AN IMPROVEMENT OF THE JOHNSON BOUND FOR SUBSPACE CODES

MICHAEL KIERMAIER AND SASCHA KURZ

ABSTRACT. Subspace codes, i.e., sets of subspaces of a finite ambient vector space, are applied in random linear network coding. Here we give improved upper bounds based on the Johnson bound and a connection to divisible codes, which is presented in a purely geometrical way. In part, our result is based on a characterization of the lengths of full-length q^r-divisible $\mathbb{F}_q$-linear codes.

This complements a recent approach for upper bounds on the maximum size of partial spreads based on projective q^r-divisible $\mathbb{F}_q$-linear codes.

1. INTRODUCTION

Let $\mathbb{F}_q$ be the finite field with q elements, where $q > 1$ is a prime power. By $\mathbb{F}_q^v$ we denote the v-dimensional vector space over $\mathbb{F}_q$, where $v \geq 1$. The set of all subspaces of $\mathbb{F}_q^v$, ordered by the incidence relation $\subseteq$, is called *$(v-1)$-dimensional projective geometry over* $\mathbb{F}_q$ and denoted by $\mathrm{PG}(v-1,\mathbb{F}_q) = \mathrm{PG}(\mathbb{F}_q^v)$. It forms a finite modular geometric lattice with meet $X \wedge Y = X \cap Y$, join $X \vee Y = X + Y$, and rank function $X \mapsto \dim(X)$. We will use the term *k-subspace* to denote a k-dimensional vector subspace of $\mathbb{F}_q^v$. The set of all k-subspaces of $V = \mathbb{F}_q^v$ will be denoted by $\begin{bmatrix} V \\ k \end{bmatrix}_q$. Its cardinality is given by the Gaussian binomial coefficient

$$\begin{bmatrix} v \\ k \end{bmatrix}_q = \begin{cases} \frac{(q^v-1)(q^{v-1}-1)\cdots(q^{v-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} & \text{if } 0 \leq k \leq v; \\ 0 & \text{otherwise.} \end{cases}$$

The geometry $\mathrm{PG}(v-1,\mathbb{F}_q)$ serves as input and output alphabet of the so-called *linear operator channel (LOC)* – a model for information transmission in coded packet networks subject to noise [17]. The relevant metrics on the LOC are given by the *subspace distance* $d_S(X,Y) := \dim(X+Y) - \dim(X \cap Y) = 2 \cdot \dim(X+Y) - \dim(X) - \dim(Y)$, which can also be seen as the graph-theoretic distance in the Hasse diagram of $\mathrm{PG}(v-1,\mathbb{F}_q)$, and the *injection distance* $d_I(X,Y) := \max\{\dim(X), \dim(Y)\} - \dim(X \cap Y)$. A set $\mathscr{C}$ of subspaces of $\mathbb{F}_q^v$ is called a *subspace code*. For $\#\mathscr{C} \geq 2$, the *minimum (subspace) distance* of $\mathscr{C}$ is given by $d = \min\{d_S(X,Y) \mid X,Y \in \mathscr{C}, X \neq Y\}$. If all elements of $\mathscr{C}$ have the same dimension, we call $\mathscr{C}$ a *constant-dimension code*. For a constant-dimension code $\mathscr{C}$ we have $d_S(X,Y) = 2d_I(X,Y)$ for all $X,Y \in \mathscr{C}$, so that we can restrict attention to the subspace distance, which has to be even. By $\mathrm{A}_q(v,d;k)$ we denote the maximum possible cardinality of a constant-dimension-k code in $\mathbb{F}_q^v$ with minimum subspace distance at least d. Like in the classical case of codes in the Hamming metric, the determination of the exact value or bounds for $\mathrm{A}_q(v,d;k)$ is a central problem. In this paper we will present some improved upper bounds. For a broader background we refer to [8, 9] and for the latest numerical bounds to the online tables at `http://subspacecodes.uni-bayreuth.de` [12].

For a subspace $U \leq \mathbb{F}_q^v$, the orthogonal subspace with respect to some fixed non-degenerate symmetric bilinear form will be denoted $U^\perp$. It has dimension $\dim(U^\perp) = v - \dim(U)$. For $U, W \leq \mathbb{F}_q^v$, we get that $\mathrm{d_S}(U,W) = \mathrm{d_S}(U^\perp, W^\perp)$. So, $\mathrm{A}_q(v,d;k) = \mathrm{A}_q(v,d;v-k)$ and we can assume $0 \leq k \leq \frac{v}{2}$ in the following. If $d > 2k$, then $\mathrm{A}_q(v,d;k) = 1$. Furthermore, we have $\mathrm{A}_q(v,2;k) = \begin{bmatrix} v \\ k \end{bmatrix}_q$. Things get more interesting for $v, d \geq 4$ and $k \geq 2$.

Let $\mathscr{C}$ be a constant-dimension-k code in $\mathbb{F}_q^v$ with minimum distance d. For every point P, i.e., 1-subspace, of $\mathbb{F}_q^v$ we can consider the quotient geometry $\mathrm{PG}(\mathbb{F}_q^v/P)$ to deduce that at most $\mathrm{A}_q(v-1,d;k-1)$ elements of $\mathscr{C}$ contain P. Since $\mathrm{PG}(\mathbb{F}_q^v)$ contains $\begin{bmatrix} v \\ 1 \end{bmatrix}_q$ points and every k-subspace contains $\begin{bmatrix} k \\ 1 \end{bmatrix}_q$ points, we obtain

$$\mathrm{A}_q(v,d;k) \le \left\lfloor \frac{\begin{bmatrix} v \\ 1 \end{bmatrix}_q \cdot \mathrm{A}_q(v-1,d;k-1)}{\begin{bmatrix} k \\ 1 \end{bmatrix}_q} \right\rfloor = \left\lfloor \frac{q^v-1}{q^k-1} \cdot \mathrm{A}_q(v-1,d;k-1) \right\rfloor, \qquad (1)$$

which was named Johnson type bound II in [27]. Recursively applied, we obtain

$$\mathrm{A}_q(v,d;k) \le \left\lfloor \frac{q^v-1}{q^k-1} \cdot \left\lfloor \frac{q^{v-1}-1}{q^{k-1}-1} \cdot \left\lfloor \cdots \left\lfloor \frac{q^{v'+1}-1}{q^{d/2+1}-1} \cdot \mathrm{A}_q(v',d;d/2) \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor, \qquad (2)$$

where $v' = v-k+d/2$.

In the case $d = 2k$, any two codewords of $\mathscr{C}$ intersect trivially, meaning that each point of $\mathrm{PG}(\mathbb{F}_q^v)$ is covered by at most a single codeword. These codes are better known as *partial k-spreads*. If all the points are covered, we have $\#\mathscr{C} = \begin{bmatrix} v \\ 1 \end{bmatrix}_q / \begin{bmatrix} k \\ 1 \end{bmatrix}_q$ and $\mathscr{C}$ is called a *k-spread*. From the work of Segre in 1964 [23, §VI] we know that k-spreads exist if and only if k divides v. Upper bounds for the size of a partial k-spreads are due to Beutelspacher [2] and Drake & Freeman [7] and date back to 1975 and 1979, respectively. Starting from [18] several recent improvements have been obtained. Currently the tightest upper bounds, besides k-spreads, are given by a list of 21 sporadic 1-parametric series and the following two theorems stated in [19]:

Theorem 1. *For integers $r \ge 1$, $t \ge 2$, $u \ge 0$, and $0 \le z \le \begin{bmatrix} r \\ 1 \end{bmatrix}_q/2$ with $k = \begin{bmatrix} r \\ 1 \end{bmatrix}_q + 1 - z + u > r$ we have $\mathrm{A}_q(v,2k;k) \le lq^k + 1 + z(q-1)$, where $l = \frac{q^{v-k}-q^r}{q^k-1}$ and $v = kt+r$.*

Theorem 2. *For integers $r \ge 1$, $t \ge 2$, $y \ge \max\{r,2\}$, $z \ge 0$ with $\lambda = q^y$, $y \le k$, $k = \begin{bmatrix} r \\ 1 \end{bmatrix}_q + 1 - z > r$, $v = kt+r$, and $l = \frac{q^{v-k}-q^r}{q^k-1}$, we have $\mathrm{A}_q(v,2k;k) \le$*

$$lq^k + \left\lceil \lambda - \frac{1}{2} - \frac{1}{2}\sqrt{1+4\lambda\left(\lambda - (z+y-1)(q-1)-1\right)} \right\rceil.$$

The special case $z = 0$ in Theorem 1 covers the breakthrough $\mathrm{A}_q(kt+r,2k;k) = 1 + \sum_{s=1}^{t-1} q^{sk+r}$ for $0 < r < k$ and $k > \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ by Năstase and Sissokho [22] from 2016, which itself covers the result of Beutelspacher. The special case $y = k$ in Theorem 2 covers the result by Drake & Freeman. A contemporary survey of the best known upper bounds for partial spreads can be found in [16].

Using the tightest known upper bounds for the sizes of partial k-spreads, there are only two known cases with $d < 2k$ where Inequality (2) is not sharp: $\mathrm{A}_2(6,4;3) = 77 < 81$ [15] and $\mathrm{A}_2(8,6;4) = 257 < 289$ [14, 11]. For the details how the proposed upper bounds for constant-dimension codes relate to Inequality (2) we refer the interested reader to [1, 13]. The two mentioned improvements of Inequality (2) involve massive computer calculations. In contrast to that, the improvements in this article are based on a self-contained theoretical argument and do not need any external computations.

The remaining part of this paper is organized as follows. In Section 2 we consider q^r-divisible multisets of points which are defined by the property $\#(\mathscr{P} \cap H) \equiv \#\mathscr{P} \pmod{q^r}$ for every hyperplane H. The set of possible cardinalities is completely characterized in Theorem 4 and used to conclude upper bounds for $\mathrm{A}_q(v,d;k)$ in Theorem 3. While it is possible to formulate the entire approach in geometrical terms, the underlying structure can possibly be best understood in terms of q^r-divisible linear codes and the linear programming method, which is the topic of Section 3. We draw a short conclusion in Section 4.

2. Main result

A multiset $\mathscr{S}$ on a base set X can be identified with its characteristic function $\chi_X : X \to \mathbb{N}_0$, mapping x to the multiplicity of x in $\mathscr{S}$. The *cardinality* of $\mathscr{S}$ is $\#\mathscr{S} = \sum_{x\in X} \chi_{\mathscr{S}}(x)$.

The multiset union $\mathscr{S} \uplus \mathscr{S}'$ of two multisets $\mathscr{S}$ and $\mathscr{S}'$ is given by the sum $\chi_{\mathscr{S}} + \chi_{\mathscr{S}'}$ of the corresponding characteristic functions. The q-fold repetition $q\mathscr{S}$ of a multiset $\mathscr{S}$ is given by the characteristic function $q\chi_{\mathscr{S}}$.

Let V be a vector space over $\mathbb{F}_q$ of finite dimension v. We call every 1-subspace of V a *point* and every $(v-1)$-subspace of V a *hyperplane* in V. For a multiset of points $\mathscr{P}$ in V and a hyperplane $H \leq V$, we define the restricted multiset $\mathscr{P} \cap H$ via its characteristic function

$$\chi_{\mathscr{P}\cap H}(P) = \begin{cases} \chi_{\mathscr{P}}(P) & \text{if } P \leq H; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\#(\mathscr{P} \cap H) = \sum_{P \in \left[{H \atop 1}\right]_q} \chi_{\mathscr{P}}(P)$.

Definition 1. Let $\mathscr{P}$ be a multiset of points in V and $r \in \{0, \ldots, v-1\}$. If

$$\#(\mathscr{P} \cap H) \equiv \#\mathscr{P} \pmod{q^r}$$

for every hyperplane $H \leq V$, then $\mathscr{P}$ is called *q^r-divisible*.

If we speak of a q^r-divisible multiset $\mathscr{P}$ of points without specifying the ambient space V or its dimension v, we assume that the points in $\mathscr{P}$ are contained in an ambient space V of a suitable finite dimension v. This is justified by the following lemma:

Lemma 1. *Let $V_1 < V_2$ be $\mathbb{F}_q$-vector spaces and $\mathscr{P}$ a multiset of points in V_1. Then $\mathscr{P}$ is q^r-divisible in V_1 if and only if $\mathscr{P}$ is q^r-divisible in V_2.*

Proof. Assume that $\mathscr{P}$ is q^r-divisible in V_1. Let H be a hyperplane of V_2. Then $\#(\mathscr{P} \cap H) = \#(\mathscr{P} \cap (H \cap V_1))$. $H \cap V_1$ is either V_1 or a hyperplane in V_1. In the first case, the expression equals $\#\mathscr{P}$, and in the second case, it is congruent to $\#\mathscr{P} \pmod{q^r}$ by q^r-divisibility of $\mathscr{P}$ in V_1.

Now assume that $\mathscr{P}$ is q^r-divisible in V_2, and let H' be a hyperplane of V_1. There is a hyperplane H in V_2 such that $H \cap V_1 = H'$. So $\#(\mathscr{P} \cap H') = \#(\mathscr{P} \cap H) \equiv \#\mathscr{P} \pmod{q^r}$ by q^r-divisibility of $\mathscr{P}$ in V_2. □

Lemma 2. *Let $\mathscr{P}$ be a q^r-divisible multiset of points in V and U a subspace of V of codimension $j \in \{0, \ldots, r\}$. Then the restriction $\mathscr{P} \cap U$ is a q^{r-j}-divisible multiset in U.*

Proof. By induction, it suffices to consider the case $j = 1$. Let W be a hyperplane of U, that is a subspace of V of codimension 2. There are $q+1$ hyperplanes $H_1, \ldots, H_{q+1}$ in V containing W (U being one of them). From the q^r-divisibility of $\mathscr{P}$ we get

$$(q+1)\#\mathscr{P} \equiv \sum_{i=1}^{q+1} \#(\mathscr{P} \cap H_i) = q \cdot \#(\mathscr{P} \cap W) + \#\mathscr{P} \pmod{q^r}.$$

Hence $q \cdot \#(\mathscr{P} \cap W) \equiv q \cdot \#\mathscr{P} \equiv q \cdot \#(\mathscr{P} \cap U) \pmod{q^r}$ and thus

$$\#(\mathscr{P} \cap W) \equiv \#(\mathscr{P} \cap U) \pmod{q^{r-1}}.$$

□

Lemma 3. *(a) For a k-subspace $U \leq V$ with $k \geq 1$, the set $\left[{U \atop 1}\right]_q$ of points contained in U is q^{k-1}-divisible.*

(b) For q^r-divisible multisets $\mathscr{P}$ and $\mathscr{P}'$ in V, the multiset union $\mathscr{P} \uplus \mathscr{P}'$ is q^r-divisible.

(c) The q-fold repetition of a q^r-divisible multiset $\mathscr{P}$ is q^{r+1}-divisible.

Proof. For part (a), let H be a hyperplane of $\mathbb{F}_q^v$. The dimension formula gives $\dim(U \cap H) \in \{k, k-1\}$. So $\#(\left[{U \atop 1}\right]_q \cap H)$ is either $\left[{k \atop 1}\right]_q$ or $\left[{k-1 \atop 1}\right]_q$. This implies

$$\#\left(\begin{bmatrix} U \\ 1 \end{bmatrix}_q \cap H\right) \equiv \begin{bmatrix} k \\ 1 \end{bmatrix}_q = \#\begin{bmatrix} U \\ 1 \end{bmatrix}_q \pmod{q^{k-1}}.$$

Parts (b) and (c) are clear from looking at the characteristic functions. □

A subspace $U \le V$ is commonly identified with the set $\begin{bmatrix} U \\ 1 \end{bmatrix}_q$ of points covered by U. With that identification, Lemma 3(a) simply states that every k-subspace is q^{k-1}-divisible. For a multiset $\mathscr{U}$ of subspaces of V, we will call the multiset union $\biguplus_{U\in\mathscr{U}} \begin{bmatrix} U \\ 1 \end{bmatrix}_q$ the *associated multiset of points*.[1]

Lemma 4. *Let $\mathscr{U}$ be a multiset of subspaces of V and k the smallest dimension among the subspaces in $\mathscr{U}$. Let $\mathscr{P} = \biguplus_{U\in\mathscr{U}} \begin{bmatrix} U \\ 1 \end{bmatrix}_q$ be the associated multiset of points. If $k \ge 1$, then*

$$\#\mathscr{P} \equiv \#(\mathscr{P} \cap H) \pmod{q^{k-1}}.$$

for all hyperplanes H of V.

Proof. Apply Lemma 3(a) and (b). □

Corollary 1. *Let $\mathscr{C}$ be a constant-dimension-k code in V with $k \ge 1$. Then the associated multiset of points is q^{k-1}-divisible.*

Note that Corollary 1 does not depended on the minimum distance of the code. It will be invoked indirectly by the following complement-type construction.

Corollary 2. *If a multiset of points $\mathscr{P}$ in V is q^r-divisible with $r < v$ and satisfies $\chi_{\mathscr{P}}(P) \le \lambda$ for all points $P \in \begin{bmatrix} V \\ 1 \end{bmatrix}_q$, then the complementary multiset $\bar{\mathscr{P}}$ defined by $\chi_{\bar{\mathscr{P}}}(P) = \lambda - \chi_{\mathscr{P}}(P)$ is also q^r-divisible.*

Proof. By Lemma 3(a), $\begin{bmatrix} V \\ 1 \end{bmatrix}_q$ is q^{v-1}-divisible. By $r < v$, it is q^r-divisible. Now the result follows from $\chi_{\bar{\mathscr{P}}} = \lambda \chi_{\begin{bmatrix} V \\ 1 \end{bmatrix}_q} - \chi_{\mathscr{P}}$. □

The remainder of the Euclidean division of an integer a by an integer $b \ge 1$ is an integer in the range $\{0, \ldots, b-1\}$. It will be denoted by a mod b.

Theorem 3. *For $\delta \in \mathbb{Z}$, we define*

$$m(\delta) = \left(\left(\begin{bmatrix} v \\ 1 \end{bmatrix}_q \cdot \mathrm{A}_q(v-1, d; k-1) \right) \bmod \begin{bmatrix} k \\ 1 \end{bmatrix}_q \right) + \delta \cdot \begin{bmatrix} k \\ 1 \end{bmatrix}_q.$$

If there exists no q^{k-1}-divisible multiset of points in $\mathbb{F}_q^v$ of cardinality $m(\delta)$, then

$$\mathrm{A}_q(v, d; k) \le \left\lfloor \frac{\begin{bmatrix} v \\ 1 \end{bmatrix}_q \cdot \mathrm{A}_q(v-1, d; k-1)}{\begin{bmatrix} k \\ 1 \end{bmatrix}_q} \right\rfloor - \delta - 1 = \left\lfloor \frac{q^v - 1}{q^k - 1} \cdot \mathrm{A}_q(v-1, d; k-1) \right\rfloor - \delta - 1.$$

Proof. Let $\mathscr{C}$ be a code with cardinality $\#\mathscr{C} = \left\lfloor \frac{\begin{bmatrix} v \\ 1 \end{bmatrix}_q \cdot \mathrm{A}_q(v-1,d;k-1)}{\begin{bmatrix} k \\ 1 \end{bmatrix}_q} \right\rfloor - \delta$ and matching parameters. Let $\mathscr{P}$ be the associated multiset of points. As in the reasoning for the Johnson bound (1), the maximum multiplicity of $\mathscr{P}$ is at most $\lambda = \mathrm{A}_q(v-1, d; k-1)$. Let $\bar{\mathscr{P}}$ be the complementary multiset as in Lemma 2. Then

$$\#\bar{\mathscr{P}} = \mathrm{A}_q(v-1, d; k-1) \cdot \begin{bmatrix} v \\ 1 \end{bmatrix}_q - \begin{bmatrix} k \\ 1 \end{bmatrix}_q \left(\left\lfloor \frac{\begin{bmatrix} v \\ 1 \end{bmatrix}_q \cdot \mathrm{A}_q(v-1, d; k-1)}{\begin{bmatrix} k \\ 1 \end{bmatrix}_q} \right\rfloor - \delta \right) = m(\delta)$$

and by Corollary 1 and Lemma 2, $\bar{\mathscr{P}}$ is q^{k-1}-divisible. This contradicts the assertion of the Theorem that no q^r-divisible multiset of size $m(\delta)$ exists. □

Remark 1. Theorem 3 can always be applied with $\delta = -1$ since there is no q^r-divisible multiset of points of size $m(-1) < 0$. The resulting bound is precisely the Johnson bound (1). To get the best possible improvement, we are looking for the largest possible δ such that no q^r-divisible multiset of points of size $m(\delta)$ exists. By Lemma 3(b) and (a), this optimal $\delta_{\max}$ is characterized by the property that there is no q^r-divisible multiset of size $m(\delta_{\max})$,

[1] In the expression $\biguplus_{U\in\mathscr{U}}$, U is repeated according to its multiplicity in the multiset $\mathscr{U}$.

but there is a q^r-divisible multiset of size $m(\delta_{\max}+1)$. Denoting the "sharpened" rounding down as

$$\left\{\frac{{v \brack 1}_q \cdot \mathrm{A}_q(v-1,d;k-1)}{{k \brack 1}_q}\right\} = \left\lfloor\frac{{v \brack 1}_q \cdot \mathrm{A}_q(v-1,d;k-1)}{{k \brack 1}_q}\right\rfloor - \delta_{\max} - 1,$$

the improved Johnson bound of Theorem 3 can simply be written as

$$\mathrm{A}_q(v,d;k) \le \left\{\frac{{v \brack 1}_q \cdot \mathrm{A}_q(v-1,d;k-1)}{{k \brack 1}_q}\right\}.$$

With $v' = v-k+d/2$, iterated application yields

$$\mathrm{A}_q(v,d;k) \le \left\{\frac{q^v-1}{q^k-1}\cdot\left\{\frac{q^{v-1}-1}{q^{k-1}-1}\cdot\left\{\ldots\left\{\frac{q^{v'+1}-1}{q^{d/2+1}-1}\cdot \mathrm{A}_q(v',d;d/2)\right\}\ldots\right\}\right\}\right\},$$

which is an improvement of (2).

In view of Theorem 3 it is worthwhile to study the possible cardinalities of q^r-divisible multisets of points.

Lemma 5. *If $\mathscr{P}_1$ and $\mathscr{P}_2$ are q^r-divisible multisets, then there exists a q^r-divisible multiset of cardinality $\#\mathscr{P}_1+\#\mathscr{P}_2$.*

Proof. Let V_1 and V_2 be the ambient space of $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively. Thus, both multisets of points can be embedded in $V_1 \times V_2$. By Lemma 3(a), their multiset union is a q^r-divisible multiset of cardinality $\#\mathscr{P}_1+\#\mathscr{P}_2$. □

Lemma 6. *Let $r \in \mathbb{N}_0$ and $i \in \{0,\ldots,r\}$, there is a q^r-divisible multiset of points of cardinality*

$$s_q(r,i) := q^i \cdot {r-i+1 \brack 1}_q = \frac{q^{r+1}-q^i}{q-1} = \sum_{j=i}^{r} q^j = q^i + q^{i+1} + \ldots + q^r.$$

Proof. A suitable multiset of points is given by the q^i-fold repetition of an $(r-i+1)$-subspace. □

As a consequence of the last two lemmas, all $n = \sum_{i=0}^{r} a_i s_q(r,i)$ with $a_i \in \mathbb{N}_0$ are realizable cardinalities of q^r-divisible multisets of points. As $s_q(r,r) = q^r$ and $s_q(r,0) = 1+q+q^2+\ldots+q^r$ are coprime, for fixed q and r there is only a finite set of cardinalities which is not realizable as a q^r-divisible multiset.

Our goal is to show Theorem 4, which says that actually all possible cardinalities are of the above form.

The numbers $s_q(r,i)$ have the property that they are divisible by q^i, but not by q^{i+1}. This allows us to create kind of a positional system upon the sequence of base numbers

$$S_q(r) = (s_q(r,0), s_q(r,1), \ldots, s_q(r,r)).$$

Lemma 7. *Let $n \in \mathbb{Z}$ and $r \in \mathbb{N}_0$. There exist $a_0,\ldots,a_{r-1} \in \{0,1,\ldots,q-1\}$ and $a_r \in \mathbb{Z}$ with $n = \sum_{i=0}^{r} a_i s_q(r,i)$. Moreover this representation is unique.*

Proof. One checks that Algorithm 1 computes such a representation. The iterations of the loop gradually compute the (only) choice for $a_0,a_1,\ldots,a_{r-1} \in \{0,\ldots,q-1\}$ to make the representation $\sum_{i=0}^{r} a_i s_q(r,i)$ fit modulo $q,q^2,\ldots,q^r$.

For uniqueness, assume that there is a different representation $n = \sum_{i=0}^{r} b_i s_q(r,i)$ with $b_0,\ldots,b_{r-1} \in \{0,\ldots,q-1\}$ and $b_r \in \mathbb{Z}$. Let t be the smallest index i with $a_i \neq b_i$. Then

$$(a_t-b_t)s_q(r,t) = \sum_{i=t+1}^{r} (b_i-a_i)s_q(r,i).$$

As $s_q(r,i)$ is divisible by q^i but not by q^{i+1}, the right hand side is divisible by q^{t+1}, but the left hand side is not, which is a contradiction. □

Algorithm 1
Data: $n \in \mathbb{Z}$, field size q, exponent r
Result: representation $n = \sum_{i=0}^{r} a_i s_q(r,i)$ with $a_0, \ldots, a_{r-1} \in \{0, \ldots, q-1\}$ and $a_r \in \mathbb{Z}$
$m \leftarrow n$
for $i \leftarrow 0$ **to** $r-1$ **do**
    $a_i \leftarrow m \bmod q$
    $m \leftarrow (m - a_i \cdot \left[\begin{smallmatrix} r-i+1 \\ 1 \end{smallmatrix}\right]_q)/q$
end
$a_r \leftarrow m$

Definition 2. The unique representation $n = \sum_{i=0}^{r} a_i s_q(r,i)$ of Lemma 7 will be called the *$S_q(r)$-adic expansion* of n. The number a_r will be called the *leading coefficient* and the number $\sigma = \sum_{i=0}^{r} a_i$ will be called the *cross sum* of the $S_q(r)$-adic expansion.

Example 1. For $q = 3$, $r = 3$, we have $S_3(3) = (40, 39, 36, 27)$. For $n = 137$, Algorithm 1 computes

$$\begin{aligned}
m &\leftarrow 137, \\
a_0 &\leftarrow 137 \bmod 3 = 2, \\
m &\leftarrow \left(137 - 2 \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix}_3\right)/3 = (137 - 2 \cdot 40)/3 = 19, \\
a_1 &\leftarrow 19 \bmod 3 = 1, \\
m &\leftarrow \left(19 - 1 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}_3\right)/3 = (19 - 1 \cdot 13)/3 = 2, \\
a_2 &\leftarrow 2 \bmod 3 = 2, \\
m &\leftarrow \left(2 - 2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}_3\right)/3 = (2 - 2 \cdot 4)/3 = -2, \\
a_3 &\leftarrow -2.
\end{aligned}$$

Therefore, the $S_3(3)$-adic expansion of 137 is

$$137 = 2 \cdot 40 + 1 \cdot 39 + 2 \cdot 36 + (-2) \cdot 27.$$

The leading coefficient is $a_3 = -2$, and the cross sum is $2 + 1 + 2 + (-2) = 3$.

We prepare one more lemma for the proof of Theorem 4, which guarantees the existence of a hyperplane containing not too many points of $\mathcal{P}$ by an averaging argument.

Lemma 8. *Let $\mathcal{P}$ be a non-empty multiset of points. Then there exists a hyperplane H with $\#(\mathcal{P} \cap H) < \frac{\#\mathcal{P}}{q}$.*

Proof. Let V be a suitable ambient space of $\mathcal{P}$ of finite dimension v. Summing over all hyperplanes H gives $\sum_{H \in \left[\begin{smallmatrix} V \\ v-1 \end{smallmatrix}\right]_q} \#(\mathcal{P} \cap H) = \#\mathcal{P} \cdot \left[\begin{smallmatrix} v-1 \\ 1 \end{smallmatrix}\right]_q$, so that we obtain on average

$$\frac{\#\mathcal{P} \cdot \left[\begin{smallmatrix} v-1 \\ 1 \end{smallmatrix}\right]_q}{\left[\begin{smallmatrix} v \\ 1 \end{smallmatrix}\right]_q} = \frac{\#\mathcal{P} \cdot \left[\begin{smallmatrix} v-1 \\ 1 \end{smallmatrix}\right]_q}{q\left[\begin{smallmatrix} v-1 \\ 1 \end{smallmatrix}\right]_q + 1} < \frac{\#\mathcal{P}}{q}$$

points of $\mathcal{P}$ per hyperplane. Choosing a hyperplane H that minimizes $\#(\mathcal{P} \cap H)$ completes the proof. $\square$

Theorem 4. *Let $n \in \mathbb{Z}$ and $r \in \mathbb{N}_0$. The following are equivalent:*
(i) There exists a q^r-divisible multiset of points of cardinality n.
(ii) The leading coefficient of the $S_q(r)$-adic expansion of n is non-negative.

Proof. The implication "(ii) $\Rightarrow$ (i)" follows from Lemma 5 and 6.

The main part of the proof is the verification of "(i) $\Rightarrow$ (ii)". The statement is clear for $r = 0$ and $n \leq 0$, so we may assume $r \geq 1$ and $n \geq 1$.

Let $\mathscr{P}$ be a q^r-divisible multiset of points of size $n = \#\mathscr{P} \geq 1$. Let $n = \sum_{i=0}^{r} a_i s_q(r,i)$ with $a_0, \ldots, a_{r-1} \in \{0, 1, \ldots, q-1\}$ and $a_r \in \mathbb{Z}$ be the $S_q(r)$-adic expansion of n (see Lemma 7) and $\sigma = \sum_{i=0}^{r} a_i$ its cross sum.

Let H be a hyperplane in V and $m = \#(\mathscr{P} \cap H)$. By the q^r-divisibility of $\mathscr{P}$ we have $n - m = \tau q^r$ with $\tau \in \mathbb{Z}$. Using $s_q(r,i) = s_q(r-1,i) + q^r$, we get

$$m = n - \tau q^r = \sum_{i=0}^{r-1} a_i (s_q(r-1,i) + q^r) + a_r q^r - \tau q^r$$

$$= \sum_{i=0}^{r-1} a_i s_q(r-1,i) + (\sigma - \tau) q^r \tag{3}$$

$$= \sum_{i=0}^{r-2} a_i s_q(r-1,i) + (a_{r-1} + q(\sigma - \tau)) q^{r-1}. \tag{4}$$

By Lemma 2, $\mathscr{P} \cap H$ is a q^{r-1}-divisible multiset of size m, and line (4) is the $S_q(r-1)$-adic expansion of m. Hence by induction over r, we get that $a_{r-1} + q(\sigma - \tau) \geq 0$. So $q(\sigma - \tau) \geq -a_{r-1} > -q$, implying that $\sigma - \tau > -1$ and thus $\sigma \geq \tau$.

By Lemma 8, we may chose H such that $m < \frac{n}{q}$. Thus, using the expression for m from line (3) together with $q s_q(r-1,i) = s_q(r,i+1)$ and $s_q(r,i) - s_q(r,i+1) = q^i$, we get

$$0 < n - qm = \sum_{i=0}^{r} a_i s_q(r,i) - \sum_{i=0}^{r-1} a_i s_q(r,i+1) - (\sigma - \tau) q^{r+1}$$
$$= \sum_{i=0}^{r-1} a_i q^i + a_r q^r - (\sigma - \tau) q^{r+1} \leq \sum_{i=0}^{r-1} (q-1) q^i + a_r q^r = (q^r - 1) + a_r q^r < (1 + a_r) q^r.$$

Therefore $1 + a_r > 0$ and finally $a_r \geq 0$. $\square$

Remark 2. By Theorem 4, the $S_q(r)$-adic expansion of n provides a certificate not only for the existence, but remarkably also for the non-existence of a q^r-divisible multiset of size n.

Remark 3. The above proof shows that if $\mathscr{P}$ is a non-empty q^r-divisible multiset of size n and σ is the cross sum of the $S_q(r)$-adic expansion of n, we have $\#\mathscr{P} - \#(\mathscr{P} \cap H) = \tau q^r$ with $\tau \leq \sigma$ for every hyperplane H. In other words, the maximum weight of a full-length linear q^r-divisible code of length n over $\mathbb{F}_q$ is at most σq^r.

Remark 4. The proof of Theorem 4 uses the q^r-divisibility of $\mathscr{P}$ only in two places: For the hyperplane H containing less than the average number of points, and for invoking Lemma 2, telling us that the restriction of $\mathscr{P}$ to this hyperplane H is q^{r-1}-divisible. Restricting the requirements to what was actually needed in the proof, let us call a multiset $\mathscr{P}$ of points *weakly q^r-divisible* if $r = 0$ or if there is a hyperplane H such that $\#(\mathscr{P} \cap H) < \frac{\#\mathscr{P}}{q}$ and $\#\mathscr{P} \equiv \#(\mathscr{P} \cap H) \pmod{q^r}$ and $\mathscr{P} \cap H$ is weakly q^{r-1}-divisible. The statement of Theorem 4 is still true for weakly q^r-divisible multisets of points.

There are many more weakly q^r-divisible multisets of points than q^r-divisible ones. As an example, any multiset $\mathscr{P}$ of points of size $\#\mathscr{P} = q$ in the projective line $\operatorname{PG}(\mathbb{F}_q^2)$ is weakly q-divisible: Since $\left[{2 \atop 1}\right]_q = q + 1 > q$, the projective line contains a point P not contained in $\mathscr{P}$ which provides a suitable hyperplane H for the definition. The only q-divisible multiset of this type is a single point of multiplicity q.

Example 2. So far, the best known upper bound on $\mathrm{A}_2(9,6;4)$ has been given by the Johnson bound (1), using $\mathrm{A}_2(8,6;3) = 34$:

$$\mathrm{A}_2(9,6;4) \leq \left\lfloor \frac{2^9 - 1}{2^4 - 1} \cdot \mathrm{A}_2(8,6;3) \right\rfloor = 1158.$$

To improve that bound by Theorem 3, we are looking for the largest value of δ such that no q^{k-1}-divisible multiset of size

$$m(\delta) = \begin{bmatrix}9\\1\end{bmatrix}_2 \cdot \mathrm{A}_2(8,6;3) - \begin{bmatrix}4\\1\end{bmatrix}_2 \cdot 1158 + \begin{bmatrix}4\\1\end{bmatrix}_2 \delta = 4 + 15\delta$$

exists.

This question can be investigated with Theorem 4. We have $S_2(3) = (15,14,12,8)$. The $S_2(3)$-adic expansion of $m(1) = 4 + 1 \cdot 15$ is $19 = 1 \cdot 15 + 0 \cdot 14 + 1 \cdot 12 + (-1) \cdot 8$. As the leading coefficient -1 is negative, there is no 8-divisible multiset of points of size 19 by Theorem 4. The $S_2(3)$-adic expansion of $m(2) = 4 + 2 \cdot 15$ is $34 = 0 \cdot 15 + 1 \cdot 14 + 1 \cdot 12 + 1 \cdot 8$. As the leading coefficient 1 is not negative, there is a 8-divisible multiset of points of size 34.

So the best possible value is $\delta = 1$, for which we obtain the improved upper bound

$$\mathrm{A}_2(9,6;4) \le 1158 - 2 = 1156.$$

We look at an application to partial spreads $\mathscr{S}$, which are subspace codes with $d = 2k$. In other words, each point is covered by at most one element of $\mathscr{S}$. For $k \mid v$, it is possible to cover all the points by the existence of spreads and thus $A_q(v,2k;k) = \frac{q^v-1}{q^k-1}$.

The more involved situation is $k \nmid v$ where no spread exists. The points which remain uncovered are called *holes* of $\mathscr{S}$. By Corollary 2, the set of holes is q^{k-1}-disivible, as it is the complementary point set of $\mathscr{S}$ with $\lambda = 1$.

We write $v = tk + r$ with $r \in \{1,\ldots,k-1\}$. For $t = 1$, any to k-subspaces intersect nontrivially, so $A_q(k+r,2k;k) = 1$. For $t \ge 2$, there exists a partial $(k-1)$-spread $\mathscr{S}$ of size $\#\mathscr{S} = \sum_{i=1}^{t-1} q^{ki+r} + 1 = \frac{q^v - q^{k+r}}{q^k-1} + 1$ by [2, Th. 4.2]. This construction implies that $A_q(v,2k;k) \ge \frac{q^v-q^{k+r}}{q^k-1} + 1$. From the same article we know that this construction is optimal whenever $r = 1$ [2, Th. 4.1]. Recently, it has been shown in [22, Theorem 5] that the same is true in many more cases. In fact, this result is a direct consequence of our classification of realizable lengths of divisible codes in Theorem 4.

Corollary 3 ([22, Theorem 5]). *Let $v = tk + r$ with $r \in \{1,\ldots,k-1\}$ and $t \ge 2$. For $k > \begin{bmatrix}r\\1\end{bmatrix}_q$ we have*

$$A_q(v,2k;k) = \frac{q^v - q^{k+r}}{q^k - 1} + 1.$$

Proof. Assume that $\mathscr{S}$ is a partial $(k-1)$-spread of size $\#\mathscr{S} = \frac{q^v-q^{k+r}}{q^k-1} + 2$. Its set $\mathscr{P}$ of holes is q^{k-1}-divisible of size $\#\mathscr{P} = \begin{bmatrix}k+r\\1\end{bmatrix}_q - 2\begin{bmatrix}k\\1\end{bmatrix}_q$. We have

$$\sum_{i=0}^{k-2}(q-1)s_q(k-1,i) + \left(q \cdot (\begin{bmatrix}r\\1\end{bmatrix}_q - k + 1) - 1\right) s_q(k-1,k-1) \tag{5}$$

$$= \sum_{i=0}^{k-2}(q^k - q^i) - (k-1)q^k - q^{k-1} + q^k \cdot \begin{bmatrix}r\\1\end{bmatrix}_q$$

$$= -\left(\frac{q^{k-1}-1}{q-1} + q^{k-1}\right) + \frac{q^{k+r}-q^k}{q-1} = \frac{q^{k+r} - 2q^k + 1}{q-1} = \#\mathscr{P}.$$

So (5) is the $S_q(r)$-adic expansion of $\#\mathscr{P}$ and by Theorem 4, its leading coefficient $q \cdot (\begin{bmatrix}r\\1\end{bmatrix}_q - k + 1) - 1$ is ≥ 0. Equivalently $k \le \begin{bmatrix}r\\1\end{bmatrix}_q$, which is a contradiction. □

As a slight generalization of Corollary 3, we get

Proposition 1. *Assume that $k \nmid v$ and let $v = tk + r$ with $r \in \{1,\ldots,k-1\}$. Then*

$$A_q(v,2k;k) \le \frac{q^v - q^{k+r}}{q^k - 1} + q\left(\begin{bmatrix}r\\1\end{bmatrix}_q - k - 1\right) + 1.$$

Proof. Let $z = \begin{bmatrix} r \\ 1 \end{bmatrix}_q - k - 1$ and assume that $\mathscr{U}$ is a set of $\frac{q^v - q^{k+r}}{q^k - 1} + qz + 2$ pairwise disjoint k-spaces in $\mathbb{F}_q^v$. The set of uncovered points $\mathscr{P}$, i.e., the complementary multiset for $\lambda = 1$, has cardinality

$$\begin{aligned} \begin{bmatrix} k+r \\ 1 \end{bmatrix}_q - 2\begin{bmatrix} k \\ 1 \end{bmatrix}_q - zq\begin{bmatrix} k \\ 1 \end{bmatrix}_q &= q^k \cdot \begin{bmatrix} r \\ 1 \end{bmatrix}_q - \begin{bmatrix} k \\ 1 \end{bmatrix}_q - q^k \cdot z + z - z\begin{bmatrix} k \\ 1 \end{bmatrix}_q \\ &= -(1+u) \cdot q^k + (q-1) \cdot \sum_{i=0}^{k-2} q^i \begin{bmatrix} k-i \\ 1 \end{bmatrix}_q - zq\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q. \end{aligned}$$

Write $z = b_{k-2}q^{k-2} + \sum_{i=0}^{k-3} b_i q^i$ for integers b_i with $0 \le b_i \le q-1$ for $0 \le i \le k-3$ and $b_{k-2} \ge 0$. By construction $\mathscr{P}$ is q^{k-1}-divisible. However $\#\mathscr{P}$ equals

$$\begin{aligned} &-((1+u)q + b_{k-2}) \cdot q^{k-1} + (q-1) \cdot \sum_{i=0}^{k-2} q^i \begin{bmatrix} k-i \\ 1 \end{bmatrix}_q - \sum_{i=1}^{k-2} b_{i-1} \left(q^i \begin{bmatrix} k-i \\ 1 \end{bmatrix}_q + q^k \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_q \right) \\ &= a_0 \cdot s_q(k-1, k-1) + \sum_{i=1}^{k-2} \underbrace{(q-1-b_{i-1})}_{\in[0,q-1]} \cdot s_q(k-1, i) + (q-1) \cdot s_q(k-1, 0), \end{aligned}$$

where $a_0 = -\left(\left(1 + u + \sum_{i=0}^{k-3} b_i \begin{bmatrix} i \\ 1 \end{bmatrix}_q\right) q + b_{k-2}\right) < 0$, which is a contradiction. □

For $z = 0$, i.e, $k > \begin{bmatrix} r \\ 1 \end{bmatrix}_q$, we obtain $A_q(v, 2k; k) = \frac{q^v - q^{k+r}}{q^k - 1} + 1$ due to the known construction. For $z = \begin{bmatrix} r \\ 1 \end{bmatrix}_q - k - 1 \ge q + 1$ the upper bound can be tightened to $A_q(tk + r, 2k; k) \le \frac{q^v - q^{k+r}}{q^k - 1} + 1 + zq$ for $r \ge 1$, $t \ge 2$, $k \ge \begin{bmatrix} r \\ 1 \end{bmatrix}_q + 1 - z - q\sum_{i=0}^{k-3} b_i - b_{k-2}$ and $z \le \begin{bmatrix} r \\ 1 \end{bmatrix}_q - r$, where the b_i are as in the proof of Proposition 1. For smaller k the corresponding q^{k-1}-divisible sets indeed exist.

In analogy to the *Frobenius Coin Problem*, cf. [4], we define $\mathrm{F}_q(r)$ as the smallest integer such that a q^r-divisible multiset of cardinality n exists for all integers $n > \mathrm{F}_q(r)$. In other words, $\mathrm{F}_q(r)$ is the largest integer which is not realizable as the size of a q^r-divisible multiset of points over $\mathbb{F}_q$. If all non-negative integers are realizable then $\mathrm{F}_q(r) = -1$, which is the case for $r = 0$.

Proposition 2. *For every prime power q and $r \in \mathbb{N}_0$ we have*

$$\mathrm{F}_q(r) = r \cdot q^{r+1} - \frac{q^{r+1} - 1}{q - 1} = r \cdot q^{r+1} - \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q.$$

Proof. By Theorem 4, $\mathrm{F}_q(r)$ is the largest integer n whose $S_q(r)$-adic expansion $n = \sum_{i=0}^{r-1} a_i s_q(r, i) + a_r q^r$ has leading coefficient $a_r < 0$. Clearly, this n is given by $a_0 = \ldots = a_{r-1} = q - 1$ and $a_r = -1$, such that

$$\begin{aligned} \mathrm{F}_q(r) = \sum_{i=0}^{r-1} (q-1) s_q(r, i) - q^r &= \sum_{i=0}^{r-1} (q^{r+1} - q^i) - q^r \\ &= rq^{r+1} - \frac{q^r - 1}{q - 1} - q^r = rq^{r+1} - \frac{q^{r+1} - 1}{q - 1}. \end{aligned}$$

□

Corollary 4. *The improvement of Theorem 3 over the original Johnson bound* (1) *is at most* $(q-1)(k-1)$.

Proof. In the notation of Theorem 3, let $\delta = (q-1)(k-1)$. Then

$$m(\delta) \ge \begin{bmatrix} k \\ 1 \end{bmatrix}_q \delta = (q^k - 1)(k-1) > (k-1)q^k - (q^{k-1} + q^{k-2} + q^{k-3} + \ldots + q^0) = \mathrm{F}_q(k-1).$$

Therefore, there exists a q^r-divisible multiset of size $m(\delta)$. Hence the optimal δ for Theorem 3 is at most $(q-1)(k-1)-1$, resulting in an improvement of at most $((q-1)(k-1)-1)+1=(q-1)(k-1)$ over the original Johnson bound. $\square$

In our application of bounds for $\mathrm{A}_q(v,d;k)$ we have the additional requirement, that the q^{k-1}-divisible multiset of points of cardinality m in Theorem 3 has to embedded in $\mathbb{F}_q^v$, i.e., there is a restriction on the dimension of the ambient space. However, the constructive part of the proof of Theorem 4 shows that if a q^r-divisible multiset of cardinality n exists, then there also exists at least one q^r-divisible multiset of cardinality n in $\mathbb{F}_q^{r+1}$. Since $r+1=k\le v$, the information on the dimension gives no proper restriction.

Proposition 3. *For all prime powers $q\ge 2$ we have*

$$\begin{aligned}\mathrm{A}_q(11,6;4)&\le q^{14}+q^{11}+q^{10}+2q^7+q^6+q^3+q^2-2q+1\\&=(q^2-q+1)(q^{12}+q^{11}+q^8+q^7+q^5+2q^4+q^3-q^2-q+1).\end{aligned}$$

Proof. Since $10\equiv 1 \pmod 3$ we have $\mathrm{A}_q(10,6;3)=q^7+q^4+1$ and

$$\frac{(q^{11}-1)(q^7+q^4+1)}{q^4-1}=q^{14}+q^{11}+q^{10}+2q^7+q^6+q^3+q^2-1+\frac{q^2+2q+2}{q^3+q^2+q+1}.$$

The fraction on the right is <1 since $(q^3+q^2+q+1)-(q^2+2q+2)=q^3-q-1>0$ for all $q\ge 2$. Therefore $m(\delta)=q^2+2q+2+(q^3+q^2+q+1)\delta$ in Theorem 3.

The number $m(2q-3)=2q^4-q^3+q-1$ has the $S_q(3)$-adic expansion

$$(q-1)\cdot(q^3+q^2+q+1)+1\cdot(q^3+q^2+q)+(q-1)\cdot(q^3+q^2)+(-2)\cdot q^3$$

with negative leading coefficient -2. Therefore by Theorem 4, there is no q^3-divisible multiset of points of size $2q^4-q^3+q-1$. Now the proposed upper bound follows by Theorem 3. $\square$

Remark 5. The choice of $\delta=2q-3$ in the proof of Proposition 3 is maximal since

$$\begin{aligned}m(2q-2)&=(q^2+2q+2)+(2q-2)\cdot(q^3+q^2+q+1)\\&=2q^4+q^2+2q\\&=0\cdot(q^3+q^2+q+1)+2\cdot(q^3+q^2+q)+(q-1)\cdot(q^3+q^2)+(q-2)\cdot q^3\end{aligned}$$

has leading coefficient $q-2\ge 0$, such that by Theorem 4 there exists a q^3-divisible multiset of cardinality $m(2q-2)$.

3. Divisible codes and the linear programming method

It is well-known (see, e.g., [24, 6, Prop. 1]) that the relation $C\to\mathscr{C}$, associating with a full-length linear $[n,v]$ code C over $\mathbb{F}_q$ the n-multiset $\mathscr{C}$ of points in $\mathrm{PG}(v-1,\mathbb{F}_q)$ defined by the columns of any generator matrix, induces a one-to-one correspondence between classes of (semi-)linearly equivalent spanning multisets and classes of (semi-)monomially equivalent full-length linear codes. The importance of the correspondence lies in the fact that it relates coding-theoretic properties of C to geometric or combinatorial properties of $\mathscr{C}$ via

$$\mathrm{w}(\mathbf{aG})=n-\#\{1\le j\le n;\mathbf{a}\cdot\mathbf{g}_j=0\}=n-\#(\mathscr{C}\cap\mathbf{a}^\perp),\tag{6}$$

where w denotes the Hamming weight, $\mathbf{G}=(\mathbf{g}_1|\dots|\mathbf{g}_n)\in\mathbb{F}_q^{v\times n}$ a generating matrix of C, $\mathbf{a}\cdot\mathbf{b}=a_1b_1+\dots+a_vb_v$, and $\mathbf{a}^\perp$ is the hyperplane in $\mathrm{PG}(v-1,\mathbb{F}_q)$ with equation $a_1x_1+\dots+a_vx_v=0$.

A linear code C is said to be Δ-*divisible* ($\Delta\in\mathbb{Z}_{>1}$) if all nonzero codeword weights are multiples of Δ. They have been introduced by Ward in 1981, see [25] and [26] for a survey. So, given a q^r-divisible multiset $\mathscr{P}$ in $\mathbb{F}_q^v$ of cardinality n there is a corresponding q^r-divisible linear $[n,k]$ code C, where $k\le v$.

The famous *MacWilliams Identities*, [20]

$$\sum_{j=0}^{n-i}\binom{n-j}{i}A_j = q^{k-i}\cdot\sum_{j=0}^{i}\binom{n-j}{n-i}A_j^\perp \quad \text{for } 0\le i\le n, \tag{7}$$

relate the weight distributions (A_i), $(A_i^\perp)$ of the (primal) code C and the dual code $C^\perp = \{\mathbf{y}\in\mathbb{F}_q^n; x_1y_1+\cdots+x_ny_n = 0 \text{ for all } \mathbf{x}\in C\}$. Since the A_i and $A_i^\perp$ count codewords of weight i, they have to be non-negative integers. In our context we have $A_0 = A_0^\perp = 1$, $A_1^\perp = 0$, and $A_i = 0$ for all i that are not divisible by q^r. Treating the remaining A_i and $A_i^\perp$ as non-negative real variable one can check feasibility via linear programming, which is known as the *linear programming method* for the existence of codes, see e.g. [5, 3].

As demonstrated in e.g. [16], the average argument of Lemma 8 is equivalent to the linear programming method applied to the first two MacWilliams Identities, i.e., $i = 0,1$. So, the proof of Theorem 4 shows that invoking the other equations gives no further restrictions for the possible lengths of divisible codes. This is different in the case of partial k-spreads, i.e., the determination of $\mathrm{A}_q(v,2k;k)$. Here the multisets of points in Corollary 1 are indeed sets that correspond to projective linear codes, which are characterized by the additional condition $\mathrm{d}(C^\perp)\ge 3$, i.e., $A_2^\perp = 0$. The upper bound of Năstase and Sissokho can be concluded from the first two MacWilliams Identities, i.e., the average argument of Lemma 8, see Proposition 1. Theorem 1 and Theorem 2 are based on the first three MacWilliams Identities while also the forth MacWilliams Identity is needed for the mentioned 21 sporadic 1-parametric series listed in [19]. The characterization of the possible lengths of q^r-divisible projective linear codes is more difficult than in the non-projective case of Theorem 4. For the corresponding Frobenius number the sharpest upper bound in the binary case $q = 2$ is $\bar{\mathrm{F}}_2(r)\le 2^{2r}-2^{r-1}-1$. The lengths of projective 2- and 4-divisible linear binary codes have been completely determined, but already for projective 8-divisible codes there is a single open case, which is length 59 [10].

4. CONCLUSION

We have presented a connection between q^r-divisible linear codes and upper bounds for constant-dimension codes, which improves the best known upper bounds in many cases. The framework of q^r-divisible linear codes covers constant-dimension codes and partial spreads, while the latter substructures call for projective linear codes as a special subclass of q^r-divisible linear codes. Here, we have characterized all possible lengths of q^r-divisible codes. This problem is open in the case of projective q^r-divisible linear codes. It is very likely that more sophisticated methods from coding theory, beyond the pure application of the linear programming method, are needed in order to decide the non-existence question in a few more cases.[2] If the possible q^r-divisible codes are classified for the parameters of a desired constant-dimension code, one may continue the analysis and look at the union of the k-dimensional codewords and their restrictions. Using the language of minihypers, the authors of [21] have obtained some extendability results for constant-dimension codes. It seems worthwhile to compare and possibly combine both methods.

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[2]In this context we would like to mention that the second author recently presented the upper bound $\mathrm{A}_2(13,10;5)\le 259$ on a conference. The proof involves an application of the split-weight enumerator and the determination of the unique weight enumerator of a projective 2^3-divisible binary code of length 51, cf. [16].

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MICHAEL KIERMAIER,UNIVERSITY OF BAYREUTH, 95440 BAYREUTH, GERMANY
E-mail address: `michael.kiermaier@uni-bayreuth.de`

SASCHA KURZ, UNIVERSITY OF BAYREUTH, 95440 BAYREUTH, GERMANY
E-mail address: `sascha.kurz@uni-bayreuth.de`