Performance Estimates for Scalar and Multiobjective Model Predictive Control Schemes

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Abstract (english/german)

Abstract

Since its first formulation in the middle of the twentieth century, Model Predictive Control (MPC) has become a very well known and investigated method for feedback synthesis of optimal control problems (OCPs). The acceptance of MPC in science as well as in industry has steadily increased within the last two to three decades. Many different schemes regarding the kind of problems and variants of MPC algorithms have been investigated, accompanied by the development of fast and/or robust (distributed) optimization methods. However, it seems that only little work has been done when it comes to the systematic investigation of multiobjective (MO) OCPs and game-theoretic settings.

The main questions of this thesis is the following: Given that we choose a specific solution (Pareto-optimum, optimum or Nash equilibrium (NE)) in each MPC iteration, can the characteristics of these strategies be carried over to the closed loop? And, if yes, how can this be achieved, i.e. how do we choose the 'right' solutions? Moreover, we are interested in the behavior of the resulting closed-loop trajectories. These questions are tackled for (scalar-valued) economic MPC without terminal conditions, for MO MPC with and without terminal conditions and for stabilizing and economic stage costs, and for games, in which a NE is played in each iteration.

For economic MPC schemes without terminal conditions we provide a practical Lyapunov function (LF) and can thus prove practical asymptotic stability as well as approximately optimal performance of the MPC controller during the so called transient phase.

In the context of MO MPC we show that imposing an additional constraint on the objective functions in the iterations enables us to make statements on the MPC performance for all objective functions, and to prove convergence of the closed-loop trajectory. This procedure prevents us from calculating the Pareto front in each iteration, which generally is an expensive computation. We show that the MPC performance is determined in the very first iteration of the MPC procedure.

In noncooperative MPC we show that the mechanism developed in MO MPC – i.e. choosing the proper solution by means of constraints on the objective functions – does generally not work for NE. For the special case of affine-quadratic games sufficient conditions for the MPC closed-loop trajectory to converge are presented.

Kurzfassung

Modellprädiktive Regelung (MPC) ist eine gut untersuchte, numerische Methode zur Approximation von Optimalsteuerungsproblemen, die seit ihrer Formulierung Mitte des zwanzigsten Jahrhunderts starke Verbreitung gefunden hat. Die Akzeptanz von MPC sowohl in der Wissenschaft als auch in der industriellen Praxis hat sich in den vergangenen 20-30 Jahren stetig ausgeweitet. In dieser Zeit wurden viele verschiedene Varianten von MPC Algorithmen vorgeschlagen und untersucht und auch die Klasse von Problemen, für die MPC verwendet wird, wurde immer wieder erweitert. Begleitet wurde diese Entwicklung von Fortschritten in Bezug auf schnelle und/oder robuste (verteilte) Optimierungsalgorithmen. Trotz der aktiven Forschung im Bereich MPC gibt es wenig Resultate, die sich mit strukturellen Aussagen für MPC von multikriteriellen Optimalsteuerungsproblemen und spieltheoretischen Problemen beschäftigen.

Diesbezüglich interessieren uns in der vorliegenden Arbeit im Wesentlichen zwei Fragestellungen. Angenommen, wir wählen in jeder Iteration des MPC Algorithmus eine ausgezeichnete Kontrollfolge (z.B. ein Pareto-Optimum, Optimum oder Nash-Gleichgewicht) und wenden sie in der für MPC typischen Art an. Lassen sich daraus Aussagen ableiten, dass der MPC Regler ebenfalls (Pareto-)Optimalität auf unendlichem Horizont aufweist? Und falls dem so ist, wie wählt man in den Iterationen von MPC die richtige Kontrollfolge aus? Von weiterem Interesse ist auch das Trajektorienverhalten des geschlossenen Regelkreises. Die soeben gestellten Fragen werden wir für ökonomisches MPC ohne Endbedingungen, für multikriterielles MPC mit und ohne Endbedingungen für stabilisierende und ökonomische Kosten, sowie für dynamische Spiele, in denen Nash-Gleichgewichte gespielt werden, systematisch untersuchen.

Für ökonomisches MPC werden wir hinreichende Bedingungen für die Existenz einer Lyapunov-Funktion vorstellen, was wiederum den Nachweis von praktischer asymptotischer Stabilität und die Analyse der Regelgüte von MPC während der sogenannten Übergangsphase ermöglicht.

Bei multikriteriellem MPC zeigen wir, dass man Aussagen über die Regelgüte und die Trajektorie des geschlossenen Regelkreises treffen kann, wenn man in den Iterationen eine zusätzliche Nebenbedingung an die Kostenfunktionale stellt. Hervorzuheben ist dabei, dass wir eine garantierte Mindestregelgüte für jedes der Kostenkriterien erhalten und dass kein Berechnen der gesamten Paretofront (was im Allgemeinen sehr aufwändig ist) in den Iterationen erfolgen muss. Die Mindestregelgüte wird bereits in der ersten MPC Iteration festgelegt.

In Bezug auf nicht-kooperatives MPC zeigen wir, dass der zuvor erwähnte Auswahlmechanismus von Lösungen – nämlich das Einführen zusätzlicher Nebenbedingungen an die Kostenfunktionale – bei Nash-Gleichgewichten im Allgemeinen nicht funktioniert. Für den Spezialfall der affin-quadratischen Spiele präsentieren wir hinreichende Bedingungen, sodass Trajektorienkonvergenz des geschlossenen Regelkreises sichergestellt werden kann.

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1 | Introduction

"Life is about decisions."

Matthias Ehrgott

It is a natural idea that optimal control problems (OCPs) of real-word applications have multiple potentially conflicting objectives, see e.g. [55]. Already for one objective solving OCPs on an infinite or very long time horizon is in general numerically and theoretically intractable and is even more involved for multiobjective (MO) OCPs. A remedy for this challenge is to use Model Predictive Control (MPC) – a numerical method whose main idea (formulated e.g. in Lee and Markus [51]) is to repeatedly solve the original problem on a short(er) horizon and to implement only the first part of the solution to the system. Proceeding this way, closed-loop solutions on arbitrary horizons can be generated. Since each of the "easy-to-solve" MO optimization problems in the MPC iterations generally has an infinite number of equally optimal solutions, we are confronted with the situation that we have to repeatedly choose 'the right' solution because only one among them can be applied to the system. While some works (see e.g. [24, 50]) approach this challenge by calculating all optimal solutions and then decide based on rules or expert knowledge, other works (e.g. [6, 43, 73, 90]) avoid the incidence of multiple solutions by transforming the MO optimization problem into a scalar optimization problem, which is then solved in the MPC procedure.

In this thesis we aim to design efficient MPC algorithms, in which the calculation of all optimal solutions is not needed, and that do not rely on a certain scalarization technique. At the same time, the algorithms should guarantee properties of the closed loop, e.g. trajectory convergence as well as performance statements for all objectives. In other words we want to take the right decisions without knowing all alternatives.

In the course of this chapter we will formally introduce the concept of MPC for scalar OCPs along with control-theoretic definitions. In Section 1.2 we present the outline and contribution of this thesis.

1.1 Fundamentals of Model Predictive Control

Assume that we are given a *state space* $\mathbb{R}^n$, a *control space* $\mathbb{R}^m$ for $n,\ m \in \mathbb{N}$, and a mapping $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ defining a discrete-time *dynamical system* or *control system*

$$\begin{aligned} x(k+1, x_0) &= f(x(k, x_0), u(k)),\ k \in \mathbb{N}_0, \\ x(0, x_0) &= x_0. \end{aligned} \tag{1.1}$$

The definition of a control system will sometimes be abbreviated by $x^+ = f(x, u)$. Equation (1.1) describes how the state x of the dynamical system evolves in time k under the influence of the control u. We will use the notation $x^{\mathbf{u}}(\cdot, x_0)$ for a trajectory resulting from input $\mathbf{u}$ and with initial value x_0, in which bold characters such as $\mathbf{u}$ always denote a control sequence of control values.

If there is a *cost criterion* $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$, defined on *constraint sets* $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$, we can define the infinite-horizon OCP[1]

$$\begin{aligned} &\min_{\mathbf{u} \in \mathbb{U}^\infty} J^\infty(x_0, \mathbf{u}),\ \text{with } J^\infty(x_0, \mathbf{u}) := \sum_{k=0}^{\infty} \ell(x(k, x_0), u(k)) \\ &\text{s.t. } (1.1), \\ &\quad x^{\mathbf{u}}(k, x_0) \in \mathbb{X} \text{ for all } k \in \mathbb{N}_0. \end{aligned}$$

Using the definition

$$\mathbb{U}^\infty(x_0) := \{\mathbf{u} \in \mathbb{U}^\infty | x^{\mathbf{u}}(k, x_0) \in \mathbb{X} \text{ for all } k \in \mathbb{N}_0\},$$

we can rewrite the OCP as follows:

$$\begin{aligned} &\min_{\mathbf{u} \in \mathbb{U}^\infty(x_0)} J^\infty(x_0, \mathbf{u}) \\ &\text{s.t. } (1.1). \end{aligned} \tag{1.2}$$

The corresponding *optimal value function* $V^\infty(x_0)$ is defined as the optimal value of (1.2).

The basic idea of MPC is now to replace the infinite-horizon OCP (1.2) by a sequence of finite-horizon OCPs. For this purpose, we fix the *MPC horizon* $N \in \mathbb{N}$ and repeatedly solve

$$\begin{aligned} &\min_{\mathbf{u} \in \mathbb{U}^N} J^N(x, \mathbf{u}),\ \text{with } J^N(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell(x(k, x), u(k)) \\ &\text{s.t. } (1.1) \text{ for all } k \in \{0, \dots, N-1\}, \\ &\quad x^{\mathbf{u}}(k, x) \in \mathbb{X} \text{ for all } k \in \{0, \dots, N\}. \end{aligned} \tag{1.3}$$

Again, we summarize all constraints by defining

$$\mathbb{U}^N(x) := \{\mathbf{u} \in \mathbb{U}^N | x^{\mathbf{u}}(k, x) \in \mathbb{X} \text{ for all } k \in \{0, \dots, N\}\}$$

and define the optimal value function $V^N(x)$ to the optimization problem (1.3). Based on these definitions we can formulate a basic MPC algorithm.

[1] Throughout this section we assume that all problems are well-posed, i.e. that there exists a solution with a finite optimal value.

Algorithm 1 (Basic MPC Algorithm).
Given a dynamical system (1.1) with initial value x_0. At each time instant $n \in \mathbb{N}_0$:

(1) Set $x := x(n)$.

(2) Solve (1.3), i.e.

$$\begin{aligned} &\min_{\mathbf{u} \in \mathbb{U}^N(x)} J^N(x, \mathbf{u}) \\ &\text{s.t. } (1.1), \end{aligned}$$

and receive an optimal control sequence $\mathbf{u}^* = (u^*(0), \ldots, u^*(N-1))$.

(3) Define the MPC feedback $\mu^N(x) := u^*(0)$ and apply it to the system, i.e., $x(n+1) := f(x, \mu^N(x))$.

By applying Algorithm 1, we obtain the so called *closed-loop trajectory* that will usually be denoted by $x^{\mu^N}(\cdot, x_0)$. Apart from many theoretical as well as technical issues such as existence of (unique) optimal solutions, real-time optimization (see e.g. [91]) or state estimation (see e.g. [72]), which will not be discussed here, there are **two fundamental questions** that naturally arise when MPC is used to approximate the infinite-horizon OCP (1.2) and that we will deal with in this thesis.

1. How does the MPC closed-loop trajectory behave?

   Regarding this question, we will investigate properties such as (practical) asymptotic stability, and convergence.

2. Is the MPC *performance* in some sense optimal?

   One way (among others) to investigate this question is to compare the value

$$J^\infty(x_0, \mu^N) := \lim_{K \to \infty} \sum_{k=0}^{K-1} \ell(x^{\mu^N}(k, x_0), \mu^N(x^{\mu^N}(k, x_0))), \tag{1.4}$$

   to $V^\infty(x_0)$.

1.1.1 Basic Definitions in Control Theory

The following definitions and results, taken from the references Goebel et al. [27], Grüne and Pannek [32], Kellett [47], Michel et al. [60], Rawlings and Mayne [73], are needed in order to answer the first question. Since we will only use those concepts in the context of MPC, we will introduce them for control systems in closed loop, i.e. systems which are controlled by a feedback $\mu : \mathbb{X} \to \mathbb{U}$. In order to deal with feasibility of the closed-loop system

$$x^+ = f(x, \mu(x)) \tag{1.5}$$

with solution $x^\mu(\cdot, x_0)$ for a given initial value $x_0 \in \mathbb{X}$ we need the following definition.

Definition 1.1 (Forward Invariance)**.** *A set* $Y \subseteq \mathbb{X}$ *is said to be* forward invariant *for the closed-loop system* (1.5) *if* $f(x, \mu(x)) \in Y$ *holds for all* $x \in Y$*.*

Synonymously to forward invariance we use the notion of *recursive feasibility*. As stated in [32, Theorem 3.5], recursive feasibility of the set $\mathbb{X}$ can be ensured if a *viability* assumption holds, i.e. if for all $x \in \mathbb{X}$ there exists $u \in \mathbb{U}$ such that $f(x, u) \in \mathbb{X}$ holds. For the next definitions the following notion will be useful: For $x \in \mathbb{X}$ and $\varepsilon \in \mathbb{R}_{>0}$ we define

$$\begin{aligned} \mathcal{B}_\varepsilon(x) &:= \{y \in \mathbb{X} : \|y - x\| < \varepsilon\} \text{ and} \\ \overline{\mathcal{B}_\varepsilon(x)} &:= \{y \in \mathbb{X} : \|y - x\| \leq \varepsilon\}. \end{aligned} \tag{1.6}$$

Definition 1.2 (Local Stability)**.** *Consider the closed-loop system* (1.5) *with equilibrium* $x_* \in \mathbb{X}$*, i.e.* $f(x_*, \mu(x_*)) = x_*$*. The equilibrium is said to be* locally stable *for the closed-loop system if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $x_0 \in \mathcal{B}_\delta(x_*)$ *implies* $x^\mu(k, x_0) \in \mathcal{B}_\varepsilon(x_*)$ *for all* $k \in \mathbb{N}_0$*.*

Local stability requires that closed-loop trajectories, which start close to the equilibrium remain close to it for all time instants.

Definition 1.3 (Attraction)**.** *We say that the equilibrium* x_* *of the closed-loop system* (1.5) *is* locally attractive *if there exists* $\delta > 0$ *such that for all* $x_0 \in \mathcal{B}_\delta(x_*)$ *it holds* $\|x^\mu(k, x_0) - x_*\| \to 0$ *as* k *tends to infinity.*

The equilibrium is globally attractive *if* $\|x^\mu(k, x_0) - x_*\| \to 0$ *for* $k \to \infty$ *holds for all* $x_0 \in \mathbb{X}$*.*

The notion of attraction is usually (and also in this thesis) used synonymously to (local or global) *convergence*.

As demonstrated in [73, Appendix B], the equilibrium of a system can be globally attractive but not stable. The following property does not allow for such a behavior.

Definition 1.4 (Asymptotic Stability)**.** *An equilibrium* x_* *of the closed-loop system* (1.5) *is called* locally asymptotically stable *if it is locally stable and locally attractive. It is called* globally asymptotically stable *if it is locally stable and globally attractive.*

Very often, asymptotic stability is characterized by means of the following comparison functions.

Definition 1.5 (Comparison functions)**.**

$$\begin{aligned} \mathcal{L} &:= \{\delta : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|\, \delta \textit{ continuous and decreasing with } \lim_{k\to\infty} \delta(k) = 0\}, \\ \mathcal{K} &:= \{\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|\, \alpha \textit{ continuous, strictly increasing with } \alpha(0) = 0\}, \\ \mathcal{K}_\infty &:= \{\alpha \in \mathcal{K} \,|\, \alpha \textit{ unbounded}\}, \\ \mathcal{KL} &:= \{\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|\, \beta \textit{ continuous, } \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L}\}. \end{aligned}$$

Definition 1.6 (Asymptotic Stability - $\mathcal{KL}$ Version)**.** *An equilibrium x_* of the closed-loop system* (1.5) *is called* locally asymptotically stable *if there is $\beta \in \mathcal{KL}$ and $\delta > 0$ such that for all $x_0 \in \mathcal{B}_\delta(x_*)$ it holds $\|x^\mu(k, x_0) - x_*\| \leq \beta(\|x_0 - x_*\|, k)$ for all $k \in \mathbb{N}_0$. The function β is called* attraction rate.

If the property holds for all $x_0 \in \mathbb{X}$, the equilibrium is globally asymptotically stable.

As stated in [73], both characterizations of asymptotic stability are equivalent if the mapping f is continuous. In the context of MPC asymptotic stability in the $\mathcal{KL}$ version is usually proved by means of Lyapunov functions (LFs).

Definition 1.7 (Lyapunov Function)**.** *Consider the closed-loop system* (1.5) *with equilibrium x_* and a set $S \subseteq \mathbb{X}$. A function $V : S \to \mathbb{R}_{\geq 0}$ is a* Lyapunov function *on S for x_* if there are $\alpha_1,\ \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{K}$ such that*

$$\alpha_1(\|x - x_*\|) \leq V(x) \leq \alpha_2(\|x - x_*\|)$$

holds for all $x \in \mathbb{X}$ and

$$V(f(x, \mu(x))) \leq V(x) - \alpha_3(\|x - x_*\|)$$

holds for all $x \in S$ with $f(x, \mu(x)) \in S$.

The inequalities in Definition 1.7 require that the LF strictly decays along the closed loop as long as the quilibrium is not reached. Moreover, the LF is a measure of the distance to the equilibrium. Very commonly, a LF is interpreted as energy of the system. This is the idea why existence of a LF implies asymptotic stability as the next theorem shows. Here, we state the 'global' version, whereas a local version can be found in [32, Theorem 2.19].

Theorem 1.8 (Lyapunov function implies asymptotic stability)**.** *Suppose that $\mathbb{X}$ is forward invariant and there exists a Lyapunov function V on $\mathbb{X}$ for an equilibrium x_* and the closed-loop system* (1.5)*. Then, x_* is globally asymptotically stable in the sense of Definition 1.4 and 1.6.*

1.1.2 Stability and Performance in MPC

In past decades the majority of MPC literature was concerned with the question how to design the feedback μ^N such that the closed-loop system $x^+ = f(x, \mu^N(x))$ asymptotically stabilizes an equilibrium x_*. The main approach in many works such as [11, 28, 32, 73, 82] is to modify the optimization problem (1.3) and/or to impose assumptions on the OCP such that the optimal value function V^N for (1.3) can be proven to be a LF for x_*. A key assumption in most references is positive definiteness of the stage cost wrt the equilibrium x_*, e.g. by setting $\ell(x, u) = \|x - x_*\|^2 + \gamma\|u - u_*\|^2$ for $\gamma \in \mathbb{R}_{>0}$. Thus, for a long time the stage cost ℓ has been regarded as a design parameter for stabilizing the system. This is why we term such MPC schemes as *stabilizing MPC*. Additionally to establishing asymptotic stability of the MPC closed loop, the performance of the MPC controller has been of interest in the literature. Based on the Dynamic Programming Principle (DPP) and by means of the LF it is possible to derive upper bounds on the infinite-horizon MPC performance $J^\infty(x_0, \mu^N)$ and to make statement on the relation between the performance and the optimal value function $V^\infty(x_0)$ of (1.2), see e.g. [32] for details.

Within the last years, the stage cost ℓ has more and more been regarded as given data, which reflects an 'economic' criterion that is supposed to be minimized. In this situation it cannot be expected that the stage cost is still positive definite wrt some equilibrium and the stability analysis known from stabilizing MPC cannot be carried out. A remedy for this challenge is to require *strict dissipativity* (see [87, 88]) of the OCP, which allows for establishing a LF, see e.g. [3]. For details we refer to Chapter 2 and [29, 31, 34].

1.2 Outline and Contribution

Chapter 2 – Economic MPC without Terminal Conditions Existing results for Economic MPC schemes with terminal conditions are extended to the setting without terminal conditions. We provide sufficient conditions, under which the optimal value function to an auxiliary (the rotated) OCP serves as practical LF. Thus, we conclude practical asymptotic stability of the optimal equilibrium of the system. Moreover, the LF allows for an analysis of the so called transient phase of the MPC closed loop. In particular, it is shown that – among all controllers, which steer the trajectory into a predefined neighborhood of the equilibrium within a fixed time – the MPC controller has the approximately best performance. For two exemplary classes of control systems it is shown that they satisfy the conditions that guarantee the existence of a practical LF.

Chapter 3 – Multiobjective Optimization In this chapter we provide some elementary definitions and properties of MO optimization problems. Those are needed for our analysis of MO MPC schemes discussed in Chapters 4 and 5.

Chapter 4 – Multiobjective Stabilizing Model Predictive Control We start with our analysis of MO MPC schemes with and without terminal conditions. The setting we consider can be seen as the straightforward generalization of stabilizing MPC for OCPs with one objective, i.e. all objectives are positive definite wrt the same equilibrium. It is demonstrated that imposing terminal conditions or assumptions on the structure of Pareto-optimal solutions (POSs) yields performance guarantees for all objectives as well as convergence of the MPC closed-loop trajectories. Moreover, the MPC performance is related to POSs on the infinite horizon for all objectives. The key for obtaining these results is to impose additional constraints when solving the MO optimization problem in the MPC iteration, which guarantee that the 'right' POSs are chosen. The proposed MPC algorithms as well as the analysis are completely independent from coupling structures and methods for solving MO optimization problems.

Chapter 5 – Multiobjective Economic MPC In this chapter we generalize the results of Chapter 4 in two ways. First, we allow for economic cost criteria (as in Chapter 2) and second, we deal with the situation that the cost criteria are strictly dissipative at different equilibria. Therefore, a novel dissipativity notion for MO OCPs is presented. We show that under terminal conditions and recursive constraints (for choosing the proper POS in the MPC iterations) it is sufficient that one cost criteria be strictly dissipative wrt to the equilibrium from the terminal condition for the MPC closed-loop trajectory to converge. A performance analysis depending on the specific dissipativity property is carried out for each objective function.

For MO economic MPC without terminal conditions we explain theoretically and illustrate numerically why a new way of analysis has to be found, which is mainly due to the fact that strict dissipativity at different equilibria leads to a non-uniform turnpike behavior.

Chapter 6 – Noncooperative Model Predictive Control We investigate the MPC procedure under the assumption that different players play a noncooperative strategy – a Nash equilibrium (NE) – in each iteration. We give an explanation, why we believe that noncooperative MPC cannot be analyzed similarly to (MO) MPC. Based on the example of a linear game, we show that selecting the 'proper' NE by means on the objective functions does generally not work.

For affine-quadratic games we present sufficient conditions for the MPC closed-loop trajectory to converge. We illustrate that turnpike behavior occurs for such games. Moreover, it is demonstrated that the occuring NE are in general no POSs to the corresponding MO optimization problem.

Chapter 8 – Future Research In this chapter, we present open questions that arose during the process of working on this thesis as well as interesting topics that one could further investigate on.

2 | Economic MPC without Terminal Conditions

We start investigating Model Predictive Control (MPC) schemes for optimal control problems (OCPs) of type (1.2), in which the stage cost ℓ represents an 'economic' criterion rather than a designed cost that penalizes the distance to a desired equilibrium. This is in contrast to classical MPC schemes and originates from the wish to deal with more general OCPs, in which the stage cost is fixed by the application. Such stage costs often reflect production costs or energy consumption of the underlying process that should not be replaced by an 'artificial' cost for the following reasons. Firstly, it can be very difficult to calculate a desired steady state (because an optimization problem has to be solved) and should, thus, be avoided if possible. This aspect is also a motivation not to use terminal conditions since they always require knowledge of the desired equilibrium. Secondly, we shall measure the performance of the MPC controller in terms of the originally given economic stage cost and, as our results will show, the (approximately) best performance is achieved under the usage of this cost criterion in the MPC algorithm instead of any other cost criterion. Results of this type were – in an averaged sense – given in [1, 3].

As in economic MPC with terminal conditions (see e.g. [3, 23, 44]), the main idea of our approach is to establish a Lyapunov function (LF) such that stability can be concluded. As opposed to 'classical' MPC, where the optimal value function $V^N(x)$ corresponding to problem (1.3) serves as LF, in economic MPC a modified optimal value function is used as practical LF. This allows to prove practical asymptotic stability.

Like in many of the references, above, we assume a strict dissipativity condition which in particular implies the existence of an optimal steady state x^e, cf. [62], and which is a key ingredient in economic MPC, see [65]. For this setting, it is already known that – under appropriate conditions, for details see [29] – economic MPC without terminal constraints yields closed-loop trajectories which are approximately optimal in an averaged infinite horizon sense. Moreover, under an exponential turnpike assumption, cf. [12, 71], the trajectories converge to a neighborhood of x^e and there exists at least one time horizon for which the closed-loop trajectory is also approximately optimal in a finite-horizon sense. Since (approximate) optimality in an infinite-horizon averaged sense is in fact a rather weak optimality concept (as the trajectory may be far from optimal on any finite time interval) the latter is important because it tells us that the closed-loop trajectory when initialized away form the optimal steady state approaches this equilibrium in an approximately optimal way. In other words, the closed loop is not only optimal on average in the long run but also shows near optimal performance during its transient phase.

The results in this chapter and related results have been published in [12, 34, 35].

For simplicity of exposition we assume $\mathbb{U}^N(x) \neq \emptyset$ for all $x \in \mathbb{X}$ and $N \in \mathbb{N}$.

2.1 Preliminary Definitions and Results

In this section we first give basic definitions before introducing the systems-theoretic concept of dissipativity.

Definition 2.1 ((Optimal) Steady state/equilibrium)**.** *A pair* $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ *with* $f(x^e, u^e) = x^e$ *is called* steady state *or* equilibrium*. It is called* optimal*, if it it is a solution to the optimization problem*

$$\min_{x \in \mathbb{X}, u \in \mathbb{U}} \ell(x, u) \ s.t. \ x - f(x, u) = 0.$$

Definition 2.2 (Practical asymptotic stability)**.** *Let* $x^e \in \mathbb{X}$ *be an equilibrium for the closed-loop system, i.e.* $x^e = f(x^e, \mu(x^e))$ *for some feedback* μ*. The equilibrium is called* practically asymptotically stable *wrt* $\varepsilon \geq 0$ *on a set* $S \subseteq \mathbb{X}$ *with* $x^e \in S$ *if there exists* $\beta \in \mathcal{KL}$ *(see Definition 1.5) such that*

$$\|x^\mu(k, x_0) - x^e\| \leq \max\{\beta(\|x_0 - x^e\|, k), \varepsilon\} \tag{2.1}$$

holds for all $x_0 \in S$ *and all* $k \in \mathbb{N}$*. The equilibrium is* globally practically asymptotically stable *wrt* $\varepsilon \geq 0$ *if* (2.1) *holds on* $S = \mathbb{X}$*.*

An illustration of this definition is given in Figure 2.1. The orange curve is the function of class $\mathcal{KL}$ that is the upper bound for the closed loop $x^\mu(\cdot, x_0)$ (depicted in black) until an ε-neighborhood (in blue) of x^e is reached. Within that neighborhood the trajectory does no longer have to exhibit asymptotic behavior. A sufficient condition for practical asymptotic stability is the existence of a practical LF that is defined as follows.

Definition 2.3 (Practical LF)**.** *A function* $V : \mathbb{X} \to \mathbb{R}$ *is a* practical Lyapunov function *wrt* $\delta > 0$ *for the closed-loop system* $x^+ = f(x, \mu(x))$ *on a set* $S \subseteq \mathbb{X}$ *with* $x^e \in S$ *if there are* α_1, $\alpha_2 \in \mathcal{K}_\infty$ *and* $\alpha_3 \in \mathcal{K}$ *such that*

$$\alpha_1(\|x - x^e\|) \leq V(x) \leq \alpha_2(\|x - x^e\|) \tag{2.2}$$

holds for all $x \in \mathbb{X}$ *and*

$$V(f(x, \mu(x))) \leq V(x) - \alpha_3(\|x - x^e\|) + \delta \tag{2.3}$$

holds for all $x \in S$*.*

Theorem 2.4 (Practical LF implies pract. as. stability)**.** *Let* V *be a practical LF wrt some* $\delta > 0$ *on a set* $S \subseteq \mathbb{X}$*. Assume that either* $S = \mathbb{X}$ *or* $S = V^{-1}[0, L] := \{x \in \mathbb{X} \,|\, V(x) \leq L\}$ *for some* $L > \alpha_2(\alpha_3^{-1}(\delta)) + \delta$*. Then* x^e *is practically asymptotically stable on* S *wrt* $\varepsilon = \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}(\delta)) + \delta)$*.*

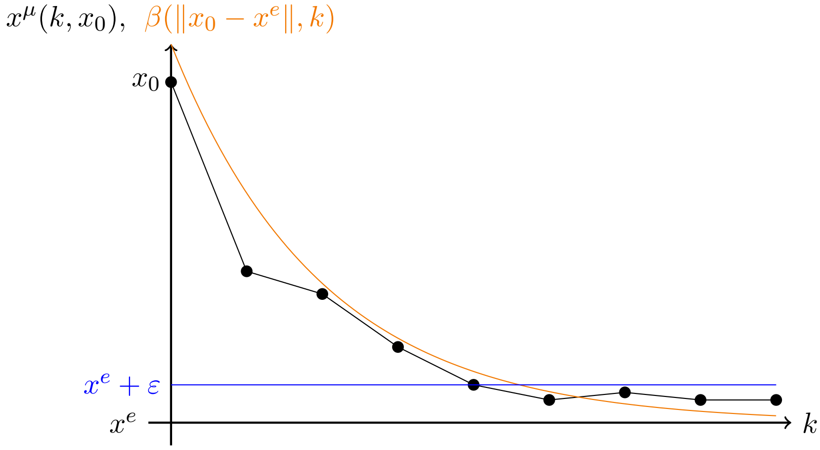

Figure 2.1: Schematic illustration of practical asymptotic stability

Proof. Inequality (2.3) and the assumption on S implies $f(x,\mu(x)) \in S$ for all $x \in S$, i.e., forward invariance of S. Define $\eta := \alpha_2(\alpha_3^{-1}(\delta)) + \delta$ and $P := V^{-1}[0,\eta]$. We claim that P is also forward invariant. To this end, we pick $x \in P$, i.e., $V(x) \leq \eta$, and distinguish two cases:

Case 1: $\alpha_3(\|x - x^e\|) \geq \delta$. In this case we get

$$V(f(x,\mu(x)) \leq V(x) - \alpha_3(\|x - x^e\|) + \delta \leq V(x) - \delta + \delta = V(x) \leq \eta$$

implying $f(x,\mu(x)) \in P$.

Case 2: $\alpha_3(\|x - x^e\|) < \delta$. In this case we get $\|x - x^e\| < \alpha_3^{-1}(\delta)$, implying $V(x) < \alpha_2(\alpha_3^{-1}(\delta))$ and thus

$$V(f(x,\mu(x)) \leq V(x) - \alpha_3(\|x - x^e\|) + \delta < \alpha_2(\alpha_3^{-1}(\delta)) + \delta = \eta$$

which again yields $f(x,\mu(x)) \in P$.

Now by continuity there exists $c > 1$ with $\alpha_2(\alpha_3^{-1}(c\delta)) \leq \eta$. For $x \in S \setminus P$ we have $V(x) \geq \eta$ and consequently $\alpha_3(\|x - x^e\|) \geq \alpha_3(\alpha_2^{-1}(V(x))) \geq \alpha_3(\alpha_2^{-1}(\eta)) \geq c\delta$ for all $x \in S \setminus P$. This implies $\alpha_3(\|x - x^e\|) - \delta \geq (1 - 1/c)\alpha_3(\|x - x^e\|)$ and thus

$$V(f(x,\mu(x))) \leq V(x) - \left(1 - \frac{1}{c}\right) \alpha_3(\|x - x^e\|)$$

for all $x \in S \setminus P$. Hence, V is a Lyapunov function on $S \setminus P$ in the sense of [32, Definition 2.18] and [32, Theorem 2.20] yields practical asymptotic stability wrt the exceptional set P. Since $x \in P$ implies $V(x) \leq \eta$ and thus $\|x - x^e\| \leq \alpha_1^{-1}(\eta) = \varepsilon$, this proves the assertion. □

The following property in systems theory was originally introduced by Willems in 1972, see [87].

Definition 2.5 ((Strict) Dissipativity)**.** *The OCP* (1.2) *or* (1.3) *is called* strictly dissipative *at* (x^e, u^e) *if there is an equilibrium* $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$*, a function* $\alpha_\ell \in \mathcal{K}_\infty$ *and a* storage function $\lambda : X \to \mathbb{R}$ *such that*

$$\min_{u \in \mathbb{U}} \tilde{\ell}(x, u) \geq \alpha_\ell(\|x - x^e\|) \tag{2.4}$$

holds for all $x \in \mathbb{X}$*, where* $\tilde{\ell}$ *denotes the* rotated stage costs

$$\tilde{\ell}(x, u) := \ell(x, u) + \lambda(x) - \lambda(f(x, u)) - \ell(x^e, u^e). \tag{2.5}$$

Based on the definition of the rotated stage cost $\tilde{\ell}$ we can formulate the *rotated OCP* that corresponds to (1.3) that is

$$\begin{aligned} &\min_{\mathbf{u} \in \mathbb{U}^N(x)} \tilde{J}^N(x, \mathbf{u}) \text{ with } \tilde{J}^N(x, \mathbf{u}) := \sum_{k=0}^{N-1} \tilde{\ell}(x(k, x), u(k)) \\ &\text{s.t. } (1.1). \end{aligned} \tag{2.6}$$

The original and the rotated cost functional are related via $\tilde{J}^N(x, \mathbf{u}) = J^N(x, \mathbf{u}) + \lambda(x) - \lambda(x(N, x)) - N\ell(x^e, u^e)$ and the optimal value function to (2.6) is called $\tilde{V}^N(x)$.

We point out that dissipativity at an equilibrium pair (x^e, u^e) implies optimality of this equilibrium (see [32, Prop. 8.9]).

From an MPC perspective one could say that dissipativity is the main ingredient that allows for a stability and performance analysis, yet this interpretation falls short of the importance of general dissipativity theory: Dissipativity is not only a property that describes how energy that is supplied to the systems, is stored within, but also allows for feedback design. Moreover, it is a sufficient and almost nessecary condition for the *turnpike property* (see [63]) that is used in the proofs in Section 2.2. We will comment on and illustrate this property in Remark 2.7.

2.2 Practical Asymptotic Stability for Economic MPC

In this section we prove that the rotated optimal value function $\tilde{V}^N$ can be used as practical LF for economic MPC schemes without terminal conditions. While the first set of inequalities in (2.2) can be obtained by strict dissipativity and structural assumptions on the underlying OCP, inequality (2.3) is in general difficult to prove. Therefore, we will first give sufficient conditions developed in [29][1] that yield (2.3) for $V = \tilde{V}^N$. Since understanding the results presented here without having studied [29] might be challenging, we will sketch the ideas in this reference and which are used here.[2] It is not our ambition to recapitulate this work, but rather to explain why the properties presented therein are sufficient to prove (2.3).

Then, in Sections 2.2.1 and 2.2.2 we present two special structures of OCPs that exhibit these conditions and thus practical asymptotic stability.

[1] We mention that the formulas taken from [29] look slightly different here since all the cost functionals in [29] are averaged, i.e., $J^N(x, \mathbf{u})$ is divided by N.

[2] A consolidated and less technical presentation of the results in this chapter can be found in [32].

Proposition 2.6 (The Lyapunov inequality)**.** *Consider the OCP* (1.3)*. We assume*

1. *Strict dissipativity at* (x^e, u^e) *and* λ *bounded on* $\mathbb{X}$.

2. (a) *There is* $C' \geq 0$ *such that* $\forall x \in \mathbb{X}, \forall \varepsilon > 0$ *the quantity*

$$Q_\varepsilon := \#\{k \in \{0, \dots, N-1\} : \|x^{\mathbf{u}^\star_{N,x}}(k,x) - x^e\| \leq \varepsilon\}$$

satisfies $Q_\varepsilon \geq N - \frac{C'}{\alpha_\ell(\varepsilon)}$*, with* α_ℓ *from strict dissipativity and* $\mathbf{u}^\star_{N,x}$ *denoting the optimal control for* $J^N(x, \mathbf{u})$.

   (b) *There is* $\tilde{C}' \geq 0$ *such that* $\forall x \in \mathbb{X}, \forall \varepsilon > 0$ *the quantity*

$$\tilde{Q}_\varepsilon := \#\{k \in \{0, \dots, N-1\} : \|x^{\tilde{\mathbf{u}}^\star_{N,x}}(k,x) - x^e\| \leq \varepsilon\}$$

satisfies $\tilde{Q}_\varepsilon \geq N - \frac{\tilde{C}'}{\alpha_\ell(\varepsilon)}$*, with* $\tilde{\mathbf{u}}^\star_{N,x}$ *denoting the optimal control for* $\tilde{J}^N(x, \mathbf{u})$.

3. *There are* $\bar{\delta} > 0, N_0 \in \mathbb{N}, \gamma_V \in \mathcal{K}_\infty$ *such that for all* $\rho \in (0, \bar{\delta}]$*, all* $N \in \mathbb{N}_{\geq N_0}$ *and all* $x \in \mathcal{B}_\rho(x^e)$ *it holds*

$$|V^N(x) - V^N(x^e)| \leq \gamma_V(\rho), \tag{2.7}$$
$$|\tilde{V}^N(x) - \tilde{V}^N(x^e)| \leq \gamma_V(\rho). \tag{2.8}$$

4. *There are* $N_1 \in \mathbb{N}, \Delta \in \mathcal{L}$ *such that*

$$\ell(x, \mu^N) \leq V^N(x) - V^N(x^{\mu^N}(1,x)) + \Delta(N) \tag{2.9}$$

holds for all $x \in \mathbb{X}, N \geq N_1 + 1$*, and* μ^N *from Algorithm 1.*

5. *The function* ℓ *is continuous and* λ *is Lipschitz continuous on* $\mathcal{B}_{\bar{\delta}}(x^e)$.

Then, inequality (2.3) *holds for* $V = \tilde{V}^N$*,* $\alpha_3 = \alpha_\ell$*,* $\mu = \mu^N$ *defined in Algorithm 1 and some*[3] $\delta = \tilde{\delta} \in \mathcal{L}$.

Proof. It was proven in [29, Thm. 7.6] that under the imposed assumptions there is $\tilde{\delta} \in \mathcal{L}$ such that for all $x \in \mathbb{X}$, $k \in \mathbb{N}$, $N \in \mathbb{N}$ sufficiently large and μ^N from Algorithm 1 we have

$$\tilde{J}^K(x, \mu^N) \leq \tilde{V}^N(x) - \tilde{V}^N(x^{\mu^N}(K,x)) + \tilde{\delta}(N). \tag{2.10}$$

For $K = 1$ and with the fact that $x^{\mu^N}(1,x) = f(x, \mu^N(x))$ we obtain

$$\tilde{V}^N(f(x, \mu^N(x)) \leq \tilde{V}^N(x) - \tilde{\ell}(x, \mu^N(x)) + \tilde{\delta}(N) \leq \tilde{V}^N(x) - \alpha_\ell(\|x - x^e\|) + \tilde{\delta}(N),$$

in which the second inequality follows from strict dissipativity. □

[3] For an upper bound for $\tilde{\delta}$ see Theorem 2.13 and Theorem 2.17.

Remark 2.7 (Turnpike property)**.** *The second property in Proposition 2.6 describes (a variant of) the* turnpike property*. It states that if we fix a distance ε to the optimal equilibrium (the so called turnpike), optimal trajectories will stay within this neighborhood for a minimum number of time instants which increases with the optimization horizon N. If we fix the horizon N and enlarge ε, we also obtain a larger number of time instants at which the optimal trajectory is close to the equilibrium. The works of Dorfman et al. [19] and von Neumann [85] are fundamental for the investigation of this property. Another prominent reference is the work McKenzie [58], while collections of turnpike theorems can be found in the works of Zaslawski, see e.g. [89]. An illustration of the turnpike property for Example 2.14 can be seen in Figure 2.2.*

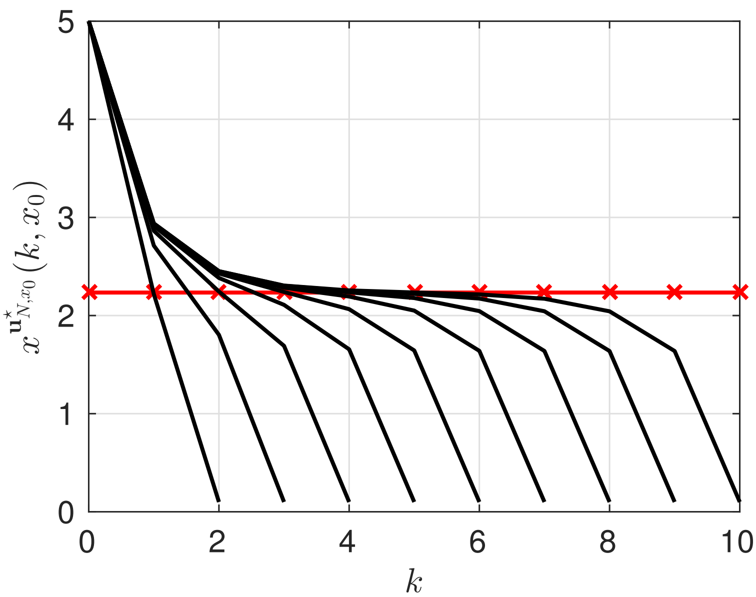

Figure 2.2: Open-loop opimal trajectories (black) for $N = 2, \ldots, 10$ and optimal equilibrium (red).

Let us now start explaining why the construction of inequality (2.3) for $V = \tilde{V}^N$ works by means of the conditions in Proposition 2.6. The aim is to establish the inequality

$$V^N(x) \leq V^{N-1}(x) + \ell(x^e, u^e) + \varepsilon \tag{2.11}$$

for a 'small' error $\varepsilon > 0$. This is helpful because the continuity assumptions (third and fifth condition in Proposition 2.6) and the turnpike property yield the following relation

$$\tilde{V}^N(x) = V^N(x) + \lambda(x) - V^N(x^e) + \nu_{x,N} \tag{2.12}$$

between the original and the modified optimal value function for some error term $\nu_{x,N}$. Thus, using first (2.12), then (2.11), the Dynamic Programming Principle (DPP)[4], and

[4]For more information on the DPP we refer to the famous books by Bertsekas [7] and Bellmann [5].

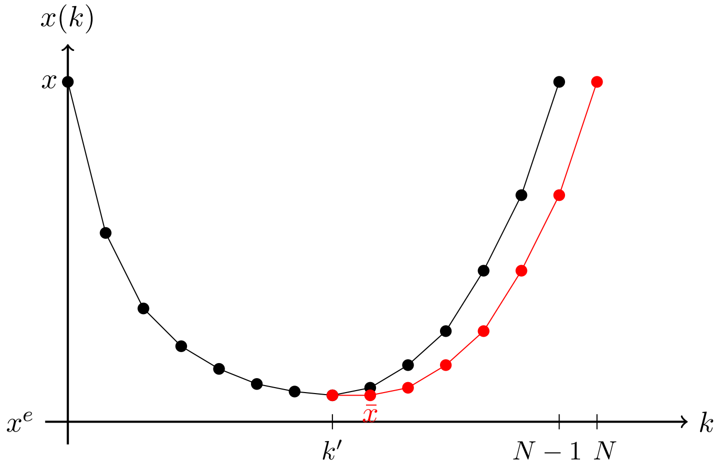

Figure 2.3: Construction of inequality (2.11)

then again (2.12) and (2.5) we get

$$\begin{aligned}
\tilde{V}^N(f(x,\mu^N(x))) &= V^N(f(x,\mu^N(x))) + \lambda(f(x,\mu^N(x))) - V^N(x^e) + \nu_{f(x,\mu^N(x)),N} \\
&\leq V^{N-1}(f(x,\mu^N(x))) + \ell(x^e,u^e) + \lambda(f(x,\mu^N(x))) - V^N(x^e) \\
&\qquad + \nu_{f(x,\mu^N(x)),N} + \varepsilon \\
&= V^N(x) - \ell(x,\mu^N(x)) + \ell(x^e,u^e) + \lambda(f(x,\mu^N(x))) - V^N(x^e) \\
&\qquad + \nu_{f(x,\mu^N(x)),N} + \varepsilon \\
&= \tilde{V}^N(x) - \lambda(x) - \ell(x,\mu^N(x)) + \ell(x^e,u^e) + \lambda(f(x,\mu^N(x))) \\
&\qquad + \nu_{x,N} + \nu_{f(x,\mu^N(x)),N} + \varepsilon \\
&= \tilde{V}^N(x) - \tilde{\ell}(x,\mu^N(x)) + \nu_{x,N} + \nu_{f(x,\mu^N(x)),N} + \varepsilon.
\end{aligned}$$

Analyzing all the error terms[5] that is $\nu_{x,N}$, $\nu_{f(x,\mu^N(x)),N}$ and ε, yields (2.3) for a $\delta = \tilde{\delta}(N)$ with $\tilde{\delta} \in \mathcal{L}$.

The remaining question is how inequality (2.11) can be established. Because of the turnpike property (second condition in Proposition 2.6), we know that optimal trajectories are close to the optimal steady state x^e for a minimum number of time instants. Let $\mathbf{u}^\star_{N-1,x}$ be an optimal control sequence of length $N-1$ for initial value x. In Figure 2.3 the corresponding open-loop optimal trajectory $x^{\mathbf{u}^\star_{N-1,x}}(\cdot,x)$ is depicted in black. At time k', where the optimal open-loop trajectory is close to x^e, we can apply a feasible control value u at the cost close to $\ell(x^e,u^e)$ and obtain a new state $\bar{x}$ at time $k'+1$ that is still close to x^e. From this point on we use the optimal control sequence of length $N-1-k'$ for initial value $\bar{x}$. This way, we have constructed a control sequence of length N for initial value x (concatenation of black and red in Figure 2.3). By local uniform continuity of the optimal value function $V^{N-1-k'}$ (third condition in Proposition (2.6)), we know that

[5] An analysis of the error terms is conducted in the references [29, 32].

$V^{N-1-k'}(x^{\mathbf{u}^\star_{N-1,x}}(k'))$ and $V^{N-1-k'}(\bar{x})$ do not differ too much. More specifically, we make our construction such that the cost $\ell(x^{\mathbf{u}^\star_{N-1,x}}(k',x),u)$ added to the value of the prolonged tail $V^{N-1-k'}(\bar{x})$ does not exceed the value of the original tail $V^{N-1-k'}(x^{\mathbf{u}^\star_{N-1,x}}(k'))$ by $\ell(x^e,u^e)+\varepsilon$. Proceeding this way and using the DPP, we obtain

$$\begin{aligned} V^N(x) &\le \sum_{k=0}^{k'-1} \ell(x^{\mathbf{u}^\star_{N-1,x}}(k,x), u^\star_{N-1,x}(k)) + \ell(x^{\mathbf{u}^\star_{N-1,x}}(k',x),u) + V^{N-1-k'}(\bar{x}) \\ &\le \sum_{k=0}^{k'-1} \ell(x^{\mathbf{u}^\star_{N-1,x}}(k,x), u^\star_{N-1,x}(k)) + V^{N-1-k'}(x^{\mathbf{u}^\star_{N-1,x}}(k',x)) + \ell(x^e,u^e) + \varepsilon \\ &= V^{N-1}(x) + \ell(x^e,u^e) + \varepsilon, \end{aligned}$$

which is the desired inequality (2.11).

2.2.1 Nonlinear Systems with Compact Constraints

We will now present the first class of OCPs for which a practical LF can be established by means of Proposition 2.6. We require the OCPs to satisfy the following assumptions.

Assumption 2.8 (Strict dissipativity)**.** *The OCP of minimizing* (1.3) *is strictly dissipative at* (x^e,u^e) *with storage function* λ *and* $\alpha_\ell \in \mathcal{K}_\infty$*.*

Assumption 2.9 (Continuity and compactness)**.** *The state and control constraint set* $\mathbb{X}$ *and* $\mathbb{U}$ *are compact, the functions* f*,* ℓ *and* λ *from Assumption 2.8 are continuous, and* λ *is Lipschitz continuous on a ball* $\mathcal{B}_\delta(x^e)$ *around* x^e*.*

We remark that under dissipativity the function $\tilde{\ell}$ is zero in (x^e,u^e). Hence, in our finite-dimensional case with $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ Assumption 2.9 implies the inequality

$$\tilde{\ell}(x,u) \le \alpha(\|x-x^e\|) + \alpha(\|u-u^e\|) \tag{2.13}$$

for all $x \in \mathbb{X}$, $u \in \mathbb{U}$ and a suitable $\alpha \in \mathcal{K}_\infty$.

Assumption 2.10 (Local controllability on $\mathcal{B}_\varepsilon(x^e)$)**.** *There is* $\varepsilon > 0$*,* $M' \in \mathbb{N}, C > 0$ *such that* $\forall x \in \mathcal{B}_\varepsilon(x^e)\ \exists \mathbf{u}_1 \in \mathbb{U}^{M'}(x), \mathbf{u}_2 \in \mathbb{U}^{M'}(x^e)$ *with*

$$x^{\mathbf{u}_1}(M',x) = x^e, x^{\mathbf{u}_2}(M',x^e) = x$$

and

$$\max\{\|x^{\mathbf{u}_1}(k,x)-x^e\|, \|x^{\mathbf{u}_2}(k,x^e)-x^e\|, \|u_1(k)-u^e\|, \|u_2(k)-u^e\|\} \le C\|x-x^e\|$$

for $k=0,1,\ldots,M'-1$*.*

Assumption 2.11 (Finite time controllability into $\mathcal{B}_\varepsilon(x^e)$)**.** *For* $\varepsilon > 0$ *from Assumption 2.10 there is* $K \in \mathbb{N}$ *such that for each* $x \in \mathbb{X}$ *there is* $k \le K$ *and* $\mathbf{u} \in \mathbb{U}^k(x)$ *with*

$$x^{\mathbf{u}}(k,x) \in \mathcal{B}_\varepsilon(x^e).$$

Assumption 2.12 (Polynomial bounds). *There are constants $C_1, C_2, p, \eta > 0$ such that*

$$C_1(\|x - x^e\|^p) \leq \tilde{\ell}(x,u) \leq C_2(\|x - x^e\|^p + \|u - u^e\|^p) \tag{2.14}$$

holds for all $x \in \mathcal{B}_\eta(x^e), u \in \mathcal{B}_\eta(u^e)$ with (x^e, u^e) and $\tilde{\ell}$ from dissipativity.

Theorem 2.13. *Consider an OCP (1.3) satisfying Assumptions 2.8–2.11. Then there exists $N_0 \in \mathbb{N}$ and functions $\delta \in \mathcal{L}$ and $\alpha_V \in \mathcal{K}_\infty$ such that the inequalities*

$$\alpha_\ell(\|x - x^e\|) \leq \tilde{V}^N(x) \leq \alpha_V(\|x - x^e\|) \tag{2.15}$$

and

$$\tilde{V}^N(f(x, \mu^N(x))) \leq \tilde{V}^N(x) - \alpha_\ell(\|x - x^e\|) + \delta(N) \tag{2.16}$$

hold for all $N \geq N_0$, $x \in \mathbb{X}$ and μ^N from Algorithm 1. In particular, the functions $\tilde{V}^N$ are practical LFs for the economic MPC closed-loop system and the closed loop is practically asymptotically stable wrt $\varepsilon \to 0$ as $N \to \infty$. If, moreover, Assumption 2.12 holds, then the function $\delta(N)$ converges to zero exponentially fast as $N \to \infty$, i.e., there are $C > 0$ and $\theta \in (0,1)$ with $\delta(N) \leq C\theta^N$.

Proof. The proof of the theorem is split into three steps. In step one we show how to obtain inequality (2.15), in step two we deal with inequality (2.16). Finally, in step three the exponential convergence of δ in (2.16) is deduced.

Step 1: Proof of (2.15)**.** Assumption 2.8 yields $\tilde{V}^N(x) \geq \alpha_\ell(\|x - x^e\|)\ \forall x \in \mathbb{X}$. The upper bound in (2.15) can be deduced from Assumptions 2.9 – 2.11 as follows.

In case $x \notin \mathcal{B}_\varepsilon(x^e)$ with ε from Assumptions 2.10, 2.11, there is a control sequence $\mathbf{u}$ that steers x into the equilibrium in at most $M' + K$ steps (M', K independent of x) and stays there for an arbitrary number of time steps. Therefore, for each $N \in \mathbb{N}$ it holds

$$\tilde{V}^N(x) \leq \tilde{J}^N(x, \mathbf{u}) \leq \tilde{J}^{M'+K}(x, \mathbf{u}) \leq (M' + K) \cdot \max_{x \in \mathbb{X}, u \in \mathbb{U}} \tilde{\ell}(x,u) =: \bar{C}.$$

In case $x \in \mathcal{B}_\varepsilon(x^e)$, there is a control sequence $\mathbf{u}_1 \in \mathbb{U}^{M'}(x)$ with $x^{\mathbf{u}_1}(M', x) = x^e$ and $\|x^{\mathbf{u}_1}(k,x) - x^e\| \leq C\|x - x^e\|$, $\|u_1(k) - u^e\| \leq C\|x - x^e\|$ for all $k = 0, \ldots, M' - 1$. Together with (2.13) this yields

$$\begin{aligned}\tilde{V}^N(x) \leq \tilde{J}^N(x, \mathbf{u}_1) \leq \tilde{J}^{M'}(x, \mathbf{u}_1) \leq \sum_{k=0}^{M'-1} \alpha(\|x^{\mathbf{u}_1}(k,x) - x^e\|) + \alpha(\|u_1(k) - u^e\|) \\ \leq 2M'\alpha(C\|x - x^e\|) =: \tilde{\alpha}(\|x - x^e\|).\end{aligned}$$

Clearly, $\tilde{\alpha} \in \mathcal{K}_\infty$. If $\tilde{\alpha}(\|x - x^e\|) \geq \bar{C}$ for $x \notin \mathcal{B}_\varepsilon(x^e)$, we get $\tilde{V}^N(x) \leq \tilde{\alpha}(\|x - x^e\|)$ for all $x \in \mathbb{X}$. Otherwise, we multiply $\tilde{\alpha}(\|x - x^e\|)$ by a constant $\bar{K}$ such that $\bar{K}\tilde{\alpha}(\|x - x^e\|) \geq \bar{C}$ for $x \notin \mathcal{B}_\varepsilon(x^e)$. Combining these considerations yields

$$\tilde{V}^N(x) \leq \alpha_V(\|x - x^e\|) \quad \text{for} \quad \alpha_V(r) := \max\{1, \bar{K}\}\tilde{\alpha}(r)$$

and hence (2.15).

Step 2: Proof of (2.16)**.** In Proposition 2.6 we have presented sufficient conditions, such that (2.16) holds. Thus, we prove that the imposed assumptions yield those five conditions.

1. Strict dissipativity holds due to Assumption 2.8, boundedness of λ on $\mathbb{X}$ follows from continuity of λ and compactness of $\mathbb{X}$ which is Assumption 2.9.

2. (a) Here, we can use [29, Thm. 5.3] (adapted to the non-averaged case), since $J^N(x, \mathbf{u}^\star_{N,x}) = V^N(x) \leq N\ell(x^e, u^e) + \tilde{V}^N(x) - \lambda(x) + \lambda(x^{\mathbf{u}^\star_{N,x}}(N))$. Compactness of $\mathbb{X}$, continuity of λ and the upper bound on $\tilde{V}^N$ from Step 1 imply the existence of $C_1 > 0$ with $J^N(x, \mathbf{u}^\star_{N,x}) \leq N\ell(x^e, u^e) + C_1$. Hence, [29, Thm. 5.3] delivers the desired estimate with $C' = C_1 + \max_{x \in \mathbb{X}} 2|\lambda(x)|$.

   (b) Proceeding analogously as in [29, Thm. 5.3] and with the help of (2.15), the desired property holds for

$$\tilde{C}' = \max_{x \in \mathbb{X}} \alpha_V(\|x - x^e\|).$$

3. Estimate (2.7) has been shown to hold in [29, Thm. 6.4] under dissipativity, a local controllability condition and boundedness of the rotated stage costs. A closer look at the proof of the theorem reveals that the latter two conditions can be substituted by (2.13), Assumption 2.10 and local Lipschitz continuity of λ.

   Estimate (2.8) can be deduced the following way: By (2.15), for each $x \in \mathbb{X}$ and $N \in \mathbb{N}$ the inequalities

$$\alpha_\ell(\|x - x^e\|) \leq \tilde{V}^N(x) \leq \alpha_V(\|x - x^e\|) \tag{2.17}$$

   hold and we conclude $\tilde{V}^N(x^e) = 0$ and thus

$$|\tilde{V}^N(x) - \tilde{V}^N(x^e)| = \tilde{V}^N(x) \leq \alpha_V(\|x - x^e\|). \tag{2.18}$$

4. Inequality (2.9) for $K = 1$ has been shown to hold for $\Delta(N) = \varepsilon(N-1)$ in [29, Thm. 4.2]. For the sake of completeness we check that the assumptions of Therorem 2.13 include those of [29, Thm. 4.2]: Condition (a) follows from continuity of ℓ and f, condition (b) is estimate (2.7) which we have shown to hold above. Condition (c) can be concluded as in [29, Thm. 5.6]. To this end, we conclude [29, Assumption 5.5] from Assumption 2.9–2.11 as follows: Similiar to Step 1 we construct an upper bound for $\tilde{J}^N(x, \mathbf{u})$. This yields an upper bound for $J^N(x, \mathbf{u})$, too, since both functionals only differ by $\lambda(x)$, $\lambda(x^{\mathbf{u}}(N, x))$ and $N\ell(x^e, u^e)$. Due to continuity of λ and compactness of $\mathbb{X}$ the λ-terms can be bounded and we can proceed as in the proof of [29, Thm. 5.6].

5. This is Assumption 2.9.

Step 3: Exponential decay of δ. In order to show that $\delta(N)$ in (2.16) converges to zero exponentially fast we shall look at the construction of $\tilde{\delta}$ in [29, Thm. 7.6], cf. the derivation of (2.10). It holds that $\tilde{\delta}(N) \leq \varepsilon(N) + 12\gamma_V(\tilde{\varepsilon}(N)) + L_\lambda \tilde{\varepsilon}(N)$ with $\varepsilon(N)$ from [29, Thm. 4.2], γ_V from (2.7), $\tilde{\varepsilon} \in \mathcal{L}$ and L_λ the Lipschitz constant of λ.

Exponential convergence of $\varepsilon(N)$ holds due to [12, Thm. 6.5] if the functions γ_V in (2.7),(2.8) and γ_ℓ, γ_f in [29, Thm. 4.2] are polynomial. This holds for γ_ℓ and γ_f due to the Assumptions 2.9, 2.10 and 2.12. Inspection of the proofs of (2.7) and (2.8) in Step 2 of this proof reveals that γ_V is polynomial if ℓ satisfies Assumption 2.12. This yields exponential convergence of $\varepsilon(N)$. To prove the assertion it is thus sufficient to show that also $\tilde{\varepsilon}(N)$ can be chosen to converge to zero exponentially fast.

In the proof of [29, Thm. 7.6], $\tilde{\varepsilon}(N)$ must be chosen such that $Q_{\tilde{\varepsilon}(N)} \geq cN$ holds for some $c \in (7/8, 1)$ for $Q_{\tilde{\varepsilon}(N)}$ from Step 2. In [12, Thm. 6.5] is was proven that the exponential turnpike property holds under the assumptions of Theorem 2.13 including Assumption 2.12. More precisely, for each $P \in \mathbb{N}$ it was shown that for $\tilde{\varepsilon}_P(N) = \tilde{K}\eta^{(N-P)/(2p)}$, $\tilde{K} > 0$, $\eta \in (0,1)$, $p > 0$, the inequality $Q_{\tilde{\varepsilon}_P(N)} \geq P$ holds. We claim that $\tilde{\varepsilon}(N) := \tilde{\varepsilon}_{\lceil cN \rceil}(N)$ satisfies the desired properties, where $\lceil cN \rceil$ denotes the smallest integer $\geq cN$: On the one hand, we have $\tilde{\varepsilon}(N) = \tilde{K}\eta^{(N-\lceil cN \rceil)/(2p)} \leq \tilde{K}\eta^{1/(2p)}\eta^{(1-c)N/(2p)}$, implying that $\tilde{\varepsilon}$ indeed decays exponentially fast. On the other hand, $Q_{\tilde{\varepsilon}_P(N)} \geq P$ directly implies the desired inequality $Q_{\tilde{\varepsilon}(N)} \geq \lceil cN \rceil \geq cN$. □

Example 2.14 (Economic growth). *Consider the one-dimensional economic growth model presented in [10], given by*

$$x^+ = u, \qquad \ell(x,u) = -\ln(Ax^\alpha - u),$$

with parameters $A = 5$ and $\alpha = 0.34$. We impose state and control constraints $\mathbb{X} = [0, 10]$ and $\mathbb{U} = [0.1, 5]$. The optimal steady state of this OCP is given by $(x^e, u^e) = (x^e, x^e)$ with $x^e \approx 2.23$ and $\ell(x^e, x^e) \approx -1.467$. The problem is known to be strictly dissipative at the optimal equilibrium with a linear storage function[6] *$\lambda(x) = 0.2306x$. In Figure 2.4 we see that the closed-loop trajectories for different horizon N all converge up to an offset to the optimal equilibrium. This offset gets smaller as N increases. This is exactly the statement of Theorem 2.13 whose assumptions are satisfied by this example.*

2.2.2 Linear Quadratic Problems

We now consider a second special setting, for which we will prove the existence of a practical Lyapunov function. Our setting is an extension to the standard linear quadratic regulator.

Assumption 2.15 (Linear quadratic problem). *The dynamics and the cost functions are given by*

$$f(x,u) = Ax + Bu + c \quad \text{and} \quad \ell(x,u) = x^T Rx + u^T Qu + s^T x + v^T u$$

[6] It was proven in [12] and [1] that linear systems with strictly convex costs are strictly dissipative with a linear storage function.

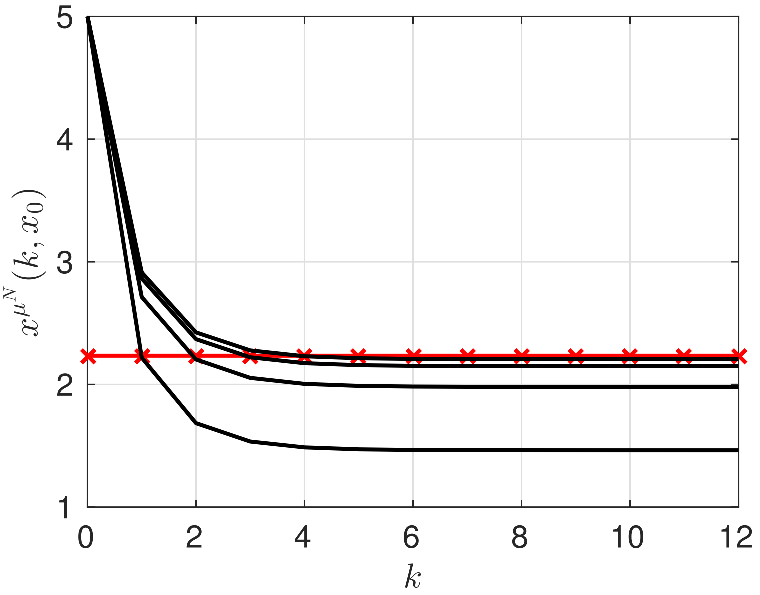

Figure 2.4: Closed-loop trajectories (black) for $N = 2, \ldots, 5$ from bottom to top and optimal equilibrium (red).

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A, B, R, Q are matrices and s, v are vectors of appropriate dimensions with R and Q symmetric and positive definite.

Assumption 2.16 (No constraints)**.** *There are no state and control constraints, i.e., $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{U} = \mathbb{R}^m$.*

Note that in this setting there exists a unique optimal steady state x^e in the sense of Definition 2.1. Moreover, [12, Prop. 4.5] shows that x^e is strictly dissipative at (x^e, u^e) with $\tilde{\ell}$ satisfying Assumption 2.12.

Theorem 2.17. *Consider an OCP* (1.2) *satisfying Assumptions 2.15 and 2.16, let x^e be the optimal steady state and μ^N the feedback from Algorithm 1. Then x^e is practically asymptotically stable on each compact subset $S \subset \mathbb{R}^n$ wrt $\varepsilon \to 0$ as $N \to \infty$ if and only if the pair (A, B) is stabilizable.*

In this case, the problem is strictly dissipative and the functions $\tilde{V}^N$ are practical LFs for the closed loop and ε converges to zero exponentially fast in N.

Proof. "$\Leftarrow$": We first show the implication "Assumptions 2.15 and 2.16 and (A, B) stabilizable $\Rightarrow$ practical asymptotic stability on each compact subset $S \subset \mathbb{R}^n$" via the existence of a practical LF. We proceed as in the proof of Theorem 2.13

Step 1: Proof of (2.15)**.** According to [12, Prop. 4.3] the affine linear quadratic problem is strictly dissipative with storage function $\lambda(x) = \nu^T x$ and $\alpha_\ell(r) = C_1 r^2$ for some vector $\nu \in \mathbb{R}^n$ and some constant $C_1 > 0$. This implies the lower bound in (2.15).

The upper bound can be concluded as follows. In the proof of [12, Prop. 4.3] it was shown that the rotated stage costs are of the form

$$\tilde{\ell}(x,u) = (x-x^e)^T R(x-x^e) + (u-u^e)^T Q(u-u^e),$$

hence there is $C_2 > 0$ such that $\tilde{\ell}(x,u) \leq C_2(\|x-x^e\|^2 + \|u-u^e\|^2)$. Since (A,B) is stabilizable, for each $x \in \mathbb{R}^n$ there exists a control sequence $\mathbf{u}$ of infinite length and constants $C_3 > 0, \sigma \in (0,1)$ independent of x, such that

$$\|x^{\mathbf{u}}(k,x) - x^e\| \leq C_3\sigma^k\|x-x^e\|, \qquad \|u(k)-u^e\| \leq C_3\sigma^k\|x-x^e\|$$

holds for all $k \geq 0$. Combining all estimates implies

$$\tilde{\ell}(x^{\mathbf{u}}(k,x), u(k)) \leq 2C_2C_3^2\sigma^{2k}\|x-x^e\|^2.$$

We obtain

$$\tilde{V}^N(x) \leq \sum_{k=0}^{\infty} \tilde{\ell}(x^{\mathbf{u}}(k,x), u(k)) \leq 2C_2C_3^2/(1-\sigma^2)\|x-x^e\|^2 =: \alpha_V(\|x-x^e\|).$$

Step 2: Proof of (2.16) We show that the assumptions of Theorem 2.17 include those of [29, Thm. 7.6] on any compact subset S of $\mathbb{R}^n$. To this end, we check the five properties listed in Proposition 2.6.

1. According to [12, Prop. 4.3] the affine linear quadratic problem is strictly dissipative with storage function $\lambda(x) = \nu^T x$ and $\alpha_\ell(r) = C_1 r^2$ for some vector $\nu \in \mathbb{R}^n$ and some constant $C_1 > 0$. This structure of the storage function yields boundedness of λ on S.

2. Both estimates, (a) and (b), can be concluded as in the proof of Theorem 2.13 as we restrict the initial state to the compact set S.

3. In order to obtain (2.7) we have a closer look at the optimal value function $V^N(x)$. In Appendix A it is shown that

$$V^N(x) = x^T P_N x + b_N^T x + d_N, \tag{2.19}$$

with P_N symmetric and positive definite and P_N is the solution of the backward Riccati iteration for the standard linear quadratic regulator.

As shown in the proof of [12, Thm. 6.2], V^N is bounded uniformly in N on the compact set S. This yields existence of constants C_S, D_S such that

$$C_S \leq V^N(x) \leq D_S \tag{2.20}$$

holds for all $x \in S$ and all $N \in \mathbb{N}$. This yields boundedness of the vector d_N. Now consider sequences $(x_i)_{i\in\mathbb{N}}$ in S and $(N_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ with $x_i^T P_{N_i} x_i \to \infty$. By (2.20) this is only possible if $b_{N_i}^T x_i \to -\infty$. Then, $(-x_i)^T P_{N_i}(-x_i) \to \infty$ and $b_{N_i}^T(-x_i) \to \infty$, too, which contradicts (2.20). Hence, there is $K > 0$ independent of N such that $0 \leq x^T P_N x \leq K\|x\|^2$ for all $x \in S$, and with the same argument there is $M > 0$ independent of N such that $|b_N^T x| \leq M\|x\|$ on S.

The bounds on $b_N^T x$ immediately imply that the entries of b_N are bounded on S. Since P_N is symmetric and positive definite its spectral norm is given by[7] $\|P_N\|_2 = \max_{\|x\|=1} x^T P_N x \leq \max_{\|x\|=1} K\|x\|^2 = K$. Therefore, the entries of P_N are bounded on S. Now, it follows from the uniformity of the deduced bounds that for all $N \in \mathbb{N}$, $x \in S$ it holds

$$\begin{aligned}
\left|V^N(x) - V^N(x^e)\right| &\leq \left|x^T P_N x - (x^e)^T P_N x^e\right| + \left|b_N^T (x - x^e)\right| \\
&\leq K \left|\|x\|^2 - \|x^e\|^2\right| + M\|x - x^e\| \\
&= K \left|(\|x\| + \|x^e\|)(\|x\| - \|x^e\|)\right| + M\|x - x^e\| \\
&\leq 2K \max\{\|x\| : x \in S\} \left|\|x\| - \|x^e\|\right| + M\|x - x^e\| \\
&\leq C\|x - x^e\|, \quad C > 0.
\end{aligned}$$

This concludes the proof of (2.7).

Inequality (2.8) can be concluded as in the proof of Theorem 2.13.

4. Again, for this property we use [29, Thm. 4.2] whose conditions are fulfilled.

5. Since λ is a linear function (cf. Step 1 of this proof) it is Lipschitz continuous on every neighborhood of the equilibrium.

Step 3: Exponential decay of δ. Completely analogous to Step 3 of the proof of Theorem 2.13 using [12, Thm. 6.2] instead of [12, Thm. 6.5].

"$\Leftarrow$": Let the closed-loop system be practically asymptotically stable on some compact subset $S \subset \mathbb{R}^n$ with $\varepsilon \to 0$ as $N \to \infty$. Then, for each $x \in S$ we can choose N large enough such that the feedback steers the closed loop into an arbitrarily small neighborhood of x^e. This implies stabilizability of (A, B). □

Example 2.18 (Continuously stirred tank reactor (CSTR) model)**.** *The second example is a linearized two-dimensional tank reactor model (Example 3.2 in [29]) with affine linear dynamics*

$$x(k+1) = \begin{pmatrix} 0.8353 & 0 \\ 0.1065 & 0.9418 \end{pmatrix} x(k) + \begin{pmatrix} 0.00457 \\ -0.00457 \end{pmatrix} u(k) + \begin{pmatrix} 0.5559 \\ 0.5033 \end{pmatrix}$$

and quadratic stage costs $\ell(x,u) = \|x\|^2 + 0.05u^2$. State and control constraints are given by $\mathbb{X} = [-100, 100]^2$ and $\mathbb{U} = [-10, 10]$. The optimal steady state of this problem is $x^e \approx (3.546, 14.653)^T$, $u^e \approx 6.163$ with cost $\ell(x^e, u^e) \approx 229.1876$. As in the previous example, we observe in Figure 2.5 that the closed-loop trajectories converge into a neighborhood of x^e which is shrinking as N increases. This confirms the result in Theorem 2.17, since the pair (A, B) in the dynamics is stabilizable and the stage costs are strictly convex.

[7] As in the proof of [78, Lemma 8.2.1].

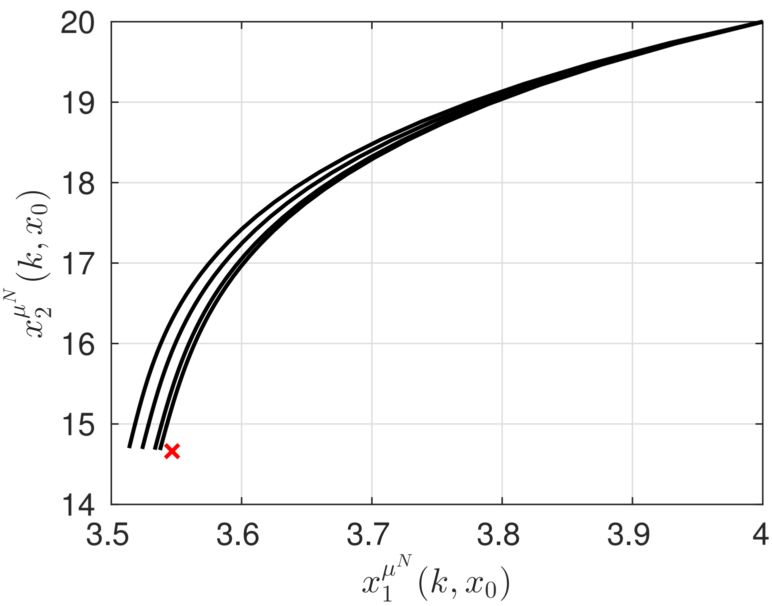

Figure 2.5: Closed-loop trajectories (black) for $N \in \{10, 12, 15, 17\}$ from left to right and optimal equilibrium (red).

2.3 Transient Performance for Economic MPC

In this section we use the results from Section 2.2 in order to prove an approximate transient optimality property of economic MPC without terminal constraints. In order to formulate the concept of transient optimality, assume that the MPC closed loop is practically asymptotically stable, implying $x^{\mu^N}(K, x) \to x^e$ as $N \to \infty$ and $K \to \infty$. Then *transient optimality* means that among all trajectories $x^{\mathbf{u}}(k, x)$ satisfying $\|x^{\mathbf{u}}(K, x) - x^e\| \leq \|x^{\mu^N}(K, x) - x^e\|$, the MPC closed-loop trajectories are those with the smallest cost $J^K(x, \mathbf{u})$ — up to an error term which vanishes as $N \to \infty$ and $\|x^{\mu^N}(K, x) - x^e\| \to 0$.

We define

$$\mathbb{U}^K_\varepsilon(x) := \{\mathbf{u} \in \mathbb{U}^K(x) \,|\, x^{\mathbf{u}}(K, x) \in \mathcal{B}_\varepsilon(x)\} \text{ and}$$
$$J^K(x, \mu^N) := \sum_{k=0}^{K-1} \ell\left(x^{\mu^N}(k, x), \mu^N(x^{\mu^N}(k, x))\right).$$

Theorem 2.19. *Assume that x^e is practically asymptotically stable on a set $S \subseteq \mathbb{X}$ wrt $\varepsilon = \varepsilon(N)$ for the economic MPC closed loop with LF $\tilde{V}^N$ satisfying* (2.15) *and* (2.16)*. Assume that there exists $\alpha_\lambda \in \mathcal{K}_\infty$ with $|\lambda(x)| \leq \alpha_\lambda(\|x - x^e\|)$ for all $x \in \mathbb{X}$. Let $\varepsilon_{K,N} := \|x^{\mu^N}(K, x) - x^e\| \leq \max\{\beta(\|x - x^e\|, K), \varepsilon(N)\}$. Then the inequality*

$$J^K(x, \mu^N) \leq \inf_{\mathbf{u} \in \mathbb{U}^K_{\varepsilon_{K,N}}(x)} J^K(x, \mathbf{u}) + \alpha_V(\varepsilon_{K,N}) + 2\alpha_\lambda(\varepsilon_{K,N}) + K\delta(N) \tag{2.21}$$

holds for all $K, N \in \mathbb{N}$ and all $x \in S$.

Proof. First, by induction from (2.16) we obtain

$$\sum_{k=0}^{K-1} \tilde{\ell}(x^{\mu^N}(k,x), \mu^N(x^{\mu^N}(k,x))) \leq \tilde{V}^N(x) - \tilde{V}^N(x^{\mu^N}(K)) + K\delta(N). \tag{2.22}$$

Second, from the DPP

$$\tilde{V}^N(x) = \inf_{\mathbf{u}\in\mathbb{U}^K(x)} \left\{ \tilde{J}^K(x,\mathbf{u}) + \tilde{V}^{N-K}(x^{\mathbf{u}}(K,x)) \right\}$$

and (2.15) we obtain for all $K \in \{1, \dots, N\}$ and $\mathbf{u} \in \mathbb{U}^K_\varepsilon(x)$

$$\tilde{J}^K(x,\mathbf{u}) = \underbrace{\tilde{J}^K(x,\mathbf{u}) + \tilde{V}^{N-K}(x^{\mathbf{u}}(K,x))}_{\geq \tilde{V}^N(x)} - \underbrace{\tilde{V}^{N-K}(x^{\mathbf{u}}(K,x))}_{\leq \alpha_V(\varepsilon)} \geq \tilde{V}^N(x) - \alpha_V(\varepsilon) \tag{2.23}$$

and we note that for $K \geq N$ non-negativity of $\tilde{\ell}$ implies the inequality $\tilde{J}^K(x,\mathbf{u}) \geq \tilde{V}^N(x)$ for all $\mathbf{u} \in \mathbb{U}^K(x)$, implying again (2.23). Third, we have

$$\sum_{k=0}^{K-1} \tilde{\ell}(x^{\mathbf{u}}(k,x), u(k)) = \tilde{J}^K(x,\mathbf{u}) = \lambda(x) + J^K(x,\mathbf{u}) - \lambda(x^{\mathbf{u}}(K,x)) \tag{2.24}$$

and $\tilde{V}^N(x) \geq 0$. Using these inequalities for all $\mathbf{u} \in \mathbb{U}^K_{\varepsilon_{K,N}}(x)$ we obtain

$$\begin{aligned}
J^K(x,\mu^N) &\overset{(2.24)}{=} \sum_{k=0}^{K-1} \tilde{\ell}(x^{\mu^N}(k,x), \mu^N(x^{\mu^N}(k,x))) - \lambda(x) + \lambda(x^{\mu^N}(K,x)) \\
&\overset{(2.22)}{\leq} \tilde{V}^N(x) - \tilde{V}^N(x^{\mu^N}(K,x)) + K\delta(N) - \lambda(x) + \lambda(x^{\mu^N}(K,x)) \\
&\overset{(2.23)}{\leq} \tilde{J}^K(x,\mathbf{u}) + \alpha_V(\varepsilon_{K,N}) - \tilde{V}^N(x^{\mu^N}(K,x)) + K\delta(N) \\
&\qquad\qquad - \lambda(x) + \lambda(x^{\mu^N}(K,x)) \\
&\overset{(2.24)}{=} J^K(x,\mathbf{u}) + \alpha_V(\varepsilon_{K,N}) - \tilde{V}^N(x^{\mu^N}(K,x)) + K\delta(N) \\
&\qquad\qquad - \lambda(x^{\mathbf{u}}(K,x)) + \lambda(x^{\mu^N}(K,x)) \\
&\leq J^K(x,\mathbf{u}) + \alpha_V(\varepsilon_{K,N}) + K\delta(N) + 2\alpha_\lambda(\varepsilon_{K,N})
\end{aligned}$$

implying the desired inequality. □

Remark 2.20. *i) Note that all assumptions of Theorem 2.19 are satisfied if either Assumptions 2.8–2.11 or Assumptions 2.15–2.16 are satisfied. In the latter case the existence of α_λ follows because in the linear quadratic setting λ is either a linear or a quadratic function, cf. [12]. Moreover, if Assumption 2.12 holds then $\delta(N)$ converges to 0 exponentially fast as $N \to \infty$, implying that the error terms on the right hand* *side of* (2.21) *converge to* 0 *if* $K, N \to \infty$ *with* $K \le cN$ *for some* $c > 0$*. In addition, in this case* $\tilde{\ell}$ *and* $\tilde{V}^N$ *have identical polynomial growth near* x^e*, implying that the convergences* $\beta(r,k) \to 0$ *as* $k \to \infty$ *and* $\varepsilon(N) \to 0$ *as* $N \to \infty$ *are exponentially fast and thus all error terms in* (2.21) *converge to* 0 *exponentially fast as* $K, N \to \infty$ *with* $K \le cN$ *for some* $c > 0$*.*

ii) Optimal trajectories that result from minimizing (1.3) *in general do not end up near* x^e*, see, e.g., the examples in [12]. Hence, for* $\mathbf{u} \in \mathbb{U}^K(x)$ *the value* $J^K(x,\mathbf{u})$ *can be much smaller than* $J^K(x,\mu_N)$ *and thus estimate* (2.21) *can only hold if we restrict the control sequences to* $\mathbf{u} \in \mathbb{U}^K_{\varepsilon_{K,N}}(x)$*. In words, the estimate states that among all trajectories converging to a neighborhood of* x^e*, the ones generated by MPC are — up to the error terms — the ones with the lowest cost* $J^K(x,\mathbf{u})$*.*

2.3.1 Numerical Simulations

In this section we illustrate the results on the transient performance of our MPC controller by means of the Examples 2.14 and 2.18.

Example 2.14

For this example, we compare the MPC controllers μ^N computed using four different cost functions:

- the original economic stage cost ℓ $\rightsquigarrow$ $\mu^{N,\text{eco}}$
- the rotated stage cost $\tilde{\ell}$ from (2.5) $\rightsquigarrow$ $\mu^{N,\text{rot}}$
- the stabilizing quadratic stage cost $\ell^{\text{stab}}(x,u) = (x - x^e)^2 + (u - u^e)^2$ $\rightsquigarrow$ $\mu^{N,\text{stab}}$
- the stabilizing quadratic stage cost

$$\begin{aligned}\ell^{\text{tayl}}(x,u) &= \ell(x^e,u^e) + \frac{1}{2}0.12125(x-x^e)^2 \\ &\quad -0.05315(x-x^e)(u-u^e) + \frac{1}{2}0.05315(u-u^e)^2,\end{aligned}$$

  whose weights were derived from a 2nd order Taylor approximation of ℓ in (x^e,u^e) $\rightsquigarrow$ $\mu^{N,\text{tayl}}$

Now, in order to investigate approximate optimal transient performance, for given N and K we calculate $J^K(x,\mu^N)$ for the different MPC controllers[8]. In Figure 2.6 we show the values for fixed $N = 5$ and varying $K = 1,\ldots,20$. One sees that the values of the cost functionals are almost parallel, which is due to the fact that the difference is mainly accumulated in the first few time steps. The value of $J^K(x,\mu^{N,\text{eco}})$ is almost identical

[8] In this comparison $J^K(x,\mu^N)$ is always evaluated using the economic cost ℓ. The different cost functions only refer to the computation of $\mu^N(x)$ in step **(2)** of the Algorithm 1.

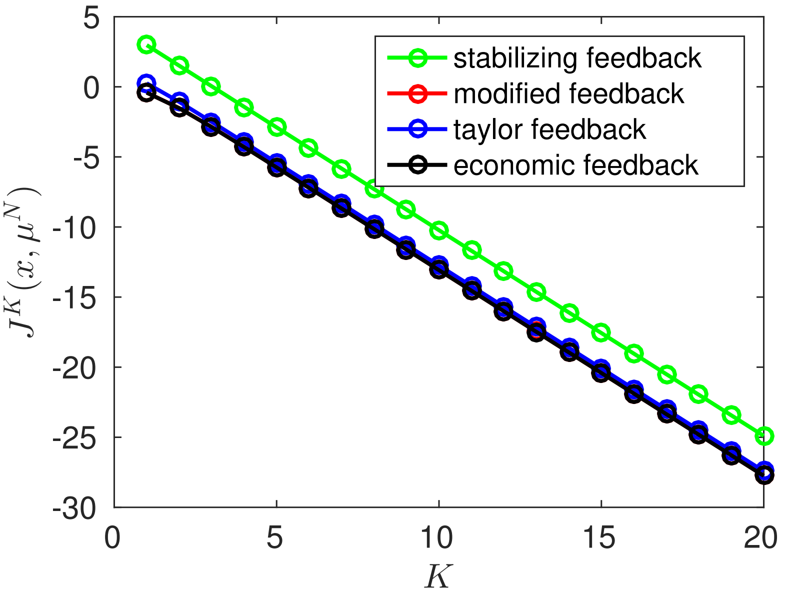

Figure 2.6: $J^K(x, \mu^N)$ for $N = 5$, $x = 0.1$ and varying K subject to different feedbacks μ^N.

to $J^K(x, \mu^{N,\text{rot}})$ and both are better than the other feedbacks. Observe that the merely practical stability of $\mu^{N,\text{eco}}$ does not have a visible effect in this comparison.

Next, we investigate two fixed values for K and varying optimization horizons N in Figure 2.7. While in Figure 2.7 left $\mu^{N,\text{eco}}$ yields the best performance for all N, Figure 2.7 right reveals that $J^K(x, \mu^{N,\text{eco}})$ might not yield the best performance for very small N, but converges to $J^K(x, \mu^{N,\text{rot}})$ as N increases and is slightly better than $\mu^{N,\text{rot}}$ and considerably better than $\mu^{N,\text{tayl}}$ and $\mu^{N,\text{stab}}$ for most values of N.

Example 2.18

For this example we only compare $\mu^{N,\text{eco}}$ and $\mu^{N,\text{rot}}$ since by [12, Proposition 4.5] the rotated costs $\tilde{\ell}$ of this problem are quadratic, i.e. $\tilde{\ell}$ coincides with the "canonical" choice of stabilizing quadratic costs ℓ^{stab} and with its 2nd order Taylor approximation ℓ^{tayl}. Our simulations show that for fixed $N = 10$ and varying $K = 1, \dots, 100$ the closed loop values for $\mu^{N,\text{eco}}$ and $\mu^{N,\text{rot}}$ are virtually indistinguishable, cf. Figure 2.8. For fixed K and varying N, Figure 2.9 shows (again) that even though the performance of $\mu^{N,\text{eco}}$ might not be the best for small N, $J^K(x, \mu^{N,\text{eco}})$ converges to $J^K(x, \mu^{N,\text{rot}})$ as N increases and $\mu^{N,\text{eco}}$ (at least slightly) outperforms $\mu^{N,\text{rot}}$ for sufficiently large N.

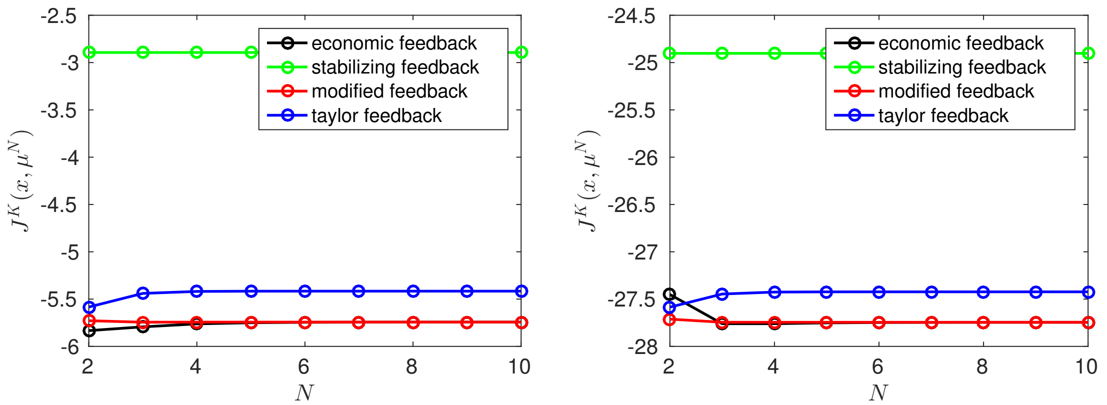

Figure 2.7: $J^K(x, \mu^N)$ for $K = 5$ (left) and $K = 20$ (right), $x = 0.1$ and varying $N = 2, \dots, 10$ with different feedbacks μ^N.

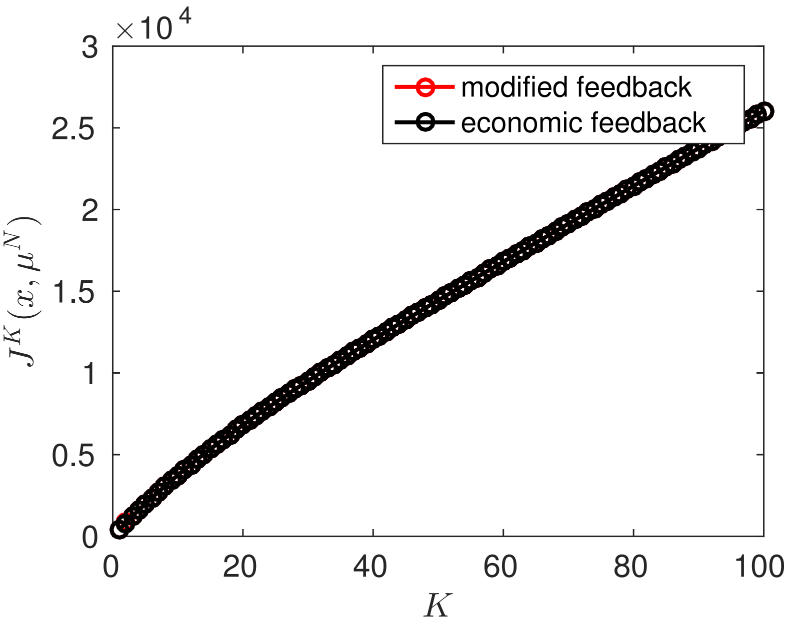

Figure 2.8: $J^K(x, \mu^N)$ for $N = 10$, $x = (4, 20)^T$ and varying K subject to different feedbacks μ^N.

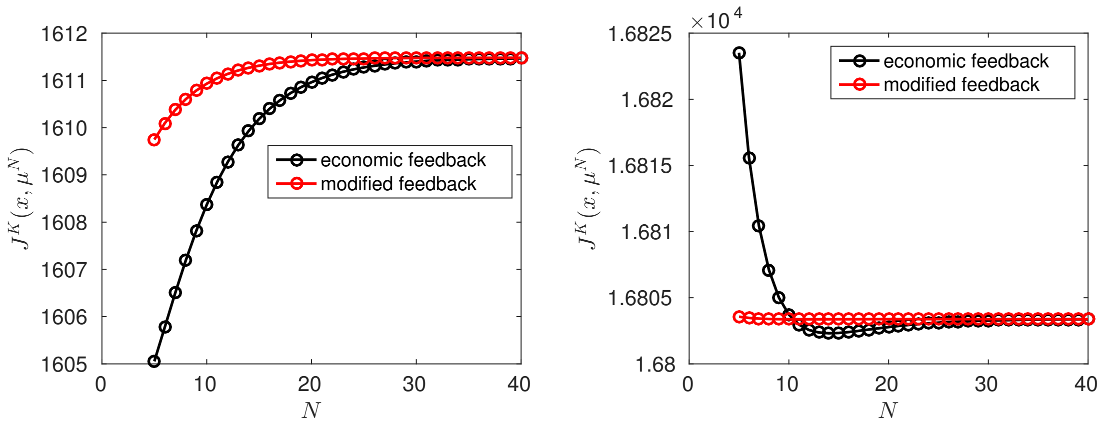

Figure 2.9: $J^K(x, \mu^N)$ for $K = 4$ (left) and $K = 60$ (right), $x = (4, 20)^T$ and varying N subject to different feedbacks μ^N.

3 | Multiobjective Optimization

The purpose of this chapter is to build the basis for multiobjective (MO) Model Predictive Control (MPC) that will be dealt with in Chapter 4 and 5. Therefore, we state our definition and notation of MO optimization problems, a corresponding notion of optimality, and we will present results on existence of solutions. Furthermore, we briefly mention the solution methods to MO optimization problems that we use in numerical experiments. The first part of this chapter was written by means of the references Ehrgott [20], Eichfelder [21], Jahn [45], Miettinen [61], Sawaragi et al. [77].

3.1 Basic Definitions and Selected Properties in Multiobjective Optimization

For functions $(h_1, \ldots, h_s) =: h : X \to \mathbb{R}^s$, $s \in \mathbb{N}$, we consider the following MO optimization problem

$$\text{“}\min_{x \in X}\text{”}\,(h_1(x), \ldots, h_s(x))\,. \tag{3.1}$$

The functions $h_1, \ldots, h_s$ are called *objective functions* and the set $X \subseteq \mathbb{R}^n$ is called *admissible set*. Of course, it would be preferable to obtain a value $h^I \in \mathbb{R}^s$ that satisfies $h_i^I = \min_{x \in X} h_i(x)$ for each $i \in \{1, \ldots, s\}$. The value h^I is called the *ideal* or *utopia value* to problem (3.1). Since it is in general not possible to achieve this value – especially when the objectives are conflicting – the 'classical' meaning of the min-operator breaks down. This is why we write the min in (3.1) in quotation marks for the moment. The appropriate notion of optimality in the context of MO optimization that we will use in this thesis, is formalized as follows.

Definition 3.1 (Pareto optimality, nondominance)**.** *A point $x^\star \in X$ is called a* Pareto-optimal solution (POS) *to the MO optimization problem* (3.1) *if there is no feasible $x \in X$ such that*

$$h_i(x) \le h_i(x^\star) \textit{ for all } i \in \{1, \ldots, s\} \textit{ and}$$
$$h_k(x) < h_k(x^\star) \textit{ for at least one } k \in \{1, \ldots, s\}.$$

The respective value $h(x^\star) := (h_1(x^\star), \ldots, h_s(x^\star))$ is called nondominated *and the set of all such values the* nondominated set *or* Pareto front*.*

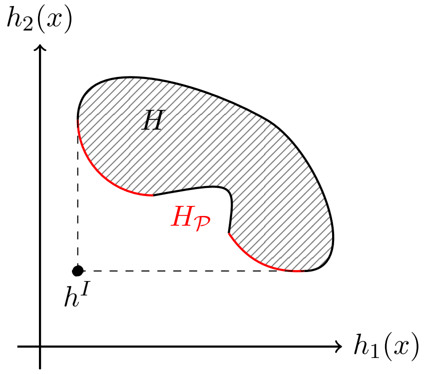

Figure 3.1: Schematic illustration of the admissible (dashed) and nondominated values (red), and the ideal value h^I of a bicriterion optimization problem.

A point $x^\star \in X$ is called a weakly POS *to the MO optimization problem* (3.1) *if there is no feasible $x \in X$ such that $h_i(x) < h_i(x^\star)$ holds for all $i \in \{1, \ldots, s\}$.*

In this section we will use the following abbreviations: $H := \{h(x)|x \in X\}$, $X_{\mathcal{P}} := \{x \in X|x \text{ is a POS to (3.1)}\}$ and $H_{\mathcal{P}} := \{h(x)|x \in X_{\mathcal{P}}\}$.

Convention: Throughout the thesis the min- and argmin-operator are in the context of MO optimization thought of as follows:

$$\min_{x \in X} (h_1(x), \ldots, h_s(x)) = H_{\mathcal{P}},$$
$$\operatorname*{argmin}_{x \in X} (h_1(x), \ldots, h_s(x)) = X_{\mathcal{P}}.$$

In Figure 3.1 we have illustrated what the ideal point and nondominated solutions to a bicriterion (i.e. $s = 2$ in (3.1)) optimization problem can look like. As depicted in the figure it is not unusual to have a whole continuum of nondominated solutions. Moreover, those solutions are always elements of the boundary of the set of admissible values (see [20] for a proof).

We point out that the nomenclature in MO optimization is not unified. POSs are also called *efficient*, *noninferior* or *Edgeworth-Pareto optimal*[1] solutions. The nondominated set which is the solution to (3.1) is – also in this thesis – sometimes referred to the *Pareto front*. It is worth mentioning that Pareto optimality is just one concept to define optimal solutions in the theory of MO optimization. A more general approach can be found in [20] or [77]. Without going into detail we just note that the dominance structure that is used in Definition 3.1 is the convex cone $\mathbb{R}^s_{\geq 0}$.

Remark 3.2 (Equivalent characterizations of POSs)**.** *$x^\star \in X$ is a POS to* (3.1) *iff*

1. *there is no $x \in X$, $x \neq x^\star$, such that $h_i(x) \lneqq h_i(x^\star)$ for all $i \in \{1, \ldots, s\}$.*

[1] Francis Y. Edgeworth (1845-1926) and Vilfredo F. Pareto (1848-1923) both contributed to the concept of optimality in MO optimization, see e.g. [61, Section 2.2].

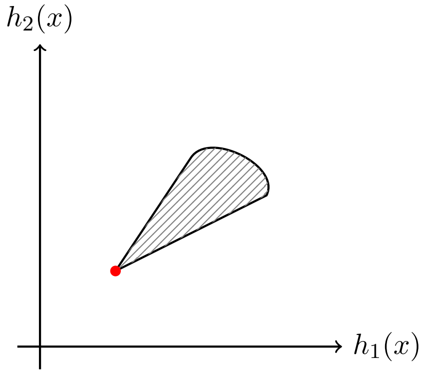

Figure 3.2: Schematic illustration of a strongly Pareto-optimal solution (red) to a bicriterion optimization problem.

2. *there is no $x \in X$ such that $h(x) - h(x^\star) \in -\mathbb{R}^s_{\geq 0} \setminus \{0\}$.*

Definition 3.3 (Strong Pareto optimality)**.** *A point $x^\star \in X$ is called a* strongly Pareto-optimal solution *to problem* (3.1) *if $h_i(x^\star) \leq h_i(x)$ holds for all $x \in X$ and all $i \in \{1, \dots, s\}$.*

Clearly, any strongly POS is Pareto optimal in the sense of Definition 3.1. Moreover, in case a strongly POS exists, its value $h(x^\star)$ coincides with the ideal or utopia value h^I. An illustration of a bicriterion optimization problem with a strongly Pareto-optimal solution can be found in Figure 3.2.

As in the theory of scalar-valued optimization, there exists the notion of *local* and *global* (Pareto-)optimal solutions (see [61]) as well as ε-(Pareto-)optimal solutions, [21]. Moreover, we point out that Definition 3.1 is related to the concept of a *strict* optimizer in scalar-valued optimization, whereas a nonstrict optimizer corresponds to the notion of a *weak* POS, [20, 21]. Since we do not put emphasis on these aspects in this thesis, we restrict ourselves to mentioning them and move on to the question of existence. As in scalar-valued optimization there exists a wide variety on sufficient conditions which ensure that the nondominated set to a MO optimization problem is not empty. In Lemma 3.6 we will state a condition that is well suited for our purposes. Furthermore, we have to deal with an issue that does not occur in scalar-valued optimization.

Imagine that we are given a minimizer to a scalar-valued optimization problem. Then, by definition, the optimal value is smaller than any feasible value. In general, this property does not hold true in MO optimization problems. In Figure 3.3 the nondominated value (red dot) has a better (smaller) value for objective one, but a worse value for objective two than the admissible value (black dot). Hence, the nondominated value does not dominate the feasible value.[2] The way to deal with this situation is not to investigate whether a POS has a smaller value than some feasible point but to ensure that for any feasible solution there is a POS with smaller value in each objective. This property is formalized as follows.

[2] Of course, this situation cannot occur for a strongly POS.

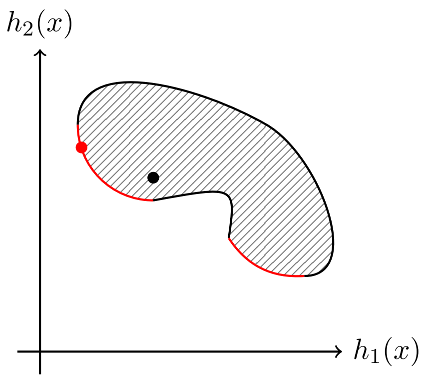

Figure 3.3: Nondominated values (red) do not always dominate arbitrary feasible points (black).

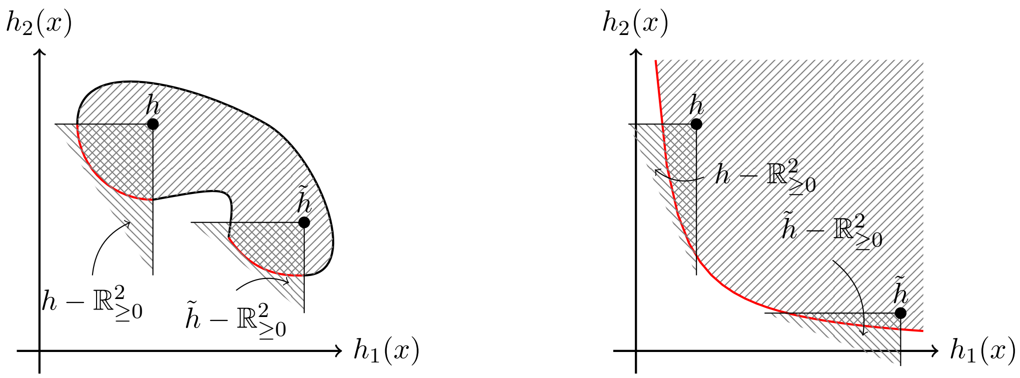

Figure 3.4: Two bicriterion optimization problems with $\mathbb{R}^s_{\geq 0}$-compact set of admissible values.

Definition 3.4 (External stability)**.** *Consider the MO optimization problem* (3.1)*. The set $H_\mathcal{P}$ is called* externally stable *if for each $h \in H$ there is $h^\star \in H_\mathcal{P}$ such that $h \in h^\star + \mathbb{R}^s_{\geq 0}$. This is equivalent to $H \subset H_\mathcal{P} + \mathbb{R}^s_{\geq 0}$.*

Existence and external stability of nondominated solutions can both be ensured using the following compactness notion.

Definition 3.5 ($\mathbb{R}^s_{\geq 0}$-compactness)**.** *A set $H \subset \mathbb{R}^s$ is said to be $\mathbb{R}^s_{\geq 0}$-*compact *if for any $h \in H$ the* section *$(h - \mathbb{R}^s_{\geq 0}) \cap H$ is compact.*

In Figure 3.4 we have illustrated $\mathbb{R}^2_{\geq 0}$-compactness for the set of admissible values of two bicriterion optimization problems. The set H on the left is already compact and thus immediately $\mathbb{R}^2_{\geq 0}$-compact, whereas the set H on the right only exhibits the weaker property of $\mathbb{R}^2_{\geq 0}$-compactness. To illustrate what a non externally stable set $H_\mathcal{P}$ can look like, let us slightly modify the example on the left in Figure 3.4 by removing a part of the boundary, see Figure 3.5. In that case, the relation $H \subset H_\mathcal{P} + \mathbb{R}^s_{\geq 0}$ is obviously wrong and externally stability is not given.

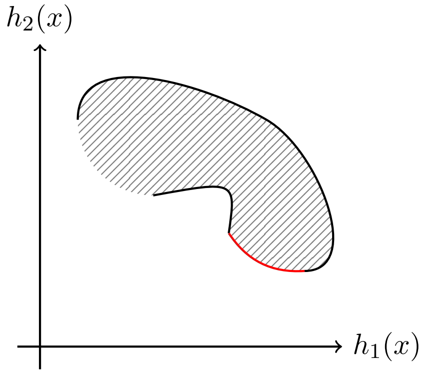

Figure 3.5: Example of a bicriterion optimization problem with $H_{\mathcal{P}}$ (red) which is not externally stable.

Lemma 3.6 (Cone-compactness implies external stability)**.** *Consider the MO optimization problem* (3.1) *and assume that H is nonempty. If H is $\mathbb{R}^s_{\geq 0}$-compact, then the set of nondominated solutions $H_{\mathcal{P}}$ is nonempty and externally stable.*

For a proof of the statements we refer the reader to [77, Section 3.2] or [20, Section 2.2]. Especially in the first reference weaker and also more general conditions that guarantee existence of POSs are presented.

3.2 Computation of Pareto-optimal Solutions

Just as in scalar-valued optimization there is a broad amount of methods to calculate solutions to MO optimization problems such as (3.1). The main idea which many of these methods share is to obtain one ore more solutions by solving an auxiliary scalar-valued optimization problem that yields a POS. Here, we do not provide an overview of existing methods. Instead, we refer to the literature (e.g. [9, 15, 20, 21, 61]) and just mention that we have used the following methods in our implementation: The Pascoletti-Serafini scalarization, see [21, 70], the method of the global criterion, see [61], the weighted sum-approach, see [20, 61], and a genetic algorithm called NSGA II, see [15]. The genetic algorithm is implemented as function `gamultiobj` in MATLAB and due to the lack of theoretical results only used as fallback. An advantage of NSGA II is that it returns a representation of the whole set of POSs and can be applied without checking any assumptions. Even though the weighted sum-approach is just a special case of the Pascoletti-Serafini scalarization, we will briefly present it for two reasons. Firstly, the weighted sum-approach seems to be the best known (and easy) solution method to MO optimization problems in general, and a popular approach to MO MPC in particular, see [6, 25, 52, 73, 81]. Secondly, there are nice theoretical results if the approach is applied to (strictly) convex MO optimization problems.

3.2.1 The Weighted Sum-Approach and Convex Problems

The basic idea of this approach is to transform the MO optimization problem (3.1) into a scalar-valued optimization. By means of weights $\mathbf{w} = (w_1, \dots, w_s)$, $w_i \in \mathbb{R}_{\geq 0}$ that satisfy the equality $\sum_{i=1}^{s} w_i = 1$, one defines the *weighted sum*

$$S(\mathbf{w}, x) := \sum_{i=1}^{s} w_i h_i(x) \tag{3.2}$$

and solves

$$\min_{x \in X} S(\mathbf{w}, x) \tag{3.3}$$

instead of (3.1). The justification for doing so is given in the following statement.

Lemma 3.7. *Consider weights $w_i \in \mathbb{R}_{\geq 0}$ satisfying $\sum_i w_i = 1$ and let $x^\star$ be a solution of* (3.3)*.*

1. *If $w_i > 0$ for all i, then $x^\star$ is a POS to* (3.1)*.*
2. *If $x^\star$ is a unique solution of* (3.3)*, then it is a POS to* (3.1)*.*

Proof.
1. $x^\star$ is a solution to (3.3) with $w_i > 0$ for all $i \in \{1, \dots, s\}$, i.e. $\sum_{i=1}^{s} w_i h_i(x^\star) \leq \sum_{i=1}^{s} w_i h_i(x)$ for all $x \in X$. Let us assume that $x^\star$ is not a POS to (3.1). By Definition 3.1 this implies the existence of $\bar{x} \in X$ such that $h_i(\bar{x}) \leq h_i(x^\star)$ for all $i \in \{1, \dots, s\}$ and $h_k(\bar{x}) < h_k(x^\star)$ for at least one $k \in \{1, \dots, s\}$. Thus, with the assumption $w_i > 0$, we have $\sum_{i=1}^{s} w_i h_i(\bar{x}) < \sum_{i=1}^{s} w_i h_i(x^\star)$, which is a contradiction.
2. Let $x^\star$ be a unique solution to (3.3), i.e. $\sum_{i=1}^{s} w_i h_i(x^\star) < \sum_{i=1}^{s} w_i h_i(x)$ for all $x \in X \setminus \{x^\star\}$. We assume that $x^\star$ is not a POS to (3.1), i.e. there is $\bar{x} \in X$ such that $h_i(\bar{x}) \leq h_i(x^\star)$ for all $i \in \{1, \dots, s\}$ and $h_k(\bar{x}) < h_k(x^\star)$ for at least one $k \in \{1, \dots, s\}$. This yields $\sum_{i=1}^{s} w_i h_i(\bar{x}) \leq \sum_{i=1}^{s} w_i h_i(x^\star)$ (because the weights are not assumed to be strictly positive) and thus contradicts the uniqueness of $x^\star$.

$\square$

Lemma 3.7 implies that by solving (3.3) with positive weights we always obtain a POS to (3.1). The converse is in general not true. If the Pareto front contains concave parts, they cannot be obtained via a weighted sum (see [20, Fig. 3.3] or [45, Fig. 11.10] for an illustration of this fact). Clearly, this aspect is the major drawback of the approach.[3] However, such a behaviour cannot occur if the Pareto front is convex.

[3] Other drawbacks of the weighted sum-approach are: It is very difficult to obtain a good (e.g. evenly spread) representation of the nondominated set by a clever choice of the weights, and even if some decision maker had clear preferences for the objectives, fixing weights might not reflect those preferences, see [13]. Thus, quite often, a sound knowledge of the nondominated set must be known a priori in order to use appropriate weights.

Definition 3.8 (Convex MO optimization problem)**.** *The MO optimization problem* (3.1) *is called* convex *if X is a convex set and all objective functions h_i, $i \in \{1, \dots, s\}$, are convex functions. It is called strictly convex if X is a convex set and all objective functions h_i, $i \in \{1, \dots, s\}$, are strictly convex functions.*

Lemma 3.9. *Let* (3.1) *be a convex problem and let $x^\star \in X_{\mathcal{P}}$. Then there exist weights $w_1, \dots, w_s \geq 0$, $\sum_i w_i = 1$, such that $x^\star$ is a solution to* (3.3)*.*

A proof of this lemma can be found in [20, Section 3.2]. We note that it is not necessary that (3.1) be a convex problem to obtain the statement in Lemma 3.9. The weaker condition of H being $\mathbb{R}^s_{\geq 0}$-convex (i.e. $H + \mathbb{R}^s_{\geq 0}$ is convex) is already sufficient, see [20, Thm. 3.5]. For strictly convex problems the previous Lemmas 3.7 and 3.9 yield the following.

Corollary 3.10. *Let* (3.1) *be a strictly convex problem. Then $x^\star$ is a POS to* (3.1) *if and only if there exist $w_i \geq 0$, $\sum_i w_i = 1$, such that $x^\star$ is a solution to* (3.3)*.*

Proof. "$\Rightarrow$": Let $x^\star$ be a POS to (3.1). Lemma 3.9 yields the existence of weights $w_i \geq 0$, $\sum_i w_i = 1$, such that $x^\star$ is a solution to (3.3).
"$\Leftarrow$": Consider arbitrary weights $w_i \geq 0$, $\sum_i w_i = 1$, and let $x^\star$ be a corresponding solution to (3.3). Our assumptions imply that (3.3) is a strictly convex optimization problem and thus, $x^\star$ is the unique solution. The second statement in Lemma 3.7 then yields that $x^\star$ is indeed a POS to (3.1). □

This means that the set of nondominated solutions can completely be characterized via a weighted sum-approach if the MO optimization problem is strictly convex. We will make use of this statement in Chapter 5.

4 | Multiobjective Stabilizing Model Predictive Control

In this chapter we move on from Model Predictive Control (MPC) for scalar-valued optimal control problems (OCPs) to MPC for multiobjective (MO) OCPs. This means we address problems that are equipped with s stage costs $\ell_1, \ldots, \ell_s : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$. Such problems can occur in (mainly) two situations. Either, we are given one system that simply has multiple objectives that have to be minimized simultaneously, or there are multiple systems or subsystems – collected in the *overall system* (1.1) – that all have their own objective(s). In the latter situation, we assume that the systems/*agents* aim for a *cooperative control* and that there is a communication structure such that the computation of common solutions is possible. This can be ensured via a *central entity* that communicates among the systems and calculates control actions.

No matter which situation, the MO counterpart of the infinite-horizon OCP (1.2) is

$$\begin{aligned}\min_{\mathbf{u}\in\mathbb{U}^\infty(x_0)} \; & (J_1^\infty(x_0,\mathbf{u}), \ldots, J_s^\infty(x_0,\mathbf{u})) \\ \text{s.t } & x(0,x_0) = x_0 \\ & x(k+1,x_0) = f(x(k,x_0),u(k)) \; \forall \; k \in \mathbb{N}_0,\end{aligned} \tag{4.1}$$

in which the min-operator is supposed to be understood in the sense of Chapter 3. In this chapter and in Chapter 5 we use the abbreviation $J^N(x,\mathbf{u}) := \big(J_1^N(x,\mathbf{u}), \ldots, J_s^N(x,\mathbf{u})\big)$, $N \in \mathbb{N} \cup \{\infty\}$, for the vector-valued cost functional.

Some results in this chapter have been published in [36].

4.1 Approaches and Challenges in Multiobjective MPC

Given the infinite-horizon MO OCP (4.1), a first idea is to transform the (now) MO optimization problem

$$\begin{aligned}\min_{\mathbf{u}\in\mathbb{U}^N(x)} \; & J^N(x,\mathbf{u}) \\ \text{s.t. } & (1.1),\end{aligned} \tag{4.2}$$

that has to be solved in step **(2)**, Algorithm 1, into a weighted sum with fixed (see, e.g. [73, Sec. 6.2.4], [25, 81]) or time-varying (see [6]) weights. Advantages of this strategy are the low computational effort and the availability of the 'standard' MPC theory, however, for the following reasons we will proceed differently:

1. The choice of weights might not represent the decision maker's preferences, see e.g. the discussion in [61, Sec. 3.1.3].

2. In general, not all Pareto-optimal solutions (POSs) can be attained (see the references mentioned in Section 3.2.1).

3. As stated in [40], we do not obtain performance estimates for every objective but only for the weighted sum.

Other approaches to handle the MO optimization problem are *hierarchical* MPC algorithms (e.g. [43]), the so called *utopia-tracking* approach in [90] and *iterative schemes*, see [56]. Apart from the last reference, they all share the idea of specifying a hierarchy among the objectives and to make use of the theory of Lyapunov stability by establishing a Lyapunov function (LF) (see Definition 1.7).

Another idea to deal with (4.1) by means of MPC, in which no prioritization of objectives has to be set, is to directly execute Algorithm 1, i.e. by just choosing an arbitrary[1] Pareto-optimal solution to the MO optimization problem (4.2), see [24, 50, 66, 69, 83]. Although this procedure might work (in terms of trajectory convergence) for the examples in this chapter, it does not allow for a 'good' performance analysis of the single objectives. This fact will be demonstrated in Section 4.4. Moreover, we will see in Chapter 5 that a random choice of POSs does not yield desirable results for more general cost criteria.

Nevertheless, our approach is conceptually closer to the latter references, since we do not want to prioritize objectives a priori. The main idea in this chapter and in Chapter 5 is to add contraints to the MO optimization in step **(2)**, Algorithm 1 that enable us to formulate performance statements for all objectives and to ensure convergence of the closed-loop trajectory. Proceeding this way (and as opposed to [6, 25, 43, 73, 81, 90]), we will not define an optimal value function that serves as Lyapunov function (LF).

In the MPC context and for $N \in \mathbb{N} \cup \{\infty\}$ we will use the following notation:

$$\begin{aligned}
\mathbb{U}^N_{\mathcal{P}}(x) &:= \{\mathbf{u} \in \mathbb{U}^N(x) | \mathbf{u} \text{ POS to } (4.2)\} \\
\mathcal{J}^N(x) &:= \{J^N(x, \mathbf{u}) | \mathbf{u} \in \mathbb{U}^N(x)\} \\
\mathcal{J}^N_{\mathcal{P}}(x) &:= \{J^N(x, \mathbf{u}) \in \mathcal{J}^N(x) | \mathbf{u} \in \mathbb{U}^N_{\mathcal{P}}(x)\}.
\end{aligned}$$

For our analysis we start with a result on POSs that resembles an aspect of the Dynamic Programming Principle (DPP).

Lemma 4.1 (Tails of POSs are POSs). *If* $\mathbf{u}^\star \in \mathbb{U}^N_{\mathcal{P}}(x)$, *then* $\mathbf{u}^{\star,K} := \mathbf{u}^\star(\cdot + K) \in \mathbb{U}^{N-K}_{\mathcal{P}}(x^{\mathbf{u}^\star}(K, x))$ *for all* $K \in \mathbb{N}_{<N}$, *where* $\mathbf{u}^\star(\cdot + K) := (u^\star(K), u^\star(K+1), \ldots, u^\star(N-1))$.

[1] Of course, the chosen solutions are not completely arbitrary, but usually subject to expert decisions. We point out that – to the best of our knowledge – performance consideration do not play a role in choosing a POS in the mentioned references.

Proof. We first note that $\mathbf{u}^\star \in \mathbb{U}^N_{\mathcal{P}}(x) \subset \mathbb{U}^N(x)$ implies $\mathbf{u}^{\star,K} \in \mathbb{U}^{N-K}(x)$, see e.g. [32, Lemma 3.12]. Let us assume that $\mathbf{u}^{\star,K}$ is not a POS of length $N-K$ for initial value $x^{\mathbf{u}^\star}(K,x)$. This implies the existence of $\mathbf{u} \in \mathbb{U}^{N-K}(x^{\mathbf{u}^\star}(K,x))$ satisfying

$$\begin{aligned}
&\forall\, i \in \{1,\dots,s\} : J_i^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}) \le J_i^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}^{\star,K}) \text{ and}\\
&\exists\, j \in \{1,\dots,s\} : J_j^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}) < J_j^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}^{\star,K}).
\end{aligned}$$

Since by definition

$$J_i^N(x,\mathbf{u}^\star) = \sum_{k=0}^{K-1} \ell_i(x^{\mathbf{u}^\star}(k,x),u^\star(k)) + J_i^{N-K}(x^{\mathbf{u}^\star}(K,x),\underbrace{\mathbf{u}^\star(\cdot+K)}_{\mathbf{u}^{\star,K}})$$

holds for all $K \in \mathbb{N}_{\le N}$, we obtain

$$\begin{aligned}
\forall\, i \in \{1,\dots,s\} :\ J_i^N(x,\mathbf{u}^\star) &\ge \sum_{k=0}^{K-1} \ell_i(x^{\mathbf{u}^\star}(k,x),u^\star(k)) + J_i^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}) \text{ and}\\
\exists\, j \in \{1,\dots,s\} :\ J_j^N(x,\mathbf{u}^\star) &= \sum_{k=0}^{K-1} \ell_j(x^{\mathbf{u}^\star}(k,x),u^\star(k)) + J_j^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}^{\star,K})\\
&> \sum_{k=0}^{K-1} \ell_j(x^{\mathbf{u}^\star}(k,x),u^\star(k)) + J_j^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}).
\end{aligned}$$

Using again [32, Lemma 3.12], it holds that the concatenated control sequence $\bar{\mathbf{u}} = (u^\star(0),\dots,u^\star(K-1),\mathbf{u})$ is contained in the set $\mathbb{U}^N(x)$, i.e. we get

$$\begin{aligned}
&\forall\, i \in \{1,\dots,s\} :\ J_i^N(x,\mathbf{u}^\star) \ge J_i^N(x,\bar{\mathbf{u}}) \text{ and}\\
&\exists\, j \in \{1,\dots,s\} :\ J_j^N(x,\mathbf{u}^\star) > J_i^N(x,\bar{\mathbf{u}}).
\end{aligned}$$

This contradicts the fact that $\mathbf{u}^\star \in \mathbb{U}^N_{\mathcal{P}}(x)$. □

In this chapter, we investigate MO MPC for 'classical' or 'stabilizing' stage costs, that is we assume the following.

Assumption 4.2 ('Classical' stage costs). *1. There is an equilibrium pair $(x_*,u_*) \in \mathbb{X}\times\mathbb{U}$, i.e., $f(x_*,u_*) = x_*$.*

2. There are $\alpha_{\ell,i} \in \mathcal{K}$ (see Def. 1.5) such that all stage costs ℓ_i, $i \in \{1,\dots,s\}$, satisfy $\min_{u\in\mathbb{U}} \ell_i(x,u) \ge \alpha_{\ell,i}(\|x-x_*\|)\ \forall x \in \mathbb{X}$.

Moreover, we require that the infinite-horizon MO OCP (4.1) that we intend to solve by means of MPC, has POSs with finite nondominated values for all objectives. Sufficient conditions that ensure this property are presented in [32, Sec. 4.1] for single-objective OCPs and can be carried over to our MO setting. Necessary and sufficient conditions for the existence of POSs on the infinite-horizon based on Pontryagin principles are derived in [42].

Remark 4.3. *In Assumption 4.2 it is required that all stage costs are positive definite wrt to the same equilibrium. This means that the sequence* $\bar{\mathbf{u}} \in \mathbb{U}^N(x_*)$ *with* $\bar{u}(k) = u_*$ *for all* $k \in \{1, \dots, N\}$*, is a strongly POS according to Def. 3.3 for the problem of minimizing* (4.2) *with initial value* $x = x_*$ *for each* $N \in \mathbb{N} \cup \{\infty\}$*.*

4.2 Multiobjective MPC with Terminal Conditions

In this section we will analyze MO MPC using terminal conditions. This means that there is a *terminal constraint set* $\mathbb{X}_0 \subseteq \mathbb{X}$ and a *terminal cost* $F_i : \mathbb{X}_0 \to \mathbb{R}_{\geq 0}$, $i \in \{1, \dots, s\}$, such that the MO optimization problem that we solve in the MPC algorithm now reads

$$\begin{aligned} \min_{\mathbf{u} \in \mathbb{U}^N(x)} \; & J^N(x, \mathbf{u}), \text{ with } J_i^N(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell_i(x(k,x), u(k)) + F_i(x(N,x)) \\ \text{s.t. } & (1.1) \\ & x(N,x) \in \mathbb{X}_0. \end{aligned} \tag{4.3}$$

Since the terminal constraint $x(N) \in \mathbb{X}_0$ can generally not be satisfied by all initial values $x \in \mathbb{X}$, we define the feasible set $\mathbb{X}_N := \{x \in \mathbb{X} | \exists \mathbf{u} \in \mathbb{U}^N : \; x(k) \in \mathbb{X}, \; k = 1, \dots, N-1, \; x(N) \in \mathbb{X}_0\}$, cf. [32, Definition 3.9] or [73, Section 2.3]. This set is assumed to be nonempty throughout this section. Only for such initial values $x \in \mathbb{X}_N$ we consider the set $\mathbb{U}^N(x)$, which in this section comprises the terminal constraint[2], i.e.

$$\mathbb{U}^N(x) := \{\mathbf{u} \in \mathbb{U}^N(x) | x(N,x) \in \mathbb{X}_0\}.$$

Assumption 4.4 (Lyapunov function terminal cost)**.** *We assume that* x_* *from Assumption 4.2 is contained in* $\mathbb{X}_0$ *and the existence of a local feedback* $\kappa : \mathbb{X}_0 \to \mathbb{U}$ *satisfying*

1. $f(x, \kappa(x)) \in \mathbb{X}_0$ *for all* $x \in \mathbb{X}_0$ *and*

2. $\forall x \in \mathbb{X}_0, \; i \in \{1, \dots, s\} : \; F_i(f(x, \kappa(x))) + \ell_i(x, \kappa(x)) \leq F_i(x)$.

Imposing Assumption 4.4 ensures that it is always possible to remain within the terminal constraint set $\mathbb{X}_0$ and that the cost of this control action is bounded from above by the original terminal cost. We note that Lemma 4.1 remains valid under the 'new' definition of the cost functionals J_i^N.

In what follows we first propose a MO MPC algorithm and prove feasibility, performance and convergence afterwards.

Algorithm 2 (Multiobjective MPC with terminal conditions).

(0) At time $n = 0$: Set $x(n) := x_0$ and choose a POS $\mathbf{u}^{\star,N}_{x(n)} \in \mathbb{U}^N_{\mathcal{P}}(x(n))$ to (4.3). Go to **(2)**.

[2] In case $N = \infty$, the set $\mathbb{U}^\infty(x)$ remains unchanged.

(1) At time $n \in \mathbb{N}$: Choose a POS $\mathbf{u}^{\star,N}_{x(n)}$ to (4.3) so that the inequalities

$$J_i^N\left(x(n), \mathbf{u}^{\star,N}_{x(n)}\right) \leq J_i^N\left(x(n), \mathbf{u}^{N}_{x(n)}\right)$$

are satisfied for all $i \in \{1, \ldots, s\}$.

(2) For $x := x^{\mathbf{u}^{\star,N}_{x(n)}}(N, x(n))$ set

$$\mathbf{u}^N_{x(n+1)} := \left(u^{\star,N}_{x(n)}(1), \ldots, u^{\star,N}_{x(n)}(N-1), \kappa(x)\right).$$

(3) Apply the feedback $\mu^N(x(n)) := u^{\star,N}_{x(n)}(0)$, set $n = n + 1$ and go to **(1)**.

Figure 4.1 schematically visualizes the choice of the POSs in step **(1)** of Algorithm 2. The bounds resulting from $\mathbf{u}^N_{x(n)}$ are visualized by dashed lines and determine the set of nondominated points that may be chosen (thick, red line). The basic idea (formalized in Lemma 4.5) is that the control sequence $\mathbf{u}^N_{x(n)}$ in step **(2)** is a POS of length $N-1$ prolonged by the local feedback from Assumption 4.4 and that the prolongation reduces the value of the objective functions. The preliminary considerations in Chapter 3 moreover

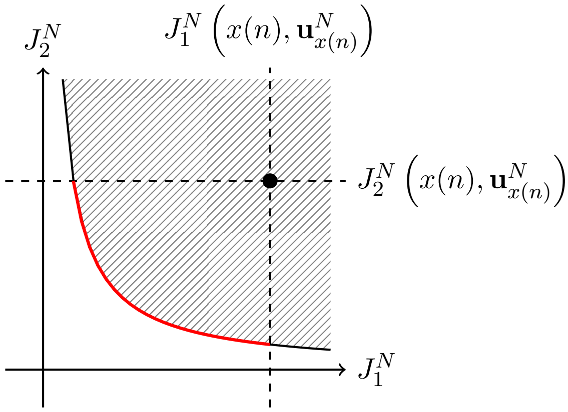

Figure 4.1: Visualization of step **(1)** in Algorithm 2 for a bicriterion OCP.

show that there are sufficient conditions, so that there is a POS with smaller objective value than the prolonged sequence (for each i).

Lemma 4.5. *Let Assumption 4.4 hold and let $\mathbf{u} \in \mathbb{U}^{N-1}(x)$, $x \in \mathbb{X}_N$. Then there exists a sequence $\mathbf{u}^N \in \mathbb{U}^N(x)$ satisfying*

$$J_i^N(x, \mathbf{u}^N) \leq J_i^{N-1}(x, \mathbf{u}) \quad \forall i \in \{1, \ldots, s\}.$$

Proof. We define $\mathbf{u}^N$ as follows. $u^N(k) := u(k)$ for $k = 0, \ldots, N-2$ and $u^N(N-1) := \kappa(\bar{x})$ from Assumption 4.4, where $\bar{x} := x^{\mathbf{u}^N}(N-1, x)$. Then $\mathbf{u}^N$ is feasible because $\mathbf{u} \in \mathbb{U}^{N-1}(x)$, and therefore, $\bar{x} \in \mathbb{X}_0$. Assumption 4.4 ensures feasibility of $\kappa(\bar{x})$ and $f(\bar{x}, \kappa(\bar{x}))$. With the definition of $\mathbf{u}^N$ we obtain the estimates

$$\begin{aligned} J_i^N(x, \mathbf{u}^N) &= \sum_{k=0}^{N-1} \ell_i(x^{\mathbf{u}^N}(k,x), \mathbf{u}^N(k)) + F_i(x^{\mathbf{u}^N}(N,x)) \\ &= \sum_{k=0}^{N-2} \ell_i(x^{\mathbf{u}^N}(k,x), \mathbf{u}^N(k)) + \ell_i(\bar{x}, \kappa(\bar{x})) + F_i(f(\bar{x}, \kappa(\bar{x}))) \\ &\le \sum_{k=0}^{N-2} \ell_i(x^{\mathbf{u}}(k,x), \mathbf{u}(k)) + F_i(\bar{x}) = J_i^{N-1}(x, \mathbf{u}). \end{aligned}$$

□

By means of our preliminary considerations we can now state our main performance result on MO stabilizing MPC with terminal conditions, which guarantees a bounded performance of the feedback μ^N defined in Algoritm 2 for all cost criteria $i \in \{1, \dots, s\}$.

Theorem 4.6 (MO MPC Performance Theorem)**.** *Consider a MO OCP* (4.3) *with system dynamics* (1.1)*, stage costs ℓ_i, $i = 1, \dots, s$, and let $N \in \mathbb{N}_{\ge 2}$. Let Assumptions 4.2 and 4.4 hold and let the set $\mathcal{J}_{\mathcal{P}}^N(x)$ be externally stable (according to Def. 3.4) for each $x \in \mathbb{X}_N$. Then, the MPC feedback $\mu^N : \mathbb{X} \to \mathbb{U}$ defined in Algorithm 2 renders the set $\mathbb{X}$ forward invariant (see Definition 1.1) and has the following infinite-horizon closed-loop performance:*

$$J_i^\infty\left(x_0, \mu^N\right) := \lim_{K\to\infty} \sum_{k=0}^{K-1} \ell_i\left(x(k), \mu^N(x(k))\right) \le J_i^N\left(x_0, \mathbf{u}_{x_0}^{\star,N}\right) \tag{4.4}$$

for all objectives $i \in \{1, \dots, s\}$, in which $\mathbf{u}_{x_0}^{\star,N}$ denotes the POS of step **(0)** *in Algorithm 2.*

Proof. **Feasibility:** The existence of the POSs in step **(0)** and **(1)** is concluded from external stability of $\mathcal{J}_{\mathcal{P}}^N(x)$. Feasibility of $\mathbf{u}_{x(n+1)}^N$ in **(2)** follows from Assumption 4.4. Recursive feasibility of $\mathbb{X}$ is an immediate consequence.
Performance: It follows from the definition of the cost functional in (4.3) that

$$J_i^N\left(x(k), \mathbf{u}_{x(k)}^{\star,N}\right) = \ell_i\left(x(k), u_{x(k)}^{\star,N}(0)\right) + J_i^{N-1}\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k)}^{\star,N}(\cdot+1)\right),$$

and hence, for arbitrary $K \in \mathbb{N}_{\ge 1}$ and all $i \in \{1, \dots, s\}$

$$\begin{aligned} &\sum_{k=0}^{K-1} \ell_i(x(k), \mu^N(x(k))) = \sum_{k=0}^{K-1} \ell_i(x(k), u_{x(k)}^{\star,N}(0)) \\ &= \sum_{k=0}^{K-1} \left[J_i^N\left(x(k), \mathbf{u}_{x(k)}^{\star,N}\right) - J_i^{N-1}\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k)}^{\star,N}(\cdot+1)\right)\right] \\ &\le \sum_{k=0}^{K-1} \left[J_i^N\left(x(k), \mathbf{u}_{x(k)}^{\star,N}\right) - J_i^N\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k+1)}^N\right)\right], \end{aligned}$$

in which the inequality follows from Lemma 4.5 in combination with Lemma 4.1, and $\mathbf{u}^{\star,N}_{x(k)}$ is the POS chosen in Algorithm 2 at time k. In step **(1)**, $\mathbf{u}^{\star,N}_{x(k+1)}$ is constructed such that

$$J_i^N\left(x(k+1), \mathbf{u}^{\star,N}_{x(k+1)}\right) \leq J_i^N\left(x(k+1), \mathbf{u}^{N}_{x(k+1)}\right).$$

Thus, we finally obtain

$$\sum_{k=0}^{K-1} \ell_i\left(x(k), \mu^N(x(k))\right) \leq J_i^N\left(x_0, \mathbf{u}^{\star,N}_{x_0}\right) - J_i^N\left(x(K), \mathbf{u}^{N}_{x(K)}\right) \leq J_i^N\left(x_0, \mathbf{u}^{\star,N}_{x_0}\right),$$

because of the positivity of J_i^N (Assumption 4.2). The expression on the left hand side of the inequalities is monotonically increasing and due to its boundedness, the limit for $K \to \infty$ exists and we conclude the assertion. □

Remark 4.7. *1. A closer look at Algorithm 2 reveals that only for $k \geq 1$ the choice of $\mathbf{u}^{\star,N}_{x(k)}$ is subject to additional constraints. The first POS $\mathbf{u}^{\star,N}_{x_0}$, which determines the bound on the performance of the MPC algorithm, can be chosen freely. Thus, the performance can be calculated a priori from a MO optimization of horizon N. This observation justifies the approach of putting major effort into generating the Pareto front in the initialization step* **(0)** *and just calculating one arbitrary solution in subsequent steps with the least possible effort.*

An overview of methods used in our implementation is given in Section 3.2. In the initialization, we usually use the Pascoletti-Serafini scalarization, and in subsequent iterations we use the method of the global criterion or a weighted sum with random weights.[3]

2. The performance result above can serve as an incentive to convince players to apply the MPC strategy.

In Theorem 4.6 we assume external stability of the sets $\mathcal{J}^N_{\mathcal{P}}(x)$ for all $x \in \mathbb{X}_N$. Since this property is difficult to verify, we now provide easily checkable conditions that are sufficient for external stability.

Lemma 4.8. *Let $\mathbb{U}$ be compact, $\mathbb{X}$ and $\mathbb{X}_0$ be closed, and f, F_i and ℓ_i be continuous. Then, the set $\mathcal{J}^N_{\mathcal{P}}(x)$ is externally stable for all $x \in \mathbb{X}_N$ and all $N \in \mathbb{N}$.*

Proof. Fix an arbitrary horizon $N \in \mathbb{N}$ and $x \in \mathbb{X}_N$.

1. It is a general assumption that $\mathbb{X}_N$ is nonempty, thus $\mathbb{U}^N(x) \neq \emptyset$ and $\mathcal{J}^N(x) \neq \emptyset$.

2. In [17] it was for even more general settings proven that the set Δ, which contains all feasible trajectories with their corresponding control sequences $(x^{\mathbf{u}}(\cdot, x), \mathbf{u})$, is compact in $Z := \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{N \text{ times}} \times \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{(N-1) \text{ times}}$. If we interpret J^N as a function

[3] We point out that using a weighted sum also yields a POS even if the Pareto front in step **(1)** is not convex.

that maps from Z to $\mathbb{R}^s_{\geq 0}$, we can conclude compactness of $\mathcal{J}^N(x)$ from compactness of Δ and continuity of ℓ_i and F_i. The stronger notion of compactness implies $\mathbb{R}^s_{\geq 0}$-compactness of $\mathcal{J}^N(x)$.

Since all conditions of Lemma 3.6 are satisfied, we conclude the assertion. □

It may be of interest to note that the conditions in Lemma 4.8, which were proven to guarantee the existence of optimal solutions to single-objective finite-horizon OCPs in [17], are also sufficient for our setting. This fact is particularly pleasing because as opposed to single-criterion OCPs we do not only need existence of optimal solutions but also the special structure of external stability.

Corollary 4.9 (Trajectory convergence)**.** *Under the assumptions of Theorem 4.6 it holds that the closed-loop trajectory $x(\cdot, x_0)$ driven by the feedback μ^N from Algorithm 2 converges to the equilibrium x_*.*

Proof. It follows from Theorem 4.6 that the sum $\sum_{k=0}^{\infty} \ell_i\left(x(k), \mu^N(x(k))\right)$ converges for each $i \in \{1, \dots, s\}$. Hence, the sequences $\left(\ell_i\left(x(k), \mu^N(x(k))\right)\right)_{k\in\mathbb{N}_0}$, $i \in \{1, \dots, s\}$, tend to zero. Together with Assumption 4.2 for arbitrary $i \in \{1, \dots, s\}$ we obtain

$$\begin{aligned} \forall \varepsilon > 0\ \exists K \in \mathbb{N}_0 : \forall k \geq K :\ \varepsilon &> |\ell_i\left(x(k), \mu^N(x(k))\right)| = \ell_i\left(x(k), \mu^N(x(k))\right) \\ &\geq \min_{u\in\mathbb{U}} \ell_i(x(k), u) \geq \alpha_{\ell,i}(\|x(k) - x_*\|), \end{aligned}$$

which is equivalent to the statement $\lim_{k\to\infty} \alpha_{\ell,i}\left(\|x(k) - x_*\|\right) = 0$ for all $i \in \{1, \dots, s\}$. Since $\alpha_{\ell,i}$ is a $\mathcal{K}$ function for each $i \in \{1, \dots, s\}$, it is continuous and it holds

$$\alpha_{\ell,i}\left(\lim_{k\to\infty} \|x(k) - x_*\|\right) = \lim_{k\to\infty} \alpha_{\ell,i}\left(\|x(k) - x_*\|\right) = 0.$$

Again, we use the fact that $\alpha_{\ell,i} \in \mathcal{K}$ for each $i \in \{1, \dots, s\}$ and conclude

$$\alpha_{\ell,i}\left(\lim_{k\to\infty} \|x(k) - x_*\|\right) = 0 \Leftrightarrow \lim_{k\to\infty} \|x(k) - x_*\| = 0.$$

□

Remark 4.10. *The result in Corollary 4.22 shows that the equilibrium x_* is globally attractive for the MPC closed-loop system in the sense of Definition 1.3. Due to the lack of a LF we are not able to prove asymptotic stability (see Definition 1.6). In Chapter 8 we will discuss an approach that might enable us to establish a LF for our MO MPC scheme.*

We have proved in Theorem 4.6 that the inequalities

$$J_i^\infty\left(x_0, \mu^N\right) \leq J_i^N\left(x_0, \mathbf{u}_{x_0}^{\star,N}\right) \quad \forall\ i \in \{1, \dots, s\}$$

hold for the MPC feedback μ^N from Algorithm 2. Usually, one would like to compare the infinite-horizon MPC cost to an expression of the form $J_i^\infty(x_0, \mathbf{u}_{x_0}^{\star,\infty})$, where $\mathbf{u}_{x_0}^{\star,\infty}$ is a POS to the MO OCP 4.1. We now show that it is, in general, not possible to bound $J_i^\infty\left(x_0, \mu^N\right)$ from above by $J_i^\infty(x_0, \mathbf{u}_{x_0}^{\star,\infty})$.

Lemma 4.11. *Let $N \in \mathbb{N}_{\geq 2}$, $x_0 \in \mathbb{X}_N$ be given. Let the assumptions of Theorem 4.6 hold and assume furthermore external stability of the set $\mathcal{J}^{\infty}_{\mathcal{P}}(x_0) := \{J^{\infty}(x_0, \mathbf{u}^{\star}) | \mathbf{u}^{\star} \in \mathbb{U}^{\infty}_{\mathcal{P}}(x_0)\}$. Then, for each $\mathbf{u}^{\star,N} \in \mathbb{U}^{N}_{\mathcal{P}}(x_0)$ there is $\mathbf{u}^{\star,\infty} \in \mathbb{U}^{\infty}_{\mathcal{P}}(x_0)$ such that the inequalities*

$$J_i^N \left(x_0, \mathbf{u}^{\star,N}\right) \geq J_i^{\infty} \left(x_0, \mathbf{u}^{\star,\infty}\right)$$

hold for all $i = 1, \ldots, s$.

Proof. For $N \in \mathbb{N}_{\geq 2}$ and $x_0 \in \mathbb{X}_N$ fix an arbitrary $\mathbf{u}^{\star,N} \in \mathbb{U}^{N}_{\mathcal{P}}(x_0)$. Define the MPC feedback μ^N according to Algorithm 2 and define $\mathbf{u} \in \mathbb{U}^{\infty}(x_0)$ via $u(k) = \mu^N(x^{\mu^N}(k))$ for $k \in \mathbb{N}_{\geq 0}$. Then, we have

$$J_i^N \left(x_0, \mathbf{u}^{\star,N}\right) \overset{\text{Thm. 4.6}}{\geq} J_i^{\infty} \left(x_0, \mu^N\right) = J_i^{\infty} \left(x_0, \mathbf{u}\right) \quad \forall\ i.$$

Since we assume external stability of the set $\mathcal{J}^{\infty}_{\mathcal{P}}(x_0)$, there exists $\mathbf{u}^{\star,\infty} \in \mathbb{U}^{\infty}_{\mathcal{P}}(x_0)$ satisfying

$$J_i^{\infty} \left(x_0, \mathbf{u}\right) \geq J_i^{\infty} \left(x_0, \mathbf{u}^{\star,\infty}\right) \quad \forall\ i.$$

This yields the assertion. □

Lemma 4.11 implies that it is not possible to bound $J_i^{\infty}\left(x_0, \mu^N\right)$ from above by $J_i^{\infty}\left(x_0, \mathbf{u}^{\star,\infty}\right)$ using Theorem 4.6. However, we will be able to show an approximate estimate of this form in Theorem 4.13. As a preparation, we first show that the trajectory corresponding to any infinite-horizon control sequence with bounded objectives gets arbitrarily close to the equilibrium x_* in a finite number of time steps.

Lemma 4.12. *Let $\delta > 0$, $x \in \mathbb{X}$ and $\mathbf{u}^{\infty} \in \mathbb{U}^{\infty}(x)$ be given. Under Assumption 4.2 and if there is $K \in \mathbb{R}_{\geq 0}$ satisfying*

$$J_i^{\infty}(x, \mathbf{u}^{\infty}) \leq K \quad \forall i \in \{1, \ldots, s\},$$

the index $\hat{k} := \min\left\{k \in \mathbb{N}_0 | x^{\mathbf{u}^{\infty}}(k) \in \overline{\mathcal{B}_{\delta}(x_*)}\right\}$ fulfills $\hat{k} \leq \frac{K}{\min_i \alpha_{\ell,i}(\delta)}$. Here, the ball $\overline{\mathcal{B}_{\delta}(x_*)}$ is defined according to (1.6).

Proof. Assume $\hat{k} > \frac{K}{\min_i \alpha_{\ell,i}(\delta)}$, then for all $i \in \{1, \ldots, s\}$ it holds

$$\begin{aligned} J_i^{\infty}\left(x, \mathbf{u}^{\infty}\right) &= \sum_{k=0}^{\hat{k}-1} \ell_i(x(k), u^{\infty}(k)) + \sum_{k=\hat{k}}^{\infty} \ell_i(x(k), u^{\infty}(k)) \\ &\geq \sum_{k=0}^{\hat{k}-1} \alpha_{\ell,i}(\|x(k) - x_*\|) > \sum_{k=0}^{\hat{k}-1} \alpha_{\ell,i}(\delta) = \hat{k} \cdot \alpha_{\ell,i}(\delta) > K, \end{aligned}$$

contradicting the assumption. □

Theorem 4.13 (Approximate infinite-horizon optimality). *Consider the MO OCP* (4.3) *and the corresponding optimal control problem on the infinite horizon* (4.1) *with the same constraints and running costs. Let the Assumptions 4.2 and 4.4 hold and assume furthermore the existence of $\sigma_i \in \mathcal{K}$ such that $F_i(x) \leq \sigma_i(\|x - x_*\|)$ holds for all $x \in \mathbb{X}_0$ and all $i \in \{1, \dots, s\}$. Consider an arbitrary initial value $x_0 \in \mathbb{X}_N$ and a sequence $\mathbf{u}^{\star,\infty} \in \mathbb{U}^\infty_{\mathcal{P}}(x_0)$ with $J_i^\infty(x_0, \mathbf{u}^{\star,\infty}) \leq C$ for all i and some $C \in \mathbb{R}_{\geq 0}$. Assume there is $\bar{N} \in \mathbb{N}$ such that the sets $\mathcal{J}^N_{\mathcal{P}}(x_0)$ are externally stable for all $N \geq \bar{N}$. Then, for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ there is $\mathbf{u}^{\star,N} \in \mathbb{U}^N_{\mathcal{P}}(x_0)$ satisfying*

$$J_i^N\left(x_0, \mathbf{u}^{\star,N}\right) \leq J_i^\infty\left(x_0, \mathbf{u}^{\star,\infty}\right) + \varepsilon \quad \forall i. \tag{4.5}$$

In particular, $\mathbf{u}^{\star,\infty}$ can be approximated arbitrarily well by μ^N from Algorithm 2 in terms of the infinite-horizon performance, that is,

$$J_i^\infty\left(x_0, \mu^N\right) \leq J_i^\infty\left(x_0, \mathbf{u}^{\star,\infty}\right) + \varepsilon. \tag{4.6}$$

Proof. Let $\varepsilon > 0$ and choose $\delta > 0$ such that $\sigma_i(\delta) \leq \varepsilon \ \forall i$ and $\overline{\mathcal{B}_\delta(x_*)} \subseteq \mathbb{X}_0$. For the sequence $\mathbf{u}^{\star,\infty} \in \mathbb{U}^\infty_{\mathcal{P}}(x_0)$ it holds $J_i^\infty\left(x_0, \mathbf{u}^{\star,\infty}\right) \leq C \ \forall i$. From Lemma 4.12 we know that the index $\hat{k} := \min\left\{k \in \mathbb{N}_0 | x^{\mathbf{u}^{\star,\infty}}(k) \in \overline{\mathcal{B}_\delta(x_*)}\right\}$ satisfies $\hat{k} \leq \frac{C}{\min_i \alpha_{\ell,i}(\delta)}$. Now let us choose $N_0 \in \mathbb{N}$ such that $N_0 \geq \max\{\hat{k}+1, \bar{N}\}$. For $N \geq N_0$ define the sequence $\mathbf{u} \in \mathbb{U}^N(x_0)$ via

$$u(k) = \begin{cases} u^{\star,\infty}(k), & k = 0, \dots, \hat{k}-1, \\ \kappa(x(k)), & k = \hat{k}, \dots, N-1, \end{cases}$$

with κ from Assumption 4.4. Since $x^{\mathbf{u}^{\star,\infty}}(\hat{k}) \in \overline{\mathcal{B}_\delta(x_*)} \subseteq \mathbb{X}_0$, κ can be applied and it holds $x^{\mathbf{u}}(N) \in \mathbb{X}_0$. From the definition of $\mathbf{u}$ we obtain

$$\begin{aligned} J_i^N\left(x_0, \mathbf{u}\right) &= \sum_{k=0}^{N-1} \ell_i(x(k), u(k)) + F_i(x(N)) \\ &= \sum_{k=0}^{\hat{k}-1} \ell_i(x(k), u^{\star,\infty}(k)) + \sum_{k=\hat{k}}^{N-1} \ell_i(x(k), \kappa(x(k))) + F_i(x(N)) \\ &\leq J_i^\infty\left(x_0, \mathbf{u}^{\star,\infty}\right) + \sum_{k=\hat{k}}^{N-1} \left[F_i(x(k)) - F_i(f(x(k), \kappa(x(k))))\right] + F_i(x(N)) \\ &= J_i^\infty\left(x_0, \mathbf{u}^{\star,\infty}\right) + F_i(x(\hat{k})) \leq J_i^\infty\left(x_0, \mathbf{u}^{\star,\infty}\right) + \sigma_i(\underbrace{\|x(\hat{k}) - x_*\|}_{\leq \delta}) \\ &\leq J_i^\infty\left(x_0, \mathbf{u}^{\star,\infty}\right) + \varepsilon. \end{aligned}$$

Due to external stability of $\mathcal{J}^N_{\mathcal{P}}(x_0)$ we conclude the existence of $\mathbf{u}^{\star,N} \in \mathbb{U}^N_{\mathcal{P}}(x_0)$ such that

$$J_i^N\left(x_0, \mathbf{u}^{\star,N}\right) \leq J_i^N\left(x_0, \mathbf{u}\right) \leq J_i^\infty\left(x_0, \mathbf{u}^{\star,\infty}\right) + \varepsilon,$$

i.e. (4.5) holds. Choosing $\mathbf{u}^{\star,N}_{x_0} = \mathbf{u}^{\star,N}$ in step **(0)** of Algorithm 2 and combining the estimates (4.4) and (4.5) yields (4.6). □

4.2.1 Endpoint Equilibrium Constraints: A Special Case

The computation of κ and $\mathbb{X}_0$ in Assumption 4.4 can be a hard task that even in the simple case of affine systems and quadratic cost functions involves the solution of linear matrix inequalities (LMIs). Thus, it is sometimes easier, though more restrictive for the set of feasible solutions, to consider the setting $\mathbb{X}_0 = \{x_*\}$ and $F_i(x_*) = 0$ for all $i \in \{1, \dots, s\}$. If the feedback κ is defined as $\kappa(x_*) = u_*$ and requiring $\ell(x_*, u_*) = 0$, this immediately yields the properties from Assumption 4.4 and, thus, also the statements in Theorem 4.6 and Corollary 4.9. In order to be able to establish a result similar to Theorem 4.13 we need some further assumptions[4] on the given OCP.

Assumption 4.14 (Local controllability with bounded costs)**.** *1. There are $\eta,\ C > 0$, $M \in \mathbb{N}$ such that for all $x \in \mathcal{B}_\eta(x_*)$ there is $\mathbf{u}_x \in \mathbb{U}^M_{\mathcal{P}}(x)$ it holds*

$$x^{\mathbf{u}_x}(M, x) = x_* \text{ with } \max\{\|x^{\mathbf{u}_x}(k, x) - x_*\|, \|u_x(k) - u_*\|\} \leq C\|x - x_*\|.$$

2. There are $\delta > 0$, $\bar{C}_i > 0$ and $p_i \in \mathbb{N}$ such that for all $x \in \mathcal{B}_\delta(x_*)$, all $u \in \mathcal{B}_\delta(u_*)$ and all $i \in \{1, \dots, s\}$ it holds

$$\ell_i(x, u) \leq \bar{C}_i(\|x - x_*\|^{p_i} + \|u - u_*\|^{p_i}).$$

We point out that the second part of Assumption 4.14 implies that all stage costs are zero in the equilibrium, i.e. $\ell_i(x_*, u_*) = 0$ for all $i \in \{1, \dots, s\}$. This requirement is needed to avoid summing up nonzero terms for an infinite time period once we have reached the equilibrium (x_*, u_*).

We remark that the first part of Assumption 4.14 is not overly restrictive in this setting since we implicitely require controllability by setting $\mathbb{X}_0 = \{x_*\}$ and assuming $\mathbb{X}_N \neq \emptyset$.

Theorem 4.15 (Approximate infinite-horizon optimality)**.** *Consider the optimal control problem* (4.3) *with $\mathbb{X}_0 = \{x_*\}$ and $F_i(x_*) = 0$ for all $i \in \{1, \dots, s\}$ and the corresponding optimal control problem on infinite horizon* (4.1) *with the same constraints and running costs. Let the Assumptions 4.2 and 4.14 hold. Consider an arbitrary initial value $x_0 \in \mathbb{X}$ and a sequence $\mathbf{u}^{\star,\infty} \in \mathbb{U}^\infty_{\mathcal{P}}(x_0)$ with $J^\infty_i(x_0, \mathbf{u}^{\star,\infty}) \leq K$ for all $i \in \{1, \dots, s\}$ and some $K \in \mathbb{R}_{\geq 0}$. Assume there is $\bar{N} \in \mathbb{N}$ such that the sets $\mathcal{J}^N_{\mathcal{P}}(x_0)$ are externally stable for all $N \geq \bar{N}$. Then, for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ there is $\mathbf{u}^{\star,N} \in \mathbb{U}^N_{\mathcal{P}}(x_0)$ satisfying*

$$J^N_i\left(x_0, \mathbf{u}^{\star,N}\right) \leq J^\infty_i\left(x_0, \mathbf{u}^{\star,\infty}\right) + \varepsilon \quad \forall i. \tag{4.7}$$

In particular, $\mathbf{u}^{\star,\infty}$ can be approximated arbitrarily well by μ^N from Algorithm 2 (using $\kappa(x_*) = u_*$) in terms of the infinite-horizon performance, that is,

$$J^\infty_i\left(x_0, \mu^N\right) \leq J^\infty_i\left(x_0, \mathbf{u}^{\star,\infty}\right) + \varepsilon \quad \forall i. \tag{4.8}$$

[4] Assumption 4.14 is a lighter, but MO version of Assumptions 2.10 and 2.12.

Proof. Let $\varepsilon > 0$ and choose $\bar{\delta} > 0$ so that $\bar{\delta} \leq \min\{\delta, \eta\}$ and $2M\bar{C}_i C(\bar{\delta})^{p_i} \leq \varepsilon$ for all $i \in \{1, \ldots, s\}$. Lemma 4.12 states that the index $k_{\bar{\delta}} := \min \left\{ k \in \mathbb{N} | x^{\mathbf{u}^{\star,\infty}}(k, x_0) \in \overline{\mathcal{B}_{\bar{\delta}}(x_*)} \right\}$ satisfies $k_{\bar{\delta}} \leq \frac{K}{\min_i \alpha_{\ell,i}(\bar{\delta})}$. Let us choose $N_0 \in \mathbb{N}$ such that $N_0 \geq \max\{\bar{N}, k_{\bar{\delta}} + M + 1\}$. For $N \geq N_0$ we define $\mathbf{u} \in \mathbb{U}^N(x_0)$ via

$$u(k) = \begin{cases} u^{\star,\infty}(k), & k = 0, \ldots, k_{\bar{\delta}} - 1, \\ u_{x_1}(k - k_{\bar{\delta}}), & k = k_{\bar{\delta}}, \ldots, k_{\bar{\delta}} + M - 1, \\ u_*, & k = k_{\bar{\delta}} + M, \ldots, N - 1, \end{cases}$$

in which $\mathbf{u}_{x_1}$ is the control sequence from Assumption 4.14 for $x_1 = x^{\mathbf{u}^{\star,\infty}}(k_{\bar{\delta}}, x_0)$. With this definition we obtain

$$\begin{aligned} J_i^N(x_0, \mathbf{u}) &= \sum_{k=0}^{N-1} \ell_i(x^{\mathbf{u}}(k, x_0), u(k)) \\ &= \sum_{k=0}^{k_{\bar{\delta}}-1} \ell_i(x^{\mathbf{u}^{\star,\infty}}(k, x_0), u^{\star,\infty}(k)) + \sum_{k=k_{\bar{\delta}}}^{k_{\bar{\delta}}+M-1} \ell_i(x^{\mathbf{u}_{x_1}}(k - k_{\bar{\delta}}, x_1), u_{x_1}(k - k_{\bar{\delta}})) \\ &\quad + \sum_{k=k_{\bar{\delta}}+M}^{N-1} \underbrace{\ell_i(x_*, u_*)}_{=0} \\ &\leq J_i^\infty(x_0, \mathbf{u}^{\star,\infty}) + \sum_{k=k_{\bar{\delta}}}^{k_{\bar{\delta}}+M-1} \bar{C}_i 2C \|x_1 - x_*\|^{p_i} \\ &= J_i^\infty(x_0, \mathbf{u}^{\star,\infty}) + M\bar{C}_i 2C \|x_1 - x_*\|^{p_i} \leq J_i^\infty(x_0, \mathbf{u}^{\star,\infty}) + \varepsilon, \end{aligned}$$

wherein the first inequality results from the fact that $x_1 \in \mathcal{B}_{\bar{\delta}}(x_*)$ in combination with Assumption 4.14. The second inequality is obtained by choice of $\bar{\delta}$. By means of external stability of the set $\mathcal{J}_{\mathcal{P}}^N(x_0)$, this chain of inequalites implies the desired estimate (4.7). Inequality (4.8) is then obtained as in the proof of Theorem 4.6. $\square$

4.2.2 A Game Theoretic Interpretation: The Bargaining Game

In this section we interpret Algorithm 2 in terms of a game theoretic concept called *bargaining problem* or *bargaining game* (see e.g. [68]). The idea of such a game is that the players define a *disagreement point* which is realized if negotiations among the players fail. In the negotiations players seek for a solution which is better than the disagreement point for each player (otherwise negotiatations fall down). If the players agree on a solution, this strategy is played. To apply the game-theoretic interpretation to our analysis, we assume that there are s players with their own criterion (i.e. ℓ_i) and that the system $x^+ = f(x, u)$ is the collection of players' systems $x_i^+ = f_i(x, u)$. Then, in our MO MPC Algorithm 2 we can interpret step **(1)** as a bargaining game, in which $\mathbf{u}_{x(k)}^N \in \mathbb{U}^N(x(k))$ is the disagreement point[5] and any solution $\mathbf{u}_{x(k)}^{\star,N} \in \mathbb{U}_{\mathcal{P}}^N(x(k))$ in step **(1)**, Algorithm 2 is a bargaining solution

[5]Though this terminology might be confusing since this point still yields the desired performance estimate and trajectory convergence.

that improves all players' objectives. Note that by definition a bargaining solution does not need to be Pareto-optimal.

In the *Nash-bargaining game* (see [67]) Nash requires more structure on solutions of the game, that is invariance under affine transformations, Pareto optimality, independence of irrelevant alternatives, and symmetry. The game theoretic interpretation of these assumptions can be found in [67]. Moreover, Nash assumes that the players are rational, have equal bargaining skills and that the game is subject to full information. The motivation for these structural requirements is the wish to define a unique solution to the bargaining game. Nash proved that if $\mathcal{J}^N(x(k))$ is convex and compact, the unique solution (the Nash-bargaining solution) satisfying the structural assumptions is given by

$$\operatorname*{argmax}_{\mathbf{u}\in\mathbb{U}^N(x(k))} \prod_{i=1}^{s} \Big(J_i^N(x(k), \mathbf{u}^N_{x(k)}) - J_i^N(x(k), \mathbf{u}) \Big) .$$

Let us note that the concept of bargaining games is a possible interpretation here, though there are valid objections against this approach for the following reason: Usually the disagreement point is thought of as a combination of strategies that all players fix individually and that they could implement independent from the other's strategies, whereas the cooperation then enables the players to commonly improve their objectives. In our approach, the control strategy $\mathbf{u}^N_{x(K)}$ is already a common strategy that yields desirable results.

The idea to interpret MO MPC as a bargaining game was also pursued in [41] for a very special class of systems. The basic idea in this reference is to implement a weighted-sum approach, in which the weights are obtained by playing a Nash-bargaining game.

4.3 Multiobjective MPC without Terminal Conditions

In this section we aim to develop performance estimates for MO MPC schemes without terminal conditions, i.e. Assumption 4.4 does no longer hold. A discussion why proceeding this way may be superior to MPC schemes with terminal conditions can be found in e.g. [32, Sec. 6.1]

Instead of imposing such terminal conditions, we follow the procedure developed in [33] for scalar-valued MPC and require the following structural property on POSs.

Assumption 4.16 (Bounds on POSs)**.** *Let an optimization horizon $N \in \mathbb{N}$ be given. For all $i \in \{1, \dots, s\}$ there exist $\gamma_i \in \mathbb{R}_{>1}$ such that the inequalities*

$$\forall x \in \mathbb{X}, \forall \mathbf{u}_x^{\star,1} \in \mathbb{U}^1_{\mathcal{P}}(x)\ \exists \mathbf{u}_x^{\star,2} \in \mathbb{U}^2_{\mathcal{P}}(x) : J_i^2(x, \mathbf{u}_x^{\star,2}) \le \gamma_i \cdot J_i^1(x, \mathbf{u}_x^{\star,1}),$$
$$\forall x \in \mathbb{X}, \forall \mathbf{u}_x^{\star,k} \in \mathbb{U}^k_{\mathcal{P}}(x) : J_i^k(x, \mathbf{u}_x^{\star,k}) \le \gamma_i \cdot \ell_i(x, u_x^{\star,k}(0))\ \forall k = 2, \dots, N$$

holds for all objectives $i \in \{1, \dots, s\}$.

Furthermore, we assume $\mathbb{U}^N(x) \neq \emptyset$ for all $x \in \mathbb{X}$ and all $N \in \mathbb{N}$ and we still impose Assumption 4.2. Assumption 4.16 requires that all POSs are in a sense structured. The second set of inequalities therein states that the values of all POSs can be expressed in terms of the stage cost of the first piece of the POS for all horizon lengths. The first set of inequalities is mainly needed as a base case for the induction in Lemma 4.18 in order to prove a relation between POS of horizon length k and $k-1$. An alternative assumption to Assumption 4.16 is to directly require the statement of Lemma 4.18. However, this seems to be even more difficult to verify.[6]

The MPC scheme we propose in this section is the following.

Algorithm 3 (Multiobjective MPC without terminal conditions).

(0) At time $n = 0$: Set $x(n) := x_0$ and choose a POS $\mathbf{u}^{\star,N}_{x(n)} \in \mathbb{U}^N_{\mathcal{P}}(x(n))$ to (4.2). Go to **(2)**.

(1) At time $n \in \mathbb{N}$: Choose a POS $\mathbf{u}^{\star,N}_{x(n)}$ to (4.2) so that the inequalities

$$J_i^N\left(x(n), \mathbf{u}^{\star,N}_{x(n)}\right) \leq \frac{\gamma_i^{N-2} + (\gamma_i - 1)^{N-1}}{\gamma_i^{N-2}} J_i^{N-1}\left(x(n), \mathbf{u}^{N-1}_{x(n)}\right)$$

are satisfied for all $i \in \{1, \ldots, s\}$.

(2) Set

$$\mathbf{u}^{N-1}_{x(n+1)} := \mathbf{u}^{\star,N}_{x(n)}(\cdot + 1).$$

(3) Apply the feedback $\mu^N(x(n)) := u^{\star,N}_{x(n)}(0)$, set $n = n+1$ and go to **(1)**.

After giving two auxiliary results, we will prove that the MPC feedback defined in Algorithm 3 has a bounded infinite-horizon performance for each objective.

Lemma 4.17. *Given $x \in \mathbb{X}$ and $\mathbf{u}^{\star,k}_x \in \mathbb{U}^k_{\mathcal{P}}(x)$ for arbitrary $k \in \{2, \ldots, N\}$. Under Assumptions 4.2 and 4.16 the inequalities*

$$J_i^{k-1}\left(f(x, u^{\star,k}_x(0)), \mathbf{u}^{\star,k}_x(\cdot + 1)\right) \leq (\gamma_i - 1)\ell_i\left(x, u^{\star,k}_x(0)\right)$$

hold for all $i \in \{1, \ldots, s\}$ and all $k \in \{2, \ldots, N\}$.

Proof. Consider an arbitrary $x \in \mathbb{X}$, $k \in \{2, \ldots, N\}$ and a POS $\mathbf{u}^{\star,k}_x \in \mathbb{U}^k_{\mathcal{P}}(x)$. Then, for all $i \in \{1, \ldots, s\}$ it holds

$$\begin{aligned} J_i^{k-1}\left(f(x, u^{\star,k}_x(0)), \mathbf{u}^{\star,k}_x(\cdot + 1)\right) &= J_i^k\left(x, \mathbf{u}^{\star,k}_x\right) - \ell_i\left(x, u^{\star,k}_x(0)\right) \\ &\leq \gamma_i \cdot \ell_i\left(x, u^{\star,k}_x(0)\right) - \ell_i\left(x, u^{\star,k}_x(0)\right), \end{aligned}$$

which shows the assertion. □

[6] A comment on the verification of Assumption 4.16 is stated before Algorithm 4.

Lemma 4.18. *Given $x \in \mathbb{X}$ and $N \in \mathbb{N}_{\geq 2}$. Let Assumptions 4.2 and 4.16 hold, assume external stability (according to Def. 3.4) of the sets $\mathcal{J}_{\mathcal{P}}^k(x)$ for all $k \in \{2, \dots, N\}$. Then, for each $k \in \{2, \dots, N\}$ and each $\mathbf{u}_x^{\star,k-1} \in \mathbb{U}_{\mathcal{P}}^{k-1}(x)$ there is $\mathbf{u}_x^{\star,k} \in \mathbb{U}_{\mathcal{P}}^k(x)$ such that*

$$\eta_{k,i} \cdot J_i^k\left(x, \mathbf{u}_x^{\star,k}\right) \leq J_i^{k-1}\left(x, \mathbf{u}_x^{\star,k-1}\right)$$

holds for all $i \in \{1, \dots, s\}$, in which $\eta_{k,i}$ is defined as

$$\eta_{k,i} = \frac{\gamma_i^{k-2}}{\gamma_i^{k-2} + (\gamma_i - 1)^{k-1}}.$$

Proof. By induction:
$k = 2$: The statement follows immediately from Assumption 4.16.
$k \to k+1$: Let $\mathbf{u}_x^{\star,k} \in \mathbb{U}_{\mathcal{P}}^k(x)$. It holds that

$$\begin{aligned}
J_i^k\left(x, \mathbf{u}_x^{\star,k}\right) &= J_i^{k-1}\left(f(x, u_x^{\star,k}(0)), \mathbf{u}_x^{\star,k}(\cdot+1)\right) + \ell_i(x, u_x^{\star,k}(0)) \\
&= J_i^{k-1}\left(f(x, u_x^{\star,k}(0)), \mathbf{u}_x^{\star,k}(\cdot+1)\right) + (\gamma_i - 1)\frac{1-\eta_{k,i}}{(\gamma_i - 1) + \eta_{k,i}}\ell_i(x, u_x^{\star,k}(0)) \\
&\quad + \left(1 - (\gamma_i - 1)\frac{1-\eta_{k,i}}{(\gamma_i - 1) + \eta_{k,i}}\right)\ell_i(x, u_x^{\star,k}(0)) \\
&\geq J_i^{k-1}\left(f(x, u_x^{\star,k}(0)), \mathbf{u}_x^{\star,k}(\cdot+1)\right) \\
&\quad + \frac{1-\eta_{k,i}}{(\gamma_i - 1) + \eta_{k,i}} \cdot J_i^{k-1}\left(f(x, u_x^{\star,k}(0)), \mathbf{u}_x^{\star,k}(\cdot+1)\right) \\
&\quad + \left(1 - (\gamma_i - 1)\frac{1-\eta_{k,i}}{(\gamma_i - 1) + \eta_{k,i}}\right)\ell_i(x, u_x^{\star,k}(0)) \\
&= \left(1 + \frac{1-\eta_{k,i}}{\gamma_i - 1 + \eta_{k,i}}\right) J_i^{k-1}\left(f(x, u_x^{\star,k}(0)), \mathbf{u}_x^{\star,k}(\cdot+1)\right) \\
&\quad + \left(1 - (\gamma_i - 1)\frac{1-\eta_{k,i}}{(\gamma_i - 1) + \eta_{k,i}}\right)\ell_i(x, u_x^{\star,k}(0)) \\
&\geq \eta_{k,i}\left(1 + \frac{1-\eta_{k,i}}{\gamma_i - 1 + \eta_{k,i}}\right) J_i^k\left(f(x, u_x^{\star,k}(0)), \mathbf{u}_{f(x,u_x^{\star,k}(0)}^{\star,k}\right) \\
&\quad + \left(1 - (\gamma_i - 1)\frac{1-\eta_{k,i}}{(\gamma_i - 1) + \eta_{k,i}}\right)\ell_i(x, u_x^{\star,k}(0)) \\
&= \frac{\eta_{k,i}\gamma_i}{\gamma_i - 1 + \eta_{k,i}}\left[J_i^k\left(f(x, u_x^{\star,k}(0)), \mathbf{u}_{f(x,u_x^{\star,k}(0)}^{\star,k}\right) + \ell_i(x, u_x^{\star,k}(0))\right] \\
&= \frac{\eta_{k,i}\gamma_i}{\gamma_i - 1 + \eta_{k,i}} J_i^{k+1}(x, \mathbf{u}_x^{k+1}), \quad \mathbf{u}_x^{k+1} := \left(u_x^{\star,k}(0), \mathbf{u}_{f(x,u_x^{\star,k}(0))}^{\star,k}\right) \\
&\geq \frac{\eta_{k,i}\gamma_i}{\gamma_i - 1 + \eta_{k,i}} J_i^{k+1}(x, \mathbf{u}_x^{\star,k+1}).
\end{aligned}$$

The first inequality holds due to Lemma 4.17 and in the second inequality we used the induction assumption. The last inequality holds due to external stability of the set $\mathcal{J}_{\mathcal{P}}^{k+1}(x)$.

Moreover, for all $i \in \{1, \dots, s\}$ we have

$$\frac{\eta_{k,i}\gamma_i}{\gamma_i - 1 + \eta_{k,i}} = \frac{\gamma_i^{k-1}/(\gamma_i^{k-2} + (\gamma_i - 1)^{k-1})}{\gamma_i - 1 + \gamma_i^{k-2}/(\gamma_i^{k-2} + (\gamma_i - 1)^{k-1})} = \frac{\gamma_i^{k-1}}{\gamma_i^{k-1} + (\gamma_i - 1)^k} = \eta_{k+1,i}.$$

□

Theorem 4.19 (Performance Theorem)**.** *Consider a MO OCP with system dynamics* (1.1)*, cost criteria ℓ_i, $i \in \{1, \dots, s\}$, and let $N \in \mathbb{N}_{\geq 2}$, and $x_0 \in \mathbb{X}$ be given. Let Assumptions 4.2 and 4.16 hold and let the sets $\mathcal{J}_{\mathcal{P}}^k(x_0)$ be externally stable for all $k \in \{2, \dots, N\}$. Let moreover $(\gamma_i - 1)^N < \gamma_i^{N-2}$ hold for all $i \in \{1, \dots, s\}$. Then, the MPC feedback $\mu^N : \mathbb{X} \to \mathbb{U}$ defined in Algorithm 3 renders the set $\mathbb{X}$ forward invariant (in the sense of Definition 1.1) and has the following infinite-horizon closed-loop performance*

$$J_i^\infty\left(x_0, \mu^N\right) \leq \frac{\gamma_i^{N-2}}{\gamma_i^{N-2} - (\gamma_i - 1)^N} \cdot J_i^N\left(x_0, \mathbf{u}_{x_0}^{\star,N}\right)$$

for all objectives $i \in \{1, \dots, s\}$. $\mathbf{u}_{x_0}^{\star,N}$ denotes the POS from step **(0)** *in Algorithm 3.*

Proof. Existence of the POSs in Algorithm 3 is obtained by Lemma 4.18 and we can thus conclude recursive feasibility of the closed-loop system. We will now prove that the MPC feedback exhibits the stated performance. For $K \in \mathbb{N}_{\geq 1}$ and all $i \in \{1, \dots, s\}$ it holds

$$\begin{aligned}
&\underbrace{\left(1 - \frac{(\gamma_i - 1)^N}{\gamma_i^{N-2}}\right)}_{>0} J_i^K(x_0, \mu^N) = \left(1 - \frac{(\gamma_i - 1)^N}{\gamma_i^{N-2}}\right) \sum_{k=0}^{K-1} \ell_i(x(k), \mu^N(x(k))) \\
&= \left(1 - \frac{(\gamma_i - 1)^N}{\gamma_i^{N-2}}\right) \sum_{k=0}^{K-1} \ell_i\left(x(k), u_{x(k)}^{\star,N}(0)\right) \\
&= \sum_{k=0}^{K-1} \left[\ell_i\left(x(k), u_{x(k)}^{\star,N}(0)\right) - \frac{(\gamma_i - 1)^N}{\gamma_i^{N-2}} \ell_i\left(x(k), u_{x(k)}^{\star,N}(0)\right)\right] \\
&\leq \sum_{k=0}^{K-1} \left[J_i^N\left(x(k), \mathbf{u}_{x(k)}^{\star,N}\right) - J_i^{N-1}\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k)}^{\star,N}(\cdot + 1)\right)\right. \\
&\qquad \left. - \frac{(\gamma_i - 1)^{N-1}}{\gamma_i^{N-2}} J_i^{N-1}\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k)}^{\star,N}(\cdot + 1)\right)\right] \\
&= \sum_{k=0}^{K-1} \left[J_i^N\left(x(k), \mathbf{u}_{x(k)}^{\star,N}\right) - J_i^{N-1}\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k)}^{\star,N}(\cdot + 1)\right) \underbrace{\left(1 + \frac{(\gamma_i - 1)^{N-1}}{\gamma_i^{N-2}}\right)}_{=\frac{\gamma_i^{N-2} + (\gamma_i - 1)^{N-1}}{\gamma_i^{N-2}}}\right],
\end{aligned}$$

in which the inequality is obtained by Lemma 4.17. In step **(1)** the POS $\mathbf{u}_{x(k)}^{\star,N}$ is chosen such that we obtain the estimates

$$\left(1 - \frac{(\gamma_i - 1)^N}{\gamma_i^{N-2}}\right) J_i^K(x_0, \mu^N) \leq J_i^N(x_0, \mathbf{u}_{x_0}^{\star,N}) - J_i^N(x(K), \mathbf{u}_{x(K)}^{\star,N}) \leq J_i^N(x_0, \mathbf{u}_{x_0}^{\star,N})$$

for all $i \in \{1, \ldots, s\}$. This concludes the assertion. $\square$

Corollary 4.20 (Infinite-horizon near optimality). *Let the assumptions of Theorem 4.19 hold for $N \in \mathbb{N}_{\geq 2}$ and $x_0 \in \mathbb{X}$ and assume that there is a POS $\mathbf{u}^{\star,\infty} \in \mathbb{U}^{\infty}_{\mathcal{P}}(x_0)$ to (4.1). Then, the estimates*

$$J_i^{\infty}(x_0, \mu^N) \leq \frac{\gamma_i^{N-2}}{\gamma_i^{N-2} - (\gamma_i - 1)^N} \cdot J_i^{\infty}(x_0, \mathbf{u}^{\star,\infty}) \quad \forall i \in \{1, \ldots, s\}$$

are obtained by applying Algorithm 3 with a proper initialization in step **(0)**.

Proof. Due to the positivity of the stage costs ℓ_i we have $J_i^{\infty}(x_0, \mathbf{u}^{\star,\infty}) \geq J_i^N(x_0, \mathbf{u}^{\star,\infty})$ for all $i \in \{1, \ldots, s\}$ and external stability of the set $\mathcal{J}^N_{\mathcal{P}}(x_0)$ guarantees the existence of $\mathbf{u}^{\star,N}_{x_0} \in \mathbb{U}^N_{\mathcal{P}}(x_0)$ such that $J_i^N(x_0, \mathbf{u}^{\star,\infty}) \geq J_i^N(x_0, \mathbf{u}^{\star,N}_{x_0})$ holds for all $i \in \{1, \ldots, s\}$. By applying $\mathbf{u}^{\star,N}_{x_0}$ in step **(0)** of Algorithm 3 we conclude $J_i^{\infty}(x_0, \mu^N) \leq \frac{\gamma_i^{N-2}}{\gamma_i^{N-2} - (\gamma_i - 1)^N} \cdot J_i^{\infty}(x_0, \mathbf{u}^{\star,\infty})$ for all objectives $i \in \{1, \ldots, s\}$. $\square$

Remark 4.21. *In all statements so far we have required Assumption 4.2 to hold. In fact, it is sufficient if $\ell_i(x, u) \geq 0$ holds for all $i \in \{1, \ldots, s\}$ to obtain the presented results. But since positive semidefinite stage costs are not sufficient for the following Corollary 4.22, we decided to impose Assumption 4.2 throughout the course of this section.*

Corollary 4.22 (Trajectory convergence). *Let the assumptions of Theorem 4.19 hold for $x_0 \in \mathbb{X}$ and $N \in \mathbb{N}$. Then, any closed-loop trajectory $x^{\mu^N}(\cdot, x_0)$ resulting from Algorithm 3 converges to x_*.*

Proof. As the proof of Corollary 4.9. $\square$

A drawback of Algorithm 3 is that finding a POS in step **(1)** is subject to constraints, which depend on the γ_i from Assumption 4.16. Checking the respective assumption is already a difficult task in the single-objective setting and is often done numerically. It is even more involved in our MO setting and can lead to large values for γ_i if the Pareto fronts have a large diameter. A possible remedy for this problem is to specify values for the γ_i, such that Assumption 4.16 holds for some POSs of each horizon length. Needless to say, this restricts our choice in each iteration of Algorithm 3. Another possibility is to find N and γ_i such that Assumption 4.16 and the inequalities in Lemma 4.18 only hold for N instead of for all $k \in \{2, \ldots, N\}$.

This is our motivation to replace the constraint in step **(1)**, Algorithm 3 by a constraint that does not explicitly depend on the knowledge of γ_i but yields the same performance result as Theorem 4.19. Thus, we are able to perform MO MPC without terminal constraints under existence theorems for the γ_i's. For this purpose we propose Algorithm 4.

Algorithm 4 (Multiobjective MPC without terminal conditions – version 2).

(0) At time $n = 0$: Set $x(n) := x_0$ and choose a POS $\mathbf{u}^{\star,N}_{x(n)} \in \mathbb{U}^N_{\mathcal{P}}(x(n))$ to (4.2). Go to **(2)**.

(1) At time $n \in \mathbb{N}$: Choose a POS $\mathbf{u}^{\star,N}_{x(n)}$ to (4.2) such that the inequalities

$$J_i^N\left(x(n), \mathbf{u}^{\star,N}_{x(n)}\right) \leq J_i^N\left(x(n), \tilde{\mathbf{u}}_{x(n)}\right)$$

are satisfied for all $i \in \{1, \dots, s\}$.

(2) For $x := x^{\mathbf{u}^{\star,N}_{x(n)}}(N-1, x(n))$ choose $\mathbf{u}^\star \in \mathbb{U}^2_{\mathcal{P}}(x)$ such that $\forall\, i \in \{1, \dots, s\}$ it holds

$$\ell_i\left(x, u^\star(0)\right) \leq \ell_i\left(x, u^{\star,N}_{x(n)}(N-1)\right). \tag{4.9}$$

Define $\tilde{\mathbf{u}}_{x(n+1)} \in \mathbb{U}^N\left(x^{\mathbf{u}^{\star,N}_{x(n)}}(1, x(n))\right)$ via

$$\tilde{u}_{x(n+1)}(k) := \begin{cases} u^{\star,N}_{x(n)}(k+1), & k = 0, \dots, N-3 \\ u^\star(k-(N-2)), & k = N-2, N-1 \end{cases}.$$

(3) Apply $\mu^N(x(n)) := u^{\star,N}_{x(n)}(0)$, set $n = n+1$ and go to **(1)**.

Lemma 4.23. *Let Assumptions 4.2 and 4.16 hold and let an initial value $x \in \mathbb{X}$ and a POS $\mathbf{u}^\star \in \mathbb{U}^N_{\mathcal{P}}(x)$ to the MO OCP (4.2) be given. Then, for all $i \in \{1, \dots, s\}$ it holds that*

$$\ell_i(x^{\mathbf{u}^\star}(N-1, x), u^\star(N-1)) \leq \left(\frac{\gamma_i - 1}{\gamma_i}\right)^{N-2} J_i^{N-1}\left(x^{\mathbf{u}^\star}(1, x), \mathbf{u}^\star(\cdot+1)\right).$$

Proof. Similar to the proof of [32, Proposition 6.19]: For each $p \in \{0, \dots, N-2\}$ and for all $i \in \{1, \dots, s\}$ it holds that

$$\sum_{k=p+1}^{N-1} \ell_i(x^{\mathbf{u}^\star}(k, x), u^\star(k)) = J_i^{N-p}(x^{\mathbf{u}^\star}(p, x), \mathbf{u}^\star(\cdot+p)) - \ell_i(x^{\mathbf{u}^\star}(p, x), u^\star(p)).$$

Since $\mathbf{u}^\star(\cdot+p)$ is a POS of length $N-p$ for initial value $x^{\mathbf{u}^\star}(p, x)$ (see Lemma 4.1), Assumption 4.16 provides the estimate

$$\begin{aligned} \sum_{k=p+1}^{N-1} \ell_i(x^{\mathbf{u}^\star}(k, x), u^\star(k)) &\leq \gamma_i \ell_i(x^{\mathbf{u}^\star}(p, x), u^\star(p)) - \ell_i(x^{\mathbf{u}^\star}(p, x), u^\star(p)) \\ &= (\gamma_i - 1)\ell_i(x^{\mathbf{u}^\star}(p, x), u^\star(p)) \\ \Rightarrow \sum_{k=p}^{N-1} \ell_i(x^{\mathbf{u}^\star}(k, x), u^\star(k)) &= \ell_i(x^{\mathbf{u}^\star}(p, x), u^\star(p)) + \sum_{k=p+1}^{N-1} \ell_i(x^{\mathbf{u}^\star}(k, x), u^\star(k)) \\ &\geq \underbrace{\left(\frac{1}{\gamma_i - 1} + 1\right)}_{=\frac{\gamma_i}{\gamma_i - 1}} \sum_{k=p+1}^{N-1} \ell_i(x^{\mathbf{u}^\star}(k, x), u^\star(k)), \quad p \in \{1, \dots, N-2\}. \end{aligned}$$

Applying this inequality inductively we obtain

$$\sum_{k=1}^{N-1} \ell_i(x^{\mathbf{u}^\star}(k,x),u^\star(k)) \geq \left(\frac{\gamma_i}{\gamma_i-1}\right)^{N-2} \ell_i(x^{\mathbf{u}^\star}(N-1,x),u^\star(N-1))$$

for all $i \in \{1,\dots,s\}$, which is the claimed estimate. □

Theorem 4.24 (Performance Theorem for Algorithm 4)**.** *Consider a MO OCP* (4.2) *with system dynamics* (1.1)*, cost criteria ℓ_i, $i \in \{1,\dots,s\}$, and let $N \in \mathbb{N}_{\geq 2}$. Let Assumptions 4.2 and 4.16 hold and let the sets $\mathcal{J}_{\mathcal{P}}^N(x)$ and $\mathcal{J}_{\mathcal{P}}^2(x)$ be externally stable for each $x \in \mathbb{X}$. Assume viability of the set $\mathbb{X}$ (see page 4). Let moreover $(\gamma_i-1)^N < \gamma_i^{N-2}$ hold for all $i \in \{1,\dots,s\}$. Then, the MPC feedback $\mu^N : \mathbb{X} \to \mathbb{U}$ defined in Algorithm 4 yields recursive feasibility of $\mathbb{X}$ and has the following infinite-horizon closed-loop performance*

$$J_i^\infty\left(x_0,\mu^N\right) \leq \frac{\gamma_i^{N-2}}{\gamma_i^{N-2}-(\gamma_i-1)^N} \cdot J_i^N\left(x_0,\mathbf{u}_{x_0}^{\star,N}\right)$$

for all objectives $i \in \{1,\dots,s\}$. $\mathbf{u}_{x_0}^{\star,N}$ denotes the POS from step **(0)** *in Algorithm 4.*

In particular, any $\mathbf{u}^{\star,\infty} \in \mathbb{U}_{\mathcal{P}}^\infty(x_0)$ that solves (4.1) *can be approximated arbitrarily well by μ^N from Algorithm 4 in terms of the infinite-horizon performance, that is,*

$$J_i^\infty\left(x_0,\mu^N\right) \leq \frac{\gamma_i^{N-2}}{\gamma_i^{N-2}-(\gamma_i-1)^N} \cdot J_i^\infty\left(x_0,\mathbf{u}^{\star,\infty}\right).$$

Proof. **Feasibility:** Step **(1)** in Algorithm 4 is feasible, because we assume external stability of the sets $\mathcal{J}_{\mathcal{P}}^N(x)$ for all $x \in \mathbb{X}$. Now let us turn to step **(2)**: The tail $u_{x(n)}^{\star,N}(N-1)$ can be prolonged by some $\hat{u} \in \mathbb{U}$ such that $\bar{\mathbf{u}} := \left(u_{x(n)}^{\star,N}(N-1),\hat{u}\right) \in \mathbb{U}^2(x)$, in which $x := x^{\mathbf{u}_{x(n)}^{\star,N}}(N-1,x(n))$, otherwise $\mathbb{U}^1\left(f\left(x,u_{x(n)}^{\star,N}(N-1)\right)\right) = \emptyset$, contradicting our viability assumption. Clearly, the control sequence $\bar{\mathbf{u}}$ satisfies the constraint (4.9). Thus, existence of a POS satisfying the constraint follows from external stability of $\mathcal{J}_{\mathcal{P}}^2(x)$.

Performance: For $n \in \mathbb{N}$ and $\tilde{\mathbf{u}}_{x(n+1)}$, $\mathbf{u}_{x(n)}^{\star,N}$, $\mathbf{u}^\star$ as defined in Algorithm 4 it holds that

$$J_i^N\left(x(n+1),\tilde{\mathbf{u}}_{x(n+1)}\right) = J_i^{N-2}\left(x(n+1),\mathbf{u}_{x(n)}^{\star,N}(\cdot+1)\right) + J_i^2\left(x^{\mathbf{u}_{x(n)}^{\star,N}}(N-1,x(n)),\mathbf{u}^\star\right).$$

Since $\mathbf{u}^\star \in \mathbb{U}_{\mathcal{P}}^2\left(x^{\mathbf{u}_{x(n)}^{\star,N}}(N-1,x(n))\right)$, Assumption 4.16 yields

$$J_i^2\left(x^{\mathbf{u}_{x(n)}^{\star,N}}(N-1,x(n)),\mathbf{u}^\star\right) \leq \gamma_i\ell_i\left(x^{\mathbf{u}_{x(n)}^{\star,N}}(N-1,x(n)),u^\star(0)\right).$$

Thus, we get

$$\begin{aligned} J_i^N\left(x(n+1), \tilde{\mathbf{u}}_{x(n+1)}\right) &\leq J_i^{N-1}\left(x(n+1), \mathbf{u}^{\star,N}_{x(n)}(\cdot+1)\right) - \ell_i\left(x^{\mathbf{u}^{\star,N}_{x(n)}}(N-1, x(n)), u^\star(0)\right) \\ &\quad + \gamma_i \ell_i\left(x^{\mathbf{u}^{\star,N}_{x(n)}}(N-1, x(n)), u^\star(0)\right) \\ &\leq J_i^{N-1}\left(x(n+1), \mathbf{u}^{\star,N}_{x(n)}(\cdot+1)\right) \\ &\quad + (\gamma_i - 1)\ell_i\left(x^{\mathbf{u}^{\star,N}_{x(n)}}\left(N-1, x(n)\right), u^{\star,N}_{x(n)}(N-1)\right), \end{aligned}$$

in which the last inequality follows from the construction in step **(2)** in Algorithm 4. If we now apply Lemma 4.23, we obtain

$$\begin{aligned} J_i^N\left(x(n+1), \tilde{\mathbf{u}}_{x(n+1)}\right) &\leq J_i^{N-1}\left(x(n+1), \mathbf{u}^{\star,N}_{x(n)}(\cdot+1)\right)\left(1 + (\gamma_i - 1)\left(\frac{\gamma_i - 1}{\gamma_i}\right)^{N-2}\right) \\ &= \frac{\gamma_i^{N-2} - (\gamma_i - 1)^{N-1}}{\gamma_i^{N-2}} J_i^{N-1}\left(x(n+1), \mathbf{u}^{\star,N}_{x(n)}(\cdot+1)\right). \end{aligned}$$

Hence, the POS in step **(1)** of Algorithm 4 satisfies the constraint in step **(1)** of Algorithm 3. This leads to the fact that the MPC feedback defined in Algorithm 4 has the same performance as the feedback defined in Algorithm 3. The second estimate follows from Corollary 4.20. □

Remark 4.25. *The values* $\frac{\gamma_i^{N-2} - (\gamma_i - 1)^{N-1}}{\gamma_i^{N-2}}$, $i \in \{1, \ldots, s\}$, *can be estimated online while executing Algorithm 4.*

4.4 Example

By means of the following example, presented in [64], we will illustrate the results of the previous sections. We consider six two-dimensional systems $x_i \in \mathbb{R}^2$, $i \in \{1, \ldots, 6\}$ that are dynamically decoupled but coupled through constraints and cost criteria. Each system is steered by a two-dimensional input $u_i \in \mathbb{R}^2$. The system dynamics and stage cost of system $i \in \{1, \ldots, 6\}$ is given by

$$\begin{aligned} x_i^+ &= \begin{pmatrix} 0.9 & 0.1 \\ -0.2 & 0.8 \end{pmatrix} x_i + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u_i + 0.1 \begin{pmatrix} x_{i,2}^2 \\ x_{i,1}^2 \end{pmatrix}, \\ \ell_i(x, u) &= x_i^T Q_i x_i + u_i^T R_i u_i + \sum_{j \in \mathcal{N}_i} (C_i x_i - C_j x_j)^T Q_{ij} (C_i x_i - C_j x_j), \end{aligned}$$

in which $\mathcal{N}_i = \{i-1, i+1\}$ for $i = 2, \ldots, 5$ and $\mathcal{N}_1 = \{2\}$, $\mathcal{N}_6 = \{5\}$ and

$$Q_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ R_i = 5Q_i, \ C_i = Q_i, \ \text{for all } i,$$

$$Q_{34} = Q_{43} = 0_{2\times 2}, \ Q_{ij} = 3Q_i \text{ otherwise.}$$

The states and controls are constrained by $\|x_i\|_\infty \leq 5$ and $\|u_i\|_\infty \leq 2$. Moreover, systems three and four are coupled by the constraint $\|x_3 - x_4\| \leq 4$. Let us first turn to the simulations that were obtained using terminal conditions. In Figure 4.2 we observe that

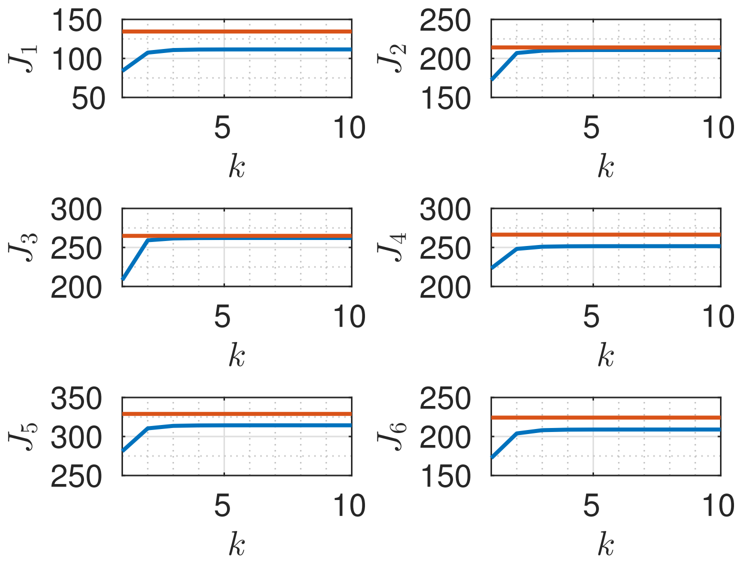

Figure 4.2: Accumulated performance of the six objectives (blue) compared to the value of the POS $\mathbf{u}_{x_0}^{\star;N}$ from step **(0)**, Algorithm 2 (red).

the accumulated performance of the MPC feedback defined in Algorithm 2 for $N = 2$ is indeed bounded from above by $J_i^N(x_0, \mathbf{u}_{x_0}^{\star;N})$ as stated in Theorem 4.6. In Corollary 4.9 convergence of the closed-loop trajectories was proven. This behavior is illustrated in Figure 4.3.

We have run Algorithm 1 for our example and have chosen an arbitrary POS in each iteration, i.e. we removed the constraints in step **(1)**. Figure 4.4 illustrates that the desired performance bound is violated[7], yet the trajectories converge to the origin. The reason for this still reasonable behavior is that all cost criteria ℓ_i are positive definite wrt the same equilibrium. This means that all objectives share the same point they want to achieve but the way to arrive there might vary among them. Thus, they converge to the point that is strongly Pareto-optimal to the problem (4.3) and the accumulated performance converges for all objectives. Such a nice behavior cannot be observed in Chapter 5, cf. Figure 5.12.

From our theoretical consideration we know that imposing the recursive contraint in step **(1)**, Algorithm 1 always yields a bounded performance, no matter what solution we choose in the iteration. As can be seen in Figure 4.5 compared to Figure 4.3 the trajectories are influenced by the choice of the POSs in the iterations.

[7] In order to obtain a violation of the bound we chose $N = 6$, because for small horizons the terminal constraint becomes so restrictive that it dominates the effect of the constraint in step **(1)**, Algorithm 1.

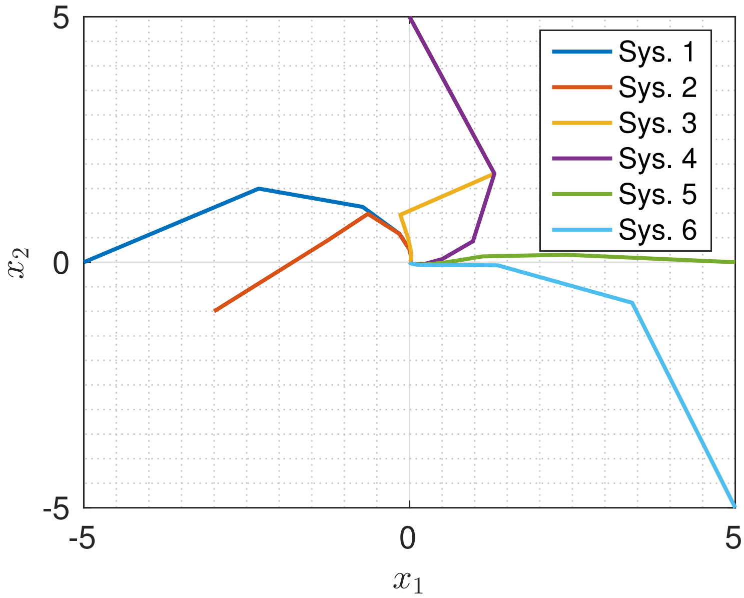

Figure 4.3: Trajectories of the six systems (phase plots).

We now present numerical simulations for our example without terminal conditions. Therefore, we have checked Assumption 4.16 numerically and used the values $(\gamma_i)_{i\in\{1,\dots,s\}} = (2.1, 1.6, 1.6, 1.5, 1.5, 1.6)$ and $N = 4$. In Figure 4.6 we have depicted the trajectories (left) and performance (right) of the MPC feedback defined in Algorithm 3. The blue lines represent the accumulated cost, the red lines the theoretical upper bound derived in Theorem 4.19, i.e.

$$\frac{\gamma_i^{N-2}}{\gamma_i^{N-2} - (\gamma_i - 1)^N} \cdot J_i^N\left(x_0, \mathbf{u}_{x_0}^{\star,N}\right).$$

Let us now assume we did not know the γ_i and therefore apply Algorithm 4 with $N = 4$ to the example. Our theoretical considerations in Theorem 4.24 guarantee that the MPC performance is bounded from above by the same bound as before. In Figure 4.7 we compare the accumulated MPC cost (blue) to the theoretical upper bound (red) using the values $(\gamma_i)_{i\in\{1,\dots,s\}} = (2.1, 1.6, 1.6, 1.5, 1.5, 1.6)$ (as before).

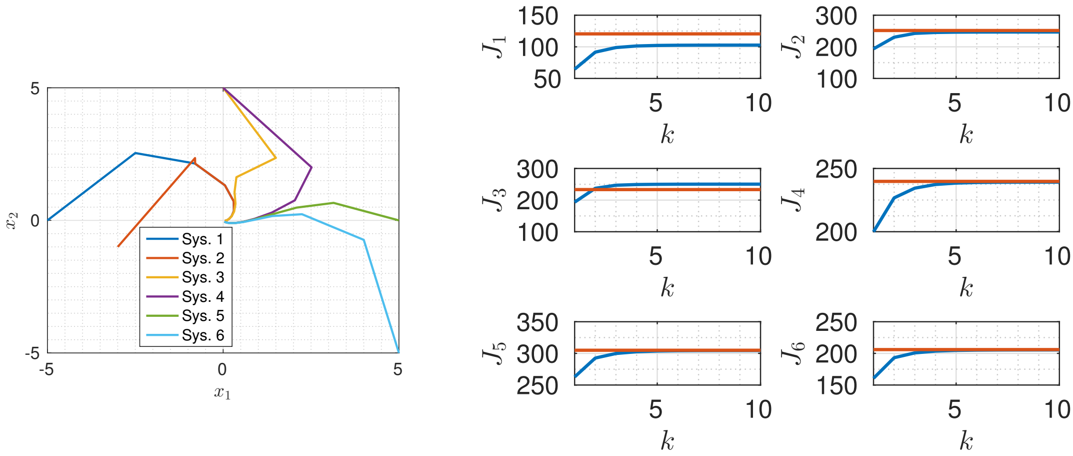

Figure 4.4: Trajectories and performance without the constraints in step **(1)**, Algorithm 2.

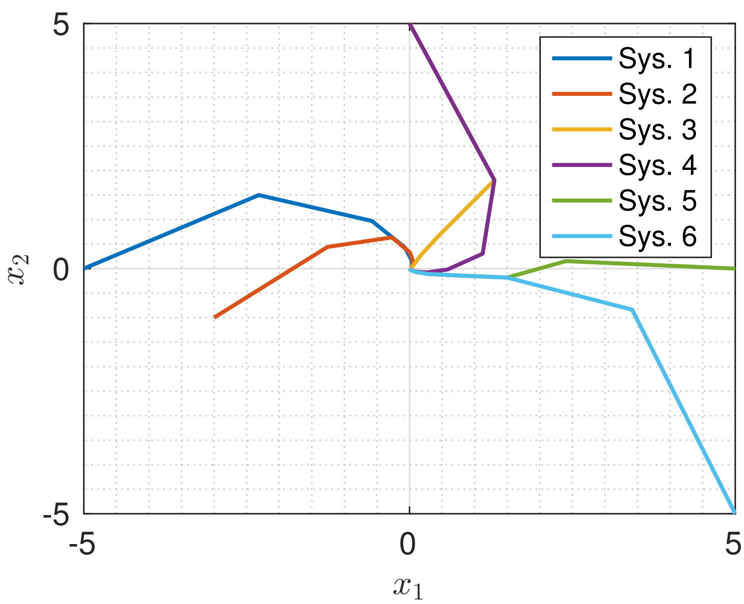

Figure 4.5: Trajectories of the six systems resulting from Algorithm 2 choosing POSs different from those in Figure 4.3.

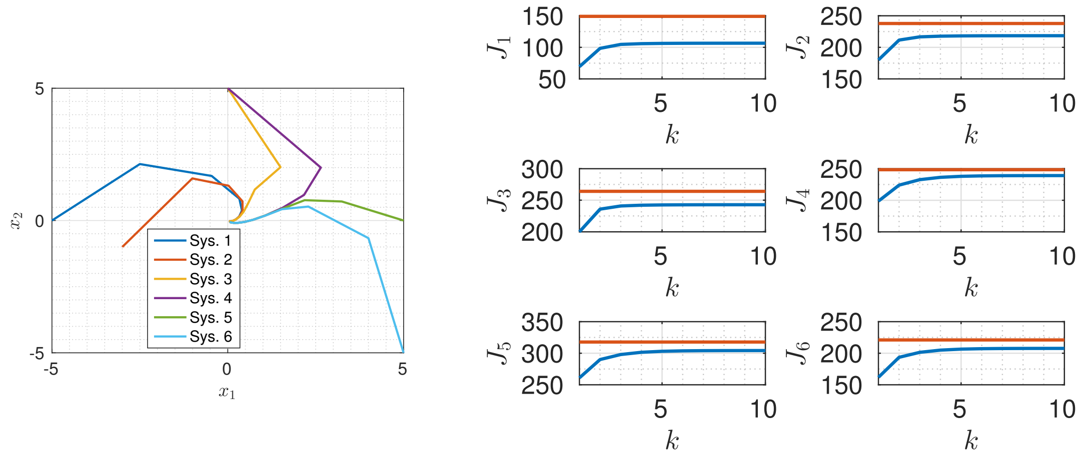

Figure 4.6: Trajectories and accumulated performance without terminal constraints using Algorithm 3.

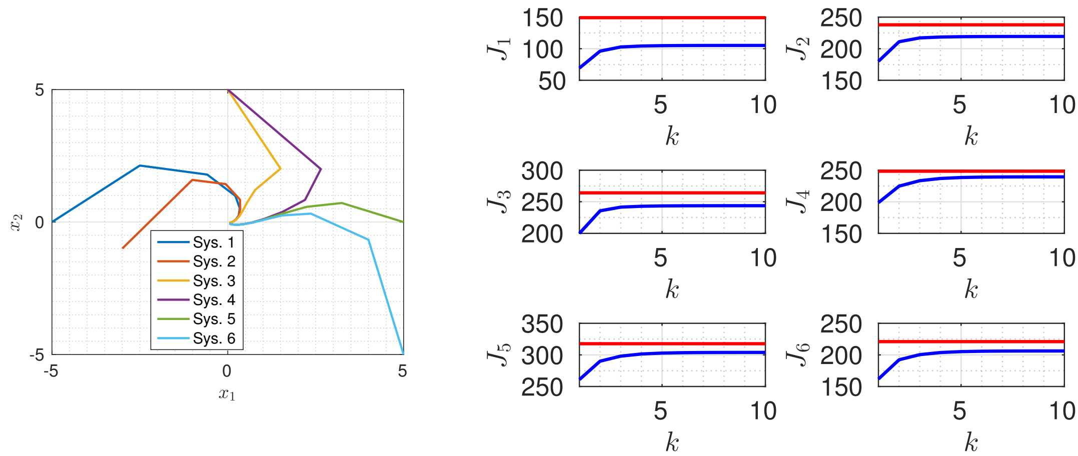

Figure 4.7: Trajectories and accumulated performance without terminal constraints using Algorithm 4.

5 | Multiobjective Economic MPC

In multiobjective (MO) economic Model Predictive Control (MPC) we entirely drop Assumption 4.2, i.e. we do not assume positive definiteness of the criteria ℓ_i wrt to an equilibrium x_*. The motivation for doing so is given in Chapter 2. The results presented here can be seen as an attempt to generalize some of the results on single-objective Economic MPC in [32, Chap. 8] and Chapter 2. Thus, a dissipativity assumption will be imposed and discussed.

As in the setting with 'classical' stage costs, we distinguish MPC schemes with and without terminal conditions and start our analysis by imposing terminal conditions.

Some of our results have been published in [36].

5.1 MO Economic MPC with Terminal Conditions

In this section we solve MO optimal control problems (OCPs) of the form (4.3). The following assumption on the terminal condition, which is an extension of [1, Assumption 6] (see also [32, Assumption 8.5]), takes the place of the Assumptions 4.2 and 4.4.

Assumption 5.1. *1. There is an equilibrium $(x_*, u_*) \in \mathbb{X} \times \mathbb{U}$ with $x_* \in \mathbb{X}_0$ and the terminal cost $F_i : \mathbb{X}_0 \to \mathbb{R}_{\geq 0}$ satisfies $F_i(x_*) = 0$ for all $i \in \{1, \dots, s\}$.*

2. There is $\kappa : \mathbb{X}_0 \to \mathbb{U}$ such that $f(x, \kappa(x)) \in \mathbb{X}_0$ and

$$F_i(f(x, \kappa(x))) \leq F_i(x) - \ell_i(x, \kappa(x)) + \ell_i(x_*, u_*)$$

holds for all $i \in \{1, \dots, s\}$.

5.1.1 Averaged Performance

By means of these conditions we can already formulate an averaged performance result for Algorithm 2 for the economic setting. The sets $\mathbb{U}^N(x)$ and $\mathbb{X}_N$ are defined as in Section 4.2. A numerical example that illustrates the results in this section will be presented in Section 5.1.4. We start our analysis with some results on the averaged infinite-horizon performance of the MPCcontroller defined in Algorithm 2 and then move on to non-averaged performance results. The necessity for considering the averaged performance is explained in [32, Sec. 8.1].

Theorem 5.2 (Averaged performance theorem). *Consider an OCP (4.3), and let $N \in \mathbb{N}_{\geq 2}$. Let Assumption 5.1 hold and let the set $\mathcal{J}_{\mathcal{P}}^N(x)$ be externally stable (according to Def. 3.4) for each $x \in \mathbb{X}_N$. We furthermore assume that there is $M \in \mathbb{R}$ such that $J_i^N(x, \mathbf{u}^\star) \geq M$ for all $x \in \mathbb{X}_N$, all $\mathbf{u}^\star \in \mathbb{U}_{\mathcal{P}}^N(x)$ and each $i \in \{1, \dots, s\}$. Then, the MPC feedback $\mu^N : \mathbb{X} \to \mathbb{U}$ defined in Algorithm 2 has the following infinite-horizon averged closed-loop performance for all objectives $i \in \{1, \dots, s\}$:*

$$\bar{J}_i^\infty(x_0, \mu^N) := \limsup_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell_i\left(x(k), \mu^N(x(k))\right) \leq \ell_i(x_*, u_*).$$

Proof. We follow the reasoning in the proof of Theorem 4.6. Feasibility of all chosen Pareto-optimal solutions (POSs) holds with the same arguments. For investigating the performance consider $K \in \mathbb{N}$ and $i \in \{1, \dots, s\}$. We obtain

$$\begin{aligned}
&\sum_{k=0}^{K-1} \ell_i(x(k), \mu^N(x(k))) \\
&= \sum_{k=0}^{K-1} \left[J_i^N\left(x(k), \mathbf{u}_{x(k)}^{\star,N}\right) - J_i^{N-1}\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k)}^{\star,N}(\cdot+1)\right)\right] \\
&\leq \sum_{k=0}^{K-1} \left[J_i^N\left(x(k), \mathbf{u}_{x(k)}^{\star,N}\right) - J_i^N\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k+1)}^{N}\right) + \ell_i(x_*, u_*)\right],
\end{aligned}$$

in which the inequality is obtained as in Lemma 4.5 for the terminal cost from Assumption 5.1 in combination with Lemma 4.1, and $\mathbf{u}_{x(k)}^{\star,N}$ is the POS chosen in Algorithm 2 at time k. In step **(1)** $\mathbf{u}_{x(k+1)}^{\star,N}$ is constructed such that

$$J_i^N\left(x(k+1), \mathbf{u}_{x(k+1)}^{\star,N}\right) \leq J_i^N\left(x(k+1), \mathbf{u}_{x(k+1)}^{N}\right)$$

holds for all $i \in \{1, \dots, s\}$. Thus, we have

$$\begin{aligned}
\sum_{k=0}^{K-1} \ell_i\left(x(k), \mu^N(x(k))\right) &\leq J_i^N\left(x_0, \mathbf{u}_{x_0}^{\star,N}\right) - J_i^N\left(x(K), \mathbf{u}_{x(K)}^{\star,N}\right) + K\ell_i(x_*, u_*) \\
&\leq J_i^N\left(x_0, \mathbf{u}_{x_0}^{\star,N}\right) - M + K\ell_i(x_*, u_*) \;\forall\, i \in \{1, \dots, s\}.
\end{aligned}$$

Taking the average and the limit superior on both sides of the inequality yields the assertion. $\square$

In Chapter 2 we have emphasized that dissipativity (see Def. 2.5) is a key ingredient for analyzing economic MPC schemes (with and without terminal conditions). To the best of our knowledge there does not exists a MO version of this systems theoretic property. Our attempt to generalize dissipativity is as follows.

Definition 5.3 ((Strict) Dissipativity)**.** *The MO OCP* (4.1) *is* strictly dissipative *with respect to ℓ_i, $i \in \{1,\dots,s\}$, at an equilibrium (x^e,u^e) if there is $\lambda_i : \mathbb{X} \to \mathbb{R}$ bounded from below with $\lambda_i(x^e) = 0$ and $\rho_i \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{X}$ and $u \in \mathbb{U}(x)$ it holds*

$$\tilde{\ell}_i(x,u) := \ell_i(x,u) - \ell_i(x^e,u^e) + \lambda_i(x) - \lambda_i(f(x,u)) \geq \rho_i(\|x - x^e\|). \tag{5.1}$$

If the inequality holds for $\rho_i \equiv 0$, we call the problem dissipative.

If the MO OCP is (strictly) dissipative wrt all ℓ_i at (x^e,u^e), we say that the problem is uniformly (strictly) dissipative *at (x^e,u^e).*[1]

If the MO OCP is strictly dissipative wrt to all ℓ_i at (x^e,u^e), Definition 5.3 seems to be the natural extension of Definition 2.5. Moreover, the common interpretation of dissipativity as a measure of how much energy is stored (and lost) in the system is still meaningful. However, we conjecture that it is more realistic that different costs criteria are (strictly) dissipative at different equilibria. While this assumption enables us to have more distinct results on MO economic MPC, it is not quite clear how to interpret this property physically. This aspect will be part of our future research on MO economic MPC.

Since in scalar-valued economic MPC, dissipativity is closely related to the optimal equilibrium/steady state as defined in Definition 2.1 (see e.g. [30, 63]), the Pareto-optimal steady states defined below seem to be the proper equivalent in the MO setting.

Definition 5.4 ((Pareto-) Optimal steady state)**.** *A steady state (x^e,u^e) is called* Pareto optimal steady state *if it is a POS to the MO optimization problem*

$$\begin{aligned} &\min(\ell_1(x,u),\dots,\ell_s(x,u)) \\ &s.t.\ \ f(x,u) - x = 0, \\ &x \in \mathbb{X},\ \ u \in \mathbb{U}(x). \end{aligned} \tag{5.2}$$

It is called optimal steady state for ℓ_i *if it is a solution to*

$$\begin{aligned} &\min \ell_i(x,u) \\ &s.t.\ \ f(x,u) - x = 0, \\ &x \in \mathbb{X},\ \ u \in \mathbb{U}(x). \end{aligned}$$

We now show that dissipativity wrt ℓ_j at a steady state (x^e,u^e) is sufficient for (x^e,u^e) to be an optimal steady state for ℓ_j and a Pareto-optimal steady state for (5.2).

Lemma 5.5 (Dissipativity implies (Pareto-)optimality)**.** *Assume that the MO OCP is dissipative wrt to ℓ_j, $j \in \{1,\dots,s\}$, at (x^e,u^e). Then, (x^e,u^e) is an optimal steady state for ℓ_j (and a weakly Pareto-optimal steady state for* (5.2)*) and for all $x_0 \in \mathbb{X}$ and $\mathbf{u} \in \mathbb{U}^\infty(x_0)$ it holds*

$$\limsup_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell_j\left(x(k,x_0),u(k)\right) \geq \ell_j(x^e,u^e).$$

[1] In that case, $\mathbf{u}$ with $u(k) \equiv u^e$ for all $k \in \mathbb{N} \cup \{\infty\}$ is a strongly POS to (4.1) and to (4.2) with initial value x^e.

Proof. Let $(\bar{x}, \bar{u})$ be an admissible steady state. It holds $\ell_j(\bar{x}, \bar{u}) - \ell_j(x^e, u^e) = \ell_j(\bar{x}, \bar{u}) - \ell_j(x^e, u^e) + \lambda_j(\bar{x}) - \lambda_j(f(\bar{x}, \bar{u})) \geq 0$. Hence, (x^e, u^e) is optimal for ℓ_j. Any point that is optimal wrt to one objective function belongs to the set of weakly POSs to the respective MO optimization problem.

Now, consider arbitrary $x_0 \in \mathbb{X}$ and $\mathbf{u} \in \mathbb{U}^\infty(x_0)$. Then, we have

$$\begin{aligned}
\frac{1}{K}\sum_{k=0}^{K-1} \ell_j\left(x(k, x_0), u(k)\right) &\geq \frac{1}{K}\sum_{k=0}^{K-1}\left[\ell_j\left(x^e, u^e\right) - \lambda_j(x(k, x_0)) + \lambda_j(x(k+1, x_0))\right] \\
&= \ell_j(x^e, u^e) + \frac{1}{K}(\lambda_j(x(K, x_0)) - \lambda_j(x_0)) \\
&\geq \ell_j(x^e, u^e) + \frac{1}{K}(M_{\lambda_j} - \lambda_j(x_0)),
\end{aligned}$$

in which M_{λ_j} is the assumed bound on λ_j (see Def. 5.3). Taking the $\limsup_{K\to\infty}$ on both sides of the inequality yields the assertion. □

Remark 5.6. *It follows immediately from Lemma 5.5 that if the MO OCP is uniformly dissipative at a steady state (x^e, u^e), then (x^e, u^e) is a strongly POS to* (5.2) *in the sense of Definition 3.3.*

So far, we have established an upper bound on the averaged performance, which depends on the terminal condition and applies to all cost criteria ℓ_i, $i \in \{1, \ldots, s\}$, and we have a lower bound that only applies to all dissipative cost criteria and which depends on the respective equilibrium that the cost criterion is dissipative at. Not suprisingly and as can be seen in the following corollary, the upper and the lower bound on the averaged infinite-horizon performance coincide for those cost criteria, which are dissipative at the steady state from the terminal condition.

Corollary 5.7. *Let Assumption 5.1 hold, let $x_0 \in \mathbb{X}_N$ and consider the MPC feedback defined in Algorithm 2. If there is a cost criterion ℓ_i that is dissipative wrt (x_*, u_*) and if the sets $\mathcal{J}^N(x)$ for all $x \in \mathbb{X}_N$ are externally stable, then it holds*

$$\bar{J}_i^\infty(x_0, \mu^N) = \inf_{\mathbf{u}\in\mathbb{U}^\infty(x_0)} \limsup_{K\to\infty} \frac{1}{K}\sum_{k=0}^{K-1} \ell_i(x(k, x_0), u(k)).$$

Proof. For all ℓ_i that are dissipative wrt to (x_*, u_*) it holds

$$\begin{aligned}
\ell_i(x_*, u_*) \geq \bar{J}_i^\infty(x_0, \mu^N) &= \limsup_{K\to\infty} \frac{1}{K}\sum_{k=0}^{K-1} \ell_i\left(x(k, x_0), \mu^N(x(k, x_0))\right) \\
&\geq \inf_{\mathbf{u}\in\mathbb{U}^\infty(x_0)} \limsup_{K\to\infty} \frac{1}{K}\sum_{k=0}^{K-1} \ell_i\left(x(k, x_0), u(k)\right) \geq \ell_i(x_*, u_*),
\end{aligned}$$

in which the first and the last inequality follow from Theorem 5.2 and Lemma 5.5. Since the left hand side and right hand side of the chain of inequalities coincide, we have equality of all expressions, and thus, obtain the assertion. □

5.1.2 Non-averaged Performance

For cost criteria that are strictly dissipative at the equilibrium from the terminal condition there is no gap between the upper and lower bound on the averaged performance. Moreover, as the next Lemma 5.8 shows, we will be able to prove that for those cost criteria we are able to establish a non-averaged infinite-horizon performance statement. Since this result will be formulated in terms of the rotated cost according to Definition 5.3, we define the following *rotated cost functionals*[2]:

$$\begin{aligned}\tilde{J}_i^N(x,\mathbf{u}) &:= \sum_{k=0}^{N-1} \tilde{\ell}_i(x(k),u(k)) + \tilde{F}_i(x(N)) \text{ with } \tilde{F}_i(x) := F_i(x) + \lambda_i(x)\\ \tilde{J}_i^\infty(x_0,\mu^N) &:= \lim_{K\to\infty} \sum_{k=0}^{K-1} \tilde{\ell}_i(x(k),\mu^N(x(k)))\end{aligned}$$

Lemma 5.8 (Non-averaged rotated performance)**.** *Let Assumption 5.1 hold, $x_0 \in \mathbb{X}_N$, and let the sets $\mathcal{J}_\mathcal{P}^N(x)$ be externally stable for all $x \in \mathbb{X}_N$. Then, for all cost criteria ℓ_i, $i \in \{1,\dots,s\}$ that are dissipative at (x_*,u_*), it holds*

$$\tilde{J}_i^\infty(x_0,\mu^N) \le \tilde{J}_i^N(x_0,\mathbf{u}_x^{\star,N}).$$

Here, μ^N is the MPC feedback defined in Algorithm 2.

Proof. Let $i \in \{1,\dots,s\}$ such that ℓ_i is strictly dissipative at (x_*,u_*). By definition, for any $x \in \mathbb{X}_N$ and $\mathbf{u} \in \mathbb{U}^N(x)$ the relation

$$\begin{aligned}&\tilde{J}_i^N(x,\mathbf{u})\\ =&\sum_{k=0}^{N-1} [\ell_i(x(k),u(k)) - \ell_i(x_*,u_*) + \lambda_i(x(k)) - \lambda_i(x(k+1))] + F_i(x(N)) + \lambda_i(x(N))\\ =&J_i^N(x,\mathbf{u}) - N\ell_i(x_*,u_*) + \lambda_i(x) \qquad (5.3)\end{aligned}$$

holds. This means that both cost functionals only differ by constants. Then, we can

[2] Of course, the rotated functionals can only be defined for those cost criteria, which are strictly dissipative at some equilibrium.

determine the performance as follows: For each $K \in \mathbb{N}$ it holds

$$\begin{aligned}
&\sum_{k=0}^{K-1} \tilde{\ell}_i(x(k,x_0),\mu^N(x(k,x_0))) \\
=&\sum_{k=0}^{K-1} \left[\ell_i(x(k,x_0),\mu^N(x(k,x_0))) - \ell_i(x_*,u_*) + \lambda_i(x(k,x_0)) - \lambda_i(x(k+1,x_0))\right] \\
=&\sum_{k=0}^{K-1} \left[J_i^N(x(k,x_0),\mathbf{u}^{\star,N}_{x(k,x_0)}) - J_i^{N-1}(x(k+1,x_0),\mathbf{u}^{\star,N}_{x(k,x_0)}(\cdot+1))\right] \\
&\quad - K\ell_i(x_*,u_*) + \lambda_i(x_0) - \lambda_i(x(K,x_0)) \\
\leq&\sum_{k=0}^{K-1} \left[J_i^N(x(k,x_0),\mathbf{u}^{\star,N}_{x(k,x_0)}) - J_i^N(x(k+1,x_0),\mathbf{u}^{N}_{x(k+1,x_0)}) + \ell_i(x_*,u_*)\right] \\
&\quad - K\ell_i(x_*,u_*) + \lambda_i(x_0) - \lambda_i(x(K,x_0)) \\
\leq& J_i^N(x_0,\mathbf{u}^{\star,N}_{x_0}) - J_i^N(x(K,x_0),\mathbf{u}^{N}_{x(K,x_0)}) + \lambda_i(x_0) - \lambda_i(x(K,x_0)) \\
=&\tilde{J}_i^N(x_0,\mathbf{u}^{\star,N}_{x_0}) - \tilde{J}_i^N(x(K,x_0),\mathbf{u}^{N}_{x(K,x_0)}) \leq \tilde{J}_i^N(x_0,\mathbf{u}^{\star,N}_{x_0}),
\end{aligned}$$

in which the first inequality holds by means of the terminal condition, the second inequality because of step **(1)** in Algorithm 2, and the last inequality holds true because dissipativity of the cost criterion as well as positive terminal costs $\tilde{F}_i$ imply positivity of the rotated cost functional. The existence of proper POSs (i.e. such POSs satisfying the constraints) is again ensured by external stability. Finally, letting K tend to infinity yields the statement. □

Lemma 5.8 can be proven more elegantly in case that the MO OCP (4.1) is uniformly strictly dissipative at (x_*,u_*) (cf. Definition 5.3): Analogous to $\mathbb{U}^N_{\mathcal{P}}(x)$ we define the set $\tilde{\mathbb{U}}^N_{\mathcal{P}}(x) = \{\mathbf{u}^\star \in \mathbb{U}^N(x) | \mathbf{u}^\star \text{ POS to (4.3) with } \tilde{J}^N(x,\mathbf{u}) \text{ instead of } J^N(x,\mathbf{u})\}$ and by relation (5.3) conclude $\tilde{\mathbb{U}}^N_{\mathcal{P}}(x) = \mathbb{U}^N_{\mathcal{P}}(x)$. This also implies that existence of POSs for the rotated problem is ensured by external stability of the sets $\mathcal{J}^N_{\mathcal{P}}(x)$.

For $x \in \mathbb{X}_0$ Assumption 5.1 yields

$$\begin{aligned}
\tilde{F}_i(f(x,\kappa(x))) &= F_i(f(x,\kappa(x))) + \lambda_i(f(x,\kappa(x))) \\
&\leq F_i(x) - \ell_i(x,\kappa(x)) + \ell_i(x_*,u_*) + \lambda_i(f(x,\kappa(x))) \\
&= F_i(x) + \lambda_i(x) - \ell_i(x,\kappa(x)) + \ell_i(x_*,u_*) - \lambda_i(x) + \lambda_i(f(x,\kappa(x))) \\
&= \tilde{F}_i(x) - \tilde{\ell}_i(x,\kappa(x)).
\end{aligned}$$

Thus, the modified OCP satisfies Assumptions 4.2 and 4.4. Hence, proceeding as in the proof of Theorem 4.6, we immediately obtain the desired inequality for all cost criteria.

At first glance the result of Lemma 5.8 might seem somehow unsatisfactory, because the performance of the MPC controller is evaluated in terms of the rotated cost instead of the original cost. The main benefit of this result lies in the fact that it enables us to prove convergence of the MPC closed-loop trajectory.

Corollary 5.9 (Convergence of the MPC closed loop)**.** *Let the assumptions of Lemma 5.8 hold and assume that there is at least one cost criterion ℓ_i that is strictly dissipative at (x_*, u_*). Then, for the MPC feedback μ^N from Algorithm 2 it holds*

$$\lim_{k\to\infty} x^{\mu^N}(k) = x_*.$$

Proof. The proof is the same as the proof of Corollary 4.9 using $\tilde{\ell}_i$ instead of ℓ_i. □

We have just shown how to obtain averaged performance and estimates for the rotated cost functional. In what follows we will deduce estimates for the non-averaged infinite-horizon MPC performance in terms of the original cost critera. Therefore, we require the following:

Assumption 5.10 (Uniform continuity)**.** *Assume there are criteria ℓ_i, $i \in \mathcal{I} \subseteq \{1,\dots,s\}$, which are strictly dissipative at (x_*, u_*) and $\ell_i(x_*, u_*) = 0$ for all $i \in \mathcal{I}$ and x_* from Assumption 5.1. We assume that for each $i \in \mathcal{I}$ there exists $\gamma_i \in \mathcal{K}_\infty$ such that for all $N \in \mathbb{N}$, for all $x \in \mathbb{X}_N$ and all $\mathbf{u}^\star \in \mathbb{U}^N_{\mathcal{P}}(x)$ it holds*

$$|J_i^N(x, \mathbf{u}^\star)| \le \gamma_i(\|x - x_*\|). \tag{5.4}$$

Property (5.4) is referred to as uniform continuity of J_i^N at x_*. This is due to the following: If $\mathcal{I} = \{1,\dots,s\}$, then for each $N \in \mathbb{N}$ the sequence $\mathbf{u} \in \mathbb{U}^N(x_*)$ with $u(k) = u_*$ for all $k \in \{0,\dots,N-1\}$ is a strongly POS to (4.3) for $x = x_*$ and it holds $\tilde{J}_i^N(x_*, \mathbf{u}) = 0$. Using (5.3) this yields $J_i^N(x_*, \mathbf{u}) = \tilde{J}_i^N(x_*, \mathbf{u}) + N\ell_i(x_*, u_*) - \lambda_i(x_*) = 0$ because of Assumption 5.10 above and Definition 5.3. Thus, in case $\mathcal{I} = \{1,\dots,s\}$ relation (5.4) in fact reads: For all $N \in \mathbb{N}$, for all $x \in \mathbb{X}_N$, all $\mathbf{u}^\star \in \mathbb{U}^N_{\mathcal{P}}(x)$ and for all $\mathbf{u}^\star_{x_*} \in \mathbb{U}^N_{\mathcal{P}}(x_*)$ it holds

$$|J_i^N(x, \mathbf{u}^\star) - J_i^N(x_*, \mathbf{u}^\star_{x_*})| \le \gamma_i(\|x - x_*\|).$$

By means of our preliminary results and Assumption 5.10 we are now ready to state our main performance result for MO economic MPC.

Theorem 5.11 (Non-averaged infinite-horizon performance)**.** *Consider $x_0 \in \mathbb{X}_N$, and let the Assumptions 5.1 and 5.10 hold and let the sets $\mathcal{J}^N_{\mathcal{P}}(x)$ be externally stable for all $x \in \mathbb{X}_N$. Then, for all $i \in \mathcal{I}$ the MPC feedback from Algorithm 2 has the following non-averaged infinite-horizon performance:*

$$J_i^\infty(x_0, \mu^N) \le J_i^N(x_0, \mathbf{u}_x^{\star,N}).$$

Proof. We proceed as in prior proofs and consider the finite-horizon performance. For each $K \in \mathbb{N}$ and all $i \in \mathcal{I}$ it holds

$$\begin{aligned}
&\sum_{k=0}^{K-1} \ell_i\left(x(k,x_0), \mu^N(x(k,x_0))\right) \\
=&\sum_{k=0}^{K-1} J_i^N\left(x(k,x_0), \mathbf{u}^\star_{x(k,x_0)}\right) - J_i^{N-1}\left(x(k+1,x_0), \mathbf{u}^\star_{x(k,x_0)}(\cdot+1)\right) \\
\le& J_i^N\left(x_0, \mathbf{u}^\star_{x_0}\right) - J_i^N\left(x(K,x_0), \mathbf{u}^\star_{x(K,x_0)}\right) + K \underbrace{\ell_i(x_*, u_*)}_{=0}.
\end{aligned}$$

Now, we combine (5.4), Corollary 5.9 and the fact that γ_i in (5.4) is a $\mathcal{K}_\infty$-function and thus continuous. This way, we get

$$\begin{aligned}\lim_{K\to\infty} |J_i^N(x(K,x_0), \mathbf{u}^\star_{x(K,x_0)})| &\leq \lim_{K\to\infty} \gamma_i(\|x(K,x_0) - x_*\|)\\ &= \gamma_i(\lim_{K\to\infty} \|x(K,x_0) - x_*\|) = 0.\end{aligned}$$

This implies the assertion. □

5.1.3 Strictly Convex MO Optimization Problems

In what follows we will be concerned with the special case of strictly convex MO OCPs. The reason for investigating this special case is twofold. Firstly, such OCPs are quite common according to [8, 77], secondly, we will be able to show that the MPC closed-loop trajectory converges to the equilibrium in Assumption 5.1 even if none of the given cost criteria ℓ_i is strictly dissipative at this equilibrium. The following Lemma 5.12 provides insights into this (probably) unexpected behavior.

Lemma 5.12. *Consider $x_0 \in \mathbb{X}_N$, let Assumption 5.1 hold, let the sets $\mathcal{J}^N_{\mathcal{P}}(x)$ be externally stable for all $x \in \mathbb{X}_N$, and assume that the MO OCP* (4.1) *is strictly dissipative wrt all ℓ_i at (possibly different) (x_i^e, u_i^e). If the dynamics* (1.1) *are affine, all ℓ_i and F_i are strictly convex and (x_*, u_*) in Assumption 5.1 is a Pareto-optimal steady state according to Definition 5.4, then the MPC feedback μ^N from Algorithm 2 yields $x^{\mu^N}(k, x_0) \to x_*$, $k \to \infty$.*

Proof. **Step 1:** We prove that there exists a convex combination ℓ_Σ of the ℓ_i, such that ℓ_Σ is strictly dissipative at (x_*, u_*).

Let $\mathcal{E}$ be the set of Pareto-optimal steady states, which includes the points (x_i^e, u_i^e) and (x_*, u_*). Due to strict convexity of the ℓ_i, the set $\mathcal{E}$ can completely be computed via a weighted sum-approach (see Corollary 3.10), i.e.

$$\mathcal{E} = \left\{ (x,u) | (x,u) = \underset{(x,u)\in\mathbb{X}\times\mathbb{U},\ x=f(x,u)}{\operatorname{argmin}} \sum_{i=1}^s w_i \ell_i(x,u),\ w_i \in \mathbb{R}_{\geq 0},\ \sum_{i=1}^s w_i = 1 \right\}.$$

According to Lemma 5.5, the points (x_i^e, u_i^e) are associated with weights $(1, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, 1)$. Since $(x_*, u_*) \in \mathcal{E}$, there are $(\hat{w}_i)_{i=1,\dots,s}$, $\hat{w}_i \geq 0$, $\sum_i \hat{w}_i = 1$, such that $(x_*, u_*) = \operatorname{argmin}_{(x,u)\in\mathbb{X}\times\mathbb{U},\ x=f(x,u)} \sum_{i=1}^s \hat{w}_i \ell_i(x,u)$. Let us define

$$\ell_\Sigma(x,u) := \sum_{i=1}^s \hat{w}_i \ell_i(x,u).$$

Clearly, ℓ_Σ is a strictly convex function with optimal steady state (x_*, u_*). It was proven in [12, Prop. 4.3][3] that (single-objective) OCPs with affine dynamics and strictly convex stage costs are strictly dissipative at the optimal steady state with a linear storage function, hence ℓ_Σ is strictly dissipative at (x_*, u_*).

[3] See also [16].

Step 2: ℓ_Σ strictly dissipative at (x_*, u_*) yields that the closed-loop trajectory tends to x_*.

Due to the convexity assumptions, we have strict convexity of the cost functionals J_i^N for all $i \in \{1, \ldots, s\}$. Thus, all POSs in step **(1)** of Algorithm 2 can be calculated via weighted sums. This means that one admissible POS is $\operatorname{argmin}_{\mathbf{u} \in \mathbb{U}^N(x)} J_\Sigma^N(x, \mathbf{u})$ subject to the constraints, in which $J_\Sigma^N(x, \mathbf{u}) := \sum_{i=1}^s \hat{w}_i J_i^N(x, \mathbf{u})$. Hence, ℓ_Σ can be included in the set of stage costs without changing the POSs in the MPC iterations and since for this weighted stage cost, the equilibrium from the terminal constraint and dissipativity coincide, we can apply Corollary 5.9 to conclude the proof. □

Remark 5.13. *1. Lemma 5.12 reveals that even though none of the original stage costs is dissipative wrt to the steady state from the terminal condition, there might be an 'artificial' criterion that is dissipative wrt this steady state. In the proof of Lemma 5.12 ℓ_Σ has that property.*

2. For the rotated stage cost $\tilde{\ell}_\Sigma$ we obtain the performance result from Lemma 5.8.

3. If the equilibrium from the terminal condition does not belong to the set $\mathcal{E}$, the MPC closed-loop trajectory does not converge towards it, see Figure 5.8.

5.1.4 Numerical Results

The Case of Uniform Dissipativity

To illustrate the results in this section we reconsider Example 2.14 (economic growth model), but this time in a MO setting. To this end we will equip the system $x^+ = u$, $\ell_1(x, u) = -\ln(Ax^\alpha - u)$ with a second cost function $\ell_2(x, u) = (x - x_1^e)^2 + 0.1(u - u_1^e)^2$ using the steady state $x_1^e = x^e \approx 2.23$ at which ℓ_1 is strictly dissipative. With this definition, the MO OCP is uniformly strictly dissipative at (x_1^e, u_1^e). We use the same constraint sets as in Example 2.14 with the additional terminal constraint $\mathbb{X}_0 = \{x_1^e\}$. This means that Assumption 5.1 holds with $x_* = x_1^e$, $\kappa \equiv u_* = x_*$ and $F_i \equiv 0$ for all $i \in \{1, \ldots, s\}$.

Let us first note that with these specifications, the sets $\mathcal{J}_\mathcal{P}^N(x)$ are externally stable for each $x \in \mathbb{X}_N$ and each $N \in \mathbb{N}$ (see Lemma 4.8). In this setting, the equilibrium from Assumption 5.1 and from uniform strict dissipativity coincide. Consequently, the bounds in Theorem 5.2, Lemma 5.5 and Corollary 5.7 coincide, too, for both cost criteria. Indeed, it can be observed in Figure 5.1 that the averaged infinite-horizon performance approaches the value $\ell_i(x_1^e, x_1^e)$ for both objectives. Convergence of the MPC closed-loop trajectory was proven in Corollary 5.9 and is illustrated in Figure 5.2. In Figure 5.3 it is shown that the non-averaged infinite-horizon performance is bounded from above by the value of the POS in step **(0)**, Algorithm 2, but need not be below this bound for all times, which is in contrast to the results in Chapter 4. In order to obtain the desired results in Figure 5.3, constants were added to ℓ_1 such that $\ell_i(x_*, u_*) = 0$ holds.

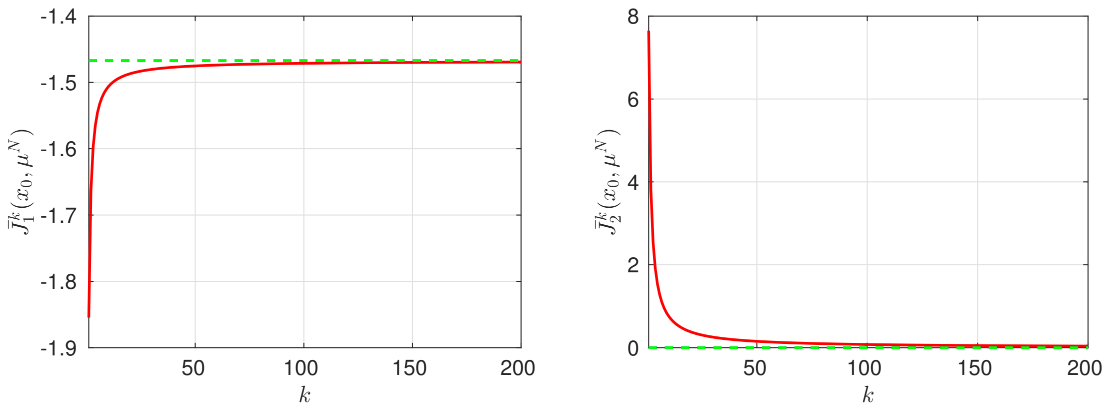

Figure 5.1: Averaged performance (red, solid) of Algorithm 2 for the MO uniformly dissipative economic growth example, using $\mathbb{X}_0 = \{x_1^e\}$, $x_0 = 5$ and $N = 3$. The dashed green lines are the values $\ell_i(x_*, u_*) = \ell_i(x_1^e, x_1^e)$.

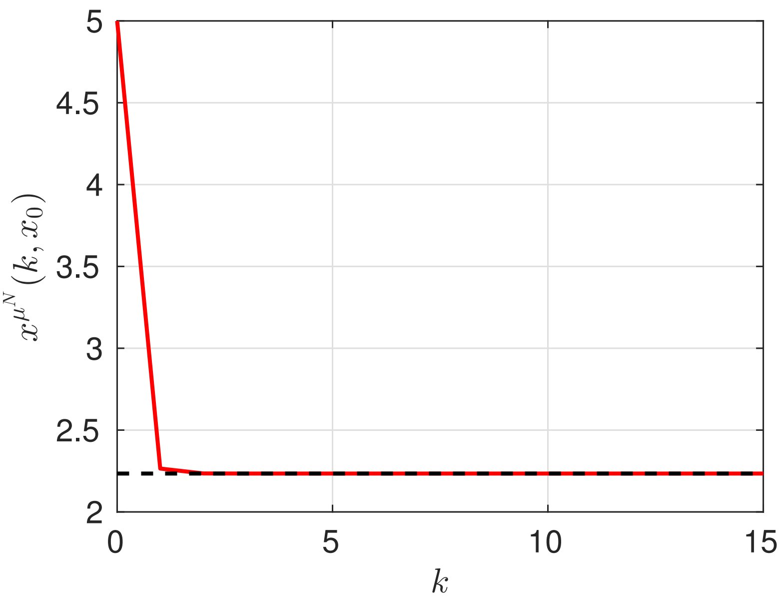

Figure 5.2: Closed-loop trajectory (red, solid) of Algorithm 2 for the uniformly strictly dissipative example using $\mathbb{X}_0 = \{x_1^e\}$ and $N = 3$, and the terminal condition (black, dashed).

The Case of Non-Uniform Dissipativity

In this section the stage cost ℓ_1 remains unchanged, whereas the second cost function is given by $\ell_2(x,u) = -\ln(A_2 x^{\alpha_2} - u)$ with $A_2 = 3$ and $\alpha_2 = 0.2$. We use the same constraints as in Example 2.14 with the additional terminal constraint $\mathbb{X}_0 = \{x_*\}$ that will vary throughout our illustrations. Again, the sets $\mathcal{J}_{\mathcal{P}}^N(x)$ are externally stable for each

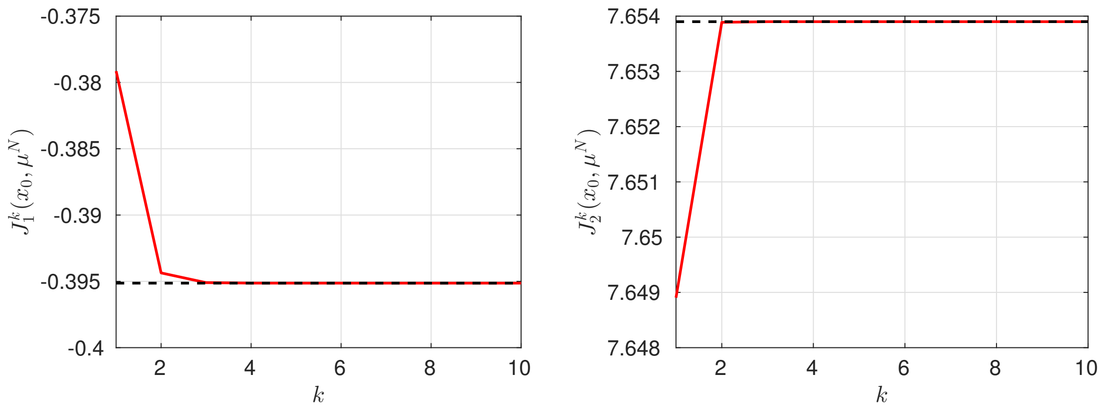

Figure 5.3: Non-averaged infinite-horizon performance (red, solid) and the value $J_i^N(x_0, \mathbf{u}^{\star,N})$ (black, dashed) for the uniformly strictly dissipative problem and $x_0 = 5$, $N = 3$.

$x \in \mathbb{X}_N$ and each $N \in \mathbb{N}$.

Let us now set $x_* = x_1^e \approx 2.23$, which is the optimal equilibrium for ℓ_1. Thus, $x_* \in \mathcal{E}$. We have already stated in Example 2.14 that the first cost criterion is strictly dissipative wrt x_1^e and since ℓ_2 has the same structure as ℓ_1 we conclude strict dissipativity (with a linear storage function λ_2) of ℓ_2 at $(x_2^e, u_2^e) = (x_2^e, x_2^e)$ with $x_2^e \approx 0.53$. In Figure 5.4

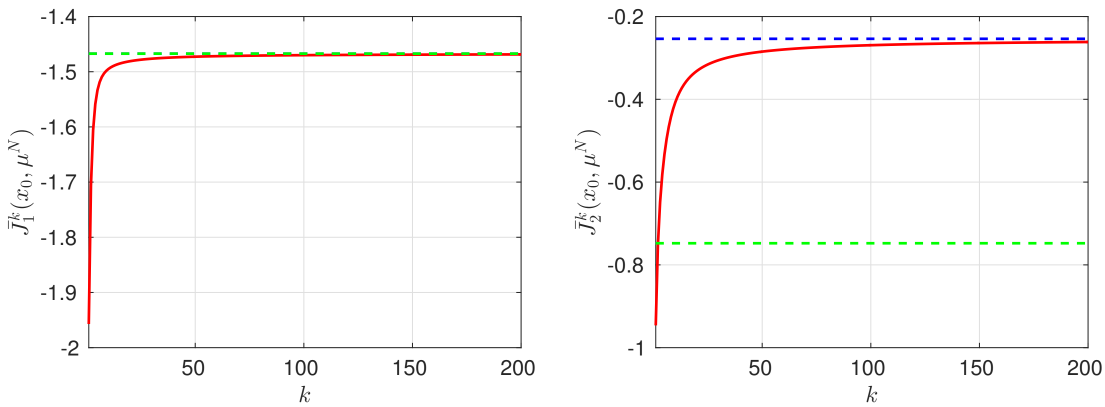

Figure 5.4: Averaged performance (red, solid) of Algorithm 2 for the MO economic growth example, using $\mathbb{X}_0 = \{x_1^e\}$, $x_0 = 5$ and $N = 3$. The dashed blue lines are the values $\ell_i(x_*, u_*) = \ell_i(x_1^e, x_1^e)$, the green lines the values $\ell_i(x_i^e, u_i^e)$.

we see that the averaged performance of Algorithm 2 for our example indeed approaches the value of the stage cost of the terminal condition $\ell_i(x_*, u_*)$ from below. This reflects the statement of Theorem 5.2. Moreover, we observe that the infinite-horizon averaged performance is lower bounded by $\ell_i(x_i^e, u_i^e)$ as proved in Lemma 5.5. For the first objective the values $\ell_1(x_*, u_*)$ and $\ell_1(x_1^e, u_1^e)$ coincide.

Since there is ℓ_i (namely ℓ_1) which is strictly dissipative at the steady state in Assumption 5.1, Corollary 5.9 applies. The convergence of the MPC closed loop is illustrated

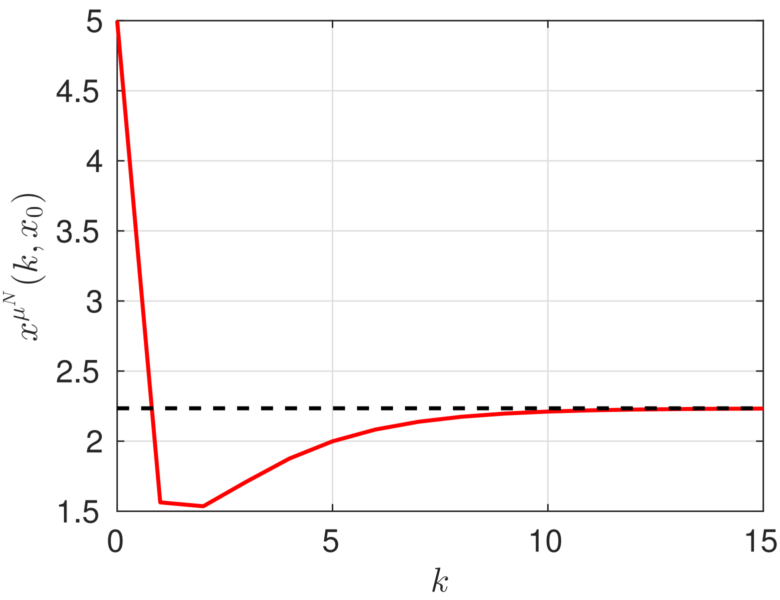

Figure 5.5: Closed-loop trajectory (red, solid) of Algorithm 2 using $\mathbb{X}_0 = \{x_1^e\}$ and $N = 3$, and the terminal condition (black, dashed).

in Figure 5.5. Let us now consider the non-averaged infinite-horizon performance of μ^N. Theorem 5.11 implies that the performance of the first cost criterion is bounded by the value of the POS in step **(0)**, Algorithm 2. Note that in order to obtain this result we have

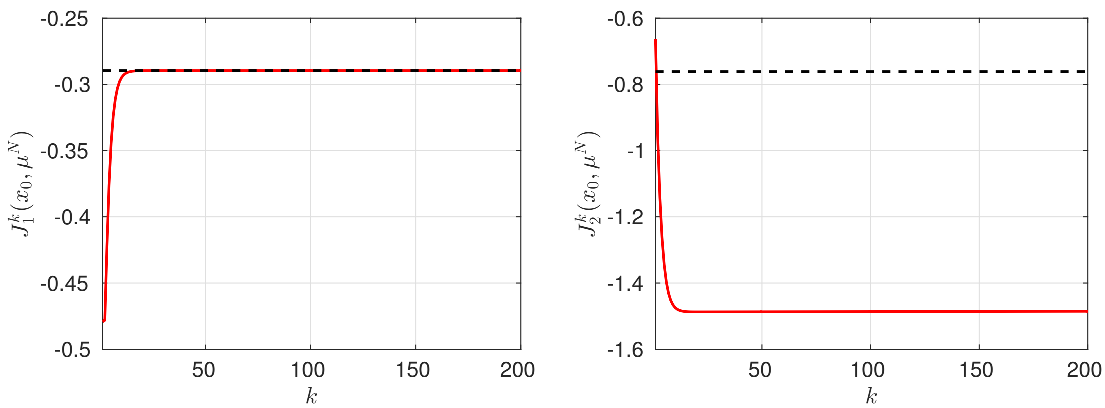

Figure 5.6: Non-averaged infinite-horizon performance (red, solid) and the value $J_i^N(x_0, \mathbf{u}^{\star,N})$ (black, dashed) for $N = 3$, $x_0 = 5$.

added constants to our stage costs, such that $\ell_i(x_*, u_*) = 0$ holds. Indeed, in Figure 5.6 we observe that only the first objective exhibits the performance estimate in Theorem 5.11.

In what follows we will illustrate the results stated in Lemma 5.12. Let us first note that the example satisfies all assumptions of the lemma. This implies that for any choice of $x_* \in \mathcal{E}$, we obtain $x^{\mu^N}(k, x_0) \to x_*$ even if neither ℓ_1 nor ℓ_2 is strictly dissipative at (x_*, u_*). In Figure 5.7 we observe that this indeed holds true for our example. Moreover, we see

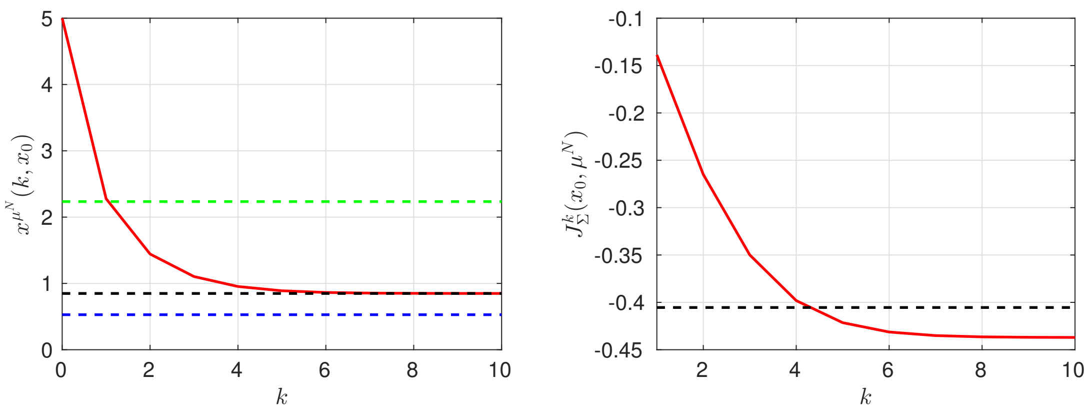

Figure 5.7: Closed-loop trajectory (red, solid) of Algorithm 2 using $\mathbb{X}_0 = \{x_*\}$ for some Pareto-optimal steady state x_* (black, dashed) and the optimal steady states for ℓ_1 and ℓ_2 (green and blue, dashed) on the left and the corresponding non-averaged performance (red) of the 'artificial' objective J_Σ^N $(N = 3)$ on the right.

that the non-averaged performance of the weighted objective J_Σ is bounded as stated in Theorem 5.11.

Let us now investigate what happens if we choose a terminal condition $x_* \in \mathbb{X}$ with $x_* \neq x_1^e$, $x_* \neq x_2^e$ and $x_* \notin \mathcal{E}$. In Figure 5.8 we used the terminal constraint $x_* = 0.1$, which is an equilibrium, but not a Pareto-optimal one. We observe that the trajectory neither converges to the terminal condition nor to any of the optimal steady states. Thus, the requirement $x_* \in \mathcal{E}$ in Lemma 5.12 is needed.

5.2 MO Economic MPC without Terminal Conditions

In this section we deal with MO OCPs with economic stage costs and without terminal conditions, i.e. we waive Assumption 5.1. In other words we aim to generalize the results in Chapter 2 to our MO setting. As in the previous section dissipativity is our main tool for analyzing such schemes.

For cost criteria ℓ_i that are strictly dissipative at (x_i^e, u_i^e), the relation between the

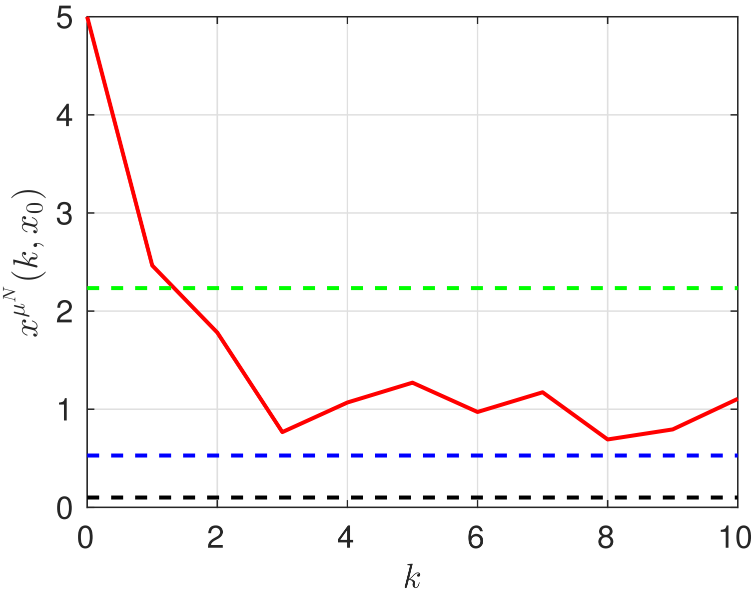

Figure 5.8: Closed-loop trajectory (red, solid) of Algorithm 2 using $\mathbb{X}_0 = \{0.1\}$ and $N = 3$ (black, dashed), and x_1^e (green, dashed), x_2^e (blue, dashed).

rotated and the original cost functional is as follows:

$$
\begin{aligned}
\tilde{J}_i^N(x, \mathbf{u}) &= \sum_{k=0}^{N-1} \tilde{\ell}_i(x(k), u(k)) \\
&= \sum_{k=0}^{N-1} \ell_i(x(k), u(k)) - \ell_i(x_i^e, u_i^e) + \lambda_i(x(k)) - \lambda_i(x(k+1)) \\
&= J_i^N(x, \mathbf{u}) - N\ell_i(x_i^e, u_i^e) + \lambda_i(x) - \lambda_i(x(N))
\end{aligned}
$$

In the absence of terminal conditions the value $\lambda_i(x(N))$ is neither fixed nor compensated for by terminal conditions, which implies that POSs to the rotated and the original MO OCP do not coincide. Consequently, an analysis as in Lemma 5.8 and below can not be conducted. In Section 2.2 we have explained that – under the turnpike property – optimal trajectories approach the optimal steady state. This behavior is exploited in the proofs in Chapter 2. Figuratively spoken one could say that the MPC closed loop is led in the right direction if the turnpike property holds.

In the presence of multiple objectives the situation can become more complex as we might be dealing with MO OCPs which are strictly dissipative at different steady states. Since dissipativity with a bounded storage function implies the turnpike property – as stated in Lemma 5.14 – this also leads to turnpike behavior wrt different steady states, see Figures 5.10, 5.11 and 5.12 for an illustration of the non-uniform turnpike property.

Lemma 5.14 (Dissipativity implies turnpike behavior)**.** *Let $i \in \{1, \dots, s\}$ be an index such that the MO OCP* (4.1) *is strictly dissipative wrt ℓ_i at (x_i^e, u_i^e) in the sense of Definition 5.3 and with λ_i bounded on $\mathbb{X}$. Then, for each $\delta > 0$ there exists $\sigma_\delta^i \in \mathcal{L}$ such that for all $x \in \mathbb{X}$, N, $P \in \mathbb{N}$ and $\mathbf{u} \in \mathbb{U}^N(x)$ with $J_i^N(x, \mathbf{u}) \leq N\ell_i(x_i^e, u_i^e) + \delta$ it holds that the set $\mathcal{D}_i(x, \mathbf{u}, P, N) := \{k \in \{0, \dots, N-1\} | \|x^{\mathbf{u}}(k, x) - x_i^e\| \geq \sigma_\delta^i(P)\}$ has at most P elements (short notation $\#\mathcal{D}_i(x, \mathbf{u}, P, N) \leq P$).*

Proof. The proof of the lemma is completely analogous to the proof of [32, Prop. 8.15] and is therefore omitted. We just mention that $\sigma_\delta^i(P) := \rho_i^{-1}((2M_i + \delta)/P)$, in which M_i is the assumed bound on λ_i. □

5.2.1 Uniformly Dissipative MO OCPs

In order to avoid turpike behavior wrt different steady states we will first consider the case of uniform dissipativity.

Assumption 5.15 (Uniform Dissipativity and Uniform Continuity)**.**

1. *The MO OCP* (4.1) *is uniformly strictly dissipative at (x^e, u^e) in the sense of Definition 5.3.*

2. *There are $\gamma_{J_i} \in \mathcal{K}_\infty$ and $\omega_i \in \mathcal{L}$ such that for all $x \in \mathbb{X}$, $N \in \mathbb{N}$, $\mathbf{u}^\star \in \mathbb{U}_{\mathcal{P}}^N(x)$ and all $\mathbf{u}_{x^e}^\star \in \mathbb{U}_{\mathcal{P}}^N(x^e)$ the inequalities*

$$|J_i^N(x, \mathbf{u}^\star) - J_i^N(x^e, \mathbf{u}_{x^e}^\star)| \leq \gamma_{J_i}(\|x - x^e\|) + \omega_i(N).$$

   hold for all $i \in \{1, \dots, s\}$ and x^e from the first assumption.

The second part of Assumption 5.15 can be seen as a counterpart of Assumption 5.10. The additional term $\omega_i(N)$ reflects the lack of a terminal condition. We conjecture that unless an explicit formula for nondominated values of the underlying problem is known, it will be a difficult task to verify this assumption.

Under the assumption of uniform strict dissipativity, the turnpike property becomes uniform, too, see Figure 5.9 for an illustration of the uniform turnpike property based on the example in Section 5.1.4 without terminal conditions. This means that $x_i^e = x^e$ for all $i \in \{1, \dots, s\}$ in Lemma 5.14. Let us now fix $\delta > 0$ and pick arbitrary $x \in \mathbb{X}$, N, $P \in \mathbb{N}$ and $\mathbf{u} \in \mathbb{U}^N(x)$ such that $J_i^N(x, \mathbf{u}) \leq N\ell_i(x^e, u^e) + \delta$ holds for all $i \in \{1, \dots, s\}$. Lemma 5.14 then yields $\#\mathcal{D}_i(x, \mathbf{u}, P, N) \leq P$ for all $i \in \{1, \dots, s\}$. The sets $\mathcal{D}_i$ refer to different distances of the trajectory to x^e, i.e. if $\sigma_\delta^i(P) \geq \sigma_\delta^j(P)$ for i, $j \in \{1, \dots, s\}$, then $\mathcal{D}_i(x, \mathbf{u}, P, N) \subseteq \mathcal{D}_j(x, \mathbf{u}, P, N)$. Hence, for $k \in \operatorname{argmin}_{i \in \{1,\dots,s\}} \sigma_\delta^i(P)$ and $j \in \operatorname{argmax}_{i \in \{1,\dots,s\}} \sigma_\delta^i(P)$ we have

$$\underbrace{\mathcal{D}_j(x, \mathbf{u}, P, N)}_{= \cap_{i \in \{1,\dots,s\}} \mathcal{D}_i(x, \mathbf{u}, P, N)} \subseteq \mathcal{D}_i(x, \mathbf{u}, P, N) \subseteq \underbrace{\mathcal{D}_k(x, \mathbf{u}, P, N)}_{= \cup_{i \in \{1,\dots,s\}} \mathcal{D}_i(x, \mathbf{u}, P, N)} \qquad \forall i \in \{1, \dots, s\}.$$

Consequently, by uniform strict dissipativity the inequalities $J_i^N(x, \mathbf{u}) \leq N\ell_i(x^e, u^e) + \delta$ ensure $\|x^{\mathbf{u}}(M, x) - x^e\| < \sigma_\delta^k(P)$ for all $M \notin \mathcal{D}_k(x, \mathbf{u}, P, N)$. We will make use of these

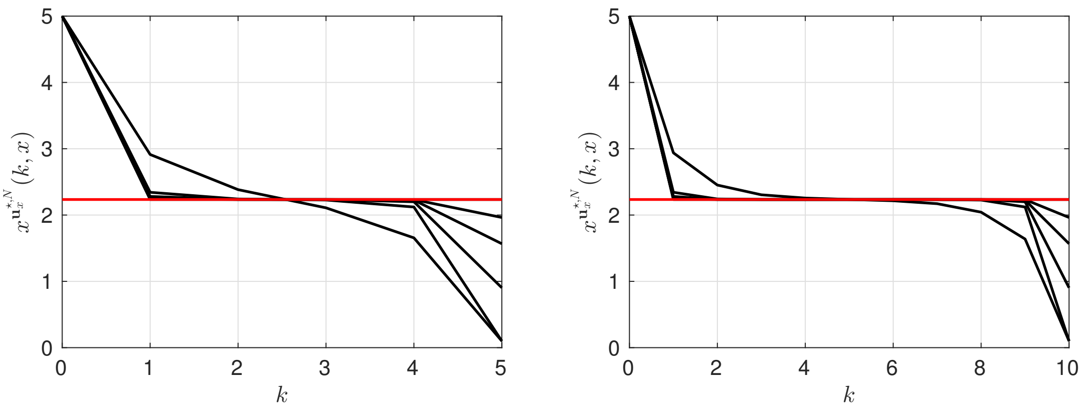

Figure 5.9: Open-loop Pareto optimal trajectories (black) for horizon $N = 5$ (left) and $N = 10$ (right) for the uniformly strictly dissipative economic growth example from Section 5.1.4 without terminal conditions, and the equilibrium (red) from dissipativity.

considerations in the next two lemmas, which serve the purpose to establish a relation between nondominated values of length N and $N-1$.

Lemma 5.16. *Let Assumption 5.15 and the assumptions of Lemma 5.14 hold. Let the set $\mathcal{J}_{\mathcal{P}}^N(x^e)$ be externally stable for all $N \in \mathbb{N}$. Then, for all $x \in \mathbb{X}$, $N \in \mathbb{N}$ and all $\mathbf{u}_x^{\star,N} \in \mathbb{U}_{\mathcal{P}}^N(x)$, $\mathbf{u}_{x^e}^{\star,N} \in \mathbb{U}_{\mathcal{P}}^N(x^e)$ the relation*

$$J_i^N(x, \mathbf{u}_x^{\star,N}) = J_i^M(x, \mathbf{u}_x^{\star,N}) + J_i^{N-M}(x^e, \mathbf{u}_{x^e}^{\star,N}) + R_{1,i}(x, M, N)$$

holds for all $i \in \{1, \dots, s\}$ with $|R_{1,i}(x, M, N)| \le \gamma_{J_i}(\sigma_\delta^k(P)) + \omega_i(N-M)$ for all $P \in \mathbb{N}$, all $M \notin \mathcal{D}_k(x, \mathbf{u}_x^{\star,N}, P, N)$ and σ_δ^k from Lemma 5.14 with $\delta = \max_{i\in\{1,\dots,s\}} (\gamma_{J_i}(\|x - x^e\|) + \omega_i(N))$ and $k \in \operatorname{argmin}_{i\in\{1,\dots,s\}} \sigma_\delta^i(P)$.

Proof. Here, and also in Lemma 5.17 we follow the reasoning of [32, Section 8.5]; however, we provide details since choosing the proper POSs might not be obvious.

For $x \in \mathbb{X}$ and $N \in \mathbb{N}$ consider the control $\mathbf{u} = (u^e, \dots, u^e) \in \mathbb{U}^N(x^e)$ with objective values $J_i^N(x^e, \mathbf{u}) = N\ell_i(x^e, u^e)$. External stability of $\mathcal{J}_{\mathcal{P}}^N(x^e)$ guarantees the existence of $\mathbf{u}_{x^e}^\star \in \mathbb{U}_{\mathcal{P}}^N(x^e)$ such that $J_i^N(x^e, \mathbf{u}_{x^e}^\star) \le N\ell_i(x^e, u^e)$ holds for all $i \in \{1, \dots, s\}$. By Assumption 5.15

$$\begin{aligned} J_i^N(x, \mathbf{u}_x^{\star,N}) &\le J_i^N(x^e, \mathbf{u}_{x^e}^\star) + \gamma_{J_i}(\|x - x^e\|) + \omega_i(N) \\ &\le N\ell_i(x^e, u^e) + \gamma_{J_i}(\|x - x^e\|) + \omega_i(N) \le \ell_i(x^e, u^e) + \delta \end{aligned}$$

holds for all $\mathbf{u}_x^{\star,N} \in \mathbb{U}_{\mathcal{P}}^N(x)$, all $i \in \{1, \dots, s\}$ and $\delta = \max_{i\in\{1,\dots,s\}} \gamma_{J_i}(\|x - x^e\|) + \omega_i(N)$. Our preliminary considerations reveal that for all $M \notin \mathcal{D}_k(x, \mathbf{u}_x^{\star,N}, P, N)$ we have $\|x^{\mathbf{u}_x^{\star,N}}(M, x) - x^e\| < \sigma_\delta^k(P)$, in which $k \in \operatorname{argmin}_{i\in\{1,\dots,s\}} \sigma_\delta^i(P)$.

By definition we have

$$\begin{aligned}J_i^N(x,\mathbf{u}_x^{\star,N}) &= J_i^M(x,\mathbf{u}_x^{\star,N}) + J_i^{N-M}(x^{\mathbf{u}_x^{\star,N}}(M,x),\mathbf{u}_x^{\star,N}(\cdot+M))\\ &= J_i^M(x,\mathbf{u}_x^{\star,N}) + J_i^{N-M}(x^e,\mathbf{u}_{x^e}^{\star,N-M})) + R_{1,i}(x,M,N)\end{aligned}$$

for $R_{1,i}(x,M,N) := J_i^{N-M}(x^{\mathbf{u}_x^{\star,N}}(M,x),\mathbf{u}_x^{\star,N}(\cdot+M)) - J_i^{N-M}(x^e,\mathbf{u}_{x^e}^{\star,N-M}))$ and arbitrary $\mathbf{u}_{x^e}^{\star,N-M} \in \mathbb{U}_{\mathcal{P}}^{N-M}(x^e)$. Using Assumption 5.15 we have the following bound on the error term: $|R_{1,i}(x,M,N)| \le \gamma_{J_i}(\|x^{\mathbf{u}_x^{\star,N}}(M,x)-x^e\|)+\omega_i(N-M) \le \gamma_{J_i}(\sigma_\delta^k(P))+\omega_i(N-M)$. □

Lemma 5.17. *Let Assumption 5.15 and the assumptions of Lemma 5.14 hold. Assume external stability of the sets $\mathcal{J}_{\mathcal{P}}^N(x)$ for all $x \in \mathbb{X}$ and $N \in \mathbb{N}$. Then, for all $x \in \mathbb{X}$, all $N \in \mathbb{N}$ and all $\mathbf{u}_x^{\star,N-1} \in \mathbb{U}_{\mathcal{P}}^{N-1}(x)$ there exists $\mathbf{u}_x^{\star,N} \in \mathbb{U}_{\mathcal{P}}^N(x)$ such that*

$$J_i^N(x,\mathbf{u}_x^{\star,N}) \le J_i^{N-1}(x,\mathbf{u}_x^{\star,N-1}) + \ell_i(x^e,u^e) + R_{2,i}(x,N)$$

holds for all $i \in \{1,\dots,s\}$ with $|R_{2,i}(x,N)| \le 2\gamma_{J_i}(\sigma_\delta^k(\lfloor N/2\rfloor)) + 2\omega_i(\lfloor N/2\rfloor)$ for σ_δ^k from Lemma 5.14 with $\delta = \max\limits_{i\in\{1,\dots,s\}}(\gamma_{J_i}(\|x-x^e\|)+\omega_i(N-1))$ and $k \in \operatorname*{argmin}\limits_{i\in\{1,\dots,s\}} \sigma_\delta^i(\lfloor N/2\rfloor)$.

Proof. For $x \in \mathbb{X}$, horizon $N-1$ and $P = \lfloor N/2\rfloor$ consider an arbitrary solution to (4.2) $\mathbf{u}_x^{\star,N-1} \in \mathbb{U}_{\mathcal{P}}^{N-1}(x)$. Since $\#\mathcal{D}_k(x,\mathbf{u}_x^{\star,N-1},P,N-1) \le \lfloor N/2\rfloor$, the set $\{0,\dots,N-2\}\setminus\mathcal{D}_k$ is non empty and we can apply Lemma 5.16 for $M \in \{0,\dots,\lfloor N/2\rfloor-1\}$. For all $\mathbf{u}_{x^e}^{\star,N-M-1} \in \mathbb{U}_{\mathcal{P}}^{N-M-1}(x^e)$ the relation

$$J_i^{N-1}(x,\mathbf{u}_x^{\star,N-1}) = J_i^M(x,\mathbf{u}_x^{\star,N-1}) + J_i^{N-M-1}(x^e,\mathbf{u}_{x^e}^{\star,N-M-1}) + R_{1,i}(x,M,N-1)$$

holds for all $i \in \{1,\dots,s\}$ and with $|R_{1,i}(x,M,N-1)| \le \gamma_{J_i}(\sigma_\delta^k(\lfloor N/2\rfloor)) + \omega_i(N-1)$ and $\delta = \max_{i\in\{1,\dots,s\}}\gamma_{J_i}(\|x-x^e\|)+\omega_i(N-1)$. Moreover, for $\bar{x} := x^{\mathbf{u}_x^{\star,N-1}}(M,x)$ it holds $\|\bar{x}-x^e\| < \sigma_\delta^k(\lfloor N/2\rfloor)$.

For $\mathbf{u}_{x^e}^{\star,N-M-1} \in \mathbb{U}_{\mathcal{P}}^{N-M-1}(x^e)$ the concatenated control $(u^e,\mathbf{u}_{x^e}^{\star,N-M-1})$ is a feasible control sequence of length $N-M$ for initial value x^e. By external stability we hence get the existence of $\mathbf{u}_{x^e}^{\star,N-M} \in \mathbb{U}_{\mathcal{P}}^{N-M}(x^e)$ such that

$$J_i^{N-M}(x^e,\mathbf{u}_{x^e}^{\star,N-M}) \le \ell_i(x^e,u^e) + J_i^{N-M-1}(x^e,\mathbf{u}_{x^e}^{\star,N-M-1})$$

holds for all $i \in \{1,\dots,s\}$. Moreover, Assumption 5.15 ensures the relation

$$|J_i^{N-M}(\bar{x},\mathbf{u}^{\star,N-M}) - J_i^{N-M}(x^e,\mathbf{u}_{x^e}^{\star,N-M})| \le \gamma_{J_i}(\|\bar{x}-x^e\|) + \omega_i(N-M)$$

for all $i \in \{1,\dots,s\}$ and all $\mathbf{u}^{\star,N-M} \in \mathbb{U}_{\mathcal{P}}^{N-M}(\bar{x})$.

Let us now define $\mathbf{u} \in \mathbb{U}^N(x)$ via

$$u(k) = \begin{cases} u_x^{\star,N-1}(k), & k = 0,\dots,M-1\\ u^{\star,N-M}(k-M), & k = M,\dots,N-1\end{cases}$$

for arbitrary $\mathbf{u}^{\star,N-M} \in \mathbb{U}_{\mathcal{P}}^{N-M}(\bar{x})$. For all $i \in \{1,\ldots,s\}$ we obtain

$$\begin{aligned} J_i^N(x,\mathbf{u}) &= J_i^M(x,\mathbf{u}_x^{\star,N-1}) + J_i^{N-M}(\bar{x},\mathbf{u}^{\star,N-M}) \\ &= J_i^M(x,\mathbf{u}_x^{\star,N-1}) + J_i^{N-M}(x^e,\mathbf{u}_{x^e}^{\star,N-M}) + \hat{R}_{1,i}(x,M,N), \end{aligned}$$

in which $|\hat{R}_{1,i}(x,M,N)| \le \gamma_{J_i}(\sigma_\delta^k(\lfloor(N/2\rfloor)) + \omega_i(\lfloor(N/2\rfloor)$.

Putting everthing together yields

$$\begin{aligned} & J_i^{N-1}(x,\mathbf{u}_x^{\star,N-1}) + \ell_i(x^e,u^e) \\ =& J_i^M(x,\mathbf{u}_x^{\star,N-1}) + J_i^{N-M-1}(x^e,\mathbf{u}_{x^e}^{\star,N-M-1}) + R_{1,i}(x,M,N-1) + \ell_i(x^e,u^e) \\ \ge& J_i^M(x,\mathbf{u}_x^{\star,N-1}) + J_i^{N-M}(x^e,\mathbf{u}_{x^e}^{\star,N-M}) + R_{1,i}(x,M,N-1) \\ =& J_i^N(x,\mathbf{u}) - \hat{R}_{1,i}(x,M,N) + R_{1,i}(x,M,N-1) \end{aligned}$$

for all $i \in \{1,\ldots,s\}$. Due to external stability there exists $\mathbf{u}_x^{\star,N} \in \mathbb{U}_{\mathcal{P}}^N(x)$ such that $J_i^N(x,\mathbf{u}_x^{\star,N}) \le J_i^N(x,\mathbf{u})$ holds for all $i \in \{1,\ldots,s\}$ and we finally obtain

$$J_i^{N-1}(x,\mathbf{u}_x^{\star,N-1}) + \ell_i(x^e,u^e) + R_{2,i}(x,N) \ge J_i^N(x,\mathbf{u}_x^{\star,N})$$

for all $i \in \{1,\ldots,s\}$ and $|R_{2,i}(x,N)| \le 2\gamma_{J_i}(\sigma_\delta^k(\lfloor N/2\rfloor)) + 2\omega_i(\lfloor N/2\rfloor)$. □

Remark 5.18. *Lemma 5.16 has been stated for $M \notin \mathcal{D}_k(x,\mathbf{u}_x^{\star,N},P,N)$ for the index $k \in \operatorname{argmin}_{i\in\{1,\ldots,s\}} \sigma_\delta^i(P)$, because this yields the smallest error term $R_{1,i}$ for all $i \in \{1,\ldots,s\}$, which also carries over to the error terms $R_{2,i}$ in Lemma 5.17. However, all the statements remain true if we pick $M \notin \mathcal{D}_i(x,\mathbf{u}_x^{\star,N},P,N)$ for an arbitrary but fixed $i \in \{1,\ldots,s\}$. In this case, the error terms depend on the value $\sigma_\delta^i(P)$.*

The following assumption similar to [30, Definition 2.2] ensures that for Pareto-optimal trajectories it is – in terms of the cost functional – not worse to move to the optimal equilibrium x^e from Assumption 5.15 than staying away from it.

Assumption 5.19 (Cheap reachability)**.** *We assume that the steady state x^e from Assumption 5.15 is* cheaply reachable, *i.e. we assume that there exists $E \in \mathbb{R}$ such that for all $x \in \mathbb{X}$, $N \in \mathbb{N}$ and all $\mathbf{u}^\star \in \mathbb{U}_{\mathcal{P}}^N(x)$ the inequality*

$$J_i^N(x,\mathbf{u}^\star) \le N\ell_i(x^e,u^e) + E$$

holds for all $i \in \{1,\ldots,s\}$.

The reason for imposing Assumption 5.19 is that up to now, the error terms in Lemmas 5.16 and 5.17 depend on the distance $\|x - x^e\|$, which can become arbitrarily large if $\mathbb{X}$ is unbounded.

Corollary 5.20. *Let Assumption 5.19 and the assumptions of Lemma 5.17 hold. Then, for all $x \in \mathbb{X}$, all $N \in \mathbb{N}$ and all $\mathbf{u}_x^{\star,N-1} \in \mathbb{U}_{\mathcal{P}}^{N-1}(x)$ there exists $\mathbf{u}_x^{\star,N} \in \mathbb{U}_{\mathcal{P}}^N(x)$ such that*

$$J_i^N(x,\mathbf{u}_x^{\star,N}) \le J_i^{N-1}(x,\mathbf{u}_x^{\star,N-1}) + \ell_i(x^e,u^e) + \delta_i(N)$$

holds for all $i \in \{1,\ldots,s\}$ with $|\delta_i(N)| \le 2\gamma_{J_i}(\sigma_E^k(\lfloor N/2\rfloor)) + 2\omega_i(\lfloor N/2\rfloor)$ for σ_E^k from Lemma 5.14, E from Assumption 5.19 and $k \in \operatorname*{argmin}_{i\in\{1,\ldots,s\}} \sigma_E^i(\lfloor N/2\rfloor)$.

Proof. If x^e is cheaply reachable, i.e. Assumption 5.19 holds, we can apply Lemma 5.17 with $\delta = E$ and $k \in \operatorname{argmin}_{i\in\{1,\dots,s\}} \sigma_E^i(\lfloor N/2 \rfloor)$ and obtain the error term $|\delta_i(N)| = |R_{2,i}(x,N)| \leq 2\gamma_{J_i}(\sigma_E^k(\lfloor N/2 \rfloor)) + 2\omega_i(\lfloor N/2 \rfloor)$, which no longer depends on x. □

By means of our preliminary considerations we are now able to formulate an algorithm for MO Economic MPC without terminal conditions. In order to execute the algorithm, the following information must be available: The functions σ_E^i from the turnpike property (see Lemma 5.14) for E from Assumption 5.19 and the functions γ_{J_i} and ω_i from Assumption 5.15.

Algorithm 5 (MO Economic MPC without Terminal Conditions).

(0) At time $n = 0$: Find $\delta_i \in \mathcal{L}$ such that $|\delta_i(N)| \leq 2\gamma_{J_i}(\sigma_E^k(\lfloor N/2 \rfloor)) + 2\omega_i(\lfloor N/2 \rfloor)$ for all $i \in \{1,\dots,s\}$. Set $x(n) := x_0$ and choose a POS $\mathbf{u}^{\star,N}_{x(n)} \in \mathbb{U}^N_{\mathcal{P}}(x(n))$ to (4.2). Go to **(2)**.

(1) At time $n \in \mathbb{N}$: Choose a POS $\mathbf{u}^{\star,N}_{x(n)}$ to (4.2) such that the inequalities

$$J_i^N\left(x(n), \mathbf{u}^{\star,N}_{x(n)}\right) \leq J_i^{N-1}\left(x(n), \mathbf{u}^{N-1}_{x(n)}\right) + \ell_i(x^e,u^e) + \delta_i(N)$$

are satisfied for all $i \in \{1,\dots,s\}$.

(2) Define $\mathbf{u}^{N-1}_{x(n+1)} := \mathbf{u}^{\star,N}_{x(n)}(\cdot + 1)$

(3) Apply $\mu^N(x(n)) := u^{\star,N}_{x(n)}(0)$, set $n = n+1$ and go to **(1)**.

Theorem 5.21 (Averaged infinite-horizon performance)**.** *Consider $x_0 \in \mathbb{X}$ and $N \in \mathbb{N}$, and let Assumptions 5.15 and 5.19 hold. Let the set $\mathcal{J}^N_{\mathcal{P}}(x)$ be externally stable for all $x \in \mathbb{X}$, assume that the storage functions λ_i are bounded on $\mathbb{X}$ for all $i \in \{1,\dots,s\}$ and that the values $J_i^N(x,\mathbf{u}^{\star,N}_x)$ are bounded from below for all $x \in \mathbb{X}$ and all $\mathbf{u}^{\star,N}_x \in \mathbb{U}^N_{\mathcal{P}}(x)$. Then, for each $N \in \mathbb{N}_{\geq 2}$ the MPC feedback μ^N defined in Algorithm 5 yields the averaged infinite-horizon performance*

$$\bar{J}_i^\infty(x_0,\mu^N) \leq \ell_i(x^e,u^e) + \delta_i(N)$$

for all cost criteria $i \in \{1,\dots,s\}$ and $\delta_i \in \mathcal{L}$ as defined in step **(0)** *of the algorithm.*

Proof. **Feasibility:** External stability of $\mathcal{J}^N_{\mathcal{P}}(x_0)$ ensures existence of POSs and thus step **(0)** in Algorithm 5 is feasible. Let us consider step **(1)**. Since $\mathbf{u}^{N-1}_{x(n)}$ is the tail of a POS (see step **(2)**), by Lemma 4.1 we know that $\mathbf{u}^{N-1}_{x(n)} \in \mathbb{U}^{N-1}_{\mathcal{P}}(x(n))$. Then we apply Lemma 5.17 to conclude feasibility of step **(1)**.

Performance:

$$
\begin{aligned}
\bar{J}_i^\infty(x_0, \mu^N) &= \limsup_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell_i(x(k), \mu^N(x(k))) \\
&= \limsup_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \left[J_i^N(x(k), \mathbf{u}^{\star,N}_{x(k)}) - J_i^{N-1}(x(k+1), \mathbf{u}^{\star,N}_{x(k)}(\cdot+1)) \right] \\
&\le \ell_i(x^e, u^e) + \delta_i(N) \\
&\qquad + \limsup_{K\to\infty} \frac{1}{K} \left(J_i^N(x_0, \mathbf{u}^{\star,N}_{x_0}) - J_i^{N-1}(x(K), \mathbf{u}^{\star,N}_{x(K-1)}(\cdot+1)) \right) \\
&\le \ell_i(x^e, u^e) + \delta_i(N) + \limsup_{K\to\infty} \frac{1}{K} \left(J_i^N(x_0, \mathbf{u}^{\star,N}_{x_0}) - M \right) \\
&= \ell_i(x^e, u^e) + \delta_i(N),
\end{aligned}
$$

in which the first inequality follows from step **(1)** in Algorithm 5 and the second inequality from the assumed bound on nondominated values. □

Theorem 5.21 states that the averaged infinite-horizon performance of the MPC controller from Algorithm 5 is bounded from above by the stage cost of the steady state (x^e, u^e) up to an error term that vanishes as N tends to infinity. This demonstrates the fact that also in the absence of terminal conditions it is possible to obtain upper bounds on the performance for each objective. The difficulty lies in the aspect that the choice of the POSs in Algorithm 5 depends on an error term, which stems from the functions in Assumption 5.15. Finding appropriate $\delta_i \in \mathcal{L}$ in Algorithm 5 seems to be a difficult task and it will be part of our future research how to tackle this problem.

5.2.2 Dissipative MO OCPs

In the last section we have given evidence that MPC schemes for MO economic OCPs require much (quantitative) knowledge on the structure of the problem, which is in general hard to obtain. In case that the MO OCP is non-uniformly strictly dissipative, i.e. the cost criteria are strictly dissipative at different Pareto-optimal steady states, the situation gets even more involved. To illustrate the difficulties, let us reconsider the example in Section 5.1.4 (which is an extension of Example 2.14):

$$
x^+ = u, \quad \ell_i(x,u) = -\ln(A_i x^{\alpha_i} - u), \; i \in \{1,2\}
$$

in which $A_1 = 5$, $\alpha_1 = 0.34$, $A_2 = 3$, $\alpha_1 = 0.2$. The constraints are given by $\mathbb{X} = [0, 10]$ and $\mathbb{U} = [0.1, 5]$. In this section, no terminal costs and constraints are imposed (i.e. $\mathbb{X}_0 = \mathbb{X}$ and $F_i \equiv 0$ for all $i \in \{1,2\}$). From previous considerations we already know that ℓ_1 is strictly dissipative at $x_1^e \approx 2.23$ and ℓ_2 at $x_2^e \approx 0.53$. The storage functions λ_i are linear functions (see Section 5.1.4) and since $\mathbb{X}$ is bounded, λ_i is bounded on $\mathbb{X}$ for $i \in \{1,2\}$. Therefore, we can apply Lemma 5.14 for $i \in \{1,2\}$ to establish that there are trajectories, which exhibit turnpike behavior wrt x_1^e as well as wrt x_2^e, if we are able to proof existence of POSs $\mathbf{u}$ such that $J_i^N(x, \mathbf{u}) \le N\ell_i(x_i^e, u_i^e) + \delta$.

Following the reasoning of (step 1 of) the proof of Theorem 2.13, for each $x \in \mathbb{X}$ and each $i \in \{1, 2\}$ we can find $\hat{\mathbf{u}} \in \mathbb{U}^N(x)$ such that $\tilde{J}_i^N(x, \hat{\mathbf{u}}) \leq \alpha_i(\|x - x_i^e\|)$ holds. Since $\tilde{J}_i^N(x, \mathbf{u}) = J_i^N(x, \mathbf{u}) - N\ell_i(x_i^e, u_i^e) + \lambda_i(x) - \lambda_i(x^{\mathbf{u}}(N, x))$, we get the estimate

$$J_i^N(x, \hat{\mathbf{u}}) \leq \alpha_i(\|x - x_i^e\|) + 2C_i + N\ell_i(x_i^e, u_i^e),$$

in which $C_i \in \mathbb{R}_{>0}$ is a bound on λ_i. In our example, the sets $\mathcal{J}_{\mathcal{P}}^N(x)$ are externally stable for each $N \in \mathbb{N}$ and each $x \in \mathbb{X}$ (see Theorem 4.8), i.e. there exists $\mathbf{u}^\star \in \mathbb{U}_{\mathcal{P}}^N(x)$ satisfying $J_i^N(x, \mathbf{u}^\star) \leq \alpha_i(\|x - x_i^e\|) + 2C_i + N\ell_i(x_i^e, u_i^e)$. Hence, $\mathbf{u}^\star$ is a POS, which exhibits turnpike behavior wrt x_i^e.

In Figure 5.10 we see that the red Pareto-optimal open-loop trajectories show turnpike behavior wrt x_1^e and the blue ones wrt x_2^e. We conjecture that one could enforce averaged

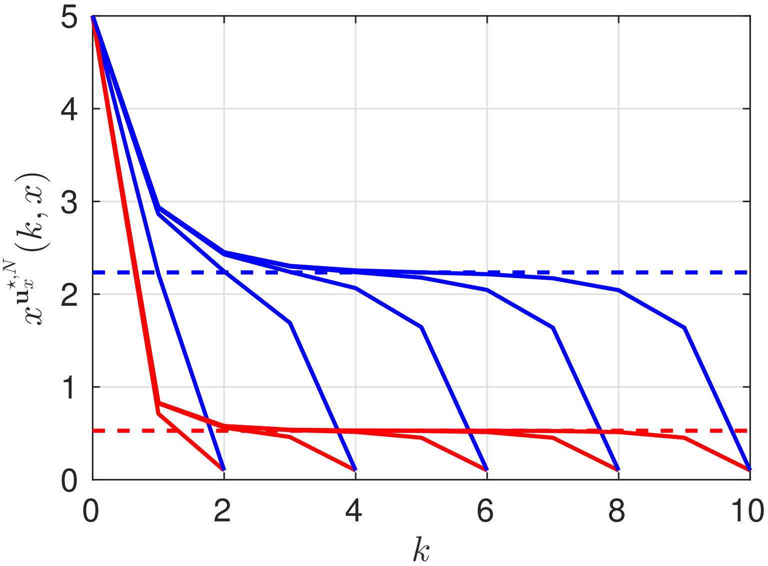

Figure 5.10: Pareto-optimal open-loop trajectories (solid) of horizon $N \in \{2, 4, 6, 8, 10\}$ and for initial value $x = 5$, and the steady states that ℓ_1 and ℓ_2 are strictly dissipative at (dashed blue and red, respectively). Blue trajectories exhibit turnpike behavior wrt x_1^e, red trajectories wrt x_2^e.

performance as stated in Theorem 5.21 if one is able to find very tight error terms because x_1^e and x_2^e are not 'too close'. The bad news is that there is a continuum of Pareto-optimal steady states and for each of them there are Pareto-optimal open-loop trajectories which exhibit turnpike behavior, see Figure 5.11. In this situation it might happen that the constraint in step **(1)** of Algorithm 5 does not only include trajectories with turnpike behavior wrt one specific steady state but also wrt to neighboring steady states. The question whether this prevents the closed loop from convergence remains open.

Numerical experiments in Figure 5.12 reveal that the closed-loop trajectory converges into the set $\mathcal{E}$ of Pareto-optimal steady states without imposing any recursive constraint

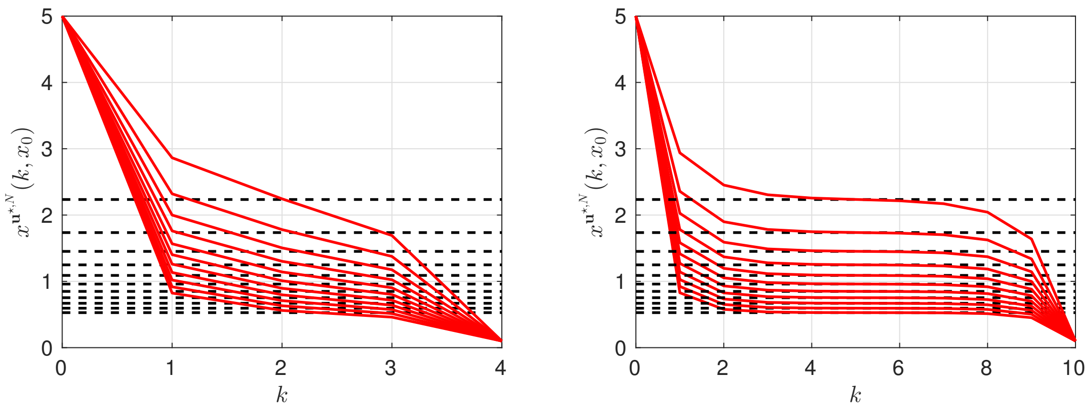

Figure 5.11: The open-loop Pareto-optimal trajectories (red, solid) exhibit turnpike behavior wrt different Pareto-optimal steady states (black, dashed) for $N = 4$ (left) and $N = 10$ (right).

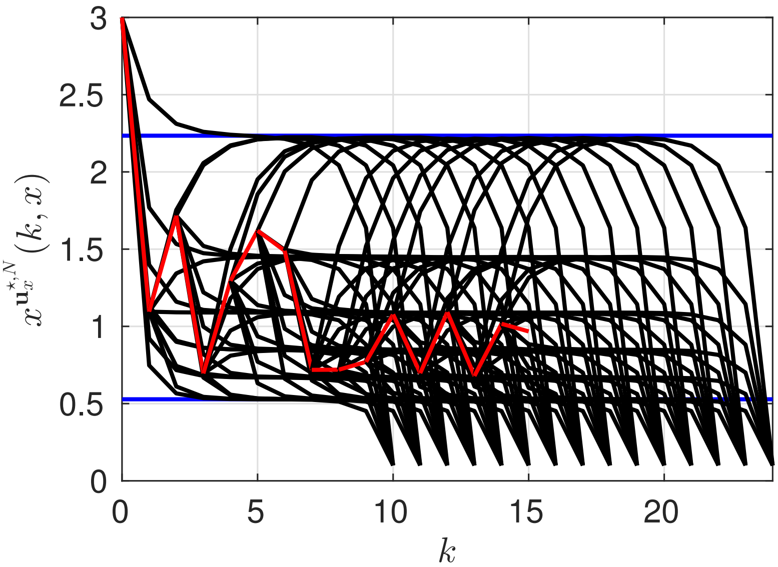

Figure 5.12: MPC closed-loop trajectory (red) and Pareto-optimal open-loop trajectories (black) without terminal conditions and recursive constraints for the non-uniformly strictly dissipative economic growth example using $N = 10$.

and by choosing an arbitrary POS to the MO optimization problem in each iteration. It will be part of future research to prove this behavior and to investigate what happens in case the set $\mathcal{E}$ is not connected (as opposed to the example here).

6 | Noncooperative Model Predictive Control

In the previous Chapters 4 and 5 we were concerned with the question how optimal control problems (OCPs) can be approximated by means of Model Predictive Control (MPC) in a cooperative fashion, which naturally led to the concept of Pareto-optimality and multiobjective (MO) optimization. In this chapter we assume that MPC is carried out *noncooperatively*. The wording noncooperative means that different players of a game are either not willing or not able to cooperate, e.g. due to a lack of trust or information or because of corporate secrets, see e.g. [4, 84]. In such a setting it cannot be expected that a (Pareto-)optimal strategy can be found, see e.g. [14, 74, 79]. The game-theoretic literature proposes different solution concepts to such noncooperative games, such as *Nash equilibria (NE)*, *subgame perfect equilibria* and *Stackelberg equilibria*, see e.g. [86]. In our analysis we focus on Nash-based MPC (i.e. MPC schemes that implement the first piece of a NE in each iteration), an approach that is also pursued in [53, 54, 75, 92]. In these references, the proposed MPC controller is usually designed and tested for a specific application. MPC based on subgame perfect equilibria is e.g. performed in [80] for smart grids.

Existence and uniqueness of NE heavily rely on the structure of the game under consideration. This is in contrast to scalar-valued and MO optimization problems, for which regularity properties such as continuity of the occuring functions and closedness or compactness of constraint sets yield existence and a certain structure of (Pareto-)optimal solutions. This is why we focus on the following setting: Let there be $s \in \mathbb{N}$ *players*, who can influence the system dynamics

$$x^+ = f(x, u_1, \dots, u_s) \tag{6.1}$$

with $f : \mathbb{R}^n \times \underbrace{\mathbb{U}_1 \times \dots \times \mathbb{U}_s}_{\mathbb{U}} \to \mathbb{R}^n$ through their inputs $u_i \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$, $i \in \{1, \dots, s\}$. The set $\mathbb{U}_i$ denotes players i's control constraint set, the state $x \in \mathbb{R}^n$ is assumed to be unconstrained. Based on player i's cost function $\ell_i : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}$ we define the cost functional for initial value $x \in \mathbb{R}^n$ and horizon $N \in \mathbb{N}$

$$J_i^N(x, \mathbf{u}_1, \dots, \mathbf{u}_s) := \sum_{k=0}^{N-1} \ell_i(x(k, x), u_1(k), \dots, u_s(k)) \tag{6.2}$$

along the solution of (6.1) resulting from $(\mathbf{u}_1, \ldots, \mathbf{u}_s) \in \mathbb{U}^N$. No terminal cost or condition will be imposed in this setting.

As in previous chapters we are mainly interested in the question whether choosing a specific solution ((Pareto-)optima in Chapters 2, 4 and 5 and NE in this chapter) in the iterations of MPC leads to closed-loop solutions with specific properties, such as approximate (Pareto-)optimality in previous chapters. Moreover, we are interested in the behavior of the closed-loop trajectories. The question how players calculate the NE will not be dealt with in this work although it certainly is a topic that should be investigated.

In what follows we will first introduce the notion of NE and explain why we believe that MPC with NE can generally not be designed such that we obtain an approximation of NE on the infinite horizon or a desired trajectory behavior. Based on a linear game, we show in Section 6.2 that choosing the 'right' NE by imposing constraints on the objective functions (as in Chapters 4 and 5) does not work. However, in Section 6.3 we present a class of games – namely affine-quadratic games – for which one can observe and prove that noncooperative MPC yields convergence of the MPC closed-loop trajectory.

6.1 Solution Concept and Some Considerations

In order to start our analysis let us first define our solution concept to the game given by (6.1) and (6.2).

Definition 6.1 (Nash equilibrium). *The control sequence $\mathbf{u}^{e,N} = (\mathbf{u}_1^{e,N}, \ldots, \mathbf{u}_s^{e,N}) \in \mathbb{U}^N$ is called a* Nash equilibrium *or* Nash strategy *for initial value $x \in \mathbb{R}$ of length $N \in \mathbb{N}$ if for all $i \in \{1, \ldots, s\}$ and all $\mathbf{u}_i^N \in \mathbb{U}_i^N$ it holds*

$$J_i^N(x, \mathbf{u}^{e,N}) \leq J_i^N(x, (\mathbf{u}_1^{e,N}, \ldots, \mathbf{u}_i^N, \ldots, \mathbf{u}_s^{e,N})).$$

The set of all NE of length N for initial value $x \in \mathbb{R}$ will be denoted by $\mathbb{U}_{\mathcal{N}}^N(x)$.

The interpretation of NE is that it is never beneficial (in terms of the cost functional) for each player to unilaterally deviate from the NE. Consequently, NE are typically regarded as 'stable' or 'reliable' solutions. Of course, this does by no means imply optimality of the NE as can be seen in the famous *prisoner's dilemma*, see e.g. [86]. The optimal solution for both prisoners is to cooperate but since this solution is not reliable (one can unilaterally change his strategy and improve his/her situation), they play the NE. Although NE are conceptually very different from optima and Pareto-optimal solutions (POSs), they also share the well-known aspect from the Dynamic Programming Principle (DPP) that tails of NE are NE. Analogous to Lemma 4.1 we give the following result.

Lemma 6.2 (Tails of NE are NE). *If $\mathbf{u}^{e,N} \in \mathbb{U}_{\mathcal{N}}^N(x)$, then $\mathbf{u}^{e,K} := \mathbf{u}^{e,N}(\cdot + K) \in \mathbb{U}_{\mathcal{N}}^{N-K}(x^{\mathbf{u}^{e,N}}(K, x))$ for all $K \in \mathbb{N}_{<N}$, in which*

$$\mathbf{u}^{e,N}(\cdot + K) := \left(u^{e,N}(K), u^{e,N}(K+1), \ldots, u^{e,N}(N-1)\right).$$

Proof. Let us assume that $\mathbf{u}^{e,K} \notin \mathbb{U}_{\mathcal{N}}^{N-K}(x^{\mathbf{u}^{e,N}}(K,x))$. This implies the existence of $i \in \{1,\dots,s\}$ and $\mathbf{u}_i \in \mathbb{U}_i^{N-K}$ so that

$$J_i^{N-K}(x^{\mathbf{u}^{e,N}}(K), \mathbf{u}^{e,K}) > J_i^{N-K}(x^{\mathbf{u}^{e,N}}(K), (\mathbf{u}_1^{e,K}, \dots, \mathbf{u}_i, \dots, \mathbf{u}_s^{e,K}))$$

holds. But since this yields

$$\begin{aligned}
J_i^N(x, \mathbf{u}^{e,N}) &= \sum_{k=0}^{K-1} \ell_i(x^{\mathbf{u}^{e,N}}(k,x), \mathbf{u}^{e,N}(k)) + J_i^{N-K}(x^{\mathbf{u}^{e,N}}(K), \mathbf{u}^{e,K}) \\
&> \sum_{k=0}^{K-1} \ell_i(x^{\mathbf{u}^{e,N}}(k,x), \mathbf{u}^{e,N}(k)) + J_i^{N-K}(x^{\mathbf{u}^{e,N}}(K), (\mathbf{u}_1^{e,K}, \dots, \mathbf{u}_i, \dots, \mathbf{u}_s^{e,K}))
\end{aligned}$$

for this i, there is a feasible[1] $\mathbf{u}_i^N$, namely $\mathbf{u}_i^N = (u_i^{e,N}(0), \dots, u_i^{e,N}(K-1), u_i(0), \dots, u_i(N-K-1))$, such that

$$J_i^N(x, \mathbf{u}^{e,N}) > J_i^N(x, (\mathbf{u}_1^{e,N}, \dots, \mathbf{u}_i^N, \dots, \mathbf{u}_s^{e,N}))$$

holds. This contradicts the fact that $\mathbf{u}^{e,N}$ is a NE of length N for initial value $x \in \mathbb{X}$. $\square$

In [79] it is shown that the in a sense converse relation does generally not hold true. In particular, the authors prove that the DPP approach of calculating NE for one time step starting at time $N-1$ and proceeding backwards in time until $k = 0$, and then putting together the solutions forward in time does not yield a NE of horizon N.

Additional to a result such as Lemma 6.2, in previous chapters we made use of the following idea: At time $n \in \mathbb{N}$ construct a feasible control sequence for time $n+1$ that produces a decay in all objective functions and then, at time $n+1$ choose a (Pareto-)optimal solution with even smaller objective values. For scalar-valued optimization problem, this choice is somewhat trivial because any optimum is smaller than any feasible value. In the presence of multiple objectives we have guaranteed the existence of a suitable POS by means of external stability (see Definition 3.4), which in turn is obtained under mild regularity assumptions, see Lemma 4.8. Using both ingredients, we were then able to upperbound the MPC closed-loop performance $\sum_{k=0}^{K-1} \ell_i(x(k), \mu^N(x(k)))$ for each objective $i \in \{1,\dots,s\}$, which led to performance estimates as well as statements for the closed-loop trajectory. To the best of our knowledge a counterpart of external stability for NE does not exist. This means that it is in general not possible to guarantee the existence of a NE which is subject to additional constraints. Moreover, in Section 6.2 we provide an example, for which we prove that imposing constraints on the objective function of the players does not yield any statements on the closed loop. Thus, we believe that a performance analysis for noncooperative MPC cannot be conducted in a similar way to scalar-valued or MO MPC. We thus propose the following noncooperative MPC algorithm, which does not contain any kind of choice (e.g. by means of recursive constraints) of the NE.

Algorithm 6 (Nash-based MPC).
Given an s-player game and a horizon $N \in \mathbb{N}$. At each time instant $n \in \mathbb{N}_0$:

[1] Feasibility of the concatenated control sequence is obtained as in Lemma 4.1.

(1) Set $x := x(n)$.

(2) Find $\mathbf{u}_x^{e,N} \in \mathbb{U}_{\mathcal{N}}^N(x)$.

(3) Apply $\mu^N(x) := u_x^{e,N}(0)$.

6.2 MPC for a Linear Game

In this section we consider a prisoner's dilemma type two-player game with one-dimensional linear dynamics

$$x^+ = x - \frac{1}{C}(u_1 + u_2) \tag{6.3}$$

with $x \in \mathbb{R}$, $C \in \mathbb{N}$ and $u_i \in [0, 1]$. The stage cost of player $i \in \{1, 2\}$ is given by

$$\ell_i(x, u_1, u_2) = x + u_i - u_j \tag{6.4}$$

for $j \in \{0, 1\}$, $j \neq i$, defining the cost functional

$$J_i^N(x, \mathbf{u}_1, \mathbf{u}_2) = \sum_{k=0}^{N-1} \ell_i(x(k), u_1(k), u_2(k))$$

of horizon $N \in \mathbb{N}$. If the game is only played for one time instant, no player will control the system because controlling is penalized in their stage costs. Depending on the parameter C and horizon length N, playing the game over multiple time stages leads to control actions because controlling the system decreases the state, which is beneficial in terms of the stage cost for both players. Moreover, any control action of one player is beneficial for the other, because it occurs in the stage cost and again reduces the state.

Lemma 6.3 (Existence of NE)**.** *The two-player game of horizon $N \in \mathbb{N}$, which is given by dynamics* (6.3) *and stage costs* (6.4)*, has a NE $\mathbf{u}_1^{e,N}$, $\mathbf{u}_2^{e,N}$. If $N < C + 1$, the NE is unique, whereas there are infinitely many NE in case $N \geq C + 1$.*

Proof. In order to calculate the NE of the two-player game of horizon $N \in \mathbb{N}$, we follow the best-response approach, see e.g. [57], i.e. we minimize J_i^N wrt $\mathbf{u}_i$ under the assumption that $\mathbf{u}_j$ is constant. Therefore, let us first consider the cost functional along the trajectory. For initial value $x_0 \in \mathbb{R}$ and all $k \in \mathbb{N}_0$ we obtain

$$x(k) = x_0 - \frac{1}{C} \sum_{l=0}^{k-1} (u_1(l) + u_2(l)),$$

and thus (using the convention $\sum_{l=0}^{-1}(\cdot) = 0$)

$$\begin{aligned} J_i^N(x_0, \mathbf{u}_1, \mathbf{u}_2) &= \sum_{k=0}^{N-1} \left[x_0 - \frac{1}{C} \sum_{l=0}^{k-1} (u_1(l) + u_2(l)) + u_i(k) - u_j(k) \right] \\ &= \sum_{k=0}^{N-1} x_0 + u_i(k) - u_j(k) - \frac{N-1-k}{C}(u_i(k) + u_j(k)) \\ &= \sum_{k=0}^{N-1} x_0 + \frac{C-N+1+k}{C} u_i(k) - \frac{C-N+1+k}{C} u_j(k). \end{aligned} \tag{6.5}$$

Minimizing J_i^N wrt $\mathbf{u}_i$ now yields

$$u_i^{e,N}(k) = \begin{cases} 0, & \frac{C-N+1+k}{C} > 0 \Leftrightarrow k > N-C-1 \\ 1, & \frac{C-N+1+k}{C} < 0 \Leftrightarrow k < N-C-1 \\ \text{can be chosen freely in } [0,1], & \frac{C-N+1+k}{C} = 0 \Leftrightarrow k = N-C-1 \end{cases}.$$

If $N < C+1$, we have $N - C - 1 < 0$, i.e. the last two cases never occur because $k \in \{0, \dots, N-1\}$. This yields a unique NE that is constantly zero for both players. If $N = C+1$, the first element of the NE can be chosen freely for both players and the last $N-1$ elements are zero. If $N > C+1$ all three cases occur, i.e. the NE of both players consist of ones, one entry that can be chosen freely and zeros in the end. Consequently, in case $N \geq C+1$ the two-player game has an infinite number of NE. $\square$

The result reflects our prior considerations, because it shows that controlling only pays off for each player as long as the planning horizon is large enough to see the positive effect on the state of the system (in terms of the stage cost).

Lemma 6.3 immediately gives the following implications on the MPC closed loop resulting from Algorithm 6. If $N < C+1$, $\mu_i^N(x) = 0$ for all $x \in \mathbb{R}$ and $i \in \{1,2\}$. This leads to the closed-loop trajectory $x^{\mu^N}(k, x_0) = x_0$ for all $k \in \mathbb{N}_0$. If $N > C+1$, by Lemma 6.3 we have $\mu_i^N(x) = u_i^{e,N}(0) = 1$ for all $i \in \{1,2\}$ and all $x \in \mathbb{R}$. This leads to the MPC closed-loop trajectory $x^{\mu^N}(k, x_0) = x_0 - \frac{2k}{C}$ for $k \in \mathbb{N}_0$, i.e. $x^{\mu^N}(k, x_0) \to -\infty$ as $k \to \infty$. In other words, in this case we obtain a unique noncooperative MPC feedback even though the corresponding open-loop NE are non-unique. In case $N = C+1$, $u_i^{e,N}(0)$ can be chosen arbitrarily and independently within the interval $[0,1]$ for both players, which leads to an unpredictable outcome of the MPC closed-loop trajectory. Since this is the only 'critical' case, one might hope that choosing the right value for $u_i^{e,N}(0)$ leads to statements on the closed-loop trajectory. Unfortunately, this can be negated as the next theorem shows.

Theorem 6.4 (MPC with recursive constraints does generally not work)**.** *Consider the two-player game of horizon $N \in \mathbb{N}$, which is given by dynamics* (6.3) *and stage costs* (6.4)*. Statements on the closed-loop trajectory resulting from Algorithm 6 cannot be obtained by including a selection of NE by means of the objective functions J_i^N in step* **(2)**.

Proof. In case $N \neq C+1$, the behavior of the closed-loop trajectory can be predicted because of the unique MPC feedback as our preliminary considerations show. Let us therefore consider the case $N = C+1$. An inspection of the NE of the game (see Lemma 6.3) gives the following implication on the corresponding objective functions.

If $N = C+1$, we have $u_i^{e,N}(0) \in [0,1]$ and $u_i^{e,N}(k) = 0$ for $k \in \{1, \dots, N-1\}$ for both players. Thus, we have

$$\begin{aligned} J_i^N(x_0, \mathbf{u}_1^{e,N}, \mathbf{u}_2^{e,N}) &= Nx_0 + \underbrace{\frac{C-N+1}{C}}_{=0}(u_i^{e,N}(0) - u_j^{e,N}(0)) \\ &= Nx_0. \end{aligned}$$

This reveals that the objective functions have the same values no matter how the players pick $u_i^{e,N}(0) \in [0,1]$. Since the choice makes a difference in terms of the closed-loop trajectory, we have thus proven that a selection of NE by means of the objective functions in step **(2)** of Algorithm 6 does not provide any information about the closed-loop trajectory. $\square$

Remark 6.5. *Very interesting about the game considered in Theorem 6.4 is the fact that not only the NE for $N = C+1$ all have the same objective function value, but also the NE for $N \neq C+1$ have this value. This can be seen from the following calculations.*

If $N < C+1$, the NE of both players are constantly zero, which in view of (6.5) *leads to*

$$\begin{aligned} J_i^N(x_0, \mathbf{u}_1^{e,N}, \mathbf{u}_2^{e,N}) &= Nx_0 + \sum_{k=0}^{N-1} \frac{C-N+1+k}{C}(u_i^{e,N}(k) - u_j^{e,N}(k)) \\ &= Nx_0 \end{aligned}$$

for both players $i \in \{1,2\}$.

If $N > C+1$, we have $\mathbf{u}_i^{e,N} = (1, \dots, 1, u_i^{e,N}(N-C-1), 0, \dots, 0)$ for both $i \in \{1,2\}$. This yields

$$\begin{aligned} J_i^N(x_0, \mathbf{u}_1^{e,N}, \mathbf{u}_2^{e,N}) = Nx_0 &+ \sum_{k=0}^{N-C-2} \frac{C-N+1+k}{C}(1-1) \\ &+ \underbrace{\frac{C-N+1+N-C-1}{C}}_{=0}(e_i^{e,N}(N-C-1) - u_j^{e,N}(N-C-1)) \\ &+ \sum_{k=N-C}^{N-1} \frac{C-N+1+k}{C}(0-0) \\ = Nx_0. \end{aligned}$$

6.3 MPC for Affine-Quadratic Games

In this section we present a class of games and sufficient conditions such that the closed-loop trajectory resulting from Algorithm 6 converges. Moreover, we illustrate by means of an example that convergence is also observed if the conditions are not satisfied. The games we consider have affine system dynamics, i.e.

$$x^+ = f(x,u) = Ax + \sum_{i=1}^{s} B_i u_i + c, \tag{6.6}$$

in which $x,\ c \in \mathbb{R}^n$, $u_i \in \mathbb{R}^{m_i}$, and A and B_i matrices of appropriate dimensions. The stage cost $\ell_i : \mathbb{R}^n \times \mathbb{R}^{m_i} \to \mathbb{R}$ of player $i \in \{1, \dots, s\}$ is given by

$$\ell_i(x,u_i) = \frac{1}{2}\left[(x - x_i^*)^T Q_i (x - x_i^*) + u_i^T R_i u_i\right], \tag{6.7}$$

for matrices R_i and Q_i of appropriate dimension. Although affine-quadratic games are under investigation in many references, e.g. in [4, 22, 46, 48] for the finite and infinite horizon, it seems that the situation of a 'true' conflict in the stage cost is not dealt with in the literature. This means that $x_i^* = 0$ for all $i \in \{1, \dots, s\}$ or all x_i^* are identical in the mentioned references. However, we will show that our game can be transformed, such that it is solvable in the same manner.

Firstly, we perform the coordinate transformation $y_i := x - x_i^*$ for all $i \in \{1, \dots, s\}$ and obtain a blown-up system

$$\begin{aligned}
\mathbb{R}^{sn} \ni y^+ &= \begin{pmatrix} y_1^+ \\ \vdots \\ y_s^+ \end{pmatrix} = \begin{pmatrix} x^+ - x_1^* \\ \vdots \\ x^+ - x_s^* \end{pmatrix} = \begin{pmatrix} Ax + \sum_{i=1}^{s} B_i u_i + c - x_1^* \\ \vdots \\ Ax + \sum_{i=1}^{s} B_i u_i + c - x_s^* \end{pmatrix} \\
&= \begin{pmatrix} Ay_1 + \sum_{i=1}^{s} B_i u_i + c + (A - Id)x_1^* \\ \vdots \\ Ay_s + \sum_{i=1}^{s} B_i u_i + c + (A - Id)x_s^* \end{pmatrix} = \bar{A}y + \sum_{i=1}^{s} \bar{B}_i u_i + \bar{c} = \bar{f}(y,u),
\end{aligned}$$

in which

$$\bar{A} := \begin{pmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{pmatrix} \in \mathbb{R}^{sn \times sn},\ \bar{B}_i := \begin{pmatrix} B_i \\ \vdots \\ B_i \end{pmatrix} \in \mathbb{R}^{sn \times m_i},\ \bar{c} := \begin{pmatrix} c + (A - Id)x_1^* \\ \vdots \\ c + (A - Id)x_s^* \end{pmatrix} \in \mathbb{R}^{sn}$$

and $\bar{Q}_i \in \mathbb{R}^{sn \times sn}$ is a zero-matrix apart from $[\bar{Q}_i]_{(i-1)*n+1 \le v,w \le i*n} = Q_i$, i.e. the ith block of size $n \times n$ on the diagonal of $\bar{Q}_i$ is Q_i. The stage costs then translate to

$$\ell_i(x,u_i) = \frac{1}{2}\left[y_i^T Q_i y_i + u_i^T R_i u_i\right] = \frac{1}{2}\left[y^T \bar{Q}_i y + u_i^T R_i u_i\right] = \bar{\ell}_i(y,u_i),$$

in which $\bar{Q}_i \in \mathbb{R}^{sn \times sn}$ is a zero-matrix apart from $[\bar{Q}_i]_{(i-1)*n+1 \le v,w \le i*n} = Q_i$, i.e. the ith block of size $n \times n$ on the diagonal of $\bar{Q}_i$ is Q_i. With these preparations we are ready to state a result on existence and uniqueness of a NE for our affine-quadratic game.

Proposition 6.6 (Existence and uniqueness of NE)**.** *Consider the s-player game on horizon $N \in \mathbb{N}$ with affine dynamics* (6.6) *and stage costs* (6.7) *with $Q_i \succeq 0$ and $R_i \succ 0$ for all $i \in \{1, \dots, s\}$. Consider the backward matrix iterations*

$$\Lambda^k = Id + \sum_{i=1}^{s} \bar{B}_i R_i^{-1} \bar{B}_i^T M_i^{k+1}, \tag{6.8}$$

$$M_i^k = \bar{Q}_i + \bar{A}^T M_i^{k+1} (\Lambda^k)^{-1} \bar{A}, \quad M_i^N = 0, \tag{6.9}$$

for $k = N-1, \dots, 0$.

If all Λ^k are invertible, then for each initial value $x \in \mathbb{X}$ there exists a unique NE $\mathbf{u}^{e,N} \in \mathbb{U}_{\mathcal{N}}^N(x)$ to the transformed system with corresponding trajectory $y^{\mathbf{u}^{e,N}}(\cdot)$ given by

$$u_i^{e,N}(k) = -R_i^{-1} \bar{B}_i^T \left[M_i^{k+1} (\Lambda^k)^{-1} \bar{A} y^{\mathbf{u}^{e,N}}(k) + \xi_i^k \right], \ k = 0, \dots, N-1, \ i = 1, \dots, s,$$

$$y^{\mathbf{u}^{e,N}}(k+1) = (\Lambda^k)^{-1} \left[\bar{A} y^{\mathbf{u}^{e,N}}(k) + \eta^k \right], \ k = 0, \dots, N-1, \ y^{\mathbf{u}^{e,N}}(0) = \begin{pmatrix} x - x_1^* \\ \vdots \\ x - x_s^* \end{pmatrix},$$

and the backward iterations

$$m_i^k = \bar{A}^T \left[m_i^{k+1} + M_i^{k+1} (\Lambda^k)^{-1} \eta^k \right], \quad m_i^N = 0, \tag{6.10}$$

$$\eta^k = \bar{c} - \sum_{i=1}^{s} \bar{B}_i R_i^{-1} \bar{B}_i^T m_i^{k+1}, \tag{6.11}$$

$$\xi_i^k = M_i^{k+1} (\Lambda^k)^{-1} \eta^k + m_i^{k+1}. \tag{6.12}$$

In the original coordinates, the Nash trajectory is given by $x^{\mathbf{u}^{e,N}}(k, x) = \left[y^{\mathbf{u}^{e,N}}(k) \right]_i + x_i^*$ for arbitrary $i \in \{1, \dots, s\}$ and all $k = 1, \dots, N$.

Proof. The proof is an adapted version of the proof of Başar and Olsder [4, Thm. 6.2] to our setting and the transformed system. First, we note that the inequality in Definition 6.1 can be formulated as a minimization problem, namely

$$J_i^N(x, \mathbf{u}^{e,N}) = \min_{\mathbf{u}_i^N} J_i^N(x, (\mathbf{u}_1^{e,N}, \dots, \mathbf{u}_i^N, \dots, \mathbf{u}_s^{e,N})) \quad \forall i \in \{1, \dots, s\}.$$

As stated in [4, Thm. 6.1], the following conditions are necessary for a control sequence to be a NE:

$$
\begin{aligned}
y^{\mathbf{u}^{e,N}}(k+1) &= \bar{A}y^{\mathbf{u}^{e,N}}(k) + \sum_{i=1}^{s} \bar{B}_i u_i^{e,N}(k) + \bar{c}, \quad y^{\mathbf{u}^{e,N}}(0) = y_0,\\
u_i^{e,N}(k) &= \\
&\operatorname*{argmin}_{u_i(k)\in\mathbb{R}^{m_i}} H_i(k, p_i(k+1), u_1^{e,N}(k), \dots, u_{i-1}^{e,N}(k), u_i(k), u_{i+1}^{e,N}(k), \dots, u_s^{e,N}(k), y^{\mathbf{u}^{e,N}}(k)),\\
p_i(k) &= \frac{\partial \bar{\ell}_i}{y}(y^{\mathbf{u}^{e,N}}(k), u_i^{e,N}(k))\\
&= \bar{Q}_i y^{\mathbf{u}^{e,N}}(k), \quad p_i(N) = 0,
\end{aligned}
\tag{6.13}
$$

in which the Hamiltonian-type functional H_i is given by

$$
\begin{aligned}
&H_i(k, p_i(k+1), u_1(k), \dots, u_s(k), y(k)) = \bar{\ell}_i(y(k), u(k)) + p_i(k+1)^T \bar{f}(y(k), u(k))\\
=&\frac{1}{2}[y(k)^T \bar{Q}_i y(k) + u_i(k)^T R_i u_i(k)] + p_i(k+1)^T \left(\bar{A}y(k) + \sum_{i=1}^{s} \bar{B}_i u_i(k) + \bar{c} \right).
\end{aligned}
$$

Since our transformation preserves the definiteness of the occuring matrices, i.e. $\bar{Q}_i \succeq 0$ and $R_i \succ 0$ for all $i \in \{1, \dots, s\}$, and since the system is still affine, we have a strictly convex problem, and the conditions (6.13) are also sufficient. Thus, it suffices to prove that the expression for $u_i^{e,N}(k)$ stated in Proposition 6.6 satisfies (6.13).

Minimization of H_i wrt $u_i(k)$ yields

$$
u_i^{e,N}(k) = -R_i^{-1}[\bar{B}_i^T p_i(k+1)]. \tag{6.14}
$$

Let us now prove the assertion by a backward induction.

Base case: $k = N-1$**.** In this case, $p_i(N) = 0$ and (6.14) gives $u_i^{e,N}(N-1) = 0$. This is accordance with our assertion, because $M_i^N = 0$ and $\xi_i^{N-1} = 0$. Using this solution, we get

$$
\begin{aligned}
y^{\mathbf{u}^{e,N}}(N) &= \bar{A}y^{\mathbf{u}^{e,N}}(N-1) + \sum_{i=1}^{s} \bar{B}_i u_i^{e,N}(N-1) + \bar{c} = \bar{A}y^{\mathbf{u}^{e,N}}(N-1) + \bar{c}\\
&\Leftrightarrow y^{\mathbf{u}^{e,N}}(N) - \bar{A}y^{\mathbf{u}^{e,N}}(N-1) = \bar{c} = (Id - \Lambda^{N-1})^{-1} y^{\mathbf{u}^{e,N}}(N) + \bar{c}\\
&\Leftrightarrow y^{\mathbf{u}^{e,N}}(N) = (\Lambda^{N-1})^{-1}(\bar{A}y^{\mathbf{u}^{e,N}}(N-1) + \bar{c}),
\end{aligned}
$$

which is again in accordance to the assertion because $\eta^{N-1} = \bar{c}$.

Inductive hypothesis. We assume that the formula for $u_i^{e,N}(k)$ holds true for $k = l+1$ and that $p_i(l+1) = M_i^{l+1} y^{\mathbf{u}^{e,N}}(l+1) + m_i^{l+1}$ (which is true for the base case).

Inductive step: $l+1 \mapsto l$**.** The relation (6.14) in combination with the hypothesis for p_i gives

$$
u_i^{e,N}(l) = -R_i^{-1}[\bar{B}_i^T (M_i^{l+1} y^{\mathbf{u}^{e,N}}(l+1) + m_i^{l+1})].
$$

Plugging this into the dynamics yields

$$\begin{aligned} y^{\mathbf{u}^{e,N}}(l+1) &= \bar{A}y^{\mathbf{u}^{e,N}}(l) - \sum_{i=1}^{s} \bar{B}_i R_i^{-1}[\bar{B}_i^T(M_i^{l+1}y^{\mathbf{u}^{e,N}}(l+1) + m_i^{l+1})] + \bar{c} \\ &= \bar{A}y^{\mathbf{u}^{e,N}}(l) + (Id + \Lambda^l)y^{\mathbf{u}^{e,N}}(l+1) + \eta^l \\ \Leftrightarrow\ y^{\mathbf{u}^{e,N}}(l+1) &= (\Lambda^l)^{-1}[\bar{A}y^{\mathbf{u}^{e,N}}(l) + \eta^l] \\ \Rightarrow\ u_i^{e,N}(l) &= -R_i^{-1}\bar{B}_i^T[M_i^{l+1}(\Lambda^l)^{-1}\bar{A}y^{\mathbf{u}^{e,N}}(l) + \xi_i^l], \end{aligned}$$

which is exactly what we have claimed. □

Remark 6.7. *1. An alternative derivation of the results, which also relies on characterizing NE by means of optimization problems, can be obtained by an approach similar to the one presented in Appendix A. In this case, NE are obtained by backward Riccati-type iterations and a value function can be calculated, see [4].*

2. As can be seen in the inductive proof, the condition that the matrices Λ^k be invertible for all $k \in \{0, \ldots, N-1\}$ is needed, because it ensures that we obtain a unique trajectory when plugging in the NE.

Theorem 6.8 (Convergence of the noncooperative MPC controller). *Let the assumptions of Proposition 6.6 hold for the affine-quadratic game of horizon $N \in \mathbb{N}$. Assume $\|\bar{A}(\Lambda^0)^{-1}\| < 1$ for an arbitrary matrix norm, in which Λ^0 is the result of the backward iteration in* (6.8)*. Moreover, assume that all eigenvalues λ of $(\Lambda^0)^{-1}\bar{A}$ fulfill $|\lambda| \leq 1$, and any λ with $|\lambda| = 1$ is* nondefective[2] *and fulfills $\lambda = 1$. Then, for each $x_0 \in \mathbb{X}$ the MPC closed-loop trajectory $x^{\mu^N}(k, x_0)$ obtained by execution of Algorithm 6 converges as k tends to infinity.*

Proof. First, we note that the backward iterations (6.8)-(6.12) only depend on the data of the game and the horizon N, but neither on the current time nor on the current state. Thus, we get the same values in (6.8)-(6.12) in each iteration of Algorithm 6. We claim that the closed-loop trajectory of the transformed system for initial value $y_0 = \begin{pmatrix} x_0 - x_1^* \\ \vdots \\ x_0 - x_s^* \end{pmatrix}$ evolves as follows:

$$y^{\mu^N}(k, y_0) = \left((\Lambda^0)^{-1}\bar{A}\right)^k y_0 + (\Lambda^0)^{-1} \sum_{l=0}^{k-1} \left(\bar{A}(\Lambda^0)^{-1}\right)^l \eta^0, \quad k \in \mathbb{N}_0$$

with the convention that $\sum_{l=0}^{-1}(\cdot) = 0$. The base case $k = 0$ is obvious. As induction hypothesis we assume that the formula holds true for $k \in \mathbb{N}$. For $y^{\mu^N}(k+1, y_0)$ we use

[2] An eigenvalue of matrix X is said to be nondefective or semisimple if it is a root of multiplicity one in the minimal polynomial of X.

the considerations from the proof of Proposition 6.6 and get (together with the induction hypothesis)

$$\begin{aligned}
y^{\mu^N}(k+1,y_0) &= (\Lambda^0)^{-1}\left[\bar{A}y^{\mu^N}(k,y_0)+\eta^0\right]\\
&= (\Lambda^0)^{-1}\bar{A}y^{\mu^N}(k,y_0)+(\Lambda^0)^{-1}\eta^0\\
&= (\Lambda^0)^{-1}\bar{A}\left[\left((\Lambda^0)^{-1}\bar{A}\right)^k y_0+(\Lambda^0)^{-1}\sum_{l=0}^{k-1}\left(\bar{A}(\Lambda^0)^{-1}\right)^l\eta^0\right]+(\Lambda^0)^{-1}\eta^0\\
&= \left((\Lambda^0)^{-1}\bar{A}\right)^{k+1} y_0+(\Lambda^0)^{-1}\left(\bar{A}(\Lambda^0)^{-1}\sum_{l=0}^{k-1}\left(\bar{A}(\Lambda^0)^{-1}\right)^l\eta^0+\eta^0\right)\\
&= \left((\Lambda^0)^{-1}\bar{A}\right)^{k+1} y_0+(\Lambda^0)^{-1}\left(\sum_{l=0}^{k-1}\left(\bar{A}(\Lambda^0)^{-1}\right)^{l+1}\eta^0+\eta^0\right)\\
&= \left((\Lambda^0)^{-1}\bar{A}\right)^{k+1} y_0+(\Lambda^0)^{-1}\left(\sum_{l=0}^{k}\left(\bar{A}(\Lambda^0)^{-1}\right)^{l}\eta^0-\eta^0+\eta^0\right)\\
&= \left((\Lambda^0)^{-1}\bar{A}\right)^{k+1} y_0+\underbrace{(\Lambda^0)^{-1}\left(\sum_{l=0}^{k}\left(\bar{A}(\Lambda^0)^{-1}\right)^{l}\eta^0\right)}_{(*)}.
\end{aligned}$$

This finishes the inductive proof. Since $\|\bar{A}(\Lambda^0)^{-1}\|<1$, the Neumann series $(*)$ converges to $(\Lambda^0)^{-1}\left(I-\bar{A}(\Lambda^0)^{-1}\right)^{-1}\eta^0=\left(\Lambda^0-\bar{A}\right)^{-1}\eta^0$, see e.g. [76]. Now consider the sequence $\left(\left((\Lambda^0)^{-1}\bar{A}\right)^k\right)_{k\in\mathbb{N}_0}$. It is well known (see e.g. [59]) that this sequence converges under the assumption that we have imposed on the eigenvalues of $(\Lambda^0)^{-1}\bar{A}$. In order to calculate the limit, let T be an invertible matrix such that $T^{-1}(\Lambda^0)^{-1}\bar{A}T=J$ and J is the Jordan canonical form of $(\Lambda^0)^{-1}\bar{A}$. From our assumption on the eigenvalues we know that $J=\begin{pmatrix} Id & 0\\ 0 & P\end{pmatrix}$, in which Id has as many rows and columns as the number of eigenvalues with $|\lambda|=1$, and P is a matrix with spectral radius strictly less than one. Thus we get

$$\left((\Lambda^0)^{-1}\bar{A}\right)^k=TJ^kT^{-1}=T\begin{pmatrix} Id & 0\\ 0 & P^k\end{pmatrix}T^{-1}$$

and $y^{\mu^N}(k,y_0)\to T\begin{pmatrix} Id & 0\\ 0 & 0\end{pmatrix}T^{-1}y_0+\left(\Lambda^0-\bar{A}\right)^{-1}\eta^0$ as $k\to\infty$. In our original coordinates this translates to

$$x^{\mu^N}(k,x_0)\to\left[T\begin{pmatrix} Id & 0\\ 0 & 0\end{pmatrix}T^{-1}\begin{pmatrix} x_0-x_1^*\\ \vdots\\ x_0-x_s^*\end{pmatrix}+\left(\Lambda^0-\bar{A}\right)^{-1}\eta^0\right]_i+x_i^*$$

for an arbitrary choice of $i\in\{1,\dots,s\}$. $\square$

Remark 6.9. *We point out that the conditions presented in Theorem 6.8 are sufficient but by no means necessary. As can be observed in our numerical simulations, the MPC closed-loop trajectory resulting from Algorithm 6 also converges if the conditions are not satisfied. Moreover, the conditions $Q_i \succeq 0$, $R_i \succ 0$ can be relaxed, see [4].*

In Chapter 4 and 5 we have analyzed MO MPC schemes and are therefore interested in the question, whether there is a relation between both concepts, cooperative and non-cooperative MPC. The next result, presented in [14], provides necessary and sufficient conditions for a NE to be Pareto-optimal in the affine-quadratic setting.

Lemma 6.10 (Pareto optimality of NE)**.** *Consider the affine-quadratic s-player game given by* (6.6) *and* (6.7) *for horizon $N \in \mathbb{N}$ and initial value $x \in \mathbb{X}$. Let A be nonsingular and $Q_i \succeq 0$, $R_i \succ 0$. A NE $\mathbf{u}^{e,N} \in \mathbb{U}^N_{\mathcal{N}}(x)$ is a POS to the MO optimization problem*

$$\min_{\mathbf{u}\in\mathbb{U}^N} \left(J_1^N(x,\mathbf{u}),\ldots,J_s^N(x,\mathbf{u})\right)$$

if and only if for all i, $j \in \{1,\ldots,s\}$ with $i \neq j$ and all $k \in \{0,\ldots,N-1\}$ it holds

$$\frac{\partial J_i^N}{\partial u_j(k)}(x,\mathbf{u}^{e,N}) = 0. \tag{6.15}$$

As stated in [14] condition (6.15) "basically boils down to the absence [...] of conflict" and is "extreme and unreasonable". Nevertheless, in the next section we demonstrate for our example that the situation, in which a NE is Pareto-optimal, may occur.

6.3.1 Numerical Example

In order to illustrate the findings of this section, we consider a very simple example of two players, who can influence the room temperature of the same room. The dynamics are given by

$$x^+ = ax + u_1 + u_2,$$

in which $x \in \mathbb{R}$ is the temperature and $u_i \in \mathbb{R}$ is the heating (or cooling) control of player i. The cost criterion

$$\ell_i(x,u_i) = \frac{1}{2}\left[(x - x_i^*)^2 + c_i u_i^2\right]$$

reflects the desired temperature x_i^* of player i as well as a penalization of the control effort (i.e. $c_i > 0$). In our simulation we use the values $x_1^* = 23$ and $x_2^* = 17$. The parameters a, $c_i \in \mathbb{R}_{>0}$ will be varied throughout our investigations and can be interpreted as follows: If $a = 1$, the temperature is constant, i.e. it is only influenced by the two players. If $a > 1$, the room heats up by itself, e.g. due to solar radiation. In case $a < 1$ there is constant loss of energy, e.g. through lack of insulation. The value c_i determines player i's motivation to control the system.

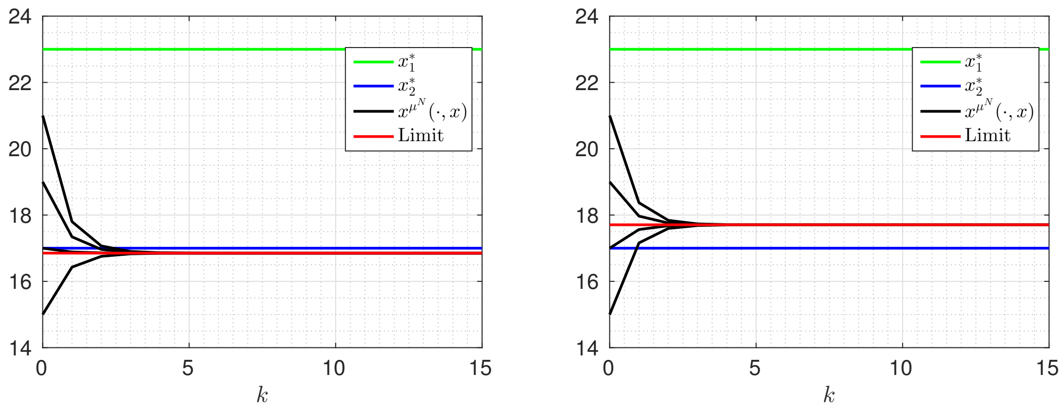

Figure 6.1: Closed-loop trajectories (black) resulting from Algorithm 6 using $a = 0.8$, $c_1 = 2$, $c_2 = 0.5$, different initial values and $N = 2$ (left) and $N = 3$ (right). The red line is the theoretically calculated limit from Theorem 6.8.

Let us first consider the case $a = 0.8$ and $c_1 = 2$, $c_2 = 0.5$, i.e. player two has a stronger incentive to control the room temperature than player one. For these parameters and $N \in \mathbb{N}_{\geq 2}$ numerical experiments show that the assumptions of Theorem 6.8 are satisfied and we expect convergence of the MPC closed-loop trajectory of Algorithm 6. In Figure 6.1 convergence of the MPC closed-loop trajectory is indeed observed. As can be seen from Figures 6.1 and 6.2, the limit of the closed loop does not depend on the initial value but on the MPC horizon N. Moreover, we observe that the limit that the trajectories converge to

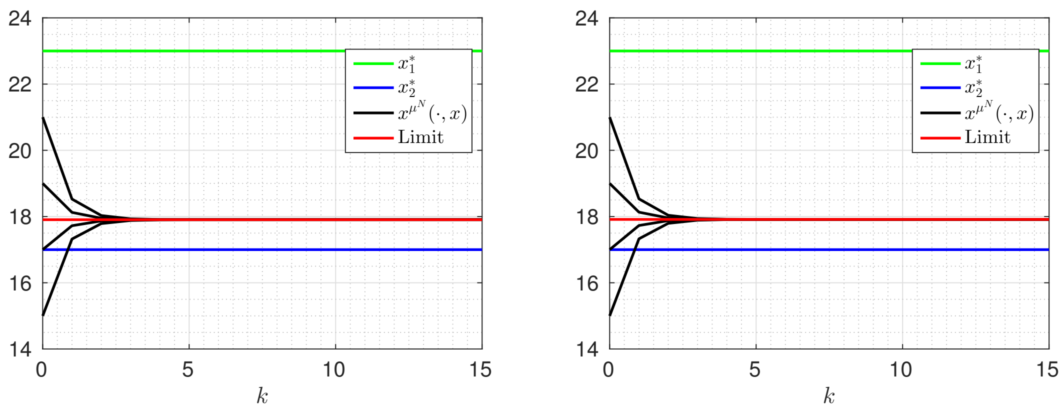

Figure 6.2: Closed-loop trajectories (black) resulting from Algorithm 6 using $a = 0.8$, $c_1 = 2$, $c_2 = 0.5$, different initial values and $N = 5$ (left) and $N = 7$ (right). The red line is the theoretically calculated limit from Theorem 6.8.

reaches a limit itself, i.e. it does not differ for $N = 5$ and $N = 7$. We would like to compare our MPC closed-loop trajectories to Nash-trajectories on the infinite horizon. Results for linear-quadratic games on the infinite horizon can be found in [4, 48]. The approaches therein require solving nonlinear matrix equations, which become involved in our situation because we are dealing with affine dynamics and they only apply to the transformed game (see the beginning of Section 6.3) of higher dimension. Thus, we compare our MPC closed-loop trajectories to Nash-solutions of the same length as the number of executed MPC iterations. In Figure 6.3 it is illustrated that the open-loop Nash-trajectory approaches a

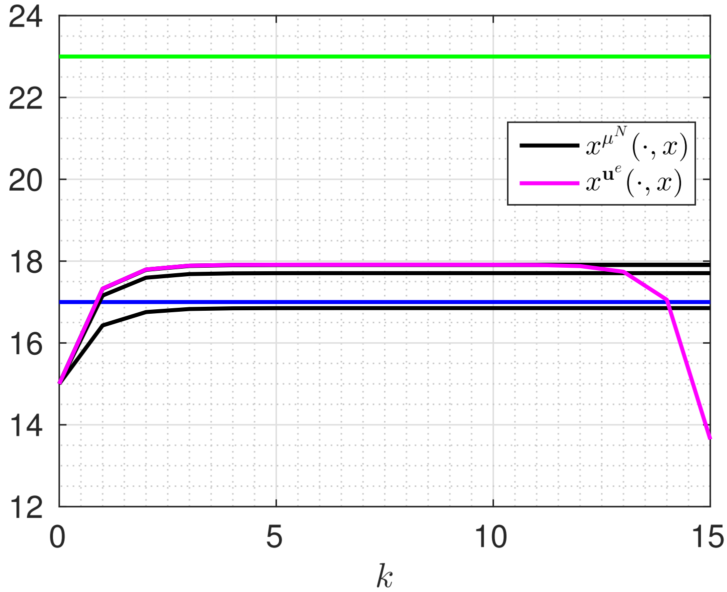

Figure 6.3: Closed-loop trajectories (black) resulting from Algorithm 6 for $N \in \{2, 3, 5, 7\}$ (bottom to top) and fixed initial value, and Nash-trajectory of horizon 15 (magenta).

value, stays there for most of the time before it eventually turns away. The MPC closed-loop trajectories get closer to the open loop as N becomes larger but do not turn away in the end. The behavior of the Nash-trajectory resembles very much the turnpike property, cf. Chapters 2 and 5. This is why we compare Nash-trajectories of different optimization horizons with each other. In Figure 6.4 we indeed obeserve turnpike behavior. As proven in [38], under regularity assumptions the turnpike property on the finite horizon is equivalent to the turnpike property on the infinite horizon, which in turn implies convergence of the trajectories. Thus, we expect the Nash-trajectory of infinite horizon to look like the MPC closed-loop for sufficiently large N. This would imply that for these parameters MPC based on Algorithm 6 indeed approximates Nash solutions on the infinite horizon in terms of trajectory behavior.

If we vary the values of c_i, the previously observed results only change quantitatively but not qualitatively.

Now let us consider the case $a = 1$. Our numerical experiments reveal that the assumptions of Theorem 6.8 are not satisfied for all $N \in \mathbb{N}_{\geq 2}$ because $\|\bar{A}(\Lambda^0)^{-1}\| \geq 1$, i.e. we cannot prove convergence of our MPC closed-loop trajectories. However, as illustrated in Figure 6.5 the MPC closed-loop trajectories still converge. We moreover see that the

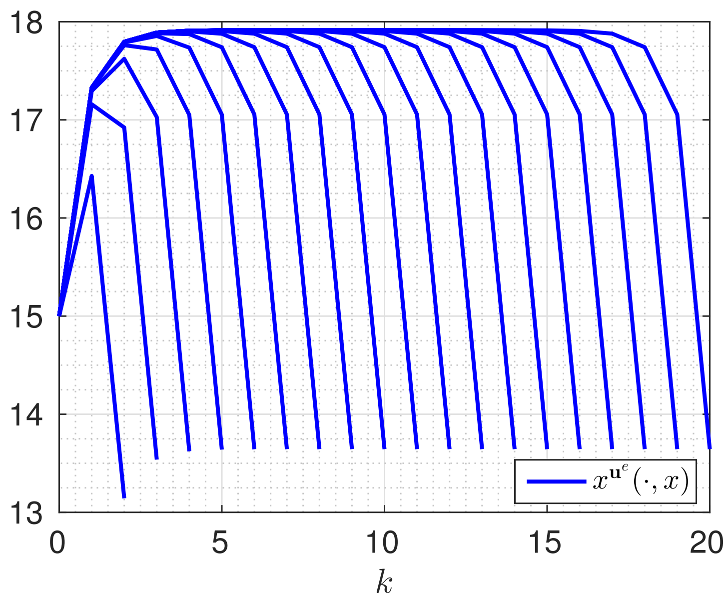

Figure 6.4: Open-loop Nash trajectories for $N = 2, \ldots, 20$ exhibit turnpike behavior.

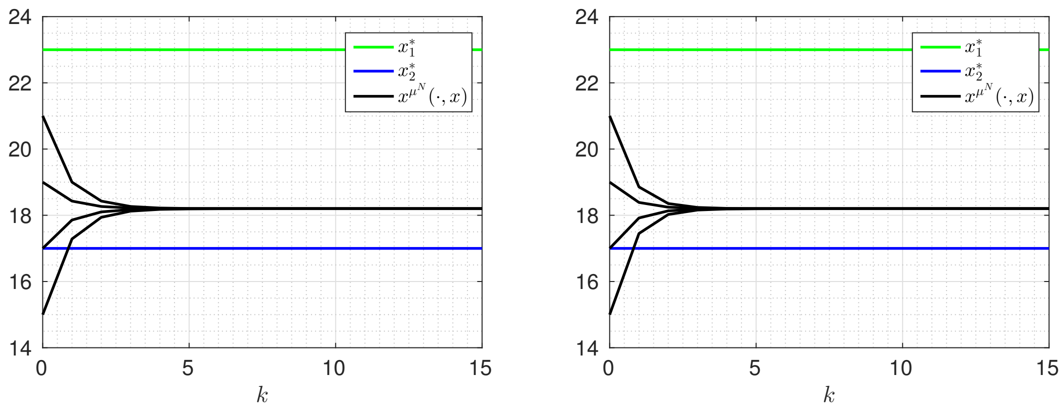

Figure 6.5: Closed-loop trajectories (black) resulting from Algorithm 6 using $a = 1$, $c_1 = 2$, $c_2 = 0.5$, different initial values and $N = 2$ (left) and $N = 7$ (right).

limit does not depend on the MPC horizon for these parameters. In Figure 6.6 the MPC closed-loop trajectories are again compared to the open-loop Nash-trajectory of the same horizon as MPC iterations. In this figure, already for small N both trajectories almost coincide. Opposed to the case $a = 0.8$, in this setting the open-loop trajectory does not turn away in the end. This is because the temperature stays constant if players do not interact, i.e. without no cost. If in contrast $a = 0.8$, player one has to counteract the contracting dynamics which is only done as long as it is beneficial in terms of the cost functional. This is why the open-loop trajectory in the end turns away in Figure 6.3 whereas it does not in

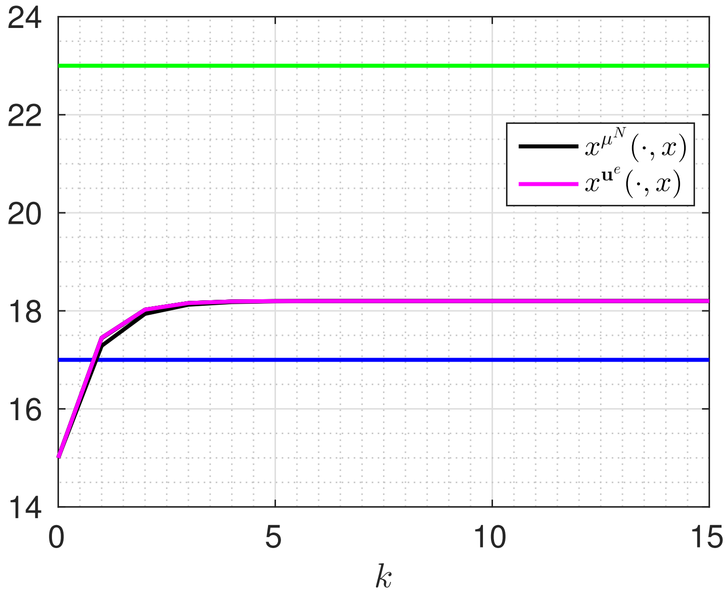

Figure 6.6: Closed-loop trajectories (black) resulting from Algorithm 6 for $N \in \{2, 3, 5, 7\}$ (bottom to top) and fixed initial value, and Nash-trajectory of horizon 15 (magenta).

Figure 6.6.

The last case we consider is the case $a = 1.1$, i.e. the uncontrolled dynamics are unstable. Again, numerical experiments show that Theorem 6.8 is not applicable because $\|\bar{A}(\Lambda^0)^{-1}\| \geq 1$ for all $N \in \mathbb{N}$. Nevertheless, in Figure 6.7 we observe the same closed-loop

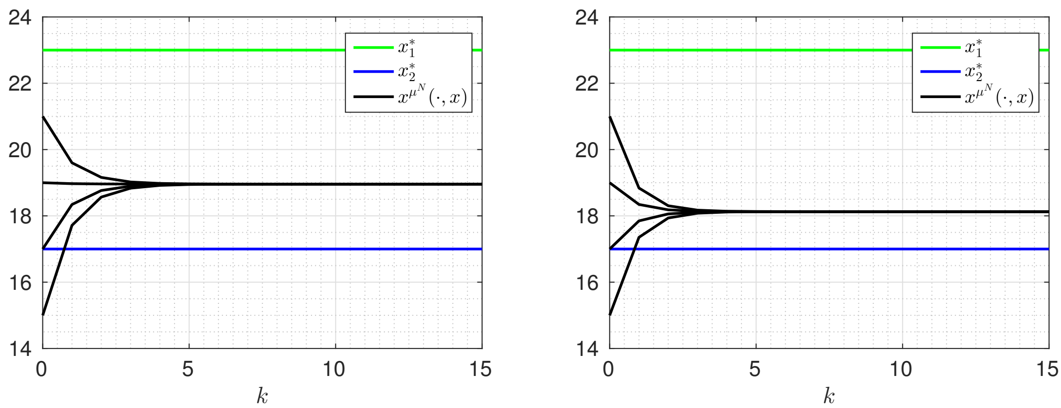

Figure 6.7: Closed-loop trajectories (black) resulting from Algorithm 6 using $a = 1.1$, $c_1 = 2$, $c_2 = 0.5$, different initial values and $N = 2$ (left) and $N = 7$ (right).

behavior as in Figures 6.1 and 6.2 for the case $a = 0.8$. If we compare MPC closed-loop trajectories to the Nash trajectory of the same length we again see that MPC approaches that solution as N increases (not displayed). Not surprisingly, we observe turnpike behavior

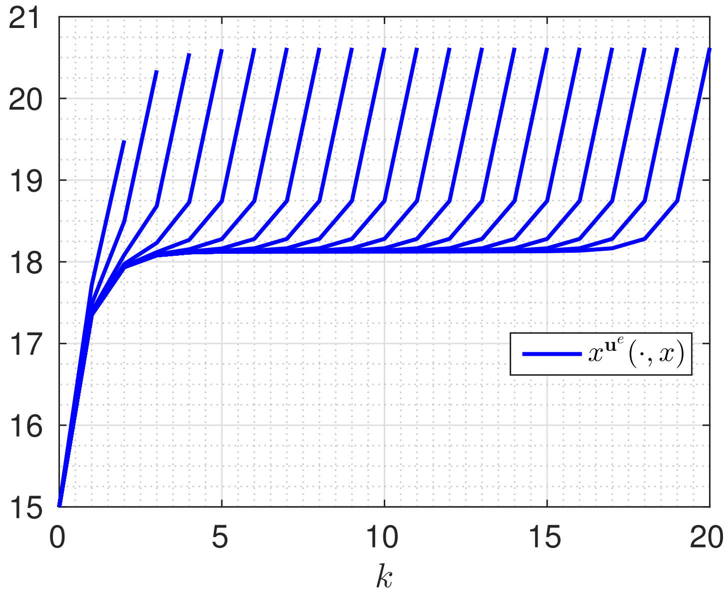

Figure 6.8: Open-loop Nash trajectories for $N = 2, \ldots, 20$ exhibit turnpike behavior.

of open-loop Nash trajectories in Figure 6.8 but this time the leaving arcs of the trajectories are going up (instead of down as in Figure 6.4), which is due to the fact that $a > 1$ and that players stop controlling the system at the end of the planning horizon.

In Lemma 6.10 we have presented conditions for NE to be Pareto-optimal. Our numerical simulations with different values for a and $N \in \mathbb{N}$ reveal that the NE obtained in Algorithm 6 are usually no POSs to the corresponding MO optimization problem, especially in case $a \geq 1$ we could not oberserve such a situation and rather get relations as depicted in Figure 6.9 left. For $a \in (0, 1)$ we were always able to find a horizon $N \in \mathbb{N}$ such that the NE in the iterations of Algorithm 6 are Pareto-optimal. Such a situation is illustrated in Figure 6.9 right.

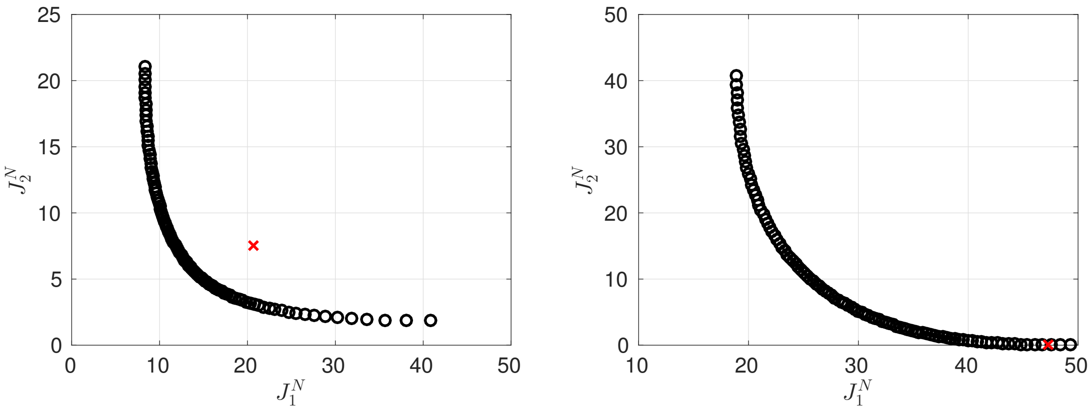

Figure 6.9: Pareto front (black circles) and NE (red cross) of our problem for $a = 1.1$ (left) and $a = 0.8$ (right), $x_0 = 15$ and $N = 2$ at iteration 5 of Algorithm 6.

7 | Implementation

All of our theoretical investigations were accompanied by numerical experiments performed in Matlab and partially supported by student assistant B.Sc. Markus Klar. The simulations in Chapter 2 rely on the implementation of Algorithm 1, for which we used the routine that can be found on http://www.nmpc-book.com/, see also Grüne and Pannek [32].

For the implementation of the multiobjective (MO) Model Predictive Control (MPC) schemes in Chapters 4 and 5 we have written and included different methods for MO optimization and the visualization of Pareto fronts in our code, i.e. the adapted Pascoletti-Serafini scalarization presented in Eichfelder [21], a weighted sum-method, the method of the global criterion, see Miettinen [61], and we made use of `NSGA II` presented in Deb [15]. This genetic algorithm is readily available in Matlab in the global optimization toolbox under the name `gamultiobj`. Our implementation is interactive in the sense that in the first iteration the whole Pareto front is approximated and visualized (if there are at most three objectives) and then the user chooses one point on the Pareto front. This way, the user can determine the upper bound on the performance for all objectives. In subsequent steps, usually only one arbitrary solution to the occuring MO optimization problems is calculated. However, one can also choose to visualize all occuring Pareto fronts.

Regarding noncooperative MPC for affine-quadratic games we have implemented the backward iteration that was presented in Proposition 6.6. In our code also the transformation of the system is done automatically. As stated in Theorem 6.8 the backward iteration neither depends on the time nor on the state and has to be performed only once. Thus, our MPC loop only consist of applying the feedback (and data storage). In order to calculate the Nash equilibria (NE) in Section 6.2 we used Maple to handle the symbolic calculations that occur when the best response-approach is persued.

8 | Future Research

8.1 Multiobjective MPC

8.1.1 Structure of Pareto-optimal Solutions and Pareto Fronts

In results such as Theorems 4.6, 4.19, 4.24 and 5.11, we have seen that the performance is determined by our choice in the very first iteration and that a calculation of the Pareto front in subsequent iterations is not needed as long as the recursive constraints are obeyed. However, it would be interesting to investigate how restrictive the recursive constraint is and how much this influences the Model Predictive Control (MPC) performance. This is closely related to the question how the Pareto fronts that we can choose from develop over time. In numerical simulation we usually observe that the Pareto front degenerates after a few iterations. In the absence of terminal conditions and for stabilizing stage costs we would also like to investigate how Pareto fronts develop for increasing optimization horizon. Our interest is triggered by the fact that we have proven in Corollary 4.20 that any Pareto-optimal solution on the infinite horizon can – in terms of the performance – be approximated arbitrarily well by MPC if the initialization in step **(0)** of Algorithm 3 is chosen correctly. The converse question, whether each initialization leads to an approximation of some Pareto-optimal solution on the infinite horizon is unanswered by now. Let us briefly give an explanation of this aspect.

In the single-objective case it follows from the definitions that if optimal solutions wrt the horizons $N,\ M \in \mathbb{N}$ with $M \geq N$ exist for initial value x, then the optimal value functions satisfy $V^M(x) \geq V^N(x)$. This leads to the following implication: If optima exist for all horizons, the infinite-horizon cost of the MPC feedback defined in Algorithm 1 fulfills (see [33] or [32])

$$J^\infty(x, \mu^N) \leq \frac{\gamma^{N-2}}{\gamma^{N-2} - (\gamma-1)^N} \cdot V^N(x) \leq \frac{\gamma^{N-2}}{\gamma^{N-2} - (\gamma-1)^N} \cdot V^\infty(x).$$

In the multiobjective (MO) setting and under the assumption of external stability for all optimization horizons, we can prove that for $M \geq N$ it holds: For all $\mathbf{u}^{\star,M} \in \mathbb{U}^M_{\mathcal{P}}(x)$ there exists $\mathbf{u}^{\star,N} \in \mathbb{U}^N_{\mathcal{P}}(x)$ such that $J^N_i(x, \mathbf{u}^{\star,N}) \leq J^M_i(x, \mathbf{u}^{\star,M})$ holds for all $i \in \{1, \ldots, s\}$. This result was used in Corollary 4.20. However, the converse statement, i.e. the question whether for all $\mathbf{u}^{\star,N} \in \mathbb{U}^N_{\mathcal{P}}(x)$ there is $\mathbf{u}^{\star,M} \in \mathbb{U}^M_{\mathcal{P}}(x)$ such that $J^N_i(x, \mathbf{u}^{\star,N}) \leq J^M_i(x, \mathbf{u}^{\star,M})$

holds for all $i \in \{1, \dots, s\}$ remains open.[1]

8.1.2 Investigation of Specific Schemes

In the presence of terminal conditions the recursive constraint depends on the local feedback on the terminal region, see Algorithm 2. So far, we have assumed that a common feedback exists (see Assumption 4.4). However, this assumption might be too strong in case of independent, cooperating systems/agents. In our future research we will therefore investigate for which types of coupling one can relax this assumption, e.g. by only requiring the existence of individual local feedbacks.

8.1.3 Towards Stability of MO MPC

Up to now, our analysis only allows for proving convergence of the MPC closed-loop trajectory, see Corollaries 4.9, 4.22 and 5.9. Of course, it would be preferable to establish asymptotic stability via a Lyapunov function (LF). We think that it might be a promising idea to consider *set-valued* LFs (see e.g. [26, 27]). Such a LF would also be beneficial in MO economic MPC for analyzing the transient behavior similar to [34] for the scalar-valued setting.

8.1.4 MO Dissipativity, MO Turnpike, and MO Economic MPC

In Chapter 5 we have proposed a MO version of (strict) dissipativity. The question, whether there is a physical interpretation of our definition (for single-criterion optimal control problems (OCPs) dissipativity is considered as a measure of energy of the system), is open and should be investigated. As we have seen, strict dissipativity at different steady states entails a non-uniform turnpike behavior. This leads to a very interesting observation: Whereas the turnpike property in the single-objective case pushes the MPC closed loop into the 'right' direction, in the MO setting we have to make sure to follow the 'right' path. The question how to enforce convergence in this setting is an interesting and challenging problem. Without imposing any constraint, in MO economic MPC we have observed in numerical simulations that the MPC closed loop converges into the set of Pareto-optimal steady states. Deriving conditions under which such a behavior can be proven could be a first step towards a convergence analysis.

8.2 Noncooperative MPC

In Section 6.2 we have shown that an additional constraint on the objective functions in the MPC iterations does generally not enable us to analyze the closed-loop behavior. Moreover, Nash equilibria (NE) seem to be less structured than (Pareto-)optimal solutions (see the explanation in Section 6.1). Thus, the tools known from scalar and MO MPC are

[1] In fact, we are aware of an example, in which this property does not hold true. Since in this counterexample the objective functions do not have the structure typical for MPC (where the objective functions are sums of the stage cost), we can not prove this property wrong in our setting.

not suitable in this setting. Consequently, new techniques need to be developed in order to analyze Nash-based MPC. We conjecture that such MPC schemes can only be analyzed for specific games.

For affine-quadratic games we have imposed conditions under which the MPC closed-loop trajectory converges and the limit can be calculated analytically. Based on the theory developed in [48] or [4], we aim to compare the Nash-based MPC solution to the infinite-horizon solution in terms of the trajectory behavior as well as in terms of the performance. Since the solution of the infinite-horizon game requires solving a nonlinear matrix equation, an analytical comparison seems to be involved. As a first step, we aim to compare the solutions for selected examples and by solving the infinite-horizon game numerically. We have observed numerically that the MPC trajectory converges even if the conditions of Theorem 6.8 are not satisfied. Thus, it is a natural wish to provide less restrictive sufficient conditions. Another interesting research topic is the investigation of turnpike properties in affine-quadratic games.

A | An Optimal Value Function for Affine-Quadratic Optimal Control Problems

In this section we prove that the optimal control problem (OCP) (1.3) with data such as in Assumption 2.15, i.e. $f(x,u) = Ax + Bu + c$, $\ell(x,u) = x^T Rx + u^T Qu + s^T x + v^T u$, $R,\ Q \succ 0$, has an optimal value function of the form $V^N(x) = x^T P_N x + b_N^T x + d_N$ with P_N symmetric and positive definite and vectors $b_N,\ d_N \in \mathbb{R}^n$. This result is needed in the proof of Theorem 2.17 in order to make statements on the optimal value function. Even though there is a lot of literature on the linear quadratic regulator for continuous- and discrete-time systems (e.g. [2, 18, 49, 78]), we did not find any result for the case of affine dynamics and costs with additive terms. This is why we provide a result on the structure of the optimal value function here. To this end, we first remark that we can eliminate the additive constant c in the system dynamics through a coordinate transformation. This does not change the structure of the stage costs, and without loss of generality (wlog) we can assume that the system is given by dynamics $x(k+1) = Ax(k) + Bu(k)$ and stage costs $\ell(x,u) = x^T Qx + u^T Ru + s^T x + v^T u$ with $R,\ Q$ symmetric and positive definite. The derivation of the result follows the reasoning of Anderson and Moore [2, Section 2.4] so we prove the statement by induction and by means of the Dynamic Programming Principle (DPP).

Claim: Let $N \in \mathbb{N}$. Then $V^N(x)$ has the structure as mentioned above with

$$\begin{aligned}
P_N &= Q + K_N^T (R + B^T P_{N-1} B) K_N + A^T P_{N-1} A,\ P_1 = Q,\\
b_N &= \left(s^T + 2\bar{K}_N^T R K_N + v^T K_N + 2\bar{K}_N B^T P_{N-1}(A + BK_N) + b_{N-1}^T (A + BK_N)\right)^T,\ b_1 = s,\\
d_N &= \bar{K}_N^T R \bar{K}_N + v^T \bar{K}_N + \bar{K}_N B^T P_{N-1} B \bar{K}_N^T + b_{N-1}^T B \bar{K}_N + d_{N-1},\ d_1 = -\frac{1}{4} v^T R^{-1} v,\\
K_N &= -(B^T P_{N-1} B + R)^{-1} \left(B^T P_{N-1} A\right),\\
\bar{K}_N &= -\frac{1}{2} (B^T P_{N-1} B + R)^{-1} \left(B^T b_{N-1} + v\right).
\end{aligned} \tag{A.1}$$

Proof. **Base case: $K = N - 1$.**

$$\begin{aligned} J^{N-K}(x(K), u(K)) &= J^1(x(N-1), u(N-1)) = \ell(x(N-1), u(N-1)) \\ &= x(N-1)^T Q x(N-1) + u(N-1)^T R u(N-1) \\ &\quad + s^T x(N-1) + v^T u(N-1) \end{aligned}$$

The partial derivative of J^1 wrt $u(N-1)$ equals zeros iff

$$u^\star(K) = -\frac{1}{2} R^{-1} v =: K_{N-K} x(K) + \bar{K}_{N-K}$$

and since the second partial derivative is positive definite, this is indeed the minimizer of J^1. This yields

$$\begin{aligned} V^{N-K}(x(K)) &= J^1(x(N-1), u^\star(N-1)) \\ &= x(N-1)^T Q x(N-1) + \left(-\frac{1}{2} R^{-1} v\right)^T R \left(-\frac{1}{2} R^{-1} v\right) \\ &\quad + s^T x(N-1) + v^T \left(-\frac{1}{2} R^{-1} v\right) \\ &= x(N-1)^T Q x(N-1) + s^T x(N-1) + \frac{1}{4} v^T R^{-1} v - \frac{1}{2} v^T R^{-1} v \\ &= x(N-1)^T P_{N-K} x(N-1) + b_{N-K}^T x(N-1) + d_{N-K}, \end{aligned}$$

with $P_{N-K} := Q$, $b_{N-K} := s$ and $d_{N-K} := -\frac{1}{4} v^T R^{-1} v$.

Inductive step: $K+1 \to K$. Prove $V^{N-K}(x(K)) = x(K)^T P_{N-K} x(K) + b_{N-K}^T x(K) + d_{N-K}$ with

$$\begin{aligned} P_{N-K} &= Q + K_{N-K}^T (R + B^T P_{N-K-1} B) K_{N-K} + A^T P_{N-K-1} A, \\ b_{N-K} &= \big(s^T + 2\bar{K}_{N-K}^T R K_{N-K} + v^T K_{N-K} + 2\bar{K}_{N-K} B^T P_{N-K-1}(A + B K_{N-K}) \\ &\qquad + b_{N-K-1}^T (A + B K_{N-K})\big)^T, \\ d_{N-K} &= \bar{K}_{N-K}^T R \bar{K}_{N-K} + v^T \bar{K}_{N-K} + \bar{K}_{N-K} B^T P_{N-K-1} B \bar{K}_{N-K}^T \\ &\qquad + b_{N-K-1}^T B \bar{K}_{N-K} + d_{N-K-1}, \end{aligned}$$

in which

$$\begin{aligned} K_{N-K} &= -(B^T P_{N-K-1} B + R)^{-1} \left(B^T P_{N-K-1} A\right), \\ \bar{K}_{N-K} &= -\frac{1}{2} (B^T P_{N-K-1} B + R)^{-1} \left(B^T b_{N-K-1} + v\right), \end{aligned}$$

using the inductive hypothesis $V^{N-K-1}(x(K+1)) = x(K+1)^T P_{N-K-1} x(K+1) + b_{N-K-1}^T x(K+1) + d_{N-K-1}$.

$$
\begin{aligned}
V^{N-K}(x(K)) &\overset{\text{DPP}}{=} \min_{u(K)} \{\ell(x(K), u(K)) + V^{N-K-1}(x(K+1))\} \\
&= \min_{u(K)} \{x(K)^T Q x(K) + u(K)^T R u(K) + s^T x(K) + v^T u(K) \\
&\qquad + V^{N-K-1}(x(K+1))\} \\
&\overset{(*)}{=} \min_{u(K)} \{x(K)^T Q x(K) + u(K)^T R u(K) + s^T x(K) + v^T u(K) \\
&\qquad + x(K+1)^T P_{N-K-1} x(K+1) + b_{N-K-1}^T x(K+1) + d_{N-K-1}\} \\
&\overset{(\#)}{=} \min_{u(K)} \{x(K)^T Q x(K) + u(K)^T R u(K) + s^T x(K) + v^T u(K) \\
&\qquad + (Ax(K) + Bu(K))^T P_{N-K-1} (Ax(K) + Bu(K)) \\
&\qquad + b_{N-K-1}^T (Ax(K) + Bu(K)) + d_{N-K-1}\},
\end{aligned}
$$

in which we used the inductive hypothesis in $(*)$ and in $(\#)$ the system dynamics. Computation of the first and second partial derivative of the last expression wrt $u(K)$ yields

$$
\begin{aligned}
u^\star(K) &= -(B^T P_{N-K-1} B + R)^{-1} \left(B^T P_{N-K-1} A x(K) + \frac{1}{2}(B^T b_{N-K-1} + v) \right) \\
&=: K_{N-K} x(K) + \bar{K}_{N-K}.
\end{aligned}
$$

We note that the inverse of $B^T P_{N-K-1} B + R$ exists due to symmetry of P_{N-K-1} and R. Plugging in the optimal control value, we get

$$
\begin{aligned}
V^{N-K}(x(K)) &= x(K)^T Q x(K) + u^\star(K)^T R u^\star(K) + s^T x(K) + v^T u^\star(K) \\
&\quad + (Ax(K) + Bu^\star(K))^T P_{N-K-1} (Ax(K) + Bu^\star(K)) \\
&\quad + b_{N-K-1}^T (Ax(K) + Bu^\star(K)) + d_{N-K-1} \\
&= x(K)^T \left[Q + K_{N-K}^T (R + B^T P_{N-K-1} B) K_{N-K} + A^T P_{N-K-1} A \right] x(K) \\
&\quad + \big[s^T + 2\bar{K}_{N-K}^T R K_{N-K} + v^T K_{N-K} + 2\bar{K}_{N-K} B^T P_{N-K-1} (A + BK_{N-K}) \\
&\qquad + b_{N-K-1}^T (A + BK_{N-K}) \big] x(K) \\
&\quad + \bar{K}_{N-K}^T R \bar{K}_{N-K} + v^T \bar{K}_{N-K} + \bar{K}_{N-K} B^T P_{N-K-1} B \bar{K}_{N-K}^T \\
&\qquad + b_{N-K-1}^T B \bar{K}_{N-K} + d_{N-K-1} \\
&=: x(K)^T P_{N-K} x(K) + b_{N-K}^T x(K) + d_{N-K}.
\end{aligned}
$$

This finishes the inductive step. □

This means that for a given horizon $N \in \mathbb{N}$, the optimal value function $V^N(x)$ can be calculated from the iteration (A.1) using the given initial conditions.

We point out that the iteration for P_N is exactly the same as for the standard linear quadratic regulator (cf. [2]). Moreover, the optimal control sequence $\mathbf{u}^\star$ for the finite-dimensional OCP of length N is affine, viz.

$$
u^\star(k) = K_{N-k} x(k) + \bar{K}_{N-k}, \quad k = 0, \ldots, N-1.
$$

The corresponding optimal trajectory is then given by

$$x(k+1, x_0) = (A + BK_{N-k})x(k, x_0) + B\bar{K}_{N-k}, \quad k = 0, \ldots, N-1.$$

Acronyms and Glossary

Acronyms

CSTR continuously stirred tank reactor

DP Dynamic Programming

DPP Dynamic Programming Principle

LF Lyapunov function

LMI linear matrix inequality

MO multiobjective

MPC Model Predictive Control

NE Nash equilibrium

OCP optimal control problem

PO Pareto optimum

POS Pareto-optimal solution

wlog without loss of generality

Glossary

$\succ$ Denotes positive definiteness of a matrix

N Optimization horizon in MPC

u Control value

$\mathbf{u}$ Sequence of control values

$\mathbf{u}_x^{\star,N}$ (Pareto-)Optimal control sequence of length N for initial value x

$\mathbf{u}_x^{e,N}$ Nash strategy of length N for initial value x

x State variable

κ Auxiliary feedback that is defined on the terminal region $\mathbb{X}_0$

$\mathcal{B}_\varepsilon(x)$ Set of states with distance to x strictly less than ε

$\mathbb{U}$ Set of admissible control values

$\mathbb{U}^N(x)$ Set of control sequences of length $N \in \mathbb{N} \cup \{\infty\}$ for initial value x, such that all state and control constraints are met

$\mathbb{X}$ Set of admissible states

$\mathbb{X}_0$ Terminal constraint set

$\mathbb{X}_N$ Set of initial values, such that there exists a control sequence, which steers the initial value into $\mathbb{X}_0$ in N steps

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(Marleen Stieler)