in a discrete time grid $\{0, h, 2h, \ldots\}$ with $h > 0$ and $t = nh$ is used. That means, the discounted SLS rule is

$$I_t^{SLS_\delta} = I_t^{L_\delta} + I_t^{S_\delta}$$

with

$$I_t^{L_\delta} = I_0^* + K f_t^{L_\delta}$$

and

$$I_t^{S_\delta} = -I_0^* - K f_t^{S_\delta}.$$

A flow diagram for the discounted SLS rule is given in Fig. 23. Note that for $\delta = 1$, this strategy is exactly the standard SLS strategy. The discounting factor δ specifies to which extent past information is used for calculating the current investment (cf. other economic discounting factors like, e.g., the game theoretic discounting factor in repeated games). The higher δ is, the more influence past information has; for $\delta = 1$, all available information is equally weighted, for δ close to zero, only the last available information is important. The discounted SLS strategy has, similar to the SLS strategy with delay, the advantage that (when $\delta < 1$) old information is not as important as new information. However, in contrast to the delay strategy, the old information loses its weight gradually, not instantaneously.

Fig. 23 about here.

All market assumptions are exactly the same as in Sections 2 and 3. In the following, I show that the robust positive expectation property holds also for the discounted SLS rule at least in two special cases, both very similar to the cases of Section 3. The analysis of the discounted SLS strategy is done analogously to the analysis of the standard rule, i.e., by the definitions of $I_t^{L_\delta}$ and $f_t^{L_\delta}$ it follows

$$\frac{I_t^{L_\delta} - I_{t-h}^{L_\delta}}{h \cdot I_{t-h}^{L_\delta}} = K\delta^{-(t-h)} \cdot \frac{p_t - p_{t-h}}{h \cdot p_{t-h}},$$

$$\mathbb{E}\left[\frac{I_t^{L_\delta} - I_{t-h}^{L_\delta}}{h \cdot I_{t-h}^{L_\delta}}\right] = K\delta^{-(t-h)}\mu_{h;t-h},$$

and

$$\mathbb{E}\left[I_t^{L_\delta}\right] = I_0^* \cdot \prod_{i=1}^{n} \left(K\delta^{-(i-1)h}\mu_{h;ih}h + 1\right).$$

Using $I_t^{L_\delta}$ it turns out:

$$\mathbb{E}\left[f_t^{L_\delta}\right] = \frac{I_0^*}{K}\left(\prod_{i=1}^{n} \left(K\delta^{-(i-1)h}\mu_{h;ih}h + 1\right) - 1\right)$$

By substituting $I_0^* \mapsto -I_0^*$ and $K \mapsto -K$ the formula for $\mathbb{E}\left[f_t^{S_\delta}\right]$ follows.

Next, I investigate whether $\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right]$ is positive. The reader may ask why I am interested in the expected sum of the discounted gain of the short and the long side of the discounted SLS strategy. I can rewrite the undiscounted gain in the following way:

$$\begin{aligned}g_t^{\ell_\delta} &= \left(f_t^{\ell_\delta} - f_{t-h}^{\ell_\delta}\right)\delta^{t-h} + \left(f_{t-h}^{\ell_\delta} - f_{t-2h}^{\ell_\delta}\right)\delta^{t-2h} + \ldots + \left(f_h^{\ell_\delta} + 0\right)1\\ &= f_t^{\ell_\delta}\delta^{t-h} + f_{t-h}^{\ell_\delta}\delta^{t-2h}\left(1-\delta^h\right) + \ldots + f_h^{\ell_\delta}\left(1-\delta^h\right)\end{aligned}$$

Since it holds

$$\mathbb{E}\left[g_t^{SLS_\delta}\right] = \mathbb{E}\left[g_t^{L_\delta} + g_t^{S_\delta}\right]$$

and the expectation operator is linear, I conclude: When $\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right] > 0$ for all t, then $\mathbb{E}\left[g_t^{SLS_\delta}\right] > 0$, too. And, what is really of interest is this inequality, i.e., the robust positive expectation property. Note: In the case $\delta = 1$, it holds that $\mathbb{E}\left[f_t^{\ell_1}\right] = \mathbb{E}\left[g_t^{\ell_1}\right]$.

Similar to the analysis of the standard rule, it holds that $\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right] > 0$ is not true for all t, all $\delta \in (0,1]$, and all $(\mu_{h;t})_t$. This is clear since I already presented a counterexample with $\delta = 1$. This means that this is not a problem of discounting the SLS strategy, it is a problem of the time-varying trend when the time axis is non-continuous, even in the standard SLS case, i.e., when $\delta = 1$.

It holds:

$$\begin{aligned}\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right] &= \frac{I_0^*}{K}\left(\prod_{i=1}^{n}\left(K\delta^{-(i-1)h}\mu_{h;ih}h + 1\right) + \prod_{i=1}^{n}\left(-K\delta^{-(i-1)h}\mu_{h;ih}h + 1\right) - 2\right)\\ &= \frac{2I_0^*}{K}\sum_{\substack{\alpha\subset\{1,\ldots,n\}\\ |\alpha|\ even\\ |\alpha|\neq 0}}\prod_{j\in\alpha}K\delta^{-(j-1)h}\mu_{h;jh}h\end{aligned}$$

The following cases where $\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right] > 0$ holds are exactly the same as for the standard SLS rule despite use of discounted gains instead of using undiscounted ones. (i) First, when $n > 1$ and $\mu_{h;t} \geq 0$ for all t and $\mu_{h;t} > 0$ for at least two points in time t, or when $\mu_{h;t} \leq 0$ for all t and $\mu_{h;t} < 0$ for at least two points in time t. (ii) Second, when letting $h \to 0$ (i.e., $n \to \infty$), one can again use the Vito-Volterra-style product integral to get

$$\mathbb{E}\left[f_t^{L_\delta} + f_t^{S_\delta}\right] = \frac{I_0^*}{K}\left(exp\left(\int_0^t K\delta^{-s}\mu(s)ds\right) + exp\left(\int_0^t -K\delta^{-s}\mu(s)ds\right) - 2\right),$$

which is positive whenever $\bar{\mu}_\delta := \int_0^t \delta^{-s}\mu(s)ds \neq 0$. See Figs. 24 and 25 for graphs of the expected discounted SLS_δ gain as functions of $\bar{\mu}_\delta$ (for varying K). And see Figs. 26 and 27 for contour plots of the expected discounted SLS_δ trading gains as functions of $K > 0$ and $\bar{\mu}_\delta$.

Figs. 24, 25, 26, and 27 about here.

6 Discussion & Conclusion

In the past, most puzzles for market efficiency came from empirical data and statistical methods. The puzzle presented in Section 3 is a purely theoretical, mathematical one. Additionally, in Section 4 some empirical evidence is given. I proved the following:

- In discrete time, the expected gain of the SLS strategy is positive when $(\bar{\mu}_{h;t})_t \geq 0 \ \forall t$ and $(\bar{\mu}_{h;t})_t > 0$ for at least two time points, or when $(\bar{\mu}_{h;t})_t \leq 0 \ \forall t$ and $(\bar{\mu}_{h;t})_t < 0$ for at least two points in time .
- In continuous time, the expected gain of the SLS strategy is positive when $\bar{\mu}(t) \neq 0$.
- The expected gain of the standard SLS rule surpasses the expected gain of a simple buy-and-hold strategy for all $t > 0$ with $\bar{\mu}_\delta(t) \notin [0, B_{eg}(K, \bar{\mu})]$ if $K > 1$, with $B_{eg}(K) \to 0$ for $K \to \infty$, and all $\bar{\mu}_\delta(t) \notin \mathbb{R}_0^+$ when $K \leq 1$ (in continuous time).
- The risk-adjusted return of the standard SLS rule is positive for all $K > 0$, $-1 < \bar{\mu} \neq 0$, and $\overline{\sigma^2} > 0$.
- The risk-adjusted return of the standard SLS rule exceeds the risk-adjusted return of a simple buy-and-hold strategy for all $-1 < \bar{\mu} < 0$ and when $K \leq 1$ for some $0 < \bar{\mu}$.
- When K and I_0^* are chosen large enough, a trader could have realized excess returns in the years 2016 and 2017 by SLS trading the DAX assets, although bid-ask-spreads, trading fees, and interest rates were taken into account.
- In discrete time, the expected gain of the discounted SLS_δ strategy for all discounting factors $\delta \in (0, 1]$ is positive when $(\bar{\mu}_{h;t})_t \geq 0 \ \forall t$ and $(\bar{\mu}_{h;t})_t > 0$ for at least two points in time, or when $(\bar{\mu}_{h;t})_t \leq 0 \ \forall t$ and $(\bar{\mu}_{h;t})_t < 0$ for at least two points in time.
- In continuous time, the expected gain of the discounted SLS_δ strategy for all discounting factors $\delta \in (0, 1]$ is positive when $\bar{\mu}_\delta(t) \neq 0$.

The findings of Section 3 mean that an SLS trader can expect positive gain (even in discrete time) on all arbitrary small intervals where the trend is not changing its sign. Only for points in time when the trend changes its sign will the SLS trader face negative expected gains. Note that the price path itself can often change its slope arbitrarily. When the trend path is to some extent smooth, and trading frequency is increased, the points in time where the trend changes its sign do carry less (or, when going to continuous time, even no) weight.

Clearly, there are some assumptions to discuss, e.g., continuous time trading. However, since the results of this work do not rely on any price path but only on the trend process, and there are high frequency trading possibilities, only a very hard non-trending assumption could invalidate these results. For example, one must assume that for every point in time with a positive (negative) price trend, for every arbitrary small interval after that point in time, there must be another point in time where the price trend is negative (positive). This would also imply that there are absolutely no identifiable trends in fundamental values. Adequate resources, perfect liquidity, the possibility of short selling, and approximately also the price-taker property can be seen as justified on modern stock exchanges when both the trader and the traded asset are big enough and I_0^* and K are chosen small enough (cf. Section 2.3).

When a person asks me to solve the puzzle, the only—more or less—satisfying answer I can give is that the risk measure is inappropriate (maybe *skewness* is better). But there are two problems: First, this idea only works when market efficiency is defined via risk-adjusted returns only (not when it is defined via expected gain). And second, I would run into a problem very similar to the joint hypotheses problem: I conjecture that for nearly every trading strategy, one can find two risk measures: one indicating that risk-adjusted returns are high, and one indicating that risk-adjusted returns are low. And the other way around, I also conjecture that for nearly all risk measures one can find two trading strategy: one that beats the market, and one that is beaten by it. No one can say whether or not the risk measure or the market efficiency hypothesis is wrong. Thus, I rely on a standard definition of risk-adjustment.

To sum up, there are three possibilities of how to solve the problem whether or not the SLS rule is beating the market (for a big, rich trader that trades small amounts of highly liquid stocks of big underlying firms). First, were I to assume that all assets are risk-neutral—and not only the market as a whole—the results would not hold. However, that would mean that at every point in time, the trend of every single stock is exactly the trend of the bond, no matter how volatile the stock. (Note: it is reasonable to assume that a high-volatile stock is riskier, and hence should have a higher trend.) Second, and a little bit weaker than the first argument, when the trends (the trends and not only the price paths) of the stocks jump in every (infinitesimally small) interval from positive to negative or vice versa, again the results will not hold. Third, and finally, we cannot adequately measure risk. This leads to a *risk including joint hypotheses problem*, because there is not one risk measure that everyone relies on. No one can say whether or not the used risk measure or the efficient market hypothesis is wrong. The last point is the most satisfying answer I can give.

Why do not all traders use the SLS rule if it really works well? Before discussing this, I mention that if all traders would use this rule, the market requirements would not be fulfilled anymore, and liquidity especially would be an issue. When all traders in a market follow the same rule (or even a very similar rule), trading volume would decrease (to zero) since all agents want to buy (sell) at the same time, but no one is selling (buying). I guess that again, risk is the answer: When most traders fear the high risk related to SLS trading, and only a very small fraction of traders use this rule, the non-SLS traders will be exploited by the SLS traders, since all market requirements are fulfilled.

At the very end of this work, I mention that the robust positive expectation property is not an arbitrage possibility. The gain is not sure, it is only in expectation. And it needs potentially a very high number of experiments, i.e., of trading processes, to realize, on average, a positive expected gain.

Acknowledgment

"Is Fama wrong?" This question was asked during my presentation at the International Symposium on Interdisciplinarity at the Università di Corsica Pasquale Paoli in Corte/Corti, France (Baumann et al., 2017). This question was the motivation for the work at hand. Therefore, I thank the audience of the track Interdisciplinarity in Economics.

Furthermore, I thank Lars Grüne (Applied Mathematics, University of Bayreuth, Germany), Bernhard Herz (International Economics and Finance, University of Bayreuth, Germany), and Michaela Baumann (Analytics Competence Center, AE-BI, NÜRNBERGER Versicherung, Germany) for their helpful support. Additionally, I thank Jane Lael for her professional editing work.

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1	ADIDAS	D:ADS
2	ALLIANZ	D:ALV
3	BASF	D:BAS
4	BAYER	D:BAYN
5	BEIERSDORF	D:BEI
6	BMW	D:BMW
7	COMMERZBANK	D:CBK
8	CONTINENTAL	D:CON
9	COVESTRO	D:1COV
10	DAIMLER	D:DAI
11	DEUTSCHE BANK	D:DBK
12	DEUTSCHE BOERSE	D:DB1
13	DEUTSCHE LUFTHANSA	D:LHA
14	DEUTSCHE POST	D:DPW
15	DEUTSCHE TELEKOM	D:DTE
16	E ON N	D:EOAN
17	FRESENIUS	D:FRE
18	FRESENIUS MED.CARE	D:FME
19	HEIDELBERGCEMENT	D:HEI
20	HENKEL	D:HEN
21	INFINEON TECHNOLOGIES	D:IFX
22	LINDE	D:LIN
23	MERCK KGAA	D:MRK
24	MUENCHENER RUCK	D:MUV2
25	RWE	D:RWE
26	SAP	D:SAP
27	SIEMENS	D:SIE
28	THYSSENKRUPP	D:TKA
29	VOLKSWAGEN	D:VOW
30	VONOVIA	D:VNA

Table 1: List of the 30 assets indexed in the DAX.

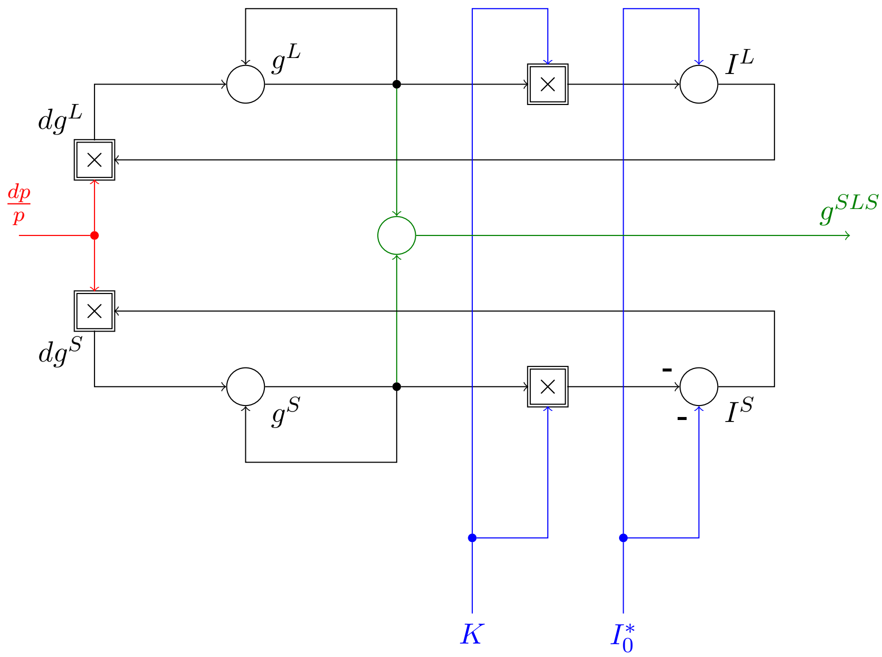

Figure 1: Flow diagram for the standard SLS controller with input (or disturbance) variable return on investment $\frac{dp}{p}$, i.e., price, and output variable gain g^{SLS}. The SLS traders' parameters are $K > 0$ and $I_0^* > 0$.

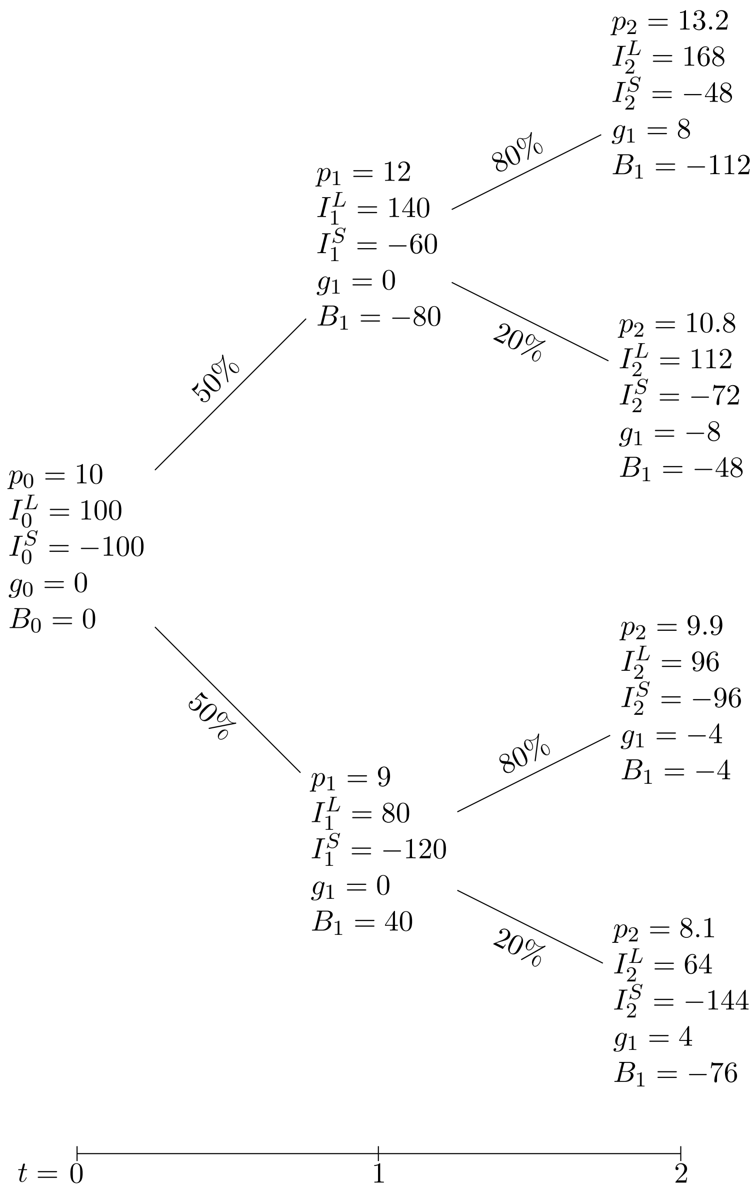

Figure 2: Small SLS trading example in a binomial tree model with three periods and trends: $\mu_1 = 5\%$ and $\mu_2 = 6\%$ and trading parameters: $I_0^* = 100$ and $K = 2$.

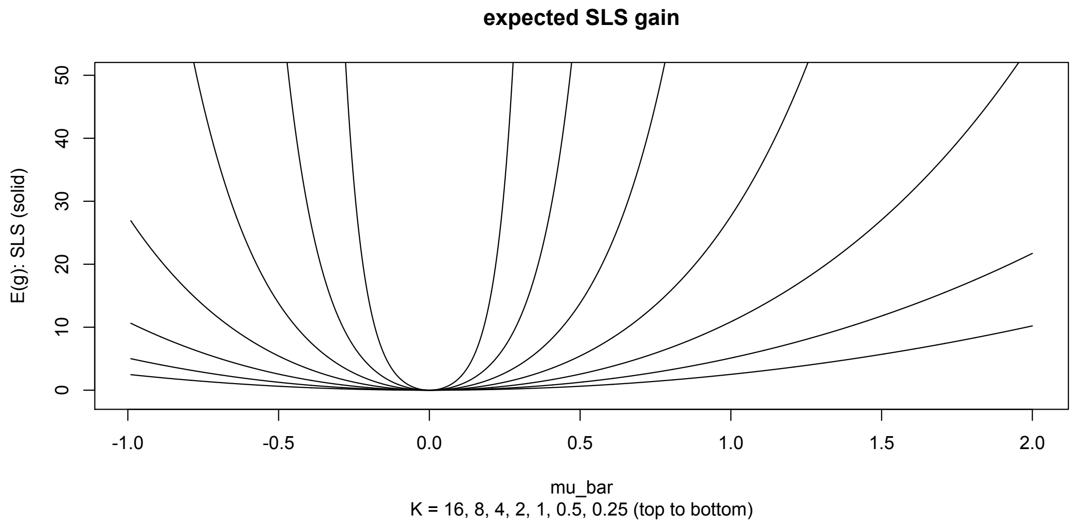

Figure 3: Expected gain of different SLS strategies with $I_0^* = 10$ and $K = 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}$. The average trend is $\bar{\mu} \in (-1, 2]$.

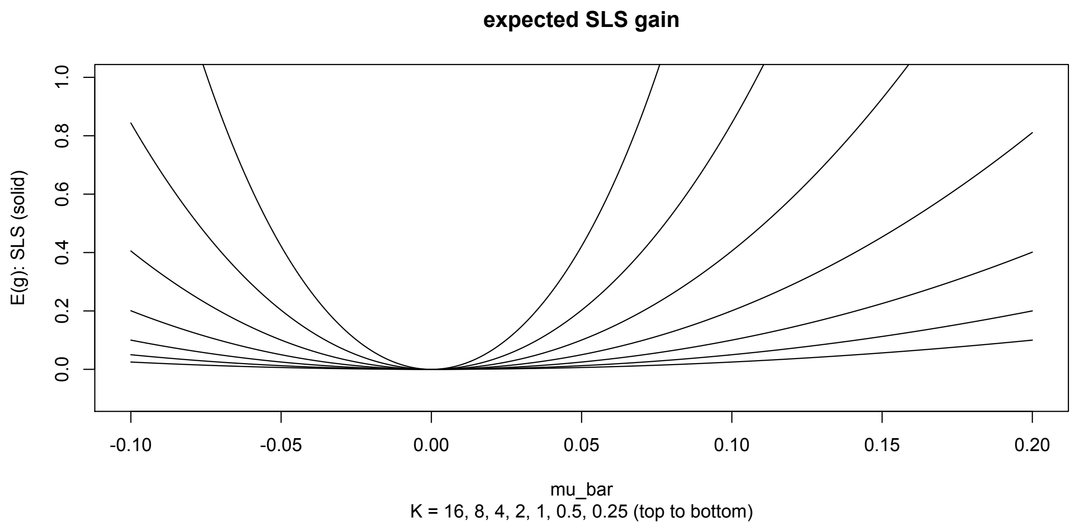

Figure 4: Expected gain of different SLS strategies with $I_0^* = 10$ and $K = 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}$. The average trend is $\bar{\mu} \in [-0.1, 0.2]$.

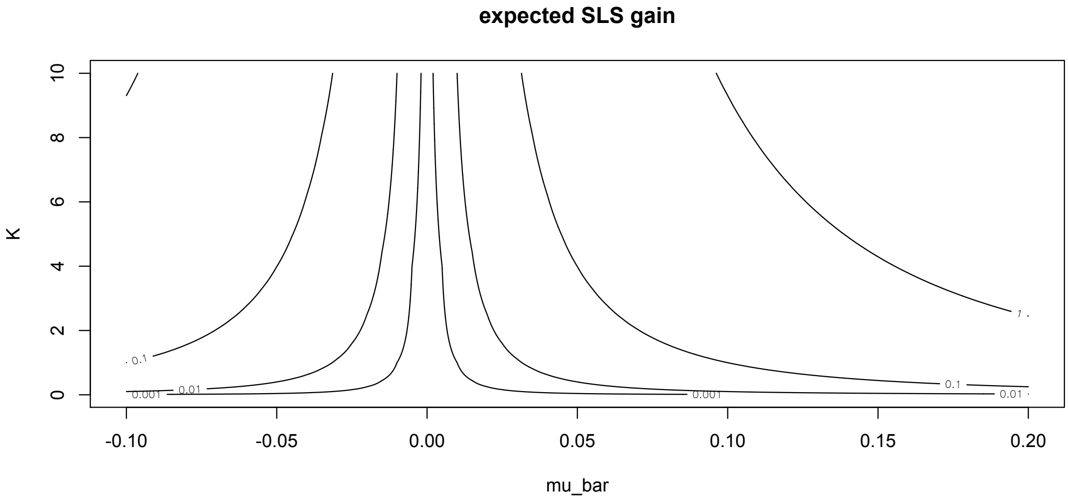

Figure 5: Contour plot of the expected gain of the SLS strategy for $K \in (0, 10]$ and $\overline{\mu} \in [-0.1, 0.2]$. The expected gain is positive for all $(K, \overline{\mu})$ with $\bar{\mu} \neq 0$.

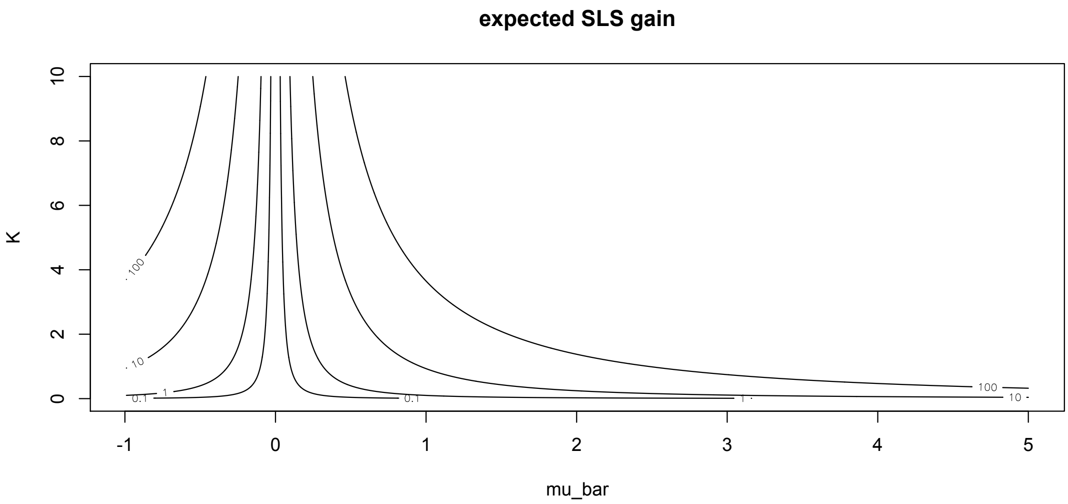

Figure 6: Contour plot of the expected gain of the SLS strategy for $K \in (0, 10]$ and $\overline{\mu} \in (-1, 5]$. The expected gain is positive for all $(K, \overline{\mu})$ with $\bar{\mu} \neq 0$.

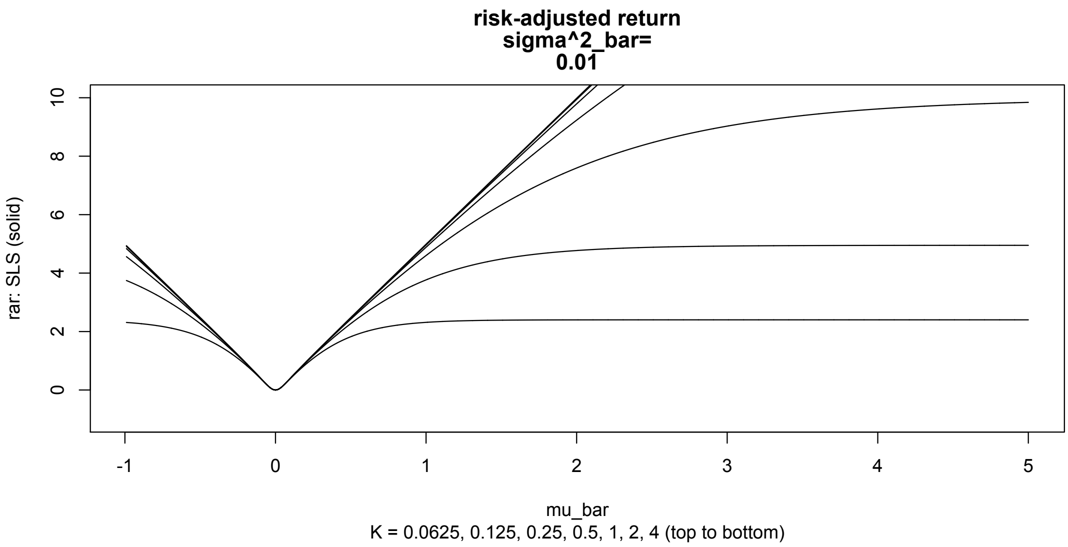

Figure 7: Risk-adjusted return of different SLS strategies with $I_0^* = 10$ and $K = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ (from top to bottom). All returns are adjusted with the respective standard deviation. The average trend is $\bar{\mu} \in (-1, 5]$, and the average volatility is $\overline{\sigma^2} = 1\%$.

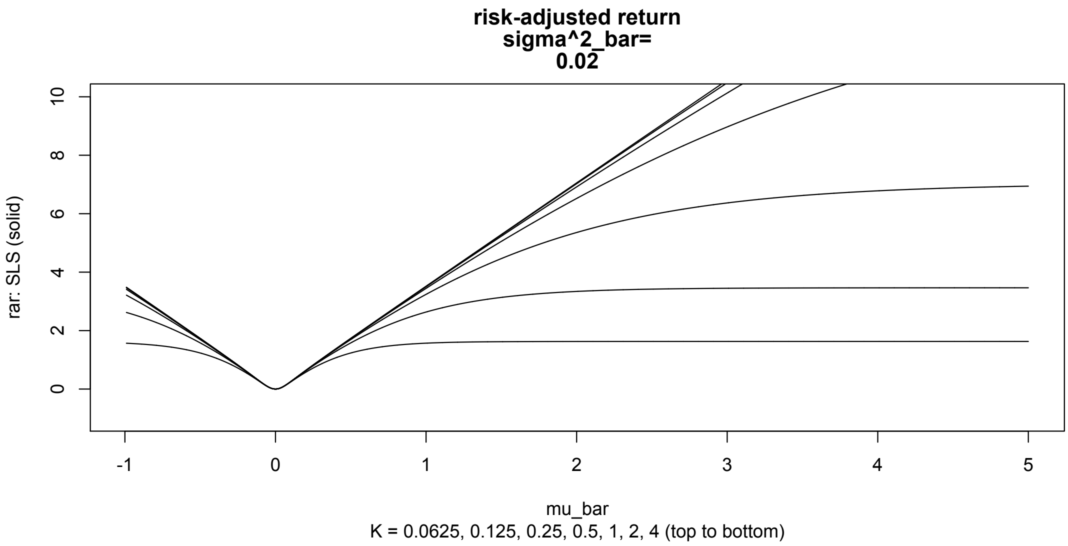

Figure 8: Risk-adjusted return of different SLS strategies with $I_0^* = 10$ and $K = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ (from top to bottom). All returns are adjusted with the respective standard deviation. The average trend is $\bar{\mu} \in (-1, 5]$, and the average volatility is $\overline{\sigma^2} = 2\%$.

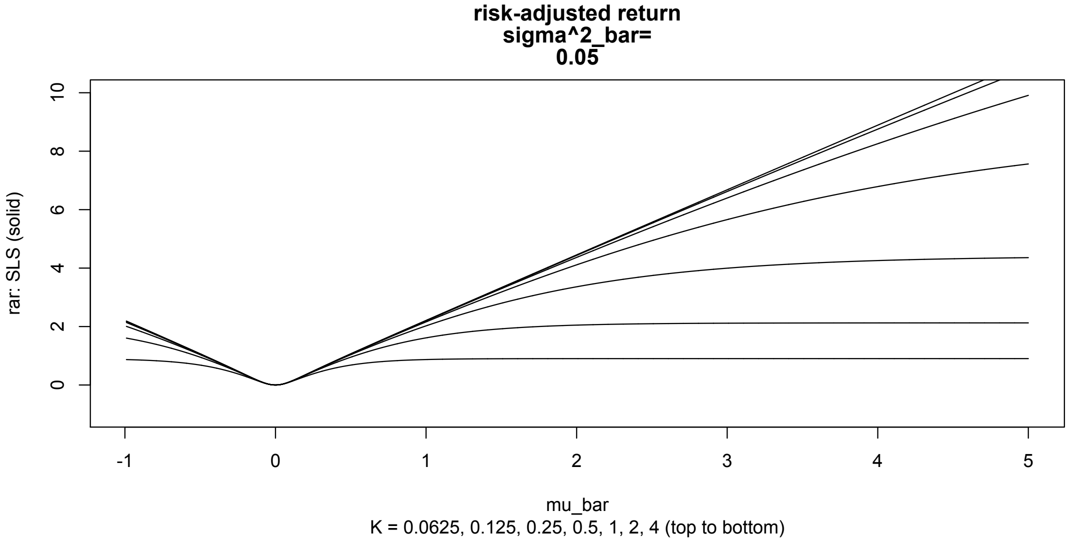

Figure 9: Risk-adjusted return of different SLS strategies with $I_0^* = 10$ and $K = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ (from top to bottom). All returns are adjusted with the respective standard deviation. The average trend is $\bar{\mu} \in (-1, 5]$, and the average volatility is $\overline{\sigma^2} = 5\%$.

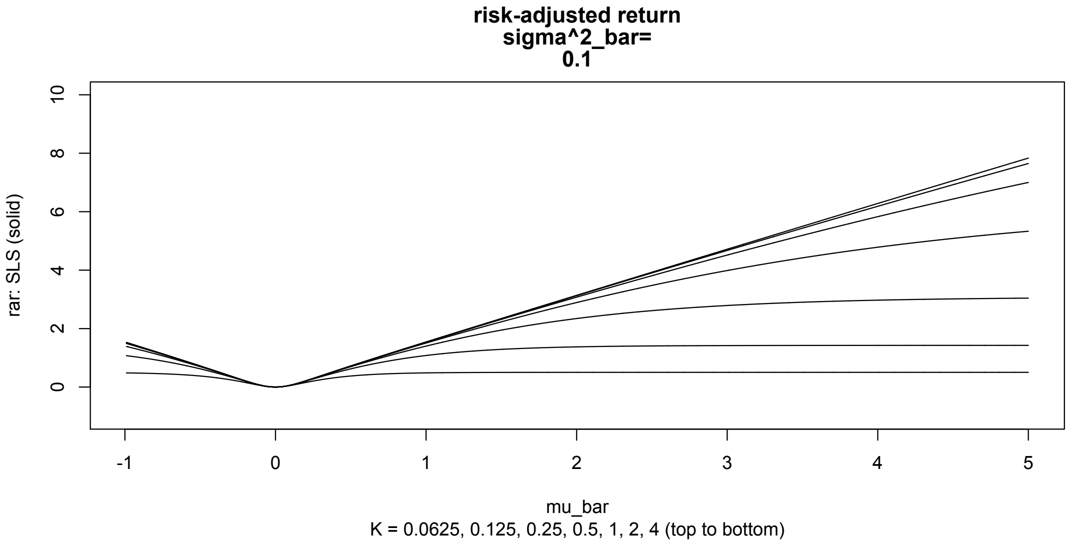

Figure 10: Risk-adjusted return of different SLS strategies with $I_0^* = 10$ and $K = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ (from top to bottom). All returns are adjusted with the respective standard deviation. The average trend is $\bar{\mu} \in (-1, 5]$, and the average volatility is $\overline{\sigma^2} = 10\%$.

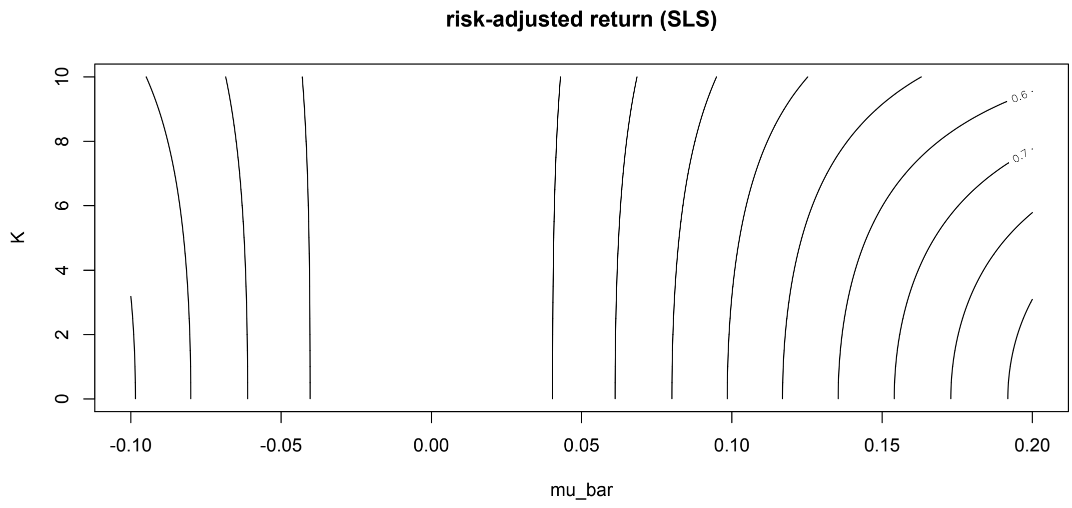

Figure 11: Contour plot of the risk-adjusted return of the SLS strategy for $K \in (0, 10]$ and $\overline{\mu} \in [-0.1, 0.2]$. For risk adjustment, I use the standard deviation. The risk-adjusted return is positive for all $(K, \overline{\mu})$ with $\bar{\mu} \neq 0$. The average volatility is $\overline{\sigma^2} = 1\%$.

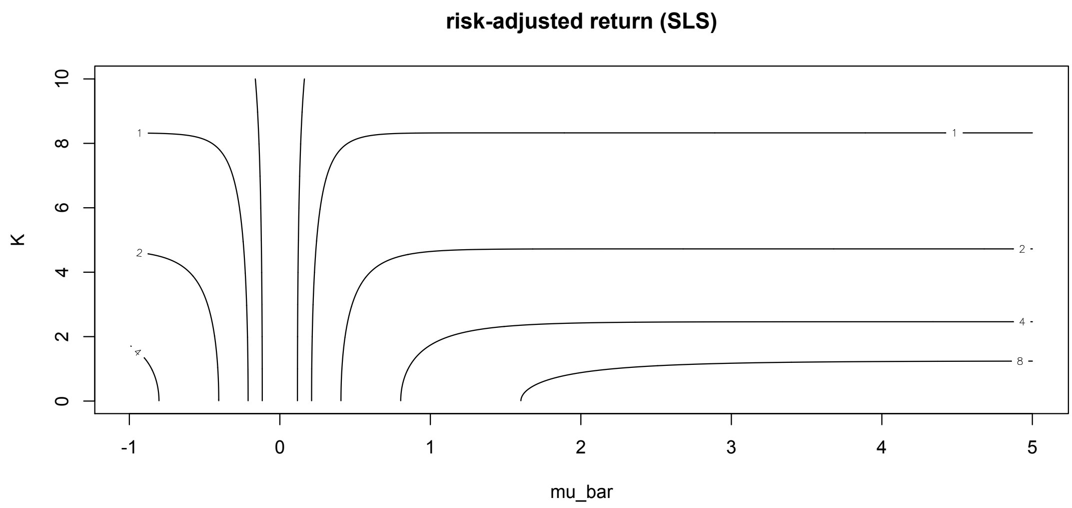

Figure 12: Contour plot of the risk-adjusted return of the SLS strategy for $K \in (0, 10]$ and $\overline{\mu} \in (-1, 5]$. For risk adjustment, I use the standard deviation. The risk-adjusted return is positive for all $(K, \overline{\mu})$ with $\bar{\mu} \neq 0$. The average volatility is $\overline{\sigma^2} = 1\%$.

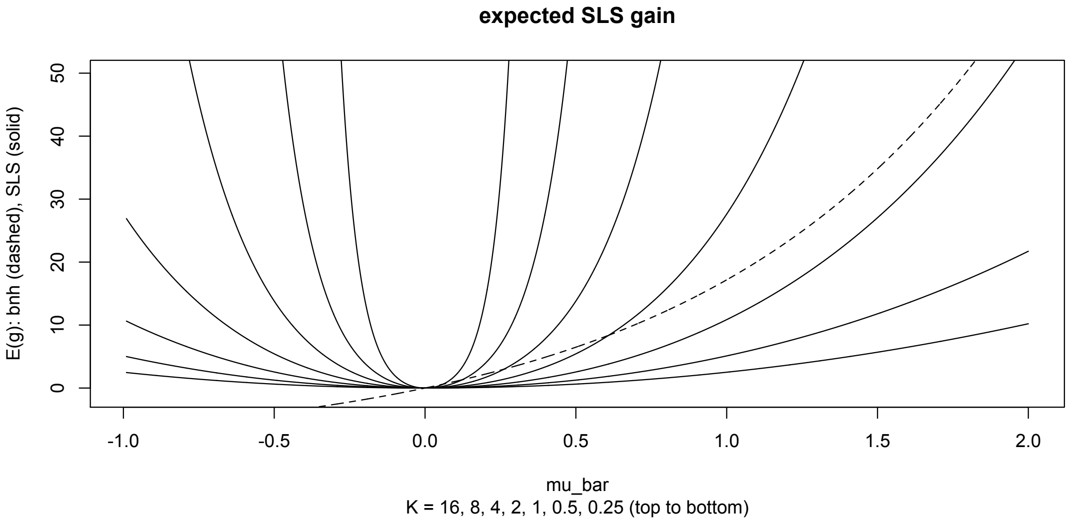

Figure 13: Expected gain of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}$ (from top to bottom) compared to the expected gain of a simple buy-and-hold strategy (dashed line) with initial investment 10. The average trend is $\bar{\mu} \in (-1, 2]$.

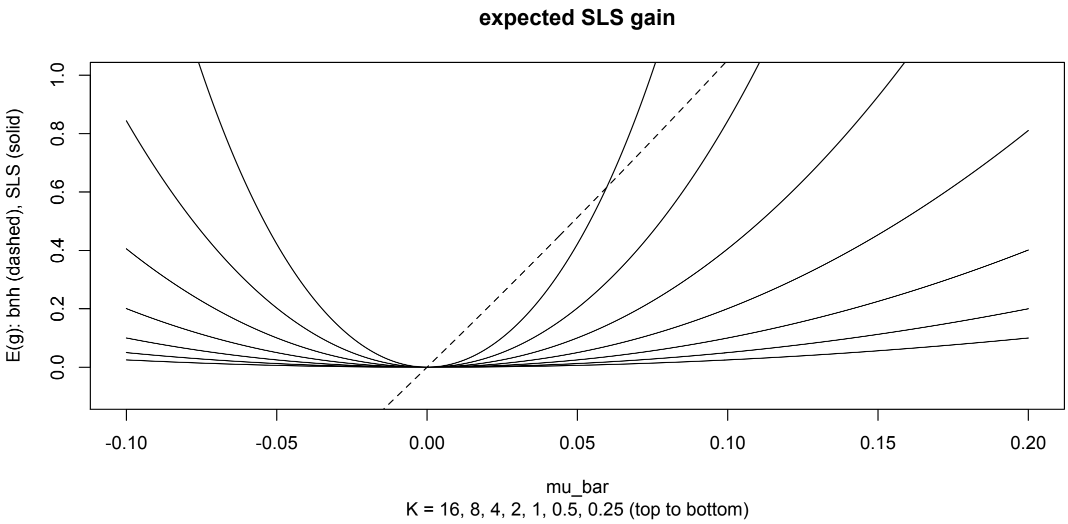

Figure 14: Expected gain of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}$ (from top to bottom) compared to the expected gain of a simple buy-and-hold strategy (dashed line) with initial investment 10. The average trend is $\bar{\mu} \in [-0.1, 0.2]$.

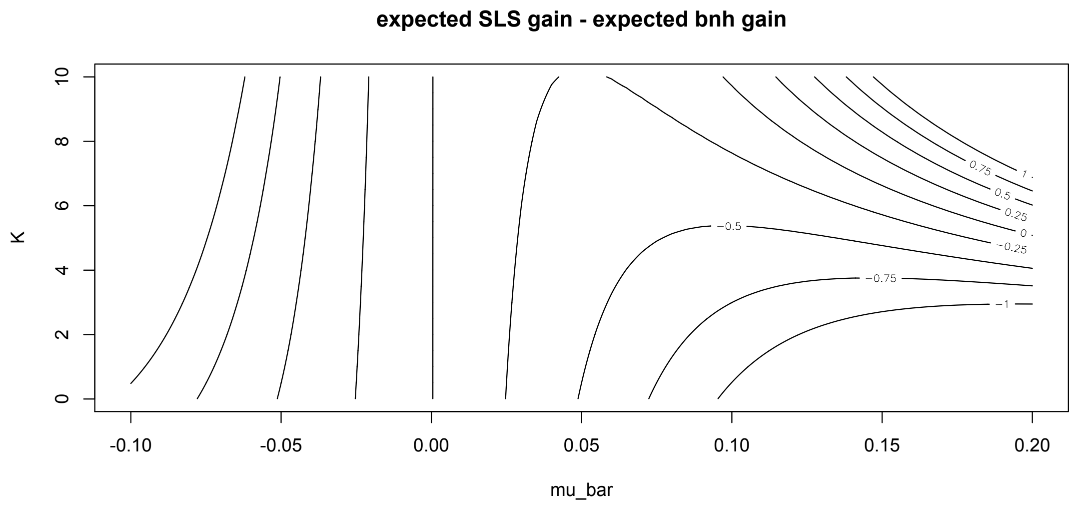

Figure 15: Contour plot of the expected difference of the gain of the SLS strategy and the bnh rule for $K \in (0, 10]$ and $\overline{\mu} \in [-0.1, 0.2]$. The expected difference is positive for all $(K, \overline{\mu})$ in the left as well as in the upper-right area.

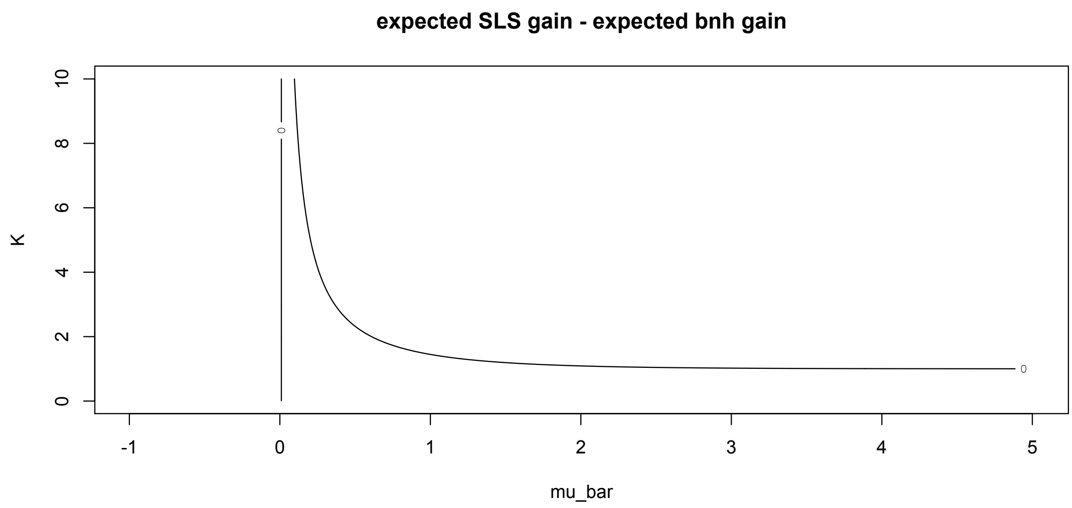

Figure 16: Contour plot of the expected difference of the gain of the SLS strategy and the bnh rule for $K \in (0, 10]$ and $\overline{\mu} \in (-1, 5]$. The expected difference is positive for all $(K, \overline{\mu})$ in the left as well as in the upper-right area.

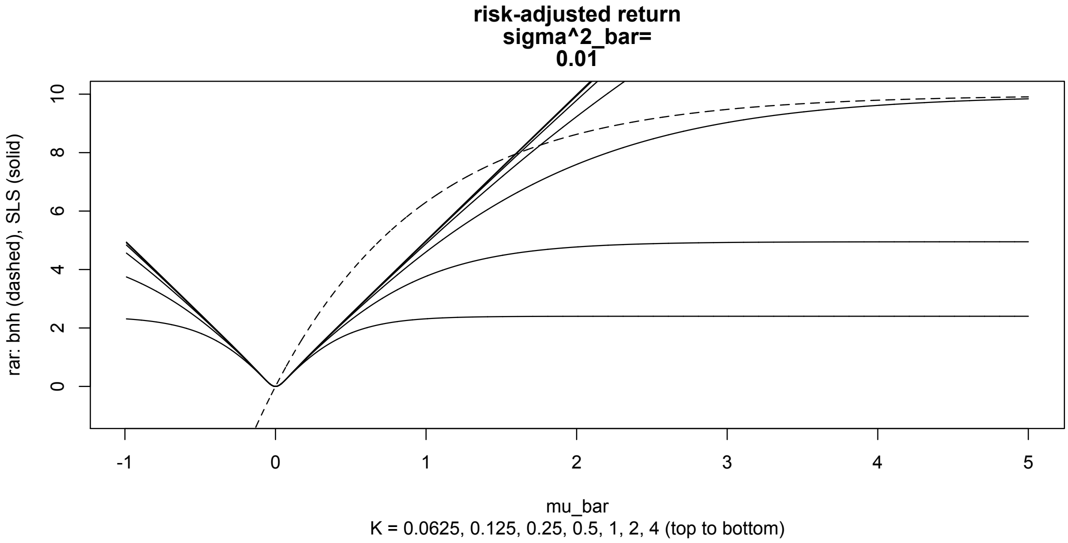

Figure 17: Risk-adjusted return of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ (from top to bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. All returns are adjusted with the respective standard deviation. The average trend is $\bar{\mu} \in (-1, 5]$, and the average volatility is $\overline{\sigma^2} = 1\%$.

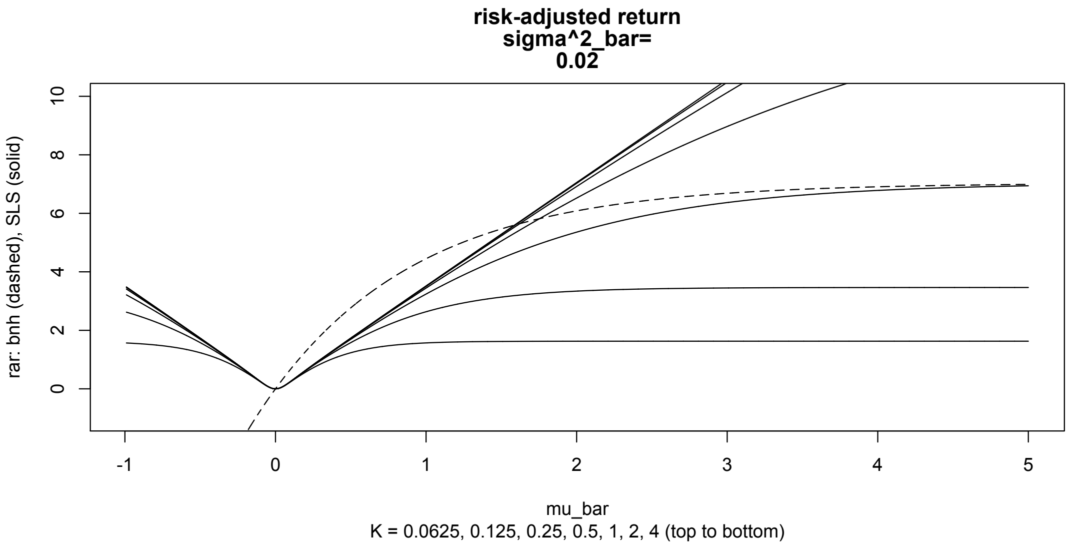

Figure 18: Risk-adjusted return of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ (from top to bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. All returns are adjusted with the respective standard deviation. The average trend is $\bar{\mu} \in (-1, 5]$, and the average volatility is $\overline{\sigma^2} = 2\%$.

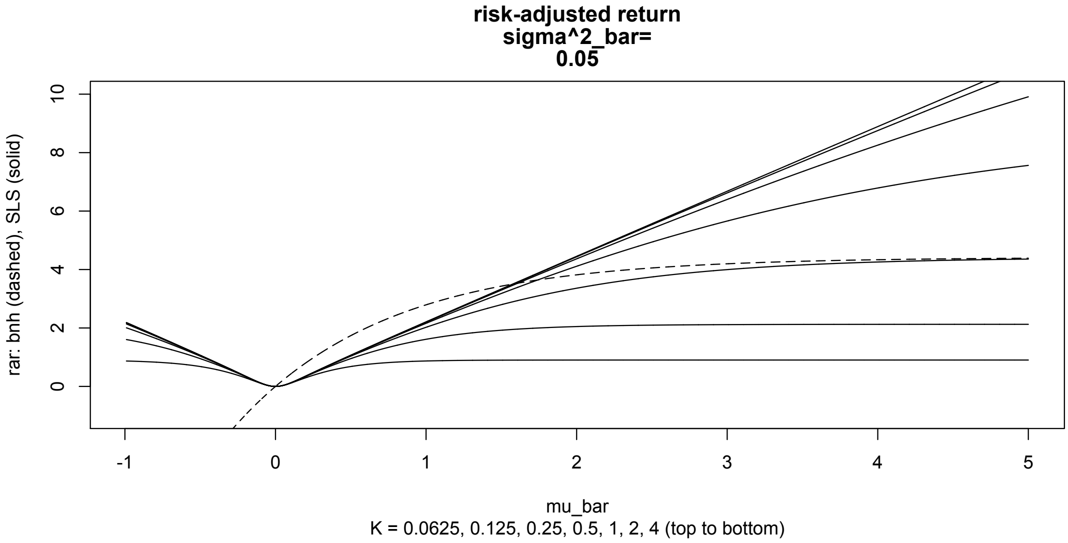

Figure 19: Risk-adjusted return of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ (from top to bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. All returns are adjusted with the respective standard deviation. The average trend is $\bar{\mu} \in (-1, 5]$, and the average volatility is $\overline{\sigma^2} = 5\%$.

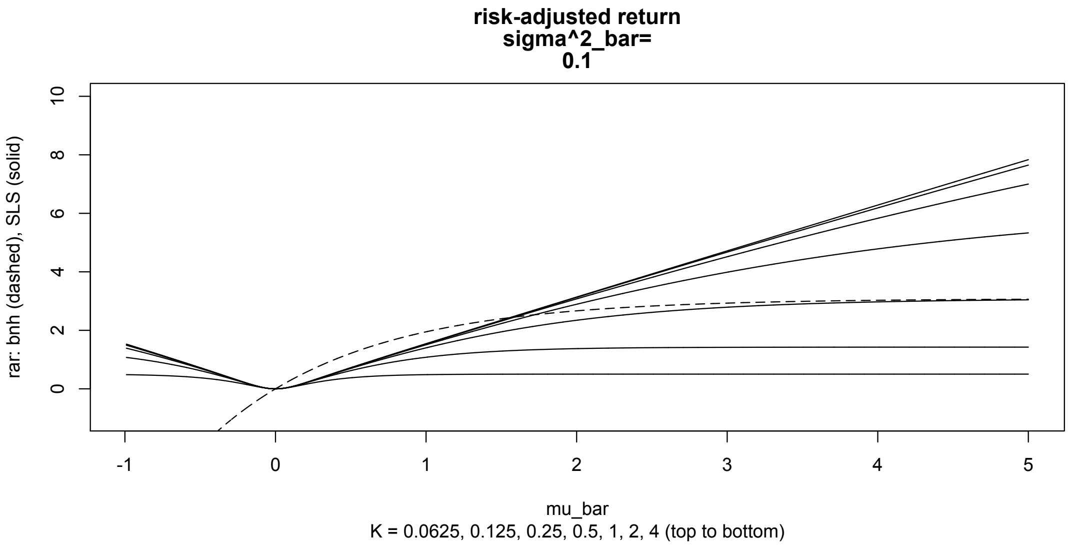

Figure 20: Risk-adjusted return of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ (from top to bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. All returns are adjusted with the respective standard deviation. The average trend is $\bar{\mu} \in (-1, 5]$, and the average volatility is $\overline{\sigma^2} = 10\%$.

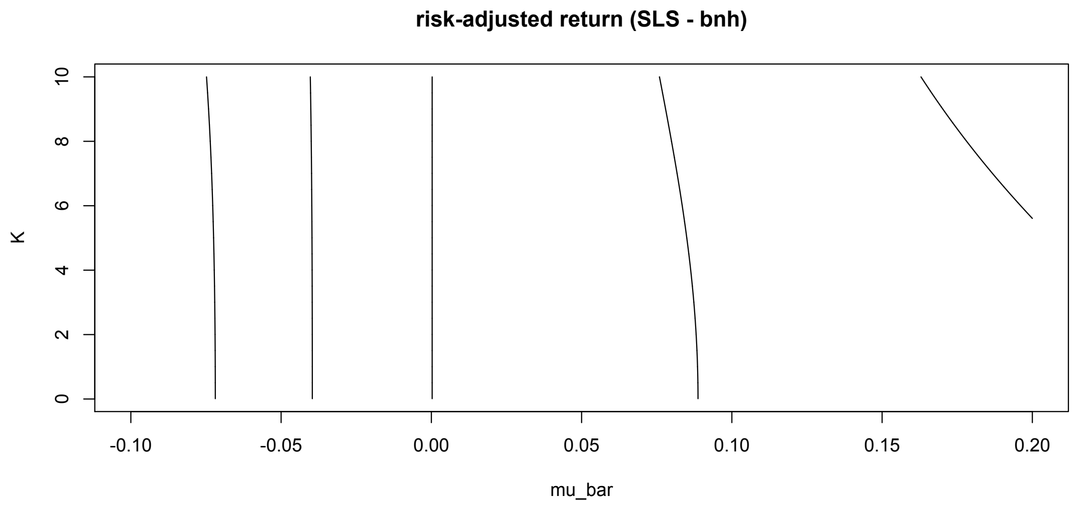

Figure 21: Contour plot of the difference of the risk-adjusted returns of the SLS rule and of a bnh rule. The average volatility is 1%. The SLS rule is dominant on the left side.

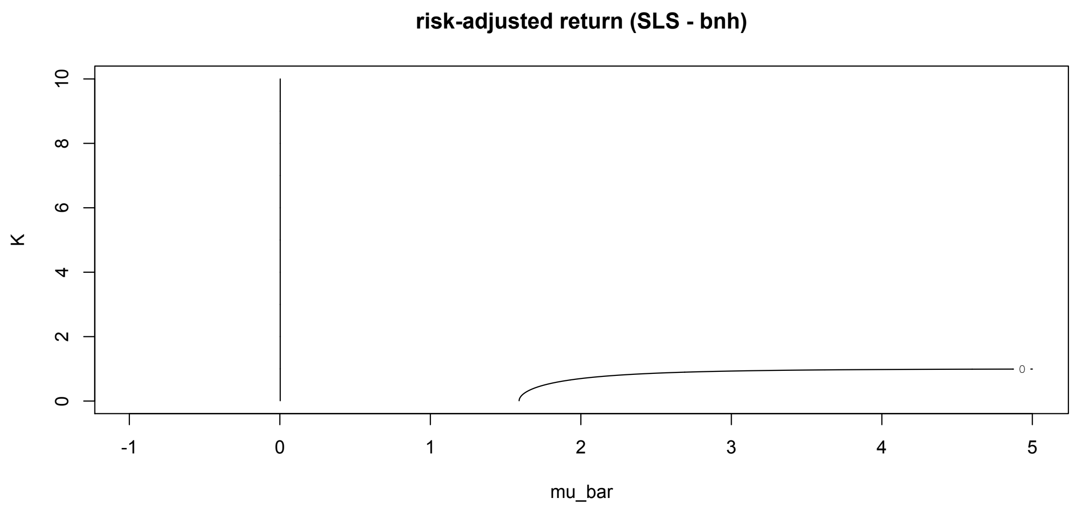

Figure 22: Contour plot of the difference of the risk-adjusted returns of the SLS rule and of a bnh rule. The average volatility is 1%. The SLS rule is dominant in the left and in the lower-right area.

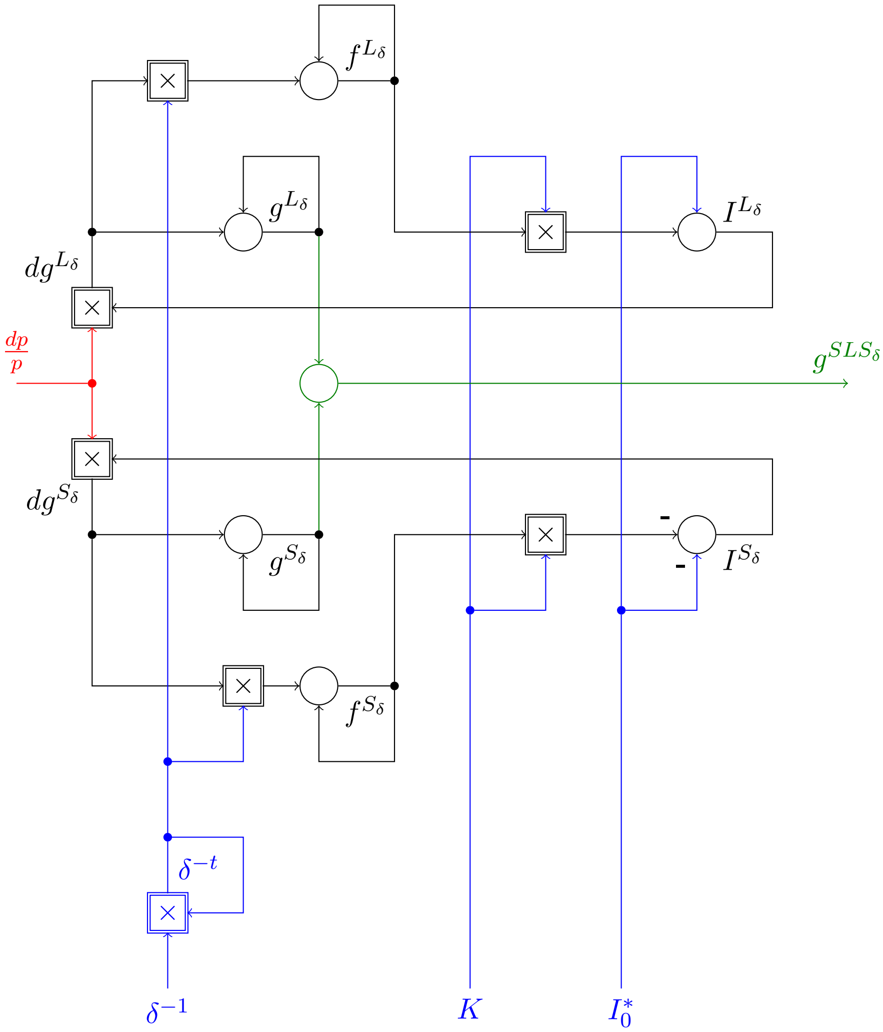

Figure 23: Flow diagram for the discounted SLS controller with input (or disturbance) variable return on investment $\frac{dp}{p}$, i.e., price, and output variable gain g^{SLS_δ}. The SLS$_\delta$ traders' parameters are $K > 0$, $I_0^* > 0$, and $\delta \in (0, 1]$.

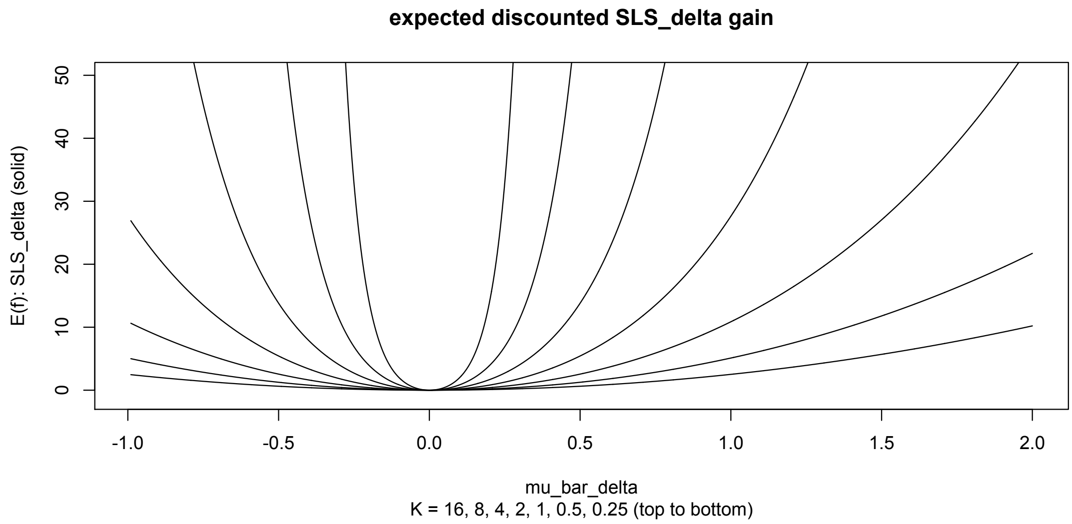

Figure 24: Expected discounted gain of different SLS_δ strategies with $I_0^* = 10$ and $K = 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}$ (from top to bottom). The average trend is $\bar{\mu}_\delta \in (-1, 2]$.

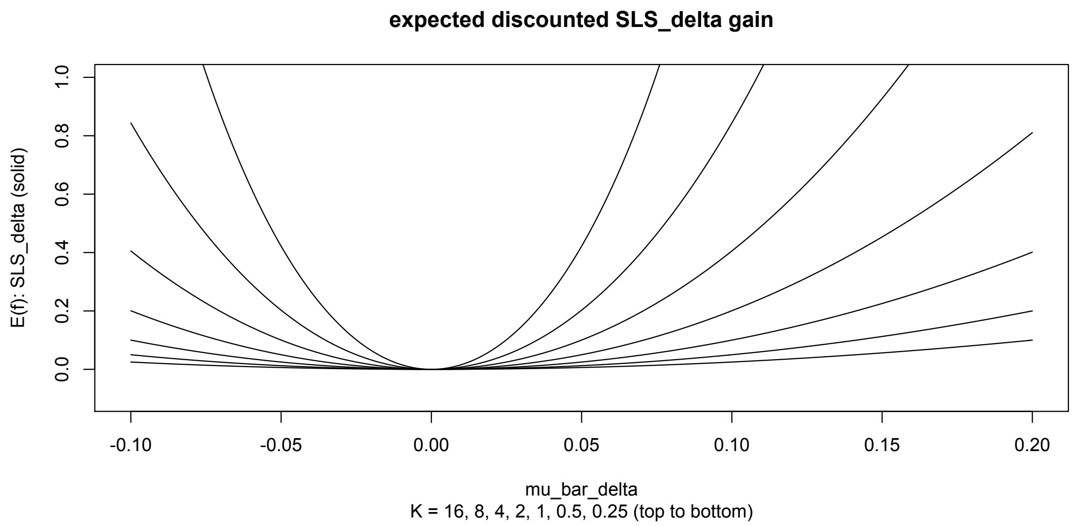

Figure 25: Expected discounted gain of different SLS_δ strategies with $I_0^* = 10$ and $K = 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}$ (from top to bottom). The average trend is $\bar{\mu}_\delta \in [-0.1, 0.2]$.

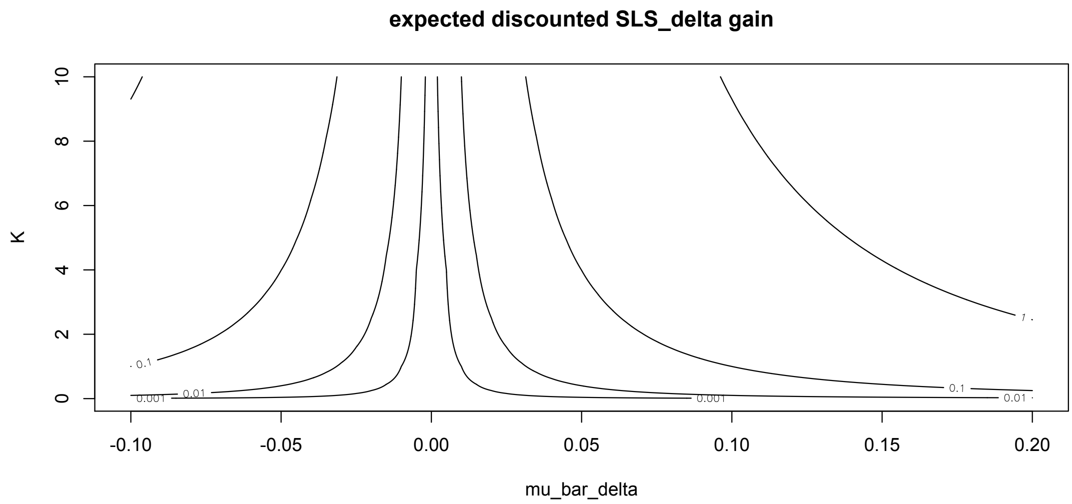

Figure 26: Contour plot of the expected discounted gain of the SLS_δ strategy for $K \in (0, 10]$ and $\overline{\mu}_\delta \in [-0.1, 0.2]$. The expected discounted gain is positive for all $(K, \overline{\mu}_\delta)$ with $\bar{\mu}_\delta \neq 0$.

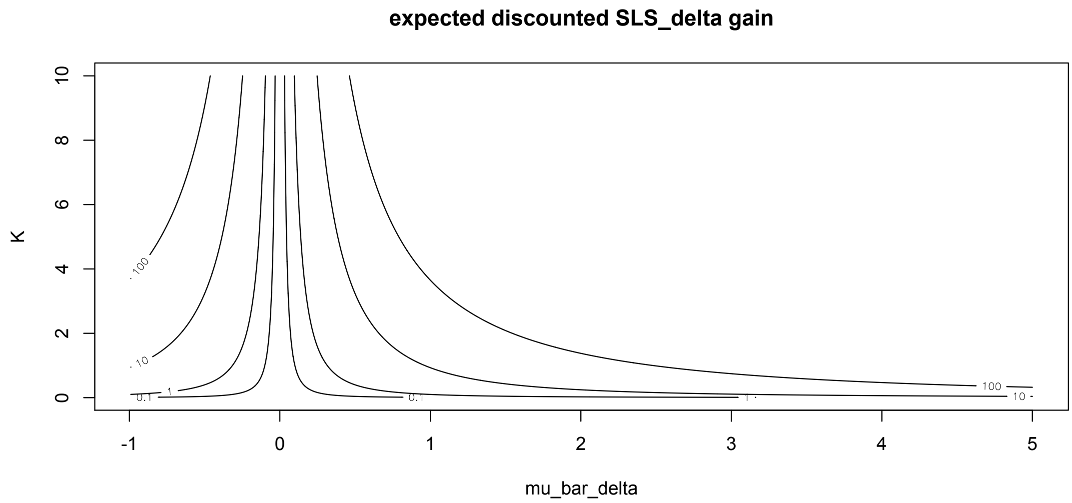

Figure 27: Contour plot of the expected discounted gain of the SLS_δ strategy for $K \in (0, 10]$ and $\overline{\mu}_\delta \in (-1, 5]$. The expected discounted gain is positive for all $(K, \overline{\mu}_\delta)$ with $\bar{\mu}_\delta \neq 0$.

D:ADS	2016	3940.43	(9813.83)	D:ADS	2017	563.09	(5080.57)
D:ALV	2016	414.34	(1660.64)	D:ALV	2017	598.92	(4460.97)
D:BAS	2016	963.51	(4878.37)	D:BAS	2017	-157.50	(2262.67)
D:BAYN	2016	252.17	(565.00)	D:BAYN	2017	79.30	(3891.56)
D:BEI	2016	-122.56	(1673.79)	D:BEI	2017	343.11	(3932.05)
D:BMW	2016	590.88	(1638.89)	D:BMW	2017	-300.53	(947.82)
D:CBK	2016	2210.57	(1013.92)	D:CBK	2017	3352.48	(8394.89)
D:CON	2016	550.15	(0)	D:CON	2017	572.92	(3924.56)
D:1COV	2016	6050.83	(11534.37)	D:1COV	2017	1066.46	(5490.10)
D:DAI	2016	431.54	(1149.03)	D:DAI	2017	-287.80	(1299.09)
D:DBK	2016	4196.28	(2116.84)	D:DBK	2017	242.12	(2924.18)
D:DB1	2016	116.38	(1807.27)	D:DB1	2017	725.72	(4570.14)
D:LHA	2016	658.69	(1549.41)	D:LHA	2017	10500.83	(15663.91)
D:DPW	2016	739.30	(4242.67)	D:DPW	2017	553.95	(4763.87)
D:DTE	2016	-136.57	(1549.40)	D:DTE	2017	-397.80	(2050.96)
D:EOAN	2016	340.38	(3508.24)	D:EOAN	2017	1133.36	(7412.42)
D:FRE	2016	537.18	(3554.21)	D:FRE	2017	-122.22	(1619.20)
D:FME	2016	185.85	(2949.88)	D:FME	2017	-77.69	(2122.15)
D:HEI	2016	941.14	(4548.62)	D:HEI	2017	-120.13	(1774.43)
D:HEN	2016	414.71	(4176.18)	D:HEN	2017	-205.56	(2574.39)
D:IFX	2016	1102.81	(4922.52)	D:IFX	2017	1374.82	(7021.69)
D:LIN	2016	769.61	(4565.62)	D:LIN	2017	240.24	(3914.84)
D:MRK	2016	471.36	(3429.72)	D:MRK	2017	-176.01	(2531.87)
D:MUV2	2016	138.26	(1445.63)	D:MUV2	2017	-57.93	(2213.70)
D:RWE	2016	1209.46	(6680.48)	D:RWE	2017	2155.15	(10414.44)
D:SAP	2016	392.64	(3467.82)	D:SAP	2017	102.72	(3462.82)
D:SIE	2016	1333.50	(5448.70)	D:SIE	2017	-52.84	(2576.32)
D:TKA	2016	1855.07	(5794.69)	D:TKA	2017	46.88	(3438.85)
D:VOW	2016	699.46	(2688.02)	D:VOW	2017	443.65	(4191.21)
D:VNA	2016	374.80	(5371.77)	D:VNA	2017	744.46	(5036.91)
mean (max)	2016	1054.07	(88811.02)	mean (max)	2017	762.80	(102604.4)
mean (max)	2016/17	908.44	(102604.4)				

Table 2: Trading gains of SLS trading for the 30 DAX stocks 2016 and 2017 as well as the maximum amount of money needed (in brackets). Parameters: $K = 2$, $I_0^* = 5,000$.

$I_0^* \setminus K$	0.5	1	1.5	2
500	-82.20 (3452.21)	-45.00 (7440.97)	-45.11 (7837.87)	-46.90 (8900.72)
1000	-111.38 (6985.3)	-50.60 (14279.65)	-29.47 (16712.75)	-11.20 (19113.39)
1500	-118.71 (9262.28)	-33.86 (19550.74)	20.94 (25313.97)	72.20 (30370.5)
2000	-118.67 (11807.72)	-0.46 (24784.13)	88.91 (32883.39)	167.47 (40641.22)

Table 3: Average trading gains for SLS trading with varying parameters for the DAX charts of 2016 and 2017.

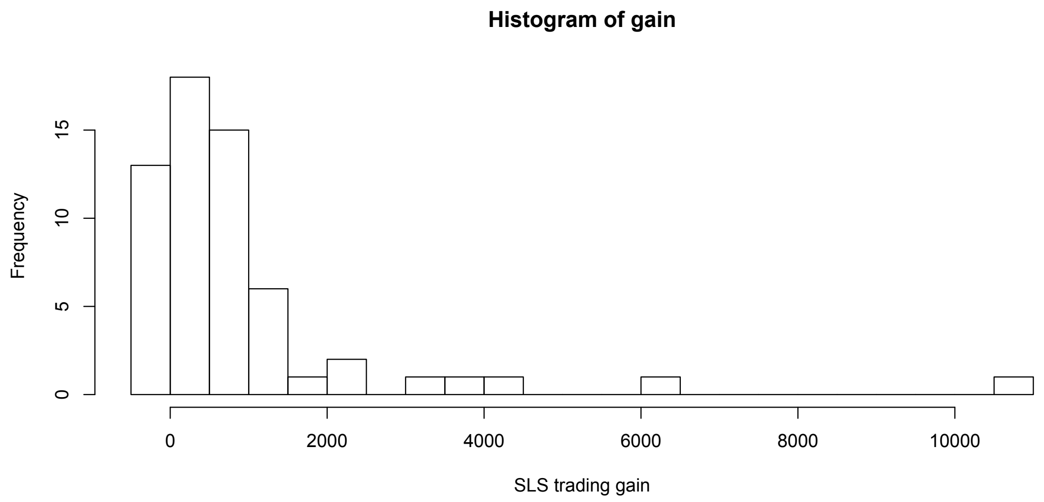

Figure 28: Histogram of the trading gains for the 30 DAX assets when SLS trading with $K = 2$ and $I_0^* = 5,000$ in 2016 and 2017.

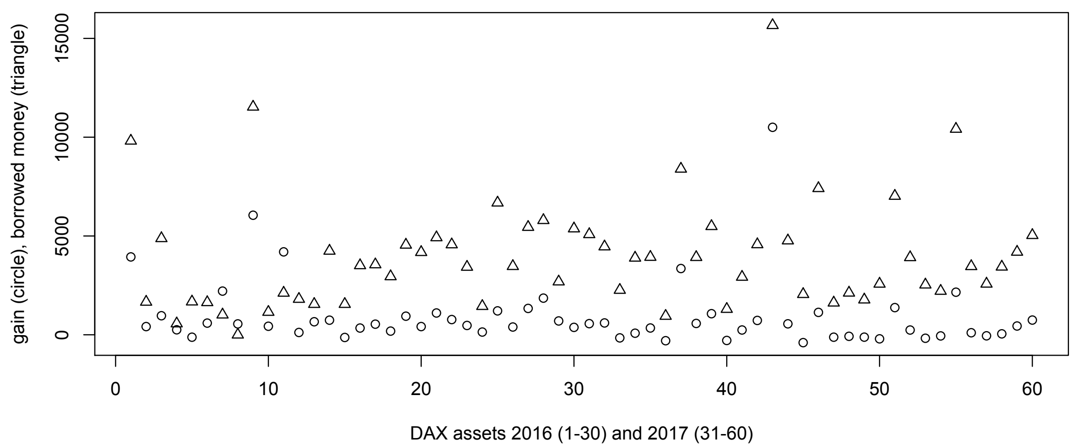

Figure 29: Trading gains for the 30 DAX assets when SLS trading with $K = 2$ and $I_0^* = 5,000$ in 2016 and 2017 (circle) and maximum amount of money needed (triangle).

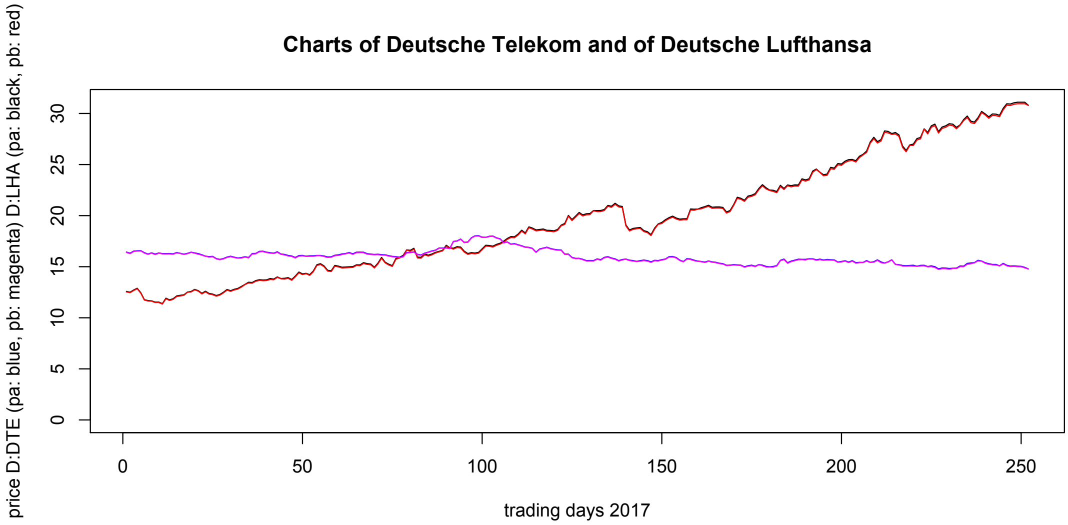

Figure 30: Charts of the bid-and-ask prices of the Deutsche Lufthansa stock and of the Deutsche Telekom stock in 2017.

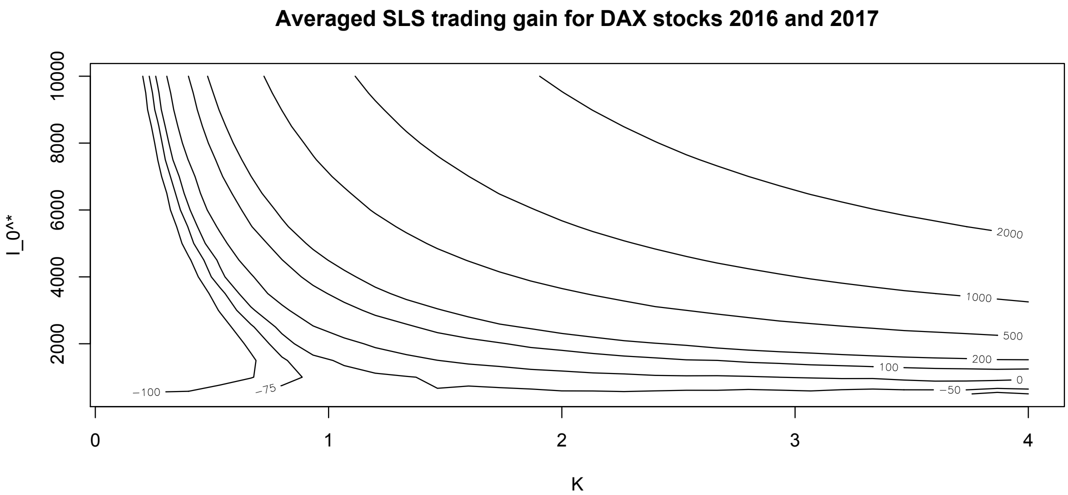

Figure 31: Contour plot of the average trading gains for the 30 DAX assets when SLS trading with $K \in (0, 4]$ and $I_0^* \in (0\ ,\ 10,000]$ for 2016 and 2017. The gain is positive on average from the middle to the upper-right area.