MCDM methods based on pairwise comparison matrices and their fuzzy extension

Dissertation
zur Erlangung des Grades eines Doktors der Wirtschaftswissenschaft
der Rechts- und Wirtschaftswissenschaftlichen Fakultät
der Universität Bayreuth

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*To my dad who has always lived for me and my sister, believed in us,
and supported us in anything we chose to do.*

Acknowledgments

Abstract

Methods based on pairwise comparison matrices (PCMs) form a significant part of multi-criteria decision making (MCDM) methods. These methods are based on structuring pairwise comparisons (PCs) of objects from a finite set of objects into a PCM and deriving priorities of objects that represent the relative importance of each object with respect to all other objects in the set. However, the crisp PCMs are not able to capture uncertainty stemming from subjectivity of human thinking and from incompleteness of information about the problem that are often closely related to MCDM problems. That is why the fuzzy extension of methods based on PCMs has been of great interest.

In order to derive fuzzy priorities of objects from a fuzzy PCM (FPCM), standard fuzzy arithmetic is usually applied to the fuzzy extension of the methods originally developed for crisp PCMs. However, such approach fails in properly handling uncertainty of preference information contained in the FPCM. Namely, reciprocity of the related PCs of objects in a FPCM and invariance of the given method under permutation of objects are violated when standard fuzzy arithmetic is applied to the fuzzy extension. This leads to distortion of the preference information contained in the FPCM and consequently to false results. Thus, the first research question of the thesis is: *"Based on a FPCM of objects, how should fuzzy priorities of these objects be determined so that they reflect properly all preference information available in the FPCM?"*

This research question is answered by introducing an appropriate fuzzy extension of methods originally developed for crisp PCMs. That is, such fuzzy extension that does not violate reciprocity of the related PCs and invariance under permutation of objects, and that does not lead to a redundant increase of uncertainty of the resulting fuzzy priorities of objects.

Fuzzy extension of three different types of PCMs is examined in this thesis - multiplicative PCMs, additive PCMs with additive representation, and additive PCMs with multiplicative representation. In particular, construction of PCMs, verifying consistency, and deriving priorities of objects from PCMs are studied in detail for each type of these PCMs. First, well-known and in practice most often applied methods based on crisp PCMs are reviewed. Afterwards, fuzzy extensions of these methods proposed in the literature are reviewed in detail and their drawbacks regarding the violation of reciprocity of the related PCs and of invariance under permutation of objects are pointed out. It is shown that these drawbacks can be overcome by properly applying constrained fuzzy arithmetic instead of standard fuzzy arithmetic to the computations. In particular, we always have to look at a FPCM as a set of PCMs with different degrees of membership to the FPCM, i.e. we always have to consider only PCs that are mutually reciprocal. Constrained fuzzy arithmetic allows us to impose the reciprocity of the related PCs as a constraint on arithmetic operations with fuzzy numbers, and its appropriate application also guarantees invariance of the methods under permutation of objects. Finally, new fuzzy extensions of the methods are proposed based on constrained fuzzy arithmetic and it is proved that these methods do not violate the reciprocity of the related PCs and are invariant under permutation of objects. Because of these desirable properties, fuzzy priorities of objects obtained by the methods proposed in this thesis reflect the preference information contained in fuzzy PCMs better in comparison to the fuzzy priorities obtained by the methods based on standard fuzzy arithmetic.

Beside the inability to capture uncertainty, methods based on PCMs are also not able to cope with situations where it is not possible or reasonable to obtain complete preference information from DMs. This problem occurs especially in the situations involving large-dimensional PCMs. When dealing with incomplete large-dimensional PCMs, compromise between reducing the number of PCs required from the DM and obtaining reasonable priorities of objects is of paramount importance. This leads to the second research question: *"How can the amount of preference information required from the DM in a large-dimensional PCM be reduced while still obtaining comparable priorities of objects?"*

This research question is answered by introducing an efficient two-phase method. Specifically, in the first phase, an interactive algorithm based on weak-consistency condition is introduced for partially filling an incomplete PCM. This algorithm is designed in such a way that minimizes the number of PCs required from the DM and provides sufficient amount of preference information at the same time. The weak-consistency condition allows for providing ranges of possible intensities of preference for every missing PC in the incomplete PCM. Thus, at the end of the first phase, a PCM containing intervals for all PCs that were not provided by the DM is obtained. Afterward, in the second phase, the methods for obtaining fuzzy priorities of objects from fuzzy PCMs proposed in this thesis within the answer to the first research question are applied to derive interval priorities of objects from this incomplete PCM. The obtained interval priorities cover all weakly consistent completions of the incomplete PCM and are very narrow. The performance of the method is illustrated by a real-life case study and by simulations that demonstrate the ability of the algorithm to reduce the number of PCs required from the DM in PCMs of dimension 15 and greater by more than 60% on average while obtaining interval priorities comparable with the priorities obtainable from the hypothetical complete PCMs.

Contents

Chapter I

Introduction

1.1 Multi-criteria decision making

Multi-criteria decision making (MCDM) is an extensive sub-discipline of operations research. The Institute for Operations Research and the Management Science defines operations research as a discipline that deals with applications of advanced analytical methods to help make better decisions (INFORMS, 2016). Applications of operations research are abundant - from scheduling airlines, over assigning employees to projects, deciding the most appropriate location for new facilities, to supply chain optimization.

The aim of MCDM, as its name suggests, is to deal with situations (problems) requiring a decision being made under the presence of a number of conflicting criteria. Decision making is regarded as a process of evaluating decision alternatives (courses of action) based on preferences of the decision maker (DM) which usually results in selecting the "best" alternative, ranking alternatives from the most preferred to the least preferred, or sorting alternatives into predefined classes. The first case is the most frequent one in practice. Criteria are measures of the objectives relevant for the decision-making problem. A DM is the subject in charge of making the decision. It can be, for example, an individual, a group of people, a family, a company, a government, etc.

Many different terms are being used in the literature besides MCDM: multi-criteria decision aiding (MCDA), multi-attribute decision making (MADM), multi-objective decision making (MODM), multi-attribute decision aiding, multi-criteria decision analysis, and more. Some researchers perceive the terms as equivalent, others distinguish between them. Zimmermann (2001) divided MCDM into MODM and MADM, where MODM focuses on problems with continuous decision spaces while MADM focuses on problems with discrete decision spaces. In this thesis, MCDM with discrete decision spaces (i.e. MADM according to Zimmermann (2001)) is of interest and it will be simply referred to as MCDM hereafter.

1.1.1 Pairwise comparison methods

A significant part of MCDM methods is based on pairwise comparison (PC). The first PC method was developed by Thurstone (1927). Since then, PC methods have undergone a great development. PCs of objects (usually criteria and alternatives) have been widely used in many well-known MCDM methods such as Analytic Hierarchy Process (AHP), Preference Ranking Organization Method for Enrichment Evaluation (PROMETHEE), Elimination and Choice Expressing Reality (ELECTRE), and their derivatives; see, e.g., Figueira et al. (2005).

PC methods are based on the idea of comparing only two objects (criteria, alternatives) at a time which is less demanding on cognitive capabilities than comparing several objects at the same time. The most usual way to represent PCs of a finite set of objects is to structure them into a pairwise comparison matrix (PCM). A PCM of n objects $o_1, \ldots, o_n$ is a square matrix $C = \{c_{ij}\}_{i,j=1}^{n}$ where c_{ij} expresses the intensity of preference of object o_i over object o_j. Clearly, PCs c_{ij} and c_{ji} are in a reciprocal relation since they express intensities of preference on the same pair of objects o_i and o_j, just in a different order. The type of the reciprocity relation depends on the type of the PCM used.

In this thesis, two types of the reciprocity relation are of interest - multiplicative reciprocity and additive reciprocity. Multiplicative reciprocity of the related PCs is inherent to multiplicative PCMs that are used to model preference information provided in form of a preference ratio while additive reciprocity of the related PCs is inherent to additive PCMs that are used to model preference information provided in form of a preference difference. Additive PCMs are further divided into additive PCMs with additive representation and additive PCMs with multiplicative representation.

PCMs have been studied in detail by many researchers, and many different methods based on multiplicative and additive PCMs have been developed to support MCDM. The methods basically consist in the following

tasks:

- constructing a PCM, i.e. comparing pairwisely objects in a PCM;

- verifying consistency (or acceptable inconsistency) of the PCM;

- deriving priorities of objects (criteria and alternatives) from the PCM;

- aggregating priorities of criteria and alternatives into final priorities of alternatives representing the final multi-criteria evaluations of alternatives.

Further extensions of the methods have been developed to deal with decision processes involving several DMs.

Each of the above mentioned tasks encounters some difficulties. In the first task, constructing a PCM, the choice of an appropriate scale for expressing the intensities of preference plays a key role. In the second task, verifying consistency of a PCM, the choice of a proper consistency condition compatible with the type of preference information provided in the PCM and with the scale chosen for making PCs is very important. In the third task, deriving priorities of objects from a PCM, it is necessary to choose such method and normalization condition that are compatible with the type of preference information contained in the PCM. In the fourth task, the choice of a suitable aggregation method is of crucial importance. Unfortunately, not all of these issues are addressed properly in the literature, which may lead to wrong results and, consequently, to low-quality decisions.

1.1.2 Incomplete pairwise comparison matrices

PC methods require complete preference information. However, in practice, it is not always possible to obtain complete preference information from the DM, e.g. due to time or cost limitations. This problem occurs especially when the DM is required to provide a large number of PCs. In such cases, PCMs of objects may be incomplete. That is why a large number of methods for dealing with incomplete large-dimensional PCMs has been proposed in the literature. These methods usually consist of two main tasks:

- identifying a set of PCs that should be provided by the DM in an incomplete PCM;

- deriving priorities from an incomplete PCM; or automatically completing the incomplete PCM (i.e. without no additional preference information from the DM) and then deriving priorities from the complete PCM.

Both tasks encounter some challenges. When identifying a set of PCs that should be provided by the DM in an incomplete large-dimensional PCM, the appropriate choice of the PCs and of the number of PCs required from the DM plays a key role. When deriving priorities of objects, the choice of an appropriate method is of crucial importance in order to derive priorities that best represent DM's preferences and the incompleteness of preference information. Not all of these issues are addressed properly in the literature which may again lead to wrong results.

1.2 Fuzzy multi-criteria decision making

The "traditional" MCDM methods were not designed to deal with uncertainty in MCDM problems. However, human world is full of uncertainties. Uncertainty in decision making has been modeled and analyzed by means of probabilistic theory and fuzzy set theory.

The aim of probabilistic theory is to capture the stochastic nature of decision making while fuzzy set theory attempts to capture the subjectivity of human behavior and human thinking. According to Dubois and Prade (1986), stochastic decision methods do not measure the imprecision in human behavior, but they are suitable for modeling incomplete knowledge about the environment surrounding humans. Contrarily, fuzzy set theory is a perfect means for modeling subjectivity (imprecision, uncertainty) which is integral to human mind. The need for modeling the subjectivity of human behavior in decision making resulted in the development of a new decision-making field - fuzzy MCDM.

There are two main phases in fuzzy MCDM (Zimmermann, 1987): phase 1 - aggregation of the performance scores with respect to all attributes for each alternative, and phase 2 - rank ordering of the alternatives according to the aggregated scores. Therefore, fuzzy MCDM methods can be classified into two categories depending on which phase they focus on: (i) fuzzy MCDM methods focusing only on the first phase (or on both phases), and (ii) fuzzy MCDM methods focusing only on the second phase. The fuzzy MCDM methods of interest in this thesis are the PC methods that belong to the first category.

The pioneering work in fuzzy MCDM was done by Bellman and Zadeh (1970) who extended the Maxmin method to fuzzy environment. However, the data in their method are expressed by crisp numbers. The first

proper fuzzy MCDM methods were developed by Baas and Kwakernaak (1977) and Kwakernaak (1979) who extended the Simple Additive Weighting method, and by Efstathiou and Rajkovic (1979) who proposed a fuzzy extension of the Multiple Attribute Utility Function method. Afterwards, many other approaches followed (see, e.g., Chen and Hwang (2005) for the review).

1.2.1   Fuzzy pairwise comparison methods

Since PCMs (both multiplicative and additive) cannot deal with uncertainty and imprecision, which is integral to human mind, their extension to fuzzy numbers and intervals has been proposed.

Most often, triangular fuzzy numbers are used for the fuzzy extension of multiplicative PCMs. Less often, trapezoidal fuzzy numbers or intervals are used for this purpose. Contrarily, unlike multiplicative PCMs, additive PCMs are usually extended to intervals rather than to fuzzy numbers. Intervals can be understood as a special case of trapezoidal fuzzy numbers where all elements from the given interval have the same degree of membership to the fuzzy set. Note that sometimes it will be explicitly distinguished between terms "fuzzy" and "interval" in this thesis. Often, however, only the term "fuzzy" will be used for the simplicity and, in such case, intervals will be understood as a particular case of fuzzy numbers, namely of trapezoidal fuzzy numbers.

The fuzzy extension of PC methods to fuzzy numbers (or intervals) basically consists in using fuzzy numbers (intervals) at the stage of entering PCs into a PCM. These fuzzy numbers can be either chosen from a predefined scale of fuzzy numbers assigned to linguistic terms expressing preference intensities, or entered expertly by the DM without linguistic representation. Having a fuzzy PCM (FPCM), i.e. a PCM with elements in form of fuzzy numbers, the aim is to obtain fuzzy priorities of these objects.

1.2.2   Critics of fuzzy extension of AHP

Saaty (2006) argued that, in AHP, "the numbers assigned to judgments are already fuzzy and making them more fuzzy does not help produce more valid outcome" (Saaty (2006), p. 457). Saaty and Tran (2007) demonstrated on several examples the invalidity of fuzzification of AHP and concluded that "one should never use fuzzy arithmetic on AHP judgment matrices" (Saaty and Tran (2007), p. 970). Clearly, when the PCs obtained from a DM are crisp, there is no gain from fuzzifying them without a good reason. However, when the PCs are vague or when linguistic terms are used to express intensities of preference on pairs of compared objects, fuzzy numbers should be applied instead of crisp numbers (Krejčí et al., 2017). Krejčí et al. (2017) further showed that by neglecting the available information about uncertainty of intensities of preferences an important part of knowledge about the decision-making problem is lost which can cause the change in raking of the alternatives of the decision-making problem.

Besides Saaty and Tran's critics, very harsh critics of the fuzzy extension of AHP were provided also by Zhü (2014). He heavily criticizes the well-known fuzzy approaches to AHP and claims fallacy of all of them. However, Zhü's critics are based mostly on misinterpreting other researchers' claims and rejecting consolidated bases of fuzzy set theory. Fedrizzi and Krejčí (2015) showed that no reliable evidence of the fallacy of AHP was provided by Zhü (2014), and thus fuzzy AHP and more in general methods utilizing multiplicative and additive FPCMs remain a valid MCDM research area.

It is clear that fuzzy AHP, and in general fuzzy MCDM methods based on multiplicative and additive FPCMs, have some critical aspects (some of them indicated by Zhü (2014)) that need to be investigated and rectified. One of them is the issue regarding the reciprocity (multiplicative or additive) of the related PCs, which is an inherent property of PCMs. Reciprocity of the related PC has been neglected in multiplicative and additive FPCMs for a long time which led to some incorrect results. Nevertheless, this drawback can be relatively easily removed by applying constrained fuzzy arithmetic instead of standard fuzzy arithmetic to the computations with fuzzy numbers (Fedrizzi and Krejčí, 2015). The following section is devoted to clarify this point.

Another issue related to the extension of methods from PCMs to FPCMs is the invariance under permutation of objects. A method is said to be invariant under permutation of objects if the result of the method does not depend on the permutation of objects compared in a PCM. Even though the invariance under permutation of objects was introduced as one of the axioms which "good" methods should meet (see Fichtner (1986); Brunelli and Fedrizzi (2015)), many methods proposed for FPCMs violate this property. Nevertheless, also the invariance under permutation can be reached by applying properly constrained fuzzy arithmetic instead of standard fuzzy arithmetic to the fuzzy extension of the methods.

1.2.3   Constrained fuzzy/interval arithmetic

Constrained fuzzy arithmetic, introduced by Klir (1997) and Klir and Pan (1998), should be applied to computations with fuzzy numbers whenever there are interactions of any type present between the fuzzy numbers. Similarly, constrained interval arithmetic, that was recently introduced by Lodwick and Jenkins (2013), should be applied to computations with intervals when interactions are present. Constrained interval arithmetic can

be seen as a special case of constrained fuzzy arithmetic since intervals are special cases of trapezoidal fuzzy numbers. Therefore, all comments and discussions on constrained fuzzy arithmetic in this thesis apply also for interval fuzzy arithmetic if not specified otherwise. Motivation for the use of constrained fuzzy arithmetic is given in the following example. For the simplicity, intervals that are (as we already know) a special case of trapezoidal fuzzy numbers are used for explanation.

Example 1. Let us assume we have exactly one liter, i.e. 1000 ml, of beer in one bottle and we want to share it with our friend. We pour half of the beer into a half-liter. Of course, it is highly improbable that we would manage to pour exactly 500 ml into the half-liter. Our friend has a look on the half-liter and estimates that there is for sure something between 450 ml and 520 ml of beer inside. What can we say about the rest of the beer in our bottle? Based on the information from our friend and on our common sense, we can say that there is for sure something between 480 ml and 550 ml in our bottle. How did we arrive to this conclusion? Well, if there is 450 ml of beer in our friend's half-liter, then in our bottle there has to be $1000 - 450 = 550$ ml left. In case there is 520 ml of beer in our friend's half-liter, then in our bottle there has to be $1000 - 520 = 480$ ml left. This is nothing else but the standard interval arithmetic; $1000 - [450, 520] = [1000 - 520, 100 - 450] = [480, 550]$.

What happens if we pour our friend's half-liter back to the one-liter bottle? Our common sense says that we have to get again the exact one liter (1000 ml) of beer, unless we spill some of it. But what mathematics is behind this very simple task? If we apply again standard interval arithmetic we know, we get $[450, 520] + [480, 550] = [450 + 480, 520 + 550] = [930, 1070]$. So this would suggest that there is something between 930 ml and 1070 ml of beer. But everyone would agree that this is not possible; there has to be 1000 ml again in our bottle. So what is going on?

The problem is that there are dependencies between the amounts of beer in our friend's half-liter and in our bottle. We had exactly 1000 ml of beer at the beginning, and we pour a part of it into a half-liter. Therefore, no matter what the exact amount x between 450 ml and 520 ml of beer in the half-liter is, we know for sure that the rest of beer in our bottle is $y = 1000 - x$ ml. When pouring the half-liter back to the bottle, we cannot forget about this dependency. However, this simple problem requires a more sophisticated approach than just standard interval arithmetic. We should correctly compute

$$z = x + y \qquad \text{where}$$
$$x \in [450, 520]$$
$$y \in [480, 550]$$
$$y = 1000 - x$$

This is nothing else than constrained (in this particular case interval) arithmetic. $\triangle$

Constrained fuzzy (or interval) arithmetic is not applicable only when sharing beer with friends. It should be applied whenever there are some dependencies between operands in arithmetic operations on fuzzy numbers (or intervals). As stated at the end of Section 1.2.2, there are dependencies between PCs in a FPCM; in particular, reciprocal relations between the related PCs. Since reciprocity of the related PCs is an inherent property of PCMs, it is necessary to extend it properly also to FPCMs.

1.3 Goal of the thesis

As already mentioned in Sections 1.2 and 1.2.1, "traditional" MCDM methods including the methods based on multiplicative PCMs as well as on additive PCMs (both with additive and multiplicative representation) are often criticized because of their inability to capture uncertainty stemming from subjectivity of human thinking and from incompleteness of information that are often closely related to MCDM problems. This uncertainty has an impact on the PCs provided by DMs in PCMs. In order to capture the uncertainty, the methods originally proposed for crisp PCMs have been extended to fuzzy numbers and intervals. The fuzzy extension often consists in simply replacing the crisp PCs in the given model by fuzzy PCs and applying standard fuzzy arithmetic to obtain the desired fuzzy priorities. However, this approach often fails in handling appropriately the uncertain preference information contained in the FPCM, which may lead to false results. Therefore, the first research question of this thesis is:

(1) *Based on a FPCM of objects, how should fuzzy priorities of these objects be determined so that they reflect properly all preference information available in the FPCM?*

In order to answer this research question, it is necessary to fully understand the meaning of PCs in a PCM and to identify inherent properties of PCMs. As discussed in Section 1.2.2, the crucial inherent property of PCMs is the reciprocity of the related PCs. The concept of reciprocity of the related PCs becomes more complex when extended to FPCMs and handling appropriately the reciprocity property becomes of key importance

in order to process correctly the preference information contained in the FPCM and to arrive to reasonable conclusions. Unfortunately, this issue is usually omitted in the literature. This leads to results (resulting fuzzy priorities of objects in particular) that are often excessively uncertain and do not reflect correctly the preference information available in the original FPCM.

Another crucial property related to both PCMs and FPCMs is the invariance of methods under permutation of objects. Many methods based on FPCMs violate this property, which leads to false results.

The drawbacks regarding the violation of the reciprocity property and of the invariance under permutation of objects can be removed by applying constrained fuzzy arithmetic to the computations instead of standard fuzzy arithmetic. Unfortunately, the use of constrained fuzzy arithmetic is still neglected in the literature on fuzzy extension of MCDM methods based on PCMs. In fact, I have encountered only one research paper dealing with this important topic - Enea and Piazza (2004).

In order to properly answer the research question (1), four tasks are pursued in the thesis:

(1.a) to critically review the well-known and in practice most often applied methods based on PCMs (multiplicative PCMs, additive PCMs with additive representation, and additive PCMs with multiplicative representation) dealing with the construction of PCMs, consistency verification, and priorities computation;

(1.b) to critically review the existing approaches to the fuzzy extension of the methods reviewed within task (1.a) and to identify their drawbacks regarding the violation of the reciprocity property and of the invariance under permutation of objects;

(1.c) to demonstrate the necessity of applying constrained fuzzy arithmetic to the fuzzy extension of MCDM methods based on PCMs in order to obtain meaningful results reflecting the preference information contained in the original FPCM and not suffering from the drawbacks identified within task (1.b);

(1.d) to propose a new fuzzy extension of the methods critically reviewed within task (1.a) by applying constrained fuzzy arithmetic that does not suffer from the drawbacks identified within task (1.b) and that reflects properly all preference information available in FPCMs.

Note that because of the excessive extent of the topic, only the first three tasks of the PC methods and their fuzzy extension will be addressed in this thesis, i.e. the construction of PCMs, consistency verification, and computation of priorities. The study of the fourth task, i.e. aggregation of the priorities of alternatives and criteria into final priorities of alternatives representing final multi-criteria evaluations of the alternatives, and of the extension of all four tasks to multiple DMs is left for future research.

Beside the inability to capture uncertainty, the "traditional" methods are also not able to cope with the situations where it is not possible or reasonable to obtain complete preference information from DMs, for example due to time or cost limitations. This problem occurs especially in the situations where large-dimensional PCMs are involved. That is why various methods for dealing with incomplete large-dimensional PCMs have been proposed in the literature.

When dealing with incomplete large-dimensional PCMs, compromise between reducing the number of PCs required from the DM as much as possible and obtaining reasonable priorities of objects is of paramount importance. Thus, the second research question of this thesis is:

(2) *How can the amount of preference information required from the DM in a large-dimensional PCM be reduced while still obtaining comparable priorities of objects?*

In order to properly answer this research question, it is necessary to define the meaning of "comparable" priorities first. In the context of incomplete large-dimensional PCMs in this thesis, by comparable priorities of objects will be meant such priorities of objects that are close enough to the actual priorities that would be obtained from the hypothetical complete PCM, i.e. such priorities that approximate the actual priorities well enough.

The research question (2) is answered in the thesis by pursuing the following two tasks:

(2.a) to propose an efficient method for partially filling an incomplete large-dimensional PCM that minimizes the number of PCs required from the DM but provides a sufficient amount of preference information;

(2.b) to propose a suitable method for deriving priorities from an incomplete large-dimen- sional PCM that reflect the incompleteness of preference information and that are "close" to the priorities obtainable from the hypothetical complete PCM.

The basic idea is to design an algorithm based on an optimal sequential choice of the PCs that should be provided by the DM and on the concept of weak consistency. Based on the weak consistency, the missing PCs in the matrix should be replaced by intervals providing ranges for the missing preference information, thus

obtaining a large-dimensional fuzzy (more precisely interval) PCM. Afterwards, methods for deriving fuzzy priorities from FPCMs that are going to be proposed in this thesis whithin task (1.d) should be applied in order to obtain interval priorities that represent properly the incompleteness of preference information contained in the original incomplete large-dimensional PCM.

Note that because of the excessive extent of the topic only the problems where the DM provides PCs in the form of crisp numbers are considered in this thesis. The generalization of the proposed method to fuzzy numbers is again left for future research.

1.4   Structure of the thesis

This thesis is divided into five chapters. This chapter, Chapter I, provides an introduction to the topic of the thesis and states the research questions and the tasks pursued in the thesis in order to provide answers to the research questions.

Chapter II provides a critical review of well-known and in real-life MCDM problems most often applied methods based on PCMs (task (1.a)). Three types of PCMs are studied in this chapter - multiplicative PCMs, additive PCMs with additive representation, and additive PCMs with multiplicative representation - and transformations between the three approaches are examined.

Chapter III reviews basic concepts from fuzzy set theory which play a key role in this thesis. Trapezoidal and triangular fuzzy numbers and intervals, that are later used for the fuzzy extension of methods based on PCMs, are defined. Standard and constrained fuzzy arithmetic are studied in detail and the necessity of applying constrained fuzzy arithmetic to arithmetic operations with fuzzy numbers in the presence of constraints on operands is emphasized.

Chapter IV is the central chapter of the thesis that provides the answer to the research question (1). In this chapter, the fuzzy extension of the methods reviewed in Chapter II is studied. In particular, the fuzzy extensions proposed in the literature are critically reviewed and their drawbacks regarding violation of the reciprocity of the related PCs and of the invariance under permutation of objects are identified (task (1.b)). Necessity of applying constrained fuzzy arithmetic to the fuzzy extension of the methods is emphasized in order to remove these drawbacks (task (1.c)) and a proper fuzzy extension of the methods is proposed afterwards (task (1.d)). In the final part of the chapter, transformations between the new methods based on constrained fuzzy arithmetic proposed for all three types of FPCMs are studied.

In Chapter V, findings from Chapter IV are utilized in order to answer the research question (2), i.e. to deal with incomplete PCMs of large dimensions. In particular, an algorithm for identifying iteratively PCs that should be provided by the DM in an incomplete large-dimensional PCM (task (2.a)) and a method for obtaining interval priorities from such an incomplete large-dimensional PCM (task (2.b)) are proposed.

Finally, Chapter VI contains discussion and perspectives for future research.

Chapter II

Pairwise comparison matrices

Part of this chapter has been published in:

Krejčí J.: **"Obtaining fuzzy priorities from additive fuzzy pairwise comparison matrices"**, *IMA Journal of Management Mathematics*, First published online: May 2016, doi: 10.1093/imaman/dpw006.

2.1 Introduction

In order to solve a MCDM problem (to choose the best alternative from a set of alternatives, to rank the alternatives from the most preferred to the least preferred, or to sort the alternatives into predefined classes), rating of objects (alternatives, criteria) is usually required from decision makers (DMs). However, DMs might have problems with rating the objects (assigning priorities to the objects) directly. That is because of the cognitive limitation and incapability to compare several objects at the same time.

This limitation can be easily overcome by providing PCs of objects and then deriving the desired rating (priorities) of objects. Using PCs allows the DM to consider only two objects at a time which is significantly less demanding than considering them all. These PCs can be conveniently structured into a pairwise comparison matrix (PCM). After, an appropriate method is applied to the PCM in order to derive priorities $w_1, \ldots, w_n$ of objects representing DM's preferences.

PCs of n objects $o_1, \ldots, o_n$ are structured into a PCM $C = \{c_{ij}\}_{i,j=1}^n$ as follows:

$$
C = \begin{matrix} & \begin{matrix} o_1 & o_2 & \ldots & o_n \end{matrix} \\ \begin{matrix} o_1 \\ o_2 \\ \vdots \\ o_n \end{matrix} & \begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{pmatrix} \end{matrix} . \tag{II.1}
$$

The PC c_{ij} in the $i-$th row and the $j-$th column of the PCM C expresses the intensity of preference of object o_i over object o_j. For easier understanding, the rows and the columns in the PCM (II.1) are labeled by the names of the compared objects. This labeling is usually omitted in the literature and, for simplicity, it will be omitted also in this thesis.

Notice that the PCs c_{ij} and c_{ji} express intensities of preference on the same pair of objects o_i and o_j; the PC c_{ij} expresses the intensity of preference of object o_i over object o_j and the PC c_{ji} expresses the intensity of preference of object o_j over object o_i. Therefore, it is obvious that these PCs are in relation. The type of relation depends on the type of representation used for expressing PCs. In this thesis, two types of relation between the PCs c_{ij} and c_{ji} are of interest: the multiplicative-reciprocity relation $c_{ji} = \frac{1}{c_{ij}}$ and the additive-reciprocity relation $c_{ji} = 1 - c_{ij}$. Reciprocity relation is an inherent property of a PCM that results naturally from the interpretation of the PCs in a PCM. Both types of the reciprocity relation will be studied in detail in the following sections.

There exists no canonical order in which to assign to n objects the labels $o_1, \ldots, o_n$. The objects can be labeled in $n!$ different ways. By changing labeling of objects in a PCM, the preference information contained in the PCM does not change; the original PCs are only permuted accordingly. Thus, it is desirable that the priorities of objects derived from a PCM are independent of the order in which the objects are associated with the rows and the columns of the PCM (Fichtner, 1986). This means that the priorities $w_1, \ldots, w_n$ of objects should not change under any permutation of the PCM C. Fichtner (1986) introduced the invariance under

permutation of objects as one of the axioms which "good" methods for deriving priorities from PCMs should meet.

Being P a permutation matrix, i.e. a square matrix with exactly one entry equal to 1 in each row and column and 0 elsewhere, $C^\pi = PCP^T$ is a permutation of C associated with P. Further, let $\mathcal{C}$ denote a certain class of PCMs. Then, invariance under permutation of methods for deriving priorities of objects from PCMs can be formally defined as follows.

Definition 1. *Let a method for deriving priorities $\underline{w} = (w_1, \ldots, w_n)^T$ of objects from PCMs in a certain class $\mathcal{C}$ be described by a function $f : \mathcal{C} \to \mathbb{R}^n$, i.e. $\underline{w} = f(C), C \in \mathcal{C}$. Then the method is said to be invariant under permutation of objects if*

$$f(PCP^T) = Pf(C), \qquad \forall C \in \mathcal{C} \text{ and for any permutation matrix } P.$$

In order to obtain reliable priorities of objects from PCMs, DMs should behave rationally when providing intensities of preference, i.e they should be consistent in their preferences. This basically means that DMs should not enter PCs into a PCM randomly without thinking carefully about their meaning but they should fully focus on this task.

Various definitions of consistency as well as inconsistency indices have been defined in the literature in order to control the consistency of PCMs. Again, it comes natural to require invariance of both the definitions of consistency and of the inconsistency indices. Brunelli and Fedrizzi (2015) even introduced the invariance under permutation of objects as one of the axioms characterizing inconsistency indices.

Definition 2. *A definition of consistency for PCMs in a certain class $\mathcal{C}$ is said to be invariant under permutation of objects if $\forall C \in \mathcal{C}$ the following holds:*

$$C \text{ consistent } \Rightarrow PCP^T \text{ consistent for every } P,$$

$$C \text{ not consistent } \Rightarrow PCP^T \text{ not consistent for any } P,$$

where P is a permutation matrix.

Definition 3. *An inconsistency index $I : \mathcal{C} \to \mathbb{R}$ defined on a certain class $\mathcal{C}$ of PCMs is said to be invariant under permutation of objects if*

$$I(PCP^T) = I(C), \qquad \forall C \in \mathcal{C} \text{ and for any permutation matrix } P.$$

As already mentioned in Section 1.1.1, there exist two basic types of PCMs in MCDM: multiplicative PCMs and additive PCMs. The additive PCMs are further divided into two types depending on the representation used. In the following sections, all three types of PCMs are defined, and well-known and most often applied consistency conditions, inconsistency indices, and methods for deriving priorities of objects from these PCMs are reviewed.

2.2 Multiplicative pairwise comparison matrices

The first bases of the theory on multiplicative PCMs were given by Saaty (1977, 1980) who introduced a complete method for supporting MCDM based on this type of PCMs called AHP. AHP covers all main issues, from structuring the problem into a hierarchy, over the construction of multiplicative PCMs, verifying their consistency, deriving priorities of objects on different levels of the hierarchy, up to the aggregation of the priorities on different levels of the hierarchy into the final priorities of decision alternatives. In this section, the methods for constructing multiplicative PCMs, verifying their consistency, and deriving priorities of objects from multiplicative PCMs are reviewed.

2.2.1 Construction of MPCMs

Definition 4. *A multiplicative pairwise comparison matrix (MPCM) of n objects $o_1, \ldots, o_n$ is a square matrix $M = \{m_{ij}\}_{i,j=1}^n$, whose elements $m_{ij}, i, j = 1, \ldots, n$, indicate the ratio of preference intensity of object o_i to that of object o_j. In other words, element m_{ij} indicates that o_i is $m_{ij}-$times as good as o_j. Further, a MPCM $M = \{m_{ij}\}_{i,j=1}^n$ has to be multiplicatively reciprocal, i.e.*

$$m_{ij} = \frac{1}{m_{ji}}, \qquad i, j = 1, \ldots, n. \tag{II.2}$$

Example 2. The matrix

$$M = \begin{pmatrix} 1 & 2 & 4 & 7 \\ \frac{1}{2} & 1 & 2 & 5 \\ \frac{1}{4} & \frac{1}{2} & 1 & 3 \\ \frac{1}{7} & \frac{1}{5} & \frac{1}{3} & 1 \end{pmatrix} \tag{II.3}$$

represents a MPCM of four objects $o_1, o_2, o_3,$ and o_4; the matrix clearly satisfies condition (II.2) of multiplicative reciprocity required in Definition 4.

According to Definition 4, element $m_{13} = 4$ indicates that object o_1 is $4-$times as good as object o_3 (or in other words: $4-$times preferred to o_3). Thus, according to the common sense, object o_3 should be $\frac{1}{4}-$times as good as object o_1 (or in other words: $4-$times less preferred to o_1), i.e. $m_{31} = \frac{1}{4}$. Therefore, multiplicative reciprocity (II.2) is a very natural property of MPCMs resulting from the interpretation of its elements.

Notice that, according to the multiplicative reciprocity, the elements on the main diagonal of the MPCM M have to be equal to 1, i.e. $m_{ii} = 1, i = 1, \ldots, 4$. This requirement is in compliance with the interpretation of the PCs in the MPCM. The element m_{ii} represents the PC of object o_i with object $o_i, i = 1, \ldots, 4$; i.e, the PC of one object with itself. It is natural that o_i is $1-$times as good as itself (or in other words: equally preferred), which results in $m_{ii} = 1$. △

As shown in Example 2, multiplicative reciprocity (II.2) is an inherent property that results naturally from the interpretation of the PCs in a MPCM. Thus, it is always sufficient to provide one of the related PCs m_{ij} and $m_{ji}, i, j \in \{1, \ldots, n\}$, in a MPCM and the second one is then determined automatically based on the multiplicative-reciprocity property (II.2).

As already mentioned in Section 1.1.1, various other terms besides "MPCM" are used in the literature. Most often they are called just "PCMs" (see e.g. Saaty (1977, 2006) etc.). Sometimes they are referred to as "reciprocal PCMs" or "reciprocal preference relations". In this thesis, the term "MPCM" is used in order to emphasize the fact that these PCMs are multiplicatively reciprocal, and to distinguish them clearly from additive PCMs (defined later in Section 2.3) that are additively reciprocal.

To make PCs of objects, Saaty (1977) suggested to use integer numbers between 1 and 9 and their reciprocals. This means that one object can be up to $9-$times preferred or more important over another one. The choice of this $9-$point scale was done based on the psychological experiments showing that humans are not able to compare simultaneously more than 5 to 9 objects (Miller, 1956).

Each integer from the scale is also assigned a linguistic term expressing the intensity of preference/importance of one object over another one. For example, number 1 represents equal preference and number 9 absolute preference. Thus, DMs can express their intensities of preference either numerically using the integers or linguistically using the assigned linguistic terms. Usually, Saaty's scale of integers $1, 3, 5, 7,$ and 9 given in Tab. II.1 is used for PCs. When there is a need for more detailed PCs, a more detailed scale with intermediate values $2, 4, 6,$ and 8 is utilized. The intermediate values are expressed using the neighboring terms and connecting them with the word "between". For example, 2 is interpreted as "between equal and weak preference".

However, even though "being $9-$times preferred" is the highest intensity of preference available in Saaty's scale to make PCs, it is not a natural maximum of intensity of preference. In fact, there is no natural maximum of number of times an object can be preferred to another one. Thus, it is quite difficult for DMs to express their intensities of preference in terms of multiples when they do not have a natural maximum value of intensity of preference available. On the other hand, number 9 in Saaty's scale is assigned the linguistic term "absolutely preferred". This may cause further confusion to DMs since "absolute preference" is naturally interpreted as the maximum possible intensity of preference. Similarly, number 3 standing for "3−times preferred" is assigned a linguistic term "weak preference". Not even this linguistic term corresponds to human intuition. Most of the DMs would probably assign "weak preference" to a number much closer to 1, which stands for equal preference, and would use the number 3 to model much stronger intensity of preference. Thus, it seems that the linguistic terms in Saaty's scale do not correspond very well to the respective numerical values that are distributed uniformly in the interval $[1, 9]$. Some further problems related to Saaty's scale will be mentioned in the following section. A detailed and interesting discussion on modeling linguistic terms from Saaty's scale is given by Stoklasa (2014).

Usually, either the term "intensity of preference" or the term "intensity of importance" is used in relation to PCs of objects. The choice of the particular term often depends on the context. When comparing alternatives with respect to a given criterion, we usually use the term "preference". When comparing criteria with respect to the goal of the decision-making problem, the term "importance" is often preferred. For the simplicity, only one of the terms will always be used hereafter.

Saaty's discrete scale $\left\{ \frac{1}{9}, \frac{1}{8}, \ldots, 1, 2, \ldots, 8, 9 \right\}$ is without doubts the most commonly used scale in practice. However, in general, any scale S of real numbers containing the neutral element 1 and such that for any element x in S also the element $\frac{1}{x}$ belongs to S can be used for the construction of MPCMs. Particular cases of such scale are the intervals $\left[\frac{1}{\sigma}, \sigma \right], \sigma > 1$, and $]0, \infty[$ (Gavalec et al., 2015).

Table II.1: Saaty's scale.

Intensity of preference	Linguistic term
1	equal preference
3	weak preference
5	strong preference
7	demonstrated preference
9	absolute preference
2, 4, 6, 8	intermediate values between the two adjacent judgments connected by word "between"

2.2.2 Consistency of MPCMs

Multiplicative reciprocity (II.2) is an inherent property of MPCMs resulting from the interpretation of the PCs in the matrices. However, requiring the multiplicative reciprocity of PCs is not sufficient to guarantee that the preference information contained in a MPCM is reasonable and that the priorities of objects derived from such a matrix are reliable, i.e. that they represent preferences of the DMs properly.

In order to guarantee reliability of the priorities obtainable from a MPCM, DMs should be consistent in their preferences when entering PCs into the MPCM. Various consistency conditions have been defined for MPCMs. A well-known and most often applied consistency condition is the traditional multiplicative-consistency condition.

2.2.2.1 Multiplicative consistency

Definition 5. *(Saaty, 1980) A MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$ is said to be multiplicatively consistent if it satisfies the multiplicative-transitivity property*

$$m_{ij} = m_{ik}m_{kj}, \qquad i,j,k = 1,\ldots,n. \tag{II.4}$$

Definition 5 of multiplicative consistency is clearly invariant under permutation of objects compared in the MPCM M.

Example 3. The MPCM

$$M = \begin{pmatrix} 1 & 2 & 4 & 8 \\ \frac{1}{2} & 1 & 2 & 4 \\ \frac{1}{4} & \frac{1}{2} & 1 & 2 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix} \tag{II.5}$$

is multiplicatively consistent according to Definition 5 since it satisfies the multiplicative-transitivity property (II.4). By permuting the MPCM M using the permutation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{II.6}$$

for example, we obtain the MPCM $M^{\pi} = PMP^{T}$ in the form

$$M^{\pi} = \begin{pmatrix} 1 & 8 & 2 & 4 \\ \frac{1}{8} & 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 4 & 1 & 2 \\ \frac{1}{4} & 2 & \frac{1}{2} & 1 \end{pmatrix}, \tag{II.7}$$

which is again multiplicatively consistent according to Definition 5. $\triangle$

The following theorem provides us with alternative ways to verify multiplicative consistency of MPCMs.

Theorem 1. *For a MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$, the following statements are equivalent:*

(i) $m_{ij} = m_{ik}m_{kj}, \qquad i,j,k = 1,\ldots,n,$

Table II.2: Random index RI.

n	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.52	0.89	1.11	1.25	1.35	1.41	1.45	1.49

(ii) $m_{ij}m_{jk}m_{ki} = 1, \qquad i,j,k = 1,\ldots,n,$

(iii) $m_{ij}m_{jk}m_{ki} = m_{ik}m_{kj}m_{ji}, \qquad i,j,k = 1,\ldots,n.$

Further, Saaty (1994) derived the following characterization of multiplicatively consistent MPCMs.

Proposition 1. *A MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$ is multiplicatively consistent if and only if there exists a positive vector $\underline{w} = (w_1,\ldots,w_n)^T$ such that*

$$m_{ij} = \frac{w_i}{w_j}, \qquad i,j = 1,\ldots,n. \tag{II.8}$$

The notation $\underline{w} = (w_1,\ldots,w_n)^T$ will be used hereafter to represent exclusively a priority vector associated with a MPCM.

According to Proposition 1, when a MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$ of n objects is multiplicatively consistent, there exist priorities $w_1,\ldots,w_n$ of objects using which we can determine precisely the original PCs $m_{ij}, i,j = 1,\ldots,n$, in the MPCM M by applying the characterization (II.8). In fact, each column of a multiplicatively consistent MPCM M is a priority vector satisfying these properties. On the other hand, when the priorities $w_1,\ldots,w_n$ of objects are available, we can construct a multiplicatively consistent MPCM of objects by applying (II.8). Notice that the multiplicative reciprocity of M is always guaranteed by applying (II.8) since $m_{ji} = \frac{w_j}{w_i} = \frac{1}{\frac{w_i}{w_j}} = \frac{1}{m_{ij}}$.

Example 4. Since the MPCM (II.5) in Example 3 is multiplicatively consistent, there exists a positive priority vector $\underline{w} = (w_1,w_2,w_3,w_4)^T$ satisfying the characterization property (II.8). It is, for example, the priority vector $\underline{w} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8})^T$. △

Multiplicative-consistency condition (II.4) seems to be reasonable when using numerical values for expressing the intensities of preference. When, for example, object o_i is $3-$times preferred to object $o_k(m_{ik} = 3)$, and object o_k is $3-$times preferred to object $o_j(m_{kj} = 3)$, then it is quite natural to require object o_i to be $9-$times preferred to object $o_j(m_{ij} = 3 \cdot 3 = 9)$. But what happens when $m_{ik} = 3$ and $m_{kj} = 5$? In order to be consistent we should write $m_{ij} = 3 \cdot 5 = 15$. However, this numerical value is out of Saaty's scale that is usually used for making PCs. Note that this problem would not occur with the scale $S =]0, \infty[$, where for any $m_{ik} \in S, m_{kj} \in S$ also $m_{ij} = m_{ik}m_{kj} \in S$.

Clearly, it is very difficult or even impossible (especially for MPCMs of large dimensions) to keep multiplicative consistency (II.4) when using only the values from limited Saaty's scale and their reciprocals. Thus, Saaty (1980) defined Consistency Index CI to measure inconsistency of MPCMs. CI is given for a MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$ as

$$CI = \frac{\lambda - n}{n - 1}, \tag{II.9}$$

where n is the number of objects compared in the matrix, and λ is the maximal eigenvalue of the matrix. CI defined by (II.9) is invariant under permutation of objects compared in the MPCM M (Brunelli and Fedrizzi, 2015). Further, Saaty (1994) showed that when the DM is absolutely consistent in his or her judgments, i.e. when the MPCM is multiplicatively consistent according to (II.4), then $\lambda = n$, and thus $CI = 0$.

Since reaching absolute multiplicative consistency is very difficult, especially with limited Saaty's scale, some degree of inconsistency is allowed. CI is required to be close to 0. However, determining the valued of CI below which MPCMs are regarded as acceptably inconsistent and above which they are regarded as inconsistent is complicated. Thus, Saaty (1980) defined Consistency Ratio CR in the form

$$CR = \frac{CI}{RI}, \tag{II.10}$$

where RI is Random Index that is computed as an average value of CI of MPCMs randomly generated from the elements of Saaty's scale separately for each order n of MPCMs. Tab. II.2 shows the values of RI (rounded to two decimal places) for MPCMs up to dimension $n = 10$ as given by Alonso and Lamata (2006).

According to Saaty (1980), a MPCM is regarded as acceptably inconsistent if $CR \leq 0.1$. A MPCM such that $CR > 0.1$ is then considered as inconsistent and the PCs in such a matrix should be reconsidered. Nevertheless, even the relaxed requirement of $CR \leq 0.1$ is quite difficult to reach, especially for MPCMs of large dimensions.

Besides CR, many other inconsistency indices have been proposed in the literature for verifying an acceptable level of inconsistency of MPCMs. Some of them are reviewed e.g. by Brunelli and Fedrizzi (2015) and by Gavalec et al. (2015).

The interpretation of the multiplicative consistency condition (II.4) does not seem very intuitive when linguistic terms from Saaty's scale are used for expressing the intensities of preference. When, for example, object o_i is weakly preferred to object o_k ($m_{ik} = 3$), and object o_k is weakly preferred to object o_j ($m_{kj} = 3$), then, according to the multiplicative consistency (II.4), it should follow that $m_{ij} = 3 \cdot 3 = 9$, which means that object o_i should be absolutely preferred to object o_j. However, absolute preference of o_i over o_j seems to be too strong in comparison to weak preferences of o_i over o_k and of o_k over o_j. Using a much smaller preference than the absolute preference would probably be more intuitive in this case. Thus, other consistency conditions respecting better the linguistic interpretation of preference intensities have been proposed in the literature. One of them is the weak-consistency condition introduced by Jandová and Talašová (2013) and by Stoklasa et al. (2013).

2.2.2.2 Weak consistency

Jandová and Talašová (2013) and Stoklasa et al. (2013) proposed the weak-consistency condition as an intuitive minimum consistency requirement for MPCMs $M = \{m_{ij}\}_{i,j=1}^{n}$. The idea is to require the preference intensity m_{ij} to be at least the maximal value of the preference intensities m_{ik} and m_{kj} for each triplet of objects $o_i, o_j, o_k,\ i,j,k \in \{1, \ldots, n\}$.

Definition 6. *(Jandová and Talašová, 2013) A MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$ is said to be weakly consistent if*

$$
\begin{aligned}
m_{ik} > 1 \ \wedge \ m_{kj} > 1 &\quad \Rightarrow \quad m_{ij} \geq \max\{m_{ik}, m_{kj}\}, \\
m_{ik} = 1 \ \wedge \ m_{kj} \geq 1 &\quad \Rightarrow \quad m_{ij} = \max\{m_{ik}, m_{kj}\}, \\
m_{ik} \geq 1 \ \wedge \ m_{kj} = 1 &\quad \Rightarrow \quad m_{ij} = \max\{m_{ik}, m_{kj}\},
\end{aligned}
\tag{II.11}
$$

holds for $i,j,k = 1, \ldots, n$.

It is obvious that Definition 6 of weak consistency is again invariant under permutation of objects compared in the MPCM M.

The weak-consistency condition is suitable especially when linguistic terms are used for expressing the intensities of preference instead of real numbers. For example, when object o_i is weakly preferred to object o_k ($m_{ik} = 3$), and object o_k is strongly preferred to object o_j ($m_{kj} = 5$), then object o_i has to be at least strongly preferred to object o_j ($m_{ij} \geq \max\{3, 5\} = 5$). Thus, the requirement of weak consistency seems to be very natural, and it provides DMs with some space for expressing their intensities of preference - a desired tolerance. Moreover, it is very easy to control while entering PCs into the MPCM.

The weak-consistency condition was also defined in more relaxed forms by Krejčí (2017b) and by Basile and D'Apuzzo (2002). Recently, Cavallo and D'Apuzzo (2016) proposed a definition of weak consistency very similar to the weak consistency in Definition 6. In this thesis, the weak-consistency condition given by Definition 6 is adopted and later applied in Chapter V in a novel method for dealing with large-dimensional PCMs.

Jandová and Talašová (2013) derived some rules equivalent to the weak-consistency condition (II.11). They are formulated in the following theorem.

Theorem 2. *(Jandová and Talašová, 2013) For a MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$, the following statements are equivalent:*

(i) M is weakly consistent according to Definition 6,

(ii) For every $i,j,k = 1, \ldots, n$:

$$
\begin{aligned}
m_{ik} < 1 \ \wedge \ m_{kj} < 1 &\quad \Rightarrow \quad m_{ij} \leq \min\{m_{ik}, m_{kj}\}, \\
m_{ik} = 1 \ \wedge \ m_{kj} \leq 1 &\quad \Rightarrow \quad m_{ij} = \min\{m_{ik}, m_{kj}\}, \\
m_{ik} \leq 1 \ \wedge \ m_{kj} = 1 &\quad \Rightarrow \quad m_{ij} = \min\{m_{ik}, m_{kj}\}.
\end{aligned}
\tag{II.12}
$$

(iii) For every $i,j,k = 1, \ldots, n$:

$$
\begin{aligned}
1 < \tfrac{1}{m_{kj}} < m_{ik} &\quad \Rightarrow \quad \tfrac{1}{m_{ik}} \leq m_{ij} \leq m_{ik}, \\
1 < m_{ik} < \tfrac{1}{m_{kj}} &\quad \Rightarrow \quad m_{kj} \leq m_{ij} < 1, \\
1 < \tfrac{1}{m_{kj}} < m_{ik} &\quad \Rightarrow \quad 1 < m_{ij} \leq m_{ik}.
\end{aligned}
\tag{II.13}
$$

(iv) For every $i, j, k = 1, \ldots, n$:

$$\frac{1}{m_{kj}} < m_{ik} < 1 \quad \Rightarrow \quad m_{kj} \leq m_{ij},$$
$$1 < m_{kj} < \frac{1}{m_{ik}} \quad \Rightarrow \quad m_{ik} \leq m_{ij} < 1, \tag{II.14}$$
$$1 < m_{kj} = \frac{1}{m_{ik}} \quad \Rightarrow \quad \frac{1}{m_{kj}} \leq m_{ij} \leq m_{kj}.$$

Further, Jandová and Talašová (2013) derived some interesting properties of weakly consistent MPCMs. For example, every weakly consistent MPCM can be permuted in such a way that the objects compared in the MPCM are ordered from the most preferred to the least preferred. In such an ordered weakly consistent MPCM, all elements above the main diagonal are greater or equal to 1, i.e. $m_{ij} \geq 1$, $i, j = 1, \ldots, n$, $i < j$. Moreover, the sequences of elements in the rows are non-decreasing and the sequences of elements in the columns are non-increasing.

Example 5. The MPCM

$$M = \begin{pmatrix} 1 & 2 & 5 & 9 \\ \frac{1}{2} & 1 & 4 & 7 \\ \frac{1}{5} & \frac{1}{4} & 1 & 6 \\ \frac{1}{9} & \frac{1}{7} & \frac{1}{6} & 1 \end{pmatrix} \tag{II.15}$$

is obviously weakly consistent according to Definition 6 since the sequences of the PCs in the rows are non-decreasing and the sequences of the PCs in the columns are non-increasing. $\triangle$

Jandová and Talašová (2013) showed that every MPCM that is multiplicatively consistent according to Definition 5 is also weakly consistent according to Definition 6. In fact, from $m_{ik} > 1 \wedge m_{kj} > 1$, it follows that $m_{ij} = m_{ik}m_{kj} \geq \max\{m_{ik}, m_{kj}\}$. For $m_{ik} = 1 \wedge m_{kj} \geq 1$, and for $m_{ik} \geq 1 \wedge m_{kj} = 1$, we get $m_{ij} = m_{ik}m_{kj} = \max\{m_{ik}, m_{kj}\}$.

The properties of weakly consistent MPCMs reviewed in this section are utilized in the novel method for dealing with large-dimensional PCMs that is described in detail in Chapter V.

2.2.3 Deriving priorities from MPCMs

Based on the given MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$ of n objects $o_1, \ldots, o_n$, it is endeavoured to derive the priority vector $\underline{w} = (w_1, \ldots, w_n)^T$ that would best represent the relative preference (or importance) of the objects with respect to the other objects in the set.

We know from the previous section that when a MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$ is multiplicatively consistent according to (II.4), then there exists a positive priority vector $\underline{w} = (w_1, \ldots, w_n)^T$ such that $m_{ij} = \frac{w_i}{w_j}$, $i, j = 1, \ldots, n$. For example, any column of the multiplicatively consistent MPCM can by used to represent the priority vector $\underline{w} = (w_1, \ldots, w_n)^T$. When the MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$ is not multiplicatively consistent, then the ratios of the priorities only estimate the PCs in the matrix, i.e.

$$m_{ij} \approx \frac{w_i}{w_j}, \qquad i, j = 1, \ldots, n. \tag{II.16}$$

Priorities $w_1, \ldots, w_n$ are given on a ratio scale. Because the ratios $\frac{w_i}{w_j}, i, j = 1, \ldots, n$, play a key role, these ratios cannot change under any normalization of the priorities. Thus, the normalization can be done only by multiplying the priorities by a constant $c > 0$ $\left(\frac{c \cdot w_i}{c \cdot w_j} = \frac{w_i}{w_j}\right)$. This means that when there exists a positive vector $\underline{w} = (w_1, \ldots, w_n)^T$ representing the priorities of objects $o_1, \ldots, o_n$, then also any vector obtained from $\underline{w}$ by the transformation

$$w_i \rightarrow c \cdot w_i, \qquad i = 1, \ldots, n, \tag{II.17}$$

where $c > 0$, represents the priorities of the objects. In the literature, the normalization condition

$$\sum_{i=1}^{n} w_i = 1, \qquad w_i \in [0, 1], \;\; i = 1, \ldots, n, \tag{II.18}$$

is usually applied in order to reach the uniqueness. This normalization condition is applied also in this thesis, and for simplicity and when no confusion arises, the normalized priorities are referred to only as priorities.

Many methods have been proposed in the literature for deriving priorities of objects from MPCMs. In this thesis, two well-known methods, the eigenvector method and the geometric-mean method, are of interest, and only the extension of these two methods to fuzzy MPCMs will be dealt with here. For a review of other methods, see, e.g., Gavalec et al. (2015).

2.2.3.1 Eigenvector method

The eigenvector method (EVM), originally proposed by Saaty (1977) for deriving priorities in AHP, is one of the oldest methods for deriving priorities of objects from MPCMs. According to this method, the priorities of objects are calculated as the components $w_1, \ldots, w_n$ of the normalized maximal eigenvector $\underline{w}$ corresponding to the maximal eigenvalue λ_{MAX} of the MPCM M.

Let A be a square matrix of size n. The scalar λ and the vector $\underline{w}$ of size n satisfying $A\underline{w} = \lambda\underline{w}$ are called the eigenvalue and the eigenvector of the matrix A, respectively. The set of all eigenvalues of matrix A is obtained as the solution to the equation $|A - \lambda I| = 0$ where I denotes the identity matrix of size n and $|.|$ denotes the determinant of a given matrix.

From Perron-Frobenius theorem it follows that for a positive matrix $A = \{a_{ij}\}_{i,j=1}^{n}$, i.e. $a_{ij} > 0$ for $i, j = 1, \ldots, n$, there exists a positive eigenvalue λ_{MAX} such that $|\lambda| < \lambda_{MAX}$ for any other eigenvalue λ of A. Such eigenvalue

$$\lambda_{MAX} = \max\{\lambda; |A - \lambda I| = 0\} \tag{II.19}$$

is called *the maximal eigenvalue* of A. Further, there exists a positive eigenvector

$$\underline{w}_{MAX} = (w_1, \ldots, w_n)^T: \quad A\underline{w}_{MAX} = \lambda_{MAX}\underline{w}_{MAX} \tag{II.20}$$

corresponding to λ_{MAX} called *the maximal eigenvector* of A. The maximal eigenvector $\underline{w}_{MAX}$ is unique up to a multiplicative constant. That is why normalization (II.18) is usually applied to obtain a unique solution - the normalized maximal eigenvector

$$\underline{w}_{MAX} = (w_1, \ldots, w_n)^T: \quad A\underline{w}_{MAX} = \lambda_{MAX}\underline{w}_{MAX}, \quad \sum_{i=1}^{n} w_i = 1. \tag{II.21}$$

A MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$ is a positive square matrix. Therefore, there always exists a positive maximal eigenvalue λ_{MAX} and a normalized positive maximal eigenvector $\underline{w}_{MAX}$. The components of this eigenvector represent the priorities of objects compared in the MPCM M. When a MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$ is multiplicatively consistent according to (II.4), then $\lambda_{MAX} = n$, and $\underline{w}_{MAX}$ satisfies (II.8). Further, it is a well-known fact that the EVM is invariant under permutation of objects in the MPCM M (see, e.g., Fichtner (1986)).

Only the maximal eigenvalues and the corresponding maximal eigenvectors are considered in this thesis. Thus, for the sake of simplicity, the lower index $_{MAX}$ is omitted and only the notation λ and $\underline{w}$ is used hereafter. Further, for later use in optimization formulas for deriving fuzzy maximal eigenvalues and normalized fuzzy maximal eigenvectors of FMPCMs in Sections 4.2.2.4 and 4.2.3.1, it is particularly useful to denote the maximal eigenvalue λ of matrix M as $EVM_\lambda(M)$ and the normalized maximal eigenvector $\underline{w}$ of matrix M corresponding to λ as $EVM_{\underline{w}}(M)$.

Further, the following property, that is later used in Section 4.2.2.4, results from the Perron-Frobenius Theorem. Let $A = \{a_{ij}\}_{i,j=1}^{n}$ and $B = \{b_{ij}\}_{i,j=1}^{n}$ be two positive matrices, and let $a_{ij} \geq b_{ij}$ for $i, j = 1, \ldots, n$ and $a_{kl} > b_{kl}$ for $k, l \in \{1, \ldots, n\}$. Then, the maximal eigenvalue of A is greater than the maximal eigenvalue of B, i.e. $EVM_\lambda(A) > EVM_\lambda(B)$.

2.2.3.2 Geometric-mean method

The geometric mean method (GMM) is another well-known method for deriving priorities of objects from MPCMs. This method gives the same results as the logarithmic least squares method (LLSM).

LLSM utilizes the characterization property (II.16) of a MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$, and it is based on the minimization of the sum of squared errors of logarithms:

$$\min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\ln m_{ij} - \ln \frac{w_i}{w_j} \right)^2 \to w_i, \qquad i = 1, \ldots, n. \tag{II.22}$$

It was shown that the optimal solution of (II.22) is always unique (up to a multiplicative constant) and can be determined by the geometric mean of the elements in the rows of the MPCM M, i.e.

$$w_i = \sqrt[n]{\prod_{j=1}^{n} m_{ij}}, \qquad i = 1, \ldots, n. \tag{II.23}$$

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By employing the normalization condition (II.18), the normalized priorities can be computed directly as

$$w_i = \frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} m_{kj}}}, \qquad i = 1, \ldots, n. \tag{II.24}$$

It can be easily verified that the GMM given by formula (II.24) is invariant under permutation of objects in the MPCM M.

Furthermore, Saaty and Vargas (1984) showed that when a MPCM is multiplicatively consistent according to (II.4), then EVM and GMM (or equivalently LLSM) lead to the same results. This means that the priority vector obtained from a multiplicatively consistent MPCM by GMM satisfies (II.8) too. When a MPCM is close to multiplicative consistency, then the methods provide very similar but not identical results (Crawford and Williams, 1985).

There have been many studies comparing EVM and GMM. Saaty and Vargas (1984) compared EVM, GMM, and the least squares method. They concluded that EVM is the only method that guarantees rank preservation (i.e., $m_{ik} \geq m_{jk}$ for all $k = 1, \ldots, n$ implies $w_i \geq w_j$) under inconsistency. Further, Saaty and Hu (1998) showed an illustrative example where the ranking of alternatives obtained by GMM differs from the ranking obtained by EVM. Based on this example, Saaty and Hu concluded that EVM is the only valid method for deriving priorities from MPCMs, in particular from inconsistent MPCMs. However, it is not acceptable to derive such a strong conclusion based on one illustrative example. Furthermore, showing that GMM leads to a ranking different from the one obtained by EVM does not surely demonstrate that GMM leads to rank reversal; this conclusion is based on the unfounded assumption that EVM provides the correct solution.

Crawford and Williams (1985) ran some simulations to compare the performance of EVM and GMM under different error distributions and metrics. The simulations suggest better performance of GMM in priorities estimation as well as in rank preservation. Other studies favoring GMM over EVM have been done, e.g., by Barzilai (1997), Blaquero et al. (2006) and Dijkstra (2013).

2.3 Additive pairwise comparison matrices

The first bases of the theory of additive PCMs were given by Orlovski (1978), Nurmi (1981), Tanino (1984), and Kacprzyk (1986). In this section, the methods for constructing additive PCMs are reviewed and two types of additive PCMs are defined - additive PCMs with additive representation and additive PCMs with multiplicative representation. Afterwards, definitions of consistency and methods for verifying consistency and deriving priorities of objects from both types of additive PCMs are reviewed.

2.3.1 Construction of APCMs

Definition 7. *An additive pairwise comparison matrix (APCM) of n objects $o_1, \ldots, o_n$ is a square matrix $A = \{a_{ij}\}_{i,j=1}^{n}$ whose elements $a_{ij}, i, j = 1, \ldots, n$, are defined on interval $[0,1]$. Further, the matrix is additively reciprocal, i.e.*

$$a_{ij} = 1 - a_{ji}, \qquad i, j = 1, \ldots, n. \tag{II.25}$$

Unlike in the case of MPCMs, there exists no widely accepted discrete scale with assigned linguistic terms for expressing the intensities of preference for APCMs. Often, the whole interval $[0,1]$ is used, i.e.

$$
\begin{aligned}
a_{ij} &= 1 & &\text{if } o_i \text{ is absolutely preferred to } o_j, \\
a_{ij} &\in]0.5, 1[& &\text{if } o_i \text{ is preferred to } o_j, \\
a_{ij} &= 0.5 & &\text{if } o_i \text{ and } o_j \text{ are indifferent/ equally preferred}, \\
a_{ij} &\in]0, 0.5[& &\text{if } o_i \text{ is less preferred to } o_j, \\
a_{ij} &= 0 & &\text{if } o_i \text{ is absolutely less preferred to } o_j.
\end{aligned}
\tag{II.26}
$$

Example 6. Matrix

$$A = \begin{pmatrix} 0.5 & 0.7 & 1 \\ 0.3 & 0.5 & 0.9 \\ 0 & 0.1 & 0.5 \end{pmatrix} \tag{II.27}$$

represents an APCM of three objects o_1, o_2, and o_3; the matrix clearly satisfies condition (II.25) of additive reciprocity required in Definition 7.

According to Definition 7, element $a_{13} = 1$ indicates that object o_1 is absolutely preferred to object o_3. Thus, according to the common sense, object o_3 should be absolutely less preferred to object o_1, i.e. $a_{31} = 0$.

Therefore, additive reciprocity (II.25) is a very natural property of APCMs resulting from the interpretation of its elements.

Notice that according to the additive reciprocity, the elements on the main diagonal of the APCM A have to be equal to 0.5, i.e. $a_{ii} = 0.5, i = 1, 2, 3$. This requirement is in compliance with the interpretation of the PCs in the APCM. The element a_{ii} represents the PC of object o_i with itself. It is natural that o_i is equally preferred to itself, which means $a_{ii} = 0.5, i = 1, 2, 3$. $\triangle$

As shown in Example 6, additive reciprocity (II.25) is an inherent property that results naturally from the interpretation of the PCs in an APCM. Thus, it is always sufficient to provide one of the related PCs a_{ij} and $a_{ji}, i, j \in \{1, \ldots, n\}$, in an APCM and the second one is then determined automatically based on the additive-reciprocity property (II.25).

Various other terms are used in the literature besides "APCM". Most often they are called "fuzzy preference relations" (see, e.g., Bezdek et al. (1978); Nurmi (1981); Tanino (1984); Kacprzyk (1986); Cabrerizo et al. (2014); Gavalec et al. (2015)), sometimes reciprocal relations (see, e.g., Baets et al. (2006); Fedrizzi and Brunelli (2009, 2010)), additively reciprocal relations (see, e.g., Fedrizzi and Giove (2013)), or reciprocal preference relations (see, e.g., Chiclana et al. (2009)). In this thesis, the term "APCM" is used in order to emphasize the fact that these PCMs are additively reciprocal, and to make an analogy to MPCMs that are multiplicatively reciprocal.

The use of the term "fuzzy preference relation" in this thesis might even be confusing or misleading. Historically, fuzzy preference relations were defined as a fuzzy extension of binary preference relations (Bezdek et al. (1978); Nurmi (1981); Tanino (1984); Kacprzyk (1986)). Binary preference relation b on a finite set of objects $O = \{o_1, \ldots, o_n\}$ is defined as $b : O \times O \rightarrow \{0, 1\}$ where $b(o_i, o_j) = 1$ if o_i is preferred to o_j, and $b(o_i, o_j) = 0$ if o_j is preferred to o_i. Fuzzy preference relation $\widetilde{b}$ on a finite set of objects $O = \{o_1, \ldots, o_n\}$ is then defined as a fuzzy set on the cartesian product $O \times O$ characterized by the membership function $\mu_{\widetilde{b}} : O \times O \rightarrow [0, 1]$ where $\mu_{\widetilde{b}}(o_i, o_j) = \alpha$ represents the degree of preference of object o_i to object o_j. (Fuzzy sets will be defined properly in Chapter III.)

In this thesis, the fuzzy extension of degrees of preference (intensities of preference) is of interest. Therefore, because the intensities of preference both in MPCMs and APCMs will be later in this thesis modeled by fuzzy numbers instead of crisp numbers, the world "fuzzy" is reserved for describing these extended versions of PCMs, i.e. they will be called fuzzy MPCMs and fuzzy APCMs. If the term "fuzzy preference relation" was used instead of "APCM", we would have to deal with the fuzzy extension of fuzzy preference relations, i.e. "fuzzy fuzzy preference relations", which would create confusion.

Similarly as for MPCMs, the preference information contained in an APCM should be reasonable in order to ensure that the priorities of objects derived from such a matrix represent properly DMs' preferences. In order to verify whether the DMs are consistent in their preferences, it is necessary to define an appropriate consistency condition for APCMs.

Two traditional and well-known consistency conditions for APCMs are the additive-consistency condition and the multiplicative-consistency condition introduced by Tanino (1984). Often, APCMs are considered as one set of matrices for which it is possible to verify both additive and multiplicative consistency. However, this approach is not correct. Each type of consistency is strictly related to a particular interpretation of the PCs in the APCM A. In fact, before constructing an APCM A, it is necessary to choose between additive and multiplicative representation (associated with additive and multiplicative consistency, respectively), which has an impact on the values of the entries in the APCM A. This means that comparing a set of n objects pairwisely in an APCM by using additive representation, and comparing the same set of objects pairwisely in an APCM by using multiplicative representation, we obtain two APCMs with different entries. Depending on the representation used, we can then verify the associated additive or multiplicative consistency. Furthermore, it is necessary to distinguish between the two representations also when deriving priorities of objects from APCMs.

The necessity of distinguishing between APCMs with additive and multiplicative representation will be much clearer as soon as the definitions of both are provided. In Section 2.3.2, APCMs with additive representation will be introduced, and in Section 2.3.3, APCMs with multiplicative representation will be dealt with.

2.3.2 Additive pairwise comparison matrices with additive representation

Definition 8. *An APCM with additive representation (APCM-A) is an APCM* $R = \{r_{ij}\}_{i,j=1}^{n}$ *satisfying the additive-reciprocity property* (II.25) *where* $r_{ij} - r_{ji}$ *indicates the difference of preference intensity of object* o_i *and of object* o_j.

2.3.2.1 Additive consistency of APCMs-A

Definition 9. *(Tanino, 1984) An APCM-A* $R = \{r_{ij}\}_{i,j=1}^{n}$ *is said to be additively consistent if it satisfies the additive-transitivity property*

$$r_{ij} = r_{ik} + r_{kj} - 0.5, \qquad i, j, k = 1, \ldots, n. \tag{II.28}$$

Definition 9 of additive consistency is invariant under permutation of objects in the APCM-A R.

Example 7. The APCM-A

$$R = \begin{pmatrix} 0.5 & 0.6 & 0.8 & 0.9 \\ 0.4 & 0.5 & 0.7 & 0.8 \\ 0.2 & 0.3 & 0.5 & 0.6 \\ 0.1 & 0.2 & 0.4 & 0.5 \end{pmatrix} \tag{II.29}$$

is additively consistent according to Definition 8 since it satisfies the additive-transitivity property (II.28). Also the permuted matrix $R^\pi = PRP^T$ obtained by applying the permutation matrix (II.6) is additively consistent. $\triangle$

The following theorem provides us with alternative ways to verify additive consistency of APCMs-A.

Theorem 3. *For an APCM-A $R = \{r_{ij}\}_{i,j=1}^n$, the following statements are equivalent:*

(i) R is additively consistent according to the additive-transitivity property (II.28),

(ii)

$$r_{ij} + r_{jk} + r_{ki} = r_{ik} + r_{kj} + r_{ji}, \qquad i, j, k = 1, \dots, n, \tag{II.30}$$

(iii)

$$r_{ij} + r_{jk} + r_{ki} = \frac{3}{2}, \qquad i, j, k = 1, \dots, n, \tag{II.31}$$

Note 1. *According to Theorem 3, the additive consistency of APCMs-A can be defined by using any of the expressions* (II.30) *and* (II.31). *These expressions will be referred to later when an extension of the definition of additive consistency to fuzzy APCMs-A is dealt with.*

Further, Tanino (1984) derived the following characterization of additively consistent APCMs-A.

Proposition 2. *(Tanino's characterization) An APCM-A $R = \{r_{ij}\}_{i,j=1}^n$ is additively consistent if and only if there exists a non-negative vector $\underline{v} = (v_1, \dots, v_n)^T$, $|v_i - v_j| \leq 1$, $i, j = 1, \dots, n$, such that*

$$r_{ij} = 0.5 \left(v_i - v_j + 1 \right), \qquad i, j = 1, \dots, n. \tag{II.32}$$

The notation $\underline{v} = (v_1, \dots, v_n)^T$ will be used hereafter to represent exclusively a priority vector associated with an APCM-A.

Proposition 2 says that when an APCM-A $R = \{r_{ij}\}_{i,j=1}^n$ of n objects is additively consistent, there exist priorities $v_1, \dots, v_n$ of objects using which we can determine precisely the original PCs $r_{ij}, i, j = 1, \dots, n$, in the APCM-A R by applying Tanino's characterization (II.32). On the other hand, when the priorities $v_1, \dots, v_n$ of objects are given such that $|v_i - v_j| \leq 1, i, j = 1, \dots, n$, we can construct an additively consistent APCM-A of objects by applying (II.32). Notice that the additive reciprocity of R is always guaranteed by applying (II.32) since $r_{ij} = 0.5 \left(v_i - v_j + 1 \right) = 1 - 0.5(v_j - v_i + 1) = 1 - r_{ji}$.

Example 8. Since the APCM-A (II.29) in Example 7 is additively consistent, there exists a non-negative priority vector $\underline{v} = (v_1, v_2, v_3, v_4)^T$ satisfying the characterization property (II.32). It is, for example, the priority vector $\underline{v} = (0.8, 0.6, 0.2, 0)^T$. $\triangle$

Moreover, Tanino's characterization (II.32) is in line with the interpretation of the differences of PCs in an APCM-A R as given in Definition 8. In particular, when an APCM-A R is additively consistent according to (II.28), then the difference of PCs indicating the difference of preference intensity of o_i and of o_j corresponds to the difference of their priorities (Krejčí, 2016), i.e.

$$r_{ij} - r_{ji} = v_i - v_j, \qquad i, j = 1, \dots, n. \tag{II.33}$$

When the APCM-A R is not additively consistent, then

$$r_{ij} - r_{ji} \approx v_i - v_j, \qquad i, j = 1, \dots, n. \tag{II.34}$$

Example 9. From the intensity of preference $r_{12} = 0.6$ in the APCM-A (II.29) we know immediately that the difference between the priorities v_1 and v_2 is 0.2 ($v_1 - v_2 = r_{12} - r_{21} = 0.6 - 0.4 = 0.2$). Similarly also for the remaining differences between the priorities. The priority vector $\underline{v} = (0.8, 0.6, 0.2, 0)^T$ shown in Example 8 satisfies this property. $\triangle$

Similarly to the multiplicative-consistency condition for MPCMs, also the additive-consistency condition (II.28) is very difficult to keep when PCs in an APCM-A R are done using the scale $[0, 1]$. Having $r_{ik} = 0.9$ and $r_{kj} = 1$, for example, it follows $r_{ij} = r_{ik} + r_{kj} - 0.5 = 1.4$, which exceeds the interval $[0, 1]$. Thus, it is possible to keep additive consistency only when using the intensities of preference close to the indifference value 0.5. This is not always reasonable or even possible (in particular for large-dimensional APCMs-A). Thus, other consistency conditions have been proposed in the literature. One of them is the weak-consistency condition proposed by Jandová et al. (2017).

The weak-consistency condition as introduced by Jandová et al. (2017) is applied in the same form both to APCMs-A and to APCMs with multiplicative representation (introduced later) without distinguishing among them, i.e. the weak-consistency condition is defined in general for APCMs. Therefore, this consistency condition will be reviewed later in Section 2.3.3.2, after introducing APCMs with multiplicative representation.

2.3.2.2 Deriving priorities from APCMs-A

Based on the given APCM-A $R = \{r_{ij}\}_{i,j=1}^n$ of n objects it is endeavoured to derive the priority vector $\underline{v} = (v_1, \ldots, v_n)^T$ that would best represent the relative preference of the objects with respect to the other objects in the set.

As discussed in the previous section, when an APCM-A $R = \{r_{ij}\}_{i,j=1}^n$ is additively consistent according to (II.28), then there exist priorities $v_1, \ldots, v_n, |v_i - v_j| \leq 1, i, j = 1, \ldots, n$, such that $r_{ij} = 0.5(v_i - v_j + 1)$. When an APCM-A $R = \{r_{ij}\}_{i,j=1}^n$ is not additively consistent, then the given expression only estimates the PCs in the matrix, i.e.

$$r_{ij} \approx 0.5(v_i - v_j + 1), \qquad i, j = 1, \ldots, n. \tag{II.35}$$

Fedrizzi and Brunelli (2010) proved that for an APCM-A $R = \{r_{ij}\}_{i,j=1}^n$ the only vector of priorities (up to an additive constant) satisfying (II.32) is $\underline{v} = (v_1, \ldots, v_n)^T$ such that

$$v_i = \frac{2}{n} \sum_{j=1}^n r_{ij}, \qquad i = 1, \ldots, n. \tag{II.36}$$

Notice that the method for deriving priorities of objects from APCMs-A by formula (II.36) is again invariant under permutation of objects.

Proposition 3. *(Krejčí, 2016) Given an APCM-A $R = \{r_{ij}\}_{i,j=1}^n$, the priorities $v_1, \ldots, v_n$ obtained from R by formula* (II.36) *are such that*

$$\sum_{i=1}^n v_i = n. \tag{II.37}$$

Proof.

$$\sum_{i=1}^n v_i = \sum_{i=1}^n \frac{2}{n} \sum_{j=1}^n r_{ij} = \frac{2}{n} \sum_{i=1}^n \sum_{j=1}^n r_{ij} = \frac{2}{n} \left(\sum_{i=1}^n r_{ii} + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n r_{ij} \right) =$$
$$\frac{2}{n} \left(\frac{n}{2} + \frac{n(n-1)}{2} \right) = n$$

$\square$

Remark 1. *Notice that the property* (II.37) *of the priorities given by* (II.36) *is independent of the additive consistency; it depends only on the additive-reciprocity condition* (II.25). *This means that also the sum of the priorities of objects obtained from an additively inconsistent APCM-A still equals n.*

As shown by Fedrizzi and Brunelli (2010), there exist infinitely many priority vectors satisfying Proposition 2. These priority vectors can be generated from (II.36) by adding an arbitrary constant. This means that for a priority vector $\underline{v} = (v_1, \ldots, v_n)^T$ satisfying Proposition 2, also any priority vector obtained from $\underline{v}$ by the transformation

$$v_i \quad \rightarrow \quad v_i + c, \qquad i = 1, \ldots, n, \tag{II.38}$$

where $c \in \mathbb{R}$, satisfies Proposition 2.

Note that it is not possible to multiply the priorities (II.36) as it is done in the case of MPCMs, where the ratios of the priorities estimate the original PCs in the matrix. In the case of APCMs-A, the original PCs r_{ij} in the APCM-A $R = \{r_{ij}\}_{i,j=1}^n$ are estimated by the differences between the priorities v_i and v_j by means of (II.35) and, thus, these differences have to remain unchanged.

In order to reach uniqueness, a normalization condition is usually applied to the priority vectors. In the case of MPCMs, the normalization condition

$$\sum_{i=1}^{n} w_i = 1, \;\; w_i \in [0,1] , \; i = 1, \ldots, n,\tag{II.39}$$

is usually applied, see Section 2.2.3. It is worth to note that the condition (II.39) is reachable independently of the requirement of multiplicative consistency of MPCMs, i.e. even the priorities obtained from a multiplicatively inconsistent MPCM can be normalized so that they satisfy (II.39).

The normalization condition (II.39) has been applied also to the priorities obtained from APCMs-A (see e.g. Xu (2004, 2007a); Xu and Chen (2008a,b); Wang and Li (2012); and the list of other papers provided by Fedrizzi and Brunelli (2009)). However, Fedrizzi and Brunelli (2009) showed that the normalization condition (II.39) is incompatible with Proposition 2.

Fedrizzi and Brunelli (2009) proposed a new normalization condition in the form

$$\min_{i=1,\ldots,n} v_i = 0, \;\; v_i \in [0,1] , \; i = 1, \ldots, n.\tag{II.40}$$

However, the normalization condition (II.40) is reachable only for additively consistent APCMs-A. For additively inconsistent APCMs-A, in general, the normalized priorities satisfying the condition $\min_{i=1,\ldots,n} v_i = 0$ do not satisfy the condition $v_i \in [0,1] , \; i = 1, \ldots, n,$ as is illustrated on the following example.

Example 10. Let us examine the priorities obtainable from the APCM-A

$$A = \left(\begin{array}{ccc} 0.5 & 0.8 & 1 \\ 0.2 & 0.5 & 0.9 \\ 0 & 0.1 & 0.5 \end{array} \right),\tag{II.41}$$

which is not additively consistent; $r_{12} + r_{23} - 0.5 = 0.8 + 0.9 - 0.5 = 1.2 \neq 1 = r_{13}$. The priorities of objects obtained by formula (II.36) are in the form $v_1 = \frac{23}{15}, v_2 = \frac{16}{15}, v_3 = \frac{6}{15}$. By applying the normalization condition $\min_{i=1,\ldots,n} v_i = 0$, we obtain normalized priorities in the form $v_1 = \frac{17}{15}, v_2 = \frac{10}{15}, v_3 = 0$. Clearly, $v_1 > 1$ which violates the normalization condition $v_i \in [0,1] , \; i = 1, \ldots, n.$ $\triangle$

Proposition 4. *(Krejčí, 2016) Given an APCM-A* $R = \{r_{ij}\}_{i,j=1}^{n}, n \geq 3$, *there exists no normalization condition of the type* (II.38) *for the priorities* (II.36) *that would guarantee the fulfillment of the property* $v_i \in [0,1] , \; i = 1, \ldots, n.$

Proof. There exist infinitely many priority vectors obtainable from (II.36) by the transformation (II.38). In order to modify the priorities so that $v_i \in [0,1] , \; i = 1, \ldots, n,$ a suitable constant c has to be added to the priorities (II.36). Further, we know that the differences between the priorities do not change by adding a constant to them; $(v_i + c) - (v_j + c) = v_i - v_j, i,j = 1, \ldots, n.$ Clearly, the priorities (II.36) could be normalized so that $v_i + c \in [0,1] , \; i = 1, \ldots, n,$ if and only if $|v_i - v_j| \leq 1, i,j = 1, \ldots, n.$ However, it will be shown that $|v_i - v_j| \leq 1, i,j = 1, \ldots, n,$ is not reachable in general.

Let $o_i, i \in \{1, \ldots, n\} ,$ be such that it is absolutely preferred to all other objects, and let $o_j, j \in \{1, \ldots, n\} ,$ be such that all other objects are absolutely preferred to o_j. Then, for $n \geq 3$

$$v_i - v_j = \frac{2}{n} \sum_{k=1}^{n} r_{ik} - \frac{2}{n} \sum_{k=1}^{n} r_{jk} = \frac{2}{n} \left((0.5 + n - 1) - (0.5 + 0) \right) = \frac{2n-2}{n} > 1.$$

$\square$

According to Proposition 4, the property $v_i \in [0,1] , \; i = 1, \ldots, n,$ cannot be guaranteed for additively inconsistent APCMs-A under any normalization condition of the type (II.38). However, as discussed in the previous section, it is difficult or even impossible to reach additive consistency of APCMs-A in many MCDM problems, especially because of the restricted scale $[0,1]$ used for expressing the intensities of preference of one compared object over another. In general, the higher the dimension of an APCM-A is, the more difficult reaching the consistency is. Even when the DM is asked to reconsider his or her preferences, it does not have to lead to an additively consistent APCM-A. Therefore, in real-life applications, priorities of objects have to be often derived from additively inconsistent APCMs-A. This calls for a normalization condition applicable also to the priorities obtained from these additively inconsistent APCMs-A. Recall that for MPCMs there is such a normalization condition - (II.39).

Since the condition $v_i \in [0, 1]$, $i = 1, \ldots, n$, is not reachable for priorities obtained from additively inconsistent APCMs-A, we may weaken the normalization condition (II.39) to

$$\sum_{i=1}^{n} v_i = 1 \tag{II.42}$$

without any further constraints on the priorities. By applying this normalization condition to the priorities obtained by formulas (II.36), we derive formulas for obtaining normalized priorities from an APCM-A as

$$v_i = \frac{2}{n} \sum_{j=1}^{n} r_{ij} - \frac{n-1}{n}, \qquad i = 1, \ldots, n. \tag{II.43}$$

Notice that the method for deriving normalized priorities from an APCM-A by formula (II.43) is again invariant under permutation of objects.

Proposition 5. *(Krejčí, 2016) Given an APCM-A $R = \{r_{ij}\}_{i,j=1}^{n}$, the priorities $v_1, \ldots, v_n$ obtained from R by formula* (II.43) *are such that*

$$\sum_{i=1}^{n} v_i = 1 \tag{II.44}$$

and

$$-1 < v_i \leq 1, \qquad i = 1, \ldots, n. \tag{II.45}$$

Proof.

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} \left(\frac{2}{n} \sum_{j=1}^{n} r_{ij} - \frac{n-1}{n} \right) = \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} - (n-1) = 1,$$

which proves (II.44).

The value of the priority $v_i, i \in \{1, \ldots, n\}$, obtained by formula (II.43) depends only on the PCs in the i-th row of the matrix, i.e. on the intensities of preference of object o_i over the other objects. To prove the inequality $v_i \leq 1$, we just need to show that the priority of object o_i will not exceed 1 even for the highest possible intensities of preference of object o_i over all other objects.

Let $o_i, i \in \{1, \ldots, n\}$, be absolutely preferred to all other objects. Then,

$$v_i = \frac{2}{n} \sum_{j=1}^{n} r_{ij} - \frac{n-1}{n} = \frac{1}{n} \left(2 \sum_{j=1}^{n} r_{ij} - n + 1 \right) = \frac{1}{n} \left(2 \left(n - 1 + 0.5 \right) - n + 1 \right) = 1.$$

Similarly, to prove the inequality $-1 < v_i$ we just need to show that the priority of object o_i will be greater than -1 even for the lowest possible intensities of preference of object o_i over all other objects. Let $o_i, i \in \{1, \ldots, n\}$, be absolutely preferred by all other objects. Then,

$$v_i = \frac{2}{n} \sum_{j=1}^{n} r_{ij} - \frac{n-1}{n} = \frac{1}{n} \left(2 \left(0 + 0.5 \right) - n + 1 \right) = \frac{2-n}{n} = -1 + \frac{2}{n} > -1$$

$\square$

Example 11. The priority vector obtainable from the APCM-A (II.41) in Example 10 by the formulas (II.43) is in the form $\underline{v} = (\frac{13}{15}, \frac{6}{15}, \frac{-4}{15})^T$. This priority vector clearly satisfies both normalization properties (II.44) and (II.45). $\triangle$

Liu et al. (2012b) showed that the normalization condition $\sum_{i=1}^{n} v_i = 1$, $v_i \in [0, 1]$, $i = 1, \ldots, n$, is reachable for the priorities (II.43) obtainable from an additively consistent APCM-A $R = \{r_{ij}\}_{i,j=1}^{n}$, if and only if

$$\min_{1 \leq i \leq n} \sum_{k=1}^{n} r_{ik} \geq \frac{n-1}{2}.$$

According to Proposition 5, some of the priorities normalized according to the normalization condition (II.42) can be negative, i.e. $v_i < 0, i \in \{1, \ldots, n\}$. To avoid these situations, Meng et al. (2016) proposed "normalization" of the priorities (II.43) as follows:

$$v_i = \max \left\{ 0, \frac{2}{n} \sum_{j=1}^{n} r_{ij} - \frac{n-1}{n} \right\}, \qquad i, = 1, \ldots, n.$$

However, such "normalization" of the priorities, similarly to the inappropriate normalization condition (II.39), distorts the preference information contained in APCMs-A, and thus it is not appropriate. Moreover, the possible negativity of some of the priorities normalized according to (II.42) is not a problem at all because the scale on which the priorities are given is an interval scale; the differences between the priorities are meaningful. For example the normalized priorities $v_1 = \frac{13}{15}, v_2 = \frac{6}{15}, v_3 = \frac{-4}{15}$ obtained in Example 11 from the APCM-A (II.41) by the formula (II.43) tell us that, e.g., $r_{23} - r_{32}$ is estimated as $v_2 - v_3 = \frac{2}{3}$ or that r_{23} is estimated as $0.5 + 0.5 \frac{2}{3} = \frac{5}{6}$.

Remark 2. *It is necessary to mention that also a more general characterization than Tanino's characterization* (II.32) *has appeared in the literature (see e.g. Xu et al. (2009); Liu et al. (2012b); Xu et al. (2014b)):*

$$r_{ij} = 0.5 + \beta(v_i - v_j), \qquad \beta \geq \max_{i=1,\ldots,n} \left\{ \frac{n}{2} - \sum_{j=1}^{n} r_{ij} \right\} > 0 \qquad (\text{II.46})$$

together with priorities

$$v_i = \frac{1}{n\beta} \sum_{j=1}^{n} r_{ij} - \frac{1}{2\beta} + \frac{1}{n} \qquad (\text{II.47})$$

satisfying this characterization and normalization condition $\sum_{i=1}^{n} v_i = 1, v_i \in [0,1]$. *More particularly, Xu et al.* *(2009) proposed to set* $\beta = \frac{n}{2}$ *and Xu et al. (2011) and Hu et al. (2014) assumed* $\beta = \frac{n-1}{2}$. *It is true that by assuming the characterization* (II.46) *the obtained normalized priorities* (II.47) *are always non-negative and constrained to interval* $[0,1]$. *However, the priorities miss an intuitive interpretation. In particular,* $r_{ij} - r_{ji} = 0.5 + \beta(v_i - v_j) - 0.5 - \beta(v_j - v_i) = 2\beta(v_i - v_j)$, *which would mean that the difference of priorities gives us* $\frac{1}{2\beta}$−*th of the difference between the related PCs in the APCM-A. This is quite difficult to interpret. Particularly, for* $\beta = \frac{n}{2}$ *we obtain* $v_i - v_j = \frac{1}{n}(r_{ij} - r_{ji})$, *and for* $\beta = \frac{n-1}{2}$ *we obtain* $v_i - v_j = \frac{1}{n-1}(r_{ij} - r_{ji})$. *Notice that for* $\beta = \frac{1}{2}$, *the characterization* (II.46) *equals to Tanino's characterization* (II.32) *and the corresponding priorities* (II.47) *equal to priorities* (II.43) *with a clear and intuitive interpretation* $v_i - v_j = r_{ij} - r_{ji}$.

2.3.3 Additive pairwise comparison matrices with multiplicative representation

Definition 10. *An APCM with multiplicative representation (APCM-M) is an APCM* $Q = \{q_{ij}\}_{i,j=1}^{n}, q_{ij} \in]0,1[$, *satisfying the additive-reciprocity property* (II.25) *where* $\frac{q_{ij}}{q_{ji}}$ *indicates the ratio of preference intensity of object* o_i *to that of object* o_j, *i.e* o_i *is* $\frac{q_{ij}}{q_{ji}}$−*times as good as* o_j.

The requirement of $q_{ij} \in]0,1[, i,j = 1,\ldots,n$, in the definition of an APCM-M means that none of the objects compared in the APCM-M can be absolutely preferred to another one. This requirement is necessary in order not to divide by 0 in the ratios $\frac{q_{ij}}{q_{ji}}$. Because of the additive-reciprocity property of APCMs, the constraint $q_{ij} \in]0,1[, i,j = 1,\ldots,n$, can be equivalently written as $q_{ij} > 0, i,j = 1,\ldots,n$.

2.3.3.1 Multiplicative consistency of APCMs-M

Definition 11. *(Tanino, 1984) An APCM-M* $Q = \{q_{ij}\}_{i,j=1}^{n}$ *is said to be multiplicatively consistent if it satisfies the multiplicative-transitivity property*

$$\frac{q_{ij}}{q_{ji}} = \frac{q_{ik}}{q_{ki}} \frac{q_{kj}}{q_{jk}}, \qquad i,j,k = 1,\ldots,n. \qquad (\text{II.48})$$

Definition 11 of multiplicative consistency is invariant under permutation of objects in the APCM-M Q.

Example 12. The APCM-M

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{3}{5} & \frac{3}{5} & \frac{3}{4} \\ \frac{2}{5} & \frac{1}{2} & \frac{1}{2} & \frac{2}{3} \\ \frac{2}{5} & \frac{1}{2} & \frac{1}{2} & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} \qquad (\text{II.49})$$

is multiplicatively consistent according to Definition 11 since it satisfies the multiplicative-transitivity property (II.48). Furthermore, also the permuted matrix $Q^\pi = PQP^T$ obtained by using the permutation matrix (II.6) is multiplicatively consistent. △

The following theorem provides us with alternative ways to verify multiplicative consistency of APCMs-M.

Theorem 4. *For an APCM-M $Q = \{q_{ij}\}_{i,j=1}^n, q_{ij} > 0, i, j = 1, \ldots, n$, the following statements are equivalent:*

(i) Q is multiplicatively consistent according to Tanino's multiplicative-transitivity property (II.48)*,*

(ii)

$$q_{ij}q_{jk}q_{ki} = q_{ik}q_{kj}q_{ji}, \qquad i, j, k = 1, \ldots, n, \tag{II.50}$$

(iii)

$$\frac{q_{ij}}{q_{ji}}\frac{q_{jk}}{q_{kj}}\frac{q_{ki}}{q_{ik}} = 1, \qquad i, j, k = 1, \ldots, n, \tag{II.51}$$

(iv)

$$\frac{q_{ij}}{q_{ji}}\frac{q_{jk}}{q_{kj}}\frac{q_{ki}}{q_{ik}} = \frac{q_{ik}}{q_{ki}}\frac{q_{kj}}{q_{jk}}\frac{q_{ji}}{q_{ij}}, \qquad i, j, k = 1, \ldots, n. \tag{II.52}$$

(v)

$$q_{ij} = \frac{q_{ik}q_{kj}}{q_{ik}q_{kj} + (1 - q_{ik})(1 - q_{kj})} \qquad i, j, k = 1, \ldots, n. \tag{II.53}$$

Further, Tanino (1984) derived the following characterization of multiplicatively consistent APCMs-M.

Proposition 6. *(Tanino's characterization) An APCM-M $Q = \{q_{ij}\}_{i,j=1}^n, q_{ij} > 0, i, j = 1, \ldots, n$, is multiplicatively consistent if and only if there exists a positive vector $\underline{u} = (u_1, \ldots, u_n)^T$ such that*

$$q_{ij} = \frac{u_i}{u_i + u_j}, \qquad i, j = 1, \ldots, n. \tag{II.54}$$

The notation $\underline{u} = (u_1, \ldots, u_n)^T$ will be used hereafter to represent exclusively a priority vector associated with an APCM-M.

Proposition 6 says that when an APCM-M $Q = \{q_{ij}\}_{i,j=1}^n$ of n objects is multiplicatively consistent then there exist non-negative priorities $u_1, \ldots, u_n$ of objects using which we can determine precisely the original PCs q_{ij} in the APCM-M Q by applying Tanino's characterization (II.54). On the other hand, when the non-negative priorities $u_1, \ldots, u_n$ of objects are known, we can construct a multiplicatively consistent APCM-M of objects by applying (II.54). Notice that the additive reciprocity of Q is always guaranteed by applying (II.54) since $q_{ij} = \frac{u_i}{u_i + u_j} = \frac{1}{1 + \frac{u_j}{u_i}} = 1 - \frac{\frac{u_j}{u_i}}{1 + \frac{u_j}{u_i}} = 1 - \frac{u_j}{u_j + u_i} = 1 - q_{ji}$.

Example 13. Since the APCM-M (II.49) in Example 12 is multiplitively consistent, there exists a positive priority vector $\underline{u} = (u_1, u_2, u_3, u_4)^T$ satisfying Tanino's characterization property (II.54). It is, for example, the priority vector $\underline{u} = (1, \frac{2}{3}, \frac{2}{3}, \frac{1}{3})^T$. $\triangle$

Moreover, Tanino's characterization (II.54) is in line with the interpretation of the ratios of PCs in an APCM-M Q as given Definition 10. In particular, when an APCM-M Q is multiplicatively consistent according to (II.48), then the ratio of PCs indicating the ratio of preference intensity of o_i to that of o_j corresponds to the ratio of their priorities, i.e.

$$\frac{q_{ij}}{q_{ji}} = \frac{u_i}{u_j}, \qquad i, j = 1, \ldots, n. \tag{II.55}$$

When the APCM-M Q is not multiplicatively consistent, then

$$\frac{q_{ij}}{q_{ji}} \approx \frac{u_i}{u_j}, \qquad i, j = 1, \ldots, n. \tag{II.56}$$

Example 14. From the intensity of preference $q_{12} = \frac{3}{5}$ in the APCM-M (II.49) we know immediately that the ratio of the priorities u_1 and u_2 is $\frac{3}{2}$ $(\frac{u_1}{u_2} = \frac{q_{12}}{q_{21}} = \frac{3}{5}\frac{5}{2} = \frac{3}{2})$. Similarly also for the remaining differences between the priorities. The priority vector $\underline{u} = (1, \frac{2}{3}, \frac{2}{3}, \frac{1}{3})^T$ shown in Example 13 satisfies this property. $\triangle$

Unlike in the case of the additive-consistency condition (II.28) for APCMs-A, the interval $]0, 1[$ can never be exceeded when trying to keep the multiplicative consistency (II.48) for APCMs-M. That is why APCMs-M become of more and more interest to researchers. However, despite this advantage, the interpretation of the multiplicative consistency is not very intuitive for DMs. Further, DMs have difficulties using the open interval $]0, 1[$ for expressing the intensities of preferences since this scale has no minimum and maximum.

Various other consistency conditions with more intuitive interpretation have been introduced. One of them is the weak-consistency condition proposed by Jandová et al. (2017) which was already mentioned in relation to APCMs-A. Since the weak-consistency condition is defined for APCMs without distinguishing between APCMs-A and APCMs-M, it is reviewed separately in the following section.

2.3.3.2 Weak consistency of APCMs

Analogously to the weak consistency for MPCMs defined by Jandová and Talašová (2013), also the weak-consistency condition introduced in this section for APCMs is based on the properties that are intuitively supposed to hold.

Definition 12. *(Jandová et al., 2017) An APCM $A = \{a_{ij}\}_{i,j=1}^n$ is said to be weakly consistent if*

$$
\begin{aligned}
a_{ik} > 0.5 \;\wedge\; a_{kj} > 0.5 &\quad\Rightarrow\quad a_{ij} \geq \max\{a_{ik}, a_{kj}\}, \\
a_{ik} = 0.5 \;\wedge\; a_{kj} \geq 0.5 &\quad\Rightarrow\quad a_{ij} = \max\{a_{ik}, a_{kj}\}, \\
a_{ik} \geq 0.5 \;\wedge\; a_{kj} = 0.5 &\quad\Rightarrow\quad a_{ij} = \max\{a_{ik}, a_{kj}\},
\end{aligned}
\tag{II.57}
$$

holds for $i, j, k = 1, \ldots, n$.

It is easy to verify that Definition 12 of weak consistency is invariant under permutation of objects in the APCM A.

Definition 12 does not distinguish between APCMs-A and APCMs-M; for both types of APCMs, the weak-consistency condition is defined in the same form.

Similarly as for the weak-consistency condition (II.11) for MPCMs, weak-consistency condition (II.57) for APCMs provides an intuitive minimum consistency requirement for APCMs. For example, when object o_i is preferred to object o_k with intensity 0.8, and object o_k is preferred to object o_j with intensity 0.6, then object o_i has to be preferred to object o_j with intensity at least 0.8 ($a_{ij} \geq \max\{0.8, 0.6\} = 0.8$). Thus, the requirement of weak consistency is very intuitive, it provides DMs with some space for expressing their intensities of preference, and it is very easy to control while entering PCs into the APCM.

In this thesis, the weak-consistency condition given by Definition 12 is adopted and later applied in Chapter V in a novel method for dealing with large-dimensional PCMs. However, it is worth to note that the weak consistency given by Definition 12 is not the only weak version of consistency for APCMs. For example, even a more relaxed form of the weak-consistency condition (II.11) for APCMs, strong stochastic transitivity, was introduced already half a century ago by Luce and Suppes (1965).

In the following theorem, rules equivalent to the weakly-consistency condition (II.57) are formulated.

Theorem 5. *(Jandová et al., 2017) For an APCM $A = \{a_{ij}\}_{i,j=1}^n$, the following statements are equivalent:*

(i) A *is weakly consistent according to Definition 12,*

(ii) *For every $i, j, k = 1, \ldots, n$:*

$$
\begin{aligned}
a_{ik} < 0.5 \;\wedge\; a_{kj} < 0.5 &\quad\Rightarrow\quad a_{ij} \leq \min\{a_{ik}, a_{kj}\}, \\
a_{ik} = 0.5 \;\wedge\; a_{kj} \leq 0.5 &\quad\Rightarrow\quad a_{ij} = \min\{a_{ik}, a_{kj}\}, \\
a_{ik} \leq 0.5 \;\wedge\; a_{kj} = 0.5 &\quad\Rightarrow\quad a_{ij} = \min\{a_{ik}, a_{kj}\}.
\end{aligned}
\tag{II.58}
$$

(iii) *For every $i, j, k = 1, \ldots, n$:*

$$
\begin{aligned}
0.5 < 1 - a_{kj} < a_{ik} &\quad\Rightarrow\quad 1 - a_{ik} \leq a_{ij} \leq a_{ik}, \\
0.5 < a_{ik} < 1 - a_{kj} &\quad\Rightarrow\quad a_{kj} \leq a_{ij} < 0.5, \\
0.5 < 1 - a_{kj} < a_{ik} &\quad\Rightarrow\quad 0.5 < a_{ij} \leq a_{ik}.
\end{aligned}
\tag{II.59}
$$

(iv) *For every $i, j, k = 1, \ldots, n$:*

$$
\begin{aligned}
1 - a_{kj} < a_{ik} < 1 &\quad\Rightarrow\quad a_{kj} \leq a_{ij}, \\
0.5 < a_{kj} < 1 - a_{ik} &\quad\Rightarrow\quad a_{ik} \leq a_{ij} < 0.5, \\
0.5 < a_{kj} = 1 - a_{ik} &\quad\Rightarrow\quad 1 - a_{kj} \leq a_{ij} \leq a_{kj}.
\end{aligned}
\tag{II.60}
$$

Similarly to Definition 6 of weak consistency for MPCMs, it is possible to derive some interesting properties for APCMs weakly consistent according to Definition 12. Every weakly consistent APCM $A = \{a_{ij}\}_{i,j=1}^n$ can be permuted in such a way that the objects compared in the APCM are ordered from the most preferred to the least preferred. In such an ordered weakly consistent APCM, all the elements above the main diagonal are greater or equal to 0.5, i.e. $a_{ij} \geq 0.5$, $i, j = 1, \ldots, n$, $i < j$. Further, the sequences of elements in the rows are non-decreasing and the sequences of elements in the columns are non-increasing.

The following propositions show that the weak-consistency condition is weaker than both the additive-consistency condition for APCMs-A and the multiplicative-consistency condition for APCMs-M.

Figure II.1: Relations between MPCMs, APCMs-A, APCMs-M, and the associated definitions of consistency.
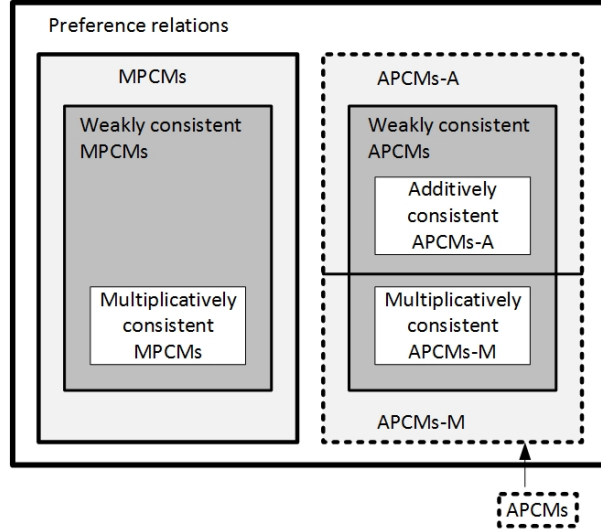
Proposition 7. *An APCM-A $R = \{r_{ij}\}_{i,j=1}^{n}$ additively consistent according to Definition 9 is also weakly consistent according to Definition 12.*

Proof. For $r_{ik} > 0.5 \wedge r_{kj} > 0.5$, we get immediately $r_{ij} = r_{ik} + r_{kj} - 0.5 > \max\{r_{ik}, r_{kj}\}$. For $r_{ik} = 0.5 \wedge r_{kj} \geq 0.5$ we get $r_{ij} = r_{ik} + r_{kj} - 0.5 = r_{kj} \geq \max\{r_{ik}, r_{kj}\}$, and analogously for $r_{ijk} \geq 0.5 \wedge r_{kj} = 0.5$. □

Proposition 8. *An APCM-M $Q = \{q_{ij}\}_{i,j=1}^{n}$ multiplicatively consistent according to Definition 11 is also weakly consistent according to Definition 12.*

Proof. It is convenient to consider the multiplicative-consistency condition in the form (II.53) for the proof. The proof for $q_{ik} = 0.5 \wedge q_{kj} \geq 0.5$ (respectively for $q_{ik} \geq 0. \wedge q_{kj} = 0.5$) is trivial:

$$q_{ij} = \frac{q_{ik}q_{kj}}{q_{ik}q_{kj} + (1 - q_{ik})(1 - q_{kj})} = \frac{0.5 \cdot q_{kj}}{0.5 \cdot q_{kj} + 0.5(1 - q_{kj})} = q_{kj} = \max\{q_{ik}, q_{kj}\}.$$

The validity for $q_{ik} > 0.5 \wedge q_{kj} > 0.5$ is demonstrated by contradiction. Without the loss of generality, let us assume $q_{ik} \geq q_{kj} > 0.5$, and suppose the proposition is false, i.e. $q_{ij} < \max\{q_{ik}, q_{kj}\}$. It follows that $q_{ij} < q_{ik}$. Thus, we have

$$\frac{q_{ik}q_{kj}}{q_{ik}q_{kj} + (1 - q_{ik})(1 - q_{kj})} < q_{ik}$$
$$\Downarrow$$
$$\frac{q_{kj}}{q_{ik}q_{kj} + (1 - q_{ik})(1 - q_{kj})} < 1$$
$$\Downarrow$$
$$q_{kj} < 2q_{ik}q_{kj} + 1 - q_{ik} - q_{kj}$$
$$\Downarrow$$
$$q_{ik} - 1 < q_{kj}(2q_{ik} - 2)$$
$$\Downarrow$$
$$q_{kj} < 0.5$$

which is in contradiction with the assumption $q_{kj} > 0.5$. □

The properties of weakly consistent APCMs described in this section are utilized in the new method for dealing with large-dimensional PCMs that is described in detail in Chapter V.

The relations between MPCMs, APCMs-A, APCMs-M, and the associated definitions of consistency reviewed in Sections 2.2 and 2.3 are represented by a diagram in Fig. II.1.

2.3.3.3 Deriving priorities from APCMs-M

As discussed in the previous section, when an APCM-M $Q = \{q_{ij}\}_{i,j=1}^{n}$ is multiplicatively consistent according to (II.48), then there exist positive priorities $u_1, \ldots, u_n$ such that $q_{ij} = \frac{u_i}{u_i + u_j}$. When the APCM-M $Q = \{q_{ij}\}_{i,j=1}^{n}$

is not multiplicatively consistent, then the given expression only estimates the PCs in the matrix, i.e.

$$q_{ij} \approx \frac{u_i}{u_i + u_j}, \qquad i,j = 1,\ldots,n. \tag{II.61}$$

Fedrizzi and Brunelli (2010) proved that, for an APCM-M $Q = \{q_{ij}\}_{i,j=1}^{n}$, the only vector of priorities (up to a multiplicative constant) satisfying (II.54) is $\underline{u} = (u_1, \ldots, u_n)^T$ such that

$$u_i = \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{q_{ji}}}, \qquad i = 1,\ldots,n. \tag{II.62}$$

This method is again invariant under permutation of objects in APCMs-M. Further, as pointed out by Xia and Xu (2011), the priorities (II.62) are such that $\prod_{i=1}^{n} u_i = 1$.

Because the priorities (II.62) can be multiplied by any positive constant, it is possible to apply the normalization condition (II.18), similarly as in the case of the priorities obtained from a MPCM. Thus, the normalized priorities can be computed from an APCM-M $Q = \{q_{ij}\}_{i,j=1}^{n}$, directly as

$$u_i = \frac{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{q_{ji}}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} \frac{q_{kj}}{q_{jk}}}}, \qquad i = 1,\ldots,n. \tag{II.63}$$

Remark 3. *Even though the representation of PCs in APCMs-M is not very intuitive for DMs, they have a great advantage over APCMs-A regarding the normalization of the priorities. In particular, the priorities obtained from these matrices can be normalized by using the widely accepted normalization condition* (II.18) *with the normalized priorities still lying in the interval* $]0,1[$. *As discussed in the previous section, this normalization condition is unreachable for priorities obtainable from APCMs-A.*

2.4 Transformations between MPCMs and APCMs

In this section, transformations between MPCMs, APCMs-A, and APCMs-M, and between the priorities obtainable from these PCMs are reviewed and discussed.

2.4.1 Transformations between MPCMs and APCMs-A

It is a well-known fact that MPCMs and APCMs-A are equivalent. Fedrizzi (1990) showed that a MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$ can be transformed into an APCM-A $R = \{r_{ij}\}_{i,j=1}^{n}$ by applying the transformation formula

$$r_{ij} = \frac{1}{2} \left(1 + \log_9 m_{ij}\right), \qquad i,j = 1,\ldots,n. \tag{II.64}$$

The values in interval $[\frac{1}{9}, 9]$ (Saaty's scale) are transformed into values in interval $[0,1]$, the multiplicative reciprocity (II.2) is transformed into the additive reciprocity (II.25), and the multiplicative consistency (II.4) is transformed into the additive consistency (II.28) by the formula (II.64) (Fedrizzi and Brunelli, 2010).

Analogously, the inverse transformation formula

$$m_{ij} = 9^{2r_{ij}-1}, \qquad i,j = 1,\ldots,n, \tag{II.65}$$

can be used to transform an APCM-A $R = \{r_{ij}\}_{i,j=1}^{n}$ into a MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$ with all its relevant properties.

Further, the following theorem is valid for the transformation of the weak-consistency condition.

Theorem 6. *Let $M = \{m_{ij}\}_{i,j=1}^{n}$ be a MPCM weakly consistent according to Definition 6. Then the APCM-A $R = \{r_{ij}\}_{i,j=1}^{n}$ obtained from M by transformations* (II.64) *is weakly consistent according to Definition 12.*

Proof. It is sufficient to show that when the weak-consistency condition (II.11) is valid for a MPCM M, then the weak-consistency condition (II.57) is valid for the APCM-A obtained from M by the transformation (II.64). By substituting (II.65) into the first part of (II.11), we obtain

$$9^{2r_{ik}-1} > 1 \;\wedge\; 9^{2r_{kj}-1} > 1 \;\Rightarrow\; 9^{2r_{ij}-1} \geq \max\left\{9^{2r_{ij}-1}, 9^{2r_{jk}-1}\right\}$$
$$\Downarrow$$
$$2r_{ik} - 1 > 0 \;\wedge\; 2r_{kj} - 1 > 0 \;\Rightarrow\; 2r_{ij} - 1 \geq \max\left\{2r_{ik} - 1, 2r_{kj} - 1\right\}$$
$$\Downarrow$$
$$r_{ik} > 0.5 \;\wedge\; r_{kj} > 0.5 \;\Rightarrow\; r_{ij} \geq \left\{r_{ik}, r_{kj}\right\},$$

which is the first part of (II.57). Analogously, for the other two parts of (II.11) we obtain the other two parts of (II.57). □

Corollary 1. *Let $R = \{r_{ij}\}_{i,j=1}^{n}$ be an APCM-A weakly consistent according to Definition 12. Then the MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$ obtained from R by transformations (II.65) is weakly consistent according to Definition 6.*

Remark 4. *The validity of Corollary 1 follows immediately from Theorem 6 by utilizing properties of an inverse function. Note that this form of representing the results is used in the whole section. This means that the transformation of a particular property is formulated in a theorem and proved only in one direction. Afterwards, each such theorem is followed by a corollary showing the transformation of the property in the opposite direction without providing the proof.*

Fedrizzi and Brunelli (2010) showed that priorities $w_1, \ldots, w_n$ of objects obtained from a MPCM M by formulas (II.23) can be transformed into the priorities $v_1, \ldots, v_n$ obtainable from the corresponding APCM-A R by (II.36) by using the transformation formula

$$v_i = 1 + \log_9 w_i, \qquad i = 1, \ldots, n. \tag{II.66}$$

The inverse transformation formula is

$$w_i = 9^{v_i - 1}, \qquad i = 1, \ldots, n. \tag{II.67}$$

Furthermore, Fedrizzi and Brunelli (2010) demonstrated that $\underline{w} = (w_1, \ldots, w_n)^T$ being a priority vector representing priorities of objects compared in a MPCM, the vector $\underline{v} = (v_1, \ldots, w_n)^T$ obtained as

$$v_i = c + \log_9 w_i, \ c \in \mathbb{R}, \qquad i = 1, \ldots, n, \tag{II.68}$$

is a priority vector representing the priorities of objects compared in the corresponding APCM-A. The inverse transformation is

$$w_i = 9^{v_i - c}, \ c \in \mathbb{R}, \qquad i = 1, \ldots, n. \tag{II.69}$$

Note that it is also possible to derive a relation between the normalized priorities (II.24) obtainable from a MPCM and the normalized priorities (II.43) obtainable from the corresponding APCM-A. In particular, the normalized priorities (II.24) can be expressed as

$$w_i = \frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} m_{kj}}} = \frac{\sqrt[n]{\prod_{j=1}^{n} 9^{2r_{ij}-1}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} 9^{2r_{kj}-1}}} = \frac{9^{\frac{1}{n}\sum_{j=1}^{n}(2r_{ij}-1)}}{\sum_{k=1}^{n} 9^{\frac{1}{n}\sum_{j=1}^{n}(2r_{kj}-1)}} =$$
$$\frac{9^{\frac{2}{n}\sum_{j=1}^{n} r_{ij} - \frac{n-1}{n} - \frac{1}{n}}}{\sum_{k=1}^{n} 9^{\frac{2}{n}\sum_{j=1}^{n} r_{kj} - \frac{n-1}{n} - \frac{1}{n}}} \overset{(\text{II.43})}{=} \frac{9^{v_i - \frac{1}{n}}}{\sum_{k=1}^{n} 9^{v_k - \frac{1}{n}}}. \tag{II.70}$$

Therefore, in order to obtain the normalized priority w_i of object o_i from the MPCM, the normalized priorities $v_j, j = 1, \ldots, n$, of all objects are necessary. This is due to the fact that the normalized priority w_i obtained by the formula (II.24) depends on all PCs in the MPCM, while the normalized priority v_i obtained by the formula (II.43) depends only on the PCs in the $i-$th row of the corresponding APCM-A.

In fact, as proved in Proposition 3, the sum of the non-normalized priorities (II.36) obtainable from an APCM-A always equals n, i.e. the sum of the priorities is independent of the PCs in the APCM-A. Thus, in order to normalize the priorities (II.36), the constant $-\frac{n-1}{n}$ is always added, i.e.

$$v_i \ \rightarrow \ v_i - \frac{n-1}{n}, \qquad i = 1, \ldots, n.$$

Contrarily, the sum of the non-normalized priorities (II.23) obtainable from a MPCM is not constant; it depends on the values of the PCs in the MPCM. Therefore, in order to normalize the priorities obtainable from a MPCM, we have to divide them by their sum, i.e.

$$w_i \ \rightarrow \ \frac{w_i}{\sum_{k=1}^{n} w_k}, \qquad i = 1, \ldots, n.$$

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Thus, it is not possible to derive a general formula that would transform a normalized priority w_i obtained by the formula (II.24) into the corresponding normalized priority v_i obtained by the formula (II.43) and vice versa. In other words, w_i is not a function of the single value v_i, but, on the contrary, it depends on the values $v_1, \ldots, v_n$, as highlighted in the expression (II.70). Nevertheless, having a particular MPCM and the corresponding APCM-A, there always exists a constant c such that the normalized priorities (II.24) can be transformed into the normalized priorities (II.43) by the transformation formula (II.68), or, in the opposite direction, by the transformation formula (II.69).

Example 15. Let us consider the APCM-A

$$R = \begin{pmatrix} 0.5 & 0.6 & 0.7 & 0.9 \\ 0.4 & 0.5 & 0.6 & 0.8 \\ 0.3 & 0.4 & 0.5 & 0.7 \\ 0.1 & 0.2 & 0.3 & 0.5 \end{pmatrix}. \tag{II.71}$$

The non-normalized priority vector obtained from the APCM-A R by the formula (II.36) is

$$\underline{v} = (1.35, 1.15, 0.95, 0.55)^T. \tag{II.72}$$

The MPCM M obtained from R by the transformation formula (II.65) is

$$M = \begin{pmatrix} 1 & 1.5518 & 2.4082 & 5.7995 \\ \frac{1}{1.5518} & 1 & 1.5518 & 3.7372 \\ \frac{1}{2.4082} & \frac{1}{1.5518} & 1 & 2.4082 \\ \frac{1}{5.7995} & \frac{1}{3.7372} & \frac{1}{2.4082} & 1 \end{pmatrix}. \tag{II.73}$$

The non-normalized priority vector obtained from the MPCM M by the formula (II.23) is

$$\underline{w} = (2.1577, 1.3904, 0.8960, 0.3720)^T. \tag{II.74}$$

The same priority vector would be obtained also by applying the transformation (II.67) to the priority vector (II.72).

Further, the normalized priority vector obtained from the APCM-A R by the formula (II.43) is $\underline{v} = (0.6, 0.4, 0.2, -0.2)^T$, and the normalized priority vector obtained from the MPCM M by the formula (II.24) is $\underline{w} = (0.4480, 0.2887, 0.1860, 0.0772)^T$. These normalized priority vectors can be transformed one into the other by using the transformations (II.68) and (II.69) with the constant $c = 0.9654$.

Notice that the APCM-A (II.71) is weakly consistent according to Definition 12. Thus, according to Corollary 1, the MPCM (II.73) is weakly consistent according to Definition 6. $\triangle$

2.4.2 Transformations between MPCMs and APCMs-M

Chiclana et al. (1998) showed that a MPCM M can be transformed into an APCM-M Q by applying the transformation formula

$$q_{ij} = \frac{m_{ij}}{1 + m_{ij}}, \qquad i, j = 1, \ldots, n. \tag{II.75}$$

The values in interval $\left[\frac{1}{9}, 9\right]$ (Saaty's scale) are transformed into values in interval $\left[\frac{1}{10}, \frac{9}{10}\right] \subset]0, 1[$, the multiplicative reciprocity (II.2) transforms to the additive reciprocity (II.25), and the multiplicative consistency (II.4) transforms to the multiplicative consistency (II.48).

Analogously, the inverse transformation formula

$$m_{ij} = \frac{q_{ij}}{q_{ji}}, \qquad i, j = 1, \ldots, n, \tag{II.76}$$

transforms an APCM-M $Q = \{q_{ij}\}_{i,j=1}^n, q_{ij} \in \left[\frac{1}{10}, \frac{9}{10}\right]$, to a MPCM $M = \{m_{ij}\}_{i,j=1}^n, m_{ij} \in \left[\frac{1}{9}, 9\right]$, with all relevant properties. Note that there is no transformation formula that would transform the interval $\left[\frac{1}{9}, 9\right]$ to the open interval $]0, 1[$ and vice versa.

Further, the following theorem is valid for the transformation of the weak-consistency condition.

Theorem 7. *Let $M = \{m_{ij}\}_{i,j=1}^n$ be a MPCM weakly consistent according to Definition 6. Then the APCM-M $Q = \{q_{ij}\}_{i,j=1}^n$ obtained from M by transformations* (II.75) *is weakly consistent according to Definition 12.*

Proof. It is sufficient to show that when the weak-consistency condition (II.11) is valid for a MPCM M, then the weak-consistency condition (II.57) is valid for the APCM-M obtained from M by the transformation (II.75). By substituting (II.76) into the first part of (II.11), we obtain

$$\frac{q_{ik}}{q_{ki}} > 1 \ \wedge \ \frac{q_{kj}}{q_{jk}} > 1 \Rightarrow \frac{q_{ij}}{q_{ji}} \geq \max\left\{\frac{q_{ik}}{q_{ki}}, \frac{q_{kj}}{q_{jk}}\right\}.$$

Further,

$$\frac{q_{ik}}{q_{ki}} > 1 \ \Leftrightarrow \ \frac{q_{ik}}{1 - q_{ik}} > 1 \ \Leftrightarrow \ q_{ik} > 1 - q_{ik} \ \Leftrightarrow \ q_{ik} > 0.5,$$

and similarly we obtain $q_{kj} > 0.5$. Then,

$$\frac{q_{ij}}{q_{ji}} \geq \max\left\{\frac{q_{ik}}{q_{ki}}, \frac{q_{kj}}{q_{jk}}\right\} \ \Leftrightarrow \ \frac{1}{q_{ji}} - 1 \geq \max\left\{\frac{1}{q_{ki}} - 1, \frac{1}{q_{jk}} - 1\right\} \ \Leftrightarrow$$

$$\frac{1}{q_{ji}} \geq \max\left\{\frac{1}{q_{ki}}, \frac{1}{q_{jk}}\right\} \ \Leftrightarrow \ q_{ji} \leq \min\{q_{ki}, q_{jk}\} \ \Leftrightarrow$$

$$1 - q_{ij} \leq \min\{1 - q_{ik}, 1 - q_{kj}\} \ \Leftrightarrow \ q_{ij} \geq \max\{q_{ik}, q_{kj}\}.$$

Thus, we obtain $q_{ik} > 0.5 \ \wedge \ q_{kj} > 0.5 \Rightarrow q_{ij} \geq \max\{q_{ik}, q_{kj}\}$, which is the first part of (II.57). Analogously, from the remaining two parts of (II.11) we obtain the remaining two parts of (II.57). $\square$

Corollary 2. *Let $Q = \{q_{ij}\}_{i,j=1}^n$ be an APCM-M weakly consistent according to Definition 12. Then the MPCM $M = \{m_{ij}\}_{i,j=1}^n$ obtained from Q by transformations* (II.76) *is weakly consistent according to Definition 6.*

Fedrizzi and Brunelli (2010) showed that the priorities $w_1, \ldots, w_n$ of objects obtained from a MPCM M by formulas (II.23) and the priorities $u_1, \ldots, u_n$ of objects obtainable from the corresponding APCM-M Q by formulas (II.62) are identical, i.e.

$$w_i = u_i, \qquad i = 1, \ldots, n. \tag{II.77}$$

Furthermore, Fedrizzi and Brunelli (2010) demonstrated that $\underline{u} = (u_1, \ldots, u_n)^T$ being a priority vector representing the priorities of objects compared in an APCM-M, the vector $\underline{w} = (w_1, \ldots, w_n)^T$ obtained as

$$w_i = c \cdot u_i, \ c > 0, \qquad i = 1, \ldots, n, \tag{II.78}$$

is a priority vector representing the priorities of objects compared in the corresponding MPCM. The inverse transformation is then

$$u_i = \frac{1}{c} \cdot w_i, \ c > 0, \qquad i = 1, \ldots, n. \tag{II.79}$$

Note that in this case it is also possible to derive a direct relation between the normalized priorities (II.24) obtainable from a MPCM and the normalized priorities (II.63) obtainable from the corresponding APCM-M. In particular,

$$w_i = \frac{\sqrt[n]{\prod_{j=1}^n m_{ij}}}{\sum_{k=1}^n \sqrt[n]{\prod_{j=1}^n m_{kj}}} \overset{\text{(II.76)}}{=} \frac{\sqrt[n]{\prod_{j=1}^n \frac{q_{ij}}{q_{ji}}}}{\sum_{k=1}^n \sqrt[n]{\prod_{j=1}^n \frac{q_{kj}}{q_{jk}}}} = u_i.$$

This means that the normalized priorities (II.24) and (II.63) obtained from the MPCM and from the corresponding APCM-M, respectively, are identical. This simple relation between the normalized priority vectors $\underline{w}$ and $\underline{u}$ was possible to obtain only because the priority vectors are normalized in the same way;

$$w_i \ \to \ \frac{w_i}{\sum_{k=1}^n w_k}, \qquad u_i \ \to \ \frac{u_i}{\sum_{k=1}^n u_k}, \qquad i = 1, \ldots, n.$$

Example 16. Let us consider the MPCM M given by (II.73). The APCM-M obtained from M by the transformation formula (II.75) is

$$Q = \begin{pmatrix} 0.5 & 0.6081 & 0.7066 & 0.8529 \\ 0.3919 & 0.5 & 0.6081 & 0.7889 \\ 0.2934 & 0.3919 & 0.5 & 0.7066 \\ 0.1471 & 0.2111 & 0.2934 & 0.5 \end{pmatrix}. \tag{II.80}$$

The non-normalized priority vector obtained from the APCM-M Q by the formula (II.62) is

$$\underline{u} = (2.1577, 1.3904, 0.8960, 0.3720)^T, \tag{II.81}$$

i.e. it is identical to the priority vector (II.74). This result is in line with the transformation formula (II.77).

Notice that the APCM-M (II.80) is weakly consistent according to Definition 12. This conclusion follows also from Corollary 2. $\triangle$

2.4.3 Transformations between APCMs-A and APCMs-M

Since there exist transformations between MPCMs and APCMs-A and between MPCMs and APCMs-M, it is clear that there exist also transformations between APCMs-A and APCMs-M. These transformations can be derived directly by composing the corresponding formulas from the previous two sections as specified in the following theorems.

Theorem 8. *An APCM-A $R = \{r_{ij}\}_{i,j=1}^{n}$ can be transformed into an APCM-M $Q = \{q_{ij}\}_{i,j=1}^{n}$ with $q_{ij} \in [\frac{1}{10}, \frac{9}{10}], i,j = 1, \ldots, n$, by transformation formula*

$$q_{ij} = \frac{9^{2r_{ij}-1}}{1 + 9^{2r_{ij}-1}}, \qquad i,j = 1, \ldots, n. \tag{II.82}$$

Proof. Because the transformation formula (II.65) transforms an APCM-A into a MPCM, and (II.75) transforms a MPCM into an APCM-M, then the composition of these formulas transforms an APCM-A into an APCM-M. By composing (II.65) and (II.75) we immediately obtain (II.82). □

Corollary 3. *An APCM-M $Q = \{q_{ij}\}_{i,j=1}^{n}$ with $q_{ij} \in [\frac{1}{10}, \frac{9}{10}], i,j = 1, \ldots, n$, can be transformed into an APCM-A $R = \{r_{ij}\}_{i,j=1}^{n}$ by transformation formula*

$$r_{ij} = \frac{1}{2}\left(1 + \log_9 \frac{q_{ij}}{q_{ji}}\right), \qquad i,j = 1, \ldots, n. \tag{II.83}$$

Theorem 9. *Let $R = \{r_{ij}\}_{i,j=1}^{n}$ be an APCM-A additively consistent according to Definition 9. Then the APCM-M $Q = \{q_{ij}\}_{i,j=1}^{n}$ obtained from R by transformations (II.82) is multiplicatively consistent according to Definition 11.*

Proof. Because the transformation formula (II.65) transforms additive consistency (II.28) of an APCM-A into multiplicative consistency (II.4) of the corresponding MPCM, and (II.75) transforms multiplicative consistency (II.4) of a MPCM into multiplicative consistency (II.48) of the corresponding APCM-M, then the composition (II.82) of these formulas transforms additive consistency of an APCM-A into multiplicative consistency of the corresponding APCM-M. □

Corollary 4. *Let $Q = \{q_{ij}\}_{i,j=1}^{n}$ be an APCM-M multiplicatively consistent according to Definition 11. Then the APCM-A $R = \{r_{ij}\}_{i,j=1}^{n}$ obtained from Q by transformations (II.83) is additively consistent according to Definition 9.*

Further, the following theorem is valid for the transformation of the weak-consistency condition.

Theorem 10. *Let $R = \{r_{ij}\}_{i,j=1}^{n}$ be an APCM-A weakly consistent according to Definition 12. Then also the APCM-M $Q = \{q_{ij}\}_{i,j=1}^{n}$ obtained from R by transformations (II.82) is weakly consistent according to Definition 12.*

Proof. Because the transformation formula (II.65) transforms the weak consistency (II.57) of an APCM-A into the weak consistency (II.11) of the corresponding MPCM, and (II.75) transforms weak consistency (II.11) of a MPCM into weak consistency (II.57) of the corresponding APCM-M, then the composition (II.82) of these formulas transforms weak consistency of an APCM-A into weak consistency of the corresponding APCM-M. □

Corollary 5. *Let $Q = \{q_{ij}\}_{i,j=1}^{n}$ be an APCM-M weakly consistent according to Definition 12. Then also the APCM-A $R = \{r_{ij}\}_{i,j=1}^{n}$ obtained from Q by transformations (II.83) is weakly consistent according to Definition 12.*

Theorem 11. *Priorities $v_1, \ldots, v_n$ of objects obtained from an APCM-A by formula (II.36) can be transformed into priorities $u_1, \ldots, u_n$ obtainable by formulas (II.62) from the corresponding APCM-M by using the transformation formula*

$$u_i = 9^{v_i-1}, \qquad i = 1, \ldots, n. \tag{II.84}$$

Proof. The proof is analogous to the proof of Theorem 8. □

Corollary 6. *Priorities $u_1, \ldots, u_n$ of objects obtained from an APCM-M by formula (II.62) can be transformed into priorities $v_1, \ldots, v_n$ obtainable by formulas (II.36) from the corresponding APCM-A by using the transformation formula*

$$v_i = 1 + \log_9 u_i, \qquad i = 1, \ldots, n. \tag{II.85}$$

Figure II.2: Transformations between MPCMs, APCMs-A, and APCMs-M.
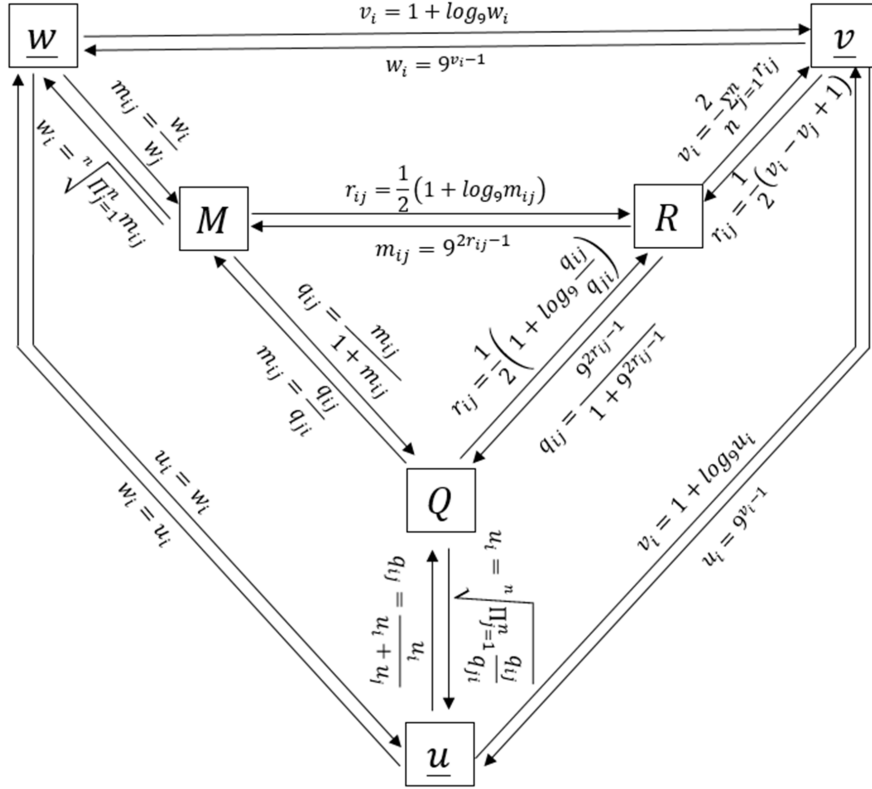
The issue with transformations between the normalized priority vectors (II.43) and (II.63) is the same as described in Section 2.4.1. That is, there is no explicit formula for transforming the normalized priority vector (II.43) obtainable from an APCM-A into the normalized priority vector (II.63) obtainable from the corresponding APCM-M and vice versa. This is again caused by the fact that the normalized priorities (II.43) depend only on PCs in the given row of the APCM-A while the normalized priorities (II.63) depend on all PCs in the APCM-M.

Example 17. Let us consider the MPCM Q given by (II.80). The APCM-A obtainable from Q by the transformation formula (II.83) is again the APCM-A R given by (II.71) with the associated non-normalized priority vector $\underline{v}$ given as (II.72). The priority vector (II.72) could be again obtained directly from the priority vector (II.81) by the transformation formula (II.85). $\triangle$

For better illustration, the transformations between MPCMs, APCMs-A, and APCMs-M and between the associated priority vectors are represented by a diagram in Fig. II.2. This diagram is a complement of the diagrams provided by Fedrizzi and Brunelli (2010).

Chapter III

Fuzzy set theory

3.1 Introduction to fuzzy sets

The traditional "crisp" set theory is based on the concept of "crisp set". A crisp set is defined in such a way that there exists a precise unambiguous distinction between the elements that belong to the crisp set and those that do not belong to the crisp set. A crisp set S on a given universe U can be defined in three basic ways (Klir and Yuan, 1995):

- Set S is defined by listing all its members, i.e. $S = \{m_1, \ldots, m_k\}$ denotes the set S whose members are $m_1, \ldots, m_k$. Only finite sets can be defined in this way.

- Set S is defined by formulating a property that is met by all its members, i.e. $S = \{x; f(x)\}$ denotes the set S on the universe U whose members $x \in U$ satisfy the property f.

- Set S is defined by providing a characteristic function $\chi_S : U \to \{0, 1\}$ declaring which elements of the universe U are members of the set and which are not as

$$\chi_S(x) = \left\{ \begin{array}{ll} 1 & \text{for } x \in S; \\ 0 & \text{for } x \notin S. \end{array} \right.$$

Example 18. The set of all even numbers on a dice can be defined as $S_1 = \{2, 4, 6\}$. The set of men who are at least 190 cm tall can be defined as $S_2 = \{x; x \geq 190\}$, where $x \in U_2$ is actually the height of a man in centimeters. The sets S_1 and S_2 are defined unambiguously; there exists a clear distinction between the elements that belong to the sets and those that do not belong there. For example, the number 1 from the universe $U_1 = \{1, 2, \ldots, 6\}$ of all numbers on a dice clearly does not belong to the set $S_1 = \{2, 4, 6\}$, i.e. $1 \notin S_1$, while the number 4 clearly belongs there, i.e. $4 \in S_1$. Similarly, a man who is 160 cm tall obviously does not belong the set of men tall at least 190 cm, i.e. $160 \notin S_2$, while a man tall 195 cm obviously belong there, i.e. $195 \in S_2$.

Now assume we want to define a set of tall men, i.e. without explicitly defining their height in cm. First of all, the meaning of the adjective "tall" depends on the context. Do we want to define the set of tall men on the universe of basketball players, horse racers, Vietnamese, Norwegians, . . . ? Further, the meaning of "tall man" is perceived subjectively by every evaluator. An evaluator who is 200 cm tall will probably not consider a 185 cm tall man as tall while an evaluator that is only 150 cm tall will probably do. Since the meaning of the adjective "tall" is vague, even for a particular universe of men and for a particular evaluator, it is very difficult to define the set of tall men. Let say I, 167 cm tall woman, am the evaluator and I attempt to define the meaning of "tall European man". Any European man over 190 cm is definitely tall for me. Any man under 170 cm is not tall. But what about the men between 170 and 190 cm tall? Are they tall or not? It is difficult to draw a line above which I perceive a European man as tall and below which I perceive a European man as not tall. For example, if I chose 180 cm as the border, a man of 181 cm would be considered tall and a man of 179 cm would be considered not tall. This is not acceptable.

The traditional "crisp" set theory is clearly not able to deal with this paradox. Thus other tools are needed. It feels very natural to describe the European men between 170 and 190 cm as "tall in some degree", i.e. by using "a partial degree of membership". This is the concept of fuzzy set theory. $\triangle$

Fuzzy set theory was initiated by Zadeh (1965, 1975a,b,c). Zadeh (1965) introduced a fuzzy set as a generalization of a crisp set with not precise boundaries.

Definition 13. *Let U be a nonempty universe. A fuzzy set $\widetilde{S}$ on U is characterized by its membership function $\mu_{\widetilde{S}}(x)$ which associates to each element $x \in U$ a real number in the interval $[0, 1]$, i.e. $\mu_{\widetilde{S}} : U \to [0, 1]$. $\mu_{\widetilde{S}}(x)$ is called the degree of membership of the element x to the fuzzy set $\widetilde{S}$.*

Figure III.1: Membership function of the fuzzy set "tall European man".
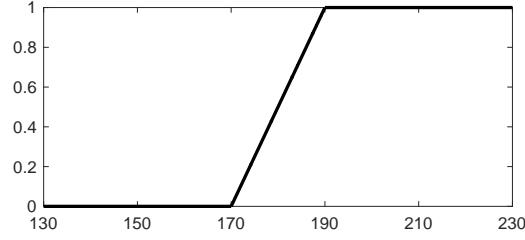
The membership function $\mu_{\widetilde{S}} : U \rightarrow [0,1]$ is a generalization of the characterization function $\chi_S : U \rightarrow \{0,1\}$ which besides the values 1 and 0 representing the total membership and the total non-membership, respectively, allows the values between 0 and 1 for expressing degrees of membership to the fuzzy set $\widetilde{S}$.

Example 19. By applying the concept of fuzzy sets, we can now easily define the meaning of "tall European men". One may define the fuzzy set $\widetilde{S}$ of tall European men by the membership function

$$\mu_{\widetilde{S}}(x) = \begin{cases} 0 & \text{for } x \leq 170, \\ \frac{x-170}{20} & \text{for } 170 < x < 190, \\ 1 & \text{for } +69 \geq 190. \end{cases}$$

The membership function is graphically represented in Fig. III.1. However, as mentioned in Example 18, the actual membership function can differ for every evaluator. $\triangle$

Note 2. *For simplicity, $\widetilde{S}(x)$ will be used hereafter to denote a membership function of a fuzzy set $\widetilde{S}$ instead of $\mu_{\widetilde{S}}(x)$.*

A fuzzy set $\widetilde{S}$ is defined uniquely by its membership function $\widetilde{S}(x) : U \rightarrow [0,1]$. Besides the membership function, also other characteristics of fuzzy sets are used to describe them.

Definition 14. *Let $\widetilde{S}$ be a fuzzy set defined on the universe U. The set $Core \, \widetilde{S} := \left\{ x \in U; \widetilde{S}(x) = 1 \right\}$ denotes the core of $\widetilde{S}$, the set $Supp \, \widetilde{S} := \left\{ x \in U; \widetilde{S}(x) > 0 \right\}$ denotes the support of $\widetilde{S}$, and the set $\widetilde{S}_{(\alpha)} := \left\{ x \in U; \widetilde{S}(x) \geq \alpha \right\}, \alpha \in]0,1]$, denotes the α−cut of $\widetilde{S}$. Fuzzy set $\widetilde{S}$ is said to be a normal fuzzy set if $\exists x \in U : \widetilde{S}(x) = 1$, i.e. if $Core \, \widetilde{S} \neq \emptyset$.*
The set of all fuzzy sets defined on $\mathbb{R}$ is denoted $\mathcal{F}(\mathbb{R})$.

Definition 15. *Let $\widetilde{S}_1, \ldots, \widetilde{S}_k$ be k fuzzy sets defined on the universes $U_1, \ldots, U_k$, respectively. The Cartesian product of $\widetilde{S}_1, \ldots, \widetilde{S}_k$ is a fuzzy set $\widetilde{S}_1 \times \cdots \times \widetilde{S}_k$ on $U_1 \times \cdots \times U_k$ with the membership function*

$$\widetilde{S}_1 \times \cdots \times \widetilde{S}_k(x_1, \ldots, x_k) = \min \left\{ \widetilde{S}_1(x_1), \ldots, \widetilde{S}_k(x_k) \right\}. \tag{III.1}$$

In order to generalize mathematical concepts for crisp sets to fuzzy sets, the extension principle was introduced.

Definition 16. *Let $\widetilde{S}_1, \ldots, \widetilde{S}_k$ be k fuzzy sets defined on the universes $U_1, \ldots, U_k$, respectively, and let $U = U_1 \times \cdots \times U_k$. Further, let f be a mapping from the universe U to the universe V, $y = f(x_1, \ldots, x_k)$. The extension principle defines the membership function of a fuzzy set $\widetilde{S}$ on V as*

$$\widetilde{S}(y) = \begin{cases} \sup \left\{ \min \left\{ \widetilde{S}_1(x_1), \ldots, \widetilde{S}_k(x_k) \right\}; (x_1, \ldots, x_k) \in U : y = f(x_1, \ldots, x_k) \right\} \\ \qquad \qquad \text{if } \{ (x_1, \ldots, x_k) \in U : y = f(x_1, \ldots, x_k) \} \neq \emptyset, \\ 0, \qquad \qquad \text{otherwise.} \end{cases} \tag{III.2}$$

Various types of fuzzy sets have been defined in the literature. Fuzzy numbers, a particular type of the fuzzy sets defined on $\mathbb{R}$ proved to be of a particular significance. "They should capture our intuitive conceptions of approximate numbers or intervals, such as "numbers that are close to a given real number" or "numbers that are around a given interval of real numbers". Such concepts are essential for characterizing states of fuzzy variables and, consequently, play an important role in many applications, including fuzzy control, decision making, approximate reasoning, optimization, and statistics with imprecise probabilities" (Klir and Yuan (1995), p. 97).

Definition 17. *A fuzzy set $\widetilde{n}$ on $\mathbb{R}$ is said to be a fuzzy number if it satisfies the following properties:*

(i) *$\widetilde{n}$ is a normal fuzzy set, i.e. $\exists x \in \mathbb{R} : \widetilde{n}(x) = 1$;*

(ii) *the $\alpha-$cuts $\widetilde{n}_{(\alpha)}$ are closed intervals for every $\alpha \in]0, 1]$;*

(iii) *the support of $\widetilde{n}$ is bounded, i.e. $\exists r_1, r_2 \in \mathbb{R} : Supp \, \widetilde{n} \subseteq [r_1, r_2]$.*

A fuzzy number $\widetilde{n}$ is said to be positive if $\exists r_1, r_2 \in \mathbb{R}^+ : Supp \, \widetilde{n} \subseteq [r_1, r_2]$. The set of all fuzzy numbers is denoted by $\mathcal{F}_N(\mathbb{R})$ and the set of all positive fuzzy numbers is denoted by $\mathcal{F}_N(\mathbb{R}^+)$.

3.2 Alpha-cut representation

In the previous section, a definition of $\alpha-$cuts of a fuzzy set was provided. In this Section, $\alpha-$cuts will be reviewed in more detail since they are of a particular usefulness in fuzzy set theory. For the sake of this section integrity, the definition of $\alpha-$cuts will be recalled here once again.

In the literature, two basic types of $\alpha-$cuts are used.

Definition 18. *Let $\widetilde{S}$ be a fuzzy set on a nonempty universe U. Then, the set $\widetilde{S}_{(\alpha)} = \left\{ x \in U; \widetilde{S}(x) \geq \alpha \right\}$ for $\alpha \in]0, 1]$ is called (weak) $\alpha-$cut (or (weak) $\alpha-$level set) of $\widetilde{S}$. The set $\widetilde{S}_{(\alpha)}^{>} = \left\{ x \in U; \widetilde{S}(x) > \alpha \right\}$ for $\alpha \in [0, 1[$ is called strong $\alpha-$cut (or strong $\alpha-$level set) of $\widetilde{S}$.*

When the membership function $\widetilde{S}(x)$ of $\widetilde{S}$ is continuous, the distinction between the (weak) $\alpha-$cuts and the strong $\alpha-$cuts is not necessary in applications. In the following, only the (weak) $\alpha-$cuts are considered.

Remark 5. *Notice that $\alpha-$cut of $\widetilde{S}$ is not defined for $\alpha = 0$. However, for later use, it is convenient to define $0-$cut of $\widetilde{S}$ as the closure[1] of the support of $\widetilde{S}$, $\widetilde{S}_{(0)} = Cl(Supp \, \widetilde{S})$.*

Theorem 12. *Let $\widetilde{S}$ be a fuzzy set on a nonempty universe U. Then its membership function $\widetilde{S}(x)$ is given as $\widetilde{S}(x) = \sup \left\{ \alpha; \ x \in \widetilde{S}_{(\alpha)}, \ \alpha \in [0, 1] \right\}, \ \forall x \in U$.*

Definition 19. *Let $\widetilde{S}$ be a fuzzy set on a nonempty universe U. Then the $\alpha-$multiple of $\widetilde{S}$ is a fuzzy set $\alpha \widetilde{S}$ on U with the membership function $(\alpha \widetilde{S})(x) = \alpha \cdot \widetilde{S}(x), \forall x \in U$.*

Definition 20. *Let $\widetilde{S}_i, i = 1, \ldots, k$, be fuzzy sets on a nonempty universe U. Then the union of $\widetilde{S}_i, i = 1, \ldots, k$, is a fuzzy set $\widetilde{S} = \bigcup_{i=1}^{k} \widetilde{S}_i$ on U with the membership function $\widetilde{S}(x) = \max \left\{ \widetilde{S}_1(x), \ldots, \widetilde{S}_k(x) \right\}, \forall x \in U$.*

Using Definition 18 of $\alpha-$cuts of a fuzzy set and Definitions 19 and 20 of the $\alpha-$multiple and of the union of fuzzy sets, respectively, it is possible to derive another representation of fuzzy sets.

Theorem 13. *Let $\widetilde{S}$ be a fuzzy set on a nonempty universe U. Then*

$$\widetilde{S} = \bigcup_{\alpha=0}^{1} \alpha \widetilde{S}_{(\alpha)}. \tag{III.3}$$

The $\alpha-$cut representation (III.3) is particularly convenient for fuzzy numbers, for which it can be defined easily by providing two functions.

Theorem 14. *Let $\widetilde{n} \in \mathcal{F}_N(\mathbb{R})$ and let $\widetilde{n}_{(\alpha)} = [n_{(\alpha)}^L, n_{(\alpha)}^U], \alpha \in [0, 1]$. Fuzzy number $\widetilde{n}$ can be determined uniquely by two functions $n^-, n^+ : [0, 1] \to \mathbb{R}$ defining the lower and upper boundary values $n_{(\alpha)}^L, n_{(\alpha)}^U$ of the $\alpha-$cuts $\widetilde{n}_{(\alpha)}, \alpha \in [0, 1]$, of $\widetilde{n}$ satisfying*

(i) *$n^- : n_{(\alpha)}^L = n^-(\alpha)$ is a bounded monotonic non-decreasing left-continuous function for $\alpha \in]0, 1]$ and right-continuous for $\alpha = 0$;*

(ii) *$n^+ : n_{(\alpha)}^U = n^+(\alpha)$ is a bounded monotonic non-increasing left-continuous function for $\alpha \in]0, 1]$ and right-continuous for $\alpha = 0$;*

(iii) *$\forall \alpha \in [0, 1] : \ n_{(\alpha)}^L \leq n_{(\alpha)}^U$.*

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[1]The closure of interval $\overline{u}$ is the smallest closed interval containing $\overline{u}$; e.g. $Cl(]1, 3[) = [1, 3]$.

3.3 Trapezoidal fuzzy numbers

In the fuzzy extension of MCDM methods, and in particular in the fuzzy extension of the MCDM methods based on PCMs, special types of fuzzy numbers are usually used; namely, triangular and trapezoidal fuzzy numbers, and intervals. All three types of fuzzy numbers are introduced in this section.

Triangular fuzzy numbers were introduced as a special case of fuzzy numbers by Laarhoven and Pedrycz (1983).

Definition 21. *A triangular fuzzy number $\widetilde{t}$ is a fuzzy number whose membership function is given as*

$$\widetilde{t}(x) = \begin{cases} \frac{x-t^L}{t^M-t^L}, & t^L < x < t^M, \\ 1, & x = t^M, \\ \frac{t^U-x}{t^U-t^M}, & t^M < x < t^U, \\ 0, & \textit{otherwise,} \end{cases} \tag{III.4}$$

where t^L and t^U are called the lower and upper boundary values of the triangular fuzzy number $\widetilde{t}$, and t^M is called the middle value of $\widetilde{t}$. Every triangular fuzzy number can be uniquely described by a triplet of these representing values; the notation $\widetilde{t} = \left(t^L, t^M, t^U\right)$ is used.

A triangular fuzzy number $\widetilde{t} = \left(t^L, t^M, t^U\right)$ is positive if $t^L > 0$. The core of $\widetilde{t} = \left(t^L, t^M, t^U\right)$ is the singleton set $Core\,\widetilde{t} = \{t^M\}$, and the support is an open interval $Supp\,\widetilde{t} =]t^L, t^U[$. The $\alpha-$cut, $\alpha \in [0,1]$, of the triangular fuzzy number $\widetilde{t} = \left(t^L, t^M, t^U\right)$ is a closed interval $\widetilde{t}_{(\alpha)} = [t^L_{(\alpha)}, t^U_{(\alpha)}]$, where $t^L_{(\alpha)} = \alpha t^M + (1-\alpha)t^L, t^U_{(\alpha)} = \alpha t^M + (1-\alpha)t^U$.

Example 20. Triangular fuzzy number $\widetilde{t} = (1, 2, 4)$, given in Fig. III.2, is positive, its core is the singleton set $Core\,\widetilde{t} = \{2\}$, and its support is the open interval $Supp\,\widetilde{t} =]1, 4[$. The $\alpha-$cuts, $\alpha \in [0,1]$, are closed intervals $\widetilde{t}_{(\alpha)} = [t^L_{(\alpha)}, t^U_{(\alpha)}]$ such that $t^L_{(\alpha)} = 1 + \alpha, t^U_{(\alpha)} = 4 - 2\alpha$. $\triangle$

A couple of years later, trapezoidal fuzzy numbers were introduced by Buckley (1985b) even though they got their name later.

Definition 22. *A trapezoidal fuzzy number $\widetilde{z}$ is a fuzzy number whose membership function is given as*

$$\widetilde{z}(x) = \begin{cases} \frac{x-z^\alpha}{z^\beta-z^\alpha}, & z^\alpha < x < z^\beta, \\ 1, & z^\beta \leq x \leq z^\gamma, \\ \frac{z^\delta-x}{z^\delta-z^\gamma}, & z^\gamma < x < z^\delta, \\ 0, & \textit{otherwise.} \end{cases} \tag{III.5}$$

Every trapezoidal fuzzy number can be uniquely described by a quadruple of its representing values; the notation $\widetilde{z} = \left(z^\alpha, z^\beta, z^\gamma, z^\delta\right)$ is used.

A trapezoidal fuzzy number $\widetilde{z} = \left(z^\alpha, z^\beta, z^\gamma, z^\delta\right)$ is positive if $z^\alpha > 0$. The core of $\widetilde{z} = \left(z^\alpha, z^\beta, z^\gamma, z^\delta\right)$ is a closed interval $Core\,\widetilde{z} = \left[z^\beta, z^\gamma\right]$, and the support is an open interval $Supp\,\widetilde{z} =]z^\alpha, z^\delta[$. The $\alpha-$cut, $\alpha \in [0,1]$, of the trapezoidal fuzzy number $\widetilde{z} = \left(z^\alpha, z^\beta, z^\gamma, z^\delta\right)$ is a closed interval $\widetilde{z}_{(\alpha)} = [z^L_{(\alpha)}, z^U_{(\alpha)}]$, where

$$z^L_{(\alpha)} = \alpha z^\beta + (1-\alpha)z^\alpha, z^U_{(\alpha)} = \alpha z^\gamma + (1-\alpha)z^\delta. \tag{III.6}$$

Figure III.2: Triangular fuzzy number $\widetilde{t} = (1, 2, 4)$.
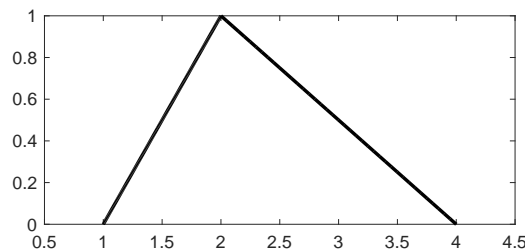
Example 21. Trapezoidal fuzzy number $\widetilde{z} = (-2.5, -2, 0, 1)$, given in Fig. III.3, is not positive since $z^\alpha = -2.5 \not> 0$. The core of $\widetilde{z}$ is the closed interval $Core\ \widetilde{z} = [-2, 0]$ and its support is the open interval $Supp\ \widetilde{z} =]-2.5, 1[$. The α−cuts, $\alpha \in [0, 1]$, are closed intervals $\widetilde{z}_{(\alpha)} = [z^L_{(\alpha)}, z^U_{(\alpha)}]$ such that $z^L_{(\alpha)} = -2.5 + 0.5\alpha$, $z^U_{(\alpha)} = 1 - \alpha$. $\qquad \triangle$

Even intervals can be understood and dealt with as fuzzy numbers.

Definition 23. *An interval $\overline{v}$ is a fuzzy number whose membership function is given as*

$$\overline{v}(x) = \begin{cases} 1, & v^L \le x \le v^U, \\ 0, & \text{otherwise.} \end{cases} \tag{III.7}$$

Every interval can be uniquely described by the pair of its lower and upper boundary values; the notation $\overline{v} = [v^L, v^U]$ is used.

An interval $\overline{v} = [v^L, v^U]$ is positive if $v^L > 0$. The core and the support of $\overline{v} = [v^L, v^U]$ are obviously identical to the interval $\overline{v}$, i.e $Core\ \overline{v} = Supp\ \overline{v} = [v^L, v^U]$. Also all the α−cuts, $\alpha \in [0, 1]$, of the interval $\overline{v} = [v^L, v^U]$ are identical to the interval, i.e. $\overline{v}_{(\alpha)} = [v^L, v^U]$.

Remark 6. *Notice that the interval $\overline{v}$ in Definition 23 is denoted by $^-$ instead of $^\sim$, which is usual for fuzzy numbers. That is because the notation $^-$ is more common than $^\sim$ to refer to intervals in the literature. Thus, the same notation was opted for in this thesis. Nevertheless, this notation does not change anything on the fact that intervals can be looked at as a particular case of fuzzy numbers.*

Not only are triangular fuzzy numbers, trapezoidal fuzzy numbers, and intervals special cases of fuzzy numbers, but triangular fuzzy numbers and intervals are also special cases of trapezoidal fuzzy numbers. In particular, a triangular fuzzy number $\widetilde{t} = (t^L, t^M, t^U)$ can be written in the form $\widetilde{t} = (t^L, t^M, t^M, t^U)$ which satisfies Definition 22 of trapezoidal fuzzy number, but the membership function given by (III.5) still has the form of a triangular fuzzy number. Analogously, an interval $\overline{v} = [v^L, v^U]$ can be written in the form $\widetilde{v} = (v^L, v^L, v^U, v^U)$ which satisfies Definition 22 of trapezoidal fuzzy number, but the membership function given by (III.5) still has the form of an interval. Thus all concepts, definitions, and arithmetic operations provided for trapezoidal fuzzy numbers are automatically applicable also to triangular fuzzy numbers and intervals.

In the rest of this section, some important terms that are going to be used in the following chapters are defined.

Definition 24. *Let $\widetilde{w}_i \in \mathcal{F}_N(\mathbb{R}), i = 1, \dots, n$. The vector $\underline{\widetilde{w}} = (\widetilde{w}_1, \dots, \widetilde{w}_n)^T \in \mathcal{F}_N(\mathbb{R})^n$ is called a fuzzy vector in $\mathcal{F}_N(\mathbb{R})^n$. The membership function of $\underline{\widetilde{w}}$ is defined as $\underline{\widetilde{w}}(\underline{w}) = \min\limits_{i=1,\dots,n} \{\widetilde{w}_i(w_i)\}$, where $\underline{w} = (w_1, \dots, w_n)^T$.*

Further, let $\widetilde{w}_i = (w^\alpha_i, w^\beta_i, w^\gamma_i, w^\delta_i) \in \mathcal{F}_N(\mathbb{R})$, $i = 1, \dots, n$, be trapezoidal fuzzy numbers. Then the fuzzy vector $\underline{\widetilde{w}} = (\widetilde{w}_1, \dots, \widetilde{w}_n)^T \in \mathcal{F}_N(\mathbb{R})^n$ can be written as $\underline{\widetilde{w}} = (\underline{w}^\alpha, \underline{w}^\beta, \underline{w}^\gamma, \underline{w}^\delta)$, where $\underline{w}^\alpha = (w^\alpha_1, \dots, w^\alpha_n)^T$, $\underline{w}^\beta = (w^\beta_1, \dots, w^\beta_n)^T$, $\underline{w}^\gamma = (w^\gamma_1, \dots, w^\gamma_n)^T$, and $\underline{w}^\delta = (w^\delta_1, \dots, w^\delta_n)^T$, are called the representing vectors of the fuzzy vector $\underline{\widetilde{w}}$.

Definition 25. *Let $\widetilde{m}_{ij} \in \mathcal{F}_N(\mathbb{R})$, $i, j = 1, \dots, n$. Then $\widetilde{M} = \{\widetilde{m}_{ij}\}^n_{i,j=1} \in \mathcal{F}_N(\mathbb{R})^{n^2}$ is called a fuzzy matrix in $\mathcal{F}_N(\mathbb{R})^{n^2}$. The membership function of $\widetilde{M}$ is defined as $\widetilde{M}(M) = \min\limits_{i,j=1,\dots,n} \{\widetilde{m}_{ij}(m_{ij})\}$, where $M = \{m_{ij}\}^n_{i,j=1}$.*

In order to defuzzify fuzzy numbers, the center-of-area defuzzification method (Takagi and Sugeno, 1985), sometimes called also the center-of-gravity method or the centroid method, is often used because of its computational simplicity and well accepted results.

Definition 26. *Let $\widetilde{z} = (z^\alpha, z^\beta, z^\gamma, z^\delta)$ be a trapezoidal fuzzy number. The center of area $COA_{\widetilde{z}}$ of $\widetilde{z}$ is defined as*

$$COA_{\widetilde{z}} = \frac{1}{3} \frac{(z^\delta)^2 + (z^\gamma)^2 - (z^\beta)^2 - (z^\alpha)^2 + z^\delta z^\gamma - z^\beta z^\alpha}{z^\delta + z^\gamma - z^\beta - z^\alpha}. \tag{III.8}$$

Figure III.3: Trapezoidal fuzzy number $\widetilde{z} = (-2.5, -2, 0, 1)$.
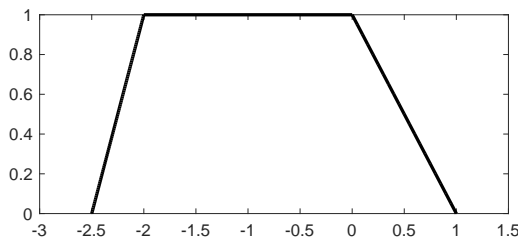
Note 3. *Note that for a triangular fuzzy number $\widetilde{t} = (t^L, t^M, t^U)$, the formula* (III.8) *is reduced to*

$$COA_{\widetilde{t}} = \frac{t^L + t^M + t^U}{3},$$

and for an interval $\overline{v} = [v^L, v^U]$, it is reduced to

$$COA_{\overline{v}} = \frac{v^L + v^U}{2}.$$

Example 22. Centers of area of the triangular fuzzy number $\widetilde{t} = (1, 2, 4)$ and of the trapezoidal fuzzy number $\widetilde{z} = (-2.5, -2, 0, 1)$ are $COA_{\widetilde{t}} = \frac{7}{3}$ and $COA_{\widetilde{z}} = -\frac{19}{22}$. $\triangle$

In the following, Ruspini's fuzzy partition (often called also fuzzy scale) is introduced as it is particularly suitable for modeling the meaning of linguistic terms from a predefined scale used for comparing objects pairwisely.

Definition 27. *(Ruspini, 1969) A set of fuzzy numbers $\widetilde{n}_1, \ldots, \widetilde{n}_k, k > 1$, defined on interval $[a, b]$ is called Ruspini's fuzzy partition of $[a, b]$, if $\widetilde{n}_i \neq \emptyset, i = 1, \ldots, k$, and $\sum_{i=1}^{k} \widetilde{n}_i(x) = 1, \forall x \in [a, b]$.*

For trapezoidal fuzzy numbers, the following proposition is valid.

Proposition 9. *A set of trapezoidal fuzzy numbers $\widetilde{z}_i = \left(z_i^\alpha, z_i^\beta, z_i^\gamma, z_i^\delta \right), i = 1, \ldots, k$, defined on interval $[a, b]$ and numbered in the conformity with their linear ordering forms Ruspini's fuzzy partition of interval $[a, b]$ if and only if*

$$\begin{aligned} z_1^\alpha = z_1^\beta &= a, \\ z_{i-1}^\gamma = z_i^\alpha, & \quad i = 2, \ldots, k, \\ z_{i-1}^\delta = z_i^\beta, & \quad i = 2, \ldots, k, \\ z_k^\gamma = z_k^\delta &= b. \end{aligned} \tag{III.9}$$

Proof. Interval $[a, b]$ can be written as the union $[a, b] = \bigcup_{i=1}^{k} [z_i^\beta, z_i^\gamma] \cup \bigcup_{i=2}^{k}]z_i^\alpha, z_i^\beta[$. For $x \in [z_i^\beta, z_i^\gamma], i \in \{1, \ldots, k\}$, it holds that $\widetilde{z}_i(x) = 1$ and $\widetilde{z}_j(x) = 0$ for $j = 1, \ldots, k, j \neq i$. Therefore, $\sum_{j=1}^{k} \widetilde{z}_j(x) = \widetilde{z}_i(x) = 1$. Further, for $x \in]z_i^\alpha, z_i^\beta[=]z_{i-1}^\gamma, z_{i-1}^\delta[, i \in \{2, \ldots, k\}$, by applying (III.4), we obtain $\widetilde{z}_i(x) = \frac{x - z_i^\alpha}{z_i^\beta - z_i^\alpha}$, $\widetilde{z}_{i-1}(x) = \frac{z_{i-1}^\delta - x}{z_{i-1}^\delta - z_{i-1}^\gamma} = \frac{z_i^\beta - x}{z_i^\beta - z_i^\alpha}$, and $\widetilde{z}_j(x) = 0$ for $j = 1, \ldots, k, j \neq i, i-1$. Therefore, for $x \in]z_i^\alpha, z_i^\beta[, i \in \{2, \ldots, k\}$, the equation $\sum_{j=1}^{k} \widetilde{z}_j(x) = \widetilde{z}_{i-1}(x) + \widetilde{z}_i(x) = \frac{z_i^\beta - x}{z_i^\beta - z_i^\alpha} + \frac{x - z_i^\alpha}{z_i^\beta - z_i^\alpha} = \frac{z_i^\beta - x + x - z_i^\alpha}{z_i^\beta - z_i^\alpha} = 1$ holds, which completes the proof. $\square$

Note 4. *Proposition 9 is a generalization of the proposition regarding Ruspini's fuzzy partition for triangular fuzzy numbers which was provided and proved by Krejčí (2017b).*

Example 23. Trapezoidal fuzzy numbers $\widetilde{z}_1 = (0, 0, 1, 2), \widetilde{z}_2 = (1, 2, 3, 4), \widetilde{z}_3 = (3, 4, 5, 5)$, given in Fig. III.4, form Ruspini's fuzzy partition of interval $[0, 5]$. $\triangle$

Figure III.4: Ruspini's fuzzy partition of interval $[0, 5]$.
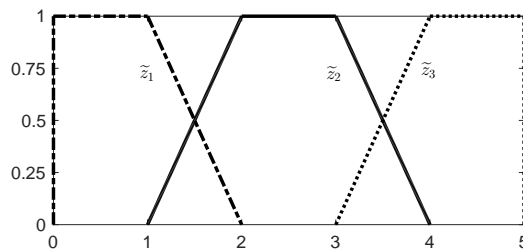
In the following chapter, a fuzzy extension of Saaty's scale given in Tab. II.1 will be proposed in such a way that the fuzzy numbers modeling the meaning of the linguistic terms from the scale and their reciprocals form Ruspini's fuzzy partition of the given interval.

As discussed in the previous chapter, the priorities of objects derived from a PCM are usually normalized. Most often, the normalization condition (II.18), $\sum_{i=1}^n w_i = 1, w_i \in [0,1], i = 1, \ldots, n$, is utilized; in particular for the priorities obtained from MPCMs and APCMs-M by the methods reviewed in the previous chapter. Recall that it was shown in Section 2.3.2.2 that the normalization condition (II.18) is not compatible with Tanino's characterization (II.32) for the priorities obtained from APCMs-A, and thus a weaker version of the normalization condition (II.18) is needed in this case.

When extending MCDM methods based on PCMs to fuzzy numbers it is necessary, besides other issues, to handle properly the fuzzy extension of the normalization condition (II.18). Many definitions of normalized interval and fuzzy vectors have been proposed in the literature; see, e.g. Chang and Lee (1995), Jiménez et al. (2003), Wang and Elhag (2006), and Sevastjanov et al. (2012). In this thesis, the approach of Wang and Elhag (2006) is considered. The reason for choosing this approach will be clarified in Chapter IV after introducing fuzzy PCMs and methods for deriving fuzzy priorities of objects from them.

Wang and Elhag (2006) provided a definition of the normalized interval vector and they extended this definition to general fuzzy numbers given by means of their $\alpha-$cuts.

Definition 28. *(Wang and Elhag, 2006) Let $\overline{w}_i = [w_i^L, w_i^U], i = 1, \ldots, n$, be intervals, $\overline{w}_i \subseteq [0,1], i = 1, \ldots, n$, and let*

$$N_{\underline{\overline{w}}} = \left\{ \underline{w} = (w_1, \ldots, w_n)^T; \ \sum_{i=1}^n w_i = 1, \ w_i^L \le w_i \le w_i^U, \ i = 1, \ldots, n \right\} \tag{III.10}$$

be a set of normalized vectors constructed from the intervals. The intervals $\overline{w}_1, \ldots, \overline{w}_n$ are said to be normalized if

 (i) *there exists at least one normalized vector $\underline{w} = (w_1, \ldots, w_n)^T$ in $N_{\underline{\overline{w}}}$;*

 (ii) *the lower and upper boundary values w_i^L and w_i^U of $\overline{w}_i, i = 1, \ldots, n$, are attainable in $N_{\underline{\overline{w}}}$.*

 The vector $\underline{\overline{w}} = (\overline{w}_1, \ldots, \overline{w}_n)^T$ of normalized intervals will be called a normalized interval vector.

Remark 7. *As already pointed out by Pavlačka (2014), the validity of the condition (ii) in Definition 28 automatically implies the validity of the condition (i). This means that the condition (i) could be omitted in Definition 28.*

According to the condition (ii), for every lower boundary value $w_i^L, i \in \{1, \ldots, n\}$, there have to exist values $w_j \in [w_j^L, w_j^U], j = 1, \ldots, n, j \ne i$, such that $w_i^L + \sum_{\substack{j=1 \\ j \ne i}}^n w_j = 1$, and analogously, for every upper boundary value $w_i^U, i \in \{1, \ldots, n\}$, there have to exist values $w_j \in [w_j^L, w_j^U], j = 1, \ldots, n, j \ne i$, such that $w_i^U + \sum_{\substack{j=1 \\ j \ne i}}^n w_j = 1$.

Wang and Elhag (2006) formulated the following theorem to verify whether a set of intervals is normalized.

Theorem 15. *(Wang and Elhag, 2006) Let $\overline{w}_i = [w_i^L, w_i^U], i = 1, \ldots, n$, be intervals, $\overline{w}_i \subseteq [0,1], i = 1, \ldots, n$. Then the intervals are normalized according to Definition 28 if and only if the inequalities*

$$w_i^L + \sum_{\substack{j=1 \\ j \ne i}}^n w_j^U \ge 1, \qquad w_i^U + \sum_{\substack{j=1 \\ j \ne i}}^n w_j^L \le 1 \tag{III.11}$$

are satisfied for all $i = 1, \ldots, n$.

Remark 8. *From the inequalities (III.11) it is obvious that a set of intervals $\overline{w}_i = [w_i^L, w_i^U], i = 1, \ldots, n$, is normalized according to Definition 28 if and only if for any $w_i \in [w_i^L, w_i^U], i \in \{1, \ldots, n\}$, there exist $w_j \in [w_j^L, w_j^U], j = 1, \ldots, n, j \ne i$, such that $\sum_{k=1}^n w_k = 1$. In other words, any value w_i from the interval $\overline{w}_i = [w_i^L, w_i^U], i \in \{1, \ldots, n\}$, is a part of a normalized vector $\underline{w} = (w_1, \ldots, w_n)^T$ belonging to the set (III.10) of normalized vectors constructed from the intervals.*

Based on Definition 28 and on Remark 8, the definition of a normalized interval vector can be extended intuitively to a definition of a normalized trapezoidal fuzzy vector.

Definition 29. *Let $\widetilde{w}_i = (w_i^\alpha, w_i^\beta, w_1^\gamma, w_1^\delta), i = 1, \ldots, n$, be trapezoidal fuzzy numbers, $\widetilde{w}_i \subseteq [0,1], i = 1, \ldots, n$. The trapezoidal fuzzy numbers $\widetilde{w}_1, \ldots, \widetilde{w}_n$ are said to be normalized if*

$$\forall w_{i\alpha} \in \widetilde{w}_{i(\alpha)} \ \exists w_{j\alpha} \in \widetilde{w}_{j(\alpha)}, j = 1, \ldots, n, j \ne i : \ w_{i\alpha} + \sum_{\substack{j=1 \\ j \ne i}}^n w_{j\alpha} = 1 \tag{III.12}$$

for all $\alpha \in [0,1]$ and $i = 1, \ldots, n$.

The vector $\underline{\widetilde{w}} = (\widetilde{w}_1, \ldots, \widetilde{w}_n)^T \in \mathcal{F}_N(\mathbb{R})^n$ of normalized trapezoidal fuzzy numbers will be called the normalized fuzzy vector.

Theorem 15 can be extended to trapezoidal fuzzy numbers as follows.

Theorem 16. *Let $\widetilde{w}_i = (w_i^\alpha, w_i^\beta, w_i^\gamma, w_i^\delta), i = 1, \ldots, n$, be trapezoidal fuzzy numbers, $\widetilde{w}_i \subseteq [0,1]$, $i = 1, \ldots, n$. Then the trapezoidal fuzzy numbers are normalized according to Definition 29 if and only if the inequalities*

$$w_i^\alpha + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\delta \geq 1, \quad w_i^\delta + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\alpha \leq 1, \quad w_i^\beta + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\gamma \geq 1, \quad w_i^\gamma + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\beta \leq 1 \tag{III.13}$$

are satisfied for all $i = 1, \ldots, n$.

Proof. First, let us show that (III.12) implies (III.13). For $\alpha = 0$, it follows from (III.12) that for $w_i^\alpha, i \in \{1, \ldots, n\}$, $\exists w_j \in [w_j^\alpha, w_j^\delta], j = 1, \ldots, n, j \neq i : w_i^\alpha + \sum_{\substack{j=1 \\ j \neq i}}^n w_j = 1$. Because $w_j^\delta \geq w_j$, then clearly $w_i^\alpha + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\delta \geq 1$. Similarly, for $w_i^\delta, i \in \{1, \ldots, n\}$, $\exists w_j \in [w_j^\alpha, w_j^\delta], j = 1, \ldots, n, j \neq i : w_i^\delta + \sum_{\substack{j=1 \\ j \neq i}}^n w_j = 1$. Because $w_j^\alpha \leq w_j$, then clearly $w_i^\delta + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\alpha \leq 1$. Analogously, for $\alpha = 1$, the inequalities $w_i^\beta + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\gamma \geq 1$ and $w_i^\gamma + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\beta \leq 1$ are derived from (III.12).

Now, let us show that (III.13) implies (III.12). From the inequalities $w_i^\delta + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\alpha \leq 1$ and $w_i^\alpha + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\delta \geq 1$, the inequalities $w_i + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\alpha \leq 1$ and $w_i + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\delta \geq 1$ follow $\forall w_i \in [w_i^\alpha, w_i^\delta]$. Therefore, $\exists w_j \in [w_j^\alpha, w_j^\delta], j = 1, \ldots, n, j \neq i : w_i + \sum_{\substack{j=1 \\ j \neq i}}^n w_j = 1$, which implies (III.12) for $\alpha = 0$. Analogously, from the inequalities $w_i^\gamma + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\beta \leq 1$ and $w_i^\beta + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\gamma \geq 1$, the inequalities $w_i + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\beta \leq 1$ and $w_i + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\gamma \geq 1$ follow $\forall w_i \in [w_i^\beta, w_i^\gamma]$. Therefore, $\exists w_j \in [w_j^\beta, w_j^\gamma], j = 1, \ldots, n, j \neq : w_i + \sum_{\substack{j=1 \\ j \neq i}}^n w_j = 1$, which implies (III.12) for $\alpha = 1$.

The proof of the validity of (III.12) for $\alpha \in]0, 1[$ is analogous; it is sufficient to show that the inequalities (III.13) hold also for the α−cuts $\widetilde{w}_{i(\alpha)} = \left[w_{i(\alpha)}^L, w_{i(\alpha)}^U\right]$ of the trapezoidal fuzzy numbers $\widetilde{w}_i = (w_i^\alpha, w_i^\beta, w_i^\gamma, w_i^\delta), i \in \{1, \ldots, n\}$, i.e.

$$w_{i(\alpha)}^L + \sum_{\substack{j=1 \\ j \neq i}}^n w_{j(\alpha)}^U \geq 1, \qquad w_{i(\alpha)}^U + \sum_{\substack{j=1 \\ j \neq i}}^n w_{j(\alpha)}^L \leq 1. \tag{III.14}$$

Then it is enough to take the α−cuts $\widetilde{w}_{i(\alpha)} = \left[w_{i(\alpha)}^L, w_{i(\alpha)}^U\right]$ of $\widetilde{w}_i, i = 1, \ldots, n$, for $[w_i^\alpha, w_i^\delta]$ in the above part of the proof.

Using the definition (III.6) of α−cuts and formulas (III.13), we obtain

$$w_{i(\alpha)}^U + \sum_{\substack{j=1 \\ j \neq i}}^n w_{j(\alpha)}^L = \alpha w_i^\gamma + (1-\alpha) w_i^\delta + \sum_{\substack{j=1 \\ j \neq i}}^n \left[\alpha w_j^\beta + (1-\alpha) w_j^\alpha\right] =$$

$$\alpha \left[w_i^\gamma + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\beta\right] + (1-\alpha) \left[w_i^\delta + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^\alpha\right] \leq \alpha + (1-\alpha) = 1$$

and analogously the inequality $w_{i(\alpha)}^L + \sum_{\substack{j=1 \\ j \neq i}}^n w_{j(\alpha)}^U \geq 1$ could be demonstrated. $\square$

Remark 9. *Note that for normalized triangular fuzzy numbers $\widetilde{w}_i = (w_i^L, w_i^M, w_i^U), i = 1, \ldots, n$, the inequalities* (III.13) *are reduced to*

$$w_i^L + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^U \geq 1, \qquad w_i^U + \sum_{\substack{j=1 \\ j \neq i}}^n w_j^L \leq 1, \qquad \sum_{i=1}^n w_i^M = 1. \tag{III.15}$$

Example 24. Triangular fuzzy numbers $\widetilde{w}_1 = (0.05, 0.1, 0.2), \widetilde{w}_2 = (0.1, 0.3, 0.4), \widetilde{w}_3 = (0.5, 0.6, 0.7)$, given in Fig. III.5, are normalized. $\triangle$

Figure III.5: Normalized triangular fuzzy numbers.
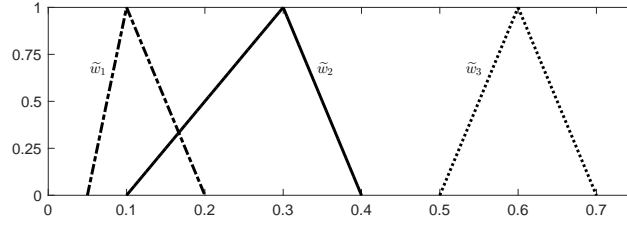
3.4 Standard fuzzy arithmetic

In Definition 16, the extension principle for fuzzy sets was formulated. The extension principle (III.2) allows us to define arithmetic operations on fuzzy numbers. In this section, standard fuzzy arithmetic is defined by using both the membership functions and the $\alpha-$cuts of fuzzy numbers. Afterwards, simplified standard fuzzy arithmetic is defined for trapezoidal fuzzy numbers.

There are four basic binary arithmetic operations on crisp numbers - addition, subtraction, multiplication, and division $(+, -, \cdot, /)$. Let $*$ denote one of them.

Definition 30. *Let* $\widetilde{n}_1, \widetilde{n}_2 \in \mathcal{F}_N(\mathbb{R})$ *and let* $* : \mathbb{R}^2 \to \mathbb{R}$ *be a binary arithmetic operation,* $z = x * y$. *Then the extension of the arithmetic operation* $*$ *to fuzzy numbers,* $* : \mathcal{F}_N(\mathbb{R})^2 \to \mathcal{F}_N(\mathbb{R})$, *is defined as* $\widetilde{n} = \widetilde{n}_1 * \widetilde{n}_2$ *with the membership function*

$$\widetilde{n}(z) = \begin{cases} \sup \left\{ \min \left\{ \widetilde{n}_1(x), \widetilde{n}_2(y) \right\}; \ (x, y) \in \mathbb{R}^2 : z = x * y \right\} \\ \qquad \text{if } \left\{ (x, y) \in \mathbb{R}^2; \ z = x * y \right\} \neq \emptyset, \\ 0, \qquad \text{otherwise.} \end{cases} \tag{III.16}$$

Note 5. *In the literature, the fuzzy extension of the binary arithmetic operations* $+, -, \cdot, /$ *to fuzzy numbers is often denoted as* $\oplus, \ominus, \odot, \oslash$. *Nevertheless, for simplicity of notation, the standard notation* $+, -, \cdot, /$ *will be used in this thesis for arithmetic operations defined on fuzzy numbers. Thus, for* $\widetilde{n}_1, \widetilde{n}_2 \in \mathcal{F}_N(\mathbb{R})$, *the notation* $\widetilde{n}_1 + \widetilde{n}_2, \widetilde{n}_1 - \widetilde{n}_2, \widetilde{n}_1 \cdot \widetilde{n}_2$ *(or simply just* $\widetilde{n}_1 \widetilde{n}_2$*), and* $\widetilde{n}_1 / \widetilde{n}_2$ *(or* $\frac{\widetilde{n}_1}{\widetilde{n}_2}$*) will be used.*

Note 6. *We know that division by* 0 *is not defined. Similarly, this limitation holds also for division of fuzzy numbers. In other words,* $\widetilde{n}_2 \in \mathcal{F}_N(\mathbb{R})$ *in Definition 30 has to be such that* $0 \notin Cl(Supp\, \widetilde{n}_2)$ *when division* $\widetilde{n}_1 / \widetilde{n}_2$ *is performed. Analogously, in the rest of this chapter, whenever arithmetic operation* $* \in \{+, -, \cdot, /\}$ *is considered on* $\widetilde{n}_1$ *and* $\widetilde{n}_2$, *it is automatically assumed that* $0 \notin Cl(Supp\, \widetilde{n}_2)$ *for the case* $* = /$.

Besides the four binary arithmetic operations, also the fuzzy extension of the $p-$th power of the variable is needed for the fuzzy extension of the methods reviewed in Chapter II.

Definition 31. *Let* $\widetilde{n} \in \mathcal{F}_N(\mathbb{R})$ *and let* $(.)^p : \mathbb{R} \to \mathbb{R}, p \in \mathbb{R}$, *be the* $p-$*th power of the variable,* $y = x^p$. *Then the extension of* $(.)^p$ *to fuzzy numbers,* $(.)^p : \mathcal{F}_N(\mathbb{R}) \to \mathcal{F}_N(\mathbb{R})$, *is defined as* $\widetilde{m} = (\widetilde{n})^p$ *with the membership function*

$$\widetilde{m}(y) = \begin{cases} \sup \left\{ \widetilde{n}(x); \ x \in \mathbb{R} : y = x^p \right\} & \text{if } \{x \in \mathbb{R}; \ y = x^p\} \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{III.17}$$

In particular, definitions of the reciprocal $\frac{1}{\widetilde{n}}$ and of the $k-$th root $\sqrt[k]{\widetilde{n}}, k \in \mathbb{N}$, of fuzzy number $\widetilde{n} \in \mathcal{F}(\mathbb{R}^+)$ are needed. Note that the reciprocal $\frac{1}{\widetilde{n}}$ is obtained by substituting $p = -1$ in Definition 31, and the $k-$th root $\sqrt[k]{\widetilde{n}}$ is obtained by substituting $p = \frac{1}{k}, k \in \mathbb{N}$, in Definition 31.

In Section 3.2, it was shown that $\widetilde{n} \in \mathcal{F}_N(\mathbb{R})$ can be represented uniquely by its $\alpha-$cuts; $\widetilde{n} = \bigcup_{\alpha=0}^{1} \alpha[n_{(\alpha)}^L, n_{(\alpha)}^U]$. This representation allows for an alternative definition of the fuzzy extension of arithmetic operations based on standard fuzzy arithmetic.

Definition 32. *Let* $\widetilde{n}_1, \widetilde{n}_2 \in \mathcal{F}_N(\mathbb{R})$ *be given by their* $\alpha-$*cuts as* $\widetilde{n}_1 = \bigcup_{\alpha=0}^{1} \alpha[n_{1(\alpha)}^L, n_{1(\alpha)}^U], \widetilde{n}_2 = \bigcup_{\alpha=0}^{1} \alpha[n_{2(\alpha)}^L, n_{2(\alpha)}^U]$. *Further, let* $* : \mathbb{R}^2 \to \mathbb{R}$ *be a binary arithmetic operation,* $z = x * y$. *Then the extension of the arithmetic operation* $*$ *to fuzzy numbers,* $* : \mathcal{F}_N(\mathbb{R})^2 \to \mathcal{F}_N(\mathbb{R})$, *is defined as* $\widetilde{n} = \widetilde{n}_1 * \widetilde{n}_2$ *with the* $\alpha-$*cut representation* $\widetilde{n} = \bigcup_{\alpha=0}^{1} \alpha[n_{(\alpha)}^L, n_{(\alpha)}^U]:$

$$\begin{aligned} n_{(\alpha)}^L &= \min \left\{ x * y; \ x \in [n_{1(\alpha)}^L, n_{1(\alpha)}^U], \ y \in [n_{2(\alpha)}^L, n_{2(\alpha)}^U] \right\}, \\ n_{(\alpha)}^U &= \max \left\{ x * y; \ x \in [n_{1(\alpha)}^L, n_{1(\alpha)}^U], \ y \in [n_{2(\alpha)}^L, n_{2(\alpha)}^U] \right\}. \end{aligned} \tag{III.18}$$

Definition 33. *Let $\widetilde{n} \in \mathcal{F}_N(\mathbb{R})$ be given by its $\alpha-$cuts as $\widetilde{n} = \bigcup_{\alpha=0}^{1} \alpha[n_{(\alpha)}^L, n_{(\alpha)}^U]$. Further, let $(.)^p : \mathbb{R} \to \mathbb{R}$ be the $p-$th power of the variable, $y = x^p$. Then the extension of $(.)^p$ to fuzzy numbers, $(.)^p : \mathcal{F}_N(\mathbb{R}) \to \mathcal{F}_N(\mathbb{R})$, is defined as $\widetilde{m} = (\widetilde{n})^p$ with the $\alpha-$cut representation $\widetilde{m} = \bigcup_{\alpha=0}^{1} \alpha[m_{(\alpha)}^L, m_{(\alpha)}^U]$:*

$$
\begin{aligned}
m_{(\alpha)}^L &= \min\left\{x^p;\ x \in [n_{(\alpha)}^L, n_{(\alpha)}^U]\right\}, \\
m_{(\alpha)}^U &= \max\left\{x^p;\ x \in [n_{(\alpha)}^L, n_{(\alpha)}^U]\right\}.
\end{aligned}
\tag{III.19}
$$

As already mentioned in Section 3.3, triangular and trapezoidal fuzzy numbers and intervals are most often used for the fuzzy extension of MCDM methods based on PCMs. This class of fuzzy numbers is used for the fuzzy extension also in this thesis. Thus, the arithmetic operations are going to be introduced in detail for this particular class of fuzzy numbers. The $\alpha-$cut representation of the fuzzy extension of arithmetic operations given by Definitions 32 and 33 is going to be used for this purpose as it is more convenient than the membership representation given by Definitions 30 and 31.

Let $\widetilde{c} = \left(c^\alpha, c^\beta, c^\gamma, c^\delta\right)$ and $\widetilde{d} = \left(d^\alpha, d^\beta, d^\gamma, d^\delta\right)$ be two trapezoidal fuzzy numbers. The sum of $\widetilde{c}$ and $\widetilde{d}$ obtained by applying the extension principle (III.18) is again a trapezoidal fuzzy number given as

$$
\widetilde{c} + \widetilde{d} = \left(c^\alpha + d^\alpha, c^\beta + d^\beta, c^\gamma + d^\gamma, c^\delta + d^\delta\right),
\tag{III.20}
$$

and their difference is a trapezoidal fuzzy number given as

$$
\widetilde{c} - \widetilde{d} = \left(c^\alpha - d^\delta, c^\beta - d^\gamma, c^\gamma - d^\beta, c^\delta - d^\alpha\right).
\tag{III.21}
$$

Unlike the sum and the difference, the product and the quotient of two trapezoidal fuzzy numbers as well as the reciprocal and the $k-$th root of a trapezoidal fuzzy number are not trapezoidal fuzzy numbers anymore when extension principle (III.18) is applied. Analogously, also the product and the quotient of two triangular fuzzy numbers as well as the reciprocal and the $k-$th root of a triangular fuzzy number are not in general triangular fuzzy numbers any more. However, for the sake of computational simplicity, it is a common practice in fuzzy MCDM based on fuzzy PCMs to approximate the results of these arithmetic operations by trapezoidal and triangular fuzzy numbers, respectively. Usually, authors do not even mention that simplified standard fuzzy arithmetic is used in their papers instead of standard fuzzy arithmetic.

According to simplified standard fuzzy arithmfetic, arithmetic operations are performed only on the representing values of trapezoidal fuzzy numbers thus obtaining representing values of the resulting trapezoidal fuzzy numbers. This means that the product and the quotient of two trapezoidal fuzzy numbers $\widetilde{c} = \left(c^\alpha, c^\beta, c^\gamma, c^\delta\right)$ and $\widetilde{d} = \left(d^\alpha, d^\beta, d^\gamma, d^\delta\right)$ are trapezoidal fuzzy numbers given as $\widetilde{p} = \widetilde{c} \cdot \widetilde{d} = \left(p^\alpha, p^\beta, p^\gamma, p^\delta\right)$ where

$$
\begin{aligned}
p^\alpha &= \min\left\{c^\alpha \cdot d^\alpha, c^\alpha \cdot d^\delta, c^\delta \cdot d^\alpha, c^\delta \cdot d^\delta\right\}, \\
p^\beta &= \min\left\{c^\beta \cdot d^\beta, c^\beta \cdot d^\gamma, c^\gamma \cdot d^\beta, c^\gamma \cdot d^\gamma\right\}, \\
p^\gamma &= \max\left\{c^\beta \cdot d^\beta, c^\beta \cdot d^\gamma, c^\gamma \cdot d^\beta, c^\gamma \cdot d^\gamma\right\}, \\
p^\delta &= \max\left\{c^\alpha \cdot d^\alpha, c^\alpha \cdot d^\delta, c^\delta \cdot d^\alpha, c^\delta \cdot d^\delta\right\},
\end{aligned}
\tag{III.22}
$$

and $\widetilde{q} = \frac{\widetilde{c}}{\widetilde{d}} = \left(q^\alpha, q^\beta, q^\gamma, q^\delta\right), 0 \notin \left[d^\alpha, d^\delta\right]$, where

$$
\begin{aligned}
q^\alpha &= \min\left\{\frac{c^\alpha}{d^\alpha}, \frac{c^\alpha}{d^\delta}, \frac{c^\delta}{d^\alpha}, \frac{c^\delta}{d^\delta}\right\}, \\
q^\beta &= \min\left\{\frac{c^\beta}{d^\beta}, \frac{c^\beta}{d^\gamma}, \frac{c^\gamma}{d^\beta}, \frac{c^\gamma}{d^\gamma}\right\}, \\
q^\gamma &= \max\left\{\frac{c^\beta}{d^\beta}, \frac{c^\beta}{d^\gamma}, \frac{c^\gamma}{d^\beta}, \frac{c^\gamma}{d^\gamma}\right\}, \\
q^\delta &= \max\left\{\frac{c^\alpha}{d^\alpha}, \frac{c^\alpha}{d^\delta}, \frac{c^\delta}{d^\alpha}, \frac{c^\delta}{d^\delta}\right\},
\end{aligned}
\tag{III.23}
$$

respectively. Analogously, the reciprocal of a trapezoidal fuzzy number $\widetilde{c} = \left(c^\alpha, c^\beta, c^\gamma, c^\delta\right), c^\alpha > 0$, is a trapezoidal fuzzy number

$$
\frac{1}{\widetilde{c}} = \left(\frac{1}{c^\delta}, \frac{1}{c^\gamma}, \frac{1}{c^\beta}, \frac{1}{c^\alpha}\right),
\tag{III.24}
$$

and the $k-$th root of $\widetilde{c} = \left(c^\alpha, c^\beta, c^\gamma, c^\delta\right), c^\alpha > 0$, is a trapezoidal fuzzy number

$$
\sqrt[k]{\widetilde{c}} = \left(\sqrt[k]{c^\alpha}, \sqrt[k]{c^\beta}, \sqrt[k]{c^\gamma}, \sqrt[k]{c^\delta}\right).
\tag{III.25}
$$

In the fuzzy extension of the MCDM methods based on MPCMs and APCMs reviewed in Chapter II, only positive fuzzy numbers are present. This enables us to further simplify the formulas (III.22) and (III.23). For $\widetilde{c} = \left(c^\alpha, c^\beta, c^\gamma, c^\delta\right) \in \mathcal{F}_N(\mathbb{R}^+)$ and $\widetilde{d} = \left(d^\alpha, d^\beta, d^\gamma, d^\delta\right) \in \mathcal{F}_N(\mathbb{R}^+)$, the formulas (III.22) and (III.23) are simplified to

$$\widetilde{c} \cdot \widetilde{d} = \left(c^\alpha \cdot d^\alpha, c^\beta \cdot d^\beta, c^\gamma \cdot d^\gamma, c^\delta \cdot d^\delta\right), \tag{III.26}$$

and

$$\frac{\widetilde{c}}{\widetilde{d}} = \left(\frac{c^\alpha}{d^\delta}, \frac{c^\beta}{d^\gamma}, \frac{c^\gamma}{d^\beta}, \frac{c^\delta}{d^\alpha}\right), \tag{III.27}$$

respectively.

The product, the quotient, the reciprocal, and the $k-$th root of trapezoidal fuzzy numbers (III.22)–(III.25), respectively, obtained by simplified standard fuzzy arithmetic have the same representing values as the results of these arithmetic operations obtained by properly applying extension principles (III.18) and (III.19), respectively, in standard fuzzy arithmetic. That means that the support and the core of the results of these arithmetic operations are determined correctly by applying simplified standard fuzzy arithmetic, and the left and right sides of the resulting fuzzy numbers are approximated by linear functions. This approximation is generally accepted as sufficient in the literature on the fuzzy extension of the MCDM methods based on PCMs.

The following example is provided in order to demonstrate better the difference between the standard fuzzy arithmetic and the simplified standard fuzzy arithmetic.

Example 25. Let us consider the trapezoidal fuzzy number $\widetilde{c} = (2, 3, 4, 6)$ and the triangular fuzzy number $\widetilde{d} = (1, 2.5, 3)$. Clearly, $\widetilde{d}$ can be written as a trapezoidal fuzzy number in the form $\widetilde{d} = (1, 2.5, 2.5, 3)$. By applying the formulas (III.20), (III.21), and (III.24)–(III.27) based on simplified standard fuzzy arithmetic to the computation with the trapezoidal fuzzy numbers $\widetilde{c}$ and $\widetilde{d}$, we obtain the trapezoidal fuzzy numbers

$$\begin{aligned}
\widetilde{c} + \widetilde{d} &= (3, 5.5, 6.5, 9), & \widetilde{c} - \widetilde{d} &= (-1, 0.5, 1.5, 5), \\
\widetilde{c} \cdot \widetilde{d} &= (2, 7.5, 10, 18), & \frac{\widetilde{c}}{\widetilde{d}} &= \left(\frac{2}{3}, \frac{6}{5}, \frac{8}{5}, 6\right), \\
\frac{1}{\widetilde{c}} &= \left(\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right), & \sqrt[2]{\widetilde{c}} &= (\sqrt[2]{2}, \sqrt[2]{3}, 2, \sqrt[2]{6}).
\end{aligned}$$

The resulting trapezoidal fuzzy numbers are represented in Fig. III.6 together with the actual results of the arithmetic operations given by the extension principles (III.18) and (III.19). The actual results of the arithmetic operations are given by a dotted line and their trapezoidal approximations are given by a solid line. Notice that the sum and the difference of the trapezoidal fuzzy numbers $\widetilde{c}$ and $\widetilde{d}$ obtained by using standard fuzzy arithmetic are again trapezoidal fuzzy numbers. In fact, as it is obvious from Fig. III.6, the results obtained by using standard fuzzy arithmetic coincide with the results obtained by using simplified standard fuzzy arithmetic. Contrarily, the results of other four arithmetic operations obtained by applying standard fuzzy arithmetic are not trapezoidal fuzzy numbers anymore. Nevertheless, as it is obvious from Fig. III.6, the trapezoidal approximations of these results obtained by applying simplified standard fuzzy arithmetic have the same support and the same core as the actual results obtained by applying standard fuzzy arithmetic. $\triangle$

Since intervals are a particular case of trapezoidal fuzzy numbers, the arithmetic operations with intervals are performed according to the formulas (III.20)–(III.25) as well. Recall that interval $\overline{c} = [c^L, c^U]$ can be easily written as trapezoidal fuzzy number $\widetilde{c} = (c^L, c^L, c^U, c^U)$. However, intervals are also a particular class of crisp sets on $\mathbb{R}$ with a well-defined interval arithmetic for performing arithmetic operations on intervals. The interval arithmetic allows us to perform arithmetic operations on intervals in a much simpler way than the fuzzy arithmetic does. Nevertheless, both interval arithmetic and standard fuzzy arithmetic applied to intervals provide the same results. Unlike for the case of triangular and trapezoidal fuzzy numbers, the results of arithmetic operations with intervals are again intervals. Thus, there is no need for defining simplified standard fuzzy (interval) arithmetic for intervals. Let $\overline{u} = [u^L, u^U]$ and $\overline{v} = [v^L, v^U]$ be two positive intervals, i.e. $u^L > 0, v^L > 0$. Then the arithmetic operations are defined by using standard interval arithmetic as

$$\overline{u} + \overline{v} = [u^L + v^L, u^U + v^U], \tag{III.28}$$

$$\overline{u} - \overline{v} = [u^L - v^U, u^U - v^L], \tag{III.29}$$

$$\overline{u} \cdot \overline{v} = [u^L \cdot v^L, u^U \cdot v^U], \tag{III.30}$$

$$\frac{\overline{u}}{\overline{v}} = \left[\frac{u^L}{v^U}, \frac{u^U}{v^L}\right], \tag{III.31}$$

Figure III.6: Arithmetic operations with trapezoidal fuzzy numbers.
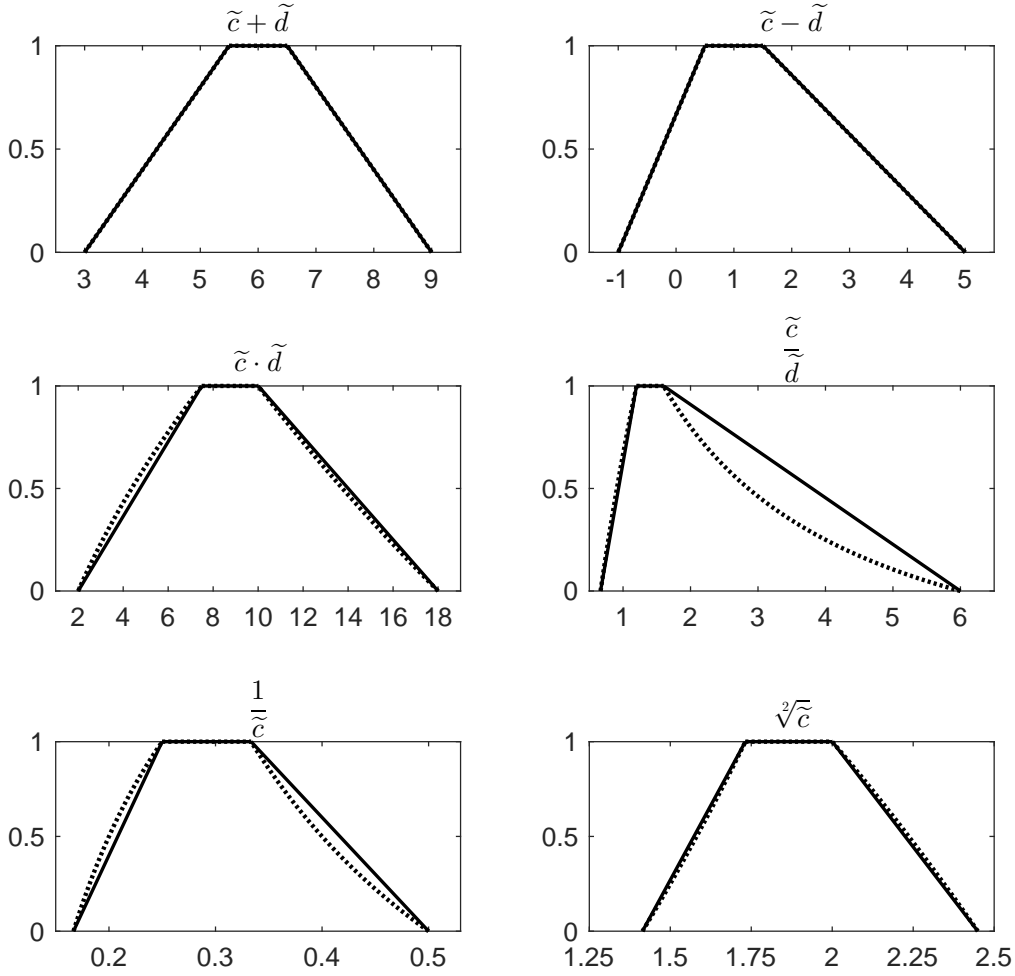
$$\frac{1}{\widetilde{u}} = \left[\frac{1}{u^U}, \frac{1}{u^L}\right], \tag{III.32}$$

$$\sqrt[k]{\widetilde{u}} = \left[\sqrt[k]{u^L}, \sqrt[k]{u^U}\right]. \tag{III.33}$$

The extension of single arithmetic operations on fuzzy numbers to functions combining the arithmetic operations is straightforward.

Definition 34. *Let* $\widetilde{n}_i \in \mathcal{F}_N(\mathbb{R}), i = 1, \ldots, k$, *and let* $f : \mathbb{R}^k \to \mathbb{R}$ *be a function defined by means of any combination of arithmetic operations,* $z = f(x_1, \ldots, x_k)$. *Then the extension of the function* f *to fuzzy numbers,* $f \colon \mathcal{F}_N(\mathbb{R})^k \to \mathcal{F}_N(\mathbb{R})$, *is defined as* $\widetilde{n} = f(\widetilde{n}_1, \ldots, \widetilde{n}_k)$ *with the membership function*

$$\widetilde{n}(z) = \begin{cases} \sup\left\{\min\left\{\widetilde{n}_1(x_1), \ldots, \widetilde{n}_k(x_k)\right\}; \ (x_1, \ldots, x_k) \in \mathbb{R}^k : z = f(x_1, \ldots, x_k)\right\} \\ \qquad \textit{if} \ \left\{(x_1, \ldots, x_k) \in \mathbb{R}^k; \ z = f(x_1, \ldots, x_k)\right\} \neq \emptyset, \\ 0, \qquad \qquad \textit{otherwise.} \end{cases} \tag{III.34}$$

Definition 35. *Let* $\widetilde{n}_i \in \mathcal{F}_N(\mathbb{R})$ *be given by their* $\alpha-$*cuts as* $\widetilde{n}_i = \bigcup_{\alpha=0}^{1} \alpha[n_{i(\alpha)}^L, n_{i(\alpha)}^U], i = 1, \ldots, k$. *Further, let* $f : \mathbb{R}^k \to \mathbb{R}$ *be a function defined by means of any combination of arithmetic operations,* $z = f(x_1, \ldots, x_k)$. *Then the extension of the function* f *to fuzzy numbers,* $f : \mathcal{F}_N(\mathbb{R})^k \to \mathcal{F}_N(\mathbb{R})$, *is defined as* $\widetilde{n} = f(\widetilde{n}_1, \ldots, \widetilde{n}_k)$

with the $\alpha-cut$ representation $\widetilde{n} = \bigcup_{\alpha=0}^{1} \alpha[n_{(\alpha)}^L, n_{(\alpha)}^U]$:

$$n_{(\alpha)}^L = \min \left\{ f(x_1, \ldots, x_k); \ x_i \in [n_{i(\alpha)}^L, n_{i(\alpha)}^U], \ i = 1, \ldots, k \right\},$$

$$n_{(\alpha)}^U = \max \left\{ f(x_1, \ldots, x_k); \ x_i \in [n_{i(\alpha)}^L, n_{i(\alpha)}^U], \ i = 1, \ldots, k \right\}. \qquad \text{(III.35)}$$

The result of applying standard fuzzy arithmetic in Definitions 34 and 35 to the computations with trapezoidal fuzzy numbers can be approximated according to simplified standard fuzzy arithmetic by a trapezoidal fuzzy number $\widetilde{n} = (n^\alpha, n^\beta, n^\gamma, n^\delta)$:

$$n^\alpha = \min \left\{ f(x_1, \ldots, x_k); \ x_i \in [n_i^\alpha, n_i^\delta], \ i = 1, \ldots, k \right\},$$

$$n^\beta = \min \left\{ f(x_1, \ldots, x_k); \ x_i \in [n_i^\beta, n_i^\gamma], \ i = 1, \ldots, k \right\},$$

$$n^\gamma = \max \left\{ f(x_1, \ldots, x_k); \ x_i \in [n_i^\beta, n_i^\gamma], \ i = 1, \ldots, k \right\}, \qquad \text{(III.36)}$$

$$n^\delta = \max \left\{ f(x_1, \ldots, x_k); \ x_i \in [n_i^\alpha, n_i^\delta], \ i = 1, \ldots, k \right\}.$$

3.5 Constrained fuzzy arithmetic

Constrained fuzzy arithmetic was introduced by Klir (1997) and Klir and Pan (1998) to handle correctly arithmetic operations on fuzzy numbers in the presence of constraints on operands. "When arithmetic operations are performed on real numbers, they follow unique rules that are independent of what is represented by the numbers involved. That is, the result of each particular arithmetic operation on real numbers depends only on the numbers involved and not on the entities represented by the numbers. As it is well known, the validity of this simple principle is also tacitly assumed in the usual interval arithmetic as well as fuzzy arithmetic" (Klir (1997), p. 167). Klir (1997) and Klir and Pan (1998) argued that this principle is valid neither for interval arithmetic nor for fuzzy arithmetic. Contrarily to standard arithmetic on real numbers, interval arithmetic and fuzzy arithmetic depend not only on the intervals or fuzzy numbers involved, but also on their meanings that impose constraints on the operands. These constraints often appear when different operands represent the same linguistic variable or when there are any relations among the operands.

Example 26. Let us consider function $f(x) = x - x$. Clearly, $f(x) = 0$ for any $x \in \mathbb{R}$ since both operands in the subtraction are the same.

Now let us examine the extension of the function f to fuzzy numbers $\widetilde{n} \in \mathcal{F}_N(\mathbb{R})$. According to standard fuzzy arithmetic (in particular Definition 30) the fuzzy extension of the function f is given as $\widetilde{m} = f(\widetilde{n}) = \widetilde{n} - \widetilde{n}$ with the membership function

$$\widetilde{m}(z) = \begin{cases} \sup \left\{ \min \left\{ \widetilde{n}(x), \widetilde{n}(y) \right\}; \ (x, y) \in \mathbb{R}^2 : z = x - y \right\} \\ \qquad \qquad \text{if } \left\{ (x, y) \in \mathbb{R}^2; \ z = x - y \right\} \neq \emptyset, \\ 0, \qquad \qquad \text{otherwise.} \end{cases} \qquad \text{(III.37)}$$

For trapezoidal fuzzy numbers $\widetilde{n} = (n^\alpha, n^\beta, n^\gamma, n^\delta) \in \mathcal{F}_N(\mathbb{R})$ in particular, we obtain

$$\widetilde{m} = f(\widetilde{n}, \widetilde{n}) = \widetilde{n} - \widetilde{n} = (n^\alpha - n^\delta, n^\beta - n^\gamma, n^\gamma - n^\beta, n^\delta - n^\alpha) \neq 0. \qquad \text{(III.38)}$$

Let us consider a trivial example: We have a bottle of water and we drink it all. How much is left? Nothing, right? How did we arrive to this simple solution? The problem can be solved by using the function f defined above. When there is 1 l of water and we drink it all then there is $f(1) = 1 - 1 = 0$ l left. When there is 650 ml and we drink it all then there is again $f(650) = 650 - 650 = 0$ ml left.

Now let us consider we do not know precisely the amount of water in the bottle. Let us say there is about 1 l, which can be described by the triangular fuzzy number $\widetilde{n} = (0.95, 1, 1.05)$, for example. How much will be now left when we drink it all? The common sense again suggests that nothing should be left. But if we apply fuzzy arithmetic to find the solution of this simple problem, we get into difficulties. By using the formula (III.38) we obtain $f(\widetilde{n}) = \widetilde{n} - \widetilde{n} = (0.95 - 1.05, 1 - 1, 1.05 - 0.95) = (-0.1, 0, 0.1)$. So according to the standard fuzzy arithmetic there should be something between -0.1 l and 0.1 l left. This conclusion is obviously nonsensical.

The problem is that we did not consider the meaning of the operands in the subtraction. From the definition of the problem, it is clear that the operands of the subtraction are in interaction. In particular, only the same values of operands are admissible; whatever the amount of water in the bottle is, we drink this exact amount. This constraint has to be considered in the computations. $\triangle$

Klir (1997) and Klir and Pan (1998) examined various types of constraints on operands in constrained fuzzy arithmetic, in particular equality, inequality, and probabilistic constraints. In this thesis, only equality constraints are of interest.

Definition 36. *Let* $\widetilde{n}_1, \widetilde{n}_2 \in \mathcal{F}_N(\mathbb{R})$ *and let* $* : \mathbb{R}^2 \to \mathbb{R}$ *be a binary arithmetic operation,* $z = x * y$. *Further, let the equality constraint* $g(x, y) = 0$ *express an interaction between the operands. Then, considering the constraint* $g(x, y) = 0$ *on the values of the operands, the extension of the arithmetic operation* $*$ *to fuzzy numbers,* $* : \mathcal{F}_N(\mathbb{R})^2 \to \mathcal{F}(\mathbb{R})$, *is defined as* $\widetilde{n} = \widetilde{n}_1 * \widetilde{n}_2$ *with the membership function*

$$\widetilde{n}(z) = \begin{cases} \sup \left\{ \min \left\{ \widetilde{n}_1(x), \widetilde{n}_2(y) \right\}; \ (x, y) \in \mathbb{R}^2 : z = x * y, \ g(x, y) = 0 \right\} \\ \qquad \text{if } \left\{ (x, y) \in \mathbb{R}^2; \ z = x * y, \ g(x, y) = 0 \right\} \neq \emptyset, \\ 0, \qquad\qquad \text{otherwise.} \end{cases} \tag{III.39}$$

Note that the result of constrained fuzzy arithmetic given by the extension principle (III.39) on fuzzy numbers is a fuzzy set but not a fuzzy number in general. This is caused by the presence of the interaction constraint $g(x, y) = 0$. To guarantee that the result of constrained fuzzy arithmetic applied to fuzzy numbers is again a fuzzy number, i.e. that $*$ in Definition 36 is such that $* : \mathcal{F}_N(\mathbb{R})^2 \to \mathcal{F}_N(\mathbb{R})$, further requirements on the constraint $g(x, y) = 0$ have to be imposed.

Some constraining requirements were given by Klir and Pan (1998). In this thesis, only a particular type of equality constraint $g(x, y) = 0$ is needed to extend appropriately the formulas defined for PCMs in Chapter II to fuzzy PCMs. The equality constraints of the type $g(x, y) = 0$ that are going to be applied in Chapter IV are such that $G = \left\{ (x, y) \in \mathbb{R}^2; \ g(x, y) = 0 \right\}$ is a connected set[2]. Being G a connected set, it is sufficient to require

$$\left\{ (x, y) \in \mathbb{R}^2; \ \widetilde{n}_1(x) = 1, \ \widetilde{n}_2(y) = 1, \ g(x, y) = 0 \right\} \neq \emptyset$$

to guarantee that the result of the constrained fuzzy arithmetic on fuzzy numbers given by the extension principle (III.39) is again a fuzzy number.

Analogously to standard fuzzy arithmetic, the $\alpha-$cut representation of fuzzy numbers can be conveniently used also to define constrained fuzzy arithmetic.

Definition 37. *Let* $\widetilde{n}_1, \widetilde{n}_2 \in \mathcal{F}_N(\mathbb{R})$ *be given by their* $\alpha-$*cuts as* $\widetilde{n}_1 = \bigcup_{\alpha=0}^1 \alpha[n_{1(\alpha)}^L, n_{1(\alpha)}^U]$, $\widetilde{n}_2 = \bigcup_{\alpha=0}^1 \alpha[n_{2(\alpha)}^L, n_{2(\alpha)}^U]$. *Further, let* $* : \mathbb{R}^2 \to \mathbb{R}$ *be a binary arithmetic operation,* $z = x * y$, *and let the equality constraint* $g(x, y) = 0$ *express an interaction between the operands. If* $G = \left\{ (x, y) \in \mathbb{R}^2; \ g(x, y) = 0 \right\}$ *is a connected set and if* $\left\{ (x, y) \in \mathbb{R}^2; \ \widetilde{n}_1(x) = 1, \ \widetilde{n}_2(y) = 1, \ g(x, y) = 0 \right\} \neq \emptyset$, *then the extension of the arithmetic operation* $*$ *to fuzzy numbers,* $* : \mathcal{F}_N(\mathbb{R})^2 \to \mathcal{F}_N(\mathbb{R})$, *is defined as* $\widetilde{n} = \widetilde{n}_1 * \widetilde{n}_2$ *with the* $\alpha-$*cut representation* $\widetilde{n} = \bigcup_{\alpha=0}^1 \alpha[n_{(\alpha)}^L, n_{(\alpha)}^U]$:

$$\begin{aligned} n_{(\alpha)}^L &= \min \left\{ x * y; \ x \in [n_{1(\alpha)}^L, n_{1(\alpha)}^U], \ y \in [n_{2(\alpha)}^L, n_{2(\alpha)}^U], \ g(x, y) = 0 \right\}, \\ n_{(\alpha)}^U &= \max \left\{ x * y; \ x \in [n_{1(\alpha)}^L, n_{1(\alpha)}^U], \ y \in [n_{2(\alpha)}^L, n_{2(\alpha)}^U], \ g(x, y) = 0 \right\}. \end{aligned} \tag{III.40}$$

As already mentioned in the previous section, simplified standard fuzzy arithmetic is commonly used in fuzzy MCDM based on fuzzy PCMs in order to keep the computational procedure simple. The results of arithmetic operations with triangular or trapezoidal fuzzy numbers are thus still triangular or trapezoidal fuzzy numbers, respectively, whose supports and cores correspond to the supports and cores of the actual results of the arithmetic operations determined precisely by applying extension principles (III.39) and (III.40). Recall that there is no need for simplified fuzzy arithmetic to perform arithmetic operations on intervals since the results are always intervals.

In order to be consistent with this approach, it is necessary to apply the simplified version of fuzzy arithmetic also when there appear any constraints on the operands in the computational procedure. This basically means that we want to reflect the constraints given on operands in the outcome of the computation, but, at the same time, we want to approximate the outcome by a triangular or trapezoidal fuzzy number, respectively, to keep the computational procedure as simple as possible still obtaining reliable results. Simplified constrained fuzzy arithmetic, a combination of simplified fuzzy arithmetic and constrained fuzzy arithmetic, is thus needed.

According to the simplified constrained fuzzy arithmetic, for two trapezoidal fuzzy numbers $\widetilde{n}_1 = (n_1^\alpha, n_1^\beta, n_1^\gamma, n_1^\delta)$, $\widetilde{n}_2 = (n_2^\alpha, n_2^\beta, n_2^\gamma, n_2^\delta) \in \mathcal{F}_N(\mathbb{R})$, only the representing values $n^\alpha, n^\beta, n^\gamma, n^\delta$ of the resulting fuzzy number $\widetilde{n} = \widetilde{n}_1 * \widetilde{n}_2$ are computed. However, unlike in the case of simplified standard fuzzy arithmetic, the representing values of $\widetilde{n}$ are not obtained by performing the arithmetic operation $*$ on the representing values of the fuzzy numbers $\widetilde{n}_1, \widetilde{n}_2$. The formulas for performing arithmetic operations on trapezoidal fuzzy numbers based on the simplified constrained fuzzy arithmetic are more complex than the formulas based on the simplified standard fuzzy arithmetic; solving optimization problems is necessary to obtain the representing values of the resulting trapezoidal fuzzy number $\widetilde{n} = (n^\alpha, n^\beta, n^\gamma, n^\delta)$.

---

[2]Set G is connected if it cannot be divided into two disjoint closed sets.

Let $* : \mathbb{R}^2 \to \mathbb{R}$ be an arithmetic operation, $z = x * y$, and let $g(x,y) = 0$ represent a constraint imposed on the operands, $G = \left\{ (x,y) \in \mathbb{R}^2; \; g(x,y) = 0 \right\}$ being a connected set. Further, let $\widetilde{n}_1 = (n_1^\alpha, n_1^\beta, n_1^\gamma, n_1^\delta), \widetilde{n}_2 = (n_2^\alpha, n_2^\beta, n_2^\gamma, n_2^\delta) \in \mathcal{F}_N(\mathbb{R})$ be trapezoidal fuzzy numbers such that $\{ (x,y) \in \mathbb{R}^2; \; \widetilde{n}_1(x) = 1, \; \widetilde{n}_2(y) = 1, \; g(x,y) = 0 \} \neq \emptyset$. Then the fuzzy extension of the arithmetic operation $*$ to trapezoidal fuzzy numbers $\widetilde{n}_1$ and $\widetilde{n}_2$ based on the simplified constrained fuzzy arithmetic is defined as $\widetilde{n} = \widetilde{n}_1 * \widetilde{n}_2$ with the representation $\widetilde{n} = (n^\alpha, n^\beta, n^\gamma, n^\delta)$:

$$
\begin{aligned}
n^\alpha &= \min \left\{ x * y; \; x \in [n_1^\alpha, n_1^\delta], \; y \in [n_2^\alpha, n_2^\delta], \; g(x,y) = 0 \right\}, \\
n^\beta &= \min \left\{ x * y; \; x \in [n_1^\beta, n_1^\gamma], \; y \in [n_2^\beta, n_2^\gamma], \; g(x,y) = 0 \right\}, \\
n^\gamma &= \max \left\{ x * y; \; x \in [n_1^\beta, n_1^\gamma], \; y \in [n_2^\beta, n_2^\gamma], \; g(x,y) = 0 \right\}, \\
n^\delta &= \max \left\{ x * y; \; x \in [n_1^\alpha, n_1^\delta], \; y \in [n_2^\alpha, n_2^\delta], \; g(x,y) = 0 \right\}.
\end{aligned}
$$
(III.41)

Difference between standard and constrained fuzzy arithmetic is demonstrated on the following illustrative example.

Example 27. Let us consider trapezoidal fuzzy number $\widetilde{c} = (0.2, 0.3, 0.4, 0.6)$ and let us compute trapezoidal fuzzy number $\widetilde{d}$ as $\widetilde{d} = 1 - \widetilde{c} = (0.4, 0.6, 0.7, 0.8)$. Clearly, trapezoidal fuzzy numbers $\widetilde{c}$ and $\widetilde{d}$ are in relation; to any value $x \in Cl(Supp \, \widetilde{c})$ corresponds a value $y \in Cl(Supp \, \widetilde{d})$ such that $y = 1 - x$.

Let us compute the product $\widetilde{e} = \widetilde{c} \cdot \widetilde{d}$. By applying the simplified standard fuzzy arithmetic, and in particular formula (III.26), we obtain trapezoidal fuzzy number $\widetilde{e}_{SS} = (0.08, 0.18, 0.28, 0.48)$. In Fig. III.7, you can compare this trapezoidal approximation with the actual outcome $\widetilde{e}_S$ of the multiplication obtainable by applying standard fuzzy arithmetic (III.18).

The relation $y = 1 - x$ between the operands represented by trapezoidal fuzzy numbers $\widetilde{c}$ and $\widetilde{d}$ was not taken into account when computing their product by using the formulas (III.26) and (III.18). Thus, the obtained results $\widetilde{e}_{SS}$ and $\widetilde{e}_S$ are both imprecise, too vague. In order to eliminate the excessive vagueness, it is necessary to apply properly the constrained fuzzy arithmetic (III.40) or, alternatively, the simplified constrained fuzzy arithmetic (III.41).

First, let us verify the requirements for the constraint on the operands. The constraint $y = 1 - x$ can be written as $g(x,y) = x + y - 1 = 0$. The set $G = \left\{ (x,y) \in \mathbb{R}^2; \; x + y - 1 = 0 \right\}$ is clearly connected and $\{ (x,y) \in \mathbb{R}^2; \; \widetilde{c}(x) = 1, \; \widetilde{d}(y) = 1, x + y - 1 = 0 \} = \{ (x, 1 - x) \in [0.3, 0.4] \times [0.6, 0.7] \} \neq \emptyset$. This guarantees that the result of the constrained fuzzy arithmetic applied to $\widetilde{c}$ and $\widetilde{d}$ is again a fuzzy number. By applying (III.41) with the function $f : \mathbb{R}^2 \to \mathbb{R}$ in the form $f(x,y) = x \cdot y$ and the constraint $g(x,y) = 0$ in the form $x + y - 1 = 0$, we obtain trapezoidal fuzzy number $\widetilde{e}_{SC} = (e^\alpha, e^\beta, e^\gamma, e^\delta)$:

$$
\begin{aligned}
e^\alpha &= \min \left\{ x \cdot y; \; x \in [0.2, 0.6], \; y \in [0.4, 0.8], \; x + y - 1 = 0 \right\} = \\
&\quad \min \left\{ x(1-x); \; x \in [0.2, 0.6] \right\} = 0.16, \\
e^\beta &= \min \left\{ x \cdot y; \; x \in [0.3, 0.4], \; y \in [0.6, 0.7], \; x + y - 1 = 0 \right\} = \\
&\quad \min \left\{ x(1-x); \; x \in [0.3, 0.4] \right\} = 0.21, \\
e^\gamma &= \max \left\{ x \cdot y; \; x \in [0.3, 0.4], \; y \in [0.6, 0.7], \; x + y - 1 = 0 \right\} = \\
&\quad \max \left\{ x(1-x); \; x \in [0.3, 0.4] \right\} = 0.24, \\
e^\delta &= \max \left\{ x \cdot y; \; x \in [0.2, 0.6], \; y \in [0.4, 0.8], \; x + y - 1 = 0 \right\} = \\
&\quad \max \left\{ x(1-x); \; x \in [0.2, 0.6] \right\} = 0.25.
\end{aligned}
$$
(III.42)

Figure III.7: Product of $\widetilde{c}$ and $\widetilde{d}$ obtained by standard fuzzy arithmetic and by simplified standard fuzzy arithmetic.
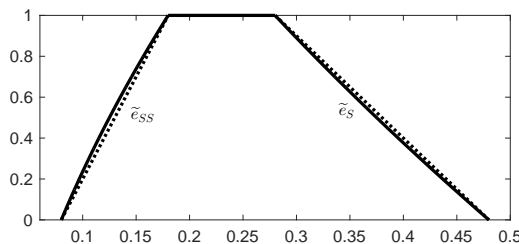
Figure III.8: Product of $\widetilde{c}$ and $\widetilde{d}$ obtained by constrained fuzzy arithmetic and by simplified constrained fuzzy arithmetic.
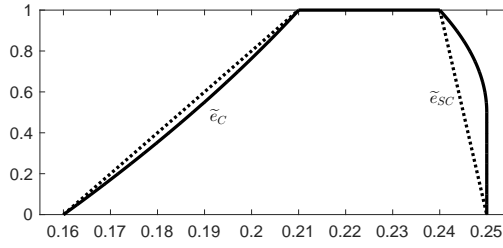
Figure III.9: Product of $\widetilde{c}$ and $\widetilde{d}$ obtained by constrained and standard fuzzy arithmetics.
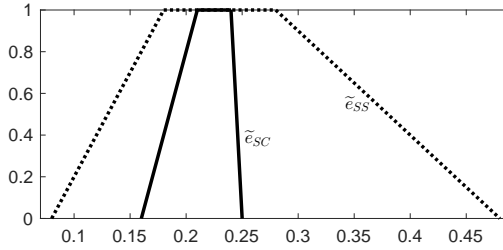
Again, trapezoidal approximation $\widetilde{e}_{SC}$ is displayed in Fig. III.8 together with the actual outcome $\widetilde{e}_C$ of the constrained fuzzy arithmetic performed by applying (III.40).

The product $\widetilde{e}_{SC} = (0.16, 0.21, 0.24, 0.25)$ obtained by simplified constrained fuzzy arithmetic is significantly less vague than the product $\widetilde{e}_{SS} = (0.08, 0.18, 0.28, 0.48)$ obtained by simplified standard fuzzy arithmetic. The difference in vagueness of both trapezoidal fuzzy numbers is even more noticeable from graphical representation, see Fig. III.9. It is clearly visible from the figure how significant the reduction of vagueness is when applying properly (simplified) constrained fuzzy arithmetic instead of (simplified) standard fuzzy arithmetic. Therefore, in order to obtain more reliable results, it is indispensable to take into account all relations between operands when performing arithmetic operations on fuzzy numbers.

$\triangle$

Example 28. In Example 26, a simple problem that cannot be solved by using standard fuzzy arithmetic was shown. Let us now apply constrained fuzzy arithmetic to the same problem. As already argued in Example 26, the operands in the subtraction $f(\widetilde{n}, \widetilde{n}) = \widetilde{n} - \widetilde{n}$ are in interaction - they are equal. This interaction can be modeled by the function $g(x, y) = x - y = 0$.

The set $G = \left\{ (x, y) \in \mathbb{R}^2; \ g(x, y) = x - y = 0 \right\}$ is connected and, further, $\{(x, y) \in \mathbb{R}^2; \ \widetilde{n}(x) = 1, \ \widetilde{n}(y) = 1, \ g(x, y) = x - y = 0\} = \left\{ (x, x) \in \mathbb{R}^2; \ x \in Core\,\widetilde{n} \right\} \neq \emptyset$ for $\widetilde{n} \in \mathcal{F}_N(\mathbb{R})$. Applying the extension principle (III.40) to the function $f(\widetilde{n}) = \widetilde{n} - \widetilde{n}$ with the entry $\widetilde{n} = (0.95, 1, 1.05)$, we obtain $\widetilde{m} = (m^L, m^M, m^U) = \widetilde{n} - \widetilde{n}$:

$$
\begin{aligned}
m^L =& \min \left\{ x - y; \ x \in [0.95, 1.05], \ y \in [0.95, 1.05], \ x - y = 0 \right\} = \\
& \min \left\{ x - x; \ x \in [0.95, 1.05] \right\} = 0, \\
m^M =& \min \left\{ x - y; \ x \in [1, 1], \ y \in [1, 1], \ x - y = 0 \right\} = \\
& \min \left\{ x - x; \ x = 1 \right\} = 0, \\
m^U =& \max \left\{ x - y; \ x \in [0.95, 1.05], \ y \in [0.95, 1.05], \ x - y = 0 \right\} = \\
& \max \left\{ x - x; \ x \in [0.95, 1.05] \right\} = 0.
\end{aligned}
$$

Thus, we finally get the correct solution to our problem. Namely, when there is about 1 l of water ($\widetilde{n} = (0.95, 1, 1.05)$) in the bottle and we drink it all, then nothing is left; $\widetilde{n} - \widetilde{n} = (0.95, 1, 1.05) - (0.95, 1, 1.05) = 0$. $\triangle$

The extension of single arithmetic operations with constraints on operands to functions combining arithmetic operations is straightforward.

Definition 38. *Let $\widetilde{n}_i \in \mathcal{F}_N(\mathbb{R}), i = 1, \ldots, k$. Further, let $f : \mathbb{R}^k \to \mathbb{R}$ be a function defined by means of any combination of arithmetic operations, $z = f(x_1, \ldots, x_k)$, and let the equality constraints $g_j(x_1, \ldots, x_k) = 0$, $j = 1, \ldots, l$, express interactions between the operands. If $G_j = \left\{ (x_1, \ldots, x_k) \in \mathbb{R}^k; \ g_j(x_1, \ldots, x_k) = 0 \right\}, \ j = 1, \ldots, l$, are connected sets and if $\left\{ (x_1, \ldots, x_k) \in \mathbb{R}^k; \ \widetilde{n}_i(x_i) = 1, \ i = 1, \ldots, k, \ g_j(x_1, \ldots, x_k) = 0, \ j = 1, \ldots, l \right\} \neq$*

$\emptyset$, *then the extension of the function f to fuzzy numbers, $f : \mathcal{F}_N(\mathbb{R})^k \to \mathcal{F}_N(\mathbb{R})$, is defined as $\widetilde{n} = f(\widetilde{n}_1, \ldots, \widetilde{n}_k)$ with the membership function*

$$\widetilde{n}(z) = \begin{cases} \sup \left\{ \min \{\widetilde{n}_1(x_1), \ldots, \widetilde{n}_k(x_k)\} ; \begin{array}{l} (x_1, \ldots, x_k) \in \mathbb{R}^k : z = f(x_1, \ldots, x_k), \\ g_j(x_1, \ldots, x_k) = 0, \ j = 1, \ldots, l \end{array} \right\} \\ \qquad\qquad\quad \text{if } \left\{ \begin{array}{l} (x_1, \ldots, x_k) \in \mathbb{R}^k; \ z = f(x_1, \ldots, x_k), \\ g_j(x_1, \ldots, x_k) = 0, \ j = 1, \ldots, l \end{array} \right\} \neq \emptyset, \\ 0, \qquad\qquad\quad\ \ \text{otherwise.} \end{cases} \tag{III.43}$$

Definition 39. *Let $\widetilde{n}_i \in \mathcal{F}_N(\mathbb{R})$ be given by their $\alpha-$cuts as $\widetilde{n}_i = \bigcup_{\alpha=0}^{1} \alpha[n_{i(\alpha)}^L, n_{i(\alpha)}^U], \ i = 1, \ldots, k$. Further, let $f : \mathbb{R}^k \to \mathbb{R}$ be a function defined by means of any combination of arithmetic operations, $z = f(x_1, \ldots, x_k)$, and let the equality constraints $g_j(x_1, \ldots, x_k) = 0, \ j = 1, \ldots, l$, express interactions between the operands. If $G_j = \{(x_1, \ldots, x_k) \in \mathbb{R}^k; \ g_j(x_1, \ldots, x_k) = 0\}, \ j = 1, \ldots, l$, are connected sets and if $\{(x_1, \ldots, x_k) \in \mathbb{R}^k; \ \widetilde{n}_i(x_i) = 1, \ i = 1, \ldots, k, \ g_j(x_1, \ldots, x_k) = 0, \ j = 1, \ldots, l\} \neq \emptyset$, then the extension of the function f to fuzzy numbers, $f : \mathcal{F}_N(\mathbb{R})^k \to \mathcal{F}_N(\mathbb{R})$, is defined as $\widetilde{n} = f(\widetilde{n}_1, \ldots, \widetilde{n}_k)$ with the $\alpha-$cut representation $\widetilde{n} = \bigcup_{\alpha=0}^{1} \alpha[n_{(\alpha)}^L, n_{(\alpha)}^U] :$*

$$\begin{aligned} n_{(\alpha)}^L &= \min \left\{ f(x_1, \ldots, x_k); \begin{array}{l} x_i \in [n_{i(\alpha)}^L, n_{i(\alpha)}^U], \ i = 1, \ldots, k, \\ g_j(x_1, \ldots, x_k) = 0, \ j = 1, \ldots, l \end{array} \right\}, \\ n_{(\alpha)}^U &= \max \left\{ f(x_1, \ldots, x_k); \begin{array}{l} x_i \in [n_{i(\alpha)}^L, n_{i(\alpha)}^U], \ i = 1, \ldots, k, \\ g_j(x_1, \ldots, x_k) = 0, \ j = 1, \ldots, l \end{array} \right\}. \end{aligned} \tag{III.44}$$

The result of applying constrained fuzzy arithmetic in Definitions 38 and 39 to the computations with trapezoidal fuzzy numbers can be approximated according to simplified constrained fuzzy arithmetic by a trapezoidal fuzzy number $\widetilde{n} = (n^\alpha, n^\beta, n^\gamma, n^\delta) :$

$$\begin{aligned} n^\alpha &= \min \left\{ f(x_1, \ldots, x_k); \begin{array}{l} x_i \in [n_i^\alpha, n_i^\delta], \ i = 1, \ldots, k, \\ g_j(x_1, \ldots, x_k) = 0, \ j = 1, \ldots, l \end{array} \right\}, \\ n^\beta &= \min \left\{ f(x_1, \ldots, x_k); \begin{array}{l} x_i \in [n_i^\beta, n_i^\gamma], \ i = 1, \ldots, k, \\ g_j(x_1, \ldots, x_k) = 0, \ j = 1, \ldots, l \end{array} \right\}, \\ n^\gamma &= \max \left\{ f(x_1, \ldots, x_k); \begin{array}{l} x_i \in [n_i^\beta, n_i^\gamma], \ i = 1, \ldots, k, \\ g_j(x_1, \ldots, x_k) = 0, \ j = 1, \ldots, l \end{array} \right\}, \\ n^\delta &= \max \left\{ f(x_1, \ldots, x_k); \begin{array}{l} x_i \in [n_i^\alpha, n_i^\delta], \ i = 1, \ldots, k, \\ g_j(x_1, \ldots, x_k) = 0, \ j = 1, \ldots, l \end{array} \right\}. \end{aligned} \tag{III.45}$$

The simplified constrained fuzzy arithmetic will be applied in the following chapter in order to preserve the reciprocity of the related PCs of objects in FPCMs. For the simplicity, the terms "standard fuzzy arithmetic" and "constrained fuzzy arithmetic" will be used hereafter. However, the terms will always refer to their simplified versions with results in the form of trapezoidal fuzzy numbers (triangular fuzzy numbers and intervals being special cases) if not specified otherwise.

Chapter IV

Fuzzy pairwise comparison matrices

Part of this chapter has been published in:

J. Krejčí: **"Obtaining fuzzy priorities from additive fuzzy pairwise comparison matrices"**, *IMA Journal of Management Mathematics*, First published online: May 2016, doi: 10.1093/imaman/dpw006.

J. Krejčí: **"Fuzzy eigenvector method for obtaining normalized fuzzy weights from fuzzy pairwise comparison matrices"**, *Fuzzy Sets and Systems*, 315(1):26-43, 2017.

J. Krejčí, O. Pavlačka and J. Talašová: **"A fuzzy extension of Analytic Hierarchy Process based on the constrained fuzzy arithmetic"**, *Fuzzy Optimization and Decision Making*, 16(1):89-110, 2017.

J. Krejčí: **"Additively reciprocal fuzzy pairwise comparison matrices and multiplicative fuzzy priorities"**, *Soft Computing*, 21(12):3177-3192, 2017.

M. Fedrizzi and J. Krejčí: **"A note on the paper 'Fuzzy Analytic Hierarchy Process: Fallacy of the Popular Methods"**, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 23(6), 2015, p. 965–970.

J. Krejčí and J. Stoklasa: **"Fuzzified AHP in the evaluation of scientific monographs"**, *Central European Journal of Operational Research* 24(2), 2015, p. 353–370.

J. Krejčí: **"Fuzzy maximal eigenvalues of fuzzy pairwise comparison matrices"**, *Proceedings of the 32nd International Conference on Mathematical Methods in Economics 2014*, Olomouc, Czech Republic, ISBN: 978-80-244-4209-9, 2014, p. 500–505.

4.1   Introduction

In Chapter II, crisp PCMs were studied in detail. However, as mentioned in Section 1.2.1, crisp PCs are not suitable for every MCDM problem. For example, when linguistic terms are used to provide intensities of preference or when the information about the problem is imprecise, fuzzy numbers seem to be more appropriate for expressing the PCs. In Chapter III, fuzzy numbers were defined together with all concepts indispensable for properly replacing crisp PCs in PCMs by fuzzy PCs in form of fuzzy numbers and for adapting the related methods accordingly. (The word "crisp" is used here to emphasize the distinction from "fuzzy".)

By a FPCM of n objects $o_1, \ldots, o_n$ we will understand a PCM whose elements are fuzzy numbers, i.e. $\widetilde{C} = \{\widetilde{c}_{ij}\}_{i,j=1}^n$, $\widetilde{c}_{ij} \in \mathcal{F}_N(\mathbb{R}), i, j = 1, \ldots, n$. Note that also a crisp PCM $C = \{c_{ij}\}_{i,j=1}^n$ is actually a FPCM since crisp numbers are a special case of fuzzy numbers.

In Section 2.1, two key properties of PCMs and of the relevant methods were identified - reciprocity of the related PCs and invariance of methods under permutation of objects. When extending crisp PCMs to FPCMs it is necessary to handle properly these two key properties.

As stated in Section 2.1, reciprocity is an inherent property of crisp PCMs resulting from the interpretation of the related PCs c_{ij} and c_{ji}. An appropriate extension of the reciprocity relation to FPCMs is of key importance for processing correctly the preference information contained in FPCMs and for deriving conclusions that reflect DM's preferences reliably. For this it is necessary to understand and interpret correctly the information contained in a FPCM. A FPCM $\widetilde{C} = \{\widetilde{c}_{ij}\}_{i,j=1}^n$ is not just a matrix with entries in form of fuzzy numbers that express uncertain preference information and that are in a reciprocity relation. It is necessary to look at the

FPCM $\widetilde{C} = \{\widetilde{c}_{ij}\}_{i,j=1}^n$ as a set of crisp PCMs $C = \{c_{ij}\}_{i,j=1}^n$ with different degrees of membership to the FPCM $\widetilde{C}$. Each such PCM $C = \{c_{ij}\}_{i,j=1}^n$ carries particular preference information and has all the properties of a PCM discussed in Chapter II. This fact has to be considered when extending to FPCMs the methods originally proposed for crisp PCMs.

Similarly as for crisp PCMs, there exists no canonical order for assigning the labels $o_1, \ldots, o_n$ to n objects. Thus, regardless of the permutation of a FPCM $\widetilde{C}$, the methods applied to FPCMs have to lead to the same results, i.e. they have to be invariant under permutation of objects. In particular, the invariance under permutation is required for definitions of consistency and inconsistency indices defined for FPCMs as well as for methods for deriving fuzzy priorities of objects from FPCMs.

Being P a permutation matrix, $\widetilde{C}^\pi = P\widetilde{C}P^T$ is a permutation of $\widetilde{C}$ associated with P. Further, let $\widetilde{\mathcal{C}}$ denote a certain class of PCMs. Then, invariance under permutation for definitions of consistency, inconsistency indices, and methods for deriving priorities of objects from FPCMs can be formally defined as follows.

Definition 40. *A definition of consistency for FPCMs in a certain class $\widetilde{\mathcal{C}}$ is invariant under permutation of objects if $\forall \widetilde{C} \in \widetilde{\mathcal{C}}$ the following holds:*

$$\widetilde{C} \text{ consistent} \Rightarrow P\widetilde{C}P^T \text{ consistent for every } P,$$

$$\widetilde{C} \text{ not consistent} \Rightarrow P\widetilde{C}P^T \text{ not consistent for any } P,$$

where P is a permutation matrix.

Definition 41. *An inconsistency index $I : \widetilde{\mathcal{C}} \to \mathcal{F}_N(\mathbb{R})$ defined on a certain class $\widetilde{\mathcal{C}}$ of FPCMs is invariant under permutation of objects if*

$$I(P\widetilde{C}P^T) = I(\widetilde{C}), \qquad \forall \widetilde{C} \in \widetilde{\mathcal{C}} \text{ and for any permutation matrix } P.$$

Definition 42. *Let a method for deriving fuzzy priorities $\underline{\widetilde{w}} = (\widetilde{w}_1, \ldots, \widetilde{w}_n)^T$ of objects from FPCMs in a certain class $\widetilde{\mathcal{C}}$ be described by a function $f : \widetilde{\mathcal{C}} \to \mathcal{F}_N(\mathbb{R})^n$, i.e. $\underline{\widetilde{w}} = f(\widetilde{C}), \widetilde{C} \in \widetilde{\mathcal{C}}$. Then the method is said to be invariant under permutation of objects if*

$$f(P\widetilde{C}P^T) = Pf(\widetilde{C}), \qquad \forall \widetilde{C} \in \widetilde{\mathcal{C}} \text{ and for any permutation matrix } P.$$

In Chapter II, three types of PCMs were studied: MPCMs, APCMs-A, and APCMs-M. In the following sections of this chapter, the fuzzy extension of all three types of PCMs and of the reviewed definitions of consistency, inconsistency idices, and methods for deriving priorities from them is studied in detail. The focus is put on preserving the reciprocity of PCs and the invariance under permutation of objects. This is achieved by applying constrained fuzzy arithmetic instead of standard fuzzy arithmetic to the fuzzy extension of the methods reviewed in Chapter II. As mentioned in Section 3.5, constrained fuzzy arithmetic allows for considering constraints on operands when performing arithmetic operations on fuzzy numbers. Thus, it enables us to introduce reciprocity of the related PCs as a constraint on fuzzy arithmetic operations with the entries of a FPCM. After introducing a proper fuzzy extension of the methods reviewed in Chapter II, it is shown that the three types of FPCMs are equivalent and transformations between the approaches are examined. Note that Gavalec et al. (2015) presented a more general framework for FPCMs. They defined FPCMs as PCMs with fuzzy elements being fuzzy numbers of an abelian linearly ordered group on $\mathbb{R}$.

4.2 Fuzzy multiplicative pairwise comparison matrices

This section deals with the fuzzy extension of the methods related to MPCMs that were reviewed in Section 2.2. In Section 4.2.1, a fuzzy MPCM is defined properly and the construction of fuzzy MPCMs is studied. Section 4.2.2 is dedicated to the consistency of fuzzy MPCMs. In particular, the fuzzy extension of the multiplicative-consistency condition (II.4) and of the consistency ratio (II.10) for verifying acceptable level of inconsistency are dealt with, and a detailed study of the fuzzy maximal eigenvalues of fuzzy MPCMs is provided. Finally, Section 4.2.3 is focused on methods for obtaining fuzzy priorities from fuzzy MPCMs, in particular on the fuzzy extension of the EVM and the GMM.

4.2.1 Construction of FMPCMs

Definition 43. *A fuzzy multiplicative pairwise comparison matrix (FMPCM) of n objects $o_1, \ldots, o_n$ is a square matrix $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$, whose elements $\widetilde{m}_{ij}, i,j = 1, \ldots, n$, are fuzzy numbers indicating the ratio of preference intensity of object o_i to that of object o_j. That is, element $\widetilde{m}_{ij}$ indicates that o_i is $\widetilde{m}_{ij}$−times as good as o_j. Further, a FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$ has to be multiplicatively reciprocal, i.e.*

$$\widetilde{m}_{ij} = \frac{1}{\widetilde{m}_{ji}}, \qquad i,j = 1, \ldots, n, \tag{IV.1}$$

and

$$\widetilde{m}_{ii} = 1, \qquad i = 1, \ldots, n. \tag{IV.2}$$

Definition 43 is very general; elements $\widetilde{m}_{ij}$ of a FMPCM $\widetilde{M}$ are meant to be arbitrary fuzzy numbers satisfying the multiplicative-reciprocity condition (IV.1). In practice, these fuzzy numbers can be triangular, trapezoidal, or general fuzzy numbers defined by their $\alpha-$cuts, intervals, or it can even be a mix of all of them. Note that even a MPCM $M = \{m_{ij}\}_{ij=1}^{n}$ given by Definition 4 is a FMPCM since crisp numbers are a special case of fuzzy numbers and since a MPCM M satisfies (IV.1) as well as (IV.2).

In the literature, it is common that FMPCMs are defined by using triangular fuzzy numbers (see, e.g., Laarhoven and Pedrycz (1983); Chang (1996); Enea and Piazza (2004); Ishizaka and Nguyen (2013)). However, some papers provide a more general approach by using trapezoidal fuzzy numbers (see, e.g., Buckley (1985a); Csutora and Buckley (2001); Ishizaka (2014)).

We already know from Section 3.4 that the reciprocal $\frac{1}{\widetilde{c}}$ of a trapezoidal fuzzy number $\widetilde{c}$ is not a trapezoidal fuzzy number any more; it is a general fuzzy number described uniquely by its $\alpha-$cuts. This means that if we provide a PC $\widetilde{m}_{ij}$ of objects o_i and o_j as a trapezoidal fuzzy number (e.g. $\widetilde{m}_{ij} = (2, 3, 4, 5)$) then, according to the multiplicative-reciprocity property (IV.1), the entry $\widetilde{m}_{ji}$ of the FMPCM $\widetilde{M}$ is not a trapezoidal fuzzy number any more. This makes the construction of FMPCMs and the related methods much more complicated. First, there is no simple way to identify uniquely a general fuzzy number $\widetilde{m}_{ji}$; it is not possible to use just a quadruple of its representing values as in the case of the trapezoidal fuzzy number $\widetilde{m}_{ij}$. Second, having some of the entries in the FMPCM $\widetilde{M}$ in the form of general fuzzy numbers, methods related to FMPCMs would become much more complex and not transparent for DMs, and the resulting fuzzy priorities of objects obtained from such FMPCMs would be general fuzzy numbers.

As already mentioned in Chapter III, this problem is solved in the literature by simply approximating the results of arithmetic operations with trapezoidal fuzzy numbers by trapezoidal fuzzy numbers, i.e. by using the simplified standard fuzzy arithmetic introduced in Section 3.4 instead of standard fuzzy arithmetic. This means that the reciprocals of trapezoidal fuzzy PCs in a FMPCM are approximated by trapezoidal fuzzy numbers. Similarly, also the results of fuzzy arithmetic operations with the entries of a trapezoidal FMPCM are approximated by trapezoidal fuzzy numbers. These trapezoidal-fuzzy-number approximations have the same support and the core as the actual results of the arithmetic operations obtainable by applying standard fuzzy arithmetic.

It is obvious that by this approximation some information contained in the original general fuzzy numbers is lost. On the other hand, this approximation allows us to keep the computational process much simpler and transparent. Thus, it is a standard procedure to approximate all results of arithmetic operations with trapezoidal fuzzy numbers by trapezoidal fuzzy numbers. This approach is so deep-rooted in the literature on the fuzzy extension of MCDM methods based on PCMs (in fact, all papers cited in this thesis apply this approach) that the authors of the research papers often do not even mention the fact that the simplified standard fuzzy arithmetic is applied to the computations in their papers instead of the standard fuzzy arithmetic. However, as far as I am aware, there are no studies showing how good or bad this approximation really is, and how much the results of fuzzy MCDM methods based on FPCMs with applied simplified standard fuzzy arithmetic vary from the hypothetical actual results obtainable by applying standard fuzzy arithmetic properly. Nevertheless, this simplification consisting in applying simplified standard fuzzy arithmetic to arithmetic operations with trapezoidal fuzzy numbers instead of standard fuzzy arithmetic is used also in this thesis in order to be in line with the main research stream.

As already mentioned in Section 2.2.1, integer numbers from Saaty's scale given in Tab. II.1 with assigned linguistic terms and their reciprocals are usually used for expressing PCs in crisp MPCMs. However, since the linguistic terms in Saaty's scale are vague, it is more natural to model their meanings by using fuzzy rather than crisp numbers. Many different approaches to the fuzzy extension of Saaty's scale have been proposed in the literature. Ishizaka and Nguyen (2013) provided a review of various fuzzy extensions of Saaty's scale applied to real-world MCDM problems. Most often, the meaning of the linguistic terms from Saaty's scale is modeled by triangular fuzzy numbers, less often by trapezoidal fuzzy numbers.

However, Saaty's scale is not always fuzzified properly. Most problems are usually related to the modeling of the linguistic term "equal preference". We have to distinguish whether o_i and o_j are the same objects or not. For $i = j$, there is obviously no uncertainty in the comparison as we compare one object with itself. Therefore, $\widetilde{m}_{ii}, i = 1, \ldots, n$, has to be set as 1, i.e. it is a crisp number. On the other hand, when two different objects o_i and o_j, $i \neq j$, are assessed to be "equally preferred", then this PC is very likely to contain some uncertainty. In such case, "equal preference" should be modeled by a fuzzy number "about 1", not necessarily by crisp number 1.

In the literature, "equal preference" is usually modeled by triangular fuzzy number $\widetilde{1} = (1, 1, c)$, $c = 2$ or $c = 3$; see the literature review provided by Ishizaka and Nguyen (2013). However, $\widetilde{1}$ defined in this way is not appropriate for modeling the meaning of the linguistic term "equal preference". If a DM assesses o_i to be equally preferred to o_j, then common sense suggests that also o_j should be equally preferred to o_i. But, if we enter PC $\widetilde{m}_{ij} = (1, 1, c)$ into a FMPCM $\widetilde{M}$, then, based on the multiplicative-reciprocity property (IV.1), it

Table IV.1: Fuzzy extension of Saaty's 5-point scale.

Intensity of preference	Linguistic term
$\left(\frac{1}{3}, 1, 3\right)$	equal preference
$(1, 3, 5)$	weak preference
$(3, 5, 7)$	strong preference
$(5, 7, 9)$	demonstrated preference
$(7, 9, 9)$	absolute preference

Table IV.2: Fuzzy extension of Saaty's 9-point scale.

Intensity of preference	Linguistic term
$\left(\frac{1}{2}, 1, 2\right)$	equal preference
$(1, 2, 3)$	between equal and weak preference
$(2, 3, 4)$	weak preference
$(3, 4, 5)$	between weak and strong preference
$(4, 5, 6)$	strong preference
$(5, 6, 7)$	between strong and demonstrated preference
$(6, 7, 8)$	demonstrated preference
$(7, 8, 9)$	between demonstrated and absolute preference
$(8, 9, 9)$	absolute preference

follows that $\widetilde{m}_{ji} = \frac{1}{(1,1,c)} = \left(\frac{1}{c}, 1, 1\right)$. Obviously, $\left(\frac{1}{c}, 1, 1\right) \neq (1, 1, c)$ for $c > 1$, and even $\left(\frac{1}{c}, 1, 1\right) < (1, 1, c)$. Thus, $\widetilde{m}_{ji} < \widetilde{m}_{ij}$, which contradicts the statement that o_i is equally preferred to o_j.

Based on the reasoning in the previous paragraph, if $\widetilde{m}_{ij} = \widetilde{1}$, then it should also hold that $\widetilde{m}_{ji} = \widetilde{1}$. Therefore, based on the multiplicative-reciprocity property (IV.1), the equality $\widetilde{1} = \frac{1}{\widetilde{1}}$ should hold. This means that $\widetilde{1} := (c^L, c^M, c^U)$ has to satisfy

$$(c^L, c^M, c^U) = \frac{1}{(c^L, c^M, c^U)}. \tag{IV.3}$$

By solving (IV.3) we obtain $\widetilde{1}$ defined as

$$\widetilde{1} = \left(\frac{1}{c}, 1, c\right), \ c \geq 1. \tag{IV.4}$$

Based on the same reasoning, if we wanted to use a trapezoidal fuzzy number instead of a triangular fuzzy number to model the linguistic term "equal preference", this would have to be in the form

$$\widetilde{1} = \left(\frac{1}{c}, \frac{1}{b}, b, c\right), \ c \geq b \geq 1. \tag{IV.5}$$

Interestingly, the appropriate representation of "equal preference" in the form (IV.4) (or (IV.5) for trapezoidal representation) has been found only in two papers (Enea and Piazza (2004) and Javanbarg et al. (2012)).

Note 7. *From now on, the term "FMPCM" will be used exclusively for a FMPCM that is given by Definition 43 and that satisfies the indispensable condition $\widetilde{1} = \frac{1}{\widetilde{1}}$ for the fuzzy number modeling the meaning of the linguistic term "equal preference", i.e. (IV.4) and (IV.5) in the case of triangular and trapezoidal fuzzy numbers, respectively.*

As for the representation of other linguistic terms from Saaty's scale, it is reasonable to define the respective fuzzy numbers and their reciprocals in such a way that they form Ruspini's fuzzy partition of interval $\left[\frac{1}{9}, 9\right]$. In this way any element in the interval $\left[\frac{1}{9}, 9\right]$ has linguistic interpretation (Stoklasa, 2014). Using triangular fuzzy numbers, such fuzzy extension of Saaty's scale with the main 5 linguistic terms is given in Tab. IV.1, while the fuzzy extension of Saaty's scale with intermediate linguistic terms included is given in Tab. IV.2 (Krejčí and Talašová, 2013).

The triangular fuzzy numbers in the fuzzy scales given in Tab. IV.1 and in Tab. IV.2 model the meanings of the corresponding linguistic terms more naturally than the crisp numbers in original Saaty's scale. However, as already emphasized in Section 2.2.1, the uniform distribution of the numerical values assigned to the linguistic terms in Saaty's scale on interval $[1, 9]$ does not seem to be appropriate (e.g. "weakly preferred" does not really correspond to "3-times preferred"). Thus, before actually modeling the meanings of the linguistic terms

by fuzzy numbers, the original scale should be revised and more intuitive meanings should be found for each linguistic term. Moreover, since every DM perceives the linguistic terms from Saaty's scale differently, the scale should be calibrated for each particular decision-making problem in cooperation with the DM. Such idea was applied e.g. by Ishizaka and Nguyen (2013) to a current bank account selection.

The idea of customizing Saaty's scale presented by Ishizaka and Nguyen (2013) is an innovative step in the direction of taking into account subjectivity of every single DM. However, the proposed process of calibration suffers from some severe drawbacks. For instance, an inappropriate form of the triangular fuzzy number modeling the meaning of the linguistic term "equal preference" is used for the calibration. In particular, Ishizaka and Nguyen (2013) suggest to model the meaning of the linguistic term "equal preference" by the triangular fuzzy number $(1, 1.1, 1.2)$. Thus, if $\widetilde{m}_{ij} = (1, 1.1, 1.2) > 1$, then $\widetilde{m}_{ji} = \frac{1}{\widetilde{m}_{ij}} = (\frac{1}{1.2}, \frac{1}{1.1}, 1) < 1$. This clearly does not model the desired preference information "objects o_i and o_j are equally preferred" but rather "o_i is about 1.1-times preferred to o_j". Thus, using the triangular fuzzy number $(1, 1.1, 1.2)$ to model "equal preference" would actually have more severe negative impact on the result of a MCDM problem than using the crisp number 1 as suggested in original Saaty's scale or the fuzzy number $\widetilde{1} = (1, 1, c), c > 1$.

Besides providing preference information linguistically by using linguistic terms from Saaty's scale, it is possible to enter expert numerical judgments into a MPCM. Especially when the information about the given problem is uncertain or incomplete, it is more appropriate to provide numerical judgments in form of fuzzy numbers rather than crisp numbers as illustrated by the following example.

Example 29. Let us assume we are searching for a new job. We have two interesting job offers, J_1 and J_2, and we evaluate them based on the expected income. Let us express our preferences by using the interval $[\frac{1}{9}, 9]$, where 1 stands for equal preference and 9 stands for absolute preference.

First, let us assume the expected income from J_1 is 2 500 € per month, and the expected income from J_2 is 1 000 € per month. For me, for example, the income 2 500 € may be 5-times preferred to 1 000 €, i.e. $m_{12} = 5$.

Now, let us assume, the income from J_1 is not known precisely, but it may be between 2 000 € and 3 000 € with the most possible income 2 500 €. In this case it would be quite difficult to express our preferences by using one crisp number, would not it? The preference information obtained from a DM in this case may have the following form. "If the income from J_1 is 2 500 €, then I prefer J_1 5-times over J_2. If the income from J_1 is 3 000 €, then J_1 is 7-times preferred to J_2. However, if the income from J_1 is only 2 000 €, then J_1 is only 4-times preferred to J_2." It is very natural to model this preference information by a triangular fuzzy number (i.e. $\widetilde{m}_{12} = (4, 5, 7)$ in this case) rather than by just one crisp number. △

Saaty (2006) argues that "what fuzziness does by wholesale change of judgments numerically without obtaining the consent of the DM for each judgment goes against the grain of what decision making is all about, namely using experts to input valid judgments to obtain valid decisions" (Saaty (2006), p. 462). I fully agree with this argument - replacing crisp numerical judgments in a PCM by fuzzy numbers blindly without the consent of the DM does not lead to valid results. When linguistic terms are used for providing PCs, the meaning of the linguistic terms should be modeled for every particular MCDM problem in cooperation with the DM, i.e. calibration of the scale should be done. Nevertheless, using linguistic scales given in Tab. IV.1 and IV.2 is sufficient for the scope of this thesis since only theoretical results are presented here without any particular MCDM application and a particular DM.

There is no doubt that by means of fuzzy numbers we can describe possible uncertainty involved in DM's judgments or incompleteness of information in a decision-making problem. For instance, by the triangular fuzzy number $\widetilde{m}_{ij} = (3, 4, 6)$ we can model the case when a DM says that o_i is about $4-$times preferred to o_j, definitely not less than $3-$times preferred and not more than $6-$times preferred to o_j. On the other hand, if the DM is sure about his/her judgment, e.g. if he/she is sure that o_i is $4-$times preferred to o_j, we can appropriately describe this information by the triangular fuzzy number $\widetilde{m}_{ij} = (4, 4, 4)$ that corresponds to the crisp number 4.

4.2.2 Multiplicative consistency of FMPCMs

As stated in Section 2.2.2, examining consistency or acceptable inconsistency of MPCMs is crucial in order to derive reliable priorities of objects. In the case of FMPCMs, this task is of the same importance. That is the reason why the fuzzy extension of definitions of consistency for FMPCMs has been studied extensively (see, e.g., Buckley (1985a); Wang et al. (2005b); Liu (2009); Liu et al. (2014); Zheng et al. (2012); Gavalec et al. (2015); Wang (2015a,b); Li et al. (2016)). Multiplicative consistency (II.4) is the basic and the most often applied consistency condition for MPCMs. Therefore, in this section, the fuzzy extension of this consistency condition is of interest.

In Section 4.2.2.1, a review of definitions of multiplicatively consistent FMPCMs proposed in the literature is given. In Section 4.2.2.2, a new fuzzy extension of the definition of multiplicative consistency based on the constrained fuzzy arithmetic is proposed. In Section 4.2.2.3, a fuzzy extension of Consistency Index (II.9) is proposed. Finally, in Section 4.2.2.4, the fuzzy maximal eigenvalue of a FMPCM is studied in detail.

4.2.2.1 Review of fuzzy extensions of multiplicative consistency

Many definitions of consistency based on the extension of multiplicative-transitivity property (II.4) have been proposed in the literature. Buckley (1985a) proposed a fuzzy extension of multiplicative consistency to FM-PCMs based on a parameter. Wang et al. (2005a) proposed a definition of multiplicative consistency for interval FMPCMs and they proposed an algorithm for deriving interval priorities from interval FMPCMs based on minimizing the inconsistency. Liu (2009) defined multiplicative consistency and acceptable inconsistency for interval FMPCMs. Liu et al. (2014) extended this definition to triangular FMPCMs and they proposed a procedure for obtaining a consistent triangular FMPCM from n-1 fuzzy PCs. Wang (2015a) showed that the definition of consistency proposed by Liu (2009) is not invariant under permutation of objects and he introduced a new definition of multiplicative consistency for interval FMPCMs. Similarly, Wang (2015b) showed that the definition of consistency proposed by Liu et al. (2014) is not invariant under permutation of objects. Afterwards, he introduced a new definition of multiplicative consistency for triangular FMPCMs and proposed formulas for converting normalized fuzzy priorities into a consistent triangular FMPCM. Gavalec et al. (2015) introduced a more general definition of multiplicative consistency for FMPCMs, so called $\alpha-$consistency, $\alpha \in [0, 1]$, which can be looked at as an acceptable-consistency definition.

In this section, the definitions of multiplicative consistency for interval, triangular, and trapezoidal FMPCMs introduced by Buckley (1985a), Wang et al. (2005a), Liu (2009), Liu et al. (2014) and Wang (2015a,b) are reviewed. Furthermore, the drawbacks in the definitions proposed by Liu (2009), Liu et al. (2014), and Wang (2015a,b) are pointed out. In particular, violation of the invariance under permutation of objects and violation of the reciprocity of PCs, which is the key property of MPCMs, are emphasized.

Buckley (1985a) defined multiplicative consistency for FMPCMs with trapezoidal fuzzy numbers.

Definition 44. *(Buckley, 1985a) Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta}\right)$, *be a trapezoidal FMPCM. Then* $\widetilde{M}$ *is said to be* multiplicatively consistent *if* $\widetilde{m}_{ik}\widetilde{m}_{kj} \approx \widetilde{m}_{ij}$*. Otherwise,* $\widetilde{M}$ *is said to be* multiplicatively inconsistent.

In Definition 44, Buckley defined the approximate equality $\approx$ as follows. Having

$$\nu(\widetilde{c} \geq \widetilde{d}) = \sup_{x \geq y} \left\{\min\left\{\widetilde{c}(x), \widetilde{d}(y)\right\}\right\},$$

the fuzzy number $\widetilde{c}$ is greater than the fuzzy number $\widetilde{d}, \widetilde{c} > \widetilde{d}$, if $\nu(\widetilde{c} \geq \widetilde{d}) = 1$ and $\nu(\widetilde{d} \geq \widetilde{c}) < \theta$, where θ is a fixed value from interval $]0, 1]$. If $\widetilde{c}$ is not greater than $\widetilde{d}$ and $\widetilde{d}$ is not greater than $\widetilde{c}$, i.e. $\min\left\{\nu(\widetilde{c} \geq \widetilde{d}), \nu(\widetilde{d} \geq \widetilde{c})\right\} \geq \theta$, then $\widetilde{c}$ and $\widetilde{d}$ are said to be approximately equal, $\widetilde{c} \approx \widetilde{d}$.

Further, Buckley (1985a) derived the following theorem for trapezoidal FMPCMs.

Theorem 17. *(Buckley, 1985a) Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta}\right)$, *be a trapezoidal FMPCM, and let* $m_{ij} \in \left[m_{ij}^{\alpha}, m_{ij}^{\delta}\right], i, j = 1, \ldots, n$. *If* $M = \{m_{ij}\}_{i,j=1}^{n}$ *is multiplicatively consistent according to* (II.4)*, then* $\widetilde{M}$ *is multiplicatively consistent according to Definition 44.*

Definition 44 is invariant under permutation of objects in trapezoidal FMPCM $\widetilde{M}$. However, it is dependent on the value of the parameter θ, and there are no studies regarding an appropriate choice of the value of θ. Further, Theorem 17 is not very helpful in verifying multiplicative consistency of trapezoidal FMPCMs. The theorem helps to identify only a small part of multiplicatively consistent trapezoidal FMPCMs. Trapezoidal FMPCMs multiplicatively consistent according to Definition 44 for which there do not exist $m_{ij} \in \left[m_{ij}^{\alpha}, m_{ij}^{\delta}\right], i, j = 1, \ldots, n$, such that $M = \{m_{ij}\}_{i,j=1}^{n}$ would be multiplicatively consistent according to (II.4) cannot be identified by using this theorem.

Another definition of consistency was proposed by Wang et al. (2005a). This definition of consistency will be studied in more detail in the following section. To distinguish easily this definition of consistency from others, consistency according to this definition will be called multiplicative weak consistency. The reason for the use of the word "weak" in the name will be clarified right after providing the definition.

Definition 45. *(Wang et al., 2005a) Let* $\overline{M} = \{\overline{m}_{ij}\}_{i,j=1}^{n}, \overline{m}_{ij} = \left[m_{ij}^{L}, m_{ij}^{U}\right]$, *be an interval FMPCM. If* $S = \left\{(w_1, \ldots, w_n) \in \mathbb{R}^n; m_{ij}^{L} \leq \frac{w_i}{w_j} \leq m_{ij}^{U}, \sum_{i=1}^{n} w_i = 1, w_i > 0, i = 1, \ldots, n\right\} \neq \emptyset$, *then* $\overline{M}$ *is said to be multiplicatively weakly consistent. Otherwise,* $\overline{M}$ *is said to be multiplicatively weakly inconsistent.*

Definition 45 of multiplicative weak consistency for interval FMPCMs is clearly based on Proposition 1 for MPCMs. According to the definition, an interval FMPCM $\overline{M}$ is multiplicatively weakly consistent if there exists a vector $\underline{w} = (w_1, \ldots, w_n)^T$ which we can use to construct a multiplicatively consistent MPCM $M^* = \{m_{ij}^*\}_{i,j=1}^{n}$ such that $m_{ij}^* \in [m_{ij}^{L}, m_{ij}^{U}], i, j = 1, \ldots, n$.

The requirement of at least one multiplicatively consistent MPCM obtainable from the interval FMPCM is very weak (that is why the name multiplicative "weak" consistency). Therefore, it is quite easy to satisfy the multiplicative-consistency condition in Definition 45 when constructing interval FMPCMs. The multiplicative-consistency conditions reviewed in the rest of this section are significantly stronger and thus much more difficult to fulfill.

Wang et al. (2005b) derived the following theorem which provides a useful tool for verifying the multiplicative weak consistency according to Definition 45.

Theorem 18. *(Wang et al., 2005b) An interval FMPCM* $\overline{M} = \{\overline{m}_{ij}\}_{i,j=1}^{n}, \overline{m}_{ij} = \left[m_{ij}^{L}, m_{ij}^{U}\right]$, *is multiplicatively weakly consistent if and only if it satisfies the inequalities*

$$\max_{k=1,\ldots,n} \left\{m_{ik}^{L} m_{kj}^{L}\right\} \leq \min_{k=1,\ldots,n} \left\{m_{ik}^{U} m_{kj}^{U}\right\}, \qquad i,j = 1,\ldots,n. \tag{IV.6}$$

The following theorem shows the relation between Definitions 44 and 45.

Theorem 19. *An interval FMPCM* $\overline{M} = \{\overline{m}_{ij}\}_{i,j=1}^{n}, \overline{m}_{ij} = \left[m_{ij}^{L}, m_{ij}^{U}\right]$, *multiplicatively weakly consistent according to Definition 45 is also multiplicatively consistent according to Definition 44.*

Proof. Let $\overline{M} = \{\overline{m}_{ij}\}_{i,j=1}^{n}, \overline{m}_{ij} = \left[m_{ij}^{L}, m_{ij}^{U}\right]$, be multiplicatively consistent according to Definition 45. Then, from Proposition 1, it follows that there exists a MPCM $M = \{m_{ij}\}_{i,j=1}^{n}, m_{ij} = \frac{w_i}{w_j} \in [m_{ij}^{L}, m_{ij}^{U}], i,j = 1,\ldots,n$, that is multiplicatively consistent according to (II.4). Thus, based on Theorem 17, intervals being a particular case of trapezoidal fuzzy numbers, it follows that $\overline{M}$ is multiplicatively consistent according to Definition 44. $\square$

Another definition of multiplicative consistency for interval FMPCMs was given by Liu (2009).

Definition 46. *(Liu, 2009) Let* $\overline{M} = \{\overline{m}_{ij}\}_{i,j=1}^{n}, \overline{m}_{ij} = \left[m_{ij}^{L}, m_{ij}^{U}\right]$, *be an interval FMPCM. Further, let MPCMs* $C = \{c_{ij}\}_{i,j=1}^{n}, D = \{d_{ij}\}_{i,j=1}^{n}$ *be constructed from the interval FMPCM* $\overline{M}$ *as*

$$c_{ij} = \begin{cases} m_{ij}^{L}, & i < j \\ 1, & i = j \\ m_{ij}^{U}, & i > j \end{cases} \qquad d_{ij} = \begin{cases} m_{ij}^{U}, & i < j \\ 1, & i = j \\ m_{ij}^{L}, & i > j \end{cases}. \tag{IV.7}$$

If the matrices C, D *are multiplicatively consistent according to* (II.4)*, then* $\overline{M}$ *is said to be* multiplicatively consistent. *Otherwise,* $\overline{M}$ *is said to be* multiplicatively inconsistent.

Later, Liu et al. (2014) extended Definition 46 of multiplicative consistency for interval FMPCMs to triangular and to trapezoidal FMPCMs.

Definition 47. *(Liu et al., 2014) Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta}\right)$, *be a trapezoidal FMPCM. Further, let MPCMs* $A = \{a_{ij}\}_{i,j=1}^{n}, B = \{b_{ij}\}_{i,j=1}^{n}, C = \{c_{ij}\}_{i,j=1}^{n}$, *and* $D = \{d_{ij}\}_{i,j=1}^{n}$ *be constructed from the trapezoidal FMPCM* $\widetilde{M}$ *as*

$$a_{ij} = \begin{cases} m_{ij}^{\alpha}, & i < j \\ 1, & i = j, \\ m_{ij}^{\delta}, & i > j \end{cases} \quad b_{ij} = \begin{cases} m_{ij}^{\beta}, & i < j \\ 1, & i = j, \\ m_{ij}^{\gamma}, & i > j \end{cases} \quad c_{ij} = \begin{cases} m_{ij}^{\gamma}, & i < j \\ 1, & i = j, \\ m_{ij}^{\beta}, & i > j \end{cases} \quad d_{ij} = \begin{cases} m_{ij}^{\delta}, & i < j \\ 1, & i = j. \\ m_{ij}^{\alpha}, & i > j \end{cases} \tag{IV.8}$$

If the matrices A, B, C, *and* D *are multiplicatively consistent according to* (II.4)*, then* $\widetilde{M}$ *is said to be multiplicatively consistent. Otherwise,* $\widetilde{M}$ *is said to be multiplicatively inconsistent.*

The definition of multiplicative consistency for triangular FMPCMs proposed by Liu et al. (2014) is actually obtainable by adjusting Definition 47 to triangular fuzzy numbers. Note that also Definition 46 of multiplicative consistency for interval FMPCMs is just a particular case of Definition 47.

Liu et al. (2014) stated that Definition 47 of multiplicative consistency naturally reflects the multiplicative-reciprocity property of trapezoidal FMPCMs since the matrices A, B, C, and D are clearly reciprocal. Furthermore, the authors claim that the definition of multiplicative consistency is closely related to Definition 5 of multiplicative consistency for MPCMs given by Saaty (1980). However, as demonstrated by Wang (2015b), Definition 47 highly depends on the ordering of objects compared in the trapezoidal FMPCM, i.e. it is not invariant under permutation of objects. Analogously, Wang (2015a) demonstrated the invalidity of Definition 46 for interval FMPCMs.

The drawback in Definition 47 is caused by the fact that the MPCMs A, B, C, and D given as (IV.8) are not invariant under permutation of objects. There is no logical justification for choosing the lower boundary values of trapezoidal fuzzy numbers above the main diagonal and the upper boundary values of the trapezoidal fuzzy

numbers below the main diagonal to construct the MPCM A. A similar problem appears also for the MPCMs B, C, and D. This way of constructing MPCMs from a trapezoidal FMPCM does not reflect naturally the multiplicative-reciprocity property of trapezoidal FMPCMs; the MPCMs A, B, C, and D change completely with a permutation of compared objects. Definition 46 of multiplicative consistency for interval FMPCMs suffers from the same drawbacks, and the same is valid also for the multiplicative consistency of triangular FMPCMs defined by Liu et al. (2014).

Clearly, Definitions 46 and 47 are not proper fuzzy extensions of Definition 5 of multiplicative consistency proposed by Saaty (1980). Properly defined multiplicative consistency for FMPCMs has to be invariant under permutation of objects and, naturally, as already pointed out by Liu et al. (2014), it has to preserve multiplicative reciprocity.

Wang (2015a,b) proposed definitions of multiplicative consistency for interval and triangular FMPCMs invariant under permutation of objects as follows.

Definition 48. *(Wang, 2015a) Let* $\overline{M} = \{\overline{m}_{ij}\}_{i,j=1}^n, \overline{m}_{ij} = \left[m_{ij}^L, m_{ij}^U\right]$, *be an interval FMPCM.* $\overline{M} = \{\overline{m}_{ij}\}_{i,j=1}^n$ *is said to be multiplicatively consistent if*

$$m_{ij}^L m_{ij}^U = m_{ik}^L m_{ik}^U m_{kj}^L m_{kj}^U, \qquad i,j,k = 1,\ldots,n. \tag{IV.9}$$

Definition 49. *(Wang, 2015b) Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n, \widetilde{m}_{ij} = \left(m_{ij}^L, m_{ij}^M, m_{ij}^U\right)$, *be a triangular FMPCM.* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$ *is said to be multiplicatively consistent if*

$$\widetilde{m}_{ij} \widetilde{m}_{jk} \widetilde{m}_{ki} = \widetilde{m}_{ik} \widetilde{m}_{kj} \widetilde{m}_{ji}, \qquad i,j,k = 1,\ldots,n. \tag{IV.10}$$

Furthermore, Wang (2015b) formulated the following theorem.

Theorem 20. *(Wang, 2015b) The following statements are equivalent for a triangular FMPCM* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$, $\widetilde{m}_{ij} = \left(m_{ij}^L, m_{ij}^M, m_{ij}^U\right)$:

(i) $\widetilde{M}$ *is multiplicatively consistent according to Definition 49,*

(ii) $m_{ij}^M = m_{ik}^M m_{kj}^M,\ m_{ij}^L m_{ij}^U = m_{ik}^L m_{ik}^U m_{kj}^L m_{kj}^U, \qquad i,j,k = 1,\ldots,n,$

(iii) $m_{ij}^M m_{jk}^M m_{ki}^M = m_{ik}^M m_{kj}^M m_{ji}^M,\ m_{ij}^L m_{jk}^L m_{ki}^L = m_{ik}^L m_{kj}^L m_{ji}^L, \qquad i,j,k = 1,\ldots,n,$

(iv) $m_{ij}^M m_{jk}^M m_{ki}^M = m_{ik}^M m_{kj}^M m_{ji}^M,\ m_{ij}^U m_{jk}^U m_{ki}^U = m_{ik}^U m_{kj}^U m_{ji}^U, \qquad i,j,k = 1,\ldots,n.$

A theorem similar to Theorem 20 could be formulated for interval FMPCMs $\overline{M} = \{\overline{m}_{ij}\}_{i,j=1}^n$ by just removing the middle values $m_{ij}^M, i,j = 1,\ldots,n$, and the associated equations.

Remark 10. *Multiplication in the formula* (IV.10) *is done according to the simplified standard fuzzy arithmetic as defined in Section 3.4, in particular by the formula* (III.26). *In fact, using the simplified standard fuzzy arithmetic, the formula* (IV.10) *can be written as (iii) and (iv) of Theorem 20. Therefore, based on Theorem 20, it follows that Definitions 48 and 49 for interval and triangular FMPCMs, respectively, are practically the same (keeping in mind that one is given for interval FMPCMs and one for triangular FMPCMs), although the multiplicative-consistency conditions* (IV.9) *and* (IV.10) *seem to have different forms.*

In order to demonstrate the inappropriateness of Definitions 48 and 49, let us analyze in more detail the requirement of multiplicative reciprocity of PCs in FMPCMs and its impact on the multiplicative-consistency condition. As stated in Section 2.2.1, multiplicative reciprocity of PCs is an inherent property of every MPCM $M = \{m_{ij}\}_{i,j=1}^n$. When, for example, the intensity of preference of object o_i over object o_j is $m_{ij} = 3$ (i.e. o_i is 3-times preferred to o_j), then the intensity of preference of object o_j over object o_i has to be $m_{ji} = \frac{1}{3}$ (i.e. o_j has to be 3-times less preferred to o_i). Therefore, multiplicative reciprocity is a reasonable property that results from the interpretation of PCs of objects in MPCMs. Because of the multiplicative reciprocity of PCs, the multiplicative-consistency condition (II.4) for MPCMs is equivalent to the statements (ii) and (iii) in Theorem 1.

Conception of the multiplicative reciprocity becomes more complicated when extended to fuzzy numbers. For a triangular FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n, \widetilde{m}_{ij} = \left(m_{ij}^L, m_{ij}^M, m_{ij}^U\right)$, multiplicative reciprocity is defined as $\widetilde{m}_{ji} = \frac{1}{\widetilde{m}_{ij}} = \left(\frac{1}{m_{ij}^U}, \frac{1}{m_{ij}^M}, \frac{1}{m_{ij}^L}\right)$. According to this property, when, e.g., the highest possible intensity of preference m_{ij}^U of object o_i over object o_j is $m_{ij}^U = 5$ (i.e. o_i is at most 5-times preferred to o_j), this means that the lowest possible intensity of preference m_{ji}^L of object o_j over object o_i is automatically $m_{ji}^L = \frac{1}{5}$ (i.e. o_j is at least 5-times less preferred to o_i). However, this is not all.

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FMPCMs carry much more information about the preference intensities. For example, when we consider any particular value m_{ij}^* of $\widetilde{m}_{ij} = \left(m_{ij}^L, m_{ij}^M, m_{ij}^U\right)$, i.e. $m_{ij}^* \in [m_{ij}^L, m_{ij}^U]$, as a possible intensity of preference of object o_i over object o_j, this intensity of preference is associated inseparably with the corresponding intensity of preference m_{ji}^* of $\widetilde{m}_{ji} = \left(m_{ji}^L, m_{ji}^M, m_{ji}^U\right)$ such that $m_{ji}^* = \frac{1}{m_{ij}^*}$; remember that m_{ij}^* and m_{ji}^* have to express the same preference information about o_i and o_j. This property results naturally from the meaning of PCs.

Wang (2015b) defined multiplicative consistency of triangular FMPCMs by fuzzifying the expression (iii) in Theorem 1 using the simplified standard fuzzy arithmetic; see Definition 49. As emphasized by Wang (2015b), the definition is invariant under permutation of objects in the triangular FMPCM. This is definitely an advantage over Definition 47 formerly proposed by Liu et al. (2014).

Since Wang (2015b) applied simplified standard fuzzy arithmetic to the computations with triangular fuzzy numbers, the expression (IV.10) is nothing else but the statements (iii) and (iv) in Theorem 20. However, the expressions $m_{ij}^L m_{jk}^L m_{ki}^L = m_{ik}^L m_{kj}^L m_{ji}^L$ and $m_{ij}^U m_{jk}^U m_{ki}^U = m_{ik}^U m_{kj}^U m_{ji}^U$ in the statements (iii) and (iv) violate the multiplicative reciprocity of PCs. For example, in the first expression, the intensity of preference m_{ij}^L of o_i over o_j and the intensity of preference m_{ji}^L of o_j over o_i are used at the same time. This clearly violates the multiplicative reciprocity of PCs since $m_{ji}^L = \frac{1}{m_{ij}^U} \neq \frac{1}{m_{ij}^L}$ (unless $m_{ij}^L = m_{ij}^U = \frac{1}{m_{ji}^L} = \frac{1}{m_{ji}^U}$).

Example 30. Let us consider the interval FMPCM

$$\overline{M} = \begin{pmatrix} 1 & \left[\frac{3}{2}, 2\right] & [x, y] \\ \left[\frac{1}{2}, \frac{2}{3}\right] & 1 & \left[\frac{3}{2}, 2\right] \\ \left[\frac{1}{y}, \frac{1}{x}\right] & \left[\frac{1}{2}, \frac{2}{3}\right] & 1 \end{pmatrix} \tag{IV.11}$$

where the PCs $\overline{m}_{13}$ and $\overline{m}_{31} = \frac{1}{\overline{m}_{13}}$ are unknown. First, let us apply Definition 48 proposed by Wang (2015a) in order to find out which values of $\overline{m}_{13}$ are allowed to preserve the multiplicative consistency.

By applying Definition 48, for $i = 1, k = 2, j = 3$, we obtain

$$xy = \frac{3}{2} \cdot 2 \cdot \frac{3}{2} \cdot 2 = 9.$$

Therefore, the interval PC $\overline{m}_{13} = [x, y], x \leq y$, can be in the form $\left[\frac{9}{y}, y\right], y \in [3, 9]$, in order to keep multiplicative consistency of $\overline{M}$. This means that even the whole interval $[1, 9]$ can be used to model the intensity of preference of o_1 over o_3, i.e. $\overline{m}_{13} = [1, 9]$.

Let us have a closer look on the intensities of preferences in such interval FMPCM. Clearly, object o_1 is preferred to object o_2 and object o_2 is preferred to object o_3 since $\overline{m}_{12} = \overline{m}_{23} = \left[\frac{3}{2}, 2\right], \frac{3}{2} > 1$. Therefore, object o_1 should be also preferred to object o_3. However, according to $\overline{m}_{13} = [1, 9]$, equal preference ($m_{13}^L = 1$) of objects o_1 and o_3 is admitted.

Moreover, the intensity of preference of object o_1 over object o_2 is at most 2 (between equal and weak preference) and also the intensity of preference of object o_2 over object o_3 is at most 2. However, the intensity of preference of object o_1 over object o_3 can be up to 9 (absolute preference) which is much higher than $2 \cdot 2 = 4$. In fact, there are no intensities of preference $m_{12} \in \left[m_{12}^L, m_{12}^U\right]$ and $m_{23} \in \left[m_{23}^L, m_{23}^U\right]$ such that $m_{12}m_{23} = 1$ or $m_{12}m_{23} = 9$.

According to Theorem 20 (properly adjusted for interval FMPCMs) the multiplicative consistency can be checked by using the equivalent multiplicative-consistency condition

$$m_{ij}^L m_{jk}^L m_{ki}^L = m_{ik}^L m_{kj}^L m_{ji}^L, \qquad i, j, k = 1, \dots, n.$$

Assuming $\overline{m}_{13} = [1, 9]$ in the interval FMPCM $\overline{M}$ given by (IV.11), this basically means to verify the multiplicative consistency of the matrix

$$M^L = \begin{pmatrix} 1 & \frac{3}{2} & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{9} & \frac{1}{2} & 1 \end{pmatrix}. \tag{IV.12}$$

by using the property (iii) of Theorem 1. However, the matrix (IV.12) is not multiplicatively reciprocal, i.e. it is not even a MPCM. Therefore, verifying its multiplicative consistency is meaningless. $\triangle$

Even though Definitions 48 and 49 of multiplicatively consistent interval and triangular FMPCMs are invariant under permutation of objects in the FMPCMs, their meaning is questionable since they violate the multiplicative reciprocity of PCs. Furthermore, when applying the simplified standard fuzzy arithmetic, as Wang (2015b) did, the condition (IV.10) is not equivalent neither to $\widetilde{m}_{ij} = \widetilde{m}_{ik}\widetilde{m}_{kj}$, $i, j, k = 1, \dots, n$, nor to

$\widetilde{m}_{ij}\widetilde{m}_{jk}\widetilde{m}_{ki} = 1$, $i, j, k = 1, \ldots, n$. The same is true for Definition 48 of multiplicative consistency for interval FMPCMs. This means that, for Definitions 48 and 49 of multiplicative consistency proposed by Wang (2015a,b), Theorem 1 cannot be extended to interval and triangular FMPCMs, respectively. Definition 5 of multiplicative consistency for MPCMs should be extended to FMPCMs in such a way that Theorem 1 can be extended to FMPCMs accordingly. This is possible to do by employing properly the multiplicative-reciprocity property.

Another serious drawback of applying the simplified standard fuzzy arithmetic to computation with PCs in triangular FMPCMs (and, of course, FMPCMs in general) is the fact that

$$\widetilde{m}_{ij}\widetilde{m}_{ji} = \left(m_{ij}^L m_{ji}^L, m_{ij}^M m_{ji}^M, m_{ij}^U m_{ji}^U\right) = \left(\frac{m_{ij}^L}{m_{ij}^U}, 1, \frac{m_{ij}^U}{m_{ij}^L}\right) \neq 1, \qquad i, j = 1, \ldots, n. \qquad \text{(IV.13)}$$

From the multiplicative-reciprocity property $\widetilde{m}_{ij} = \frac{1}{\widetilde{m}_{ji}}$ of FMPCMs, it should follow that $\widetilde{m}_{ij}\widetilde{m}_{ji} = 1$. Obviously, the expression (IV.13) violates the multiplicative-reciprocity property of PCs.

We know that constrained fuzzy arithmetic has to be applied whenever there are any interactions among operands. Clearly, there is an interaction between fuzzy PCs $\widetilde{m}_{ij}$ and $\widetilde{m}_{ji}$; from the multiplicative reciprocity of PCs it follows that any intensity of preference $m_{ij}^* \in \widetilde{m}_{ij}$ of object o_i over object o_j is associated with the corresponding intensity of preference $m_{ji}^* = \frac{1}{m_{ij}^*}$ of object o_j over object o_i. Thus, $\widetilde{c} = \widetilde{m}_{ij}\widetilde{m}_{ji}$ should be correctly computed according to the constrained fuzzy arithmetic (III.41) as $\widetilde{c} = (c^L, c^M, c^U)$:

$$\begin{aligned}
c^L \quad &= \min\left\{m_{ij}m_{ji}; \ m_{ij} \in [m_{ij}^L, m_{ij}^U], m_{ji} \in [m_{ji}^L, m_{ji}^U], m_{ji} = \frac{1}{m_{ij}}\right\} = \\
&= \min\left\{m_{ij}\frac{1}{m_{ij}}; \ m_{ij} \in [m_{ij}^L, m_{ij}^U]\right\} = 1 \\
c^M \quad &= m_{ij}^M m_{ji}^M = m_{ij}^M \frac{1}{m_{ij}^M} = 1 \\
c^U \quad &= \max\left\{m_{ij}m_{ji}; \ m_{ij} \in [m_{ij}^L, m_{ij}^U], m_{ji} \in [m_{ji}^L, m_{ji}^U], m_{ji} = \frac{1}{m_{ij}}\right\} = \\
&= \max\left\{m_{ij}\frac{1}{m_{ij}}; \ m_{ij} \in [m_{ij}^L, m_{ij}^U]\right\} = 1
\end{aligned}$$

Keeping in mind the importance of the multiplicative-reciprocity property of PCs in FMPCMs, multiplicative consistency needs to be defined accordingly so that it does not violate the multiplicative reciprocity. In the following section, two definitions of multiplicatively consistent trapezoidal FMPCMs respecting the multiplicative reciprocity of PCs and invariant under permutation of objects are proposed.

4.2.2.2 New fuzzy extension of multiplicative consistency

In this section, Definition 45 of multiplicative weak consistency given by Wang et al. (2005a) is extended to trapezoidal FMPCMs and another definition of multiplicative consistency much stronger than Definition 45 is proposed. Tools for verifying both the multiplicative weak consistency and the multiplicative consistency are provided and some properties of multiplicatively weakly consistent and multiplicatively consistent trapezoidal FMPCMs are derived. Both definitions preserve two desired properties - invariance under permutation of objects and multiplicative reciprocity of PCs.

Definition 50. *Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$, $\widetilde{m}_{ij} = \left(m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta\right)$, *be a trapezoidal FMPCM.* $\widetilde{M}$ *is said to be multiplicatively weakly consistent if there exists a positive vector* $\underline{w} = (w_1, \ldots, w_n)^T$ *such that*

$$m_{ij}^\alpha \leq \frac{w_i}{w_j} \leq m_{ij}^\delta, \qquad i, j = 1, \ldots, n. \qquad \text{(IV.14)}$$

Notice that when Definition 50 is applied to interval FMPCMs, it is identical to Definition 45 proposed by Wang et al. (2005a).

Proposition 10. *Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$, $\widetilde{m}_{ij} = \left(m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta\right)$, *be a trapezoidal FMPCM.* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$ *is multiplicatively weakly consistent according to Definition 50 if and only if there exist elements* $m_{ij}^* \in \left[m_{ij}^\alpha, m_{ij}^\delta\right]$, $i, j = 1, \ldots, n$, *such that* $M^* = \{m_{ij}^*\}_{i,j=1}^n$ *is a MPCM multiplicatively consistent according to Definition 5.*

Proof. First, let $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$, $\widetilde{m}_{ij} = \left(m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta\right)$, be a trapezoidal FMPCM multiplicatively weakly consistent according to Definition 50. Let us denote $m_{ij}^* := \frac{w_i}{w_j}$. From (IV.14), it follows that $m_{ij}^* \in \left[m_{ij}^\alpha, m_{ij}^\delta\right]$, $i, j =$

$1, \ldots, n$. Further, we have $m_{ii}^* = \frac{w_i}{w_i} = 1$, and $m_{ji}^* = \frac{w_i}{w_j} = \frac{1}{\frac{w_j}{w_i}} = \frac{1}{m_{ij}^*}$, $i, j = 1, \ldots, n$. From $[m_{ij}^\alpha, m_{ij}^\delta] \subseteq [\frac{1}{9}, 9], i, j = 1, \ldots, n$, it follows that also $m_{ij}^* \subseteq [\frac{1}{9}, 9], i, j = 1, \ldots, n$. Therefore, $M^* = \{m_{ij}^*\}_{i,j=1}^n$ is a MPCM. Finally, $m_{ik}^* m_{kj}^* = \frac{w_i}{w_k} \frac{w_k}{w_j} = \frac{w_i}{w_j} = m_{ij}^*$, $i, j, k = 1, \ldots, n$, which means that $M^* = \{m_{ij}^*\}_{i,j=1}^n$ is multiplicatively consistent according to (II.4).

In the opposite direction, let $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n, \widetilde{m}_{ij} = \left(m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta\right)$, be a trapezoidal FMPCM and let $M^* = \{m_{ij}^*\}_{i,j=1}^n, m_{ij}^* \in [m_{ij}^\alpha, m_{ij}^\delta], i, j = 1, \ldots, n$, be a MPCM multiplicatively consistent according to Definition 5. Then, from Proposition 1, it follows that there exists a vector $\underline{w} = (w_1, \ldots, w_n)^T$ such that $m_{ij}^* = \frac{w_i}{w_j}, i, j = 1, \ldots, n$. Because, $m_{ij}^* \in [m_{ij}^\alpha, m_{ij}^\delta], i, j = 1, \ldots, n$, then clearly (IV.14) holds. $\square$

Remark 11. *According to Proposition 10 and its proof, a trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n, \widetilde{m}_{ij} = \left(m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta\right)$, is multiplicatively weakly consistent if and only if there exists a multiplicatively consistent MPCM $M^* = \{m_{ij}^*\}_{i,j=1}^n$ such that $m_{ij}^* \in [m_{ij}^\alpha, m_{ij}^\delta]$. This consistency condition is quite easy to reach. That is why the consistency according to Definition 50 is called weak. Later in this section, also a much stronger definition of multiplicative consistency for trapezoidal FMPCMs will be given.*

Definition 50 of multiplicative weak consistency satisfies two desirable properties - it is invariant under permutation of objects and it preserves the multiplicative reciprocity of PCs in trapezoidal FMPCMs.

Theorem 21. *Definition 50 of multiplicative weak consistency is invariant under permutation of objects in trapezoidal FMPCMs.*

Proof. For a multiplicatively consistent trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n, \widetilde{m}_{ij} = \left(m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta\right)$, there exists a positive priority vector $\underline{w} = (w_1, \ldots, w_n)^T$, such that the inequality $m_{ij}^\alpha \leq \frac{w_i}{w_j} \leq m_{ij}^\delta$ is required to hold for every single PC $\widetilde{m}_{ij}$. By permuting the FMPCM $\widetilde{M}$ to $\widetilde{M}^\pi = P\widetilde{M}P^T$, the original PC $\widetilde{m}_{ij}$ in the $i-$th row and in the $j-$th column of $\widetilde{M}$ is moved to the $\pi(i)-$th row and to the $\pi(j)-$th column of the permuted trapezoidal FMPCM $\widetilde{M}^\pi$ as $\widetilde{m}_{\pi(i)\pi(j)}^\pi$, but still keeping $\widetilde{m}_{ij} = \widetilde{m}_{\pi(i)\pi(j)}^\pi, i, j = 1, \ldots, n$. Thus, there exists a vector $\underline{w}^\pi = (w_1^\pi, \ldots, w_n^\pi)^T$, obtained by permuting the vector $\underline{w}$, i.e. $\underline{w}^\pi = P\underline{w}$, with the components satisfying the inequalities $m_{ij}^{\pi\alpha} \leq \frac{w_i^\pi}{w_j^\pi} \leq m_{ij}^{\pi\delta}$ for every $i, j = 1, \ldots, n$. $\square$

Theorem 22. *Definition 50 of multiplicative weak consistency preserves the multiplicative reciprocity of PCs in trapezoidal FMPCMs in the sense that any fixed value $m_{ij} \in [m_{ij}^\alpha, m_{ij}^\delta], i, j \in \{1, \ldots, n\}$, representing the intensity of preference of object o_i over object o_j is associated with the corresponding value $m_{ji} \in [m_{ji}^\alpha, m_{ji}^\delta]$, representing the intensity of preference of object o_j over object o_i such that $m_{ji} = \frac{1}{m_{ij}}$.*

Proof. The existence of a priority vector $\underline{w} = (w_1, \ldots, w_n)^T$ satisfying the inequalities (IV.14) means that there exists a MPCM $M = \{m_{ij}\}_{i,j=1}^n, m_{ij} \in [m_{ij}^\alpha, m_{ij}^\delta]$, such that $m_{ij} = \frac{w_i}{w_j}, i, j = 1, \ldots, n$. M is multiplicatively reciprocal from the definition, i.e. every PC m_{ij} is associated with the PC m_{ji} such that $m_{ji} = \frac{1}{m_{ij}}$. $\square$

Remark 12. *Note that Theorem 22 does not simply state that a FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$ multiplicatively weakly consistent according to Definition 50 is multiplicatively reciprocal, i.e. $\widetilde{m}_{ji} = \frac{1}{\widetilde{m}_{ij}}, i, j = 1, \ldots, n$. The validity of this property automatically follows from Definition 43 of a FMPCM; every FMPCM is multiplicatively reciprocal, and thus, also a FMPCM that is multiplicatively weakly consistent according to Definition 50 is multiplicatively reciprocal.*

As explained on p. 56, the extension of the multiplicative-reciprocity property from MPCMs to FMPCMs does not concern only the "simple" multiplicative reciprocity of the related fuzzy PCs $\widetilde{m}_{ij}$ and $\widetilde{m}_{ji}$ in the sense that $\widetilde{m}_{ji} = \frac{1}{\widetilde{m}_{ij}}, i, j = 1, \ldots, n$. The conception of the multiplicative reciprocity becomes more complex for FMPCMs. In particular, every intensity of preference $m_{ij}^* \in \widetilde{m}_{ij}$ of object o_i over object o_j is associated inseparably with the corresponding intensity of preference $m_{ji}^* \in \widetilde{m}_{ji}$ such that $m_{ji}^* = \frac{1}{m_{ij}^*}$ since both m_{ij}^* and m_{ji}^* have to express the same preference information about the objects o_i and o_j. Theorem 22 states that Definition 50 is in accordance with this conception of multiplicative reciprocity, i.e. only multiplicatively reciprocal PCs are involved in Definition 50 of multiplicative weak consistency.

Theorem 18 for verifying multiplicative weak consistency of interval FMPCMs can be extended to trapezoidal FMPCMs as follows.

Theorem 23. *A trapezoidal FMPCM* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta} \right),$ *is multiplicatively weakly consistent according to Definition 50 if and only if*

$$\max_{k=1,\ldots,n} \left\{ m_{ik}^{\alpha} m_{kj}^{\alpha} \right\} \leq \min_{k=1,\ldots,n} \left\{ m_{ik}^{\delta} m_{kj}^{\delta} \right\}, \qquad i, j = 1, \ldots, n. \tag{IV.15}$$

Proof. The proof is the same as the proof of Theorem 18; it is sufficient to substitute $m_{ik}^{L}, m_{kj}^{L}, m_{ik}^{U}, m_{kj}^{U}$ with $m_{ik}^{\alpha}, m_{kj}^{\alpha}, m_{ik}^{\delta}, m_{kj}^{\delta},$ respectively. $\qquad\square$

The following theorem shows that it is sufficient to verify the inequality (IV.15) only for $i, j = 1, \ldots, n, \ i < j,$ thus saving half of the computations.

Theorem 24. *A trapezoidal FMPCM* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta} \right),$ *is multiplicatively weakly consistent according to Definition 50 if and only if*

$$\max_{k=1,\ldots,n} \left\{ m_{ik}^{\alpha} m_{kj}^{\alpha} \right\} \leq \min_{k=1,\ldots,n} \left\{ m_{ik}^{\delta} m_{kj}^{\delta} \right\}, \qquad i, j = 1, \ldots, n, \ i < j. \tag{IV.16}$$

Proof. It is sufficient to show that the validity of inequalities (IV.16) for $i, j = 1, \ldots, n, i < j$ implies automatically their validity for all $i, j = 1, \ldots, n,$ i.e. the validity of (IV.15). The validity of inequalities (IV.15) for $i = j$ is trivial from the definition of trapezoidal FMPCMs since

$$\max_{k=1,\ldots,n} \left\{ m_{ik}^{\alpha} m_{ki}^{\alpha} \right\} = \max_{k=1,\ldots,n} \left\{ \frac{m_{ik}^{\alpha}}{m_{ik}^{\delta}} \right\} \leq 1 \leq \min_{k=1,\ldots,n} \left\{ \frac{m_{ik}^{\delta}}{m_{ik}^{\alpha}} \right\} = \min_{k=1,\ldots,n} \left\{ m_{ik}^{\delta} m_{ki}^{\delta} \right\}.$$

Further, for $i > j,$ by using (IV.16) and the multiplicative-reciprocity properties, we get

$$\max_{k=1,\ldots,n} \left\{ m_{ik}^{\alpha} m_{kj}^{\alpha} \right\} = \max_{k=1,\ldots,n} \left\{ \frac{1}{m_{ki}^{\delta}} \frac{1}{m_{jk}^{\delta}} \right\} = \frac{1}{\min_{k=1,\ldots,n} \left\{ m_{jk}^{\delta} m_{ki}^{\delta} \right\}} \leq$$

$$\frac{1}{\max_{k=1,\ldots,n} \left\{ m_{jk}^{\alpha} m_{ki}^{\alpha} \right\}} = \min_{k=1,\ldots,n} \left\{ \frac{1}{m_{jk}^{\alpha}} \frac{1}{m_{ki}^{\alpha}} \right\} = \min_{k=1,\ldots,n} \left\{ m_{ik}^{\delta} m_{kj}^{\delta} \right\}.$$

$\qquad\square$

Remark 13. *An alternative definition of multiplicative weak consistency to Definition 50 might be formulated as follows.*
Let $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta} \right),$ *be a trapezoidal FMPCM.* $\widetilde{M}$ *is said to be multiplicatively weakly consistent if there exists a positive vector* $\underline{w} = (w_1, \ldots, w_n)^T$ *such that*

$$m_{ij}^{\beta} \leq \frac{w_i}{w_j} \leq m_{ij}^{\gamma}, \qquad i, j = 1, \ldots, n. \tag{IV.17}$$

Notice that this definition is stronger than Definition 50. In fact, every trapezoidal FMPCM multiplicatively weakly consistent according to this definition is also multiplicatively weakly consistent according to Definition 50 since (IV.17) implies (IV.14). Further, when this definition is applied to interval FMPCMs, it is again identical to Definition 45 proposed by Wang et al. (2005a).
All theorems regarding FMPCMs multiplicatively weakly consistent according to Definition 50 formulated above can be easily reformulated for FMPCMs multiplicatively weakly consistent according to this definition; it is sufficient to consider m_{ij}^{β} *and* m_{ij}^{γ} *instead of* m_{ij}^{α} *and* $m_{ij}^{\delta},$ *respectively, where appropriate.*

In the following definition, a stronger version of multiplicative consistency for trapezoidal FMPCMs is formulated.

Definition 51. *Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta} \right),$ *be a trapezoidal FMPCM.* $\widetilde{M}$ *is said to be multiplicatively consistent if for each triplet* $(i, j, k) \subseteq \{1, \ldots, n\}$ *the following holds:*

$$\forall m_{ij} \in \left[m_{ij}^{\alpha}, m_{ij}^{\delta} \right] \exists m_{ik} \in \left[m_{ik}^{\alpha}, m_{ik}^{\delta} \right] \wedge \exists m_{kj} \in \left[m_{kj}^{\alpha}, m_{kj}^{\delta} \right] : m_{ij} = m_{ik} m_{kj}, \tag{IV.18}$$

$$\forall m_{ij} \in \left[m_{ij}^{\beta}, m_{ij}^{\gamma} \right] \exists m_{ik} \in \left[m_{ik}^{\beta}, m_{ik}^{\gamma} \right] \wedge \exists m_{kj} \in \left[m_{kj}^{\beta}, m_{kj}^{\gamma} \right] : m_{ij} = m_{ik} m_{kj}. \tag{IV.19}$$

Remark 14. *Definition 51 is a natural fuzzy extension of Definition 5 of multiplicative consistency proposed by Saaty (1980). According to this definition, for any value $m_{ij} \in \widetilde{m}_{ij}, i, j \in \{1, \ldots, n\}$, there exist values $m_{ik} \in \widetilde{m}_{ik}$ and $m_{kj} \in \widetilde{m}_{kj}, k \in \{1, \ldots, n\}$, such that they satisfy the multiplicative-consistency property* (II.4). *Analogously, for any value $m_{ij} \in Core\ \widetilde{m}_{ij}, i, j \in \{1, \ldots, n\}$, there exist possible values $m_{ik} \in Core\ \widetilde{m}_{ik}$ and $m_{kj} \in Core\ \widetilde{m}_{kj}, k \in \{1, \ldots, n\}$, such that they satisfy* (II.4). *Clearly, in comparison to the multiplicative weak consistency given by Definition 50, the multiplicative consistency given by Definition 51 is very strong.*

Unlike Definitions 46 and 47 of multiplicative consistency for FMPCMs proposed by Liu (2009) and Liu et al. (2014), respectively, Definition 51 is invariant under permutation of objects compared in trapezoidal FMPCMs.

Theorem 25. *Definition 51 of multiplicative consistency is invariant under permutation of objects in trapezoidal FMPCMs.*

Proof. For a multiplicatively consistent trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$, $\widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, the conditions (IV.18) and (IV.19) are satisfied for every triplet $(i, j, k) \subseteq \{1, \ldots, n\}$. By permuting the FMPCM $\widetilde{M}$ to $\widetilde{M}^{\pi} = P\widetilde{M}P^{T}$, the original PC $\widetilde{m}_{ij}$ in the i−th row and in the j−th column of $\widetilde{M}$ moves to the $\pi(i)$−th row and to the $\pi(j)$−th column of $\widetilde{M}^{\pi}$ preserving $\widetilde{m}_{\pi(i)\pi(j)}^{\pi} = \widetilde{m}_{ij}$. Thus, by permuting $\widetilde{M}$, also the validity of the conditions (IV.18) and (IV.19) is preserved, i.e.

$$\forall m_{ij}^{\pi} \in \left[m_{ij}^{\pi\alpha}, m_{ij}^{\pi\delta}\right] \exists m_{ik}^{\pi} \in \left[m_{ik}^{\pi\alpha}, m_{ik}^{\pi\delta}\right] \ \wedge \ \exists m_{kj}^{\pi} \in \left[m_{kj}^{\pi\alpha}, m_{kj}^{\pi\delta}\right]: \ m_{ij}^{\pi} = m_{ik}^{\pi}m_{kj}^{\pi},$$

$$\forall m_{ij}^{\pi} \in \left[m_{ij}^{\pi\beta}, m_{ij}^{\pi\gamma}\right] \exists m_{ik}^{\pi} \in \left[m_{ik}^{\pi\beta}, m_{ik}^{\pi\gamma}\right] \ \wedge \ \exists m_{kj}^{\pi} \in \left[m_{kj}^{\pi\beta}, m_{kj}^{\pi\gamma}\right]: \ m_{ij}^{\pi} = m_{ik}^{\pi}m_{kj}^{\pi},$$

for every triplet $(i, j, k) \subseteq \{1, \ldots, n\}$. Thus, $\widetilde{M}^{\pi}$ is multiplicatively consistent according to Definition 51. $\square$

Further, unlike Definitions 48 and 49 of multiplicatively consistent interval and triangular FMPCMs proposed by Wang (2015a) and Wang (2015b), respectively, new Definition 51 does not violate the multiplicative reciprocity of the related PCs.

Theorem 26. *Definition 51 of multiplicative consistency preserves the multiplicative reciprocity of PCs in trapezoidal FMPCMs in the sense that any fixed value $m_{ij} \in \left[m_{ij}^{\alpha}, m_{ij}^{\delta}\right], i, j \in \{1, \ldots, n\}$, representing the intensity of preference of object o_i over object o_j is associated with the corresponding value $m_{ji} \in \left[m_{ji}^{\alpha}, m_{ji}^{\delta}\right]$ representing the intensity of preference of object o_j over object o_i such that $m_{ji} = \frac{1}{m_{ij}}$.*

Proof. It is sufficient to show that expressions (IV.18) and (IV.19) do not violate the multiplicative-reciprocity property in the sense that when two particular intensities of preference $m_{ij} \in \widetilde{m}_{ij}$ and $m_{ji} \in \widetilde{m}_{ji}$ on the pair of objects o_i and o_j are considered at the same time in the expressions (IV.18) and (IV.19), then they are such that $m_{ji} = \frac{1}{m_{ij}}$.

For a triplet $(i, j, k) \subseteq \{1, \ldots, n\}$, $i \neq j \neq k$, no reciprocals appear in expression $m_{ij} = m_{ik}m_{kj}$ for any $m_{ij} \in \left[m_{ij}^{\alpha}, m_{ij}^{\delta}\right]$. For $i = j = k$, expression (IV.18) reduces to: $\forall m_{ii} = 1 \exists m_{ii}^{*} = 1 \wedge \exists m_{ii}^{**} = 1 : 1 = 1 \cdot 1$, which again does not violate the multiplicative reciprocity. Further, for $i \neq j = k$, expression (IV.18) is as: $\forall m_{ij} \in \left[m_{ij}^{\alpha}, m_{ij}^{\delta}\right] \exists m_{ij}^{*} \in \left[m_{ij}^{\alpha}, m_{ij}^{\delta}\right] \wedge \exists m_{jj} = 1 : m_{ij} = m_{ij}^{*} \cdot 1$. This means that $m_{ij}^{*} = m_{ij}$ and, therefore, the multiplicative reciprocity is not violated. For $i = k \neq j$ the proof is analogous. Finally, for $i = j \neq k$, expression (IV.18) is as

$$\forall m_{ii} = 1 \exists m_{ik} \in \left[m_{ik}^{\alpha}, m_{ik}^{\delta}\right] \wedge \exists m_{ki}^{*} \in \left[m_{ki}^{\alpha}, m_{ki}^{\delta}\right] + : 1 = m_{ik}m_{ki}^{*}.$$

This means that $m_{ki}^{*} = \frac{1}{m_{ik}}$ and, therefore, the multiplicative reciprocity is preserved.

The proof for the expression (IV.19) is analogous. $\square$

Remark 15. *Similarly to Theorem 22, also Theorem 26 does not simply state that a FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$ multiplicatively consistent according to Definition 51 is multiplicatively reciprocal since this property automatically follows from Definition 43 of a FMPCM. Theorem 26 states that Definition 51 is in accordance with the conception of multiplicative reciprocity discussed on p. 56, i.e. only multiplicatively reciprocal PCs are involved in Definition 51 of multiplicative consistency. For more details, see Remark 12.*

By handling properly the multiplicative reciprocity of the related PCs, Theorem 1 can be extended to trapezoidal FMPCMs as follows.

Theorem 27. *For a trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$, $\widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta}\right)$, the following statements are equivalent:*

(i) $\widetilde{M}$ is multiplicatively consistent according to Definition 51.

(ii) For every $i, j, k = 1, \ldots, n$:

$$\forall m_{ij} \in \left[m_{ij}^{\alpha}, m_{ij}^{\delta}\right] \exists m_{jk} \in \left[m_{jk}^{\alpha}, m_{jk}^{\delta}\right] \wedge \exists m_{ki} \in \left[m_{ki}^{\alpha}, m_{ki}^{\delta}\right] : m_{ij} m_{jk} m_{ki} = 1, \qquad \text{(IV.20)}$$

$$\forall m_{ij} \in \left[m_{ij}^{\beta}, m_{ij}^{\gamma}\right] \exists m_{jk} \in \left[m_{jk}^{\beta}, m_{jk}^{\gamma}\right] \wedge \exists m_{ki} \in \left[m_{ki}^{\beta}, m_{ki}^{\gamma}\right] : m_{ij} m_{jk} m_{ki} = 1, \qquad \text{(IV.21)}$$

(iii) For every $i, j, k = 1, \ldots, n$:

$$\forall m_{ij} \in \left[m_{ij}^{\alpha}, m_{ij}^{\delta}\right] \exists m_{jk} \in \left[m_{jk}^{\alpha}, m_{jk}^{\delta}\right] \wedge \exists m_{ki} \in \left[m_{ki}^{\alpha}, m_{ki}^{\delta}\right] :$$
$$m_{ij} m_{jk} m_{ki} = m_{ik} m_{kj} m_{ji}, m_{ji} = \frac{1}{m_{ij}}, m_{ki} = \frac{1}{m_{ik}}, m_{jk} = \frac{1}{m_{kj}}, \quad \text{(IV.22)}$$

$$\forall m_{ij} \in \left[m_{ij}^{\beta}, m_{ij}^{\gamma}\right] \exists m_{jk} \in \left[m_{jk}^{\beta}, m_{jk}^{\gamma}\right] \wedge \exists m_{ki} \in \left[m_{ki}^{\beta}, m_{ki}^{\gamma}\right] :$$
$$m_{ij} m_{jk} m_{ki} = m_{ik} m_{kj} m_{ji}, m_{ji} = \frac{1}{m_{ij}}, m_{ki} = \frac{1}{m_{ik}}, m_{jk} = \frac{1}{m_{kj}}. \quad \text{(IV.23)}$$

Proof. From the multiplicative-reciprocity property $\widetilde{m}_{ij} = \frac{1}{\widetilde{m}_{ji}}, i, j = 1, \ldots, n$, it follows that $\forall m_{ij} \in \left[m_{ij}^{\alpha}, m_{ij}^{\delta}\right]$ $\exists m_{ji} \in \left[m_{ji}^{\alpha}, m_{ji}^{\delta}\right] : m_{ji} = \frac{1}{m_{ij}}$, and $\forall m_{ij} \in \left[m_{ij}^{\beta}, m_{ij}^{\gamma}\right] \exists m_{ji} \in \left[m_{ji}^{\beta}, m_{ji}^{\gamma}\right] : m_{ji} = \frac{1}{m_{ij}}$.

(a) First, let us show that the statements (i) and (ii) are equivalent. Because of the multiplicative-reciprocity property, (IV.18) can be equivalently written as

$$\forall m_{ij} \in \left[m_{ij}^{\alpha}, m_{ij}^{\delta}\right] \exists m_{ki} \in \left[m_{ki}^{\alpha}, m_{ki}^{\delta}\right] \wedge \exists m_{jk} \in \left[m_{jk}^{\alpha}, m_{jk}^{\delta}\right] : m_{ij} = \frac{1}{m_{jk}} \frac{1}{m_{ki}},$$

which is equivalent to (IV.20). Analogously, the equivalence of (IV.19) and (IV.21) is proved.

(b) Now, let us show that the statements (ii) and (iii) are equivalent. The expression (IV.20) can be equivalently written as

$$\forall m_{ij} \in \left[m_{ij}^{\alpha}, m_{ij}^{\delta}\right] \exists m_{jk} \in \left[m_{jk}^{\alpha}, m_{jk}^{\delta}\right] \wedge \exists m_{ki} \in \left[m_{ki}^{\alpha}, m_{ki}^{\delta}\right] : m_{ij}^2 m_{jk}^2 m_{ki}^2 = 1. \qquad \text{(IV.24)}$$

Because all fuzzy numbers in a trapezoidal FMPCM are positive, the second power can be removed from expression $m_{ij}^2 m_{jk}^2 m_{ki}^2 = 1$, which means that (IV.20) is equivalent to (IV.22). Analogously, the equivalence of (IV.21) and (IV.23) is proved.

$\square$

The following theorems give us useful tools for verifying multiplicative consistency of trapezoidal FMPCMs.

Theorem 28. *A trapezoidal FMPCM* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta}\right),$ *is multiplicatively consistent according to Definition 51 if and only if the inequalities*

$$m_{ij}^{\alpha} \geq m_{ik}^{\alpha} m_{kj}^{\alpha}, \qquad m_{ij}^{\delta} \leq m_{ik}^{\delta} m_{kj}^{\delta}, \qquad \text{(IV.25)}$$
$$m_{ij}^{\beta} \geq m_{ik}^{\beta} m_{kj}^{\beta}, \qquad m_{ij}^{\gamma} \leq m_{ik}^{\gamma} m_{kj}^{\gamma}, \qquad \text{(IV.26)}$$

hold for every $i, j, k = 1, \ldots, n, \ i < j, \ k \neq i, j$.

Proof. It is sufficient to demonstrate the equivalence of the expressions (IV.25) and (IV.18). The demonstration of the equivalence of (IV.26) and (IV.19) is analogous.

First, let us demonstrate that when the inequalities (IV.25) hold for every $i, j, k = 1, \ldots, n, \ i < j, \ k \neq i, j$, then they hold for every $i, j, k = 1, \ldots, n$. The inequalities (IV.25) are always satisfied for $i, j, k = 1, \ldots, n$, such that $i = j \neq k$, or $i \neq j = k$, or $j \neq k = i$, or $i = j = k$:

$$m_{ik}^{\alpha} m_{ki}^{\alpha} = \frac{m_{ik}^{\alpha}}{m_{ik}^{\delta}} \leq 1 = m_{ii}^{\alpha}, \qquad m_{ik}^{\delta} m_{ki}^{\delta} = \frac{m_{ik}^{\delta}}{m_{ik}^{\alpha}} \geq 1 = m_{ii}^{\delta},$$

$$m_{ij}^{\alpha} m_{jj}^{\alpha} = m_{ij}^{\alpha}, \qquad m_{ij}^{\delta} m_{jj}^{\delta} = m_{ij}^{\delta},$$

$$m_{ii}^{\alpha} m_{ij}^{\alpha} = m_{ij}^{\alpha}, \qquad m_{ii}^{\delta} m_{ij}^{\delta} = m_{ij}^{\delta},$$

$$m_{ii}^\alpha m_{ii}^\alpha = 1 = m_{ii}^\alpha, \qquad m_{ii}^\delta m_{ii}^\delta = 1 = m_{ij}^\delta.$$

Further, when the inequalities (IV.25) are satisfied for $i, j, k = 1, \ldots, n,\ i < j,\ k \neq i, j$, then they are satisfied also for $j, i, k = 1, \ldots, n,\ j > i,\ k \neq i, j$:

$$m_{jk}^\alpha m_{ki}^\alpha = \frac{1}{m_{kj}^\delta} \frac{1}{m_{ik}^\delta} = \frac{1}{m_{ik}^\delta m_{kj}^\delta} \leq \frac{1}{m_{ij}^\delta} = m_{ji}^\alpha,$$

$$m_{jk}^\delta m_{ki}^\delta = \frac{1}{m_{kj}^\alpha} \frac{1}{m_{ik}^\alpha} = \frac{1}{m_{ik}^\alpha m_{kj}^\alpha} \geq \frac{1}{m_{ij}^\alpha} = m_{ji}^\delta.$$

To finalize the proof, it is sufficient to show that the inequalities (IV.25) are equivalent to the condition (IV.18) for every $i, j, k = 1, \ldots, n$. First, let $\widetilde{M}$ be a trapezoidal FMPCM multiplicatively consistent according to Definition 51. Then, for $m_{ij} := m_{ij}^\alpha \ \exists m_{ik} \in \left[m_{ik}^\alpha, m_{ik}^\delta\right] \wedge \exists m_{kj} \in \left[m_{kj}^\alpha, m_{kj}^\delta\right] : m_{ij}^\alpha = m_{ik}m_{kj}$. Since $m_{ik} \geq m_{ik}^\alpha, m_{kj} \geq m_{kj}^\alpha$, then clearly $m_{ij}^\alpha \geq m_{ik}^\alpha m_{kj}^\alpha$. Analogously, for $m_{ij} := m_{ij}^\delta \ \exists m_{ik} \in \left[m_{ik}^\alpha, m_{ik}^\delta\right] \wedge \exists m_{kj} \in \left[m_{kj}^\alpha, m_{kj}^\delta\right] : m_{ij}^\delta = m_{ik}m_{kj}$. Since $m_{ik} \leq m_{ik}^\delta, m_{kj} \leq m_{kj}^\delta$, then $m_{ij}^\delta \leq m_{ik}^\delta m_{kj}^\delta$.

Second, let (IV.25) be valid for a trapezoidal FMPCM $\widetilde{M}$. Then, from inequalities (IV.25) we get $\forall m_{ij} \in \left[m_{ij}^\alpha, m_{ij}^\delta\right] : m_{ik}^\alpha m_{kj}^\alpha \leq m_{ij} \leq m_{ik}^\delta m_{kj}^\delta$ and, therefore, (IV.18) is satisfied. □

Theorem 29. *A trapezoidal FMPCM* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n, \widetilde{m}_{ij} = \left(m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta\right)$, *is multiplicatively consistent according to Definition 51 if and only if the inequalities*

$$m_{ij}^\alpha \geq \max_{\substack{k=1,\ldots,n \\ k \neq i,j}} \left\{m_{ik}^\alpha m_{kj}^\alpha\right\}, \qquad m_{ij}^\delta \leq \min_{\substack{k=1,\ldots,n \\ k \neq i,j}} \left\{m_{ik}^\delta m_{kj}^\delta\right\}, \tag{IV.27}$$

$$m_{ij}^\beta \geq \min_{\substack{k=1,\ldots,n \\ k \neq i,j}} \left\{m_{ik}^\beta m_{kj}^\beta\right\}, \qquad m_{ij}^\gamma \leq \min_{\substack{k=1,\ldots,n \\ k \neq i,j}} \left\{m_{ik}^\gamma m_{kj}^\gamma\right\}, \tag{IV.28}$$

hold for every $i, j = 1, \ldots, n,\ i < j$.

Proof. The inequalities (IV.27) and (IV.28) follow immediately from Theorem 28. □

In the following example, Definition 51 of multiplicative consistency for trapezoidal FMPCMs is confronted with Definitions 46 and 48. In particular, it is demonstrated how the drawbacks regarding the dependence of Definition 46 on permutation of objects and violation of multiplicative-reciprocity property in Definition 48 are removed by Definition 51. Further, multiplicative weak consistency according to Definition 50 is examined.

Example 31. Let us examine the interval FMPCM $\overline{M}$ of objects o_1, o_2, o_3 given as

$$\overline{M} = \begin{pmatrix} 1 & \left[\frac{2}{5}, \frac{2}{3}\right] & \left[\frac{1}{5}, \frac{2}{3}\right] \\ \left[\frac{3}{2}, \frac{5}{2}\right] & 1 & \left[\frac{1}{2}, 1\right] \\ \left[\frac{3}{2}, 5\right] & \left[1, 2\right] & 1 \end{pmatrix}. \tag{IV.29}$$

and its permutation $\overline{M}^\pi = P\overline{M}P^T$ given as

$$\overline{M}^\pi = \begin{pmatrix} 1 & \left[\frac{1}{2}, 1\right] & \left[\frac{3}{2}, \frac{5}{2}\right] \\ \left[1, 2\right] & 1 & \left[\frac{3}{2}, 5\right] \\ \left[\frac{2}{5}, \frac{2}{3}\right] & \left[\frac{1}{5}, \frac{2}{3}\right] & 1 \end{pmatrix} \tag{IV.30}$$

which is obtained from $\overline{M}$ by applying the permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{IV.31}$$

As pointed out by Wang (2015a) and Wang (2015b) and mentioned in Section 4.2.2.1, Definitions 46 and 47 of multiplicative consistency for interval and trapezoidal FMPCMs, respectively, are not invariant under permutation of objects. In fact, the interval FMPCM (IV.29) results to be multiplicatively consistent, and its permutation (IV.30) results to be inconsistent according to Definition 46.

Now, let us apply Definition 51 to the interval FMPCM (IV.29). By using Theorem 28, the interval FMPCM (IV.29) is judged as multiplicatively consistent since it satisfies the inequalities (IV.25); see Tab. IV.3. Also

Table IV.3: Inequality conditions (IV.25) for the interval FMPCM (IV.29).

$i < j:$	$m_{ij}^L \geq m_{ik}^L m_{kj}^L$	$m_{ij}^U \leq m_{ik}^U m_{kj}^U$
$1, 2:$	$\frac{2}{5} \geq \frac{1}{5} \cdot 1$	$\frac{2}{3} \leq \frac{2}{3} \cdot 2$
$1, 3:$	$\frac{1}{5} \geq \frac{2}{5} \cdot \frac{1}{2}$	$\frac{2}{3} \leq \frac{2}{3} \cdot 1$
$2, 3:$	$\frac{1}{2} \geq \frac{3}{2} \cdot \frac{1}{5}$	$1 \leq \frac{5}{2} \cdot \frac{2}{3}$

Table IV.4: Inequality conditions (IV.25) for the permuted interval FMPCM (IV.30).

$i < j:$	$m_{ij}^L \geq m_{ik}^L m_{kj}^L$	$m_{ij}^U \leq m_{ik}^U m_{kj}^U$
$1, 2:$	$\frac{1}{2} \geq \frac{3}{2} \cdot \frac{1}{5}$	$1 \leq \frac{5}{2} \cdot \frac{2}{3}$
$1, 3:$	$\frac{3}{2} \geq \frac{1}{2} \cdot \frac{3}{2}$	$\frac{5}{2} \leq 1 \cdot 5$
$2, 3:$	$\frac{3}{2} \geq 1 \cdot \frac{3}{2}$	$5 \leq 2 \cdot \frac{5}{2}$

the permuted interval FMPCM (IV.30) satisfies the inequalities (IV.25); see Tab. IV.4. Therefore, it is again judged as multiplicatively consistent. Moreover, from Theorem 25 it follows that any permutation of the interval FMPCM (IV.29) is multiplicatively consistent according to Definition 51.

In Example 30, it was demonstrated that Definition 48 violates the multiplicative reciprocity of PCs. In fact, by using the property (iii) of Theorem 20 (more precisely a version of the theorem adapted for interval FMPCMs), the multiplicative consistency of the matrix

$$M^L = \begin{pmatrix} 1 & \frac{2}{5} & \frac{1}{5} \\ \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{3}{2} & 1 & 1 \end{pmatrix} \tag{IV.32}$$

is checked in order to verify multiplicative consistency of the interval FMPCM (IV.29) according to Definition 48. The matrix (IV.32) is not multiplicatively reciprocal, and thus it is not even a MPCM. Therefore, verifying its consistency is nonsensical.

According to Theorem 26, the multiplicative-reciprocity property is preserved by new Definition 51. This basically means that by taking any value from any interval PC in the interval FMPCM (IV.29), there exist values in the remaining interval PCs such that they form a multiplicatively consistent MPCM. Let us examine the triplet $i = 1, j = 2, k = 3$ of indices and let us consider the value $m_{12} = \frac{1}{2} \in \left[\frac{2}{5}, \frac{2}{3}\right]$. Then, according to (IV.18), there exist values $m_{13} \in \left[\frac{1}{5}, \frac{2}{3}\right]$ and $m_{32} \in [1, 2]$ such that $\frac{1}{2} = m_{13} m_{32}$. It is, for example, $m_{13} = \frac{1}{4}, m_{32} = 2$. The multiplicative reciprocity is clearly not violated. More interestingly, let us consider the triplet $i = 1, j = 1, k = 2$. Then, according to (IV.18), there exist values $m_{12} \in \left[\frac{2}{5}, \frac{2}{3}\right]$ and $m_{21} \in \left[\frac{3}{2}, \frac{5}{2}\right]$ such that $1 = m_{12} m_{21}$. This equality is satisfied by any value $m_{12} \in \left[\frac{2}{5}, \frac{2}{3}\right]$ and the corresponding value $m_{21} \in \left[\frac{3}{2}, \frac{5}{2}\right]$ such that $m_{21} = \frac{1}{m_{12}}$, which again preserves the multiplicative reciprocity.

By verifying the inequalities (IV.16) we also find out that the interval FMPCM (IV.29) is multiplicatively consistent according to Definition 50; see Tab. IV.5. Analogously it could be shown that also the permuted interval FMPCM (IV.30) is multiplicatively weakly consistent according to Definition 50. $\triangle$

In the following example, Definition 51 of multiplicative consistency is applied to an incomplete FMPCM in order to identify a missing PC.

Table IV.5: Inequality conditions (IV.16) for the interval FMPCM (IV.29).

$i < j:$	$\displaystyle\max_{k=1,\ldots,n}\left\{m_{ik}^L m_{kj}^L\right\}$	$\leq$	$\displaystyle\min_{k=1,\ldots,n}\left\{m_{ik}^U m_{kj}^U\right\}$
$1, 2:$	$\max\left\{\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right\}$	$\leq$	$\min\left\{\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\right\}$
$1, 3:$	$\max\left\{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right\}$	$\leq$	$\min\left\{\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right\}$
$2, 3:$	$\max\left\{\frac{3}{10}, \frac{1}{2}, \frac{1}{2}\right\}$	$\leq$	$\min\left\{\frac{5}{3}, 1, 1\right\}$

Example 32. Let us consider the interval FMPCM in the form (IV.11) with unknown interval PCs $\overline{m}_{13}$ and $\overline{m}_{31} = \frac{1}{\overline{m}_{13}}$ examined in Example 30. It was shown in example 30 that applying Definition 48 to identify the missing PCs leads to unreasonable results.

Now let us apply Definition 51 to the interval FMPCM (IV.11). By using Theorem 29, we obtain the following:

$$
\begin{aligned}
i = 1, j = 2: \quad & \tfrac{3}{2} \geq x \cdot \tfrac{1}{2} = \tfrac{x}{2} \Rightarrow x \leq 3, \quad && 2 \leq y \cdot \tfrac{2}{3} \Rightarrow y \geq 3 \\
i = 1, j = 3: \quad & x \geq \tfrac{3}{2} \cdot \tfrac{3}{2} = \tfrac{9}{4} \Rightarrow x \geq \tfrac{9}{4}, \quad && y \leq 2 \cdot 2 \Rightarrow y \leq 4 \\
i = 2, j = 3: \quad & \tfrac{3}{2} \geq \tfrac{1}{2} \cdot x = \tfrac{x}{2} \Rightarrow x \leq 3, \quad && 2 \leq \tfrac{2}{3} \cdot y \Rightarrow y \geq 3
\end{aligned}
$$

Therefore, the interval FMPCM (IV.11) is multiplicatively consistent according to Definition 51 if $\overline{m}_{13} = [x, y], x \leq y$, is such that $x \in \left[\frac{9}{4}, 3\right], y \in [3, 4]$. This means that the lowest possible intensity of preference of object o_1 over object o_3 is at least $\frac{9}{4} > 1$, i.e. object o_1 is preferred to object o_3. Moreover, the highest possible intensity of preference of object o_1 over object o_3 is 4, which is reachable under the multiplicative consistency condition for $m_{12} = 2 \in \left[\frac{3}{2}, 2\right], m_{23} = 2 \in \left[\frac{3}{2}, 2\right]$. $\triangle$

In the rest of this section, some interesting properties of multiplicatively weakly consistent and multiplicatively consistent trapezoidal FMPCMs are examined. The following theorem shows the relation between Definition 51 of multiplicative consistency and Definition 50 of multiplicative weak consistency.

Theorem 30. *Let $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, be a trapezoidal FMPCM. If $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$ is multiplicatively consistent according to Definition 51, then it is also multiplicatively weakly consistent according to Definition 50.*

Proof. The statement follows immediately from Theorem 29. In particular, the inequality (IV.16) is obtained immediately from the inequalities (IV.27). $\square$

Remark 16. *According to Theorem 30, when a trapezoidal FMPCM is multiplicatively consistent according to Definition 51 then it is also automatically multiplicatively weakly consistent according to Definition 50. However, this does not hold the other way around. Clearly, the definition of multiplicative weak consistency is much weaker than the definition of multiplicative consistency; it only requires existence of one crisp multiplicatively consistent MPCM obtainable by combining particular elements from the closures of the supports of the trapezoidal fuzzy numbers in the trapezoidal FMPCM. Thus, the set of all trapezoidal FMPCMs multiplicatively consistent according to Definition 51 is a proper subset of the set of all trapezoidal FMPCMs multiplicatively weakly consistent according to Definition 50.*

In the following example, the multiplicative consistency given by Definition 51 and the multiplicative weak consistency given by Definition 50 are examined.

Example 33. Let us consider the trapezoidal FMPCM

$$
\widetilde{M} = \begin{pmatrix}
1 & (2,3,4,5) & (2,3,3,4) & (1,1,2,3) \\
\left(\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right) & 1 & (1,2,3,3) & \left(1, \frac{3}{2}, 2, 2\right) \\
\left(\frac{1}{4}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}\right) & \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, 1\right) & 1 & \left(\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right) \\
\left(\frac{1}{3}, \frac{1}{2}, 1, 1\right) & \left(\frac{1}{2}, \frac{1}{2}, \frac{2}{3}, 1\right) & \left(1, \frac{5}{4}, \frac{5}{3}, \frac{5}{2}\right) & 1
\end{pmatrix}.
\tag{IV.33}
$$

$\widetilde{M}$ is not multiplicatively consistent according to new Definition 51 of multiplicative consistency since inequalities (IV.27) are violated; e.g.,

$$
\max_{k=2,3} \{m_{1k}^{\alpha} m_{k4}^{\alpha}\} = \max\left\{2, \frac{4}{5}\right\} = 2 \not\leq 1 = m_{14}^{\alpha}.
$$

However, $\widetilde{M}$ is multiplicatively consistent according to Definition 50 since inequalities (IV.16) are satisfied; see Tab. IV.6. Therefore, according to Definition 50, there exists at least one multiplicatively consistent MPCM $M = \left\{\frac{w_i}{w_j}\right\}_{i,j=1}^{n}$, such that $m_{ij}^{\alpha} \leq \frac{w_i}{w_j} \leq m_{ij}^{\delta}$, $i, j = 1, \ldots, n$, $\sum_{i=1}^{n} w_i = 1$. It is, for example,

$$
M = \begin{pmatrix}
1 & 2 & 2 & 2 \\
\frac{1}{2} & 1 & 1 & 1 \\
\frac{1}{2} & 1 & 1 & 1 \\
\frac{1}{2} & 1 & 1 & 1
\end{pmatrix},
$$

with the priority vector $\underline{w} = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)^T$. $\triangle$

Table IV.6: Inequality conditions (IV.16) for the interval FMPCM (IV.33).

$i < j :$	$\max\limits_{k=1,\dots,4} \left\{ m_{ik}^{\alpha} m_{kj}^{\alpha} \right\}$	$\leq$	$\min\limits_{k=1,\dots,4} \left\{ m_{ik}^{\delta} m_{kj}^{\delta} \right\}$
$1, 2 :$	$\max\left\{ 2, 2, \frac{2}{3}, \frac{1}{2} \right\}$	$\leq$	$\min\{5, 5, 4, 3\}$
$1, 3 :$	$\max\{2, 2, 2, 1\}$	$\leq$	$\min\left\{4, 15, 4, \frac{15}{2}\right\}$
$1, 4 :$	$\max\left\{1, 2, \frac{4}{5}, 1\right\}$	$\leq$	$\min\{3, 10, 4, 3\}$
$2, 3 :$	$\max\left\{\frac{2}{5}, 1, 1, 1\right\}$	$\leq$	$\min\{2, 3, 3, 5\}$
$2, 4 :$	$\max\left\{\frac{1}{5}, 1, \frac{2}{5}, 1\right\}$	$\leq$	$\min\left\{\frac{3}{2}, 2, 3, 2\right\}$
$3, 4 :$	$\max\left\{\frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{2}{5}\right\}$	$\leq$	$\min\left\{\frac{3}{2}, 2, 1, 1\right\}$

Theorem 31. *Let $\widetilde{M}$ be a trapezoidal FMPCM multiplicatively weakly consistent according to Definition 50. A trapezoidal FMPCM $\widetilde{M}^*$ constructed by eliminating the l-th row and the l-th column, $l \in \{1, \dots, n\}$, of $\widetilde{M}$ is again multiplicatively weakly consistent.*

Proof. For $\widetilde{M}$, the inequalities (IV.16) are valid for every $i, j, k = 1, \dots, n$. After eliminating the l-th row and the l-th column of $\widetilde{M}$, the inequalities (IV.16) are still valid for every remaining $i, j, k \in \{1, \dots, n\} \setminus \{l\}$. Therefore, the new trapezoidal FMPCM $\widetilde{M}^*$ is still multiplicatively weakly consistent. $\qquad\square$

The same holds also for multiplicatively consistent trapezoidal FMPCMs.

Theorem 32. *Let $\widetilde{M}$ be a trapezoidal FMPCM multiplicatively consistent according to Definition 51. A trapezoidal FMPCM $\widetilde{M}^*$ constructed by eliminating the l-th row and the l-th column, $l \in \{1, \dots, n\}$, of $\widetilde{M}$ is again multiplicatively consistent.*

Proof. For $\widetilde{M}$, the inequalities (IV.27) and (IV.28) are valid for each $i, j, k \in \{1, \dots, n\}$. After eliminating the l-th row and the l-th column of $\widetilde{M}$, (IV.27) and (IV.28) are still valid for each remaining $i, j, k \in \{1, \dots, n\} \setminus \{l\}$. Therefore, the new trapezoidal FMPCM $\widetilde{M}^*$ is still multiplicatively consistent. $\qquad\square$

Remark 17. *Theorems 31 and 32 are useful in situations when the set of objects compared pairwisely is being reduced. According to the theorems, elimination of one or more objects has no impact on the multiplicative or multiplicative weak consistency of fuzzy PCs of the remaining objects.*

The following theorems provide results regarding aggregation of multiplicatively and multiplicatively weakly consistent trapezoidal FMPCMs, which are particularly useful in group decision making.

Theorem 33. *Let $\widetilde{M}^1 = \left\{ \widetilde{m}_{ij}^1 \right\}_{i,j=1}^n, \widetilde{m}_{ij}^1 = (m_{ij}^{1\alpha}, m_{ij}^{1\beta}, m_{ij}^{1\gamma}, m_{ij}^{1\delta})$, and $\widetilde{M}^2 = \left\{ \widetilde{m}_{ij}^2 \right\}_{i,j=1}^n, \widetilde{m}_{ij}^2 = (m_{ij}^{2\alpha}, m_{ij}^{2\beta}, m_{ij}^{2\gamma}, m_{ij}^{2\delta})$, be trapezoidal FMPCMs multiplicatively weakly consistent according to Definition 50. Then $\widetilde{M} = \left\{ \widetilde{m}_{ij} \right\}_{i,j=1}^n, \widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, such that*

$$m_{ij}^{\alpha} = (m_{ij}^{1\alpha})^{\epsilon}(m_{ij}^{2\alpha})^{1-\epsilon}, \qquad m_{ij}^{\beta} = (m_{ij}^{1\beta})^{\epsilon}(m_{ij}^{2\beta})^{1-\epsilon},$$
$$m_{ij}^{\gamma} = (m_{ij}^{1\gamma})^{\epsilon}(m_{ij}^{2\gamma})^{1-\epsilon}, \qquad m_{ij}^{\delta} = (m_{ij}^{1\delta})^{\epsilon}(m_{ij}^{2\delta})^{1-\epsilon},$$

is a multiplicatively weakly consistent trapezoidal FMPCM for any $\epsilon \in [0, 1]$.

Proof. First, let us show that $\widetilde{M}$ is a trapezoidal FMPCM. For $i = 1, \dots, n$, we get

$$m_{ii}^{\alpha} = (m_{ii}^{1\alpha})^{\epsilon}(m_{ii}^{2\alpha})^{1-\epsilon} = 1^{\epsilon}1^{1-\epsilon} = 1, \quad m_{ii}^{\delta} = (m_{ii}^{1\delta})^{\epsilon}(m_{ii}^{2\delta})^{1-\epsilon} = 1^{\epsilon}1^{1-\epsilon} = 1.$$

Similarly, $m_{ii}^{\beta} = 1, m_{ii}^{\gamma} = 1$, and thus, $\widetilde{m}_{ii} = 1, i = 1, \dots, n$. Further,

$$m_{ij}^{\alpha} = (m_{ij}^{1\alpha})^{\epsilon}(m_{ij}^{2\alpha})^{1-\epsilon} = \left(\frac{1}{m_{ji}^{1\delta}}\right)^{\epsilon}\left(\frac{1}{m_{ji}^{2\delta}}\right)^{1-\epsilon} = \frac{1}{(m_{ji}^{1\delta})^{\epsilon}(m_{ji}^{2\delta})^{1-\epsilon}} = \frac{1}{m_{ji}^{\delta}},$$

$$m_{ij}^{\delta} = (m_{ij}^{1\delta})^{\epsilon}(m_{ij}^{2\delta})^{1-\epsilon} = \left(\frac{1}{m_{ji}^{1\alpha}}\right)^{\epsilon}\left(\frac{1}{m_{ji}^{2\alpha}}\right)^{1-\epsilon} = \frac{1}{(m_{ji}^{1\alpha})^{\epsilon}(m_{ji}^{2\alpha})^{1-\epsilon}} = \frac{1}{m_{ji}^{\alpha}},$$

and analogously we obtain $m_{ij}^{\beta} = \frac{1}{m_{ji}^{\gamma}}, m_{ij}^{\gamma} = \frac{1}{m_{ji}^{\beta}}$. Therefore, $\widetilde{m}_{ij} = \frac{1}{\widetilde{m}_{ji}}, i, j = 1, \dots, n$.

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Second, let us show that $\widetilde{M}$ is multiplicatively weakly consistent. It is sufficient to prove inequalities (IV.16). Since (IV.16) is valid for FMPCMs $\widetilde{M}^1$ and $\widetilde{M}^2$, we obtain

$$\max_{k=1,\ldots,n}\left\{m_{ik}^{\alpha}m_{kj}^{\alpha}\right\} = \max_{k=1,\ldots,n}\left\{\left(m_{ik}^{1\alpha}\right)^{\epsilon}\left(m_{ik}^{2\alpha}\right)^{1-\epsilon}\left(m_{kj}^{1\alpha}\right)^{\epsilon}\left(m_{kj}^{2\alpha}\right)^{1-\epsilon}\right\} \leq$$

$$\max_{k=1,\ldots,n}\left\{(m_{ik}^{1\alpha}m_{kj}^{1\alpha})^{\epsilon}\right\}\max_{k=1,\ldots,n}\left\{(m_{ik}^{2\alpha}m_{kj}^{2\alpha})^{1-\epsilon}\right\} \leq$$

$$\min_{k=1,\ldots,n}\left\{(m_{ik}^{1\delta}m_{kj}^{1\delta})^{\epsilon}\right\}\min_{k=1,\ldots,n}\left\{(m_{ik}^{2\delta}m_{kj}^{2\delta})^{1-\epsilon}\right\} \leq$$

$$\min_{k=1,\ldots,n}\left\{\left(m_{ik}^{1\delta}\right)^{\epsilon}\left(m_{ik}^{2\delta}\right)^{1-\epsilon}\left(m_{kj}^{1\delta}\right)^{\epsilon}\left(m_{kj}^{2\delta}\right)^{1-\epsilon}\right\} = \min_{k=1,\ldots,n}\left\{m_{ik}^{\delta}m_{kj}^{\delta}\right\}$$

which proves the theorem. $\square$

Theorem 33 can be further extended to the aggregation of $p \geq 2$ multiplicatively weakly consistent trapezoidal FMPCMs as follows.

Theorem 34. *Let $\widetilde{M}^{\tau} = \left\{\widetilde{m}_{ij}^{\tau}\right\}_{i,j=1}^{n}, \widetilde{m}_{ij}^{\tau} = (m_{ij}^{\tau\alpha}, m_{ij}^{\tau\beta}, m_{ij}^{\tau\gamma}, m_{ij}^{\tau\delta}), \tau = 1, \ldots, p$, be trapezoidal FMPCMs multiplicatively weakly consistent according to Definition 50. Then $\widetilde{M} = \left\{\widetilde{m}_{ij}\right\}_{i,j=1}^{n}$, such that*

$$\widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta}\right) = \left(\prod_{\tau=1}^{p}\left(m_{ij}^{\tau\alpha}\right)^{\epsilon_{\tau}}, \prod_{\tau=1}^{p}(m_{ij}^{\tau\beta})^{\epsilon_{\tau}}, \prod_{\tau=1}^{p}\left(m_{ij}^{\tau\gamma}\right)^{\epsilon_{\tau}}, \prod_{\tau=1}^{p}\left(m_{ij}^{\tau\delta}\right)^{\epsilon_{\tau}}\right),$$

is a multiplicatively weakly consistent trapezoidal FMPCM for any $\epsilon_{\tau} \in [0,1], \tau = 1, \ldots, p$, with $\sum_{\tau=1}^{p}\epsilon_{\tau} = 1$.

Proof. The proof is analogous to the proof of Theorem 33. $\square$

Similar theorems are formulated also for multiplicatively consistent trapezoidal FMPCMs.

Theorem 35. *Let $\widetilde{M}^1 = \left\{\widetilde{m}_{ij}^1\right\}_{i,j=1}^{n}, \widetilde{m}_{ij}^1 = \left(m_{ij}^{1\alpha}, m_{ij}^{1\beta}, m_{ij}^{1\gamma}, m_{ij}^{1\delta}\right)$, and $\widetilde{M}^2 = \left\{\widetilde{m}_{ij}^2\right\}_{i,j=1}^{n}, \widetilde{m}_{ij}^2 = \left(m_{ij}^{2\alpha}, m_{ij}^{2\beta}, m_{ij}^{2\gamma}, m_{ij}^{2\delta}\right)$, be trapezoidal FMPCMs multiplicatively consistent according to Definition 51. Then $\widetilde{M} = \left\{\widetilde{m}_{ij}\right\}_{i,j=1}^{n}$, such that*

$$m_{ij}^{\alpha} = (m_{ij}^{1\alpha})^{\epsilon}(m_{ij}^{2\alpha})^{1-\epsilon}, \qquad m_{ij}^{\beta} = (m_{ij}^{1\beta})^{\epsilon}(m_{ij}^{2\beta})^{1-\epsilon},$$

$$m_{ij}^{\gamma} = (m_{ij}^{1\gamma})^{\epsilon}(m_{ij}^{2\gamma})^{1-\epsilon}, \qquad m_{ij}^{\delta} = (m_{ij}^{1\delta})^{\epsilon}(m_{ij}^{2\delta})^{1-\epsilon},$$

is a multiplicatively consistent trapezoidal FMPCM for any $\epsilon \in [0,1]$.

Proof. From Theorem 33 we already know that $\widetilde{M}$ is a FMPCM. It remains to show that $\widetilde{M}$ is multiplicatively consistent. It is sufficient to prove inequalities (IV.25) and (IV.26). Since (IV.25) are valid for the FMPCMs $\widetilde{M}^1$ and $\widetilde{M}^2$, we obtain

$$\begin{aligned}m_{ik}^{\alpha}m_{kj}^{\alpha} = \quad &(m_{ik}^{1\alpha})^{\epsilon}(m_{ik}^{2\alpha})^{1-\epsilon}(m_{kj}^{1\alpha})^{\epsilon}(m_{kj}^{2\alpha})^{1-\epsilon} = \\ &(m_{ik}^{1\alpha}m_{kj}^{1\alpha})^{\epsilon}(m_{ik}^{2\alpha}m_{kj}^{2\alpha})^{1-\epsilon} \leq (m_{ij}^{1\alpha})^{\epsilon}(m_{ij}^{2\alpha})^{1-\epsilon} = m_{ij}^{\alpha},\end{aligned}$$

$$\begin{aligned}m_{ik}^{\delta}m_{kj}^{\delta} = \quad &(m_{ik}^{1\delta})^{\epsilon}(m_{ik}^{2\delta})^{1-\epsilon}(m_{kj}^{1\delta})^{\epsilon}(m_{kj}^{2\delta})^{1-\epsilon} = \\ &(m_{ik}^{1\delta}m_{kj}^{1\delta})^{\epsilon}(m_{ik}^{2\delta}m_{kj}^{2\delta})^{1-\epsilon} \geq (m_{ij}^{1\delta})^{\epsilon}(m_{ij}^{2\delta})^{1-\epsilon} = m_{ij}^{\delta}.\end{aligned}$$

Analogously, the validity of inequalities (IV.26) is proved. $\square$

Theorem 35 can be further extended to the aggregation of $p \geq 2$ multiplicatively consistent trapezoidal FMPCMs as follows.

Theorem 36. *Let $\widetilde{M}^{\tau} = \left\{\widetilde{m}_{ij}^{\tau}\right\}_{i,j=1}^{n}, \widetilde{m}_{ij}^{\tau} = \left(m_{ij}^{\tau\alpha}, m_{ij}^{\tau\beta}, m_{ij}^{\tau\gamma}, m_{ij}^{\tau\delta}\right), \tau = 1, \ldots, p$, be trapezoidal FMPCMs multiplicatively consistent according to Definition 51. Then $\widetilde{M} = \left\{\widetilde{m}_{ij}\right\}_{i,j=1}^{n}$, such that*

$$\widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta}\right) = \left(\prod_{\tau=1}^{p}\left(m_{ij}^{\tau\alpha}\right)^{\epsilon_{\tau}}, \prod_{\tau=1}^{p}(m_{ij}^{\tau\beta})^{\epsilon_{\tau}}, \prod_{\tau=1}^{p}\left(m_{ij}^{\tau\gamma}\right)^{\epsilon_{\tau}}, \prod_{\tau=1}^{p}\left(m_{ij}^{\tau\delta}\right)^{\epsilon_{\tau}}\right),$$

is a multiplicatively consistent trapezoidal FMPCM for any $\epsilon_{\tau} \in [0,1], \tau = 1, \ldots, p$, with $\sum_{\tau=1}^{p}\epsilon_{\tau} = 1$.

Proof. The proof is similar to the proof of Theorem 35. $\square$

Reaching full consistency is not always managable in practice. Often, even when DMs are asked to reconsider their inconsistent preference information they are not able to provide a consistent FMPCM. That is why the problem of measuring acceptable inconsistency of FMPCMs has been addressed in the literature.

4.2.2.3 Fuzzy Consistency Index and Fuzzy Consistency Ratio

A number of inconsistency indices for measuring an acceptable level of inconsistency of MPCMs has been proposed in the literature. One of the well-known and most often applied ones is the Consistency Index (II.9) proposed by Saaty (1980). Strangely, the problem of verifying an acceptable level of inconsistency of FMPCMs has been quite neglected in the literature. Very often, authors dealing with the fuzzy extension of methods based on MPCMs do not address the issue of (in)consistency at all; see, e.g., the well-known theoretical articles by Laarhoven and Pedrycz (1983); Chang (1996); Enea and Piazza (2004). Similarly, also most real-application articles do not address this important issue.

In some real-application articles, the authors verify the acceptable inconsistency of a FMPCM by means of CR only for the crisp MPCM $M = \left\{m_{ij}^M\right\}_{i,j=1}^n$ constructed from the middle values of the triangular FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$, $\widetilde{m}_{ij} = (m_{ij}^L, m_{ij}^M, m_{ij}^U)$, $i,j = 1, \ldots, n$; see, e.g., Tesfamariam and Sadiq (2006); Pan (2008); Vahidnia et al. (2009). However, by verifying acceptable inconsistency of just one particular matrix of crisp numbers obtained from the original FMPCM (in this case the middle values of the triangular fuzzy numbers) the uncertainty modeled by the fuzzy numbers in the FMPCM is neglected. This is inconsistent with the original intention to model the incompleteness of information as well as the linguistically expressed preference information by fuzzy numbers.

A similar approach was considered by Zheng et al. (2012), although they suggested to compute CR for the crisp matrix $M^* = \{m_{ij}^*\}_{i,j=1}^n$ whose elements are obtained from the trapezoidal FMPCM $\widetilde{M} = \{m_{ij}\}_{i,j=1}^n$, $\widetilde{m}_{ij} = (m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta)$, by formula

$$m_{ij}^* = \frac{m_{ij}^\alpha + 2m_{ij}^\beta + 2m_{ij}^\gamma + m_{ij}^\delta}{6}, \qquad i,j = 1, \ldots, n. \tag{IV.34}$$

It is worth to note that the obtained matrix M^* is in general not multiplicatively reciprocal; for example, for $\widetilde{m}_{ij} = (2,3,4,6)$ and $\widetilde{m}_{ji} = \left(\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right)$, we obtain $m_{ij}^* = \frac{22}{6}$ and $m_{ji}^* = \frac{11}{36} \neq \frac{22}{6} = \frac{1}{m_{ij}^*}$. Thus, verifying acceptable inconsistency of such a matrix is meaningless.

Liu (2009) proposed a method for verifying acceptable level of inconsistency (he actually calls it acceptable consistency) of interval FMPCMs. For an interval FMPCM $\overline{M} = \{\overline{m}_{ij}\}_{i,j=1}^n$, $\overline{m}_{ij} = [m_{ij}^L, m_{ij}^U]$, Liu (2009) constructs two MPCMs C and D by using (IV.7) and verifies their acceptable inconsistency by comparing their Consistency Ratio (II.10) with the boundary value 0.1, i.e. $CR \leq 0.1$. When both MPCMs C and D are acceptably inconsistent, then also the interval FMPCM $\overline{M}$ is said to be acceptably inconsistent. When at least one of the MPCMs C and D is not acceptably inconsistent, then also the interval FMPCM $\overline{M}$ is considered as inconsistent. However, this method, similarly to Definition 46 of multiplicative consistency for interval FMPCMs proposed by Liu (2009) is not invariant under permutation of objects.

Another index of inconsistency for FMPCMs was proposed by Ramík and Korviny (2010). This index is based on the idea of measuring distance of the FMPCM from the closest fuzzy matrix of ratios of fuzzy priorities. The advantage of this approach is that, unlike the two approaches mentioned above, it takes into account the uncertainty present in the FMPCM.

Since in the focus of this thesis is the fuzzy extension of well-known and most often applied methods based on PCMs, only CI and CR and their extension to FMPCMs are dealt with here.

As already reviewed in Section 2.2.2, using the maximal eigenvalue of a MPCM, Saaty defined the Consistency Index CI and the Consistency Ratio CR to verify an acceptable level of inconsistency of the matrix. In order to verify an acceptable level of inconsistency of FMPCMs, these two measures should be fuzzified properly.

In order to compute CI, it is necessary to know the maximal eigenvalue of the given MPCM. Having a FMPCM whose entries are fuzzy numbers, it is natural to compute the maximal eigenvalue of this matrix in form of a fuzzy number as well. This fuzzy maximal eigenvalue will then substitute the crisp maximal eigenvalue in the formula (II.9), which can be easily fuzzified in order to obtain CI in the form of a fuzzy number.

The question is how to obtain the fuzzy maximal eigenvalue of a FMPCM. This task is not as simple as it might seem at first sight. Several methods for deriving the fuzzy maximal eigenvalue from a FMPCM have been proposed in the literature, but, as it will be shown in the following section, these methods suffer from severe drawbacks. An appropriate method for deriving the fuzzy maximal eigenvalue is indispensable not only for verifying acceptable inconsistency of a FMPCM by using the Fuzzy Consistency Ratio, but it is also necessary for the fuzzy extension of the EVM for deriving fuzzy priorities of objects from a FMPCM. Thus, later in the following section, particular attention will be paid to the problem of deriving properly the fuzzy maximal eigenvalue from a FMPCM.

For a trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$, Fuzzy Consistency Index $\widetilde{CI}$ is defined in the form

$$\widetilde{CI} = \frac{\widetilde{\lambda} - n}{n - 1} = \left(\frac{\lambda^\alpha - n}{n - 1}, \frac{\lambda^\beta - n}{n - 1}, \frac{\lambda^\gamma - n}{n - 1}, \frac{\lambda^\delta - n}{n - 1}\right), \tag{IV.35}$$

where $\widetilde{\lambda} = \left(\lambda^\alpha, \lambda^\beta, \lambda^\gamma, \lambda^\delta \right)$ is the fuzzy maximal eigenvalue of $\widetilde{M}$ (for whose computation the formulas will be provided in the following section). Fuzzy Consistency Ratio $\widetilde{CR}$ is defined as

$$\widetilde{CR} = \frac{\widetilde{CI}}{RI} = \left(\frac{\lambda^\alpha - n}{RI\,(n-1)}, \frac{\lambda^\beta - n}{RI\,(n-1)}, \frac{\lambda^\gamma - n}{RI\,(n-1)}, \frac{\lambda^\delta - n}{RI\,(n-1)} \right). \tag{IV.36}$$

Notice that Random Index RI used in the formula (IV.36) is the same as RI used for crisp MPCMs; the values of RI are given in Tab. II.2. Analogously as in the case of crisp MPCMs, we need to compare $\widetilde{CR}$ with the boundary value 0.1 in order to decide whether the FMPCM is acceptably inconsistent or not. This might be done easily by defuzzifying the trapezoidal fuzzy number $\widetilde{CR}$ by the center-of-area defuzzification method using formula (III.8) first, and then comparing the center of area $COA_{\widetilde{CR}}$ of $\widetilde{CR}$ with the boundary value 0.1. Thus, the FMPCM $\widetilde{M}$ is said to be acceptably inconsistent if

$$COA_{\widetilde{CR}} \leq 0.1. \tag{IV.37}$$

4.2.2.4 Fuzzy maximal eigenvalue of a FMPCM

As reviewed in Section 2.2, the maximal eigenvalue of a MPCM is utilized in the consistency index (II.9) to verify acceptable multiplicative inconsistency of a MPCM and in the EVM method (II.21) to obtain normalized priorities of objects from a MPCM. In this section, extension of the maximal eigenvalue to the fuzzy maximal eigenvalue of a FMPCM is studied. The formulas for obtaining the fuzzy maximal eigenvalue of a FMPCM formerly proposed in the literature are reviewed, deficiencies of these formulas are pointed out, and then new formulas are proposed. Subsequently, properties of the new fuzzy maximal eigenvalue are discussed.

Csutora and Buckley (2001) proposed formulas for obtaining $\alpha-$cuts, $\alpha \in [0,1]$, of the fuzzy maximal eigenvalue of a FMPCM. For the sake of simplicity, this thesis is limited to trapezoidal fuzzy numbers. Therefore, the formulas of Csutora and Buckley (2001) will be shown for trapezoidal fuzzy numbers here.

By applying the trapezoidal approximation, the representing values of the fuzzy maximal eigenvalue $\widetilde{\lambda}_S = (\lambda_S^\alpha, \lambda_S^\beta, \lambda_S^\gamma, \lambda_S^\delta)$ (the lower index S stands for standard fuzzy arithmetic which is applied to obtain the fuzzy maximal eigenvalue) of a trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$, $\widetilde{m}_{ij} = (m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta)$, is obtained as

$$\lambda_S^\alpha = EVM_\lambda(M^\alpha), \quad \text{where } M^\alpha = \left\{ m_{ij}^\alpha \right\}_{i,j=1}^n, \tag{IV.38}$$

$$\lambda_S^\beta = EVM_\lambda(M^\beta), \quad \text{where } M^\beta = \left\{ m_{ij}^\beta \right\}_{i,j=1}^n, \tag{IV.39}$$

$$\lambda_S^\gamma = EVM_\lambda(M^\gamma), \quad \text{where } M^\gamma = \left\{ m_{ij}^\gamma \right\}_{i,j=1}^n, \tag{IV.40}$$

$$\lambda_S^\delta = EVM_\lambda(M^\delta), \quad \text{where } M^\delta = \left\{ m_{ij}^\delta \right\}_{i,j=1}^n. \tag{IV.41}$$

This means that the lower boundary value λ_S^α of the fuzzy maximal eigenvalue $\widetilde{\lambda}_S$ of a trapezoidal FMPCM $\widetilde{M}$ is computed as the maximal eigenvalue of the matrix $M^\alpha = \left\{ m_{ij}^\alpha \right\}_{i,j=1}^n$, whose elements are the lower boundary values of the trapezoidal fuzzy numbers from the trapezoidal FMPCM $\widetilde{M}$. Analogously, the upper boundary value λ_S^δ is computed as the maximal eigenvalue of the matrix of the upper boundary values of the trapezoidal fuzzy numbers from the trapezoidal FMPCM $\widetilde{M}$, similarly for the representing values λ_S^β and λ_S^γ of the resulting fuzzy maximal eigenvalue $\widetilde{\lambda}_S$.

Clearly, none of the matrices $M^\alpha, M^\beta, M^\gamma$, and M^δ is multiplicatively reciprocal. Csutora and Buckley (2001) observed this fact, but they did not consider it to be a flaw. Also Wang and Chin (2006), who adopted formulas (IV.38)–(IV.41) in their method for obtaining the fuzzy priorities of objects from FMPCMs, did not realize the flaw.

However, as already emphasized, multiplicative reciprocity is a key property of MPCMs, and thus it has to be preserved also under the fuzzy extension. Unless $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$, $\widetilde{m}_{ij} = \left(m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta \right)$, is a crisp MPCM, it is meaningless to consider the matrix $M^\alpha = \left\{ m_{ij}^\alpha \right\}_{i,j=1}^n$ of the lower boundary values of $\widetilde{M}$ for the computation of the fuzzy maximal eigenvalue $\widetilde{\lambda}_S$ (in particular its lower boundary value λ_S^α). Since $M^\alpha = \left\{ m_{ij}^\alpha \right\}_{i,j=1}^n$ does not satisfy the multiplicative-reciprocity property (II.2), it is not even a MPCM, and thus it does not reflect the preference information provided by the DM in the original trapezoidal FMPCM $\widetilde{M}$. The same holds also for the matrices $M^\beta = \left\{ m_{ij}^\beta \right\}_{i,j=1}^n$, $M^\gamma = \left\{ m_{ij}^\gamma \right\}_{i,j=1}^n$, and $M^\delta = \left\{ m_{ij}^\delta \right\}_{i,j=1}^n$.

Despite the violation of the multiplicative reciprocity of PCs, we have to acknowledge that the method proposed by Csutora and Buckley (2001) is at least invariant under permutation of objects in the FMPCM $\widetilde{M}$. By permuting $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$ to $\widetilde{M}^{\pi} = P\widetilde{M}P^{T}$, using any permutation matrix P, the matrices $M^{\alpha}, M^{\beta}, M^{\gamma}$, and M^{δ} used in the formulas (IV.38)–(IV.41) are permuted in the same way, which does not have any impact on the resulting maximal eigenvalues $\lambda_{S}^{\alpha}, \lambda_{S}^{\beta}, \lambda_{S}^{\gamma}$, and λ_{S}^{δ}. Thus, the resulting fuzzy maximal eigenvalue $\widetilde{\lambda}_{S}$ remains unchanged under any permutation of objects compared in the FMPCM $\widetilde{M}$.

Example 34. Let us apply the approach for obtaining the fuzzy maximal eigenvalue proposed by Csutora and Buckley (2001) to the FMPCM

$$\widetilde{M} = \begin{pmatrix} 1 & (2,2,3,4) & (2,4,5,8) \\ \left(\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{1}{2}\right) & 1 & (4,5,6,7) \\ \left(\frac{1}{8},\frac{1}{5},\frac{1}{4},\frac{1}{2}\right) & \left(\frac{1}{7},\frac{1}{6},\frac{1}{5},\frac{1}{4}\right) & 1 \end{pmatrix}. \tag{IV.42}$$

By applying the formulas (IV.38)–(IV.41), we actually compute the maximal eigenvalues $\lambda^{\alpha}, \lambda^{\beta}, \lambda^{\gamma}$, and λ^{δ} of the matrices

$$M^{\alpha} = \begin{pmatrix} 1 & 2 & 2 \\ \frac{1}{4} & 1 & 4 \\ \frac{1}{8} & \frac{1}{7} & 1 \end{pmatrix}, \quad M^{\beta} = \begin{pmatrix} 1 & 2 & 4 \\ \frac{1}{3} & 1 & 5 \\ \frac{1}{5} & \frac{1}{6} & 1 \end{pmatrix}, \quad M^{\gamma} = \begin{pmatrix} 1 & 3 & 5 \\ \frac{1}{2} & 1 & 6 \\ \frac{1}{4} & \frac{1}{5} & 1 \end{pmatrix}, \quad M^{\delta} = \begin{pmatrix} 1 & 4 & 8 \\ \frac{1}{2} & 1 & 7 \\ \frac{1}{2} & \frac{1}{4} & 1 \end{pmatrix}, \tag{IV.43}$$

respectively. The fuzzy maximal eigenvalue obtained by the formulas (IV.38)–(IV.41) is in the form $\widetilde{\lambda}_{S} = (2.4376, 2.8680, 3.4480, 4.4739)$.

However, as we can see from (IV.43), none of the matrices $M^{\alpha}, M^{\beta}, M^{\gamma}$, and M^{δ} is multiplicatively reciprocal. This means that they are not MPCMs. Therefore, computing their maximal eigenvalues in order to verify the acceptable consistency or to derive priorities is nonsensical as these matrices do not represent the preference information contained in the FMPCM (IV.42).   △

Also Ishizaka (2014) used formulas (IV.38)–(IV.41) to obtain the fuzzy maximal eigenvalue of a trapezoidal FMPCM. However, in his method, a particular approach for the construction of the trapezoidal FMPCM was employed. In order to distinguish the method proposed by Ishizaka (2014) from the method proposed by Csutora and Buckley (2001), the FMPCM and the fuzzy maximal eigenvalue obtained by formulas (IV.38)–(IV.41) in the approach of Ishizaka (2014) will be denoted $\widetilde{M}_{I} = \{\widetilde{m}_{Iij}\}_{i,j=1}^{n}$, and $\widetilde{\lambda}_{I} = \left(\lambda_{I}^{\alpha}, \lambda_{I}^{\beta}, \lambda_{I}^{\gamma}, \lambda_{I}^{\delta}\right)$, respectively.

Ishizaka (2014) constructed the multiplicative reciprocals of trapezoidal fuzzy numbers $\widetilde{m}_{Iij} = (m_{Iij}^{\alpha}, m_{Iij}^{\beta}, m_{Iij}^{\gamma}, m_{Iij}^{\delta}), i, j = 1, \ldots, n, \ i < j$, in the trapezoidal FMPCM $\widetilde{M}_{I} = \{\widetilde{m}_{Iij}\}_{i,j=1}^{n}$ as $\widetilde{m}_{Iji} = \left(\frac{1}{m_{Iij}^{\alpha}}, \frac{1}{m_{Iij}^{\beta}}, \frac{1}{m_{Iij}^{\gamma}}, \frac{1}{m_{Iij}^{\delta}}\right)$. Thus, $\widetilde{m}_{Iji}$ does not represent a (trapezoidal) fuzzy number anymore since $\frac{1}{m_{Iij}^{\alpha}} \geq \frac{1}{m_{Iij}^{\beta}} \geq \frac{1}{m_{Iij}^{\gamma}} \geq \frac{1}{m_{Iij}^{\delta}}$; it is just a quadruple of real numbers. This means that Ishizaka's approach violates even the widely accepted approach to the construction of FMPCMs.

Furthermore, the resulting fuzzy maximal eigenvalue $\widetilde{\lambda}_{I}$ is not invariant under permutation of objects in the FMPCM $\widetilde{M}$, which will be demonstrated on an illustrative example. Moreover, because of the inappropriate construction of the reciprocals of the fuzzy numbers in the FMPCM $\widetilde{M}_{I}$, it is not even guaranteed that the fuzzy maximal eigenvalue $\widetilde{\lambda}_{I}$ obtained by formulas (IV.38)–(IV.41) is a fuzzy number; in general, it is just a quadruple of real numbers similarly as for the reciprocals of $\widetilde{m}_{Iij}, \ i, j = 1, \ldots, n, \ i < j$, in $\widetilde{M}_{I} = \{\widetilde{m}_{Iij}\}_{i,j=1}^{n}$.

Example 35. Let us apply the method for obtaining the fuzzy maximal eigenvalue proposed by Ishizaka (2014) to the FMPCM $\widetilde{M}$ in the form (IV.42) examined in Example 34.

The corresponding matrix $\widetilde{M}_{I}$ utilized in the approach of Ishizaka (2014) is given as

$$\widetilde{M}_{I} = \begin{pmatrix} 1 & (2,2,3,4) & (2,4,5,8) \\ \left(\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{4}\right) & 1 & (4,5,6,7) \\ \left(\frac{1}{2},\frac{1}{4},\frac{1}{5},\frac{1}{8}\right) & \left(\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7}\right) & 1 \end{pmatrix}. \tag{IV.44}$$

Obviously, the elements below the main diagonal of the matrix $\widetilde{M}_{I}$ are not trapezoidal fuzzy numbers as they do not satisfy Definition 22. In fact, $m_{Iji}^{\alpha} \geq m_{Iji}^{\beta} \geq m_{Iji}^{\gamma} \geq m_{Iji}^{\delta}, i, j = 1, \ldots, n, \ i < j$. The fuzzy maximal eigenvalue obtained by formulas (IV.38)–(IV.41) from the matrix $\widetilde{M}_{I}$ is in the form $\widetilde{\lambda}_{I} = (3.2174, 3.0940, 3.1851, 3.1769)$, which again does not satisfy Definition 22 of a trapezoidal fuzzy number.

Let us now permute the FMPCM (IV.42) to $\widetilde{M}^\pi$ by applying the permutation matrix P in the form (IV.31). The corresponding permuted matrix $\widetilde{M}_I^\pi$ is in the form

$$\widetilde{M}_I^\pi = \begin{pmatrix} 1 & (4,5,6,7)\left(\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{1}{2}\right) \\ \left(\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7}\right) & 1 & \left(\frac{1}{8},\frac{1}{5},\frac{1}{4},\frac{1}{2}\right) \\ (4,3,2,2) & (8,5,4,2) & 1 \end{pmatrix}. \tag{IV.45}$$

The fuzzy maximal eigenvalue $\widetilde{\lambda}_I^\pi$ obtained from this permuted matrix is in the form $\widetilde{\lambda}_I^\pi = (3.0536, 3.1356, 3.1356, 3.4357)$. We see that $\widetilde{\lambda}_I^\pi \neq \widetilde{\lambda}_I$, which means that the method for obtaining the fuzzy maximal eigenvalue from a FMPCM proposed by Ishizaka (2014) is not invariant under permutation of objects in FMPCMs. $\triangle$

It is obvious that the reciprocals of the fuzzy numbers in the FMPCM have to be constructed properly, as given in Definition 43 of a FMPCM. Furthermore, it is necessary to consider the multiplicative reciprocity of PCs in a FMPCM also in the process of deriving the fuzzy maximal eigenvalue since it is an inherent property of FMPCMs (see discussions on p. 49 and p. 56). For this it is necessary to apply constrained fuzzy arithmetic (III.43) instead of standard fuzzy arithmetic (III.34) when extending the formula (II.19) for obtaining the maximal eigenvalue of a MPCM to fuzzy numbers.

By applying simplified constrained fuzzy arithmetic (III.45), the fuzzy maximal eigenvalue $\widetilde{\lambda}_C = (\lambda_C^\alpha, \lambda_C^\beta, \lambda_C^\gamma, \lambda_C^\delta)$ (the lower index C stands for applying constrained fuzzy arithmetic) of a trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$, $\widetilde{m}_{ij} = (m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta)$, is obtained as:

$$\lambda_C^\alpha = \min\left\{ EVM_\lambda(M); \begin{array}{l} M = \{m_{rs}\}_{r,s=1}^n, m_{rs} \in \left[m_{rs}^\alpha, m_{rs}^\delta\right], \\ m_{sr} = \frac{1}{m_{rs}}, \ r,s = 1,\dots,n \end{array} \right\}, \tag{IV.46}$$

$$\lambda_C^\beta = \min\left\{ EVM_\lambda(M); \begin{array}{l} M = \{m_{rs}\}_{r,s=1}^n, m_{rs} \in \left[m_{rs}^\beta, m_{rs}^\gamma\right], \\ m_{sr} = \frac{1}{m_{rs}}, \ r,s = 1,\dots,n \end{array} \right\}, \tag{IV.47}$$

$$\lambda_C^\gamma = \max\left\{ EVM_\lambda(M); \begin{array}{l} M = \{m_{rs}\}_{r,s=1}^n, m_{rs} \in \left[m_{rs}^\beta, m_{rs}^\gamma\right], \\ m_{sr} = \frac{1}{m_{rs}}, \ r,s = 1,\dots,n \end{array} \right\}, \tag{IV.48}$$

$$\lambda_C^\delta = \max\left\{ EVM_\lambda(M); \begin{array}{l} M = \{m_{rs}\}_{r,s=1}^n, m_{rs} \in \left[m_{rs}^\alpha, m_{rs}^\delta\right], \\ m_{sr} = \frac{1}{m_{rs}}, \ r,s = 1,\dots,n \end{array} \right\}. \tag{IV.49}$$

By using constrained fuzzy arithmetic (III.45) in the formulas (IV.46)–(IV.49) in order to reflect multiplicative reciprocity of the related PCs, all redundant vagueness is eliminated from the fuzzy maximal eigenvalue $\widetilde{\lambda}_C$, which is the advantage over the method proposed by Csutora and Buckley (2001). Thus, $\widetilde{\lambda}_C$ represents the actual fuzzy maximal eigenvalue (more precisely its best trapezoidal approximation) of a trapezoidal FMPCM. Further, unlike the method proposed by Ishizaka (2014), the new method is invariant under permutation of objects.

Theorem 37. *Let $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$ be a trapezoidal FMPCM. The fuzzy maximal eigenvalue $\widetilde{\lambda}_C$ of $\widetilde{M}$ obtained by the formulas (IV.46)–(IV.49) is invariant under permutation of objects in FMPCMs.*

Proof. We already know from Section 2.2.3.1 that the maximal eigenvalue $\lambda = EVM_\lambda(M)$ is invariant under permutation of objects in a MPCM M, i.e. $EVM_\lambda(M) = EVM_\lambda(PMP^T)$ for every permutation matrix P. Thus also the maximal eigenvalue of any MPCM M constructed from the elements from the closures of the supports of the trapezoidal fuzzy numbers $\widetilde{m}_{ij} = (m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta)$ in the FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$ is invariant under permutation of objects in M. Therefore, also the minimum λ_C^α and the maximum λ_C^δ of these maximal eigenvalues obtained by the formulas (IV.46) and (IV.49), respectively, are invariant under permutation of objects. Similarly, also λ_C^β and λ_C^γ obtained by the formulas (IV.47) and (IV.48), respectively, are invariant under permutation of objects. Thus, it results that the fuzzy maximal eigenvalue $\widetilde{\lambda}_C = (\lambda_C^\alpha, \lambda_C^\beta, \lambda_C^\gamma, \lambda_C^\delta)$ is invariant under permutation of objects in FMPCMs. $\square$

Using the properties of the maximal eigenvalues reviewed in Section 2.2.3.1, it is possible to derive some properties of the fuzzy maximal eigenvalue of a trapezoidal FMPCM obtained by the new formulas (IV.46)–(IV.49) as well as of the fuzzy maximal eigenvalue obtained by the formulas (IV.38)–(IV.41) in the approaches proposed by Csutora and Buckley (2001) and by Ishizaka (2014). The case when the trapezoidal FMPCM

$\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta}\right)$, is a crisp MPCM, i.e. $m_{ij}^{\alpha} = m_{ij}^{\delta}, i, j = 1, \ldots, n$, is not interesting regarding the properties of the fuzzy maximal eigenvalue as this is simply a crisp number $\lambda \geq n$. Without loss of generality, let us consider trapezoidal FMPCMs $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta}\right)$, for which $\exists k, l \in \{1, \ldots, n\}: m_{kl}^{\alpha} < m_{kl}^{\beta} < m_{kl}^{\gamma} < m_{kl}^{\delta}$.

Being $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta}\right)$, a positive trapezoidal FMPCM, i.e. $m_{ij}^{\alpha} > 0, i, j = 1, \ldots, n$, the maximal eigenvalue of any matrix constructed from the elements from the closures of the supports of its fuzzy elements is positive too. Since the representing values of the fuzzy maximal eigenvalue $\widetilde{\lambda}_C = \left(\lambda_C^{\alpha}, \lambda_C^{\beta}, \lambda_C^{\gamma}, \lambda_C^{\delta}\right)$ are obtained from MPCMs that are multiplicatively reciprocal (see formulas (IV.46)–(IV.49)), the inequalities $\lambda_C^{\alpha} \geq n, \lambda_C^{\beta} \geq n, \lambda_C^{\gamma} \geq n, \lambda_C^{\delta} \geq n$ necessarily hold.

The lower boundary value λ_C^{α} and the upper boundary value λ_C^{δ} of the fuzzy maximal eigenvalue $\widetilde{\lambda}_C$ are obtained as the minimum and the maximum, respectively, of function EVM_{λ} defined on the closures of the supports of the trapezoidal fuzzy numbers in the trapezoidal FMPCM. Analogously, λ_C^{β} and λ_C^{γ} are obtained as the minimum and the maximum, respectively, of function EVM_{λ} defined on the cores of the trapezoidal fuzzy numbers in the trapezoidal FMPCM. Therefore, the inequalities $\lambda_C^{\alpha} \leq \lambda_C^{\beta} \leq \lambda_C^{\gamma} \leq \lambda_C^{\delta}$ necessarily hold. Thus, overall, for the fuzzy maximal eigenvalue obtained by formulas (IV.46)–(IV.49), the inequalities $n \leq \lambda_C^{\alpha} \leq \lambda_C^{\beta} \leq \lambda_C^{\gamma} \leq \lambda_C^{\delta}$ hold.

In the special case where there exists a multiplicatively consistent MPCM obtainable by combining particular elements from the cores of the trapezoidal fuzzy numbers $\widetilde{m}_{ij}, i, j = 1, \ldots, n$, in the trapezoidal FMPCM $\widetilde{M}$, the representing values of the fuzzy maximal eigenvalue $\widetilde{\lambda}_C = \left(\lambda_C^{\alpha}, \lambda_C^{\beta}, \lambda_C^{\gamma}, \lambda_C^{\delta}\right)$ are in the form $n = \lambda_C^{\alpha} = \lambda_C^{\beta} < \lambda_C^{\gamma} < \lambda_C^{\delta}$. In the case where there does not exist a multiplicatively consistent MPCM obtainable by combining elements from the cores of the trapezoidal fuzzy numbers $\widetilde{m}_{ij}, i, j = 1, \ldots, n$, but where there exist elements in the closures of the supports of the trapezoidal fuzzy numbers $\widetilde{m}_{ij}, i, j = 1, \ldots, n$, in $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$ such that they form a multiplicatively consistent MPCM, the representing values of the fuzzy maximal eigenvalue of the FMPCM are in the form $n = \lambda_C^{\alpha} < \lambda_C^{\beta} < \lambda_C^{\gamma} < \lambda_C^{\delta}$.

Because the inequalities $m_{ij}^{\alpha} \leq m_{ij}^{\beta} \leq m_{ij}^{\gamma} \leq m_{ij}^{\delta}$ hold for $i, j = 1, \ldots, n$, and because $\exists k, l \in \{1, \ldots, n\}: m_{kl}^{\alpha} < m_{kl}^{\beta} < m_{kl}^{\gamma} < m_{kl}^{\delta}$, then, clearly, the inequalities $\lambda_S^{\alpha} < \lambda_S^{\beta} < \lambda_S^{\gamma} < \lambda_S^{\delta}$ hold for the fuzzy maximal eigenvalue $\widetilde{\lambda}_S = (\lambda_S^{\alpha}, \lambda_S^{\beta}, \lambda_S^{\gamma}, \lambda_S^{\delta})$ obtained by the formulas (IV.38)–(IV.41) in Csutora and Buckley's method.

The properties mentioned above do not hold for the fuzzy maximal eigenvalue $\widetilde{\lambda}_I = \left(\lambda_I^{\alpha}, \lambda_I^{\beta}, \lambda_I^{\gamma}, \lambda_I^{\delta}\right)$ of the particular FMPCM $\widetilde{M}_I = \{\widetilde{m}_{Iij}\}_{i,j=1}^{n}$ obtained by the formulas (IV.38)–(IV.41) in Ishizaka's approach. Since there are $\frac{n^2-n}{2}$ elements $\widetilde{m}_{Iij}$ in the FMPCM $\widetilde{M}_I = \{\widetilde{m}_{Iij}\}_{i,j=1}^{n}, \widetilde{m}_{Iij} = \left(m_{Iij}^{\alpha}, m_{Iij}^{\beta}, m_{Iij}^{\gamma}, m_{Iij}^{\delta}\right)$, such that the inequalities $m_{Iij}^{\alpha} \leq m_{Iij}^{\beta} \leq m_{Iij}^{\gamma} \leq m_{Iij}^{\delta}$ do not hold for them, then also the inequalities $\lambda_I^{\alpha} < \lambda_I^{\beta} < \lambda_I^{\gamma} < \lambda_I^{\delta}$ cannot be guaranteed for the resulting fuzzy maximal eigenvalue $\widetilde{\lambda}_I$. Thus, in general, the resulting quadruple $\widetilde{\lambda}_I = \left(\lambda_I^{\alpha}, \lambda_I^{\beta}, \lambda_I^{\gamma}, \lambda_I^{\delta}\right)$ does not represent a fuzzy number, which is a very serious flaw of the method.

There exists a very interesting relation between the fuzzy maximal eigenvalues $\widetilde{\lambda}_C = (\lambda_C^{\alpha}, \lambda_C^{\beta}, \lambda_C^{\gamma}, \lambda_C^{\delta})$ and $\widetilde{\lambda}_S = (\lambda_S^{\alpha}, \lambda_S^{\beta}, \lambda_S^{\gamma}, \lambda_S^{\delta})$. Since λ_S^{α} is the maximal eigenvalue of $M^{\alpha} = \{m_{ij}^{\alpha}\}_{i,j=1}^{n}$ and λ_C^{α} is the maximal eigenvalue of a multiplicatively reciprocal matrix $M^{*} = \{m_{ij}^{*}\}_{i,j=1}^{n}, m_{ij}^{*} \in [m_{ij}^{\alpha}, m_{ij}^{\delta}], m_{ij}^{\alpha} \leq m_{ij}^{*}, i, j = 1, \ldots, n$, with at least one strict inequality, then the inequality $\lambda_S^{\alpha} < \lambda_C^{\alpha}$ follows from the Perron-Frobenius Theorem. Analogously, since λ_S^{δ} is the maximal eigenvalue of $M^{\delta} = \{m_{ij}^{\delta}\}_{i,j=1}^{n}$ and λ_C^{δ} is the maximal eigenvalue of a multiplicatively reciprocal matrix $M^{**} = \{m_{ij}^{**}\}_{i,j=1}^{n}, m_{ij}^{**} \in [m_{ij}^{\alpha}, m_{ij}^{\delta}], m_{ij}^{\delta} \geq m_{ij}^{**}, i, j = 1, \ldots, n$, with at least one strict inequality, then also the inequality $\lambda_C^{\delta} < \lambda_S^{\delta}$ holds. Therefore, the support of the fuzzy maximal eigenvalue $\widetilde{\lambda}_C$ of a given FMPCM $\widetilde{M}$ is a proper subset of the support of the fuzzy maximal eigenvalue $\widetilde{\lambda}_S$ of $\widetilde{M}$, i.e. $]\lambda_C^{\alpha}, \lambda_C^{\delta}[\subset]\lambda_S^{\alpha}, \lambda_S^{\delta}[$. The same holds also for the cores of the fuzzy maximal eigenvalues $\widetilde{\lambda}_C$ and $\widetilde{\lambda}_S$, i.e. $[\lambda_C^{\beta}, \lambda_C^{\gamma}] \subset [\lambda_S^{\beta}, \lambda_S^{\gamma}]$. Thus, by employing the multiplicative-reciprocity condition in formulas (IV.46)–(IV.49), all unfeasible combinations of elements from the supports of the fuzzy numbers in the FMPCM are eliminated. As a consequence, the resulting fuzzy maximal eigenvalue $\widetilde{\lambda}_C$ is less vague than the original fuzzy maximal eigenvalue $\widetilde{\lambda}_S$ obtained by the formulas (IV.38)–(IV.41) proposed by Csutora and Buckley (2001) where the multiplicative reciprocity of PCs is violated.

The properties of the fuzzy maximal eigenvalues derived above are valid for a trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, such that $m_{ij}^{\alpha} \leq m_{ij}^{\beta} \leq m_{ij}^{\gamma} \leq m_{ij}^{\delta}, i, j = 1, \ldots, n$, and $\exists k, l \in \{1, \ldots, n\}: m_{kl}^{\alpha} < m_{kl}^{\beta} < m_{kl}^{\gamma} < m_{kl}^{\delta}$. This means that triangular FMPCMs were excluded from the analysis above. Therefore, some interesting properties that appear only for the fuzzy maximal eigenvalues of triangular FMPCMs will be shown here. Without loss of generality, let as assume a triangular FMPCM $\widetilde{M} =$

Figure IV.1: Fuzzy maximal eigenvalues $\widetilde{\lambda}$ and $\widetilde{\lambda}_C$ of the FMPCM (IV.50).
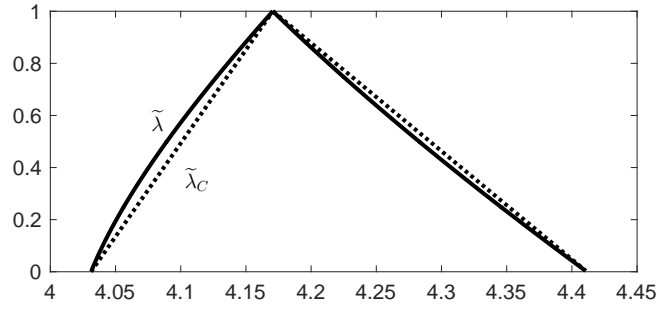
$\{\widetilde{m}_{ij}\}_{i,j=1}^{n}$, $\widetilde{m}_{ij} = (m_{ij}^L, m_{ij}^M, m_{ij}^U)$, such that $m_{ij}^L \leq m_{ij}^M \leq m_{ij}^U$, $i,j = 1, \ldots, n$, and $\exists k, l \in \{1, \ldots, n\} : m_{kl}^L < m_{kl}^M < m_{kl}^U$. Since $M^M = \{m_{ij}^M\}_{i,j=1}^n$ is multiplicatively reciprocal, the inequality $\lambda_S^M \geq n$ holds for its maximal eigenvalue $\widetilde{\lambda}_S = (\lambda_S^L, \lambda_S^M, \lambda_S^U)$. In the special case where $M^M = \{m_{ij}^M\}_{i,j=1}^n$ is multiplicatively consistent according to (II.4), the equality $\lambda_S^M = n$ occurs. Furthermore, in such case, the representing values of the fuzzy maximal eigenvalue $\widetilde{\lambda}_C = (\lambda_C^L, \lambda_C^M, \lambda_C^U)$ are in the form $n = \lambda_C^L = \lambda_C^M < \lambda_C^U$.

In the following three examples, the fuzzy maximal eigenvalues $\widetilde{\lambda}_C$ of three different FMPCMs are examined. In particular, based on the analysis above, three main types of the fuzzy maximal eigenvalues $\widetilde{\lambda}_C$ are identified and studied. For the simplicity of presentation, triangular FMPCMs are considered. In addition, in each example, the fuzzy maximal eigenvalues $\widetilde{\lambda}_S$ and $\widetilde{\lambda}_I$ obtained by formulas (IV.38)–(IV.41) in the approaches proposed by Csutora and Buckley (2001) and by Ishizaka (2014) are computed and confronted with the fuzzy maximal eigenvalue λ_C obtained by the formulas (IV.46)–(IV.49).

Example 36. Let us consider the triangular FMPCM of four objects o_1, o_2, o_3, and o_4 given as

$$\widetilde{M} = \begin{pmatrix} 1 & (2,3,4) & (4,5,6) & (8,9,9) \\ \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right) & 1 & (2,3,4) & (6,7,8) \\ \left(\frac{1}{6}, \frac{1}{5}, \frac{1}{4}\right) & \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right) & 1 & (4,5,6) \\ \left(\frac{1}{9}, \frac{1}{9}, \frac{1}{8}\right) & \left(\frac{1}{8}, \frac{1}{7}, \frac{1}{6}\right) & \left(\frac{1}{6}, \frac{1}{5}, \frac{1}{4}\right) & 1 \end{pmatrix}. \tag{IV.50}$$

The fuzzy maximal eigenvalue obtained by the formulas (IV.46)–(IV.49) is $\widetilde{\lambda}_C = (4.0312, 4.1707, 4.4115)$. Since $\lambda_C^L = 4.0312 > n = 4$, it is clear that there does not exist a single MPCM $M^* = \{m_{ij}^*\}_{i,j=1}^4$, $m_{ij}^* \in [m_{ij}^L, m_{ij}^U]$, $i, j = 1, \ldots, 4$, that would be multiplicatively consistent according to (II.4). According to Theorem 37, by permuting the FMPCM $\widetilde{M}$, the corresponding fuzzy maximal eigenvalue $\widetilde{\lambda}_C$ obtained by the formulas (IV.46)–(IV.49) remains unchanged.

Note that the triangular fuzzy number $\widetilde{\lambda}_C = (4.0312, 4.1707, 4.4115)$ is only a triangular approximation of the actual fuzzy maximal eigenvalue of the FMPCM $\widetilde{M}$. The actual fuzzy maximal eigenvalue $\widetilde{\lambda}$ obtained by applying properly constrained fuzzy arithmetic (III.43), i.e., $\widetilde{\lambda} = \bigcup_{\alpha=0}^{1} \alpha[\lambda_{(\alpha)}^L, \lambda_{(\alpha)}^U]$ such that

$$\lambda_{(\alpha)}^L = \min \left\{ EVM_{MAX}(M); \begin{array}{l} M = \{m_{rs}\}_{r,s=1}^n, \ m_{rs} \in [m_{rs(\alpha)}^L, m_{rs(\alpha)}^U], \\ m_{sr} = \frac{1}{m_{rs}}, \ r, s = 1, \ldots, n \end{array} \right\},$$

$$\lambda_{(\alpha)}^U = \max \left\{ EVM_{MAX}(M); \begin{array}{l} M = \{m_{rs}\}_{r,s=1}^n, \ m_{rs} \in [m_{rs(\alpha)}^L, m_{rs(\alpha)}^U], \\ m_{sr} = \frac{1}{m_{rs}}, \ r, s = 1, \ldots, n \end{array} \right\}, \tag{IV.51}$$

and its triangular approximation $\widetilde{\lambda}_C = (4.0312, 4.1707, 4.4115)$ are displayed in Fig. IV.1. It is clear from Fig. IV.1 that the actual fuzzy maximal eigenvalue $\widetilde{\lambda}$ is not triangular (but such a result was expected since even a product of two triangular fuzzy numbers is not a triangular fuzzy number anymore). However, the triangular fuzzy maximal eigenvalue $\widetilde{\lambda}_C = (4.0312, 4.1707, 4.4115)$ is a sufficient approximation since the lower and upper boundary values and the middle value of the fuzzy maximal eigenvalue $\widetilde{\lambda}$ are computed correctly.

The fuzzy maximal eigenvalue $\widetilde{\lambda}_S$ obtained from the FMPCM (IV.50) by the formulas (IV.38)–(IV.41) is in the form $\widetilde{\lambda}_S = (3.5653, 4.1707, 4.9446)$, and thus it is obviously much vaguer than $\widetilde{\lambda}_C = (4.0312, 4.1707, 4.4115)$. The huge difference in vagueness of both fuzzy maximal eigenvalues is even more noticeable from graphical

Figure IV.2: Fuzzy maximal eigenvalues $\widetilde{\lambda}_C$ and $\widetilde{\lambda}_S$ of the FMPCM (IV.50).
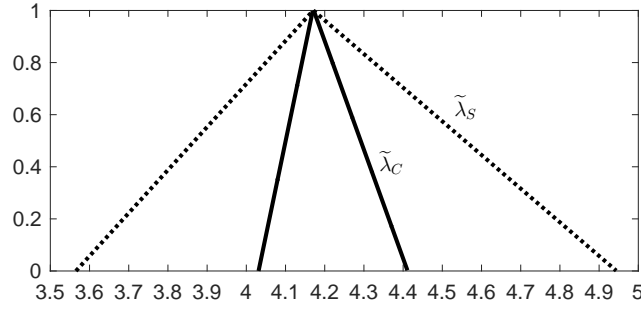
Figure IV.3: Fuzzy maximal eigenvalues $\widetilde{\lambda}$ and $\widetilde{\lambda}_C$ of the FMPCM (IV.52) .
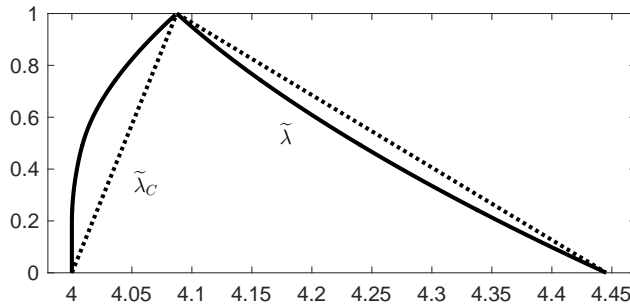
representation, see Fig. IV.2. By applying the approach of Ishizaka (2014), the fuzzy maximal eigenvalue $\widetilde{\lambda}_I$ obtained by the formulas (IV.38)–(IV.41) from the matrix $\widetilde{M}_I$ corresponding to the FMPCM (IV.50) is in the form $\widetilde{\lambda}_I = (4.0458, 4.1707, 4.3675)$. It is only a coincidence that $\widetilde{\lambda}_I$ is such that $\lambda_I^L \leq \lambda_I^M \leq \lambda_I^U$; in general, this property is not satisfied. $\triangle$

Example 37. Let us consider the FMPCM $\widetilde{M}$

$$\widetilde{M} = \begin{pmatrix} 1 & (1,2,3) & (2,3,4) & (3,4,5) \\ \left(\frac{1}{3},\frac{1}{2},1\right) & 1 & (1,2,3) & (3,4,5) \\ \left(\frac{1}{4},\frac{1}{3},\frac{1}{2}\right) & \left(\frac{1}{3},\frac{1}{2},1\right) & 1 & (2,3,4) \\ \left(\frac{1}{5},\frac{1}{4},\frac{1}{3}\right) & \left(\frac{1}{5},\frac{1}{4},\frac{1}{3}\right) & \left(\frac{1}{4},\frac{1}{3},\frac{1}{2}\right) & 1 \end{pmatrix}. \tag{IV.52}$$

The fuzzy maximal eigenvalue obtained by the formulas (IV.46)–(IV.49) is $\widetilde{\lambda}_C = (4, 4.0875, 4.4453)$. Since $\lambda_C^L = 4 = n$, it is clear that there exists a MPCM $M^* = \{m_{ij}^*\}_{i,j=1}^4, m_{ij}^* \in [m_{ij}^L, m_{ij}^U], i, j = 1, \ldots, n$, that is multiplicatively consistent according to (II.4). It is, for example,

$$M^* = \begin{pmatrix} 1 & 1 & 2 & 5 \\ 1 & 1 & 2 & 5 \\ \frac{1}{2} & \frac{1}{2} & 1 & 2.5 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{2.5} & 1 \end{pmatrix}, \tag{IV.53}$$

but there exist infinitely many of them.

Again, the triangular fuzzy number $\widetilde{\lambda}_C = (4, 4.0875, 4.4453)$ is only a triangular approximation of the actual fuzzy maximal eigenvalue of the FMPCM $\widetilde{M}$. The actual fuzzy maximal eigenvalue $\widetilde{\lambda} = \bigcup_{\alpha=0}^1 \alpha[\lambda_{(\alpha)}^L, \lambda_{(\alpha)}^U]$ obtainable by the formula (IV.51) is given in Fig. IV.3 together with the triangular fuzzy maximal eigenvalue $\widetilde{\lambda}_C = (4, 4.0875, 4.4453)$.

Notice the particular form of the actual fuzzy maximal eigenvalue $\widetilde{\lambda}$ in Fig. IV.3; the degree of membership of λ^L is not 0. This is caused by the fact that there exist infinitely many multiplicatively consistent MPCMs $M = \{m_{ij}\}_{i,j=1}^n$ (i.e. their maximal eigenvalue equals 4) obtainable from the FMPCM $\widetilde{M}$. The degree of

Figure IV.4: Fuzzy maximal eigenvalues $\widetilde{\lambda}_C$ and $\widetilde{\lambda}_S$ of the FMPCM (IV.52).
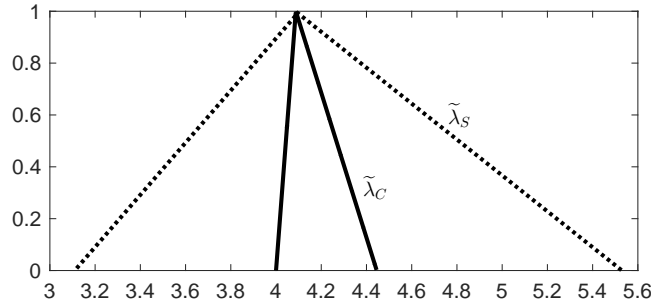
membership of some of these MPCMs to the FMPCM $\widetilde{M}$ (computed according to Definition 25) is non-zero. The degree of membership of the maximal eigenvalue $\lambda^L = 4$ to the actual fuzzy maximal eigenvalue $\widetilde{\lambda}$ is 0.1843, i.e. $\widetilde{\lambda}(\lambda^L) = 0.1843$. The corresponding multiplicatively consistent MPCM M^* (i.e. the MPCM M^* such that $\widetilde{M}(M^*) = 0.1843$) is in the form

$$M^* = \begin{pmatrix} 1 & 1.1887 & 2.1843 & 4.7681 \\ \frac{1}{1.1887} & 1 & 1.8372 & 4.0121 \\ \frac{1}{2.1843} & \frac{1}{1.8372} & 1 & 2.1843 \\ \frac{1}{4.7681} & \frac{1}{4.0121} & \frac{1}{2.1843} & 1 \end{pmatrix}. \tag{IV.54}$$

The fuzzy maximal eigenvalue $\widetilde{\lambda}_S$ obtained from the FMPCM (IV.52) by the formulas (IV.38)–(IV.41) is in the form $\widetilde{\lambda}_S = (3.1056, 4.0875, 5.5250)$, and thus it is again significantly vaguer than $\widetilde{\lambda}_C = (4, 4.0875, 4.4453)$. The huge difference in vagueness of both fuzzy maximal eigenvalues is even more noticeable from graphical representation, see Fig. IV.4. By applying the approach of Ishizaka (2014), the fuzzy maximal eigenvalue $\widetilde{\lambda}_I$ obtained by the formulas (IV.38)–(IV.41) from the matrix $\widetilde{M}_I$ corresponding to the FMPCM (IV.52) is in the form $\widetilde{\lambda}_I = (4.1031, 4.0875, 4.1407)$. Thus, $\widetilde{\lambda}_I$ is not a fuzzy number since $\lambda_I^L > \lambda_I^M$. $\triangle$

Example 38. Let us consider the FMPCM

$$\widetilde{M} = \begin{pmatrix} 1 & \left(\frac{1}{2},1,2\right) & (1,2,3) & (5,6,7) \\ \left(\frac{1}{2},1,2\right) & 1 & (1,2,3) & (5,6,6) \\ \left(\frac{1}{3},\frac{1}{2},1\right) & \left(\frac{1}{3},\frac{1}{2},1\right) & 1 & (2,3,4) \\ \left(\frac{1}{7},\frac{1}{6},\frac{1}{5}\right) & \left(\frac{1}{6},\frac{1}{6},\frac{1}{5}\right) & \left(\frac{1}{4},\frac{1}{3},\frac{1}{2}\right) & 1 \end{pmatrix} \tag{IV.55}$$

The fuzzy maximal eigenvalue obtained by the formulas (IV.46)–(IV.49) is $\widetilde{\lambda}_C = (4, 4, 4.2961)$. Since $\lambda_C^L = \lambda_C^M = 4 = n$, it is clear that MPCM $M^M = \{m_{ij}^M\}_{i,j=1}^4$ is multiplicatively consistent according to (II.4). However, there exist infinitely many multiplicatively consistent MPCMs obtainable from the FMPCM (IV.55).

Again, the triangular fuzzy number $\widetilde{\lambda}_C = (4, 4, 4.2961)$ is only a sufficient triangular approximation of the actual fuzzy maximal eigenvalue $\widetilde{\lambda}$ of the FMPCM $\widetilde{M}$. The actual fuzzy maximal eigenvalue $\widetilde{\lambda}$ obtainable by the formula (IV.51) is given together with its triangular approximation $\widetilde{\lambda}_C$ in Fig. IV.5.

The fuzzy maximal eigenvalue $\widetilde{\lambda}_S$ obtained from the FMPCM (IV.52) by the formulas (IV.38)–(IV.41) is in the form $\widetilde{\lambda}_S = (3.0815, 4, 5.6300)$, and thus it is obviously much vaguer than $\widetilde{\lambda}_C = (4, 4, 4.2961)$. The huge difference in vagueness of both fuzzy maximal eigenvalues is even more noticeable from graphical representation, see Fig. IV.6. By applying the approach of Ishizaka (2014), the fuzzy maximal eigenvalue $\widetilde{\lambda}_I$ obtained by the formulas (IV.38)–(IV.41) from the matrix $\widetilde{M}_I$ corresponding to the FMPCM (IV.52) is in the form $\widetilde{\lambda}_I = (4.1674, 4, 4.0972)$. Thus, as in the previous example, $\widetilde{\lambda}_I$ is again not a fuzzy number. $\triangle$

Since the fuzzy maximal eigenvalue $\widetilde{\lambda}_C$ obtained by the formulas (IV.46)–(IV.49) has all desired properties, it can be used in the formulas (IV.35) and (IV.36) for computing $\widetilde{CI}$ and $\widetilde{CR}$, respectively, in order to verify acceptable inconsistency of FMPCMs.

Theorem 38. *Fuzzy consistency index $\widetilde{CI}$ given by the formula* (IV.35) *with the fuzzy maximal eigenvalue $\widetilde{\lambda}_C$ given by the formulas* (IV.46)–(IV.49) *is invariant under permutation of objects in FMPCMs.*

Figure IV.5: Fuzzy maximal eigenvalues $\widetilde{\lambda}$ and $\widetilde{\lambda}_C$ of the FMPCM (IV.55).
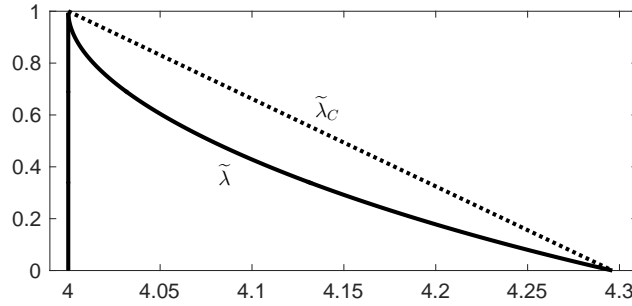
Figure IV.6: Fuzzy maximal eigenvalues $\widetilde{\lambda}_C$ and $\widetilde{\lambda}_S$ of the FMPCM (IV.55).
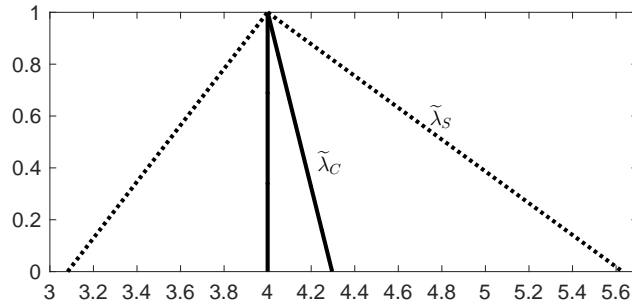
Proof. According to Theorem 37, the fuzzy maximal eigenvalue $\widetilde{\lambda}_C = (\lambda_C^\alpha, \lambda_C^\beta, \lambda_C^\gamma, \lambda_C^\delta)$ obtained by the formulas (IV.46)–(IV.49) is invariant under permutation of objects in FMPCMs. Thus, also the expressions $\frac{\lambda_C^\alpha - n}{n-1}, \frac{\lambda_C^\beta - n}{n-1}, \frac{\lambda_C^\gamma - n}{n-1}$, and $\frac{\lambda_C^\delta - n}{n-1}$ in the formula (IV.35) are invariant under permutation of objects. This means that $\widetilde{CI}$ given by the formula (IV.35) is invariant under permutation of objects in FMPCMs. $\square$

4.2.3 Deriving priorities from FMPCMs

In this section, the focus is put on methods for obtaining fuzzy priorities of objects from FMPCMs. The notation $\widetilde{\underline{w}} = (\widetilde{w}_1, \ldots, \widetilde{w}_n)^T, \widetilde{w}_i = (w_i^\alpha, w_i^\beta, w_i^\gamma, w_i^\delta), i = 1, \ldots, n,$ will be used hereafter to represent exclusively a fuzzy priority vector associated with a FMPCM.

Analogously as for MPCMs, the fuzzy priorities obtained from a FMPCM are usually normalized to reach uniqueness. In MPCMs theory, the normalization condition (II.18), $\sum_{i=1}^{n} w_i = 1, w_i \in [0, 1], i = 1, \ldots, n,$ is usually applied to the priorities. This normalization condition is usually extended to the fuzzy priorities $\widetilde{w}_i = (w_i^\alpha, w_i^\beta, w_i^\gamma, w_i^\delta), i = 1, \ldots, n,$ as

$$w_i^\alpha + \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j^\delta \geq 1, \quad w_i^\delta + \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j^\alpha \leq 1, \quad w_i^\beta + \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j^\gamma \geq 1, \quad w_i^\gamma + \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j^\beta \leq 1, \quad \text{(IV.56)}$$

which is in fact Definition 28 of the normalized fuzzy vector.

There exist several well-known methods for deriving priorities of objects from FMPCMs. Laarhoven and Pedrycz (1983) proposed the fuzzy logarithmic least squares method for obtaining fuzzy priorities of objects from FMPCMs. Later, fuzzy extension of the GMM was proposed by Buckley (1985a) in order to obtain fuzzy priorities. Chang (1996) introduced the extent analysis method to obtain crisp priorities from FMPCMs. However, this method has been severely criticized, especially because the priorities determined by this method do not represent the relative importance of objects; see, e.g., Wang et al. (2008). Despite this criticism, the extent analysis method seems to be the most popular in practice, mainly because of its computational simplicity; see Kubler et al. (2016). Csutora and Buckley (2001) proposed a fuzzy extension of the EVM in order to obtain fuzzy priorities of objects. Mikhailov (2003) proposed a fuzzy preference programming method for obtaining crisp priorities from FMPCMs.

In this thesis only the methods based on the fuzzy extension of well-known methods originally developed for PCMs are of interest. In Section 2.2.3, two famous methods for obtaining priorities from MPCMs were

reviewed - the EVM and the GMM. The fuzzy extension of these two methods to FMPCMs is studied in detail in the following two subsections.

4.2.3.1 Fuzzy extension of the eigenvector method

This section focuses on the methods for obtaining the fuzzy maximal eigenvector corresponding to the fuzzy maximal eigenvalue of a FMPCM. The methods proposed by Csutora and Buckley (2001), Wang and Chin (2006), and Ishizaka (2014) are reviewed an their drawbacks regarding the violation of the multiplicative reciprocity of the related PCs and the invariance under permutation of objects are pointed out. Afterwards, a new method for deriving the fuzzy maximal eigenvector corresponding to the fuzzy maximal eigenvalue obtained by the formulas proposed in the previous section is introduced.

Csutora and Buckley (2001) proposed a procedure for obtaining the lower and upper boundary values of α-cuts of the fuzzy maximal eigenvector corresponding to the fuzzy maximal eigenvalue $\widetilde{\lambda}_S$ of a given FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$, $\widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$. Simplification to trapezoidal representation is used here to review their method.

Csutora and Buckley (2001) computed the normalized maximal eigenvectors $\underline{w}^{\alpha*}, \underline{w}^{\beta*}, \underline{w}^{\gamma*}, \underline{w}^{\delta*}$ corresponding to the representing values of the fuzzy maximal eigenvalue $\widetilde{\lambda}_S = (\lambda_S^{\alpha}, \lambda_S^{\beta}, \lambda_S^{\gamma}, \lambda_S^{\delta})$ obtained by the formulas (IV.38)–(IV.41) as follows:

$$\underline{w}^{\alpha*} = (w_1^{\alpha*}, \ldots, w_n^{\alpha*})^T : \qquad \underline{w}^{\alpha*} = EVM_{\underline{w}}(M^{\alpha}), \ M^{\alpha} = \left\{m_{ij}^{\alpha}\right\}_{i,j=1}^{n}, \tag{IV.57}$$

$$\underline{w}^{\beta*} = (w_1^{\beta*}, \ldots, w_n^{\beta*})^T : \qquad \underline{w}^{\beta*} = EVM_{\underline{w}}(M^{\beta}), \ M^{\beta} = \left\{m_{ij}^{\beta}\right\}_{i,j=1}^{n}, \tag{IV.58}$$

$$\underline{w}^{\gamma*} = \left(w_1^{\gamma*}, \ldots, w_n^{\gamma*}\right)^T : \qquad \underline{w}^{\gamma*} = EVM_{\underline{w}}(M^{\gamma}), \ M^{\gamma} = \left\{m_{ij}^{\gamma}\right\}_{i,j=1}^{n}, \tag{IV.59}$$

$$\underline{w}^{\delta*} = \left(w_1^{\delta*}, \ldots, w_n^{\delta*}\right)^T : \qquad \underline{w}^{\delta*} = EVM_{\underline{w}}(M^{\delta}), \ M^{\delta} = \left\{m_{ij}^{\delta}\right\}_{i,j=1}^{n}. \tag{IV.60}$$

Since the maximal eigenvectors $\underline{w}^{\alpha*}, \underline{w}^{\beta*}, \underline{w}^{\gamma*}, \underline{w}^{\delta*}$ are normalized, it is clear that the resulting fuzzy vector $\widetilde{w} = (\underline{w}^{\alpha*}, \underline{w}^{\beta*}, \underline{w}^{\gamma*}, \underline{w}^{\delta*})$ is not a vector of trapezoidal fuzzy numbers; the inequalities $w_i^{\alpha*} \leq w_i^{\beta*} \leq w_i^{\gamma*} \leq w_i^{\delta*}$ are not satisfied for each $i = 1, \ldots, n$ (unless $\widetilde{M}$ is a crisp MPCM). Thus, Csutora and Buckley (2001) proposed to adjust the resulting maximal eigenvectors in the following way. First, the normalized maximal eigenvector $\underline{w}^M = (w_1^M \ldots, w_n^M)^T, \sum_{i=1}^{n} w_i^M = 1$, of the MPCM $M^M = \{m_{ij}^M\}_{i,j=1}^{n}, m_{ij}^M = \sqrt{m_{ij}^{\beta} m_{ij}^{\gamma}}$, is computed. Afterwards, the fuzzy maximal eigenvector $\widetilde{w} = \left(\underline{w}^{\alpha}, \underline{w}^{\beta}, \underline{w}^{\gamma}, \underline{w}^{\delta}\right), \underline{w}^{\alpha} = (w_1^{\alpha}, \ldots, w_n^{\alpha})^T, \underline{w}^{\beta} = (w_1^{\beta}, \ldots, w_n^{\beta})^T, \underline{w}^{\gamma} = (w_1^{\gamma}, \ldots, w_n^{\gamma})^T, \underline{w}^{\delta} = (w_1^{\delta}, \ldots, w_n^{\delta})^T$ corresponding to the fuzzy maximal eigenvalue $\widetilde{\lambda}_S$ is obtained as

$$k^{\beta} = \min \left\{ \frac{w_i^M}{w_i^{\beta*}}; i = 1, \ldots, n \right\} \rightarrow \underline{w}^{\beta} = k^{\beta} \underline{w}^{\beta*}, \tag{IV.61}$$

$$k^{\gamma} = \max \left\{ \frac{w_i^M}{w_i^{\gamma*}}; i = 1, \ldots, n \right\} \rightarrow \underline{w}^{\gamma} = k^{\gamma} \underline{w}^{\gamma*}, \tag{IV.62}$$

$$k^{\alpha} = \min \left\{ \frac{w_i^{\beta}}{w_i^{\alpha*}}; i = 1, \ldots, n \right\} \rightarrow \underline{w}^{\alpha} = k^{\alpha} \underline{w}^{\alpha*}, \tag{IV.63}$$

$$k^{\delta} = \max \left\{ \frac{w_i^{\gamma}}{w_i^{\delta*}}; i = 1, \ldots, n \right\} \rightarrow \underline{w}^{\delta} = k^{\delta} \underline{w}^{\delta*}. \tag{IV.64}$$

Note that the fuzzy maximal eigenvector $\widetilde{w} = \left(\underline{w}^{\alpha}, \underline{w}^{\beta}, \underline{w}^{\gamma}, \underline{w}^{\delta}\right)$ is a vector of trapezoidal fuzzy numbers $\widetilde{w}_i = \left(w_i^{\alpha}, w_i^{\beta}, w_i^{\gamma}, w_i^{\delta}\right), i = 1, \ldots, n$, i.e. it can be written as $\widetilde{w} = (\widetilde{w}_1, \ldots, \widetilde{w}_n)^T$. However, for the convenience of representation, the different notation is used in the method given by the formulas (IV.57)–(IV.64).

Example 39. Let us examine the method for obtaining the fuzzy maximal eigenvector proposed by Csutora

and Buckley (2001) on the trapezoidal FMPCM

$$\widetilde{M} = \begin{pmatrix} 1 & (1,1,2,3) & (2,2.5,3,4) & (4,6,7,8) \\ \left(\frac{1}{3},\frac{1}{2},1,1\right) & 1 & (3,4,4,5) & (4,5,6,6) \\ \left(\frac{1}{4},\frac{1}{3},\frac{1}{2.5},\frac{1}{2}\right) & \left(\frac{1}{5},\frac{1}{4},\frac{1}{4},\frac{1}{3}\right) & 1 & (1,2,2,3) \\ \left(\frac{1}{8},\frac{1}{7},\frac{1}{6},\frac{1}{4}\right) & \left(\frac{1}{6},\frac{1}{6},\frac{1}{5},\frac{1}{4}\right) & \left(\frac{1}{3},\frac{1}{2},\frac{1}{2},1\right) & 1 \end{pmatrix} \tag{IV.65}$$

of four objects o_1, o_2, o_3, and o_4.

The fuzzy maximal eigenvalue obtained from the FMPCM (IV.65) by the formulas (IV.38)–(IV.41) is $\widetilde{\lambda}_S = (3.1250, 3.7752, 4.4053, 5.5024)$. The fuzzy priorities of objects obtained by the formulas (IV.57)–(IV.64) are given as

$$\begin{aligned} \widetilde{w}_1 &= (0.3747, 0.3838, 0.4843, 0.5574), \\ \widetilde{w}_2 &= (0.3284, 0.3657, 0.4057, 0.4057), \\ \widetilde{w}_3 &= (0.1053, 0.1254, 0.1254, 0.1497), \\ \widetilde{w}_4 &= (0.0643, 0.0643, 0.0659, 0.0867). \end{aligned} \tag{IV.66}$$

$\triangle$

Ishizaka (2014) proposed another approach for obtaining the fuzzy maximal eigenvector of a trapezoidal FMPCM. In this approach, the fuzzy maximal eigenvector $\widetilde{\underline{w}}_I = \left(\underline{w}_I^\alpha, \underline{w}_I^\beta, \underline{w}_I^\gamma, \underline{w}_I^\delta\right)$, similarly to the fuzzy maximal eigenvalue $\widetilde{\lambda}_I$ in Section 4.2.2.4, is obtained from the fuzzy matrix $\widetilde{M}_I = \{\widetilde{m}_{Iij}\}_{i,j=1}^n$, $\widetilde{m}_{Iij} = \left(m_{Iij}^\alpha, m_{Iij}^\beta, m_{Iij}^\gamma, m_{Iij}^\delta\right)$, such that $\widetilde{m}_{Iji} = \left(\frac{1}{m_{Iij}^\alpha}, \frac{1}{m_{Iij}^\beta}, \frac{1}{m_{Iij}^\gamma}, \frac{1}{m_{Iij}^\delta}\right)$, $i < j$. In particular, the fuzzy maximal eigenvector $\widetilde{\underline{w}}_I = \left(\underline{w}_I^\alpha, \underline{w}_I^\beta, \underline{w}_I^\gamma, \underline{w}_I^\delta\right)$ is obtained as

$$\underline{w}_I^\alpha = (w_{I1}^\alpha, \ldots, w_{In}^\alpha)^T : \qquad \underline{w}_I^\alpha = EVM_{\underline{w}}(M_I^\alpha), \ M_I^\alpha = \{m_{Iij}^\alpha\}_{i,j=1}^n, \tag{IV.67}$$

$$\underline{w}_I^\beta = (w_{I1}^\beta, \ldots, w_{In}^\beta)^T : \qquad \underline{w}_I^\beta = EVM_{\underline{w}}(M_I^\beta), \ M_I^\beta = \{m_{Iij}^\beta\}_{i,j=1}^n, \tag{IV.68}$$

$$\underline{w}_I^\gamma = (w_{I1}^\gamma, \ldots, w_{In}^\gamma)^T : \qquad \underline{w}_I^\gamma = EVM_{\underline{w}}(M_I^\gamma), \ M_I^\gamma = \{m_{Iij}^\gamma\}_{i,j=1}^n, \tag{IV.69}$$

$$\underline{w}_I^\delta = (w_{I1}^\delta, \ldots, w_{In}^\delta)^T : \qquad \underline{w}_I^\delta = EVM_{\underline{w}}(M_I^\delta), \ M_I^\delta = \{m_{Iij}^\delta\}_{i,j=1}^n. \tag{IV.70}$$

However, similarly as for $\widetilde{\lambda}_I$ obtained from the FMPCM $\widetilde{M}_I$, also the elements $\widetilde{w}_{Ii} = \left(w_{Ii}^\alpha, w_{Ii}^\beta, w_{Ii}^\gamma, w_{Ii}^\delta\right)$, $i = 1, \ldots, n$, of $\widetilde{\underline{w}}_I$ are not even fuzzy numbers in general, just quadruples of real numbers. This is caused not only by the inappropriate form of the fuzzy maximal eigenvalue $\widetilde{\lambda}_I$ used in the formulas (IV.67)–(IV.70) but also by the inappropriate normalization of the maximal eigenvectors $\underline{w}_I^\alpha, \underline{w}_I^\beta, \underline{w}_I^\gamma$, and $\underline{w}_I^\delta$. Since $\sum_{i=1}^n w_{Ii}^\alpha = 1, \sum_{i=1}^n w_{Ii}^\beta = 1, \sum_{i=1}^n w_{Ii}^\gamma = 1, \sum_{i=1}^n w_{Ii}^\delta = 1$, then clearly, the inequalities $w_{Ii}^\alpha \leq w_{Ii}^\beta \leq w_{Ii}^\gamma \leq w_{Ii}^\delta$, $i = 1, \ldots, n$, cannot be guaranteed. Moreover, since Ishizaka's method for obtaining the fuzzy maximal eigenvalue $\widetilde{\lambda}_I$ reviewed in Section 4.2.2.4 is not invariant under permutation of objects in the FMPCM, it is obvious that also the method for obtaining the fuzzy maximal eigenvector $\widetilde{\underline{w}}_I$ is not invariant under permutation of objects.

Example 40. Let us examine the method for obtaining the fuzzy maximal eigenvector proposed by Ishizaka (2014) on the trapezoidal FMPCM (IV.65) from Example 39. The corresponding matrix $\widetilde{M}_I$ used in Ishizaka's approach is

$$\widetilde{M}_I = \begin{pmatrix} 1 & (1,1,2,3) & (2,2.5,3,4) & (4,6,7,8) \\ \left(1,1,\frac{1}{2},\frac{1}{3}\right) & 1 & (3,4,4,5) & (4,5,6,6) \\ \left(\frac{1}{2},\frac{1}{2.5},\frac{1}{3},\frac{1}{4}\right) & \left(\frac{1}{3},\frac{1}{4},\frac{1}{4},\frac{1}{5}\right) & 1 & (1,2,2,3) \\ \left(\frac{1}{4},\frac{1}{6},\frac{1}{7},\frac{1}{8}\right) & \left(\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{6}\right) & \left(1,\frac{1}{2},\frac{1}{2},\frac{1}{3}\right) & 1 \end{pmatrix}. \tag{IV.71}$$

The fuzzy maximal eigenvalue obtained from (IV.71) by the formulas (IV.38)–(IV.41) is $\widetilde{\lambda}_I = (4.0458, 4.0319, 4.0863, 4.2184)$. Obviously, $\widetilde{\lambda}_I$ is not a fuzzy number since $\lambda_I^\alpha = 4.0458 \not\leq 4.0319 = \lambda_I^\beta$. The corresponding fuzzy

priorities of objects obtained by the formulas (IV.67)–(IV.70) are given as

$$\widetilde{w}_{I1} = (0.3606, 0.3837, 0.4753, 0.5379),$$
$$\widetilde{w}_{I2} = (0.3946, 0.4151, 0.3466, 0.3088),$$
$$\widetilde{w}_{I3} = (0.1376, 0.1311, 0.1182, 0.1055),$$
$$\widetilde{w}_{I4} = (0.1072, 0.0701, 0.0599, 0.0478).$$

(IV.72)

Notice that $\widetilde{w}_{I2}, \widetilde{w}_{I3}$, and $\widetilde{w}_{I4}$ are not trapezoidal fuzzy numbers since the inequalities $w^\alpha_{Ii} \leq w^\beta_{Ii} \leq w^\gamma_{Ii} \leq w^\delta_{Ii}, i = 2, 3, 4$, are not satisfied.

Let us now permute the trapezoidal FMPCM (IV.65) to $\widetilde{M^\pi} = P\widetilde{M}P^T$ by using the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(IV.73)

The corresponding matrix in Ishizaka's approach is

$$\widetilde{M}^\pi_I = \begin{pmatrix} 1 & \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, 1\right) & \left(\frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{4}\right) & \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}\right) \\ (3, 2, 2, 1) & 1 & \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2.5}, \frac{1}{2}\right)\left(\frac{1}{5}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3}\right) \\ (8, 7, 6, 4)(4, 3, 2.5, 2) & 1 & (1, 1, 2, 3) \\ (6, 6, 5, 4) & (5, 4, 4, 3) & \left(1, 1, \frac{1}{2}, \frac{1}{3}\right) & 1 \end{pmatrix}.$$

(IV.74)

The fuzzy maximal eigenvalue obtained from (IV.74) by the formulas (IV.38)–(IV.41) is $\widetilde{\lambda}^\pi_I = (4.0720, 4.0133, 4.1193, 4.2606)$. The corresponding fuzzy priorities of objects obtained by the formulas (IV.67)–(IV.70) are given as

$$\widetilde{w}^\pi_{I\pi(1)} = (0.4171, 0.4017, 0.4560, 0.4738),$$
$$\widetilde{w}^\pi_{I\pi(2)} = (0.4191, 0.4162, 0.3472, 0.4738),$$
$$\widetilde{w}^\pi_{I\pi(3)} = (0.1119, 0.1201, 0.1295, 0.1332),$$
$$\widetilde{w}^\pi_{I\pi(4)} = (0.0519, 0.0620, 0.0674, 0.0997).$$

(IV.75)

Again, $\widetilde{\lambda}^\pi_I, \widetilde{w}^\pi_{I1}, \widetilde{w}^\pi_{I2}$ are not fuzzy numbers. Further, we see that $\widetilde{\lambda}^\pi_I \neq \widetilde{\lambda}_I$ and $\widetilde{w}^\pi_{I\pi(i)} \neq \widetilde{w}_{Ii}, i = 1 \ldots, 4$, which means that the method is not invariant under permutation of objects. $\triangle$

Wang and Chin (2006) revised the method for obtaining the fuzzy maximal eigenvector proposed by Csutora and Buckley (2001). They argued that the fuzzy maximal eigenvector obtained by formulas (IV.57)–(IV.64) is not normalized according to Definition 29, and they proposed a new procedure for obtaining the normalized fuzzy maximal eigenvector corresponding to the fuzzy maximal eigenvalue $\widetilde{\lambda}_S$ obtained by formulas (IV.38)–(IV.41). First, formulas (IV.57)–(IV.60) for obtaining the normalized maximal eigenvectors $\underline{w}^{\alpha*}, \underline{w}^{\beta*}, \underline{w}^{\gamma*}, \underline{w}^{\delta*}$ corresponding to the maximal eigenvalues $\lambda^\alpha_S, \lambda^\beta_S, \lambda^\gamma_S, \lambda^\delta_S$ obtained by formulas (IV.38)–(IV.41), respectively, are applied. Afterwards, the normalization constants $k^\alpha, k^\beta, k^\gamma$, and k^δ are searched for to obtain $\underline{w}^\alpha = k^\alpha \underline{w}^{\alpha*}, \underline{w}^\beta = k^\beta \underline{w}^{\beta*}, \underline{w}^\gamma = k^\gamma \underline{w}^{\gamma*}$, and $\underline{w}^\delta = k^\delta \underline{w}^{\delta*}$ so that $\widetilde{w} = (\underline{w}^\alpha, \underline{w}^\beta, \underline{w}^\gamma, \underline{w}^\delta)$ is a normalized fuzzy vector according to (III.13), i.e. the inequalities

$$k^\alpha w^{\alpha*}_i + \sum_{j=1, j\neq i}^n k^\delta w^{\delta*}_j \geq 1, \qquad k^\delta w^{\delta*}_i + \sum_{j=1, j\neq i}^n k^\alpha w^{\alpha*}_j \leq 1,$$

$$k^\beta w^{\beta*}_i + \sum_{j=1, j\neq i}^n k^\gamma w^{\gamma*}_j \geq 1, \qquad k^\gamma w^{\gamma*}_i + \sum_{j=1, j\neq i}^n k^\beta w^{\beta*}_j \leq 1,$$

hold for every $i = 1, \ldots, n$. The inequalities can be further written as

$$k^\alpha w^{\alpha*}_i + k^\delta \left(1 - w^{\delta*}_i\right) \geq 1, \qquad k^\delta w^{\delta*}_i + k^\alpha \left(1 - w^{\alpha*}_i\right) \leq 1,$$

$$k^\beta w^{\beta*}_i + k^\gamma \left(1 - w^{\gamma*}_i\right) \geq 1, \qquad k^\gamma w^{\gamma*}_i + k^\beta (1 - w^{\beta*}_i) \leq 1.$$

Obviously, $0 \leq k^\alpha \leq k^\beta \leq 1 \leq k^\gamma \leq k^\delta$.

In order not to lose any information obtained in the fuzzy maximal eigenvector, parameters $k^\alpha, k^\beta, k^\gamma$, and k^δ are chosen in such a form that the supports of the trapezoidal fuzzy numbers in the fuzzy maximal eigenvector

are as wide as possible. It means that the parameters k^α and k^β are minimized and the parameters k^γ and k^δ are maximized. The parameters $k^\alpha, k^\beta, k^\gamma$, and k^δ can be obtained as the solution of the following linear programming problem (Wang and Chin, 2006):

$$
\begin{aligned}
J^* = \max \quad & k^\alpha + \delta_1 + \delta_2 + \delta_3 \\
s.t. \quad & k^\delta \left(1 - w_i^{\delta*}\right) + k^\alpha w_i^{\alpha*} \geq 1, \\
& k^\delta w_i^{\delta*} + k^\alpha \left(1 - w_i^{\alpha*}\right) \leq 1, \\
& k^\gamma \left(1 - w_i^{\gamma*}\right) + k^\beta w_i^{\beta*} \geq 1, \\
& k^\gamma w_i^{\gamma*} + k^\beta (1 - w_i^{\beta*}) \leq 1, \qquad i = 1, \dots, n. \\
& k^\delta w_i^{\delta*} - k^\gamma w_i^{\gamma*} - \delta_1 \geq 0, \\
& k^\gamma w_i^{\gamma*} - k^\beta w_i^{\beta*} - \delta_2 \geq 0, \\
& k^\beta w_i^{\beta*} - k^\alpha w_i^{\alpha*} - \delta_3 \geq 0, \\
& k^\alpha, \delta_1, \delta_2, \delta_3 \geq 0,
\end{aligned}
\tag{IV.76}
$$

Readers can refer to Wang and Chin (2006) and to Krejčí (2017a) for more details. The notation $\widetilde{\underline{w}}_S = (\widetilde{w}_{S1}, \dots, \widetilde{w}_{Sn})^T, \widetilde{w}_{Si} = (w_{Si}^\alpha, w_{Si}^\beta, w_{Si}^\gamma, w_{Si}^\delta)$ will be used hereafter to refer to the normalized fuzzy maximal eigenvector obtained by the formulas (IV.57)–(IV.60) and (IV.76) corresponding to the fuzzy maximal eigenvalue $\widetilde{\lambda}_S$ (the lower index S stands for standard fuzzy arithmetic that is used in the formulas (IV.38)–(IV.41)).

It is true that the supports of the trapezoidal fuzzy numbers $\widetilde{w}_{Si} = (w_{Si}^\alpha, w_{Si}^\beta, w_{Si}^\gamma, w_{Si}^\delta)$, $i = 1, \dots, n$, in the fuzzy vector $\widetilde{\underline{w}}_S$ obtained by the method proposed by Wang and Chin (2006) are as wide as possible still meeting the condition (III.13) of normalized fuzzy numbers. Moreover, the method for deriving the normalized fuzzy maximal eigenvector proposed by Wang and Chin (2006) is invariant under permutation of objects. However, the fuzzy vector $\widetilde{\underline{w}}_S$ does not represent the actual normalized fuzzy maximal eigenvector of the FMPCM $\widetilde{M}$. First of all, as already shown in Section 4.2.2.4, the fuzzy maximal eigenvalue $\widetilde{\lambda}_S$ obtained by formulas (IV.38)–(IV.41) does not represent the actual fuzzy maximal eigenvalue of a FMPCM $\widetilde{M}$ since it violates the multiplicative reciprocity of PCs. Thus, it should not be used in the process of deriving the fuzzy maximal eigenvectors of FMPCMs. However, just simply replacing the fuzzy maximal eigenvalue $\widetilde{\lambda}_S$ by the fuzzy maximal eigenvalue $\widetilde{\lambda}_C$ in the method for deriving the normalized fuzzy maximal eigenvector is not sufficient.

A severe drawback independent of the formulas for obtaining the fuzzy maximal eigenvalue is that the normalized fuzzy vector $\widetilde{\underline{w}}_S$ obtained by the formulas (IV.57)–(IV.60) and (IV.76) does not "consist" of normalized eigenvectors. For example, the eigenvector $\underline{w}^\alpha = k^\alpha \underline{w}^{\alpha*} = (k^\alpha w_1^{\alpha*}, \dots, k^\alpha w_n^{\alpha*})^T$ obtained as the maximal eigenvector of the matrix $M^\alpha = \left\{ m_{ij}^\alpha \right\}_{i,j=1}^n$ corresponding to the maximal eigenvalue λ_S^α is not normalized, $\sum_{i=1}^n k^\alpha w_i^{\alpha*} < 1$. This drawback would occur even if the eigenvector $\underline{w}^\alpha$ was obtained as the maximal eigenvector of the MPCM $M^* = \left\{ m_{ij} \right\}_{i,j=1}^n$ corresponding to the maximal eigenvalue λ_C^α obtainable by the formula (IV.46). The same drawback occurs also for the eigenvector $\underline{w}^\beta = k^\beta \underline{w}^{\beta*} = (k^\beta w_1^{\beta*}, \dots, k^\beta w_n^{\beta*})^T$, i.e. $\sum_{i=1}^n k^\beta w_i^{\beta*} < 1$, unless $M^\beta = M^\gamma$. Analogously, also the eigenvectors $\underline{w}^\delta = k^\delta \underline{w}^{\delta*}$ and $\underline{w}^\gamma = k^\gamma \underline{w}^{\gamma*}$ are such that $\sum_{i=1}^n k^\delta w_i^{\delta*} > 1$ and $\sum_{i=1}^n k^\gamma w_i^{\gamma*} > 1$, unless $M^\beta = M^\gamma$.

Example 41. Let us examine the method for obtaining the fuzzy maximal eigenvector proposed by Wang and Chin (2006) on the trapezoidal FMPCM (IV.65) from Example 39. The fuzzy priorities obtained by the formulas (IV.57)–(IV.60) and (IV.76) are

$$
\begin{aligned}
\widetilde{w}_{S1} &= (0.2584, 0.2969, 0.5704, 0.6565), \\
\widetilde{w}_{S2} &= (0.2265, 0.2829, 0.4778, 0.4778), \\
\widetilde{w}_{S3} &= (0.0726, 0.0970, 0.1477, 0.1763), \\
\widetilde{w}_{S4} &= (0.0444, 0.0497, 0.0777, 0.1022).
\end{aligned}
\tag{IV.77}
$$

It can be easily verified by using (III.13) that the fuzzy priorities are normalized.

In the approach of Wang and Chin (2006), as well as in the original approach of Csutora and Buckley (2001), the representing values of the fuzzy priorities $\widetilde{w}_{Si}$ are obtained from matrices that are not multiplicatively reciprocal. For example, the lower boundary values $w_{Si}^\alpha, i = 1, \dots, 4$, are obtained as the components of the maximal eigenvector of matrix

$$
M^\alpha = \begin{pmatrix} 1 & 1 & 2 & 4 \\ \frac{1}{3} & 1 & 3 & 4 \\ \frac{1}{4} & \frac{1}{5} & 1 & 1 \\ \frac{1}{8} & \frac{1}{6} & \frac{1}{3} & 1 \end{pmatrix}
\tag{IV.78}
$$

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which does not keep multiplicative reciprocity of the related PCs. The meaning of the maximal eigenvectors obtained from such matrices is questionable. This violation of multiplicative reciprocity of PCs occurs both in the approach of Wang and Chin (2006) as well as in the approach of Csutora and Buckley (2001).

Further, the fuzzy vector $\widetilde{w}_S = (\widetilde{w}_{S1}, \widetilde{w}_{S2}, \widetilde{w}_{S3}, \widetilde{w}_{S4})^T$ with the components given as (IV.77) does not really represent a normalized fuzzy maximal eigenvector of the trapezoidal FMPCM (IV.65) since it does not consist of normalized eigenvectors. For example, the vector $\underline{w}_S^\alpha = (0.2584, 0.2265, 0.0726, 0.0444)^T$ obtained as the maximal eigenvector of the matrix (IV.78) is not normalized; $\sum_{i=1}^4 w_{Si}^\alpha = 0.6019 < 1$. $\triangle$

As emphasized repeatedly, it is necessary to consider the multiplicative reciprocity of the related PCs when performing operations on the elements of a FMPCM in order to reflect properly the preference information contained in the FMPCM. Thus, similar to the formulas for obtaining the fuzzy maximal eigenvalue $\widetilde{\lambda}_C$ of a FMPCM, it is necessary to apply constrained fuzzy arithmetic to the fuzzy extension of the formula (II.21) in order to obtain the normalized fuzzy maximal eigenvector of a FMPCM. For a trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n, \widetilde{m}_{ij} = (m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta)$, the components $\widetilde{w}_{Ci} = (w_{Ci}^\alpha, w_{Ci}^\beta, w_{Ci}^\gamma, w_{Ci}^\delta)$, $i = 1, \ldots, n$, of the normalized fuzzy maximal eigenvector $\widetilde{w}_C$ (the lower index C stands for the applied concept of constrained fuzzy arithmetic (III.45)) should be obtained as:

$$w_{Ci}^\alpha = \min \left\{ w_i \; ; \; \begin{array}{l} \underline{w} = (w_1, ..., w_i, ..., w_n)^T, \ \underline{w} = EVM_{\underline{w}}(M), \ M = \{m_{rs}\}_{r,s=1}^n, \\ m_{rs} \in [m_{rs}^\alpha, m_{rs}^\delta], \ m_{sr} = \frac{1}{m_{rs}}, \ r, s = 1, \ldots, n \end{array} \right\}, \tag{IV.79}$$

$$w_{Ci}^\beta = \min \left\{ w_i \; ; \; \begin{array}{l} \underline{w} = (w_1, ..., w_i, ..., w_n)^T, \ \underline{w} = EVM_{\underline{w}}(M), \ M = \{m_{rs}\}_{r,s=1}^n, \\ m_{rs} \in [m_{rs}^\beta, m_{rs}^\gamma], \ m_{sr} = \frac{1}{m_{rs}}, \ r, s = 1, \ldots, n \end{array} \right\}, \tag{IV.80}$$

$$w_{Ci}^\gamma = \max \left\{ w_i \; ; \; \begin{array}{l} \underline{w} = (w_1, ..., w_i, ..., w_n)^T, \ \underline{w} = EVM_{\underline{w}}(M), \ M = \{m_{rs}\}_{r,s=1}^n, \\ m_{rs} \in [m_{rs}^\beta, m_{rs}^\gamma], \ m_{sr} = \frac{1}{m_{rs}}, \ r, s = 1, \ldots, n \end{array} \right\}, \tag{IV.81}$$

$$w_{Ci}^\delta = \max \left\{ w_i \; ; \; \begin{array}{l} \underline{w} = (w_1, ..., w_i, ..., w_n)^T, \ \underline{w} = EVM_{\underline{w}}(M), \ M = \{m_{rs}\}_{r,s=1}^n, \\ m_{rs} \in [m_{rs}^\alpha, m_{rs}^\delta], \ m_{sr} = \frac{1}{m_{rs}}, \ r, s = 1, \ldots, n \end{array} \right\}. \tag{IV.82}$$

Theorem 39. *The fuzzy priorities* $\widetilde{w}_{Ci} = (w_{Ci}^\alpha, w_{Ci}^\beta, w_{Ci}^\gamma, w_{Ci}^\delta), i = 1, \ldots, n$, *obtained from a FMPCM* $\widetilde{M}$ *by the formulas* (IV.79)–(IV.82) *are normalized.*

Proof. It is sufficient to prove that the fuzzy priorities $\widetilde{w}_{Ci}, i = 1, \ldots, n$, satisfy the inequalities (III.13). From the formula (IV.79), it follows that w_{Ci}^α was obtained as the $i-$th component of the normalized maximal eigenvector of one particular MPCM $M^{\alpha i} = \{m_{pq}\}_{p,q=1}^n, m_{pq} \in [m_{pq}^\alpha, m_{pq}^\delta], p, q = 1, \ldots, n$. Let us denote by $w_k^{\alpha i}$ the priorities of objects $o_k, k \neq i$, obtainable from the same MPCM $M^{\alpha i}$, i.e. $(w_1^{\alpha i}, \ldots, w_{Ci}^\alpha, \ldots, w_n^{\alpha i})^T$ is the normalized maximal eigenvector of $M^{\alpha i}$. Obviously, $w_{Ci}^\alpha + \sum_{\substack{k=1 \\ k \neq i}}^n w_k^{\alpha i} = 1$, and $w_k^{\alpha i} \in [w_{Ck}^\alpha, w_{Ck}^\delta], k \neq i$. From this, it follows that $w_{Ci}^\alpha + \sum_{\substack{k=1 \\ k \neq i}}^n w_{Ck}^\delta \geq 1$. The remaining inequalities in (III.13) are proved analogously. $\square$

Remark 18. *According to Theorem 39, the fuzzy maximal eigenvector* $\widetilde{w}_C$ *obtained from a FMPCM by the formulas* (IV.79)–(IV.82) *is normalized in the sense of Definition 29, i.e. we can really call* $\widetilde{w}_C$ *the normalized fuzzy maximal eigenvector. Notice that the normality of the fuzzy maximal eigenvector was reached naturally by just properly applying constrained fuzzy arithmetic to the fuzzy extension of the formula* (II.21) *for obtaining the normalized maximal eigenvector of a MPCM; no forced normalization was done as in the case of normalizing the fuzzy maximal eigenvector in the method proposed by Wang and Chin (2006). Therefore, the normalized fuzzy vector given by Definition 29 is a natural counterpart of the normalized crisp vector given by* (II.18).

Theorem 40. *The fuzzy extension of the EVM based on the formulas* (IV.79)–(IV.82) *is invariant under permutation of objects in FMPCMs.*

Proof. It is sufficient to show that for a given object $o_i, i \in \{1, \ldots, n\}$, its priority $\widetilde{w}_{Ci}$ corresponding to the $i-$th component of the normalized maximal eigenvector $\widetilde{w}$ does not change under permutation of objects in the FMPCM $\widetilde{M}$.

From the invariance of the EVM reviewed in Section 2.2.3.1, it follows that the normalized maximal eigenvector $\underline{w}$ of the given MPCM M does not change under any permutation $M^\pi = PMP^T$ of M, but it is just permuted accordingly to the normalized maximal eigenvector $\underline{w}^\pi$. This means that each component $w_i, i \in \{1, \ldots, n\}$, of the normalized maximal eigenvector $\underline{w}$ is equal to the corresponding component $w_{\pi(i)}^\pi$ of the permuted normalized maximal eigenvector $\underline{w}^\pi$. Therefore, neither the minimum w_{Ci}^α nor the maximum w_{Ci}^δ of the component w_i of the normalized maximal eigenvector $\underline{w}$ over all MPCMs obtainable from the closures of the supports of the trapezoidal fuzzy numbers in the trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n, \widetilde{m}_{ij} =$

Figure IV.7: Fuzzy maximal eigenvectors $\widetilde{\underline{w}}_C$ and $\widetilde{\underline{w}}_S$ of the FMPCM (IV.65).
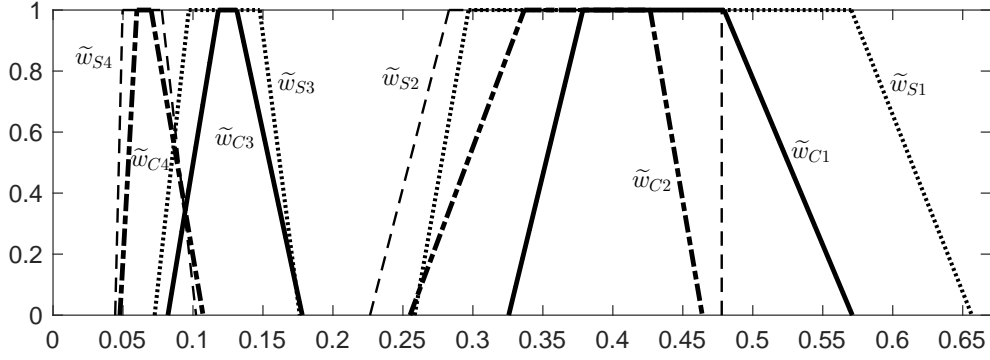
$(m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta)$, change under permutation of objects in the FMPCM $\widetilde{M}$. Analogously, also the minimum w_{Ci}^β and the maximum w_{Ci}^γ of the component w_i of the normalized maximal eigenvector $\underline{w}$ over all MPCMs obtainable from the cores of the trapezoidal fuzzy numbers in the trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$ do not change under permutation. Therefore, the fuzzy priority $\widetilde{w}_{Ci} = (w_{Ci}^\alpha, w_{Ci}^\beta, w_{Ci}^\gamma, w_{Ci}^\delta)$ obtained by the formulas (IV.79)–(IV.82) does not change under permutation of objects in FMPCMs (it is only permuted accordingly), which concludes the proof. $\qquad\square$

Example 42. Let us apply the new method given by formulas (IV.79)–(IV.82) to the FMPCM (IV.65). The fuzzy maximal eigenvalue obtained by the formulas (IV.46)–(IV.49) is $\widetilde{\lambda}_C = (4, 4.0104, 4.1245, 4.4747)$ and the normalized fuzzy maximal eigenvector $\widetilde{\underline{w}}_C$ obtained by the formulas (IV.79)–(IV.82) is given as

$$
\begin{aligned}
\widetilde{w}_{C1} &= (0.3256, 0.3786, 0.4795, 0.5711), \\
\widetilde{w}_{C2} &= (0.2551, 0.3367, 0.4267, 0.4639), \\
\widetilde{w}_{C3} &= (0.0821, 0.1182, 0.1313, 0.1781), \\
\widetilde{w}_{C4} &= (0.0478, 0.0599, 0.0701, 0.1072).
\end{aligned}
\tag{IV.83}
$$

The fuzzy priorities are normalized, i.e. they satisfy the inequalities (III.13).

The normalized fuzzy maximal eigenvector $\widetilde{\underline{w}}_C$ given by (IV.83) differs significantly from the fuzzy maximal eigenvector $\widetilde{\underline{w}}_S$ given by (IV.77) that was obtained by the formulas (IV.57)–(IV.60) and (IV.76). For easier comparison, both fuzzy maximal eigenvectors are displayed in Fig. IV.7.

Let us now examine the lower boundary value $w_{C1}^\alpha = 0.3256$ of $\widetilde{w}_{C1}$. It was obtained as the solution of the optimization problem (IV.79), in particular as the first component of the normalized maximal eigenvector $\underline{w} = (0.3256, 0.4631, 0.1451, 0.0662)^T$ of the MPCM

$$
M^\alpha = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 1 & 5 & 6 \\ \frac{1}{2} & \frac{1}{5} & 1 & 3 \\ \frac{1}{4} & \frac{1}{6} & \frac{1}{3} & 1 \end{pmatrix}.
\tag{IV.84}
$$

Similarly any other element from the closure of the support of any fuzzy priority $\widetilde{w}_{Ci}, i = 1, \ldots, 4$, is an element of a normalized maximal eigenvector corresponding to a MPCM obtainable from the closures of the supports of the trapezoidal fuzzy numbers in the FMPCM (IV.65). $\qquad\triangle$

4.2.3.2 Fuzzy extension of the geometric-mean method

In this section, the fuzzy extension of the GMM to FMPCMs is dealt with. The methods proposed by Buckley (1985a) and by Liu (2009) are reviewed and their drawbacks regarding the violation of multiplicative reciprocity and of invariance under permutation are pointed out. Afterwards, the formulas proposed by Enea and Piazza (2004) based on constrained fuzzy arithmetic are analyzed and some interesting properties are derived.

Buckley (1985a) proposed a fuzzy extension of the GMM to compute $\alpha-$cuts of the fuzzy priorities of objects from FMPCMs. Trapezoidal representation is again used here to review the method.

Buckley (1985a) first computed the geometric mean $\widetilde{g}_i = (g_i^\alpha, g_i^\beta, g_i^\gamma, g_i^\delta), i = 1, \ldots, n$, of the elements in each row of the FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n, \widetilde{m}_{ij} = (m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta)$, by applying standard fuzzy arithmetic to

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the fuzzy extension of the formula (II.23). Thus, the representing values of the geometric means $\widetilde{g}_i, i = 1, \ldots, n$, of the elements in the rows of the FMPCM $\widetilde{M}$ are computed as

$$g_i^\alpha = \sqrt[n]{\prod_{j=1}^n m_{ij}^\alpha}, \quad g_i^\beta = \sqrt[n]{\prod_{j=1}^n m_{ij}^\beta}, \quad g_i^\gamma = \sqrt[n]{\prod_{j=1}^n m_{ij}^\gamma}, \quad g_i^\delta = \sqrt[n]{\prod_{j=1}^n m_{ij}^\delta}. \tag{IV.85}$$

The geometric means $\widetilde{g}_i = (g_i^\alpha, g_i^\beta, g_i^\gamma, g_i^\delta), i = 1, \ldots, n$, represent the non-normalized fuzzy priorities of objects compared in the FMPCM $\widetilde{M}$. Notice that simplified standard fuzzy arithmetic (III.36) is used in (IV.85).

Afterwards, Buckley (1985a) divided each geometric mean by their sum, analogously to the formula (II.24), in order to normalize the fuzzy priorities of objects Standard fuzzy arithmetic was again used for this purpose. Thus, according to Buckley (1985a), the trapezoidal fuzzy priorities $\widetilde{w}_{Si} = (w_{Si}^\alpha, w_{Si}^\beta, w_{Si}^\gamma, w_{Si}^\delta), i = 1, \ldots, n$, (the lower index S stands for the standard fuzzy arithmetic that is applied to the formulas) are obtained from a trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n, \widetilde{m}_{ij} = (m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta)$, as

$$w_{Si}^\alpha = \frac{\sqrt[n]{\prod_{j=1}^n m_{ij}^\alpha}}{\sum_{k=1}^n \sqrt[n]{\prod_{j=1}^n m_{kj}^\delta}}, \tag{IV.86}$$

$$w_{Si}^\beta = \frac{\sqrt[n]{\prod_{j=1}^n m_{ij}^\beta}}{\sum_{k=1}^n \sqrt[n]{\prod_{j=1}^n m_{kj}^\gamma}}, \tag{IV.87}$$

$$w_{Si}^\gamma = \frac{\sqrt[n]{\prod_{j=1}^n m_{ij}^\gamma}}{\sum_{k=1}^n \sqrt[n]{\prod_{j=1}^n m_{kj}^\beta}}, \tag{IV.88}$$

$$w_{Si}^\delta = \frac{\sqrt[n]{\prod_{j=1}^n m_{ij}^\delta}}{\sum_{k=1}^n \sqrt[n]{\prod_{j=1}^n m_{kj}^\alpha}}. \tag{IV.89}$$

It is a well-known fact that the fuzzy extension of the GMM given by the formulas (IV.86)–(IV.89) is invariant under permutation of objects.

The fuzzy priorities $\widetilde{w}_{Si}, i = 1, \ldots, n$, obtained by the formulas (IV.86)–(IV.89) are not normalized according to Definition 28 as they do not satisfy the inequalities (III.13). In fact, they are not even constrained to the interval $[0, 1]$. Thus, Buckley (1985a) suggested to multiply all fuzzy priorities $\widetilde{w}_{Si}, i = 1, \ldots, n$, by a suitable normalization constant $c < 1$ in order to limit them to the interval $[0, 1]$. However, such fuzzy priorities still do not satisfy the inequalities (III.13), i.e. they are not normalized.

Furthermore, the representing values $w_{Si}^\alpha, w_{Si}^\beta, w_{Si}^\gamma, w_{Si}^\delta$ given by (IV.86)–(IV.89), respectively, are not obtained from MPCMs. In particular, in the formula (IV.86), the upper boundary values m_{kj}^δ of all PCs $\widetilde{m}_{kj}, k, j = 1, \ldots, n$, are used. This violates the multiplicative reciprocity of PCs since $m_{kj}^\delta \neq \frac{1}{m_{jk}^\delta}$ (unless $\widetilde{m}_{kj}$ is a crisp number). In addition, also the lower boundary values m_{ij}^α of the PCs $\widetilde{m}_{ij}, j = 1, \ldots, n$, in the i−th row of the FMPCM are present in the formula. This even violates the extension principle (III.2) since two different values, in particular m_{ij}^α and $m_{ij}^\delta, j = 1, \ldots, n$, of one variable are used in the formula at the same time. The formulas (IV.87), (IV.88), and (IV.89) suffer from the same drawbacks.

Liu (2009) proposed the following extension of the GMM to interval FMPCMs. For an interval FMPCM $\overline{M} = \{\overline{m}_{ij}\}_{i,j=1}^n, \overline{m}_{ij} = [m_{ij}^L, m_{ij}^U]$, he constructed two MPCMs $C = \{c_{ij}\}_{i,j=1}^n$ and $D = \{d_{ij}\}_{i,j=1}^n$ by applying (IV.7). Afterwards, he derived non-normalized priorities $w_i(C)$ and $w_i(D), i = 1, \ldots, n$, of objects from these MPCMs C and D, respectively, by using the formula (II.23). The interval priorities $\overline{w}_i = [w_i^L, w_i^U], i = 1, \ldots, n$, were then determined as

$$w_i^L = \min\{w_i(C), w_i(D)\}, \qquad w_i^U = \max\{w_i(C), w_i(D)\}. \tag{IV.90}$$

This method, similarly to Definitions 46 and 47 of multiplicative consistency for interval and triangular FMPCMs proposed by Liu (2009) and Liu et al. (2014), respectively, reviewed already in Section 4.2.2.1, is not invariant under permutation of objects. This drawback is illustrated on the following example.

Example 43. Let us apply the method for obtaining interval priorities proposed by Liu (2009) to the interval FMPCM (IV.29). The interval priorities of objects obtained by the formula (IV.90) are

$$\overline{w}_1 = [0.4309, 0.7631], \quad \overline{w}_2 = [1.0772, 1.1447], \quad \overline{w}_3 = [1.1447, 2.1544].$$

By applying the formula (IV.90) to the permuted interval FMPCM (IV.30), the interval priorities of objects are obtained as

$$\overline{w}^\pi_{(1)} = [0.4309, 0.7631], \quad \overline{w}^\pi_{(2)} = [0.9086, 1.3572], \quad \overline{w}^\pi_{(3)} = [1.4422, 1.7100].$$

We see that $\overline{w}^\pi_{(2)} \neq \overline{w}_2$ and $\overline{w}^\pi_{(3)} \neq \overline{w}_3$, which demonstrates that the method for obtaining interval priorities from interval FMPCMs proposed by Liu (2009) is not invariant under permutation of objects.

The non-normalized interval priorities obtained from the interval FMPCM (IV.29) by the formulas (IV.85) are

$$\overline{w}_1 = [0.4309, 0.7631], \quad \overline{w}_2 = [0.9086, 1.3572], \quad \overline{w}_3 = [1.1447, 2.1544].$$

The same interval priorities are obtained by the formulas (IV.85) also from the permuted interval FMPCM (IV.30), i.e. $\overline{w}^\pi_{(i)} = \overline{w}_i, \ i = 1, 2, 3$. $\triangle$

Note that the formulas (IV.85) for obtaining non-normalized fuzzy priorities do not violate the multiplicative-reciprocity property as well as the invariance under permutation of objects. That follows from the absence of mutually reciprocal PCs in the formulas. Therefore, use of standard fuzzy arithmetic instead of constrained fuzzy arithmetic is sufficient here. Contrarily, the formulas (IV.86)–(IV.89) violate the multiplicative reciprocity and the formulas (IV.90) violate the invariance under permutation of objects. The reason is that the formulas (IV.86)–(IV.89) are based on standard fuzzy arithmetic instead of constrained fuzzy arithmetic, which is in this case indispensable, whereas the formulas (IV.90) are not even based on standard fuzzy arithmetic.

In order to handle properly the multiplicative reciprocity of PCs, and thus automatically ensuring also the invariance under permutation, it is necessary to follow the same approach as in the previous section where the fuzzy extension of the EVM was dealt with. This means that constrained fuzzy arithmetic has to be applied to the fuzzy extension of the formula (II.24) instead of standard fuzzy arithmetic in order to respect the multiplicative reciprocity of PCs in the FMPCM. Enea and Piazza (2004) realized this necessity and proposed appropriate fuzzy extension of the formula (II.24) to triangular FMPCM.

Extending the method proposed by Enea and Piazza (2004) to trapezoidal FMPCMs $\widetilde{M} = \{\widetilde{m}_{ij}\}^n_{i,j=1}, \widetilde{m}_{ij} = (m^\alpha_{ij}, m^\beta_{ij}, m^\gamma_{ij}, m^\delta_{ij})$, fuzzy priorities $\widetilde{w}_{Ci} = \left(w^\alpha_{Ci}, w^\beta_{Ci}, w^\gamma_{Ci}, w^\delta_{Ci} \right), i = 1, \ldots, n$, (the lower index C stands for the applied concept of constrained fuzzy arithmetic (III.45)) are obtained in the following form:

$$w^\alpha_{Ci} = \min \left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} m_{kj}}} ; \begin{array}{l} m_{rs} \in \left[m^\alpha_{rs}, m^\delta_{rs} \right], \\ m_{sr} = \frac{1}{m_{rs}}, \\ r, s = 1, \ldots, n \end{array} \right\}, \tag{IV.91}$$

$$w^\beta_{Ci} = \min \left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} m_{kj}}} ; \begin{array}{l} m_{rs} \in \left[m^\beta_{rs}, m^\gamma_{rs} \right], \\ m_{sr} = \frac{1}{m_{rs}}, \\ r, s = 1, \ldots, n \end{array} \right\}, \tag{IV.92}$$

$$w^\gamma_{Ci} = \max \left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} m_{kj}}} ; \begin{array}{l} m_{rs} \in \left[m^\beta_{rs}, m^\gamma_{rs} \right], \\ m_{sr} = \frac{1}{m_{rs}}, \\ r, s = 1, \ldots, n \end{array} \right\}, \tag{IV.93}$$

$$w^\delta_{Ci} = \max \left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} m_{kj}}} ; \begin{array}{l} m_{rs} \in \left[m^\alpha_{rs}, m^\delta_{rs} \right], \\ m_{sr} = \frac{1}{m_{rs}}, \\ r, s = 1, \ldots, n \end{array} \right\}. \tag{IV.94}$$

Theorem 41. *The fuzzy priorities $\widetilde{w}_{Ci} = (w^\alpha_{Ci}, w^\beta_{Ci}, w^\gamma_{Ci}, w^\delta_{Ci}), i = 1, \ldots, n$, obtained from a FMPCM $\widetilde{M}$ by the formulas* (IV.91)–(IV.94) *are normalized.*

Proof. It is sufficient to prove that the fuzzy priorities $\widetilde{w}_{Ci}, i = 1, \ldots, n$, satisfy the inequalities (III.13). From the formula (IV.91), it follows that w^α_{Ci} was obtained by applying the formula (II.24) to one particular MPCM $M^{\alpha i} = \{m_{pq}\}^n_{p,q=1}, m_{pq} \in [m^\alpha_{pq}, m^\delta_{pq}], p, q = 1, \ldots, n$. Let $w^{\alpha i}_k$ denote the priorities of objects $o_k, k \neq i$, obtainable by the formula (II.24) from the same MPCM $M^{\alpha i}$. Obviously, $w^\alpha_{Ci} + \sum_{\substack{k=1 \\ k \neq i}}^{n} w^{\alpha i}_k = 1$, and $w^{\alpha i}_k \in [w^\alpha_{Ck}, w^\delta_{Ck}], k \neq i$. From this, it follows that $w^\alpha_{Ci} + \sum_{\substack{k=1 \\ k \neq i}}^{n} w^\delta_{Ck} \geq 1$. The remaining inequalities in (III.13) are proved analogously. $\square$

Remark 19. *According to Theorem 41, the fuzzy priorities $\widetilde{w}_{Ci}$, $i = 1, \ldots, n$, obtained from a FMPCM by the formulas (IV.91)–(IV.94) are normalized in the sense of Definition 29. Notice that the normality of the fuzzy priorities was reached naturally by just properly applying constrained fuzzy arithmetic to the fuzzy extension of the formula (II.24) for obtaining normalized priorities from a MPCM; no forced normalization was needed, unlike in the case of the "normalization" of the fuzzy priorities (IV.86)–(IV.89) as suggested by Buckley (1985a).*

Theorem 42. *The fuzzy extension of the GMM based on the formulas (IV.91)–(IV.94) is invariant under permutation of objects in FMPCMs.*

Proof. It is sufficient to show that for a given object o_i, $i \in \{1, \ldots, n\}$, its priority $\widetilde{w}_{Ci}$ obtained by the formulas (IV.91)–(IV.94) does not change under permutation of objects in a FMPCM $\widetilde{M}$.

From the invariance of the GMM reviewed in Section 2.2.3.2, it follows that the priority w_i of object o_i determined by the formula (II.24) from the given MPCM M does not change under any permutation $M^\pi = PMP^T$ of M, it is just permuted accordingly. This means that the priority w_i obtained from M is equal to the corresponding priority $w^\pi_{\pi(i)}$ obtained from M^π.

Therefore, also the minimum w^α_{Ci} and the maximum w^δ_{Ci} of the priority w_i of object o_i obtained by (II.24) over all MPCMs obtainable from the closures of the supports of the trapezoidal fuzzy numbers in the trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}^n_{i,j=1}$, $\widetilde{m}_{ij} = (m^\alpha_{ij}, m^\beta_{ij}, m^\gamma_{ij}, m^\delta_{ij})$, do not change. Analogously, also the minimum w^β_{Ci} and the maximum w^γ_{Ci} of the priority w_i of object o_i obtained by (II.24) over all MPCMs obtainable from the cores of the trapezoidal fuzzy numbers in the trapezoidal FMPCM $\widetilde{M} = \{\widetilde{m}_{ij}\}^n_{i,j=1}$ do not change.

Therefore, the fuzzy priority $\widetilde{w}_{Ci} = (w^\alpha_{Ci}, w^\beta_{Ci}, w^\gamma_{Ci}, w^\delta_{Ci})$ obtained by the formulas (IV.91)–(IV.94) does not change under permutation of objects in FMPCMs (it is only permuted accordingly), which concludes the proof. $\square$

In the following example, the difference between the fuzzy priorities obtained by formulas (IV.86)–(IV.89) and by formulas (IV.91)–(IV.94) is illustrated, and the drawbacks in the formulas (IV.86)–(IV.89) are demonstrated.

Example 44. Let us consider the FMPCM $\widetilde{M}$ given by (IV.65). The fuzzy priorities of objects obtained from this FMPCM by the fuzzy extension of the GMM proposed both by Buckley (1985a) and by Enea and Piazza (2004) are given in Tab. IV.7. The fuzzy priorities obtained by formulas (IV.91)–(IV.94) are significantly less uncertain than the fuzzy priorities obtained by formulas (IV.86)–(IV.89). Moreover, the closures of the supports of the fuzzy priorities obtained by formulas (IV.91)–(IV.94) are the proper subsets of the closures of the supports of the fuzzy priorities obtained by formulas (IV.86)–(IV.89). Analogously, the cores of the fuzzy priorities obtained by formulas (IV.91)–(IV.94) are proper subsets of the cores of the fuzzy priorities obtained by formulas (IV.86)–(IV.89). This is caused by using constrained fuzzy arithmetic (that preserves the multiplicative reciprocity of PCs) in the formulas (IV.91)–(IV.94).

Table IV.7: Fuzzy priorities of objects obtained from the FMPCM (IV.65).

Fuzzy priorities obtained by formulas (IV.86)–(IV.89)	Fuzzy priorities obtained by formulas (IV.91)–(IV.94)
$\widetilde{w}_{S1} = (0.2469, 0.3401, 0.5399, 0.8114)$	$\widetilde{w}_{C1} = (0.3294, 0.3789, 0.4795, 0.5689)$
$\widetilde{w}_{S2} = (0.2076, 0.3073, 0.4694, 0.6067)$	$\widetilde{w}_{C2} = (0.2539, 0.3350, 0.4262, 0.4647)$
$\widetilde{w}_{S3} = (0.0694, 0.1104, 0.1418, 0.2180)$	$\widetilde{w}_{C3} = (0.0821, 0.1188, 0.1309, 0.1767)$
$\widetilde{w}_{S4} = (0.0424, 0.0571, 0.0762, 0.1296)$	$\widetilde{w}_{C4} = (0.0496, 0.0614, 0.0703, 0.1069)$

Further, let us illustrate inappropriateness of the formulas (IV.86)–(IV.89) for computing the fuzzy priorities of objects from a FMPCM. For this purpose, let us see how the lower boundary value w^α_{S1} of the fuzzy priority $\widetilde{w}_{S1}$ was obtained. The intensities of preference figuring in the formula (IV.86) for obtaining the lower boundary value $w^\alpha_{S1} = 0.2469$ are highlighted in bold in the FMPCM

$$\widetilde{M} = \begin{pmatrix} \mathbf{1} & (\mathbf{1}, 1, 2, \mathbf{3}) & (\mathbf{2}, 2.5, 3, \mathbf{4}) & (\mathbf{4}, 6, 7, \mathbf{8}) \\ \left(\frac{1}{3}, \frac{1}{2}, 1, \mathbf{1}\right) & 1 & (3, 4, 4, \mathbf{5}) & (4, 5, 6, \mathbf{6}) \\ \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2.5}, \mathbf{\frac{1}{2}}\right) & \left(\frac{1}{5}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3}\right) & 1 & (1, 2, 2, \mathbf{3}) \\ \left(\frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \mathbf{\frac{1}{4}}\right) & \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}\right) & \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \mathbf{1}\right) & 1 \end{pmatrix}. \tag{IV.95}$$

Figure IV.8: Fuzzy priority vectors $\widetilde{\underline{w}}_C$ and $\widetilde{\underline{w}}_S$ of the FMPCM (IV.65).
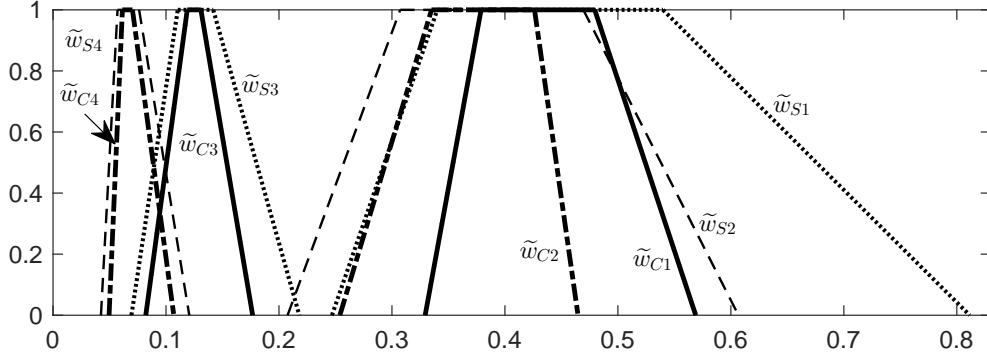
From (IV.95) we see that two different intensities of preference of object o_1 over the other objects are used in the formula (IV.86) at the same time. Moreover, the multiplicative reciprocity of PCs in the matrix is violated. This adds redundant vagueness to the fuzzy priorities, which leads to the distortion of the information contained in the FMPCM (IV.65).

Contrarily, the lower boundary value $w_{C1}^\alpha = 0.3294$ of the fuzzy priority $\widetilde{w}_{C1}$ was obtained from a multiplicatively reciprocal matrix, in particular from the MPCM

$$
M = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 1 & 5 & 6 \\ \frac{1}{2} & \frac{1}{5} & 1 & 3 \\ \frac{1}{4} & \frac{1}{6} & \frac{1}{3} & 1 \end{pmatrix}.
\tag{IV.96}
$$

In the same way, it could be shown that all representing values of all four fuzzy priorities were obtained from MPCMs by the formulas (IV.91)–(IV.94). △

In general, the optimization problems solved in (IV.91)–(IV.94) have $n^2 - n$ variables and $\frac{n^2-n}{2}$ multiplicative-reciprocity constraints (the number of variables and multiplicative-reciprocity constraints gets reduced when crisp numbers are present above and below the main diagonal of the FMPCM). Thus, the computational complexity of the optimization problems increases rapidly with an increasing dimension n. However, the following theorem shows that the optimization problems (IV.91)–(IV.94) can be simplified significantly. First, the multiplicative-reciprocity constraints can be incorporated into the objective functions. Second, when w_{Ci}^α is computed, the variables m_{ij}, $j = 1, \ldots, n$, can be fixed as the lower boundary values of the trapezoidal fuzzy numbers in the i-th row of the FMPCM, i.e. as $m_{ij} := m_{ij}^\alpha$. Analogously, also for the representing values $w_{Ci}^\beta, w_{Ci}^\gamma$, and w_{Ci}^δ. In this way, the number of variables is reduced from $n^2 - n$ to $\frac{n^2-n}{2} - (n-1)$.

Theorem 43. *Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^n$, $\widetilde{m}_{ij} = \left(m_{ij}^\alpha, m_{ij}^\beta, m_{ij}^\gamma, m_{ij}^\delta\right)$, *be a trapezoidal FMPCM. The optimization problems* (IV.91)–(IV.94) *can be simplified for* $i = 1, \ldots, n$ *in the following way:*

$$
w_{Ci}^\alpha = \frac{\sqrt[n]{\prod_{j=1}^n m_{ij}^\alpha}}{\sqrt[n]{\prod_{j=1}^n m_{ij}^\alpha} + \max\left\{ \sum_{\substack{k=1 \\ k \neq i}}^n \sqrt[n]{\frac{1}{m_{ik}^\alpha} \prod_{\substack{l=1 \\ l \neq i}}^{k-1} \frac{1}{m_{lk}} \prod_{\substack{l=k+1 \\ l \neq i}}^n m_{kl}} \; ; \; \begin{array}{l} m_{rs} \in \left[m_{rs}^\alpha, m_{rs}^\delta\right], \\ r = 1, \ldots, n-1, \\ s = r+1, \ldots, n, \\ r, s \neq i \end{array} \right\}},
\tag{IV.97}
$$

$$
w_{Ci}^\beta = \frac{\sqrt[n]{\prod_{j=1}^n m_{ij}^\beta}}{\sqrt[n]{\prod_{j=1}^n m_{ij}^\beta} + \max\left\{ \sum_{\substack{k=1 \\ k \neq i}}^n \sqrt[n]{\frac{1}{m_{ik}^\beta} \prod_{\substack{l=1 \\ l \neq i}}^{k-1} \frac{1}{m_{lk}} \prod_{\substack{l=k+1 \\ l \neq i}}^n m_{kl}} \; ; \; \begin{array}{l} m_{rs} \in \left[m_{rs}^\beta, m_{rs}^\gamma\right], \\ r = 1, \ldots, n-1, \\ s = r+1, \ldots, n, \\ r, s \neq i \end{array} \right\}},
\tag{IV.98}
$$

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$$w_{Ci}^{\gamma} = \frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}^{\gamma}}}{\sqrt[n]{\prod_{j=1}^{n} m_{ij}^{\gamma}} + \min\left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1}{m_{ik}^{\gamma}} \prod_{\substack{l=1 \\ l \neq i}}^{k-1} \frac{1}{m_{lk}} \prod_{\substack{l=k+1 \\ l \neq i}}^{n} m_{kl}} \ ; \begin{array}{l} m_{rs} \in \left[m_{rs}^{\beta}, m_{rs}^{\gamma} \right], \\ r = 1, \ldots, n-1, \\ s = r+1, \ldots, n, \\ r, s \neq i \end{array} \right\}}, \tag{IV.99}$$

$$w_{Ci}^{\delta} = \frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}^{\delta}}}{\sqrt[n]{\prod_{j=1}^{n} m_{ij}^{\delta}} + \min\left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1}{m_{ik}^{\delta}} \prod_{\substack{l=1 \\ l \neq i}}^{k-1} \frac{1}{m_{lk}} \prod_{\substack{l=k+1 \\ l \neq i}}^{n} m_{kl}} \ ; \begin{array}{l} m_{rs} \in \left[m_{rs}^{\alpha}, m_{rs}^{\delta} \right], \\ r = 1, \ldots, n-1, \\ s = r+1, \ldots, n, \\ r, s \neq i \end{array} \right\}}. \tag{IV.100}$$

Proof. First, let us show that the formulas (IV.91) and (IV.97) are identical. For any $i \in \{1, \ldots, n\}$, the formula (IV.91) can be written in the following way:

$$w_{Ci}^{\alpha} = \min\left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}}}{\sqrt[n]{\prod_{j=1}^{n} m_{ij}} + \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1}{m_{ik}} \prod_{\substack{j=1 \\ j \neq i}}^{n} m_{kj}}} \ ; \begin{array}{l} m_{rs} \in \left[m_{rs}^{\alpha}, m_{rs}^{\delta} \right], \\ m_{sr} = \frac{1}{m_{rs}}, \\ r, s = 1, \ldots, n, \end{array} \right\}.$$

Let us denote $\quad x_i := \sqrt[n]{\prod_{j=1}^{n} m_{ij}}, \quad$ and $\quad y_i := \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1}{m_{ik}} \prod_{\substack{j=1 \\ j \neq i}}^{n} m_{kj}}.$

Obviously, $x_i > 0$ for $m_{is} \in \left[m_{is}^{\alpha}, m_{is}^{\delta} \right]$, $s = 1, \ldots, n$, and $y_i > 0$ for $m_{rs} \in \left[m_{rs}^{\alpha}, m_{rs}^{\delta} \right]$, $m_{sr} = 1/m_{rs}$, $r, s = 1, \ldots, n$. Further, let us denote $f_i := \frac{x_i}{x_i + y_i}$. Then $\frac{\partial f_i}{\partial x_i} = \frac{y_i}{(x_i + y_i)^2} > 0$, and $\frac{\partial f_i}{\partial y_i} = \frac{-x_i}{(x_i + y_i)^2} < 0$. Hence, f_i is an increasing function of x_i and a decreasing function of y_i. It means that for minimizing the function f_i, we have to minimize x_i and maximize y_i. The function x_i is increasing in all the variables. Therefore,

$$x_i' := \min\left\{ x_i \ ; \ m_{ij} \in \left[m_{ij}^{\alpha}, m_{ij}^{\delta} \right], \ j = 1, \ldots, n \right\} = \sqrt[n]{\prod_{j=1}^{n} m_{ij}^{\alpha}}.$$

The function y_i is decreasing in the variables $m_{i1}, \ldots, m_{in}$. Therefore,

$$y_i' := \max\left\{ y_i \ ; \begin{array}{l} m_{rs} \in \left[m_{rs}^{\alpha}, m_{rs}^{\delta} \right], m_{sr} = \frac{1}{m_{rs}}, \\ r, s = 1, \ldots, n \end{array} \right\}$$

$$= \left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1}{m_{ik}^{\alpha}} \prod_{\substack{j=1 \\ j \neq i}}^{n} m_{kj}} \ ; \begin{array}{l} m_{rs} \in \left[m_{rs}^{\alpha}, m_{rs}^{\delta} \right], m_{sr} = \frac{1}{m_{rs}}, \\ r, s = 1, \ldots, n \end{array} \right\}$$

Finally, thanks to the reciprocity of $\widetilde{M}$, we can also replace all the elements m_{sr}, $r, s = 1, \ldots, n$, $r < s$, i.e. the elements below the main diagonal, by the reciprocals $1/m_{rs}$ of the corresponding elements m_{rs} above the main diagonal. By that we obtain formula (IV.97).

Analogously, it can be demonstrated that (IV.92) is equivalent to (IV.98), (IV.93) is equivalent to (IV.99), and (IV.94) is equivalent (IV.100). $\qquad \square$

In Example 44, it was pointed out that the cores and the closures of the supports of the fuzzy priorities in Tab. IV.7 obtained by formulas (IV.91)–(IV.94) are proper subsets of the cores and closures of the supports of the fuzzy priorities obtained by formulas (IV.86)–(IV.89), respectively. The following theorem shows that this property is valid in general for trapezoidal FMPCMs with at least one entry that is not a crisp number.

Theorem 44. *Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$, $\widetilde{m}_{ij} = \left(m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta}\right)$, *be a trapezoidal FMPCM. Further, let* $\widetilde{w}_{S1}, \ldots, \widetilde{w}_{Sn}$ *be trapezoidal fuzzy priorities obtained by formulas (IV.86)–(IV.89), and* $\widetilde{w}_{C1}, \ldots, \widetilde{w}_{Cn}$ *be trapezoidal fuzzy priorities obtained by formulas (IV.91)–(IV.94). If there exists at least one* $\widetilde{m}_{kl}$, $k, l \in \{1, \ldots, n\}$, *such that* $m_{kl}^{\alpha} < m_{kl}^{\delta}$, *then*

$$[w_{Ci}^{\alpha}, w_{Ci}^{\delta}] \subset [w_{Si}^{\alpha}, w_{Si}^{\delta}] \quad \text{and} \quad [w_{Ci}^{\beta}, w_{Ci}^{\gamma}] \subseteq [w_{Si}^{\beta}, w_{Si}^{\gamma}], \quad i = 1, \ldots, n. \tag{IV.101}$$

The equality

$$[w_{Ci}^{\beta}, w_{Ci}^{\gamma}] = [w_{Si}^{\beta}, w_{Si}^{\gamma}], \quad i = 1, \ldots, n, \tag{IV.102}$$

occurs only when $m_{ij}^{\beta} = m_{ij}^{\gamma}$ *for all* $i, j = 1, \ldots, n$.

Proof. First, let us demonstrate the validity of $[w_{Ci}^{\alpha}, w_{Ci}^{\delta}] \subset [w_{Si}^{\alpha}, w_{Si}^{\delta}], i = 1 \ldots, n$. Presence of at least one fuzzy number $\widetilde{m}_{kl} = (m_{kl}^{\alpha}, m_{kl}^{\beta}, m_{kl}^{\gamma}, m_{kl}^{\delta})$ that is not a crisp number, i.e. $m_{kl}^{\alpha} < m_{kl}^{\delta}$, implies the strict inequalities

$$\sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\prod_{j=1}^{n} m_{kj}^{\alpha}} < \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1}{m_{ik}^{\alpha}} \prod_{\substack{l=1 \\ l \neq i}}^{k-1} \frac{1}{m_{lk}} \prod_{\substack{l=k+1 \\ l \neq i}}^{n} m_{kl}} < \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\prod_{j=1}^{n} m_{kj}^{\delta}}, \quad \begin{array}{l} m_{kl} \in [m_{kl}^{\alpha}, m_{kl}^{\delta}], \\ m_{lk} = \frac{1}{m_{kl}}. \end{array} \tag{IV.103}$$

Thus, for any $i \in \{1, \ldots, n\}$:

$$w_{Si}^{\alpha} \overset{(\text{IV.86})}{=} \frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}^{\alpha}}}{\sqrt[n]{\prod_{j=1}^{n} m_{ij}^{\delta}} + \sum_{k=1, k \neq i}^{n} \sqrt[n]{\prod_{j=1}^{n} m_{kj}^{\delta}}} \overset{(\text{IV.103})}{<}$$

$$\frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}^{\alpha}}}{\sqrt[n]{\prod_{j=1}^{n} m_{ij}^{\alpha}} + \max \left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1}{m_{ik}^{\alpha}} \prod_{\substack{l=1 \\ l \neq i}}^{k-1} \frac{1}{m_{lk}} \prod_{\substack{l=k+1 \\ l \neq i}}^{n} m_{kl}} ; \begin{array}{l} m_{rs} \in [m_{rs}^{\alpha}, m_{rs}^{\beta}], \\ r = 1, \ldots, n-1, \\ s = r+1, \ldots, n, \\ r, s \neq i \end{array} \right\}} \overset{(\text{IV.97})}{=} w_{Ci}^{\alpha},$$

and

$$w_{Si}^{\delta} \overset{(\text{IV.88})}{=} \frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}^{\delta}}}{\sqrt[n]{\prod_{j=1}^{n} m_{ij}^{\alpha}} + \sum_{k=1, k \neq i}^{n} \sqrt[n]{\prod_{j=1}^{n} m_{kj}^{\alpha}}} \overset{(\text{IV.103})}{>}$$

$$\frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}^{\delta}}}{\sqrt[n]{\prod_{j=1}^{n} m_{ij}^{\delta}} + \min \left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1}{m_{ik}^{\delta}} \prod_{\substack{l=1 \\ l \neq i}}^{k-1} \frac{1}{m_{lk}} \prod_{\substack{l=k+1 \\ l \neq i}}^{n} m_{kl}} ; \begin{array}{l} m_{rs} \in [m_{rs}^{\alpha}, m_{rs}^{\beta}], \\ r = 1, \ldots, n-1, \\ s = r+1, \ldots, n, \\ r, s \neq i \end{array} \right\}} \overset{(\text{IV.99})}{=} w_{Ci}^{\delta}.$$

Analogously, validity of $[w_{Ci}^{\beta}, w_{Ci}^{\gamma}] \subseteq [w_{Si}^{\beta}, w_{Si}^{\gamma}], i = 1, \ldots, n$, is demonstrated.

Further, let us prove the validity of (IV.102). The equality $m_{ij}^{\beta} = m_{ij}^{\gamma}, i, j = 1, \ldots, n$, means that the trapezoidal FMPCM $\widetilde{M}$ is reduced to a triangular FMPCM. The domains of all variables in the formulas (IV.92) and (IV.93) are thus reduced to singletons. This basically means that the formula (IV.92) is reduced to the formula (IV.87) and the formula (IV.93) is reduced to the formula (IV.88). Thus, $w_{Ci}^{\beta} = w_{Si}^{\beta}$ and $w_{Ci}^{\gamma} = w_{Si}^{\gamma}, i = 1, \ldots, n$. Furthermore, the formulas (IV.87) and (IV.88) give the same results since they have the same entries $m_{ij}^{\beta} = m_{ij}^{\gamma}$. Thus, $w_{Ci}^{\beta} = w_{Si}^{\beta} = w_{Ci}^{\gamma} = w_{Si}^{\gamma}, i = 1, \ldots, n$. $\qquad \square$

4.3 Fuzzy additive pairwise comparison matrices

Similarly as in the case of MPCMs, also the PCs in APCMs are done either expertly or by using linguistic terms from a predefined scale. Especially when information about the decision-making problem is incomplete

or imprecise, it is more appropriate to provide the expert numerical judgments in form of fuzzy numbers rather than crisp numbers. The use of fuzzy numbers is more natural also when modeling the meaning of linguistic terms from a predefined scale (see the discussion in Section 4.2.1). Thus, in this section, the fuzzy extension of APCMs is dealt with. In particular, in Section 4.3.1, the construction of fuzzy APCMs is studied and the fuzzy extension of APCMs-A and of APCMs-M is defined. In Section 4.3.2, fuzzy APCMs-A are then studied in detail and Section 4.3.3 is focused on fuzzy APCMs-M.

4.3.1 Construction of FAPCMs

In Section 4.2, the fuzzy extension of MPCMs was dealt with. As it is obvious from the literature review provided in Section 4.2, MPCMs are most often extended to fuzzy numbers (usually triangular or trapezoidal), less often to intervals. That is probably because fuzzy numbers allow for more subtle modeling of preference intensities by using different degrees of membership. Intervals do not provide this option; all elements in the given interval have the same degree of membership.

The situation is different with APCMs. In the literature, the trend is to extend APCMs to intervals rather than to fuzzy numbers in general. The reason behind this might be the fact that APCMs are already perceived as fuzzy having the elements defined on interval $[0, 1]$. Thus, the intensity of preference of each PC in the APCM is looked at as a degree of membership to a fuzzy set, and modeling a degree of membership by a fuzzy number is not a common practice.

As already mentioned in Section 2.3.1, APCMs are often called "*fuzzy* preference relations" in the literature. Therefore, another reason why APCMs are usually extended to intervals rather than to fuzzy numbers in general might be the issue of terminology. Interval extension of a fuzzy preference relation is simply called "interval fuzzy preference relation" which probably seems to be acceptable (something "fuzzy" is extended to intervals). Extending fuzzy preference relations to fuzzy numbers sounds somehow wrong; "*fuzzy* preference relations" are already *fuzzy* and by extending them to fuzzy numbers we would obtain "*fuzzy fuzzy* preference relations"?

As shown in Section 2.4, MPCMs, APCMs-A, and APCMs-M are equivalent; each representation can be transformed into the others. This means that an APCM defined on interval $[0, 1]$ can be transformed into a MPCM defined on interval $[\frac{1}{9}, 9]$ (or $[\frac{1}{S}, S], S > 1$, in general) and vice versa. It will be shown later that there exist transformations also between the fuzzy extensions of MPCMs, APCMs-A, and APCMs-M. Therefore, using fuzzy numbers for one representation of preference information (in this case MPCMs), it does not seem reasonable to avoid using fuzzy numbers for other representations of preference information (APCMs-A and APCMs-M). If there is a need for more subtle modeling of preference intensities in MPCMs by using fuzzy numbers, why should these not be used for modeling preference intensities in APCMs as well? Why should the imprecision of information and vagueness of human judgment be modeled solely by intervals rather than by fuzzy numbers in general in the case of APCMs?

In this section, the methods related to APCMs will be extended not only to interval FAPCMs but, more in general, to fuzzy APCMs.

Definition 52. *A fuzzy additive pairwise comparison matrix (FAPCM) of n objects $o_1, ..., o_n$ is a square matrix $\widetilde{A} = \{\widetilde{a}_{ij}\}_{i,j=1}^{n}$, whose elements $\widetilde{a}_{ij}, i, j = 1, \ldots, n$, are fuzzy numbers defined on interval $[0, 1]$. Further, the matrix is additively reciprocal, i.e.*

$$\widetilde{a}_{ij} = 1 - \widetilde{a}_{ji}, \qquad i, j = 1, \ldots, n, \tag{IV.104}$$

and

$$\widetilde{a}_{ii} = 0.5, \qquad i, j = 1, \ldots, n. \tag{IV.105}$$

Definition 52 of a FAPCM is very general; elements $\widetilde{a}_{ij}$ of $\widetilde{A}$ are arbitrary fuzzy numbers satisfying the additive-reciprocity condition (IV.104). In practice, these fuzzy numbers are usually intervals, less often triangular or trapezoidal fuzzy numbers. In general, more types of fuzzy numbers and intervals can be present in a FAPCM at the same time. Even an APCM $A = \{a_{ij}\}_{i,j=1}^{n}$ given by Definition 7 is a FAPCM since crisp numbers are a special case of fuzzy numbers and since an APCM A satisfies (IV.105).

As already mentioned above, intervals are most often used in the literature to model intensities of preference in FAPCMs. Interval representation will be preserved when reviewing the existing approaches to interval extension of APCMs. However, in order to be coherent with Section 4.2 on the fuzzy extension of MPCMs, the new approaches and formulas related to the fuzzy extension of APCMs proposed in this thesis will be presented for trapezoidal fuzzy numbers. Since intervals are a special case of trapezoidal fuzzy numbers, there will be no difficulties in confronting the new approaches and formulas presented for trapezoidal fuzzy numbers with the original approaches and formulas presented for intervals.

Being $\widetilde{a}_{ij} = (a_{ij}^{\alpha}, a_{ij}^{\beta}, a_{ij}^{\gamma}, a_{ij}^{\delta})$ a trapezoidal fuzzy number, then also $\widetilde{a}_{ji} = 1 - \widetilde{a}_{ij}$ is a trapezoidal fuzzy number. Thus, unlike in the case of FMPCMs, there is no need for simplified fuzzy arithmetic when constructing a FAPCM $\widetilde{A}$.

When constructing a FAPCM by using a scale of predefined linguistic terms, it is necessary to model appropriately the meaning of the linguistic term "equal preference". Similarly as in the case of FMPCMs, we have to distinguish whether o_i and o_j are the same objects or not. For $i = j$, it is necessary to set $\widetilde{a}_{ii} = 0.5, i = 1, \ldots, n$, because there is no vagueness in the PC; we compare one object with itself. On the other hand, when two different objects are assessed as "equally preferred", there is very likely to be some vagueness contained in such PC. Therefore, in this case, "equal preference" should be modeled by a fuzzy number "about 0.5", i.e. $\widetilde{0.5}$, not necessarily by crisp number 0.5. Furthermore, similarly as in the case of FMPCMs, it is again necessary to preserve $\widetilde{0.5} = 1 - \widetilde{0.5}$ (a detailed reasoning behind this requirement is given in Section 4.2.1). Therefore, the fuzzy number $\widetilde{0.5} := (c^\alpha, c^\beta, c^\gamma, c^\delta)$ has to satisfy

$$(c^\alpha, c^\beta, c^\gamma, c^\delta) = 1 - (c^\alpha, c^\beta, c^\gamma, c^\delta). \tag{IV.106}$$

By solving (IV.106), we obtain $\widetilde{0.5}$ defined as

$$\widetilde{0.5} = (1 - c, 1 - b, b, c), \qquad 0.5 \le b \le c \le 1. \tag{IV.107}$$

Note 8. *From now on, by a FAPCM will be meant a FAPCM given by Definition 52 satisfying the indispensable condition $\widetilde{0.5} = 1 - \widetilde{0.5}$ for the fuzzy number $\widetilde{0.5}$ modeling the meaning of the linguistic term "equal preference", i.e. (IV.107) in the case of trapezoidal fuzzy numbers.*

Similarly as in the case of FMPCMs, also for FAPCMs it seems to be reasonable to define the fuzzy numbers modeling the meanings of linguistic terms in a predefined scale in such a way that they form Ruspini's fuzzy partition of interval $[0, 1]$. Defining a particular fuzzy scale is, however, out of the scope of the thesis as this should be done in cooperation with the particular DM. In the illustrative examples in this thesis only expertly defined fuzzy PCs without assigned linguistic terms will be used.

Analogously to APCMs, also in the case of FAPCMs it is necessary to distinguish between FAPCMs with additive representation and FAPCMs with multiplicative representation depending on the represention used when constructing a FAPCM.

Definition 53. *A FAPCM with additive representation (FAPCM-A) is a FAPCM $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^n$, where $\widetilde{r}_{ij} - \widetilde{r}_{ji}$ indicates the difference of preference intensity of object o_i and of object o_j.*

Definition 54. *A FAPCM with multiplicative representation (FAPCM-M) is a FAPCM $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n, \widetilde{q}_{ij} \in]0, 1[$, where $\frac{\widetilde{q}_{ij}}{\widetilde{q}_{ji}}$ indicates the ratio of preference intensity of object o_i to that of object o_j, i.e. o_i is $\frac{\widetilde{q}_{ij}}{\widetilde{q}_{ji}}$ –times as good as o_j.*

4.3.2 Fuzzy additive pairwise comparison matrices with additive representation

In this Section, the fuzzy extension of the methods related to APCMs-A reviewed in Section 2.3.2 is dealt with. In particular, Section 4.3.2.1 is dedicated to the extension of additive-consistency condition (II.28) to FAPCMs-A and Section 4.3.2.2 is focused on methods for obtaining fuzzy priorities from FAPCMs-A.

4.3.2.1 Additive consistency of FAPCMs-A

In this section, additive consistency of FAPCMs-A is studied. First, in Section 4.3.2.1.1, definitions of additive consistency for interval FAPCMs-A based on Tanino's characterization (II.32) proposed in the literature are reviewed and some drawbacks of the definitions are pointed out. Afterwards, in Section 4.3.2.1.2, two new definitions of additive consistency for FAPCMs-A are proposed.

4.3.2.1.1 Review of fuzzy extensions of additive consistency

Many definitions of consistency for interval FAPCMs-A have been proposed in the literature. These definitions are most often based on interval extension of additive-transitivity property (II.28) and related Tanino's characterization (II.32), or, alternatively, on interval extension of the more general characterization (II.46) reviewed in Remark 2 of Section 2.3.2.2.

Xu (2007a) introduced a weak version of additive consistency for interval FAPCMs-A based on Tanino's characterization (II.32). Wang and Li (2012) proposed another definition of additive consistency based on Tanino's additive-transitivity property (II.28). Qian et al. (2014) proposed a method for constructing additively consistent interval FAPCMs-A satisfying definition proposed by Wang and Li (2012) from inconsistent interval FAPCMs-A. Another definition of additive consistency for interval FAPCMs-A based on Tanino's additive-transitivity property (II.28) was proposed by Liu et al. (2012a). Xu et al. (2014a) proposed another definition of additive consistency for interval FAPCMs-A and introduced a method for completing incomplete interval

FAPCMs-A based on this additive-consistency condition. Wang (2014) pointed out that the additive consistency defined by Xu et al. (2014a) is not invariant under permutation of objects. Further, they approved the definition of additive consistency proposed by Wang and Li (2012) and derived some further properties. Wang et al. (2012) introduced a definition of consistency using a particular characterization based on logarithms. However, the interpretation of such characterization based on logarithms was not clarified.

As discussed in Remark 2, priorities (II.47) corresponding to more general characterization (II.46) lack intuitive representation. Therefore, in this section, the focus is put only on traditional Tanino's characterization (II.32). Definitions of additively consistent interval FAPCMs-A based on interval extension of Tanino's characterization (II.32) are reviewed in detail and some drawbacks are identified. In particular, it will be shown that some definitions are not invariant under permutation of objects in interval FAPCMs-A and some violate additive reciprocity of PCs of objects. Afterwards, it will be shown that the drawbacks can be eliminated by employing the constrained fuzzy arithmetic.

Xu (2007b) defined the additive consistency of interval FAPCMs-A as follows:

Definition 55. *(Xu, 2007b) Let $\overline{R} = \{\overline{r}_{ij}\}_{i,j=1}^{n}, \overline{r}_{ij} = \left[r_{ij}^{L}, r_{ij}^{U}\right]$, be an interval FAPCM-A. If there exists a vector $\underline{v} = (v_1, \ldots, v_n)^T, \sum_{i=1}^{n} v_i = 1, v_i \geq 0, i = 1, \ldots, n$, such that*

$$r_{ij}^{L} \leq 0.5(v_i - v_j + 1) \leq r_{ij}^{U}, \qquad i, j = 1, \ldots, n, \tag{IV.108}$$

then $\overline{R}$ is called an additively consistent interval FAPCM-A.

Xu and Chen (2008a) formulated the following theorem in order to verify additive consistency of interval FAPCMs-A according to Definition 55.

Theorem 45. *An interval FAPCM-A $\overline{R} = \{\overline{r}_{ij}\}_{i,j=1}^{n}, \overline{r}_{ij} = \left[r_{ij}^{L}, r_{ij}^{U}\right]$, is additively consistent according to Definition 55 if and only if the solution of the optimization model*

$$
\begin{aligned}
J^* = \min \quad & \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (d_{ij}^{-} + d_{ij}^{+}) \\
s.t. \quad & 0.5(v_i - v_j + 1) + d_{ij}^{L} \geq r_{ij}^{L}, \quad i, j = 1, \ldots, n, \ i < j, \\
& 0.5(v_i - v_j + 1) - d_{ij}^{U} \leq r_{ij}^{U}, \quad i, j = 1, \ldots, n, \ i < j, \\
& \sum_{i=1}^{n} v_i = 1, \ v_i \geq 0, \qquad\qquad i = 1, \ldots, n, \\
& d_{ij}^{L}, d_{ij}^{U} \geq 0, \qquad\qquad\qquad i, j = 1, \ldots, n, \ i < j,
\end{aligned}
\tag{IV.109}
$$

is $J^* = 0$.

Definition 55 of additive consistency for interval FAPCMs-A is clearly based on Proposition 2 for APCMs-A. According to the definition, an interval FAPCM-A $\overline{R} = \{\overline{r}_{ij}\}_{i,j=1}^{n}, \overline{r}_{ij} = \left[r_{ij}^{L}, r_{ij}^{U}\right]$, is additively consistent if there exists a vector $\underline{v} = (v_1, \ldots, v_n)^T$ using which we can construct an additively consistent APCM-A $R^* = \{r_{ij}^*\}_{i,j=1}^{n}$ such that $r_{ij}^* \in [r_{ij}^{L}, r_{ij}^{U}], i, j = 1, \ldots, n$. However, extension of Proposition 2 to Definition 55 is not done appropriately. The vector $\underline{v} = (v_1, \ldots, v_n)^T$ in Proposition 2 is such that $|v_i - v_j| \leq 1, i, j = 1, \ldots, n$, while in Definition 55, the vector should satisfy the normalization condition

$$\sum_{i=1}^{n} v_i = 1, \ v_i \geq 0, \ i = 1, \ldots, n. \tag{IV.110}$$

The same normalization condition is employed also in the optimization model (IV.109).

The incompatibility of the normalization condition (IV.110) and Tanino's characterization (II.32) for APCMs-A was demonstrated by Fedrizzi and Brunelli (2009). They showed that for some additively consistent APCMs-A, there exists no vector satisfying the normalization condition (IV.110). The inappropriateness of Definition 55 of additive consistency is demonstrated on the following illustrative example.

Example 45. Let us examine the interval FAPCM-A

$$
\overline{R} = \begin{pmatrix}
0.5 & [0.5, 0.6] & [0.8, 0.9] & [0.9, 1] \\
[0.4, 0.5] & 0.5 & [0.6, 0.7] & [0.6, 0.9] \\
[0.1, 0.2] & [0.3, 0.4] & 0.5 & [0.5, 0.7] \\
[0, 0.1] & [0.1, 0.4] & [0.3, 0.5] & 0.5
\end{pmatrix}.
\tag{IV.111}
$$

By solving the optimization model (IV.109), we obtain $J^* = 0.1$. Thus, based on Theorem 45, the interval FAPCM-A $\overline{R}$ is not additively consistent according to Definition 55. This means that there does not exist an APCM-A $R^* = \{r_{ij}^*\}_{i,j=1}^n$ obtainable from the interval FAPCM-A (IV.111) that would be additively consistent according to Definition 9. However, this conclusion is wrong.

There do exist additively consistent APCMs-A obtainable from (IV.111); one of them is, for example,

$$R^* = \begin{pmatrix} 0.5 & 0.6 & 0.8 & 1 \\ 0.4 & 0.5 & 0.7 & 0.9 \\ 0.2 & 0.3 & 0.5 & 0.7 \\ 0 & 0.1 & 0.3 & 0.5 \end{pmatrix} \tag{IV.112}$$

with the corresponding priority vector $\underline{v} = (1.45, 1.25, 0.85, 0.45)^T$ satisfying $|v_i - v_j| \leq 1, i, j = 1, \ldots, n$. Because there does not exist a priority vector corresponding to the APCM-A R^* that would satisfy the inappropriate normalization condition (IV.110), it could not have been revealed by solving the optimization model (IV.109). Thus, by utilizing Definition 55 in real-life applications, a DM who provided the interval FAPCM-A (IV.111) would be judged as inconsistent in his or her preferences even though the opposite is true. $\triangle$

Because the normalization condition (IV.110) is not compatible with Tanino's characterization for APCMs-A, it is clearly not reasonable to employ it in the definition of additively consistent interval FAPCMs-A (APCMs-A being a special case of interval FAPCMs-A where $r_{ij}^L = r_{ij}^U, i, j = 1, \ldots, n$).

Liu et al. (2012a) defined additive consistency in the following way.

Definition 56. *(Liu et al., 2012a) Let* $\overline{R} = \{\overline{r}_{ij}\}_{i,j=1}^n, \overline{r}_{ij} = [r_{ij}^L, r_{ij}^U]$, *be an interval FAPCM-A.* $\overline{R}$ *is called additively consistent if the APCMs-A* $C = \{c_{ij}\}_{i,j=1}^n$ *and* $D = \{d_{ij}\}_{i,j=1}^n$:

$$c_{ij} = \begin{cases} r_{ij}^L, & i < j \\ 0.5, & i = j \\ r_{ij}^U, & i > j \end{cases}, \qquad d_{ij} = \begin{cases} r_{ij}^U, & i < j \\ 0.5, & i = j \\ r_{ij}^L, & i > j \end{cases}, \qquad i, j = 1, \ldots, n, \tag{IV.113}$$

are additively consistent according to Definition 9.

However, Definition 56 is not invariant under permutation of objects. This serious drawback is demonstrated on the following example.

Example 46. Let us consider the interval FAPCM-A $\overline{R}$ in the form

$$\overline{R} = \begin{pmatrix} 0.5 & [0.6, 0.7] & [0.8, 1] \\ [0.3, 0.4] & 0.5 & [0.7, 0.8] \\ [0, 0.2] & [0.2, 0.3] & 0.5 \end{pmatrix}. \tag{IV.114}$$

The corresponding APCMs-A C and D given by (IV.113) are in the form

$$C = \begin{pmatrix} 0.5 & 0.6 & 0.8 \\ 0.4 & 0.5 & 0.7 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}, \qquad D = \begin{pmatrix} 0.5 & 0.7 & 1 \\ 0.3 & 0.5 & 0.8 \\ 0 & 0.2 & 0.5 \end{pmatrix}.$$

Both C and D satisfy the additive-transitivity condition (II.28), which means that they are additively consistent according to Definition 9. Therefore, according to Definition 56, the interval FAPCM-A $\overline{R}$ is additively consistent.

Now, let us permute the interval FAPCM-A (IV.114) by using the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{IV.115}$$

to the interval FAPCM-A $\overline{R}^\pi = P\overline{R}P^T$:

$$\overline{R}^\pi = \begin{pmatrix} 0.5 & [0, 0.2] & [0.2, 0.3] \\ [0.8, 1] & 0.5 & [0.6, 0.7] \\ [0.7, 0.8] & [0.3, 0.4] & 0.5 \end{pmatrix}. \tag{IV.116}$$

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The corresponding APCMs-A C^π and D^π given by (IV.113) are in the form

$$C^\pi = \begin{pmatrix} 0.5 & 0 & 0.2 \\ 1 & 0.5 & 0.6 \\ 0.8 & 0.4 & 0.5 \end{pmatrix}, \qquad D^\pi = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.8 & 0.5 & 0.7 \\ 0.7 & 0.3 & 0.5 \end{pmatrix}.$$

We can see that neither C^π nor D^π satisfies the additive-transitivity condition (II.28), and thus, according to Definition 56, the interval FAPCM-A $\overline{R}^\pi$ is not additively consistent.

Obviously, the PCs of objects in the interval FAPCMs-A $\overline{R}$ and $\overline{R}^\pi$ are the same; they are just provided in different orders. Therefore, also the conclusion about the additive consistency should be the same for both interval FAPCMs-A. However, as demonstrated, according to Definition 56, the interval FAPCM-A $\overline{R}$ results to be additively consistent while the interval FAPCM-A $\overline{R}^\pi$ results to be additively inconsistent. $\triangle$

Another definition of additive consistency for interval FAPCMs-A was proposed by Wang and Li (2012).

Definition 57. *(Wang and Li, 2012) Let $\overline{R} = \{\overline{r}_{ij}\}_{i,j=1}^n, \overline{r}_{ij} = [r_{ij}^L, r_{ij}^U]$, be an interval FAPCM-A. $\overline{R}$ is called additively consistent if it satisfies the additive-transitivity condition*

$$\overline{r}_{ij} + \overline{r}_{jk} + \overline{r}_{ki} = \overline{r}_{kj} + \overline{r}_{ji} + \overline{r}_{ik}, \qquad i, j, k = 1, \ldots, n. \tag{IV.117}$$

Wang and Li (2012) defined the additive-transitivity condition (IV.117) by using the standard interval arithmetic, i.e. addition is done according to formula (III.20). Therefore, the equation (IV.117) is nothing else but the equations

$$\begin{aligned} r_{ij}^L + r_{jk}^L + r_{ki}^L &= r_{kj}^L + r_{ji}^L + r_{ik}^L, \\ r_{ij}^U + r_{jk}^U + r_{ki}^U &= r_{kj}^U + r_{ji}^U + r_{ik}^U, \end{aligned} \qquad i, j, k = 1, \ldots, n. \tag{IV.118}$$

Further, Wang and Li (2012) formulated the following proposition.

Proposition 11. *(Wang and Li, 2012) Let $\overline{R} = \{\overline{r}_{ij}\}_{i,j=1}^n$ be an interval FAPCM-A. If there exists a normalized interval vector $\overline{v} = (\overline{v}_1, \ldots, \overline{v}_n)^T, \overline{v}_i = [v_i^L, v_i^U], \ i = 1, \ldots, n$, such that*

$$\overline{r}_{ij} = [0.5(v_i^L - v_j^U + 1), 0.5(v_i^U - v_j^L + 1)],$$

then $\overline{R}$ is additively consistent according to Definition 57.

According to Proposition 11, a crisp APCM-A $R = \{r_{ij}\}_{i,j=1}^n$, which is a particular case of interval FAPCMs-A, is additively consistent according to Definition 2 if there exists a priority vector $\underline{v} = (v_1, \ldots, v_n)^T$ satisfying Tanino's characterization (II.32) and the normalization condition (IV.110). However, the normalization condition (IV.110) is incompatible with Tanino's Proposition 2 (see Fedrizzi and Brunelli (2009)).

Furthermore, Wang and Li (2012) pointed out that "due to the fact that $\overline{a} - \overline{a}$ does not always yield 0, we cannot derive $\overline{r}_{ij} + \overline{r}_{ji} = 1$ any more" (Wang and Li (2012), p.183). For example, for $\overline{r}_{ij} = [0.7, 0.8]$, we obtain $\overline{r}_{ij} + \overline{r}_{ji} = [0.7, 0.8] + [0.2, 0.3] = [0.9, 1.1] \neq 1$. Wang and Li (2012) conclude that "due to the possibility of $\overline{a} - \overline{a} \neq 0$, which makes it impossible to manipulate an interval-valued equation by moving terms from one side to the other, (IV.117) may not necessarily be able to produce equation

$$\overline{r}_{ij} = \overline{r}_{ik} - \overline{r}_{jk} + 0.5 \tag{IV.119}$$

in contrast to the case of regular APCMs-A where these two expressions are equivalent" (Wang and Li (2012), p.183). This is why Wang and Li (2012) defined the additive consistency of interval FAPCMs-A (Definition 57) by extending the property (II.30) of APCMs-A.

However, it is necessary to point out here that Wang and Li's assertion that "we cannot derive $\overline{r}_{ij} + \overline{r}_{ji} = 1$" is not true. At the end of this section, it will be demonstrated that the validity of equation $\overline{r}_{ij} + \overline{r}_{ji} = 1$ can be easily achieved by applying appropriately the constrained fuzzy arithmetic instead of the standard fuzzy arithmetic (or equivalently by applying constrained interval arithmetic (Lodwick and Jenkins (2013)) instead of the standard interval arithmetic in this case). Furthermore, it will be shown that Definition 57 of additive consistency is inappropriate since it violates the additive reciprocity of PCs. Before doing that, let us finalize the literature review.

Definition 58. *(Xu et al., 2014a) Let $\overline{R} = \{\overline{r}_{ij}\}_{i,j=1}^n, \overline{r}_{ij} = [r_{ij}^L, r_{ij}^U]$, be an interval FAPCM-A. $\overline{R}$ is called an additively consistent interval FAPCM-A if*

$$\overline{r}_{ij} + \overline{r}_{jk} = \overline{r}_{ik} + [0.5, 0.5], \qquad i < j < k, \ i, j, k = 1, \ldots, n. \tag{IV.120}$$

Wang (2014) demonstrated that Definition 58 of additive consistency is dependent on objects labeling, i.e. it is not invariant under permutation of objects. Further, Wang (2014) adopted Definition 57 of additive consistency since it is invariant under permutation of objects and derived the following theorem.

Theorem 46. *(Wang, 2014) Let* $\overline{R} = \{\overline{r}_{ij}\}_{i,j=1}^{n}, \overline{r}_{ij} = \left[r_{ij}^{L}, r_{ij}^{U}\right],$ *be an interval FAPCM-A.* $\overline{R}$ *is additively consistent (according to Definition 57) if and only if*

$$r_{ij}^{L} + r_{ij}^{U} - (r_{ik}^{L} + r_{ik}^{U}) = r_{lj}^{L} + r_{lj}^{U} - (r_{lk}^{L} + r_{lk}^{U}), \qquad i, j, k, l = 1, \ldots, n. \qquad \text{(IV.121)}$$

Based on Theorem 46, Wang (2014) formulated also the following proposition.

Proposition 12. *(Wang, 2014) Let* $\overline{R} = \{\overline{r}_{ij}\}_{i,j=1}^{n}, \overline{r}_{ij} = \left[r_{ij}^{L}, r_{ij}^{U}\right],$ *be an interval FAPCM-A.* $\overline{R}$ *is additively consistent (according to Definition 57) if and only if*

$$r_{ij}^{L} + r_{ij}^{U} + r_{jk}^{L} + r_{jk}^{U} + r_{ki}^{L} + r_{ki}^{U} = 3, \qquad i, j, k = 1, \ldots, n. \qquad \text{(IV.122)}$$

In order to demonstrate the inappropriateness of Definition 57 and thus also of Theorem 46 and Proposition 12, let us analyze in more detail the requirement of additive reciprocity of PCs in APCMs and interval FAPCMs. As stated in Section 2.3.1, the additive reciprocity of PCs is an inherent property of APCMs. Because of the additive-reciprocity property, the PCs $r_{ii}, i = 1, \ldots, n$, in an APCM are always equal to 0.5 standing for equal preference. This result is very natural since the PC r_{ii} expresses the intensity of preference of object o_i over itself (clearly, any object has to be equally preferred to itself). Further, because of the additive reciprocity of PCs, the additive-transitivity property (II.28) for APCMs-A is equivalent to statements (ii) and (iii) in Theorem 3.

Conception of additive reciprocity becomes more complicated when extended to intervals. For an interval FAPCM $\overline{R} = \{\overline{r}_{ij}\}_{i,j=1}^{n}, \overline{r}_{ij} = \left[r_{ij}^{L}, r_{ij}^{U}\right]$, the additive reciprocity is defined as $\overline{r}_{ji} = 1 - \overline{r}_{ij} = [1 - r_{ij}^{U}, 1 - r_{ij}^{L}]$. According to this property, when, e.g., the highest possible intensity of preference r_{ij}^{U} of object o_i over object o_j is $r_{ij}^{U} = 0.9$, this means that the lowest possible intensity of preference $r_{ji}^{L} = 1 - r_{ij}^{U}$ of object o_j over object o_i is automatically $r_{ji}^{L} = 0.1$. However, this is not all.

FAPCMs carry more information about the preference intensities. In particular, any value $r_{ij}^{*} \in \overline{r}_{ij} = \left[r_{ij}^{L}, r_{ij}^{U}\right]$ expressing a possible intensity of preference of object o_i over object o_j is associated with a corresponding intensity of preference $r_{ji}^{*} \in \overline{r}_{ji} = \left[r_{ji}^{L}, r_{ji}^{U}\right]$ such that $r_{ji}^{*} = 1 - r_{ij}^{*}$; r_{ij}^{*} and r_{ji}^{*} express the same preference information about o_i and o_j. This property results naturally from the meaning of PCs in an interval FAPCM. The same holds for trapezoidal FAPCMs in general.

Wang and Li (2012) and Wang (2014), similarly to other researchers whose work was reviewed in this section, applied the standard interval arithmetic to the computations with intervals. This means that the additive-consistency condition (IV.117) is equivalent to equations (IV.118) which are equivalent to (IV.121) and (IV.122). However, the equations (IV.118), (IV.121), and (IV.122) do not preserve the additive reciprocity of PCs. In equations (IV.118), each intensity of preference r_{pq}^{L} or $r_{pq}^{U}, p, q = 1, \ldots, n$, always appears in a pair with the intensity of preference r_{qp}^{L} or r_{qp}^{U}, respectively. And since $r_{pq}^{L} = 1 - r_{qp}^{U} \neq 1 - r_{qp}^{L}$ (unless $r_{pq}^{L} = r_{pq}^{U}$), the additive reciprocity of PCs is violated. Similarly, in equations (IV.121) and (IV.122), each intensity of preference $r_{pq}^{L}, p, q = 1, \ldots, n$, appears in a pair with the intensity of preference r_{pq}^{U}. This again violates the additive reciprocity since $r_{pq}^{U} = 1 - r_{qp}^{L}$ and $r_{pq}^{L} \neq 1 - r_{qp}^{L}$. The problem is for better understanding illustrated on the following example.

Example 47. Let us examine the additive consistency of the interval FAPCM-A $\overline{R}$ given by (IV.114) by applying Definition 57. The expressions (IV.118) mean that we construct matrices $R^{L} = \left\{r_{ij}^{L}\right\}_{i=1}^{n}$ and $R^{U} = \left\{r_{ij}^{U}\right\}_{i=1}^{n}$ from the interval FAPCM-A (IV.114) as

$$R^{L} = \begin{pmatrix} 0.5 & 0.6 & 0.8 \\ 0.3 & 0.5 & 0.7 \\ 0 & 0.2 & 0.5 \end{pmatrix}, \qquad R^{U} = \begin{pmatrix} 0.5 & 0.7 & 1 \\ 0.4 & 0.5 & 0.8 \\ 0.2 & 0.3 & 0.5 \end{pmatrix},$$

and we verify their additive consistency by utilizing the property (II.30). However, we can easily see that neither R^{L} nor R^{U} is additively reciprocal, which means that both R^{L} and R^{U} are not even APCMs-A according to Definition 8. Therefore, it is nonsensical to verify their "additive consistency". △

As already mentioned in the discussion following Definition 57, Wang and Li (2012) claim that it is not possible to derive the equality $\overline{r}_{ij} + \overline{r}_{ji} = 1$ due to the fact that $\overline{a} - \overline{a}$ does not always yield 0. Obviously, for an interval FAPCM-A $\overline{R} = \{\overline{r}_{ij}\}_{i,j=1}^{n}, \overline{r}_{ij} = [r_{ij}^{L}, r_{ij}^{U}]$, by applying the standard fuzzy arithmetic, we obtain

$$\begin{aligned} \overline{r}_{ij} + \overline{r}_{ji} &= [r_{ij}^{L}, r_{ij}^{U}] + [r_{ji}^{L}, r_{ji}^{U}] = [r_{ij}^{L}, r_{ij}^{U}] + [1 - r_{ij}^{U}, 1 - r_{ij}^{L}] = \\ &= [1 + r_{ij}^{L} - r_{ij}^{U}, 1 + r_{ij}^{U} - r_{ij}^{L}] \neq 1, \end{aligned}$$

unless $r_{ij}^L = r_{ij}^U$. Similarly for $\overline{a} = [a^L, a^U]$, we obtain

$$\overline{a} - \overline{a} = [a^L, a^U] - [a^L, a^U] = [a^L - a^U, a^U - a^L] \neq 0, \qquad \text{unless } a^L = a^U.$$

However, applying the standard fuzzy arithmetic in this case is not appropriate; the constrained fuzzy arithmetic needs to be applied whenever there are any interactions among operands. Obviously, there is an interaction between interval PCs $\overline{r}_{ij}$ and $\overline{r}_{ji}$; from the additive reciprocity of PCs it follows that any intensity of preference $r_{ij}^* \in \overline{r}_{ij}$ of object o_i over object o_j is associated with the corresponding intensity of preference $r_{ji}^* = 1 - r_{ij}^*$ of object o_j over object o_i. Thus, $\overline{b} = \overline{r}_{ij} + \overline{r}_{ji}$ should be correctly computed according to the constrained fuzzy arithmetic (III.40) as $\overline{b} = [b^L, b^U]$:

$$\begin{aligned}
b^L &= \min\left\{r_{ij} + r_{ji}; r_{ij} \in [r_{ij}^L, r_{ij}^U], r_{ji} \in [r_{ji}^L, r_{ji}^U], r_{ji} = 1 - r_{ij}\right\} = \\
&= \min\left\{r_{ij} + 1 - r_{ij}; r_{ij} \in [r_{ij}^L, r_{ij}^U]\right\} = 1, \\
b^U &= \max\left\{r_{ij} + r_{ji}; r_{ij} \in [r_{ij}^L, r_{ij}^U], r_{ji} \in [r_{ji}^L, r_{ji}^U], r_{ji} = 1 - r_{ij}\right\} = \\
&= \max\left\{r_{ij} + 1 - r_{ij}; r_{ij} \in [r_{ij}^L, r_{ij}^U]\right\} = 1,
\end{aligned}$$

where the constraint $g(r_{ij}, r_{ji}) = 0$ is in the form $r_{ji} = 1 - r_{ij}$ (equivalently written as $r_{ij} + r_{ji} - 1 = 0$).

Analogously, $\overline{c} = \overline{a} - \overline{a}$ should be computed as $\overline{c} = [c^L, c^U]$:

$$\begin{aligned}
c^L &= \min\left\{a_1 - a_2; a_1 \in [a^L, a^U], a_2 \in [a^L, a^U], a_1 = a_2\right\} = \\
&= \min\left\{a_1 - a_1; a_1 \in [a^L, a^U]\right\} = 0, \\
c^U &= \max\left\{a_1 - a_2; a_1 \in [a^L, a^U], a_2 \in [a^L, a^U], a_1 = a_2\right\} = \\
&= \max\left\{a_1 - a_1; a_1 \in [a^L, a^U]\right\} = 0.
\end{aligned}$$

Keeping in mind the importance of the additive-reciprocity property of PCs in FAPCMs-A, additive consistency needs to be defined accordingly so that it does not violate the additive reciprocity. In the following section, two definitions of additively consistent trapezoidal FAPCMs-A respecting the additive reciprocity of PCs and invariant under permutation of objects are provided.

4.3.2.1.2 New fuzzy extension of additive consistency

In this section, additive weak consistency for trapezoidal FAPCMs-A is defined by removing the mistake in Definition 55 of additive consistency for interval FAPCMs-A. Further, another definition of additive consistency much stronger than the definition of additive weak consistency is proposed. Tools for verifying both the additive weak consistency and additive consistency are provided and some properties of both additively weakly consistent and additively consistent FAPCMs-A are derived. Both new definitions are invariant under permutation and preserve the additive reciprocity of PCs.

An additively weakly consistent trapezoidal FAPCM-A is defined as follows.

Definition 59. *Let $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^n, \widetilde{r}_{ij} = \left(r_{ij}^\alpha, r_{ij}^\beta, r_{ij}^\gamma, r_{ij}^\delta\right)$, be a trapezoidal FAPCM-A. $\widetilde{R}$ is said to be additively weakly consistent if there exists a vector $\underline{v} = (v_1, \ldots, v_n)^T, |v_i - v_j| \leq 1, i, j = 1, \ldots, n$, such that*

$$r_{ij}^\alpha \leq 0.5(v_i - v_j + 1) \leq r_{ij}^\delta, \qquad i, j = 1, \ldots, n. \tag{IV.123}$$

Definition 59 is obtained by correcting the error concerning the normalization of vector $\underline{v} = (v_1, \ldots, v_n)^T$ in Definition 55. As already mentioned in the previous section, the normalization condition (IV.110) used in Definition 55 is not compatible with Tanino's characterization (II.32). Therefore, the original normalization condition $\underline{v} = (v_1, \ldots, v_n)^T, |v_i - v_j| \leq 1, i, j = 1, \ldots, n$, used in Tanino's characterization was employed in Definition 59 instead. Definition 59 is confronted with Definition 55 in the following example.

Example 48. Let us examine the additive weak consistency of the interval FAPCM-A $\widetilde{R}$ given by (IV.111). It was shown in Example 45 that $\widetilde{R}$ is wrongly judged as additively inconsistent according to Definition 55. By applying Definition 59, we can find out that there exists a vector $\underline{v} = (v_1, v_2, v_3, v_4)^T$ satisfying the appropriate normalization condition $|v_i - v_j| \leq 1, i, j = 1, \ldots, 4$. It is, for example, the vector $\underline{v} = (1.45, 1.25, 0.85, 0.45)^T$ with the corresponding additively consistent APCM-A given as (IV.112). The reader can easily verify that this vector satisfies the inequalities (IV.123). Thus, the interval FAPCM-A $\widetilde{R}$ given by (IV.111) is correctly judged as additively weakly consistent. $\triangle$

Based on Proposition 2 for FAPCMs-A, the following proposition can be formulated for interval FAPCMs-A.

Proposition 13. *Let* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = \left(r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta}\right)$, *be a trapezoidal FAPCM-A.* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$ *is additively weakly consistent according to Definition 59 if and only if there exist elements* $r_{ij}^{*} \in \left[r_{ij}^{\alpha}, r_{ij}^{\delta}\right], i, j = 1, \dots, n$, *such that* $R^{*} = \{r_{ij}^{*}\}_{i,j=1}^{n}$ *is an APCM-A additively consistent according to Definition 9.*

Proof. First, let $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = \left(r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta}\right)$, be a trapezoidal FAPCM-A additively weakly consistent according to Definition 59. Let us denote $r_{ij}^{*} := 0.5(v_i - v_j + 1)$. From (IV.123), it follows that $r_{ij}^{*} \in \left[r_{ij}^{\alpha}, r_{ij}^{\delta}\right], i, j = 1, \dots, n$. Further, we have $r_{ii}^{*} = 0.5(v_i - v_i + 1) = 0.5$, $i = 1, \dots, n$, and $r_{ji}^{*} = 0.5(v_j - v_i + 1) = 1 - 0.5(v_i - v_j + 1) = 1 - r_{ij}^{*}$, $i, j = 1, \dots, n$. From $[r_{ij}^{\alpha}, r_{ij}^{\delta}] \subseteq [0,1], i, j = 1, \dots, n$, it follows that $r_{ij}^{*} \in [0,1], i, j = 1, \dots, n$. Therefore, $R^{*} = \{r_{ij}^{*}\}_{i,j=1}^{n}$ is an APCM-A. Finally, $r_{ik}^{*} + r_{kj}^{*} - 0.5 = 0.5(v_i - v_k + 1) + 0.5(v_k - v_j + 1) - 0.5 = 0.5(v_i - v_j + 1) = r_{ij}^{*}$, $i, j, k = 1, \dots, n$, which means that $R^{*} = \{r_{ij}^{*}\}_{i,j=1}^{n}$ is additively consistent according to (II.28).

In the opposite direction, let $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = \left(r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta}\right)$, be a trapezoidal FAPCM-A and let $R^{*} = \{r_{ij}^{*}\}_{i,j=1}^{n}, r_{ij}^{*} \in \left[r_{ij}^{\alpha}, r_{ij}^{\delta}\right], i, j = 1, \dots, n$, be an APCM-A additively consistent according to Definition 9. Then, from Proposition 2, it follows that there exists a vector $\underline{v} = (v_1, \dots, v_n)^{T}, |v_i - v_j| \le 1, i, j = 1, \dots, n$, such that $r_{ij}^{*} = 0.5(v_i - v_j + 1), i, j = 1, \dots, n$. Because, $r_{ij}^{*} \in \left[r_{ij}^{\alpha}, r_{ij}^{\delta}\right], i, j = 1, \dots, n$, then clearly (IV.123) holds. $\square$

Remark 20. *According to Proposition 13 and its proof, a trapezoidal FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = \left(r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta}\right)$, *is additively weakly consistent if and only if there exists an additively consistent APCM-A* $R^{*} = \{r_{ij}^{*}\}_{i,j=1}^{n}$ *such that* $r_{ij}^{*} \in \left[r_{ij}^{\alpha}, r_{ij}^{\delta}\right]$. *This consistency condition is quite easy to reach. That is why the consistency according to Definition 59 is called weak. Later in this section, also a much stronger definition of additive consistency for trapezoidal FAPCMs-A will be given.*

Definition 59 of additive weak consistency satisfies two desirable properties - invariance under permutation of objects and additive reciprocity of PCs in interval FAPCMs-A.

Theorem 47. *Definition 59 of additive weak consistency is invariant under permutation of objects in trapezoidal FAPCMs-A.*

Proof. There exists a priority vector $\underline{v} = (v_1, \dots, v_n)^{T}, |v_i - v_j| \le 1, i, j = 1, \dots, n$, such that the inequality $r_{ij}^{\alpha} \le 0.5(v_i - v_j + 1) \le r_{ij}^{\delta}$ is required to hold for every single PC $\widetilde{r}_{ij} = \left(r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta}\right)$ in an additively consistent trapezoidal FAPCM-A $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$. By permuting the FAPCM-A $\widetilde{R}$ to $\widetilde{R}^{\pi} = P\widetilde{R}P^{T}$, the original PC $\widetilde{r}_{ij}$ in the i-th row and in the j-th column of $\widetilde{R}$ is moved to the $\pi(i)$-th row and to the $\pi(j)$-th column of the permuted trapezoidal FAPCM-A $\widetilde{R}^{\pi}$ as $\widetilde{r}_{\pi(i)\pi(j)}^{\pi}$, but still keeping $\widetilde{r}_{ij} = \widetilde{r}_{\pi(i)\pi(j)}^{\pi}, i, j = 1, \dots, n$. Thus, there exists a vector $\underline{v}^{\pi} = (v_1^{\pi}, \dots, v_n^{\pi})^{T}$, obtained by permuting the vector $\underline{v}$, i.e. $\underline{v}^{\pi} = P\underline{v}$, with the components satisfying the inequalities $r_{ij}^{\pi\alpha} \le 0.5(v_i^{\pi} - v_j^{\pi} + 1) \le r_{ij}^{\pi\delta}$ as well as the normalization condition $|v_i^{\pi} - v_j^{\pi}| \le 1$ for every $i, j = 1, \dots, n$. $\square$

Theorem 48. *Definition 59 of additive weak consistency does not violate the additive reciprocity of PCs in a trapezoidal FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = (r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta})$, *in the sense that any fixed value* $r_{ij} \in [r_{ij}^{\alpha}, r_{ij}^{\delta}], i, j \in \{1, \dots, n\}$, *representing the intensity of preference of object* o_i *over object* o_j *is associated with the corresponding value* $r_{ji} \in [r_{ji}^{\alpha}, r_{ji}^{\delta}]$ *representing the intensity of preference of object* o_j *over object* o_i *such that* $r_{ji} = 1 - r_{ij}$.

Proof. The existence of the priority vector $\underline{v} = (v_1, \dots, v_n)^{T}$ satisfying the inequalities (IV.123) means that there exists an APCM-A $R = \{r_{ij}\}_{i,j=1}^{n}, r_{ij} \in [r_{ij}^{\alpha}, r_{ij}^{\delta}]$, such that $r_{ij} = 0.5(v_i - v_j + 1), i, j = 1, \dots, n$. R is additively reciprocal from the definition, i.e. every PC r_{ij} is associated with the PC r_{ji} such that $r_{ji} = 1 - r_{ij}$. $\square$

Remark 21. *Note that Theorem 48 does not simply state that a FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$ *additively weakly consistent according to Definition 59 is additively reciprocal, i.e.* $\widetilde{r}_{ji} = 1 - \widetilde{r}_{ij}, i, j = 1, \dots, n$. *The validity of this property automatically follows from Definition 52 of a FAPCM; every FAPCM is additively reciprocal, and thus, also a FAPCM-A that is additively weakly consistent according to Definition 59 is additively reciprocal.*

As explained on p. 94, the extension of the additive-reciprocity property from APCMs to FAPCMs does not concern only the "simple" additive reciprocity of the related fuzzy PCs $\widetilde{r}_{ij}$ *and* $\widetilde{r}_{ji}$ *in the sense that* $\widetilde{r}_{ji} = 1 - \widetilde{r}_{ij}, i, j = 1, \dots, n$. *The conception of the additive reciprocity becomes more complex for FAPCMs. In particular, every possible intensity of preference* $r_{ij}^{*} \in \widetilde{r}_{ij}$ *of object* o_i *over object* o_j *is associated inseparably with the corresponding possible intensity of preference* $r_{ji}^{*} \in \widetilde{r}_{ji}$ *such that* $r_{ji}^{*} = 1 - r_{ij}^{*}$ *since both* r_{ij}^{*} *and* r_{ji}^{*} *have to express the same preference information about the objects* o_i *and* o_j. *Theorem 48 states that Definition 59 is in accordance with this conception of additive reciprocity, i.e. that only additively reciprocal PCs are involved in Definition 59 of additive weak consistency.*

The following theorems provide useful tools for verifying additive weak consistency of trapezoidal FAPCMs-A.

Theorem 49. *A trapezoidal FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = \left(r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta}\right),$ *is additively weakly consistent according to Definition 59 if and only if*

$$\max_{k=1,\ldots,n} \left\{r_{ik}^{\alpha} + r_{kj}^{\alpha} - 0.5\right\} \leq \min_{k=1,\ldots,n} \left\{r_{ik}^{\delta} + r_{kj}^{\delta} - 0.5\right\}, \qquad i,j = 1,\ldots,n. \tag{IV.124}$$

Proof. From the inequalities (IV.123), it follows that $r_{ik}^{\alpha} \leq 0.5(v_i - v_k + 1) \leq r_{ik}^{\delta}$ and $r_{kj}^{\alpha} \leq 0.5(v_k - v_j + 1) \leq r_{kj}^{\delta}$. Thus, for every $k \in \{1,\ldots,n\}$:

$$r_{ik}^{\alpha} + r_{kj}^{\alpha} \leq 0.5(v_i - v_k + 1) + 0.5(v_k - v_j + 1) \leq r_{ik}^{\delta} + r_{kj}^{\delta}$$

$$r_{ik}^{\alpha} + r_{kj}^{\alpha} - 0.5 \leq 0.5(v_i - v_j + 1) \leq r_{ik}^{\delta} + r_{kj}^{\delta} - 0.5.$$

From this we obtain

$$\max_{k=1,\ldots,n} \left\{r_{ik}^{\alpha} + r_{kj}^{\alpha} - 0.5\right\} \leq 0.5(v_i - v_j + 1) \leq \min_{k=1,\ldots,n} \left\{r_{ik}^{\delta} + r_{kj}^{\delta} - 0.5\right\},$$

and thus (IV.124) holds for every $i,j = 1,\ldots,n$.

In the opposite direction, let (IV.124) hold. Then, for every $i,j,k \in \{1,\ldots,n\}$:

$$r_{ij}^{\alpha} \leq \max_{k=1,\ldots,n} \left\{r_{ik}^{\alpha} + r_{kj}^{\alpha} - 0.5\right\} \leq \min_{k=1,\ldots,n} \left\{r_{ik}^{\delta} + r_{kj}^{\delta} - 0.5\right\} \leq r_{ij}^{\delta}.$$

Thus, for every $i,j,k \in \{1,\ldots,n\}$:

$$\exists r_{ij}^{*} \in \left[\max_{k=1,\ldots,n} \left\{r_{ik}^{\alpha} + r_{kj}^{\alpha} - 0.5\right\}, \min_{k=1,\ldots,n} \left\{r_{ik}^{\delta} + r_{kj}^{\delta} - 0.5\right\}\right] \subseteq [r_{ij}^{\alpha}, r_{ij}^{\delta}]$$

$$\wedge \exists r_{ik}^{*} \in [r_{ik}^{\alpha}, r_{ik}^{\delta}] \wedge \exists r_{kj}^{*} \in [r_{kj}^{\alpha}, r_{kj}^{\delta}] : r_{ij}^{*} = r_{ik}^{*} + r_{kj}^{*} - 0.5.$$

This means that $R^{*} = \{r_{ij}^{*}\}_{i,j=1}^{n}$ is an APCM-A. Thus, according to Proposition 2, there exists a vector $\underline{v} = (v_1,\ldots,v_n)^T, |v_i - v_j| \leq 1, i,j = 1,\ldots,n,$ such that $r_{ij}^{*} = 0.5(v_i - v_j + 1)$. Since $r_{ij}^{*} \in [r_{ij}^{\alpha}, r_{ij}^{\delta}], i,j = 1,\ldots,n,$ we obtain the inequality (IV.123). □

The following theorem shows that it is sufficient to verify the inequality (IV.124) only for $i,j = 1,\ldots,n, i < j,$ thus saving half of the computations.

Theorem 50. *A trapezoidal FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = \left(r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta}\right),$ *is additively weakly consistent according to Definition 59 if and only if*

$$\max_{k=1,\ldots,n} \left\{r_{ik}^{\alpha} + r_{kj}^{\alpha} - 0.5\right\} \leq \min_{k=1,\ldots,n} \left\{r_{ik}^{\delta} + r_{kj}^{\delta} - 0.5\right\}, \qquad i,j = 1,\ldots,n, i < j. \tag{IV.125}$$

Proof. It is sufficient to show that the validity of inequalities (IV.125) for $i,j = 1,\ldots,n, i < j,$ implies automatically their validity for all $i,j = 1,\ldots,n,$ i.e. the validity of (IV.124). The validity of inequalities (IV.124) for $i = j$ is trivial from the definition of trapezoidal FAPCMs-A since

$$\max_{k=1,\ldots,n} \left\{r_{ik}^{\alpha} + r_{ki}^{\alpha} - 0.5\right\} = \max_{k=1,\ldots,n} \left\{r_{ik}^{\alpha} + 1 - r_{ik}^{\delta} - 0.5\right\} \leq 0.5 \leq$$

$$\min_{k=1,\ldots,n} \left\{r_{ik}^{\delta} + 1 - r_{ik}^{\alpha} - 0.5\right\} = \min_{k=1,\ldots,n} \left\{r_{ik}^{\delta} + r_{ki}^{\delta} - 0.5\right\}.$$

Further, for $i > j,$ by using (IV.125) and the additive-reciprocity properties, we obtain

$$\max_{k=1,\ldots,n} \left\{r_{ik}^{\alpha} + r_{kj}^{\alpha} - 0.5\right\} = \max_{k=1,\ldots,n} \left\{1 - r_{ki}^{\delta} + 1 - r_{jk}^{\delta} - 0.5\right\} =$$

$$1 - \min_{k=1,\ldots,n} \left\{r_{jk}^{\delta} + r_{ki}^{\delta} - 0.5\right\} \leq 1 - \max_{k=1,\ldots,n} \left\{r_{jk}^{\alpha} + r_{ki}^{\alpha} - 0.5\right\} =$$

$$\min_{k=1,\ldots,n} \left\{1 - 1 + r_{kj}^{\delta} - 1 + r_{ik}^{\delta} + 0.5\right\} = \min_{k=1,\ldots,n} \left\{r_{ik}^{\delta} + r_{kj}^{\delta} - 0.5\right\}.$$

□

Remark 22. *An alternative definition of additive weak consistency to Definition 59 might be formulated as follows.*

Let $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = \left(r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta}\right)$, be a trapezoidal FAPCM-A. $\widetilde{R}$ is said to be additively weakly consistent if there exists a vector $\underline{v} = (v_1, \ldots, v_n)^T$, $|v_i - v_j| \leq 1$, $i, j = 1, \ldots, n$, such that

$$r_{ij}^{\beta} \leq 0.5(v_i - v_j + 1) \leq r_{ij}^{\gamma}, \qquad i, j = 1, \ldots, n. \tag{IV.126}$$

Notice that this definition is stronger than Definition 59. In fact, every trapezoidal FAPCM-A additively weakly consistent according to this definition is also additively weakly consistent according to Definition 59 since (IV.126) *automatically implies* (IV.123).

All theorems regarding FAPCMs-A additively weakly consistent according to Definition 59 formulated above can be easily reformulated for FAPCMs-A additively weakly consistent according to this definition; it is sufficient to consider r_{ij}^{β} and r_{ij}^{γ} instead of r_{ij}^{α} and r_{ij}^{δ}, respectively, where appropriate.

In the following definition, a stronger version of additive consistency for trapezoidal FAPCMs-A is formulated.

Definition 60. *Let $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = \left(r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta}\right)$, be a trapezoidal FAPCM-A. $\widetilde{R}$ is said to be additively consistent if for each triplet $(i, j, k) \subseteq \{1, \ldots, n\}$ the following holds:*

$$\forall r_{ij} \in \left[r_{ij}^{\alpha}, r_{ij}^{\delta}\right] \exists r_{ik} \in \left[r_{ik}^{\alpha}, r_{ik}^{\delta}\right] \wedge \exists r_{kj} \in \left[r_{kj}^{\alpha}, r_{kj}^{\delta}\right] : r_{ij} = r_{ik} + r_{kj} - 0.5, \tag{IV.127}$$

$$\forall r_{ij} \in \left[r_{ij}^{\beta}, r_{ij}^{\gamma}\right] \exists r_{ik} \in \left[r_{ik}^{\beta}, r_{ik}^{\gamma}\right] \wedge \exists r_{kj} \in \left[r_{kj}^{\beta}, r_{kj}^{\gamma}\right] : r_{ij} = r_{ik} + r_{kj} - 0.5. \tag{IV.128}$$

Remark 23. *Definition 60 is a natural fuzzy extension of Definition 9 of additive consistency proposed by Tanino (1984). According to this definition, for any possible value $r_{ij} \in \widetilde{r}_{ij}, i, j \in \{1, \ldots, n\}$, there exist possible values $r_{ik} \in \widetilde{r}_{ik}$ and $r_{kj} \in \widetilde{r}_{kj}, k \in \{1, \ldots, n\}$, such that they satisfy the additive-transitivity property* (II.28). *Analogously, for any possible value $r_{ij} \in Core\,\widetilde{r}_{ij}, i, j \in \{1, \ldots, n\}$, there exist possible values $r_{ik} \in Core\,\widetilde{r}_{ik}$ and $r_{kj} \in Core\,\widetilde{r}_{kj}, k \in \{1, \ldots, n\}$, such that they satisfy* (II.28). *Clearly, in comparison to the additive weak consistency given by Definition 59, the additive consistency given by Definition 60 is very strong.*

Unlike Definitions 56 and 58 of additively consistent interval FAPCMs-A proposed by Liu et al. (2012a) and Xu et al. (2014a), respectively, new Definition 60 is invariant under permutation of objects compared in FAPCMs-A.

Theorem 51. *Definition 60 of additive consistency is invariant under permutation of objects in trapezoidal FAPCMs-A.*

Proof. For a trapezoidal FAPCM-A $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = (r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta})$, additively consistent according to Definition 60, the conditions (IV.127) and (IV.128) are satisfied for every triplet $(i, j, k) \subseteq \{1, \ldots, n\}$. By permuting the FAPCM-A $\widetilde{R}$ to $\widetilde{R}^{\pi} = P\widetilde{R}P^T$, the original PC $\widetilde{r}_{ij}$ in the $i-$th row and in the $j-$th column of $\widetilde{R}$ moves to the $\pi(i)-$th row and to the $\pi(j)-$th column of $\widetilde{R}^{\pi}$ preserving $\widetilde{r}_{\pi(i)\pi(j)}^{\pi} = \widetilde{r}_{ij}$. Thus, by permuting $\widetilde{R}$, also the validity of the conditions (IV.127) and (IV.128) is preserved, i.e.

$$\forall r_{ij}^{\pi} \in \left[r_{ji}^{\pi\alpha}, r_{ij}^{\pi\delta}\right] \exists r_{ik}^{\pi} \in \left[r_{ik}^{\pi\alpha}, r_{ik}^{\pi\delta}\right] \wedge \exists r_{kj}^{\pi} \in \left[r_{kj}^{\pi\alpha}, r_{kj}^{\pi\delta}\right] : r_{ij}^{\pi} = r_{ik}^{\pi} + r_{kj}^{\pi} - 0.5,$$

$$\forall r_{ij}^{\pi} \in \left[r_{ji}^{\pi\beta}, r_{ij}^{\pi\gamma}\right] \exists r_{ik}^{\pi} \in \left[r_{ik}^{\pi\beta}, r_{ik}^{\pi\gamma}\right] \wedge \exists r_{kj}^{\pi} \in \left[r_{kj}^{\pi\beta}, r_{kj}^{\pi\gamma}\right] : r_{ij}^{\pi} = r_{ik}^{\pi} + r_{kj}^{\pi} - 0.5,$$

for every triplet $(i, j, k) \in \{1, \ldots, n\}$. Thus, $\widetilde{R}^{\pi}$ is additively consistent according to Definition 60. $\square$

Further, unlike Definition 57 of additively consistent interval FAPCMs-A proposed by Wang and Li (2012), and the equivalent conditions derived by Wang (2014), Definition 60 does not violate the additive reciprocity of PCs.

Theorem 52. *Definition 60 of additive consistency preserves the additive reciprocity of PCs in trapezoidal FAPCMs-A in the sense that any fixed value $r_{ij} \in [r_{ij}^{\alpha}, r_{ij}^{\delta}], i, j \in \{1, \ldots, n\}$, representing the intensity of preference of object o_i over object o_j is associated with the corresponding value $r_{ji} \in [r_{ji}^{\alpha}, r_{ji}^{\delta}]$ representing the intensity of preference of object o_j over object o_i such that $r_{ji} = 1 - r_{ij}$.*

Proof. It is sufficient to show that expressions (IV.127) and (IV.128) do not violate the additive-reciprocity property in the sense that when two particular intensities of preference $r_{ij} \in \widetilde{r}_{ij}$ and $r_{ji} \in \widetilde{r}_{ji}$ on the pair of objects o_i and o_j are considered at the same time in the expressions (IV.127) and (IV.128), then they are such that $r_{ji} = 1 - r_{ij}$.

For a triplet $(i, j, k) \subseteq \{1, \ldots, n\}$, $i \neq j \neq k$, no reciprocals appear in the expression $r_{ij} = r_{ik} + r_{kj} - 0.5$ for any $r_{ij} \in [r_{ij}^\alpha, r_{ij}^\delta]$. For $i = j = k$, the expression (IV.127) reduces to: $\forall r_{ii} = 0.5 \, \exists r_{ii}^* = 0.5 \land \exists r_{ii}^{**} = 0.5 : \; 0.5 = 0.5 + 0.5 - 0.5$, which again does not violate the additive reciprocity. Further, for $i \neq j = k$, the expression (IV.127) is as: $\forall r_{ij} \in [r_{ij}^\alpha, r_{ij}^\delta] \, \exists r_{ij}^* \in [r_{ij}^\alpha, r_{ij}^\delta] \land \exists r_{jj} = 0.5 : \; r_{ij} = r_{ij}^* + 0.5 - 0.5$. This means that $r_{ij} = r_{ij}$ and, therefore, the additive reciprocity is not violated. For $i = k \neq j$ the proof is analogous. Finally, for $i = j \neq k$, the expression (IV.127) is as:

$$\forall r_{ij} = 0.5 \, \exists r_{ik} \in [r_{ik}^\alpha, r_{ik}^\delta] \land \exists r_{ki}^* \in [r_{ki}^\alpha, r_{ki}^\delta] : \; 0.5 = r_{ik} + r_{ki}^* - 0.5.$$

This means that $r_{ik} = 1 - r_{ki}^*$ and, thus, the additive reciprocity is again preserved.

The proof for the expression (IV.128) is analogous. $\qquad\square$

Remark 24. *Similarly to Theorem 48, also Theorem 52 does not simply state that a FAPCM-A $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^n$ additively consistent according to Definition 60 is additively reciprocal since this property automatically follows from Definition 52 of a FAPCM. Theorem 52 states that Definition 60 is in accordance with the conception of additive reciprocity discussed on p. 94, i.e. only additively reciprocal PCs are involved in Definition 60 of additive consistency. For more details, see Remark 21.*

By handling properly the additive-reciprocity property of PCs, Theorem 3 can be easily extended to trapezoidal FAPCMs-A as follows.

Theorem 53. *For a trapezoidal FAPCM-A $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^n$, $\widetilde{r}_{ij} = \left(r_{ij}^\alpha, r_{ij}^\beta, r_{ij}^\gamma, r_{ij}^\delta \right)$, the following statements are equivalent:*

(i) $\widetilde{R}$ is additively consistent according to Definition 60.

(ii) For every $i, j, k = 1, \ldots, n$:

$$\forall r_{ij} \in \left[r_{ij}^\alpha, r_{ij}^\delta \right] \, \exists r_{jk} \in \left[r_{jk}^\alpha, r_{jk}^\delta \right] \; \land \; \exists r_{ki} \in \left[r_{ki}^\alpha, r_{ki}^\delta \right] : \; r_{ij} + r_{jk} + r_{ki} = r_{ik} + r_{kj} + r_{ji},$$
$$r_{ij} = 1 - r_{ji}, \; r_{jk} = 1 - r_{kj}, \; r_{ki} = 1 - r_{ik}. \quad \text{(IV.129)}$$

$$\forall r_{ij} \in \left[r_{ij}^\beta, r_{ij}^\gamma \right] \, \exists r_{jk} \in \left[r_{jk}^\beta, r_{jk}^\gamma \right] \; \land \; \exists r_{ki} \in \left[r_{ki}^\beta, r_{ki}^\gamma \right] : \; r_{ij} + r_{jk} + r_{ki} = r_{ik} + r_{kj} + r_{ji},$$
$$r_{ij} = 1 - r_{ji}, \; r_{jk} = 1 - r_{kj}, \; r_{ki} = 1 - r_{ik}. \quad \text{(IV.130)}$$

(iii) For every $i, j, k = 1, \ldots, n$:

$$\forall r_{ij} \in \left[r_{ij}^\alpha, r_{ij}^\delta \right] \, \exists r_{jk} \in \left[r_{jk}^\alpha, r_{jk}^\delta \right] \land \exists r_{ki} \in \left[r_{ki}^\alpha, r_{ki}^\delta \right] : r_{ij} + r_{jk} + r_{ki} = \frac{3}{2}. \quad \text{(IV.131)}$$

$$\forall r_{ij} \in \left[r_{ij}^\beta, r_{ij}^\gamma \right] \, \exists r_{jk} \in \left[r_{jk}^\beta, r_{jk}^\gamma \right] \land \exists r_{ki} \in \left[r_{ki}^\beta, r_{ki}^\gamma \right] : r_{ij} + r_{jk} + r_{ki} = \frac{3}{2}. \quad \text{(IV.132)}$$

Proof. From the additive-reciprocity property $\widetilde{r}_{ij} = 1 - \widetilde{r}_{ji}, i, j = 1, \ldots, n$, it follows that $\forall r_{ij} \in \left[r_{ij}^\alpha, r_{ij}^\delta \right] \, \exists r_{ji} \in \left[r_{ji}^\alpha, r_{ji}^\delta \right] : r_{ji} = 1 - r_{ij}$, and $\forall r_{ij} \in \left[r_{ij}^\beta, r_{ij}^\gamma \right] \, \exists r_{ji} \in \left[r_{ji}^\beta, r_{ji}^\gamma \right] : r_{ji} = 1 - r_{ij}$.

(a) First, let us show that the statements (i) and (iii) are equivalent. Because of the reciprocity property, (IV.127) can be equivalently written as

$$\forall r_{ij} \in \left[r_{ij}^\alpha, r_{ij}^\delta \right] \, \exists r_{jk} \in \left[r_{jk}^\alpha, r_{jk}^\delta \right] \land \exists r_{ki} \in \left[r_{ki}^\alpha, r_{ki}^\delta \right] : r_{ij} = (1 - r_{ki}) + (1 - r_{jk}) + 0.5,$$

which is equivalent to (IV.131). Analogously, the equivalence of (IV.128) and (IV.132) is proved.

(b) Now, let us show that the statements (ii) and (iii) are equivalent. Because of the reciprocity property, (IV.129) can be equivalently written as

$$\forall r_{ij} \in \left[r_{ij}^\alpha, r_{ij}^\delta \right] \, \exists r_{jk} \in \left[r_{jk}^\alpha, r_{jk}^\delta \right] \land \exists r_{ki} \in \left[r_{ki}^\alpha, r_{ki}^\delta \right] :$$
$$r_{ij} + r_{jk} + r_{ki} = (1 - r_{ki}) + (1 - r_{jk}) + (1 - r_{ij}),$$

which is equivalent to (IV.131). Analogously, the equivalence of (IV.130) and (IV.132) is proved.

$\qquad\square$

The following theorems give us useful tools for verifying additive consistency of trapezoidal FAPCMs-A.

Theorem 54. *A trapezoidal FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = \left(r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta}\right),$ *is additively consistent according to Definition 60 if and only if the inequalities*

$$r_{ij}^{\alpha} \geq r_{ik}^{\alpha} + r_{kj}^{\alpha} - 0.5, \qquad r_{ij}^{\delta} \leq r_{ik}^{\delta} + r_{kj}^{\delta} - 0.5, \tag{IV.133}$$

$$r_{ij}^{\beta} \geq r_{ik}^{\beta} + r_{kj}^{\beta} - 0.5, \qquad r_{ij}^{\gamma} \leq r_{ik}^{\gamma} + r_{kj}^{\gamma} - 0.5, \tag{IV.134}$$

hold for every $i, j, k = 1, \ldots, n, \ i < j, \ k \neq i, j.$

Proof. It is sufficient to demonstrate the equivalence of the expressions (IV.133) and (IV.127). The demonstration of the equivalence of (IV.134) and (IV.128) is analogous.

First, let us demonstrate that when the inequalities (IV.133) hold for every $i, j, k = 1, \ldots, n, i < j, k \neq i, j,$ then they hold for every $i, j, k = 1, \ldots, n.$ The inequalities (IV.133) are always satisfied for $i, j, k = 1, \ldots, n$ such that $i = j \neq k,$ or $i \neq j = k,$ or $j \neq k = i,$ or $i = j = k:$

$$r_{ik}^{\alpha} + r_{ki}^{\alpha} - 0.5 = 0.5 - (r_{ik}^{\delta} - r_{ik}^{\alpha}) \leq 0.5 = r_{ii}^{\alpha}, \ \ r_{ik}^{\delta} + r_{ki}^{\delta} - 0.5 = 0.5 + (r_{ik}^{\delta} - r_{ik}^{\alpha}) \geq 0.5 = r_{ii}^{\delta},$$

$$r_{ij}^{\alpha} + r_{jj}^{\alpha} - 0.5 = r_{ij}^{\alpha}, \ \ r_{ij}^{\delta} + r_{jj}^{\delta} - 0.5 = r_{ij}^{\delta},$$

$$r_{ii}^{\alpha} + r_{ij}^{\alpha} - 0.5 = r_{ij}^{\alpha}, \ \ r_{ii}^{\delta} + r_{ij}^{\delta} - 0.5 = r_{ij}^{\delta},$$

$$r_{ii}^{\alpha} + r_{ii}^{\alpha} - 0.5 = 0.5 = r_{ii}^{\alpha}, \ \ r_{ii}^{\delta} + r_{ii}^{\delta} - 0.5 = 0.5 = r_{ii}^{\delta}.$$

Further, when the inequalities (IV.133) are satisfied for $i, j, k = 1, \ldots, n, \ i < j, \ k \neq i, j,$ then they are satisfied also for $j, i, k = 1, \ldots, n, \ j > i, \ k \neq i, j:$

$$r_{jk}^{\alpha} + r_{ki}^{\alpha} - 0.5 = 1 - r_{ik}^{\delta} + 1 - r_{kj}^{\delta} - 0.5 = 1 - (r_{ik}^{\delta} + r_{kj}^{\delta} - 0.5) \leq 1 - r_{ij}^{\delta} = r_{ji}^{\alpha},$$

$$r_{jk}^{\delta} + r_{ki}^{\delta} - 0.5 = 1 - r_{ik}^{\alpha} + 1 - r_{kj}^{\alpha} - 0.5 = 1 - (r_{ik}^{\alpha} + r_{kj}^{\alpha} - 0.5) \geq 1 - r_{ij}^{\alpha} = r_{ji}^{\delta}.$$

To finalize the proof, it is sufficient to show that the inequalities (IV.133) are equivalent to the condition (IV.127) for every $i, j, k = 1, \ldots, n.$ First, let $\widetilde{R}$ be a trapezoidal FAPCM-A additively consistent according to Definition 60. Then for $r_{ij} := r_{ij}^{\alpha} \exists r_{ik} \in \left[r_{ik}^{\alpha}, r_{ik}^{\delta}\right] \wedge \exists r_{kj} \in \left[r_{kj}^{\alpha}, r_{kj}^{\delta}\right] : r_{ij}^{\alpha} = r_{ik} + r_{kj} - 0.5.$ Since $r_{ik} \geq r_{ik}^{\alpha}, r_{kj} \geq r_{kj}^{\alpha},$ then clearly $r_{ij}^{\alpha} \geq r_{ik}^{\alpha} + r_{ji}^{\alpha} - 0.5.$ Analogously, for $r_{ij} := r_{ij}^{\delta} \exists r_{ik} \in \left[r_{ik}^{\alpha}, r_{ik}^{\delta}\right] \wedge \exists r_{kj} \in \left[r_{kj}^{\alpha}, r_{kj}^{\delta}\right] : r_{ij}^{\delta} = r_{ik} + r_{kj} - 0.5.$ Since $r_{ik} \leq r_{ik}^{\delta}, r_{kj} \leq r_{kj}^{\delta},$ then clearly $r_{ij}^{\delta} \leq r_{ik}^{\delta} + r_{kj}^{\delta} - 0.5.$

Second, let (IV.133) be valid for a trapezoidal FAPCM-A $\widetilde{R}.$ From inequalities (IV.133) we get $\forall r_{ij} \in \left[r_{ij}^{\alpha}, r_{ij}^{\delta}\right] : r_{ik}^{\alpha} + r_{kj}^{\alpha} - 0.5 \leq r_{ij} \leq r_{ik}^{\delta} + r_{kj}^{\delta} - 0.5$ and, therefore, (IV.127) is satisfied. $\qquad \square$

Theorem 55. *A trapezoidal FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = \left(r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta}\right),$ *is additively consistent according to Definition 60 if and only if*

$$r_{ij}^{\alpha} \geq \max_{\substack{k=1,\ldots,n \\ k \neq i,j}} \left\{r_{ik}^{\alpha} + r_{kj}^{\alpha} - 0.5\right\}, \quad r_{ij}^{\delta} \leq \min_{\substack{k=1,\ldots,n \\ k \neq i,j}} \left\{r_{ik}^{\delta} + r_{kj}^{\delta} - 0.5\right\}, \tag{IV.135}$$

$$r_{ij}^{\beta} \geq \max_{\substack{k=1,\ldots,n \\ k \neq i,j}} \left\{r_{ik}^{\beta} + r_{kj}^{\beta} - 0.5\right\}, \quad r_{ij}^{\gamma} \leq \min_{\substack{k=1,\ldots,n \\ k \neq i,j}} \left\{r_{ik}^{\gamma} + r_{kj}^{\gamma} - 0.5\right\}, \tag{IV.136}$$

hold for every $i, j = 1, \ldots, n, \ i < j.$

Proof. The inequatilies (IV.135) and (IV.136) follow immediately from Theorem 54. $\qquad \square$

In the following example, Definition 60 of additive consistency is confronted with Definitions 56, 57, and 58. In particular, it is demonstrated how the drawbacks regarding the dependence of Definitions 56 and 58 on permutation of objects and violation of the additive-reciprocity property in Definition 57 are removed by Definition 60.

Example 49. Let us examine the interval FAPCM-A given by (IV.114). In Example 46, it was demonstrated that Definition 56 is not invariant under permutation of objects since the interval FAPCM-A (IV.114) is judged as additively consistent while its permutation (IV.116) is judged as additively inconsistent.

Similarly, the interval FAPCM-A (IV.114) is judged additively consistent also according to Definition 58 since the expression (IV.120) is valid for $i = 1, j = 2, k = 3 : [0.6, 0.7] + [0.7, 0.8] = [0.8, 1] + [0.5, 0.5].$ The permuted interval FAPCM-A (IV.116) is, however, judged as additively inconsistent since the expression (IV.120) is not valid: $[0, 0.2] + [0.6, 0.7] \neq [0.2, 0.3] + [0.5, 0.5].$

Table IV.8: Inequality conditions (IV.133) for the interval FAPCM-A (IV.114).

$i < j:$	$r_{ij}^L \geq r_{ik}^L + r_{kj}^L - 0.5$	$r_{ij}^U \leq r_{ik}^U + r_{kj}^U - 0.5$
$1,2:$	$0.6 \geq 0.8 + 0.2 - 0.5$	$0.7 \leq 1 + 0.3 - 0.5$
$1,3:$	$0.8 \geq 0.6 + 0.7 - 0.5$	$1 \leq 0.7 + 0.8 - 0.5$
$2,3:$	$0.7 \geq 0.3 + 0.8 - 0.5$	$0.8 \leq 0.4 + 1 - 0.5$

Table IV.9: Inequality conditions (IV.133) for the permuted interval FAPCM-A (IV.116).

$i < j:$	$r_{ij}^L \geq r_{ik}^L + r_{kj}^L - 0.5$	$r_{ij}^U \leq r_{ik}^U + r_{kj}^U - 0.5$
$1,2:$	$0 \geq 0.2 + 0.3 - 0.5$	$0.2 \leq 0.3 + 0.4 - 0.5$
$1,3:$	$0.2 \geq 0 + 0.6 - 0.5$	$0.3 \leq 0.2 + 0.7 - 0.5$
$2,3:$	$0.6 \geq 0.8 + 0.2 - 0.5$	$0.7 \leq 1 + 0.3 - 0.5$

Now let us apply Definition 60 to the interval FAPCM-A (IV.114). By using Theorem 54, the interval FAPCM-A (IV.114) is judged additively consistent since it satisfies the inequalities (IV.133); see Tab. IV.8. Also the permuted interval FAPCM-A (IV.116) satisfies the inequalities (IV.133); see Tab. IV.9. Therefore, it is again judged as additively consistent. Moreover, from Theorem 51 it follows that any permutation of the interval FAPCM-A (IV.114) is additively consistent.

In Example 47, it was demonstrated that Definition 57 violates the additive reciprocity of PCs. According to Theorem 52, the additive-reciprocity property is preserved in new Definition 60. This basically means that by taking any value from any interval PC in the interval FAPCM-A (IV.114), there exist values in the remaining interval PCs such that they form an additively consistent APCM-A. Let us examine the triplet $i = 1, j = 2, k = 3$ of indices and let us consider the value $r_{12} = 0.65 \in [0.6, 0.7]$. Then, according to (IV.127), there exist values $r_{13} \in [0.8, 1]$ and $r_{32} \in [0.2, 0.3]$ such that $0.65 = r_{13} + r_{32} - 0.5$. It is, for example, $r_{13} = 0.9, r_{32} = 0.25$. The additive reciprocity is clearly not violated. More interestingly, let us consider the triplet $i = 1, j = 1, k = 2$. Then, according to (IV.127), there exist values $r_{12} \in [0.6, 0.7]$ and $r_{21} \in [0.3, 0.4]$ such that $0.5 = r_{12} + r_{21} - 0.5$. This equality is satisfied by any value $r_{12} \in [0.6, 0.7]$ and the corresponding value $r_{21} \in [0.3, 0.4]$ such that $r_{21} = 1 - r_{12}$, which again preserves the additive reciprocity. $\triangle$

In the rest of this section, some interesting properties of additively weakly consistent and additively consistent trapezoidal FAPCMs-A are examined. The following theorem shows the relation between Definition 60 of additive consistency and Definition 59 of additive weak consistency.

Theorem 56. *Let* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^n , \widetilde{r}_{ij} = \left(r_{ij}^\alpha, r_{ij}^\beta, r_{ij}^\gamma, r_{ij}^\delta\right)$, *be a trapezoidal FAPCM-A. If* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^n$ *is additively consistent according to Definition 60, then it is also additively weakly consistent according to Definition 59.*

Proof. The statement follows immediately from Theorem 55. In particular, the inequality (IV.125) is obtained immediately from the inequalities (IV.135). $\square$

Remark 25. *According to Theorem 56, when a trapezoidal FAPCM-A is additively consistent according to Definition 60, then it is also automatically additively weakly consistent according to Definition 59. However, this does not hold true the other way around. Clearly, the definition of additive weak consistency is much weaker then the definition of additive consistency; it only requires existence of one crisp additively consistent APCM-A obtainable by combining particular elements from the closures of the supports of the trapezoidal fuzzy numbers in the trapezoidal FAPCM-A. Thus, the set of all trapezoidal FAPCMs-A additively consistent according to Definition 60 is a proper subset of the set of all trapezoidal FAPCMs-A additively weakly consistent according to Definition 59.*

Example 50. Let us examine additive consistency and additive weak consistency of the trapezoidal FAPCM-A

$$\widetilde{R} = \begin{pmatrix} 0.5 & (0.5, 0.6, 0.65, 0.7) & (0.7, 0.8, 0.9, 0.95) & (0.85, 0.9, 0.95, 1) \\ (0.3, 0.35, 0.4, 0.5) & 0.5 & (0.6, 0.65, 0.7, 0.7) & (0.6, 0.7, 0.8, 0.9) \\ (0.05, 0.1, 0.2, 0.3) & (0.3, 0.3, 0.35, 0.4) & 0.5 & (0.5, 0.55.0.6, 0.7) \\ (0, 0.05, 0.1, 0.15) & (0.1, 0.2, 0.3, 0.4) & (0.3, 0.4, 0.45, 0.5) & 0.5 \end{pmatrix}. \qquad \text{(IV.137)}$$

Table IV.10: Condition (IV.125) for the trapezoidal FAPCM-A (IV.137).

$i < j$:	$\max\limits_{k=1,\dots,4}\left\{r_{ik}^{\alpha} + r_{kj}^{\alpha} - 0.5\right\}$	$\leq$	$\min\limits_{k=1,\dots,4}\left\{r_{ik}^{\delta} + r_{kj}^{\delta} - 0.5\right\}$
$1,2$:	$\max\left\{0.5, 0.5, 0.45, 0.45\right\}$	$\leq$	$\min\left\{0.7, 0.7, 0.8, 0.9\right\}$
$1,3$:	$\max\left\{0.7, 0.6, 0.7, 0.65\right\}$	$\leq$	$\min\left\{0.9, 0.95, 0.9, 1\right\}$
$1,4$:	$\max\left\{0.85, 0.6, 0.7, 0.85\right\}$	$\leq$	$\min\left\{1, 1.1, 1.1, 1\right\}$
$2,3$:	$\max\left\{0.5, 0.6, 0.6, 0.4\right\}$	$\leq$	$\min\left\{0.9, 0.75, 0.75, 0.9\right\}$
$2,4$:	$\max\left\{0.65, 0.6, 0.6, 0.6\right\}$	$\leq$	$\min\left\{1, 0.9, 0.95, 0.9\right\}$
$3,4$:	$\max\left\{0.45, 0.35, 0.5, 0.5\right\}$	$\leq$	$\min\left\{0.8, 0.8, 0.7, 0.7\right\}$

Let us, for example, use Theorem 54 to verify the additive consistency of $\widetilde{R}$. We find out that $\widetilde{R}$ is not additively consistent because it violates the inequalities (IV.133) and (IV.134); for example, $r_{13}^{\delta} = 0.95 \not\leq r_{12}^{\delta} + r_{23}^{\delta} - 0.5 = 0.9$.

Even though $\widetilde{R}$ is not additively consistent, it can still be at least additively weakly consistent. Let us verify that by using Theorem 50. According to Tab. IV.10, the condition (IV.125) is satisfied, and thus $\widetilde{R}$ is additively weakly consistent.

A vector satisfying the inequalities (IV.123) in Definition 59 is, for example, $\underline{v} = (1.45, 1.25, 0.85, 0.45)^{T}$. The corresponding APCM-A R^{*} is in the form

$$R^{*} = \begin{pmatrix} 0.5 & 0.6 & 0.8 & 1 \\ 0.4 & 0.5 & 0.7 & 0.9 \\ 0.2 & 0.3 & 0.5 & 0.7 \\ 0 & 0.1 & 0.3 & 0.5 \end{pmatrix}. \tag{IV.138}$$

$\triangle$

Theorem 57. *Let $\widetilde{R}$ be a trapezoidal FAPCM-A additively weakly consistent according to Definition 59. The trapezoidal FAPCM-A $\widetilde{R}^{*}$ constructed from $\widetilde{R}$ by eliminating the l-th row and the l-th column, $l \in \{1, \dots, n\}$, is again additively weakly consistent.*

Proof. For $\widetilde{R}$, the inequalities (IV.123) are valid for every $i, j = 1, \dots, n$. After eliminating the l-th row and the l-th column of $\widetilde{R}$, (IV.123) is still valid for every remaining $i, j \in \{1, \dots, n\} \setminus \{l\}$. Therefore, the new trapezoidal FAPCM-A $\widetilde{R}^{*}$ is still additively weakly consistent. □

The same holds also for additively consistent trapezoidal FAPCMs-A.

Theorem 58. *Let $\widetilde{R}$ be a trapezoidal FAPCM-A additively weakly consistent according to Definition 60. The trapezoidal FAPCM-A $\widetilde{R}^{*}$ constructed from $\widetilde{R}$ by eliminating the l-th row and the l-th column, $l \in \{1, \dots, n\}$, is again additively consistent.*

Proof. For $\widetilde{R}$, the inequalities (IV.127) and (IV.128) are valid for every $i, j, k = 1, \dots, n$. After eliminating the l-th row and the l-th column of $\widetilde{R}$, (IV.127) and (IV.128) is still valid for every remaining $i, j, k \in \{1, \dots, n\} \setminus \{l\}$. Therefore, the new trapezoidal FAPCM-A $\widetilde{R}^{*}$ is additively consistent. □

Remark 26. *Theorems 57 and 58 are useful in situations when the set of objects compared pairwisely is being reduced. According to the theorems, elimination of one or more objects has no impact on the additive or additive weak consistency of fuzzy PCs of the remaining objects.*

The following theorems provide some results regarding aggregation of additively and additively weakly consistent trapezoidal FAPCMs-A into one trapezoidal FAPCM-A, which are particularly useful in group decision making.

Theorem 59. *Let $\widetilde{R}^{1} = \left\{\widetilde{r}_{ij}^{1}\right\}_{i,j=1}^{n}, \widetilde{r}_{ij}^{1} = \left(r_{ij}^{1\alpha}, r_{ij}^{1\beta}, r_{ij}^{1\gamma}, r_{ij}^{1\delta}\right)$, and $\widetilde{R}^{2} = \left\{\widetilde{r}_{ij}^{2}\right\}_{i,j=1}^{n}, \widetilde{r}_{ij}^{2} = \left(r_{ij}^{2\alpha}, r_{ij}^{2\beta}, r_{ij}^{2\gamma}, r_{ij}^{2\delta}\right)$, be trapezoidal FAPCMs-A additively weakly consistent according to Definition 59. Then $\widetilde{R} = \left\{\widetilde{r}_{ij}\right\}_{i,j=1}^{n}, \widetilde{r}_{ij} = \left(r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta}\right)$, such that*

$$r_{ij}^{\alpha} = \epsilon r_{ij}^{1\alpha} + (1-\epsilon) r_{ij}^{2\alpha}, \qquad r_{ij}^{\beta} = \epsilon r_{ij}^{1\beta} + (1-\epsilon) r_{ij}^{2\beta},$$
$$r_{ij}^{\gamma} = \epsilon r_{ij}^{1\gamma} + (1-\epsilon) r_{ij}^{2\gamma}, \qquad r_{ij}^{\delta} = \epsilon r_{ij}^{1\delta} + (1-\epsilon) r_{ij}^{2\delta},$$

is an additively weakly consistent trapezoidal FAPCM-A for any $\epsilon \in [0, 1]$.

Proof. First, let us show that $\widetilde{R}$ is a trapezoidal FAPCM-A. For $i = 1, \ldots, n$ we get

$$r_{ii}^\alpha = \epsilon r_{ii}^{1\alpha} + (1-\epsilon)r_{ii}^{2\alpha} = 0.5\epsilon + 0.5(1-\epsilon) = 0.5,$$
$$r_{ii}^\delta = \epsilon r_{ii}^{1\delta} + (1-\epsilon)r_{ii}^{2\delta} = 0.5\epsilon + 0.5(1-\epsilon) = 0.5.$$

Similarly, $r_{ii}^\beta = 0.5, r_{ii}^\gamma = 0.5$, and thus, $\widetilde{r}_{ii} = 0.5, i = 1, \ldots, n$. Further, for $i \neq j$ we have

$$r_{ij}^\alpha = \epsilon r_{ij}^{1\alpha} + (1-\epsilon)r_{ij}^{2\alpha} = \epsilon(1 - r_{ji}^{1\delta}) + (1-\epsilon)(1 - r_{ji}^{2\delta}) = 1 - \left[\epsilon r_{ji}^{1\delta} + (1-\epsilon)r_{ji}^{2\delta}\right] = 1 - r_{ji}^\delta,$$
$$r_{ij}^\delta = \epsilon r_{ij}^{1\delta} + (1-\epsilon)r_{ij}^{2\delta} = \epsilon(1 - r_{ji}^{1\alpha}) + (1-\epsilon)(1 - r_{ji}^{2\alpha}) = 1 - \left[\epsilon r_{ji}^{1\alpha} + (1-\epsilon)r_{ji}^{2\alpha}\right] = 1 - r_{ji}^\alpha,$$

and analogously we obtain $r_{ij}^\beta = 1 - r_{ji}^\gamma, r_{ij}^\gamma = 1 - r_{ji}^\beta$. Therefore, $\widetilde{r}_{ij} = 1 - \widetilde{r}_{ji}, i, j = 1, \ldots, n$. Finally,

$$r_{ij}^\alpha = \epsilon r_{ij}^{1\alpha} + (1-\epsilon)r_{ij}^{2\alpha} \geq \epsilon \cdot 0 + (1-\epsilon) \cdot 0 = 0$$
$$r_{ij}^\delta = \epsilon r_{ij}^{1\delta} + (1-\epsilon)r_{ij}^{2\delta} = \epsilon + (1-\epsilon) = 1,$$

i.e. $[r_{ij}^\alpha, r_{ij}^\delta] \subseteq [0, 1]$.

Second, let us show that $\widetilde{R}$ is additively weakly consistent. It is sufficient to prove inequalities (IV.125). Since (IV.125) is valid for FAPCMs-A $\widetilde{R}^1$ and $\widetilde{R}^2$, we obtain

$$\max_{k=1,\ldots,n} \left\{r_{ik}^\alpha + r_{kj}^\alpha - 0.5\right\} = \max_{k=1,\ldots,n} \left\{\epsilon r_{ik}^{1\alpha} + (1-\epsilon)r_{ik}^{2\alpha} + \epsilon r_{kj}^{1\alpha} + (1-\epsilon)r_{kj}^{2\alpha} - 0.5\right\} \leq$$

$$\max_{k=1,\ldots,n} \left\{\epsilon[r_{ik}^{1\alpha} + r_{kj}^{1\alpha} - 0.5]\right\} + \max_{k=1,\ldots,n} \left\{(1-\epsilon)[r_{ik}^{2\alpha} + r_{kj}^{2\alpha} - 0.5]\right\} \leq$$

$$\min_{k=1,\ldots,n} \left\{\epsilon[r_{ik}^{1\delta} + r_{kj}^{1\delta} - 0.5]\right\} + \min_{k=1,\ldots,n} \left\{(1-\epsilon)[r_{ik}^{2\delta} + r_{kj}^{2\delta} - 0.5]\right\} \leq$$

$$\min_{k=1,\ldots,n} \left\{\epsilon r_{ik}^{1\delta} + (1-\epsilon)r_{ik}^{2\delta} + \epsilon r_{kj}^{1\delta} + (1-\epsilon)r_{kj}^{2\delta} - 0.5\right\} = \min_{k=1,\ldots,n} \left\{r_{ik}^\delta + r_{kj}^\delta - 0.5\right\}$$

which proves the theorem. $\qquad\square$

Theorem 59 can be further extended to the aggregation of $p \geq 2$ additively weakly consistent trapezoidal FAPCMs-A as follows.

Theorem 60. *Let $\widetilde{R}^\tau = \{\widetilde{r}_{ij}^\tau\}_{i,j=1}^n, \widetilde{r}_{ij}^\tau = \left(r_{ij}^{\tau\alpha}, r_{ij}^{\tau\beta}, r_{ij}^{\tau\gamma}, r_{ij}^{\tau\delta}\right), \tau = 1, \ldots, p$, be trapezoidal FAPCMs-A additively weakly consistent according to Definition 59. Then $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^n$ such that*

$$\widetilde{r}_{ij} = \left(r_{ij}^\alpha, r_{ij}^\beta, r_{ij}^\gamma, r_{ij}^\delta\right) = \left(\sum_{\tau=1}^p \epsilon_\tau r_{ij}^{\tau\alpha}, \sum_{\tau=1}^p \epsilon_\tau r_{ij}^{\tau\beta}, \sum_{\tau=1}^p \epsilon_\tau r_{ij}^{\tau\gamma}, \sum_{\tau=1}^p \epsilon_\tau r_{ij}^{\tau\delta}\right), \tag{IV.139}$$

is an additively weakly consistent trapezoidal FAPCM-A for any $\epsilon_\tau \in [0, 1], \tau = 1, \ldots, p$, with $\sum_{\tau=1}^p \epsilon_\tau = 1$.

Proof. The proof is analogous to the proof of Theorem 59. $\qquad\square$

Similar theorems are formulated also for additively consistent trapezoidal FAPCMs-A.

Theorem 61. *Let $\widetilde{R}^1 = \{\widetilde{r}_{ij}^1\}_{i,j=1}^n, \widetilde{r}_{ij}^1 = \left(r_{ij}^{1\alpha}, r_{ij}^{1\beta}, r_{ij}^{1\gamma}, r_{ij}^{1\delta}\right)$, and $\widetilde{R}^2 = \{\widetilde{r}_{ij}^2\}_{i,j=1}^n, \widetilde{r}_{ij}^2 = \left(r_{ij}^{2\alpha}, r_{ij}^{2\beta}, r_{ij}^{2\gamma}, r_{ij}^{2\delta}\right)$, be trapezoidal FAPCMs-A additively consistent according to Definition 60. Then $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^n, \widetilde{r}_{ij} = \left(r_{ij}^\alpha, r_{ij}^\beta, r_{ij}^\gamma, r_{ij}^\delta\right)$, such that*

$$r_{ij}^\alpha = \epsilon r_{ij}^{1\alpha} + (1-\epsilon)r_{ij}^{2\alpha}, \qquad r_{ij}^\beta = \epsilon r_{ij}^{1\beta} + (1-\epsilon)r_{ij}^{2\beta},$$
$$r_{ij}^\gamma = \epsilon r_{ij}^{1\gamma} + (1-\epsilon)r_{ij}^{2\gamma}, \qquad r_{ij}^\delta = \epsilon r_{ij}^{1\delta} + (1-\epsilon)r_{ij}^{2\delta},$$

is an additively consistent trapezoidal FAPCM-A for any $\epsilon \in [0, 1]$.

Proof. From Theorem 59, we already know that $\widetilde{R}$ is a trapezoidal FAPCM-A. Therefore, we only need to show that $\widetilde{R}$ is additively consistent according to Definition 60.

It is sufficient to prove the inequalities (IV.133) and (IV.134). Since (IV.133) are valid for interval FAPCMs-A $\widetilde{R}^1$ and $\widetilde{R}^2$, we obtain

$$\begin{aligned} r_{ik}^\alpha + r_{kj}^\alpha - 0.5 &= \left[\epsilon r_{ik}^{1\alpha} + (1-\epsilon)r_{ik}^{2\alpha}\right] + \left[\epsilon r_{kj}^{1\alpha} + (1-\epsilon)r_{kj}^{2\alpha}\right] - 0.5 \\ &= \epsilon(r_{ik}^{1\alpha} + r_{kj}^{1\alpha} - 0.5) + (1-\epsilon)(r_{ik}^{2\alpha} + r_{kj}^{2\alpha} - 0.5) \\ &\leq \epsilon r_{ij}^{1\alpha} + (1-\epsilon)r_{ij}^{2\alpha} = r_{ij}^\alpha, \end{aligned}$$

$$\begin{aligned} r_{ik}^\delta + r_{kj}^\delta - 0.5 &= \left[\epsilon r_{ik}^{1\delta} + (1-\epsilon)r_{ik}^{2\delta}\right] + \left[\epsilon r_{kj}^{1\delta} + (1-\epsilon)r_{kj}^{2\delta}\right] - 0.5 \\ &= \epsilon(r_{ik}^{1\delta} + r_{kj}^{1\delta} - 0.5) + (1-\epsilon)(r_{ik}^{2\delta} + r_{kj}^{2\delta} - 0.5) \\ &\geq \epsilon r_{ij}^{1\delta} + (1-\epsilon)r_{ij}^{2\delta} = r_{ij}^\delta. \end{aligned}$$

Analogously, the validity of inequalities (IV.134) is proved. $\qquad\square$

Table IV.11: Inequality conditions (IV.135) for the interval FAPCM-A (IV.140).

$i < j:$	r_{ij}^{α}	$\geq$	$\max\limits_{k=2,3}\left\{r_{ik}^{\alpha}+r_{kj}^{\alpha}-0.5\right\}$	r_{ij}^{δ}	$\leq$	$\min\limits_{k=2,3}\left\{r_{ik}^{\delta}+r_{kj}^{\delta}-0.5\right\}$
$1,2:$	0.5	$\geq$	$\max\{0.5, 0.4\}$	0.6	$\leq$	$\min\{0.7, 0.7\}$
$1,3:$	0.6	$\geq$	$\max\{0.5, 0.5\}$	0.7	$\leq$	$\min\{0.7, 0.75\}$
$1,4:$	0.7	$\geq$	$\max\{0.65, 0.7\}$	0.85	$\leq$	$\min\{0.9, 0.9\}$
$2,3:$	0.5	$\geq$	$\max\{0.5, 0.45\}$	0.6	$\leq$	$\min\{0.7, 0.7\}$
$2,4:$	0.65	$\geq$	$\max\{0.6, 0.6\}$	0.8	$\leq$	$\min\{0.85, 0.8\}$
$3,4:$	0.6	$\geq$	$\max\{0.5, 0.55\}$	0.7	$\leq$	$\min\{0.75, 0.8\}$

Theorem 61 can be further extended to the aggregation of $p \geq 2$ additively consistent interval FAPCMs-A as follows.

Theorem 62. *Let* $\widetilde{R}^{\tau} = \{\widetilde{r}_{ij}^{\tau}\}_{i,j=1}^{n}, \widetilde{r}_{ij}^{\tau} = \left(r_{ij}^{\tau\alpha}, r_{ij}^{\tau\beta}, r_{ij}^{\tau\gamma}, r_{ij}^{\tau\delta}\right), \tau = 1, \ldots, p,$ *be trapezoidal FAPCMs-A additively consistent according to Definition 60. Then* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$ *such that*

$$\widetilde{r}_{ij} = \left(r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta}\right) = \left(\sum_{\tau=1}^{p} \epsilon_{\tau} r_{ij}^{\tau\alpha}, \sum_{\tau=1}^{p} \epsilon_{\tau} r_{ij}^{\tau\beta}, \sum_{\tau=1}^{p} \epsilon_{\tau} r_{ij}^{\tau\gamma}, \sum_{\tau=1}^{p} \epsilon_{\tau} r_{ij}^{\tau\delta}\right),$$

is an additively consistent trapezoidal FAPCM-A for any $\epsilon_{\tau} \in [0,1], \tau = 1, \ldots, p,$ *with* $\sum_{\tau=1}^{p} \epsilon_{\tau} = 1.$

Proof. The proof is analogous to the proof of Theorem 61. $\square$

Example 51. Let us assume that two DMs m_1 and m_2 compared pairwisely four objects in interval FAPCMs-A $\widetilde{R}^1$ and $\widetilde{R}^2$, respectively, as follows:

$$\widetilde{R}^1 = \begin{pmatrix} 0.5 & [0.5, 0.6] & [0.6, 0.7] & [0.7, 0.85] \\ [0.4, 0.5] & 0.5 & [0.5, 0.6] & [0.65, 0.8] \\ [0.3, 0.4] & [0.4, 0.5] & 0.5 & [0.6, 0.7] \\ [0.15, 0.3] & [0.2, 0.35] & [0.3, 0.4] & 0.5 \end{pmatrix}, \tag{IV.140}$$

$$\widetilde{R}^2 = \begin{pmatrix} 0.5 & [0.5, 0.6] & [0.55, 0.7] & [0.8, 1] \\ [0.4, 0.5] & 0.5 & [0.55, 0.65] & [0.7, 0.9] \\ [0.3, 0.45] & [0.35, 0.45] & 0.5 & [0.6, 0.8] \\ [0, 0.2] & [0.1, 0.3] & [0.2, 0.4] & 0.5 \end{pmatrix}. \tag{IV.141}$$

First, let us verify their additive consistency by applying Theorem 55. From Tab. IV.11 we see that the inequality conditions (IV.135) are satisfied for the interval FAPCM-A $\widetilde{R}^1$. Therefore, $\widetilde{R}^1$ is additively consistent according to Definition 60. Additive consistency of the interval FAPCM-A $\widetilde{R}^2$ is verified in an analogous way.

Because the interval FAPCMs-A $\widetilde{R}^1$ and $\widetilde{R}^2$ are additively consistent according to Definition 60, then, based on Theorem 56, they are also additively weakly consistent according to Definition 59.

Let us now aggregate the interval FAPCMs-A (IV.140) and (IV.141) into one interval FAPCM-A $\widetilde{R}$ representing the preferences of both DMs. Let us use the formula (IV.139) with the importance of the DM m_1 given as $\epsilon_1 = 0.4$ and the importance of the DM m_2 given as $\epsilon_2 = 0.6$, $\epsilon_1 + \epsilon_2 = 1$. The resulting interval FAPCM-A $\widetilde{R}$ is in the form

$$\widetilde{R} = \begin{pmatrix} 0.5 & [0.5, 0.6] & [0.57, 0.7] & [0.76, 0.94] \\ [0.4, 0.5] & 0.5 & [0.53, 0.63] & [0.68, 0.86] \\ [0.3, 0.43] & [0.37, 0.47] & 0.5 & [0.6, 0.76] \\ [0.06, 0.24] & [0.14, 0.32] & [0.24, 0.4] & 0.5 \end{pmatrix}. \tag{IV.142}$$

According to Theorem 55, the interval FAPCM-A (IV.142) is again additively consistent (the reader can again verify that the inequalities (IV.135) are satisfied). Further, based on Theorem 56, it is also additively weakly consistent. $\triangle$

4.3.2.2 Deriving priorities from FAPCMs-A

In this section, the focus is put on methods for obtaining fuzzy priorities of objects from FAPCMs-A. The notation $\widetilde{\underline{v}} = (\widetilde{v}_1, \ldots, \widetilde{v}_n)^T, \widetilde{v}_i = (v_i^{\alpha}, v_i^{\beta}, v_i^{\gamma}, v_i^{\delta}), i = 1, \ldots, n,$ will be used hereafter to represent exclusively a fuzzy priority vector associated with a FAPCM-A.

In the literature, various methods have been proposed to obtain interval priorities from interval FAPCMs-A. Most of these methods are based on linear programming models rather than on interval arithmetic. Xu (2007a) and Xu and Chen (2008a) proposed linear programming models for obtaining interval priorities of objects from interval FAPCMs-A additively consistent according to Definition 55. The solution of the models is a set of priority vectors satisfying the inequalities (IV.108) and the normalization condition (IV.110). Further, Xu and Chen (2008a) proposed a modification of these linear programming models to obtain interval priorities also from additively inconsistent interval FAPCMs-A. The modification is based on introducing a set of deviation variables relaxing the inequalities (IV.108). However, as already mentioned in Section 2.3.2.2, the normalization condition (IV.110) is not compatible with Tanino's characterization (II.32). Similar linear programming models for obtaining interval priorities from interval FAPCMs-A were proposed by Wang and Li (2012). However, they again employed the inappropriate normalization condition (IV.110) in their models. Hu et al. (2014) later proposed a modification of the linear programming models introduced by Xu and Chen (2008a) by replacing Tanino's characterization with the characterization (II.46), and Xu et al. (2014b) generalized the models by adding a parameter into the characterization. Wang et al. (2012) introduced linear programming models for obtaining interval priorities using a particular characterization based on logarithms.

The number of papers proposing linear programming models based on Tanino's characterization (II.32) or on alternative characterizations is extensive. Nevertheless, the focus of this thesis is put only on the methods based on applying fuzzy arithmetic to the fuzzy extension of Tanino's characterization.

From Section 2.3.2.2, we know that the only appropriate formula (up to addition of a constant) for obtaining priorities from APCMs-A, compatible with Tanino's characterization (II.32) is the formula (II.36). So far, I have not encountered any research paper dealing with the fuzzy extension of this formula to FAPCMs-A by applying fuzzy arithmetic. As far as I am aware, the only approach for obtaining interval priorities from interval FAPCMs-A based on interval arithmetic is the approach proposed by Liu et al. (2012a).

Liu et al. (2012a) proposed to transform an interval FAPCM-A into an interval FMPCM and then to compute interval priorities by using the formulas (IV.90). Putting aside the fact that such interval priorities are not invariant under permutation of objects, it is important to realize that priorities obtained in such a way do not reflect the preference information in the interval FAPCMs-A by means of differences; they reflect, instead, by means of ratios, the preference information contained in the corresponding interval FMPCMs. It is very important to realize this difference. Nevertheless, Liu et al. (2012a) completely omitted this issue.

The necessity of distinguishing between these two types of priorities was emphasized by Krejčí (2017b), who proposed a method based on the constrained fuzzy arithmetic for obtaining "multiplicative" fuzzy priorities from triangular FAPCMs-A. Note that the method for obtaining "multiplicative" fuzzy priorities proposed by Krejčí (2017b) is invariant under permutation of objects and preserves the additive reciprocity of PCs as it is based on the formulas (IV.97)–(IV.100).

In this section, the fuzzy extension of the formula (II.36) to FAPCMs-A is introduced and properties of the new formulas are discussed. Further, it is demonstrated that the new formulas preserve two desirable properties - additive reciprocity of PCs and invariance under permutation of objects.

We already know from Section 2.3.2.2 that the usual normalization condition (II.39), $\sum_{i=1}^{n} v_i = 1$, $v_i \in [0,1], i = 1, \ldots, n$, is not reachable for the priorities obtainable from APCMs-A. Thus, Fedrizzi and Brunelli (2009) proposed the normalization condition (II.40), $\min_{i \in \{1,\ldots,n\}} v_i = 0$, $v_i \in [0,1], i = 1, \ldots, n$, for the priorities obtainable from APCMs-A. However, as demonstrated in Section 2.3.2.2, when an APCM-A is not additively consistent, reaching the property (II.40) is not guaranteed anymore. Therefore, the weakened normalization condition (II.42), $\sum_{i=1}^{n} v_i = 1$, was introduced in Section 2.3.2.2 for the priorities obtainable from APCMs-A in order to make an analogy to the normalization condition (II.39) for the priorities obtainable from MPCMs and from APCMs-M. In this section, the normalization condition (II.42) is extended to fuzzy priorities obtainable from FAPCMs-A.

First, let us start with the fuzzy extension of the formulas (II.36) for obtaining non-normalized priorities from APCMs-A. By applying constrained fuzzy arithmetic (III.45) to the formula (II.36), the representing values of the fuzzy priorities $\widetilde{v}_i = \left(v_i^\alpha, v_i^\beta, v_i^\gamma, v_i^\delta\right), i = 1, \ldots, n$, obtained from a FAPCM-A $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = \left(r_{ij}^\alpha, r_{ij}^\beta, r_{ij}^\gamma, r_{ij}^\delta\right)$, are given as:

$$v_i^\alpha = \min \left\{ \frac{2}{n} \sum_{j=1}^{n} r_{ij}; \begin{array}{l} r_{pq} \in \left[r_{pq}^\alpha, r_{pq}^\delta\right], \\ r_{pq} = 1 - r_{qp}, \\ p, q = 1, \ldots, n \end{array} \right\}, \tag{IV.143}$$

$$v_i^\beta = \min \left\{ \frac{2}{n} \sum_{j=1}^{n} r_{ij}; \begin{array}{l} r_{pq} \in \left[r_{pq}^\beta, r_{pq}^\gamma\right], \\ r_{pq} = 1 - r_{qp}, \\ p, q = 1, \ldots, n \end{array} \right\}, \tag{IV.144}$$

$$v_i^\gamma = \max \left\{ \frac{2}{n} \sum_{j=1}^{n} r_{ij}; \begin{array}{l} r_{pq} \in \left[r_{pq}^\beta, r_{pq}^\gamma \right], \\ r_{pq} = 1 - r_{qp}, \\ p, q = 1, \ldots, n \end{array} \right\}, \qquad \text{(IV.145)}$$

$$v_i^\delta = \max \left\{ \frac{2}{n} \sum_{j=1}^{n} r_{ij}; \begin{array}{l} r_{pq} \in \left[r_{pq}^\alpha, r_{pq}^\delta \right], \\ r_{pq} = 1 - r_{qp}, \\ p, q = 1, \ldots, n \end{array} \right\}. \qquad \text{(IV.146)}$$

Because the function optimized in the formulas (IV.143)–(IV.146) is increasing in all variables, the formulas can be further simplified so that no optimization is needed:

$$v_i^\alpha = \frac{2}{n} \sum_{j=1}^{n} r_{ij}^\alpha, \quad v_i^\beta = \frac{2}{n} \sum_{j=1}^{n} r_{ij}^\beta, \quad v_i^\gamma = \frac{2}{n} \sum_{j=1}^{n} r_{ij}^\gamma, \quad v_i^\delta = \frac{2}{n} \sum_{j=1}^{n} r_{ij}^\delta. \qquad \text{(IV.147)}$$

Remark 27. *It is worth to note that the elimination of the optimization problems in the formulas (IV.143)–(IV.146) and their replacement by very simple formulas (IV.147) was possible to do only because the constraints of the optimization problems have no effect on the optima; the additive-reciprocity condition $r_{ij} = 1 - r_{ji}$ has no influence since only the PCs from the $i-$th row of the FAPCM-A are present in the optimized function. Thus, in this particular case, the formulas (IV.143)–(IV.146) based on the constrained fuzzy arithmetic actually give the same results as the formulas (IV.147) based on the standard fuzzy arithmetic.*

Notice that also the formulas (IV.85) for computing fuzzy priorities from FMPCMs proposed by Buckley (1985a) reviewed in Section 4.2.3.2 have this simple form. Also in this case the multiplicative-reciprocity condition $m_{ij} = \frac{1}{m_{ji}}$ would have no impact since only the PCs from the $i-$th row of the FMPCM are present in the optimized function. Thus, the standard fuzzy arithmetic is sufficient here.

Usually, however, the formulas based on the constrained fuzzy arithmetic cannot be simplified to standard fuzzy arithmetic by simply eliminating the constraints and thus avoiding solving an optimization problem; the formulas (IV.91)–(IV.94) for obtaining normalized fuzzy priorities from FMPCMs serve as an example.

There are interactions between the fuzzy priorities $\widetilde{v}_i, i = 1, \ldots, n$, obtained by formulas (IV.147). The property (II.37) valid for the priorities obtained from an APCM-A by the formulas (II.36) is extended to the fuzzy priorities as

$$\forall v_i \in \widetilde{v}_{i(\alpha)} \, \exists v_j \in \widetilde{v}_{j(\alpha)}, j = 1, \ldots, n, j \neq i : \quad v_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j = n, \qquad \text{(IV.148)}$$

for all $\alpha \in [0, 1]$ and $i = 1, \ldots, n$. This interaction property will be formulated properly and proved later. First, the following proposition is needed.

Proposition 14. *The interaction property (IV.148) between the trapezoidal fuzzy numbers $\widetilde{v}_i = \left(v_i^\alpha, v_i^\beta, v_i^\gamma, v_i^\delta \right)$, $i = 1, \ldots, n$, is valid if and only if*

$$v_i^\alpha + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\delta \geq n, \qquad v_i^\delta + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\alpha \leq n, \qquad v_i^\beta + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\gamma \geq n, \qquad v_i^\gamma + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\beta \leq n. \qquad \text{(IV.149)}$$

Proof. First, let us show that (IV.148) implies (IV.149). For $\alpha = 0$, it follows from (IV.148) that for $v_i^\alpha, i \in \{1, \ldots, n\}$, $\exists v_j \in \left[v_j^\alpha, v_j^\delta \right], j = 1, \ldots, n, j \neq i : v_i^\alpha + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j = n$. Because $v_j^\delta \geq v_j$, then clearly $v_i^\alpha + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\delta \geq n$. Similarly, for $v_i^\delta, i \in \{1, \ldots, n\}$, $\exists v_j \in \left[v_j^\alpha, v_j^\delta \right], j = 1, \ldots, n, j \neq i : v_i^\delta + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j = n$. Because $v_j^\alpha \leq v_j$, then clearly $v_i^\delta + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\alpha \leq n$. Analogously, for $\alpha = 1$, it follows from (IV.148) that for $v_i^\beta, i \in \{1, \ldots, n\}$, $\exists v_j \in \left[v_j^\beta, v_j^\gamma \right], j = 1, \ldots, n, j \neq i : v_i^\beta + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j = n$. Because $v_j^\gamma \geq v_j$, then clearly $v_i^\beta + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\gamma \geq n$. Similarly, for $v_i^\gamma, i \in \{1, \ldots, n\}$, $\exists v_j \in \left[v_j^\beta, v_j^\gamma \right], j = 1, \ldots, n, j \neq i : v_i^\gamma + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j = n$. Because $v_j^\beta \leq v_j$, then clearly $v_i^\gamma + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\beta \leq n$.

Now, let us show that (IV.149) implies (IV.148). From the inequalities $v_i^\delta + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\alpha \leq n$ and $v_i^\alpha + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\delta \geq n$, the inequalities $v_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\alpha \leq n$ and $v_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\delta \geq n$ follow $\forall v_i \in \left[v_i^\alpha, v_i^\delta \right]$. Therefore, $\exists v_j \in \left[v_j^\alpha, v_j^\delta \right] : v_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j = n$, which implies (IV.148) for $\alpha = 0$. Analogously, from the inequalities $v_i^\gamma + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\beta \leq n$ and

$v_i^\beta + \sum_{\substack{j=1 \\ j\neq i}}^{n} v_j^\gamma \geq n$, the inequalities $v_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} v_j^\beta \leq n$ and $v_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} v_j^\gamma \geq n$ follow $\forall v_i \in \left[v_i^\beta, v_i^\gamma \right]$. Therefore, $\exists v_j \in \left[v_j^\beta, v_j^\gamma \right] : v_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} v_j = n$, which implies (IV.148) for $\alpha = 1$.

The proof of the validity of (IV.148) for $\alpha \in \,]0,1[$ is analogous; it is sufficient to show that the inequalities (IV.149) hold also for the $\alpha-$cuts $\widetilde{v}_{i(\alpha)} = \left[v_{i(\alpha)}^L, v_{i(\alpha)}^U \right]$ of the trapezoidal fuzzy numbers $\widetilde{v}_i = (v_i^\alpha, v_i^\beta, v_i^\gamma, v_i^\delta)$, i.e.

$$v_{i(\alpha)}^L + \sum_{\substack{j=1 \\ j\neq i}}^{n} v_{j(\alpha)}^U \geq n, \qquad v_{i(\alpha)}^U + \sum_{\substack{j=1 \\ j\neq i}}^{n} v_{j(\alpha)}^L \leq n. \tag{IV.150}$$

Then it is enough to take the $\alpha-$cuts $\widetilde{v}_{i(\alpha)} = \left[v_{i(\alpha)}^L, v_{i(\alpha)}^U \right]$ of $\widetilde{v}_i, i = 1, \ldots, n$, for $\left[v_i^\alpha, v_i^\delta \right], i = 1, \ldots, n$, in the above part of the proof.

Using the definition (III.6) of $\alpha-$cuts and formulas (IV.149), we have

$$v_{i(\alpha)}^U + \sum_{\substack{j=1 \\ j\neq i}}^{n} v_{j(\alpha)}^L = \alpha v_i^\gamma + (1-\alpha)v_i^\delta + \sum_{\substack{j=1 \\ j\neq i}}^{n} \left[\alpha v_j^\beta + (1-\alpha)v_j^\alpha \right] =$$

$$\alpha \left[v_i^\gamma + \sum_{\substack{j=1 \\ j\neq i}}^{n} v_j^\beta \right] + (1-\alpha) \left[v_i^\delta + \sum_{\substack{j=1 \\ j\neq i}}^{n} v_j^\alpha \right] \leq \alpha n + (1-\alpha)n = n$$

and analogously the inequality $v_{i(\alpha)}^L + \sum_{\substack{j=1 \\ j\neq i}}^{n} v_{j(\alpha)}^U \geq n$ could be demonstrated. $\square$

Now, by utilizing Proposition 14, we can formulate and prove the following theorem.

Theorem 63. *Let $\widetilde{v}_i = \left(v_i^\alpha, v_i^\beta, v_i^\gamma, v_i^\delta \right), i = 1, \ldots, n$, be trapezoidal fuzzy priorities obtained from a FAPCM-A by formulas* (IV.147). *Then the property* (IV.148) *holds for all $\alpha \in [0,1]$ and $i = 1, \ldots, n$.*

Proof. By utilizing Proposition 14, it is sufficient to show that the fuzzy priorities obtained by formulas (IV.147) satisfy (IV.149).

$$v_i^\alpha + \sum_{\substack{j=1 \\ j\neq i}}^{n} v_j^\delta = \frac{2}{n} \sum_{k=1}^{n} r_{ik}^\alpha + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{2}{n} \sum_{k=1}^{n} r_{jk}^\delta = \frac{2}{n} \left(\sum_{k=1}^{n} r_{ik}^\alpha + \sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{k=1}^{n} r_{jk}^\delta \right) =$$

$$\frac{2}{n} \left(0.5n + (n-1) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{\substack{k=1 \\ k\neq i \\ k\neq j}}^{n} r_{jk}^\delta \right) \geq \frac{2}{n} \left(0.5n + (n-1) + \frac{(n-1)(n-2)}{2} \right) = n$$

$$v_i^\delta + \sum_{\substack{j=1 \\ j\neq i}}^{n} v_j^\alpha = \frac{2}{n} \sum_{k=1}^{n} r_{ik}^\delta + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{2}{n} \sum_{k=1}^{n} r_{jk}^\alpha = \frac{2}{n} \left(\sum_{k=1}^{n} r_{ik}^\delta + \sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{k=1}^{n} r_{jk}^\alpha \right) =$$

$$\frac{2}{n} \left(0.5n + (n-1) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{\substack{k=1 \\ k\neq i \\ k\neq j}}^{n} r_{jk}^\alpha \right) \leq \frac{2}{n} \left(0.5n + (n-1) + \frac{(n-1)(n-2)}{2} \right) = n$$

Validity of the remaining two inequalities is proved in an analogous way. $\square$

The interaction property (IV.148) corresponds to the fact that, for any $i \in \{1, \ldots, n\}$, and for any value $v_i \in \widetilde{v}_{i(\alpha)}, \alpha \in [0,1]$, there exist values $v_j \in \widetilde{v}_{j(\alpha)}, j \neq i$, such that they are all obtained by the formula (II.36) from the same APCM-A R belonging to the FAPCM-A $\widetilde{R}$. According to Proposition 3, these priorities are such that $\sum_{i=1}^{n} v_i = n$. The following example is given to illustrate better this interaction property.

Example 52. Let us consider the FAPCM-A

$$\widetilde{R} = \begin{pmatrix} 0.5 & (0.6, 0.7, 0.8, 0.9) & (0.8, 0.9, 0.9, 1) \\ (0.1, 0.2, 0.3, 0.4) & 0.5 & (0.5, 0.6, 0.7, 0.8) \\ (0, 0.1, 0.1, 0.2) & (0.2, 0.3, 0.4, 0.5) & 0.5 \end{pmatrix}. \tag{IV.151}$$

The fuzzy priorities of objects obtained by formulas (IV.147) are

$$\widetilde{v}_1 = \left(\frac{19}{15}, \frac{21}{15}, \frac{22}{15}, \frac{24}{15}\right), \quad \widetilde{v}_2 = \left(\frac{11}{15}, \frac{13}{15}, \frac{15}{15}, \frac{17}{15}\right), \quad \widetilde{v}_3 = \left(\frac{7}{15}, \frac{9}{15}, \frac{10}{15}, \frac{12}{15}\right). \tag{IV.152}$$

Let us fix, for example, the upper boundary value $v_1^\delta = \frac{24}{15}$ of $\widetilde{v}_1$ and let us show that there exist priorities $v_2 \in \widetilde{v}_2$ and $v_3 \in \widetilde{v}_3$ such that all three priorities are obtained from the same APCM-A belonging to the FAPCM-A (IV.151) and $\sum_{i=1}^{3} v_i = 3$.

In order not to violate the additive reciprocity of the related PCs, v_1^δ must have been obtained from the matrix

$$\begin{pmatrix} 0.5 & 0.9 & 1 \\ 0.1 & 0.5 & ... \\ 0 & ... & 0.5 \end{pmatrix} \tag{IV.153}$$

since $\frac{2}{3}(0.5 + 0.9 + 1) = \frac{24}{15} = v_1^\delta$. Notice that in order to obtain the priority $v_i, i \in \{1, \ldots, n\}$, by formula (II.36) we do not need to know all the PCs in the APCM-A; the PCs in the $i-$th row are sufficient. The possible values v_2, v_3 of the fuzzy priorities $\widetilde{v}_2, \widetilde{v}_3$ corresponding to the possible value v_1 are then obtainable from APCMs-A

$$\begin{pmatrix} 0.5 & 0.9 & 1 \\ 0.1 & 0.5 & x \\ 0 & 1-x & 0.5 \end{pmatrix}, \qquad x \in [0.5, 0.8],$$

in order to preserve the additive reciprocity of PCs. The sum of the priorities obtained from such matrices is always equal to 3:

$$\sum_{i=1}^{3} v_i = \frac{24}{15} + \frac{2}{3}(0.1 + 0.5 + x) + \frac{2}{3}(0 + (1 - x) + 0.5) = 3$$

$\triangle$

Let us now focus on the fuzzy extension of the weakened normalization condition (II.42) introduced in Section 2.3.2.2 for the priorities obtainable from APCMs-A. By applying constrained fuzzy arithmetic (III.45) to the fuzzy extension of the formula (II.43), the normalized fuzzy priorities $\widetilde{v}_i = \left(v_i^\alpha, v_i^\beta, v_i^\gamma, v_i^\delta\right), i = 1, \ldots, n$, are derived from a FAPCM-A $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = \left(r_{ij}^\alpha, r_{ij}^\beta, r_{ij}^\gamma, r_{ij}^\delta\right)$, as:

$$v_i^\alpha = \min \left\{ \frac{2}{n}\sum_{j=1}^{n} r_{ij} - \frac{n-1}{n}; \begin{array}{l} r_{pq} \in [r_{pq}^\alpha, r_{pq}^\delta], \\ r_{pq} = 1 - r_{qp}, \\ p, q = 1, \ldots, n \end{array} \right\}, \tag{IV.154}$$

$$v_i^\beta = \min \left\{ \frac{2}{n}\sum_{j=1}^{n} r_{ij} - \frac{n-1}{n}; \begin{array}{l} r_{pq} \in [r_{pq}^\beta, r_{pq}^\gamma], \\ r_{pq} = 1 - r_{qp}, \\ p, q = 1, \ldots, n \end{array} \right\}, \tag{IV.155}$$

$$v_i^\gamma = \max \left\{ \frac{2}{n}\sum_{j=1}^{n} r_{ij} - \frac{n-1}{n}; \begin{array}{l} r_{pq} \in [r_{pq}^\beta, r_{pq}^\gamma], \\ r_{pq} = 1 - r_{qp}, \\ p, q = 1, \ldots, n \end{array} \right\}, \tag{IV.156}$$

$$v_i^\delta = \max \left\{ \frac{2}{n}\sum_{j=1}^{n} r_{ij} - \frac{n-1}{n}; \begin{array}{l} r_{pq} \in [r_{pq}^\alpha, r_{pq}^\delta], \\ r_{pq} = 1 - r_{qp}, \\ p, q = 1, \ldots, n \end{array} \right\}. \tag{IV.157}$$

Analogously as in the case of the formulas (IV.143)–(IV.146), also the formulas (IV.154)–(IV.157) can be simplified so that no optimization is needed:

$$v_i^\alpha = \frac{2}{n}\sum_{j=1}^{n} r_{ij}^\alpha - \frac{n-1}{n}, \qquad v_i^\beta = \frac{2}{n}\sum_{j=1}^{n} r_{ij}^\beta - \frac{n-1}{n},$$
$$v_i^\gamma = \frac{2}{n}\sum_{j=1}^{n} r_{ij}^\gamma - \frac{n-1}{n}, \qquad v_i^\delta = \frac{2}{n}\sum_{j=1}^{n} r_{ij}^\delta - \frac{n-1}{n}. \tag{IV.158}$$

The normalization property (II.44) valid for the priorities (II.43) obtained from an APCM-A is extended to the fuzzy priorities (IV.158) as

$$\forall v_i \in \widetilde{v}_{i(\alpha)} \ \exists v_j \in \widetilde{v}_{j(\alpha)}, j = 1, \ldots, n, j \neq i : \quad v_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j = 1, \tag{IV.159}$$

for all $\alpha \in [0,1]$ and $i = 1, \ldots, n$. Similarly to Proposition 14 and Theorem 63, the following properties are valid.

Proposition 15. *The interaction property* (IV.159) *between the trapezoidal fuzzy numbers* $\widetilde{v}_i = \left(v_i^\alpha, v_i^\beta, v_i^\gamma, v_i^\delta \right)$, $i = 1, \ldots, n$, *is valid if and only if*

$$v_i^\alpha + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\delta \geq 1, \quad v_i^\delta + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\alpha \leq 1, \quad v_i^\beta + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\gamma \geq 1, \quad v_i^\gamma + \sum_{\substack{j=1 \\ j \neq i}}^{n} v_j^\beta \leq 1, \tag{IV.160}$$

for all $\alpha \in [0,1]$ *and* $i = 1, \ldots, n$.

Proof. The proof is analogous to the proof of Proposition 14. $\qquad\square$

Theorem 64. *Let* $\widetilde{v}_i = \left(v_i^\alpha, v_i^\beta, v_i^\gamma, v_i^\delta \right)$, $i = 1, \ldots, n$, *be fuzzy priorities obtained from a FAPCM-A by formulas* (IV.158). *Then the property* (IV.159) *holds for all* $\alpha \in [0,1]$ *and* $i = 1, \ldots, n$.

Proof. The proof is analogous to the proof of Theorem 63. $\qquad\square$

Remark 28. *The interaction property* (IV.159) *is in fact the interaction property* (III.12) *from Definition 29 of the normalized fuzzy vector. However, the fuzzy priorities obtainable from a FAPCM-A by the formulas* (IV.158) *satisfying the property* (IV.159) *are not constrained to the interval* $[0,1]$ *as it is required in Definition 29. Therefore, they are not normalized in the sense of Definition 29. Nevertheless, for the simplicity, they will be called "normalized" here (keeping in mind the slight difference in the definitions).*

Proposition 16. *Let* $\widetilde{v}_i = \left(v_i^\alpha, v_i^\beta, v_i^\gamma, v_i^\delta \right)$, $i = 1, \ldots, n$, *be normalized fuzzy priorities obtained by formulas* (IV.158). *Then*

$$-1 < \widetilde{v}_i \leq 1, \qquad i = 1, \ldots, n. \tag{IV.161}$$

Proof. It is sufficient to prove inequalities $-1 < v_i^\alpha$ and $v_i^\delta \leq 1, i = 1, \ldots, n$. The proof is analogous to the proof of Proposition 5. $\qquad\square$

Remark 29. *As mentioned in Section 2.3.2.2, for an APCM-A, any vector derived from the priority vector* (II.36) *by adding an arbitrary constant, i.e. by the transformation* (II.38), *is again a priority vector. For the case of FAPCMs-A, the formulas* (IV.158) *for obtaining normalized fuzzy priorities are in fact obtained from the formulas* (IV.147) *by adding constant* $-\frac{n-1}{n}$. *That is, the shape of the trapezoidal fuzzy numbers and the distances between them remain unchanged by applying the normalization condition* (IV.159); *the whole set of trapezoidal fuzzy numbers is just shifted back on the scale of real numbers by* $-\frac{n-1}{n}$.

Example 53. Let us consider the FAPCM-A (IV.151). The fuzzy priorities obtained by formulas (IV.147) are given as (IV.152), and the normalized fuzzy priorities obtained by formulas (IV.158) are given as

$$\widetilde{v}_{N1} = \left(\frac{9}{15}, \frac{11}{15}, \frac{12}{15}, \frac{14}{15} \right), \ \widetilde{v}_{N2} = \left(\frac{1}{15}, \frac{3}{15}, \frac{5}{15}, \frac{7}{15} \right), \ \widetilde{v}_{N3} = \left(\frac{-3}{15}, \frac{-1}{15}, 0, \frac{2}{15} \right). \tag{IV.162}$$

The fuzzy priorities (IV.152) and the normalized fuzzy priorities (IV.162) are depicted in Fig. IV.9. It is evident from the figure that the normalized fuzzy priorities have the same shape as the original non-normalized fuzzy priorities; they are just moved backwards by $-\frac{2}{3}$. $\qquad\triangle$

Theorem 65. *The method for obtaining the normalized fuzzy priorities of objects from FAPCMs-A by using the formulas* (IV.158) *is invariant under permutation of objects in FAPCMs-A.*

Proof. It is sufficient to show that for a given object $o_i, i \in \{1, \ldots, n\}$, its priority $\widetilde{v}_i$ obtained by the formulas (IV.158) does not change under permutation of objects in a FAPCM-A $\widetilde{R}$.

From the invariance of the formula (II.43) reviewed in Section 2.3.2.2, it follows that the priority v_i of object o_i determined by the formula (II.43) from the given APCM-A R does not change under any permutation $R^\pi = PRP^T$ of R, it is just permuted accordingly. This means that the priority v_i obtained from R is equal to the corresponding priority $v_{\pi(i)}^\pi$ obtained from R^π.

From this it follows that the priorities $v_i^\alpha, v_i^\beta, v_i^\gamma$, and v_i^δ obtained by the formulas (IV.158) from a FAPCM-A $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}, \widetilde{r}_{ij} = (r_{ij}^\alpha, r_{ij}^\beta, r_{ij}^\gamma, r_{ij}^\delta)$, do not change under permutation (they are just permuted accordingly). Therefore, the fuzzy priority $\widetilde{v}_i = (v_i^\alpha, v_i^\beta, v_i^\gamma, v_i^\delta)$ does not change for any permutation of objects in a FAPCM-A (it is only permuted accordingly), which concludes the proof. $\qquad\square$

Figure IV.9: Fuzzy priorities of the FAPCM-A (IV.151).
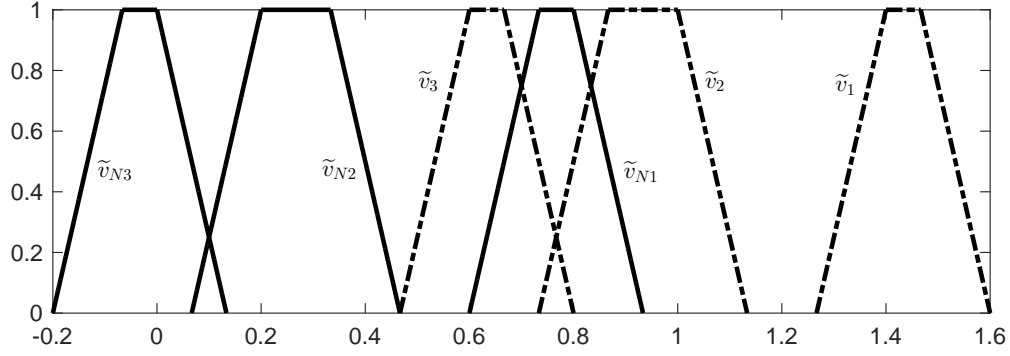
4.3.3 Fuzzy additive pairwise comparison matrices with multiplicative representation

In this Section, the fuzzy extension of the methods related to APCMs-M reviewed in Section 2.3.3 is dealt with. In particular, Section 4.3.3.1 is dedicated to the extension of the multiplicative-consistency condition (II.48) to FAPCMs-M and Section 4.3.3.2 is focused on methods for obtaining fuzzy priorities of objects from FAPCMs-M.

4.3.3.1 Multiplicative consistency of FAPCMs-M

In this section, multiplicative consistency of FAPCMs-M is studied. First, definitions of multiplicative consistency for interval FAPCMs-M based on Tanino's characterization (II.54) proposed in the literature are reviewed and some drawbacks of the definitions are pointed out in Section 4.3.3.1.1. Afterwards, in Section 4.3.3.1.2, a new definition of multiplicative consistency for FAPCMs-M is proposed.

4.3.3.1.1 Review of fuzzy extensions of multiplicative consistency

In order to examine consistency of interval FAPCMs-M, many definitions of consistency have been proposed in the literature. These definitions are mostly based on interval extension of multiplicative-transitivity property (II.48) and related Tanino's characterization (II.54).

Xu and Chen (2008a) proposed a weak version of multiplicative consistency for interval FAPCMs-M based on Tanino's characterization. Xia and Xu (2011) defined perfect multiplicative consistency for interval FAPCMs-M based on an extension of Tanino's multiplicative-transitivity property and discussed its properties. Wang and Li (2012) proposed another definition of multiplicative consistency for interval FAPCMs-M based on an interval extension of a property equivalent to Tanino's multiplicative-transitivity property. Wu and Chiclana (2014a) defined multiplicative consistency of interval FAPCMs-M based on a direct extension of Tanino's multiplicative-transitivity property to intervals and proposed a consistency index for measuring inconsistency of interval FAPCMs-M. Wu and Chiclana (2014b) proposed another extension of Tanino's multiplicative-transitivity property to define multiplicative consistency of interval FAPCMs-M, and they further extended the definition to intuitionistic FAPCMs-M.

In this section, all these definitions of multiplicatively consistent interval FAPCMs-M are reviewed in detail and some drawbacks are pointed out. In particular, it is shown that some definitions are not invariant under permutation of objects in interval FAPCMs-M and some violate additive reciprocity of PCs of objects. Afterwards, it is shown that the drawbacks can be eliminated by employing the constrained fuzzy arithmetic into computations instead of the standard fuzzy arithmetic.

Xu and Chen (2008) defined a weak version of multiplicative consistency for interval FAPCMs-M. This definition of consistency will be studied in more detail in the following section. In order to distinguish easily this definition of consistency from others, consistency according to this definition will be simply called multiplicative weak consistency.

Definition 61. *(Xu and Chen, 2008a) Let $\overline{Q} = \left\{ \overline{q}_{ij} \right\}_{i,j=1}^{n}, \overline{q}_{ij} = \left[q_{ij}^{L}, q_{ij}^{U} \right]$, be an interval FAPCM-M. If there exists a positive vector $\underline{u} = (u_1, \ldots, u_n)^T$ such that*

$$q_{ij}^{L} \leq \frac{u_i}{u_i + u_j} \leq q_{ij}^{U}, \qquad i,j = 1, \ldots, n, \tag{IV.163}$$

then $\overline{Q}$ is called a multiplicatively weakly consistent interval FAPCM-M.

Definition 61 of multiplicative weak consistency for interval FAPCMs-M is clearly based on Proposition 6 for APCMs-M. According to the definition, an interval FAPCM-M $\overline{Q}$ is multiplicatively multiplicatively consistent if there exists a vector $\underline{u} = (u_1, \ldots, u_n)^T$ using which we can construct a multiplicatively consistent APCM-M $Q^* = \{q_{ij}^*\}_{i,j=1}^n$ such that $q_{ij}^* \in \left[q_{ij}^L, q_{ij}^U\right], i,j = 1, \ldots, n$.

The requirement of at least one multiplicatively consistent APCM-M obtainable from the interval FAPCM-M is very weak. Therefore, it is quite easy to satisfy the multiplicative-consistency condition in Definition 61 when constructing an interval FAPCM-M. The multiplicative consistency conditions reviewed in the rest of this section are significantly stronger and thus much more difficult to fulfill.

Xia and Xu (2011) defined multiplicative consistency for interval FAPCMs-M as follows.

Definition 62. *(Xia and Xu, 2011) Let $\overline{Q} = \{\overline{q}_{ij}\}_{i,j=1}^n, \overline{q}_{ij} = \left[q_{ij}^L, q_{ij}^U\right]$, be an interval FAPCM-M. $\overline{Q}$ is called multiplicatively consistent if the APCMs-M $C = \{c_{ij}\}_{i,j=1}^n, D = \{d_{ij}\}_{i,j=1}^n$ such that*

$$c_{ij} = \begin{cases} q_{ij}^L, & i < j \\ 0.5, & i = j \\ q_{ij}^U, & i > j \end{cases}, \qquad d_{ij} = \begin{cases} q_{ij}^U, & i < j \\ 0.5, & i = j \\ q_{ij}^L, & i > j \end{cases}, \qquad i,j = 1, \ldots, n, \tag{IV.164}$$

are multiplicatively consistent according to (II.53).

However, Definition 62 is not invariant under permutation of objects in the interval FAPCM-M. This serious drawback is demonstrated on the following example.

Example 54. Let us consider interval FAPCM-M $\overline{Q}$ of three objects o_1, o_2, and o_3 in the form

$$\overline{Q} = \begin{pmatrix} \frac{1}{2} & \left[\frac{1}{2}, \frac{3}{5}\right] & \left[\frac{3}{5}, \frac{6}{7}\right] \\ \left[\frac{2}{5}, \frac{1}{2}\right] & \frac{1}{2} & \left[\frac{3}{5}, \frac{4}{5}\right] \\ \left[\frac{1}{7}, \frac{2}{5}\right] & \left[\frac{1}{5}, \frac{2}{5}\right] & \frac{1}{2} \end{pmatrix}. \tag{IV.165}$$

The corresponding APCMs-M C and D given by (IV.164) are in the form

$$C = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{3}{5} \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{2} \end{pmatrix}, \qquad D = \begin{pmatrix} \frac{1}{2} & \frac{3}{5} & \frac{6}{7} \\ \frac{2}{5} & \frac{1}{2} & \frac{4}{5} \\ \frac{1}{7} & \frac{1}{5} & \frac{1}{2} \end{pmatrix}.$$

Both APCMs-M C and D satisfy the property (II.53), which means that they are multiplicatively consistent according to Definition 11. Therefore, according to Definition 62, the interval FAPCM-M $\overline{Q}$ is multiplicatively consistent.

Now, let us permute the interval FAPCM-M $\overline{Q}$ to the interval FAPCM-M $\overline{Q}^\pi = P\overline{Q}P^T$ by using the permutation matrix (IV.115):

$$\overline{Q}^\pi = \begin{pmatrix} \frac{1}{2} & \left[\frac{1}{7}, \frac{2}{5}\right] & \left[\frac{1}{5}, \frac{2}{5}\right] \\ \left[\frac{3}{5}, \frac{6}{7}\right] & \frac{1}{2} & \left[\frac{1}{2}, \frac{3}{5}\right] \\ \left[\frac{3}{5}, \frac{4}{5}\right] & \left[\frac{2}{5}, \frac{1}{2}\right] & \frac{1}{2} \end{pmatrix}. \tag{IV.166}$$

The corresponding APCMs-M C^π and D^π given by (IV.164) are in the form

$$C^\pi = \begin{pmatrix} \frac{1}{2} & \frac{1}{7} & \frac{1}{5} \\ \frac{6}{7} & \frac{1}{2} & \frac{1}{2} \\ \frac{4}{5} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad D^\pi = \begin{pmatrix} \frac{1}{2} & \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{1}{2} & \frac{3}{5} \\ \frac{3}{5} & \frac{2}{5} & \frac{1}{2} \end{pmatrix},$$

and they clearly do not satisfy the property (II.53). Therefore, according to Definition 62, the interval FAPCM-M $\overline{Q}^\pi$ is not multiplicatively consistent.

Obviously, the PCs of objects o_1, o_2, and o_3 in interval FAPCMs-M $\overline{Q}$ and $\overline{Q}^\pi$ are the same, they vary only in the order in which they are associated with the rows and the columns of the interval FAPCM-M. Therefore, also the conclusion about the multiplicative consistency should be the same for both interval FAPCMs-M. However, as demonstrated, the interval FAPCM-M $\overline{Q}$ is judged as multiplicatively consistent while the interval FAPCM-M $\overline{Q}^\pi$ results to be inconsistent according to Definition 62. △

Wang and Li (2012) proposed the following definition of multiplicative consistency for interval FAPCMs-M.

Definition 63. *(Wang and Li, 2012) Let $\overline{Q} = \{\overline{q}_{ij}\}_{i,j=1}^{n}, \overline{q}_{ij} = [q_{ij}^{L}, q_{ij}^{U}]$, be an interval FAPCM-M. $\overline{Q}$ is called multiplicatively consistent if the multiplicative-transitivity condition*

$$\frac{\overline{q}_{ji}}{\overline{q}_{ij}} \frac{\overline{q}_{kj}}{\overline{q}_{jk}} \frac{\overline{q}_{ik}}{\overline{q}_{ki}} = \frac{\overline{q}_{jk}}{\overline{q}_{kj}} \frac{\overline{q}_{ij}}{\overline{q}_{ji}} \frac{\overline{q}_{ki}}{\overline{q}_{ik}}, \qquad i, j, k = 1, \dots, n, \tag{IV.167}$$

is satisfied.

Wang and Li (2012) applied the standard interval arithmetic to extend the multiplicative consistency of APCMs-M to interval FAPCMs-M in Definition 63. Therefore, the equation (IV.167) is equivalent to the equation

$$\frac{q_{ji}^{L} q_{kj}^{L} q_{ik}^{L}}{q_{ij}^{U} q_{jk}^{U} q_{ki}^{U}} = \frac{q_{jk}^{L} q_{ij}^{L} q_{ki}^{L}}{q_{kj}^{U} q_{ji}^{U} q_{ik}^{U}}, \qquad i, j, k = 1, \dots, n. \tag{IV.168}$$

Beside claiming that $\overline{q}_{ij} + \overline{q}_{ji} = 1$ cannot be reached, as reviewed in Section 4.3.2.1.1, Wang and Li (2012) pointed out that "due to the possibility of $\frac{\overline{a}}{\overline{a}} \neq 1$ for intervals, (IV.167) is not equivalent to

$$\frac{\overline{q}_{ik}}{\overline{q}_{ki}} \frac{\overline{q}_{kj}}{\overline{q}_{jk}} = \frac{\overline{q}_{ij}}{\overline{q}_{ji}}, \qquad i, j, k = 1, \dots, n, \tag{IV.169}$$

as in the case of regular APCMs-M" (Wang and Li (2012), p. 183). However, it is necessary to clarify here that, in contrast to Wang and Li's assertion, it is possible to ensure easily the validity of $\frac{\overline{a}}{\overline{a}} = 1$. At the end of this section, it will be demonstrated that the validity of equation $\frac{\overline{a}}{\overline{a}} = 1$ can be easily achieved by applying appropriately the constrained fuzzy arithmetic instead of the standard fuzzy arithmetic. Furthermore, it will be shown that Definition 63 of multiplicative consistency is inappropriate since it violates the additive reciprocity of PCs. Before doing that, let us finalize the literature review.

Wu and Chiclana (2014a) and Wu and Chiclana (2014b) defined multiplicative transitivity for interval FAPCMs-M. However, since the multiplicative-transitivity property is normally used to define the multiplicative consistency (and this is what is done in this thesis), the expression "multiplicative consistency" will be used in their definition instead of "multiplicative transitivity" in order to keep the same terminology.

Definition 64. *(Wu and Chiclana, 2014a) Let $\overline{Q} = \{\overline{q}_{ij}\}_{i,j=1}^{n}, \overline{q}_{ij} = [q_{ij}^{L}, q_{ij}^{U}]$, be an interval FAPCM-M. $\overline{Q}$ is called multiplicatively consistent if*

$$\frac{\overline{q}_{ki}}{\overline{q}_{ik}} = \frac{\overline{q}_{kj}}{\overline{q}_{jk}} \frac{\overline{q}_{ji}}{\overline{q}_{ij}}, \qquad i < j < k, \ i, j, k = 1, \dots, n. \tag{IV.170}$$

Wu and Chiclana (2014a) defined multiplicatively consistent interval FAPCMs-M by extending the multiplicative-transitivity property (II.48). For the extension, they used the standard interval arithmetic. Therefore, Wu and Chiclana (2014a) derived that the equation (IV.170) is equivalent to the equations

$$\begin{aligned} \frac{1}{q_{ik}^{L}} - 1 &= \left(\frac{1}{q_{ij}^{L}} - 1\right) \left(\frac{1}{q_{jk}^{L}} - 1\right), \\ \frac{1}{q_{ik}^{U}} - 1 &= \left(\frac{1}{q_{ij}^{U}} - 1\right) \left(\frac{1}{q_{jk}^{U}} - 1\right), \end{aligned} \quad i < j < k, \ i, j, k = 1, \dots, n, \tag{IV.171}$$

which can be further written as

$$\begin{aligned} q_{ik}^{L} &= \frac{q_{ij}^{L} q_{jk}^{L}}{q_{ij}^{L} q_{jk}^{L} + (1 - q_{ij}^{L})(1 - q_{jk}^{L})}, \\ q_{ik}^{U} &= \frac{q_{ij}^{U} q_{jk}^{U}}{q_{ij}^{U} q_{jk}^{U} + (1 - q_{ij}^{U})(1 - q_{jk}^{U})}, \end{aligned} \quad i < j < k, \ i, j, k = 1, \dots, n. \tag{IV.172}$$

However, it will be demonstrated on the following example that Definition 64, similarly to Definition 62, is not invariant under permutation of objects in interval FAPCMs-M.

Example 55. Let us consider again the interval FAPCM-M $\overline{Q}$ of three objects o_1, o_2, and o_3 in the form (IV.165). In order to evaluate whether $\overline{Q}$ is multiplicatively consistent according to Definition 64, we only need to verify whether the equations (IV.172) are satisfied for $i = 1, j = 2, k = 3$. Since

$$\frac{q_{12}^{L} q_{23}^{L}}{q_{12}^{L} q_{23}^{L} + (1 - q_{12}^{L})(1 - q_{23}^{L})} = \frac{\frac{1}{2} \frac{3}{5}}{\frac{1}{2} \frac{3}{5} + \frac{1}{2} \frac{2}{5}} = \frac{3}{5} = q_{13}^{L}$$

and

$$\frac{q_{12}^{U} q_{23}^{U}}{q_{12}^{U} q_{23}^{U} + (1 - q_{12}^{U})(1 - q_{23}^{U})} = \frac{\frac{3}{5} \frac{4}{5}}{\frac{3}{5} \frac{4}{5} + \frac{2}{5} \frac{1}{5}} = \frac{6}{7} = q_{13}^{U},$$

we can conclude that the interval FAPCM-M (IV.165) is multiplicatively consistent according to Definition 64.

Now, let us consider the permuted interval FAPCM-M $\overline{Q}^{\pi}$ given by (IV.166). By verifying equations (IV.172) for $i = 1, j = 2, k = 3$, we obtain

$$\frac{q_{12}^{*L} q_{23}^{*L}}{q_{12}^{*L} q_{23}^{*L} + (1 - q_{12}^{*L})(1 - q_{23}^{*L})} = \frac{\frac{1}{7} \frac{1}{2}}{\frac{1}{7} \frac{1}{2} + \frac{6}{7} \frac{1}{2}} = \frac{1}{7} \neq q_{13}^{*L}$$

and

$$\frac{q_{12}^{*U} q_{23}^{*U}}{q_{12}^{*U} q_{23}^{*U} + (1 - q_{12}^{*U})(1 - q_{23}^{*U})} = \frac{\frac{2}{5} \frac{3}{5}}{\frac{2}{5} \frac{3}{5} + \frac{3}{5} \frac{2}{5}} = \frac{1}{2} \neq q_{13}^{*U},$$

which means that the permuted interval FAPCM-M (IV.166) is not multiplicatively consistent according to Definition 64. Therefore, it results that Definition 64 of multiplicative consistency is not invariant under permutation of objects in interval FAPCMs-M. $\triangle$

Definition 65. *(Wu and Chiclana, 2014b) Let* $\overline{Q} = \{\overline{q}_{ij}\}_{i,j=1}^{n}, \overline{q}_{ij} = [q_{ij}^{L}, q_{ij}^{U}]$, *be an interval FAPCM-M.* $\overline{Q}$ *is called multiplicatively consistent if*

$$\begin{aligned} q_{ij}^{L} q_{jk}^{L} q_{ki}^{L} &= q_{ik}^{L} q_{kj}^{L} q_{ji}^{L}, \\ q_{ij}^{U} q_{jk}^{U} q_{ki}^{U} &= q_{ik}^{U} q_{kj}^{U} q_{ji}^{U}, \end{aligned} \qquad i, j, k = 1, \ldots, n. \tag{IV.173}$$

The equations (IV.173) are equivalent to the equation

$$\overline{q}_{ij} \overline{q}_{jk} \overline{q}_{ki} = \overline{q}_{ik} \overline{q}_{kj} \overline{q}_{ji}, \qquad i, j, k = 1, \ldots, n, \tag{IV.174}$$

when using standard fuzzy arithmetic (and in particular the formula (III.26) for the multiplication of trapezoidal fuzzy numbers). Therefore, Definition 65 suffers again from the same drawbacks as Definition 63; the additive reciprocity of PCs is violated. For a detailed discussion on the issue of additive reciprocity, see p. 94.

The problem is caused by the fact that both Wang and Li (2012) and Wu and Chiclana (2014b), similarly to other researchers whose work has been reviewed in this section, applied the standard interval arithmetic to the computations with intervals. This means that the multiplicative-consistency condition (IV.167) is equivalent to the equation (IV.168) and the multiplicative-consistency condition (IV.173) is equivalent to the equation (IV.174). Similarly to the additive-consistency conditions (IV.117), (IV.121), and (IV.122) described in Section 4.3.2.1.1, none of the multiplicative-consistency conditions (IV.167) and (IV.173) preserves the additive reciprocity of PCs. This drawback is demonstrated on the following example.

Example 56. Let us examine Definitions 63 and 65 of multiplicative consistency on the interval FAPCM-M $\overline{Q}$ given by (IV.165). The expressions (IV.173) basically mean that we construct matrices $Q^{L} = \{q_{ij}^{L}\}_{i=1}^{n}$ and $Q^{U} = \{q_{ij}^{U}\}_{i=1}^{n}$ from the interval FAPCM-M (IV.165) as

$$Q^{L} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{3}{5} \\ \frac{2}{5} & \frac{1}{2} & \frac{3}{5} \\ \frac{1}{7} & \frac{1}{5} & \frac{1}{2} \end{pmatrix}, \qquad Q^{U} = \begin{pmatrix} \frac{1}{2} & \frac{3}{5} & \frac{6}{7} \\ \frac{2}{5} & \frac{1}{2} & \frac{4}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{2} \end{pmatrix},$$

and we verify their multiplicative consistency by utilizing the property (II.50). However, we can easily see that neither Q^{L} nor Q^{U} is additively reciprocal, which means that both Q^{L} and Q^{U} are not even APCMs-M according to Definition 10. Therefore, it is nonsensical to verify their "multiplicative consistency".

An analogous drawback appears also when using Definition 63. In addition, two values of each intensity of preference $\overline{q}_{ij}, i, j = 1, 2, 3, i \neq j$ appear in the expression (IV.168) at the same time. For example, the intensities $\frac{1}{2}$ and $\frac{3}{5}$ of preference of object o_1 over object o_2 are considered at the same time, which is nonsensical. $\triangle$

As already mentioned in the discussion following Definition 63, Wang and Li (2012) claim that it is not possible to obtain the equality $\frac{\overline{a}}{\overline{a}} = 1$. Obviously, for $\overline{a} = [a^{L}, a^{U}]$, we obtain

$$\frac{\overline{a}}{\overline{a}} = \frac{[a^{L}, a^{U}]}{[a^{L}, a^{U}]} = \left[\frac{a^{L}}{a^{U}}, \frac{a^{U}}{a^{L}} \right] \neq 1, \qquad \text{unless } a^{L} = a^{U}.$$

Applying the standard fuzzy arithmetic in this case is not appropriate; the constrained fuzzy arithmetic needs to be applied whenever there are any interactions among fuzzy numbers. The presence of interactions

in the expression $\frac{\overline{a}}{\overline{a}}$ is clear since the intervals in the expression represent the same variable. Therefore, $\overline{d} = \frac{\overline{a}}{\overline{a}}$ should be computed by using the constrained fuzzy arithmetic (III.40) as $\overline{d} = [d^L, d^U]$:

$$
\begin{aligned}
d^L \quad &= \min\left\{\tfrac{a_1}{a_2}; a_1 \in [a^L, a^U], a_2 \in [a^L, a^U], a_1 = a_2\right\} = \\
&= \min\left\{\tfrac{a_1}{a_1}; a_1 \in [a^L, a^U]\right\} = 1, \\
d^U \quad &= \max\left\{\tfrac{a_1}{a_2}; a_1 \in [a^L, a^U], a_2 \in [a^L, a^U], a_1 = a_2\right\} = \\
&= \max\left\{\tfrac{a_1}{a_1}; a_1 \in [a^L, a^U]\right\} = 1.
\end{aligned}
$$

Keeping in mind the importance of the additive-reciprocity property of PCs in interval FAPCMs-M, multiplicative consistency needs to be defined accordingly so that it does not violate the additive reciprocity. In the following section, two definitions of multiplicatively consistent trapezoidal FAPCMs-M keeping the additive reciprocity of PCs and invariant under permutation of objects are proposed.

4.3.3.1.2 New fuzzy extension of multiplicative consistency

In this section, Definition 61 of multiplicative weak consistency given by Xu and Chen (2008a) is extended to trapezoidal FAPCMs-M and another definition of multiplicative consistency much stronger than Definition 61 is proposed. Tools for verifying both the multiplicative weak consistency and the multiplicative consistency are provided and some properties of multiplicatively weakly consistent and multiplicatively consistent trapezoidal FAPCMs-M are derived. Both definitions preserve two desired properties - invariance under permutation of objects and additive reciprocity of the related PCs.

Definition 66. *Let* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n, \widetilde{q}_{ij} = (q_{ij}^\alpha, q_{ij}^\beta, q_{ij}^\gamma, q_{ij}^\delta)$, *be a trapezoidal FAPCM-M.* $\widetilde{Q}$ *is said to be multiplicatively weakly consistent if there exists a positive vector* $\underline{u} = (u_1, \ldots, u_n)^T$ *such that*

$$
q_{ij}^\alpha \le \frac{u_i}{u_i + u_j} \le q_{ij}^\delta, \qquad i, j = 1, \ldots, n. \tag{IV.175}
$$

Note that when Definition 66 is applied to interval FAPCMs-M, it is identical to Definition 61 proposed by Xu and Chen (2008a).

Proposition 17. *Let* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n, \widetilde{q}_{ij} = (q_{ij}^\alpha, q_{ij}^\beta, q_{ij}^\gamma, q_{ij}^\delta)$, *be a trapezoidal FAPCM-M.* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n$ *is multiplicatively weakly consistent according to Definition 66 if and only if there exist elements* $q_{ij}^* \in [q_{ij}^\alpha, q_{ij}^\delta], i, j = 1, \ldots, n$, *such that* $Q^* = \{q_{ij}^*\}_{i,j=1}^n$ *is an APCM-M multiplicatively consistent according to Definition 11.*

Proof. First, let $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n, \widetilde{q}_{ij} = (q_{ij}^\alpha, q_{ij}^\beta, q_{ij}^\gamma, q_{ij}^\delta)$, be a trapezoidal FAPCM-M multiplicatively weakly consistent according to Definition 66. Let us denote $q_{ij}^* := \frac{u_i}{u_i + u_j}$. From (IV.175) it follows that $q_{ij}^* \in [q_{ij}^\alpha, q_{ij}^\delta], i, j = 1, \ldots, n$. Further, we have $q_{ii}^* = \frac{u_i}{u_i + u_i} = 0.5$ and $q_{ji}^* = \frac{u_j}{u_j + u_i} = 1 - \frac{u_i}{u_i + u_j} = 1 - q_{ij}^*, i, j = 1, \ldots, n$. From $[q_{ij}^\alpha, q_{ij}^\delta] \subseteq]0, 1[, i, j = 1, \ldots, n$, it follows that $q_{ij}^* \in]0, 1[, i, j = 1, \ldots, n$. Therefore, $Q^* = \{q_{ij}^*\}_{i,j=1}^n$ is a FAPCM-M.

Finally,

$$
\frac{q_{ik}^* \, q_{kj}^*}{q_{ki}^* \, q_{jk}^*} = \frac{\frac{u_i}{u_i + u_k} \frac{u_k}{u_k + u_j}}{\frac{u_k}{u_k + u_i} \frac{u_j}{u_j + u_k}} = \frac{u_i}{u_j} = \frac{\frac{u_i}{u_i + u_j}}{\frac{u_j}{u_j + u_i}} = \frac{q_{ij}^*}{q_{ji}^*}, \qquad i, j, k = 1, \ldots, n,
$$

which means that $Q^* = \{q_{ij}^*\}_{i,j=1}^n$ is multiplicatively consistent according to (II.48).

In the opposite direction, let $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n, \widetilde{q}_{ij} = (q_{ij}^\alpha, q_{ij}^\beta, q_{ij}^\gamma, q_{ij}^\delta)$, be a trapezoidal FAPCM-M and let $Q^* = \{q_{ij}^*\}_{i,j=1}^n, q_{ij}^* \in [q_{ij}^\alpha, q_{ij}^\delta], i, j = 1, \ldots, n$, be an APCM-M multiplicatively consistent according to (II.48). Then, from Proposition 6, it follows that there exists a vector $\underline{u} = (u_1, \ldots, u_n)^T, u_i > 0, i, j = 1, \ldots, n$, such that $q_{ij}^* = \frac{u_i}{u_i + u_j}, i, j = 1, \ldots, n$. Because, $q_{ij}^* \in [q_{ij}^\alpha, q_{ij}^\delta], i, j = 1, \ldots, n$, then (IV.175) holds. $\square$

Remark 30. *According to Proposition 17 and its proof, a trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n, \widetilde{q}_{ij} = (q_{ij}^\alpha, q_{ij}^\beta, q_{ij}^\gamma, q_{ij}^\delta)$, *is multiplicatively weakly consistent if and only if there exists a multiplicatively consistent APCM-M* $Q^* = \{q_{ij}^*\}_{i,j=1}^n$ *such that* $q_{ij}^* \in [q_{ij}^\alpha, q_{ij}^\delta]$. *This consistency condition is quite easy to reach. That is why the multiplicative consistency according to Definition 66 is called weak. Later in this section, a much stronger definition of multiplicative consistency for trapezoidal FAPCMs-M will be given.*

Definition 66 of multiplicative weak consistency satisfies two desirable properties - invariance under permutation of objects and additive reciprocity of PCs in trapezoidal FAPCMs-M.

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Theorem 66. *Definition 66 of multiplicative weak consistency is invariant under permutation of objects in FAPCMs-M.*

Proof. For a multiplicatively weakly consistent trapezoidal FAPCM-M $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = \left(q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta} \right)$, there exists a positive priority vector $\underline{u} = (u_1, \ldots, u_n)^T$, such that the inequality $q_{ij}^{\alpha} \leq \frac{u_i}{u_i + u_j} \leq q_{ij}^{\delta}$ is required to hold for every single PC $\widetilde{q}_{ij}$. By permuting the FAPCM-M $\widetilde{Q}$ to $\widetilde{Q}^{\pi} = P\widetilde{Q}P^T$, the original PC $\widetilde{q}_{ij}$ in the $i-$th row and in the $j-$th column of $\widetilde{Q}$ is moved to the $\pi(i)-$th row and to the $\pi(j)-$th column of the permuted trapezoidal FAPCM-M $\widetilde{Q}^{\pi}$ as $\widetilde{q}_{\pi(i)\pi(j)}^{\pi}$, but still keeping $\widetilde{q}_{ij} = \widetilde{q}_{\pi(i)\pi(j)}^{\pi}$, $i, j = 1, \ldots, n$. Thus, there exists a vector $\underline{u}^{\pi} = (u_1^{\pi}, \ldots, u_n^{\pi})^T$, obtained by permuting the vector $\underline{u}$, i.e. $\underline{u}^{\pi} = P\underline{u}$, with the components satisfying the inequalities $q_{ij}^{\pi\alpha} \leq \frac{u_i}{u_i + u_j} \leq q_{ij}^{\pi\delta}$ for every $i, j = 1, \ldots, n$. $\qquad\square$

Theorem 67. *Definition 66 of multiplicative weak consistency does not violate the additive reciprocity of PCs in trapezoidal FAPCMs-M in the sense that any fixed value $q_{ij} \in [q_{ij}^{\alpha}, q_{ij}^{\delta}], i, j \in \{1, \ldots, n\}$, representing the intensity of preference of object o_i over object o_j is associated with the corresponding values $q_{ji} \in [q_{ji}^{\alpha}, q_{ji}^{\delta}]$ representing the intensity of preference of object o_j over object o_i such that $q_{ji} = 1 - q_{ij}$.*

Proof. The existence of the positive priority vector $\underline{u} = (u_1, \ldots, u_n)^T$ satisfying the inequalities (IV.175) means that there exists an APCM-M $Q = \{q_{ij}\}_{i,j=1}^{n}$, $q_{ij} \in [q_{ij}^{\alpha}, q_{ij}^{\delta}]$, such that $q_{ij} = \frac{u_i}{u_i + u_j}$, $i, j = 1, \ldots, n$. Q is additively reciprocal from the definition, i.e. every PC q_{ij} is associated with the PC q_{ji} such that $q_{ji} = 1 - q_{ij}$. $\qquad\square$

Remark 31. *Note that Theorem 67 does not simply state that a FAPCM-M $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$ multiplicatively weakly consistent according to Definition 66 is additively reciprocal, i.e. $\widetilde{q}_{ji} = 1 - \widetilde{q}_{ij}, i, j = 1, \ldots, n$. Theorem 67 states that only additively reciprocal PCs are involved in Definition 66 of multiplicative weak consistency, which is in accordance with the conception of additive reciprocity discussed on p. 94. For more detailed discussion, see Remark 21.*

The following theorems provide useful tools for verifying multiplicative weak consistency of trapezoidal FAPCMs-M.

Theorem 68. *A trapezoidal FAPCM-M $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = \left(q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta} \right)$, is multiplicatively weakly consistent according to Definition 66 if and only if*

$$\max_{k=1,\ldots,n} \left\{ \frac{q_{ik}^{\alpha} q_{kj}^{\alpha}}{q_{ik}^{\alpha} q_{kj}^{\alpha} + (1 - q_{ik}^{\alpha})(1 - q_{kj}^{\alpha})} \right\} \leq \min_{k=1,\ldots,n} \left\{ \frac{q_{ik}^{\delta} q_{kj}^{\delta}}{q_{ik}^{\delta} q_{kj}^{\delta} + (1 - q_{ik}^{\delta})(1 - q_{kj}^{\delta})} \right\}, \tag{IV.176}$$
$$i, j = 1, \ldots, n.$$

Proof. From the inequalities (IV.175), it follows that $q_{ik}^{\alpha} \leq \frac{u_i}{u_i + u_k} \leq q_{ik}^{\delta}$ and $q_{kj}^{\alpha} \leq \frac{u_k}{u_k + u_j} \leq q_{kj}^{\delta}$. Thus, $\forall k \in \{1, \ldots, n\}$ the following inequalities hold:

$$q_{ik}^{\alpha} q_{kj}^{\alpha} \leq \frac{u_i}{u_i + u_k} \frac{u_k}{u_k + u_j} \leq q_{ik}^{\delta} q_{kj}^{\delta},$$

$$1 - q_{ik}^{\alpha} \geq \frac{u_k}{u_i + u_k} \geq 1 - q_{ik}^{\delta}, \quad 1 - q_{kj}^{\alpha} \geq \frac{u_j}{u_k + u_j} \geq 1 - q_{kj}^{\delta},$$

$$(1 - q_{ik}^{\alpha})(1 - q_{kj}^{\alpha}) \geq \frac{u_k}{u_i + u_k} \frac{u_j}{u_k + u_j} \geq (1 - q_{ik}^{\delta})(1 - q_{kj}^{\delta}).$$

Putting all together we obtain $\forall k \in \{1, \ldots, n\}$

$$\frac{q_{ik}^{\alpha} q_{kj}^{\alpha}}{q_{ik}^{\alpha} q_{kj}^{\alpha} + (1 - q_{ik}^{\alpha})(1 - q_{kj}^{\alpha})} \leq \frac{u_i}{u_i + u_j} \leq \frac{q_{ik}^{\delta} q_{kj}^{\delta}}{q_{ik}^{\delta} q_{kj}^{\delta} + (1 - q_{ik}^{\delta})(1 - q_{kj}^{\delta})},$$

and thus (IV.176) holds.

In the opposite direction, let (IV.176) hold. Then, $\forall i, j, k \in \{1, \ldots, n\}$:

$$q_{ij}^{\alpha} \leq \begin{array}{l} \max\limits_{k=1,\ldots,n} \left\{ \dfrac{q_{ik}^{\alpha} q_{kj}^{\alpha}}{q_{ik}^{\alpha} q_{kj}^{\alpha} + (1 - q_{ik}^{\alpha})(1 - q_{kj}^{\alpha})} \right\} \leq \\[3ex] \min\limits_{k=1,\ldots,n} \left\{ \dfrac{q_{ik}^{\delta} q_{kj}^{\delta}}{q_{ik}^{\delta} q_{kj}^{\delta} + (1 - q_{ik}^{\delta})(1 - q_{kj}^{\delta})} \right\} \leq q_{ij}^{\delta}. \end{array}$$

Thus, $\forall i, j, k \in \{1, \ldots, n\}$:

$$\exists q_{ij}^* \in \left[\max_{k=1,\ldots,n} \left\{ \frac{q_{ik}^\alpha q_{kj}^\alpha}{q_{ik}^\alpha q_{kj}^\alpha + (1-q_{ik}^\alpha)(1-q_{kj}^\alpha)} \right\}, \min_{k=1,\ldots,n} \left\{ \frac{q_{ik}^\delta q_{kj}^\delta}{q_{ik}^\delta q_{kj}^\delta + (1-q_{ik}^\delta)(1-q_{kj}^\delta)} \right\} \right]$$

$$\wedge \exists q_{ik}^* \in [q_{ik}^\alpha, q_{ik}^\delta] \wedge \exists q_{kj}^* \in [q_{kj}^\alpha, q_{kj}^\delta] : q_{ij}^* = q_{ik}^* + q_{kj}^* - 0.5.$$

This means that $Q^* = \left\{ q_{ij}^* \right\}_{i,j=1}^n$ is an APCM-M. Thus, according to Proposition 6, there exists a positive vector $\underline{u} = (u_1, \ldots, u_n)^T$ such that $q_{ij}^* = \frac{u_i}{u_i+u_j}$. Since $q_{ij}^* \in [q_{ij}^\alpha, q_{ij}^\delta], i, j = 1, \ldots, n$, we obtain the inequality (IV.175). □

The following theorem shows that it is sufficient to verify the inequality (IV.176) only for $i, j = 1, \ldots, n, \ i < j$, thus saving half of the computations.

Theorem 69. *A trapezoidal FAPCM-M $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n, \widetilde{q}_{ij} = \left(q_{ij}^\alpha, q_{ij}^\beta, q_{ij}^\gamma, q_{ij}^\delta \right)$, is multiplicatively weakly consistent according to Definition 66 if and only if*

$$\max_{k=1,\ldots,n} \left\{ \frac{q_{ik}^\alpha q_{kj}^\alpha}{q_{ik}^\alpha q_{kj}^\alpha + (1 - q_{ik}^\alpha)(1 - q_{kj}^\alpha)} \right\} \leq \min_{k=1,\ldots,n} \left\{ \frac{q_{ik}^\delta q_{kj}^\delta}{q_{ik}^\delta q_{kj}^\delta + (1 - q_{ik}^\delta)(1 - q_{kj}^\delta)} \right\}, \tag{IV.177}$$
$$i, j = 1, \ldots, n, \ i < j.$$

Proof. It is sufficient to show that the validity of inequalities (IV.177) for $i, j = 1, \ldots, n, i < j$, implies automatically their validity for all $i, j = 1, \ldots, n$, i.e. the validity of (IV.176). The validity of inequalities (IV.176) for $i = j$ is trivial from the definition of trapezoidal FAPCMs-M since

$$\max_{k=1,\ldots,n} \left\{ \frac{q_{ik}^\alpha q_{ki}^\alpha}{q_{ik}^\alpha q_{ki}^\alpha + (1 - q_{ik}^\alpha)(1 - q_{ki}^\alpha)} \right\} = \max_{k=1,\ldots,n} \left\{ \frac{q_{ik}^\alpha q_{ki}^\alpha}{q_{ik}^\alpha q_{ki}^\alpha + q_{ki}^\delta q_{ik}^\delta} \right\} \leq 0.5 \leq$$

$$\min_{k=1,\ldots,n} \left\{ \frac{q_{ik}^\delta q_{ki}^\delta}{q_{ik}^\delta q_{ki}^\delta + q_{ki}^\alpha q_{ik}^\alpha} \right\} = \min_{k=1,\ldots,n} \left\{ \frac{q_{ik}^\delta q_{ki}^\delta}{q_{ik}^\delta q_{ki}^\delta + (1 - q_{ik}^\delta)(1 - q_{ki}^\delta)} \right\}.$$

Further, for $i > j$, by using (IV.177) and the additive-reciprocity properties, we obtain

$$\max_{k=1,\ldots,n} \left\{ \frac{q_{ik}^\alpha q_{kj}^\alpha}{q_{ik}^\alpha q_{kj}^\alpha + (1-q_{ik}^\alpha)(1-q_{kj}^\alpha)} \right\} = \max_{k=1,\ldots,n} \left\{ 1 - \frac{q_{ki}^\delta q_{jk}^\delta}{q_{ki}^\delta q_{jk}^\delta + (1-q_{ki}^\delta)(1-q_{jk}^\delta)} \right\} =$$

$$1 - \min_{k=1,\ldots,n} \left\{ \frac{q_{jk}^\delta q_{ki}^\delta}{q_{jk}^\delta q_{ki}^\delta + (1-q_{jk}^\delta)(1-q_{ki}^\delta)} \right\} \leq 1 - \max_{k=1,\ldots,n} \left\{ \frac{q_{jk}^\alpha q_{ki}^\alpha}{q_{jk}^\alpha q_{ki}^\alpha + (1-q_{jk}^\alpha)(1-q_{ki}^\alpha)} \right\} =$$

$$\min_{k=1,\ldots,n} \left\{ 1 - \frac{q_{jk}^\alpha q_{ki}^\alpha}{q_{jk}^\alpha q_{ki}^\alpha + (1-q_{jk}^\alpha)(1-q_{ki}^\alpha)} \right\} = \min_{k=1,\ldots,n} \left\{ \frac{q_{ik}^\delta q_{kj}^\delta}{q_{ik}^\delta q_{kj}^\delta + (1-q_{ik}^\delta)(1-q_{kj}^\delta)} \right\}.$$

□

Remark 32. *An alternative definition of multiplicative weak consistency to Definition 66 might be formulated as follows.*
Let $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n, \widetilde{q}_{ij} = \left(q_{ij}^\alpha, q_{ij}^\beta, q_{ij}^\gamma, q_{ij}^\delta \right)$, be a trapezoidal FAPCM-M. $\widetilde{Q}$ is said to be multiplicatively weakly consistent if there exists a positive vector $\underline{u} = (u_1, \ldots, u_n)^T$ such that

$$q_{ij}^\beta \leq \frac{u_i}{u_i + u_j} \leq q_{ij}^\gamma, \qquad i, j = 1, \ldots, n. \tag{IV.178}$$

Notice that this definition is stronger than Definition 66. In fact, every trapezoidal FAPCM-M multiplicatively weakly consistent according to this definition is also multiplicatively weakly consistent according to Definition 66 since (IV.178) automatically implies (IV.175). Furthermore, when this definition is applied to interval FAPCMs-M, it is again identical to Definition 61 proposed by Xu and Chen (2008a).
All theorems regarding FAPCMs-M multiplicatively weakly consistent according to Definition 66 formulated above can be easily reformulated for FAPCMs-M multiplicatively weakly consistent according to this definition; it is sufficient to consider q_{ij}^β and q_{ij}^γ instead of q_{ij}^α and q_{ij}^δ, respectively, where appropriate.

In the following definition, a stronger version of mutiplicative consistency for trapezoidal FAPCMs-M is formulated.

Definition 67. *Let* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}, \widetilde{q}_{ij} = \left(q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta} \right)$, *be a trapezoidal FAPCM-M.* $\widetilde{Q}$ *is said to be multiplicatively consistent if for each triplet* $(i, j, k) \subseteq \{1, \ldots, n\}$ *the following holds:*

$$\forall q_{ij} \in \left[q_{ij}^{\alpha}, q_{ij}^{\delta} \right] \exists q_{ik} \in \left[q_{ik}^{\alpha}, q_{ik}^{\delta} \right] \ \wedge \ \exists q_{kj} \in \left[q_{kj}^{\alpha}, q_{kj}^{\delta} \right] : \ q_{ij} = \frac{q_{ik} q_{kj}}{q_{ik} q_{kj} + (1 - q_{ik})(1 - q_{kj})}, \tag{IV.179}$$

$$\forall q_{ij} \in \left[q_{ij}^{\beta}, q_{ij}^{\gamma} \right] \exists q_{ik} \in \left[q_{ik}^{\beta}, q_{ik}^{\gamma} \right] \ \wedge \ \exists q_{kj} \in \left[q_{kj}^{\beta}, q_{kj}^{\gamma} \right] : \ q_{ij} = \frac{q_{ik} q_{kj}}{q_{ik} q_{kj} + (1 - q_{ik})(1 - q_{kj})}. \tag{IV.180}$$

Remark 33. *Definition 67 provides a natural extension of multiplicative consistency from APCMs-M to trapezoidal FAPCMs-M. According to this definition, for any possible value* $q_{ij} \in \widetilde{q}_{ij}, i, j \in \{1, \ldots, n\}$, *there exist possible values* $q_{ik} \in \widetilde{q}_{ik}$ *and* $q_{kj} \in \widetilde{q}_{kj}, k \in \{1, \ldots, n\}$, *such that they satisfy* (II.53)*, which is equivalent to the multiplicative-transitivity property* (II.48) *for APCMs-M. Analogously, for any possible value* $q_{ij} \in Core\, \widetilde{q}_{ij}$, $i, j \in \{1, \ldots, n\}$, *there exist possible values* $q_{ik} \in Core\, \widetilde{q}_{ik}$ *and* $q_{kj} \in Core\, \widetilde{q}_{kj}, k \in \{1, \ldots, n\}$, *such that they satisfy* (II.53)*. Clearly, in comparison to the multiplicative weak consistency given by Definition 66, the multiplicative consistency given by Definition 67 is very strong.*

Note that any of the properties (II.48)–(II.52) *could have been used in Definition 67 instead of the property* (II.53) *as they are all equivalent (see Theorem 4).*

Unlike Definitions 62 and 64 of multiplicatively consistent interval FAPCMs-M proposed by Xia and Xu (2011) and by Wu and Chiclana (2014a), respectively, new Definition 67 is invariant under permutation of objects compared in FAPCMs-M.

Theorem 70. *Definition 67 of multiplicative consistency is invariant under permutation of objects in FAPCMs-M.*

Proof. For a multiplicatively consistent trapezoidal FAPCM-M $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}, \widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, the conditions (IV.179) and (IV.180) are satisfied for every triplet $(i, j, k) \subseteq \{1, \ldots, n\}$. By permuting the FAPCM-M $\widetilde{Q}$ to $\widetilde{Q}^{\pi} = P\widetilde{Q}P^{T}$, the original PC $\widetilde{q}_{ij}$ in the i–th row and in the j–th column of $\widetilde{Q}$ moves to the $\pi(i)$–th row and to the $\pi(j)$–th column of $\widetilde{Q}^{\pi}$ preserving $\widetilde{q}_{\pi(i)\pi(j)}^{\pi} = \widetilde{q}_{ij}$. Thus, by permuting $\widetilde{Q}$, also the validity of the conditions (IV.179) and (IV.180) is preserved, i.e.

$$\forall q_{ij}^{\pi} \in \left[q_{ij}^{\pi\alpha}, q_{ij}^{\pi\delta} \right] \exists q_{ik}^{\pi} \in \left[q_{ik}^{\pi\alpha}, q_{ik}^{\pi\delta} \right] \wedge \exists q_{kj}^{\pi} \in \left[q_{kj}^{\pi\alpha}, q_{kj}^{\pi\delta} \right] : \ q_{ij}^{\pi} = \frac{q_{ik}^{\pi} q_{kj}^{\pi}}{q_{ik}^{\pi} q_{kj}^{\pi} + (1 - q_{ik}^{\pi})(1 - q_{kj}^{\pi})},$$

$$\forall q_{ij}^{\pi} \in \left[q_{ij}^{\pi\beta}, q_{ij}^{\pi\gamma} \right] \exists q_{ik}^{\pi} \in \left[q_{ik}^{\pi\beta}, q_{ik}^{\pi\gamma} \right] \wedge \exists q_{kj}^{\pi} \in \left[q_{kj}^{\pi\beta}, q_{kj}^{\pi\gamma} \right] : \ q_{ij}^{\pi} = \frac{q_{ik}^{\pi} q_{kj}^{\pi}}{q_{ik}^{\pi} q_{kj}^{\pi} + (1 - q_{ik}^{\pi})(1 - q_{kj}^{\pi})},$$

for every triplet $(i, j, k) \subseteq \{1, \ldots, n\}$. Thus, $\widetilde{Q}^{\pi}$ is multiplicatively consistent according to Definition 67. $\square$

Further, unlike Definitions 63 and 65 of multiplicatively consistent interval FAPCMs-M proposed in Wang and Li (2012) and Wu and Chiclana (2014b), respectively, Definition 67 does not violate the additive reciprocity of PCs.

Theorem 71. *Definition 67 of multiplicative consistency does not violate the additive reciprocity of PCs in trapezoidal FAPCMs-M in the sense that any fixed value* $q_{ij} \in [q_{ij}^{\alpha}, q_{ij}^{\delta}], i, j \in \{1, \ldots, n\}$, *representing the intensity of preference of object* o_i *over object* o_j *is associated with the corresponding value* $q_{ji} \in [q_{ji}^{\alpha}, q_{ji}^{\delta}]$ *representing the intensity of preference of object* o_j *over object* o_i *such that* $q_{ji} = 1 - q_{ij}$.

Proof. It is sufficient to show that expressions (IV.179) and (IV.180) do not violate the additive-reciprocity property in the sense that when two particular intensities of preference $q_{ij} \in \widetilde{q}_{ij}$ and $q_{ji} \in \widetilde{q}_{ji}$ on the pair of objects o_i and o_j are considered at the same time in the expressions (IV.179) and (IV.180), then they are such that $q_{ji} = 1 - q_{ij}$.

For a triplet $(i, j, k) \subseteq \{1, \ldots, n\}$, $i \neq j \neq k$, no reciprocals appear in expression

$$q_{ij} = \frac{q_{ik} q_{kj}}{q_{ik} q_{kj} + (1 - q_{ik})(1 - q_{kj})}$$

for any $q_{ij} \in \left[q_{ij}^{\alpha}, q_{ij}^{\delta} \right]$. For $i = j = k$, expression (IV.179) reduces to: $\forall q_{ii} = 0.5 \, \exists q_{ii}^{*} = 0.5 \wedge \exists q_{ii}^{**} = 0.5 : 0.5 = \frac{0.5 \cdot 0.5}{0.5 \cdot 0.5 + 0.5 \cdot 0.5}$, which again does not violate the additive reciprocity. Further, for $i \neq j = k$, expression (IV.179) is as:

$$\forall q_{ij} \in \left[q_{ij}^{\alpha}, q_{ij}^{\delta} \right] \exists q_{ij}^{*} \in \left[q_{ij}^{\alpha}, q_{ij}^{\delta} \right] \wedge \exists q_{jj} = 0.5 : q_{ij} = \frac{q_{ij}^{*} \cdot 0.5}{q_{ij}^{*} \cdot 0.5 + (1 - q_{ij}^{*}) \cdot 0.5}.$$

This means that $q_{ij}^* = q_{ij}$ and, therefore, the additive reciprocity is not violated. For $i = k \neq j$ the proof is analogous. Finally, for $i = j \neq k$, expression (IV.179) is as

$$\forall q_{ii} = 0.5 \, \exists q_{ik} \in \left[q_{ik}^\alpha, q_{ik}^\delta\right] \wedge \exists q_{ki}^* \in \left[q_{ki}^\alpha, q_{ki}^\delta\right] : 0.5 = \frac{q_{ik} q_{ki}^*}{q_{ik} q_{ki}^* + (1 - q_{ik})(1 - q_{ki}^*)}.$$

This means that $q_{ki}^* = 1 - q_{ik}$ and, therefore, the additive reciprocity is preserved.

The proof for the expression (IV.180) is analogous. □

Remark 34. *Similarly to Theorem 67, also Theorem 71 does not simply state that a FAPCM-M $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n$ multiplicatively consistent according to Definition 67 is additively reciprocal since this property automatically follows from Definition 52 of a FAPCM. Theorem 67 states that only additively reciprocal PCs are involved in Definition 67 of multiplicative onsistency, which is in accordance with the conception of additive reciprocity discussed on p. 94. For more detailed discussion, see Remark 21.*

By handling properly the additive-reciprocity property of PCs, Theorem 4 can be easily extended to interval FAPCMs-M as follows.

Theorem 72. *For a trapezoidal FAPCM-M $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n$, $\widetilde{q}_{ij} = \left(q_{ij}^\alpha, q_{ij}^\beta, q_{ij}^\gamma, q_{ij}^\delta\right)$, the following statements are equivalent:*

(i) $\widetilde{Q}$ *is multiplicatively consistent according to Definition 67.*

(ii) For every $i, j, k = 1, \ldots, n$:

$$\forall q_{ij} \in \left[q_{ij}^\alpha, q_{ij}^\delta\right] \exists q_{jk} \in \left[q_{jk}^\alpha, q_{jk}^\delta\right] \wedge \exists q_{ki} \in \left[q_{ki}^\alpha, q_{ki}^\delta\right] : q_{ij} q_{jk} q_{ki} = q_{ik} q_{kj} q_{ji},$$
$$q_{ij} = 1 - q_{ji}, \ q_{jk} = 1 - q_{kj}, \ q_{ki} = 1 - q_{ik}. \quad \text{(IV.181)}$$

$$\forall q_{ij} \in \left[q_{ij}^\beta, q_{ij}^\gamma\right] \exists q_{jk} \in \left[q_{jk}^\beta, q_{jk}^\gamma\right] \wedge \exists q_{ki} \in \left[q_{ki}^\beta, q_{ki}^\gamma\right] : q_{ij} q_{jk} q_{ki} = q_{ik} q_{kj} q_{ji},$$
$$q_{ij} = 1 - q_{ji}, \ q_{jk} = 1 - q_{kj}, \ q_{ki} = 1 - q_{ik}. \quad \text{(IV.182)}$$

(iii) For every $i, j, k = 1, \ldots, n$:

$$\forall q_{ij} \in \left[q_{ij}^\alpha, q_{ij}^\delta\right] \exists q_{jk} \in \left[q_{jk}^\alpha, q_{jk}^\delta\right] \wedge \exists q_{ki} \in \left[q_{ki}^\alpha, q_{ki}^\delta\right] : \frac{q_{ij}}{q_{ji}} \frac{q_{jk}}{q_{kj}} \frac{q_{ki}}{q_{ik}} = 1,$$
$$q_{ij} = 1 - q_{ji}, \ q_{jk} = 1 - q_{kj}, \ q_{ki} = 1 - q_{ik}. \quad \text{(IV.183)}$$

$$\forall q_{ij} \in \left[q_{ij}^\beta, q_{ij}^\gamma\right] \exists q_{jk} \in \left[q_{jk}^\beta, q_{jk}^\gamma\right] \wedge \exists q_{ki} \in \left[q_{ki}^\beta, q_{ki}^\gamma\right] : \frac{q_{ij}}{q_{ji}} \frac{q_{jk}}{q_{kj}} \frac{q_{ki}}{q_{ik}} = 1,$$
$$q_{ij} = 1 - q_{ji}, \ q_{jk} = 1 - q_{kj}, \ q_{ki} = 1 - q_{ik}. \quad \text{(IV.184)}$$

(iv) For every $i, j, k = 1, \ldots, n$:

$$\forall q_{ij} \in \left[q_{ij}^\alpha, q_{ij}^\delta\right] \exists q_{jk} \in \left[q_{jk}^\alpha, q_{jk}^\delta\right] \wedge \exists q_{ki} \in \left[q_{ki}^\alpha, q_{ki}^\delta\right] : \frac{q_{ij}}{q_{ji}} \frac{q_{jk}}{q_{kj}} \frac{q_{ki}}{q_{ik}} = \frac{q_{ik}}{q_{ki}} \frac{q_{kj}}{q_{jk}} \frac{q_{ji}}{q_{ij}},$$
$$q_{ij} = 1 - q_{ji}, \ q_{jk} = 1 - q_{kj}, \ q_{ki} = 1 - q_{ik}. \quad \text{(IV.185)}$$

$$\forall q_{ij} \in \left[q_{ij}^\beta, q_{ij}^\gamma\right] \exists q_{jk} \in \left[q_{jk}^\beta, q_{jk}^\gamma\right] \wedge \exists q_{ki} \in \left[q_{ki}^\beta, q_{ki}^\gamma\right] : \frac{q_{ij}}{q_{ji}} \frac{q_{jk}}{q_{kj}} \frac{q_{ki}}{q_{ik}} = \frac{q_{ik}}{q_{ki}} \frac{q_{kj}}{q_{jk}} \frac{q_{ji}}{q_{ij}},$$
$$q_{ij} = 1 - q_{ji}, \ q_{jk} = 1 - q_{kj}, \ q_{ki} = 1 - q_{ik}. \quad \text{(IV.186)}$$

(v) For every $i, j, k = 1, \ldots, n$:

$$\forall q_{ij} \in \left[q_{ij}^\alpha, q_{ij}^\delta\right] \exists q_{jk} \in \left[q_{jk}^\alpha, q_{jk}^\delta\right] \wedge \exists q_{ki} \in \left[q_{ki}^\alpha, q_{ki}^\delta\right] : \frac{q_{ij}}{q_{ji}} = \frac{q_{ik}}{q_{ki}} \frac{q_{kj}}{q_{jk}},$$
$$q_{ij} = 1 - q_{ji}, \ q_{jk} = 1 - q_{kj}, \ q_{ki} = 1 - q_{ik}. \quad \text{(IV.187)}$$

$$\forall q_{ij} \in \left[q_{ij}^\beta, q_{ij}^\gamma\right] \exists q_{jk} \in \left[q_{jk}^\beta, q_{jk}^\gamma\right] \wedge \exists q_{ki} \in \left[q_{ki}^\beta, q_{ki}^\gamma\right] : \frac{q_{ij}}{q_{ji}} = \frac{q_{ik}}{q_{ki}} \frac{q_{kj}}{q_{jk}},$$
$$q_{ij} = 1 - q_{ji}, \ q_{jk} = 1 - q_{kj}, \ q_{ki} = 1 - q_{ik}. \quad \text{(IV.188)}$$

Proof. From the additive-reciprocity property $\widetilde{q}_{ij} = 1 - \widetilde{q}_{ji}, i,j = 1,\ldots,n$, it follows that $\forall q_{ij} \in \left[q_{ij}^\alpha, q_{ij}^\delta\right] \exists q_{ji} \in \left[q_{ji}^\alpha, q_{ji}^\delta\right] : q_{ji} = 1 - q_{ij}$ and $\forall q_{ij} \in \left[q_{ij}^\beta, q_{ij}^\gamma\right] \exists q_{ji} \in \left[q_{ji}^\beta, q_{ji}^\gamma\right] : q_{ji} = 1 - q_{ij}$.

Obviously, the statements (ii), (iii) and (v) are equivalent since the expressions in the statements are obtained by a simple reordering. Therefore, it is sufficent to prove the equivalence of statements (i) and (ii), and the equivalence of statements (iii) and (iv).

(a) First, let us show that the statements (i) and (ii) are equivalent. Because of the reciprocity property, (IV.181) can be equivalently written as

$$\forall q_{ij} \in \left[q_{ij}^\alpha, q_{ij}^\delta\right] \exists q_{ik} \in \left[q_{ik}^\alpha, q_{ik}^\delta\right] \wedge \exists q_{kj} \in \left[q_{kj}^\alpha, q_{kj}^\delta\right] :$$

$$q_{ij}(1 - q_{kj})(1 - q_{ik}) = q_{ik}q_{kj}(1 - q_{ij}). \quad \text{(IV.189)}$$

The expression $q_{ij}(1 - q_{kj})(1 - q_{ik}) = q_{ik}q_{kj}(1 - q_{ij})$ can be further rewritten as

$$q_{ij} = \frac{q_{ik}q_{kj}}{q_{ik}q_{kj} + (1 - q_{ik})(1 - q_{kj})}.$$

This means that the statement (IV.181) is equivalent to (IV.179). Analogously, the equivalence of (IV.182) and (IV.180) is proved.

(b) Now, let us show that the statements (iii) and (iv) are equivalent. The expression $\frac{q_{ij}}{q_{ji}}\frac{q_{jk}}{q_{kj}}\frac{q_{ki}}{q_{ik}} = \frac{q_{ik}}{q_{ki}}\frac{q_{kj}}{q_{jk}}\frac{q_{ji}}{q_{ij}}$ in the statement (IV.185) can be equivalently written as $\left(\frac{q_{ij}}{q_{ji}}\frac{q_{jk}}{q_{kj}}\frac{q_{ki}}{q_{ik}}\right)^2 = 1$, which is equivalent to $\frac{q_{ij}}{q_{ji}}\frac{q_{jk}}{q_{kj}}\frac{q_{ki}}{q_{ik}} = 1$. Therefore, the statement (IV.185) is equivalent to the statement (IV.183). Analogously, the equivalence of (IV.186) and (IV.184) is proved.

$\square$

The following theorems give us useful tools for verifying the multiplicative consistency of trapezoidal FAPCMs-M.

Theorem 73. *A trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n, \widetilde{q}_{ij} = \left(q_{ij}^\alpha, q_{ij}^\beta, q_{ij}^\gamma, q_{ij}^\delta\right)$, *is multiplicatively consistent according to Definition 67 if and only if the inequalities*

$$q_{ij}^\alpha \geq \frac{q_{ik}^\alpha q_{kj}^\alpha}{q_{ik}^\alpha q_{kj}^\alpha + (1 - q_{ik}^\alpha)(1 - q_{kj}^\alpha)}, \quad q_{ij}^\delta \leq \frac{q_{ik}^\delta q_{kj}^\delta}{q_{ik}^\delta q_{kj}^\delta + (1 - q_{ik}^\delta)(1 - q_{kj}^\delta)}, \quad \text{(IV.190)}$$

$$q_{ij}^\beta \geq \frac{q_{ik}^\beta q_{kj}^\beta}{q_{ik}^\beta q_{kj}^\beta + (1 - q_{ik}^\beta)(1 - q_{kj}^\beta)}, \quad q_{ij}^\gamma \leq \frac{q_{ik}^\gamma q_{kj}^\gamma}{q_{ik}^\gamma q_{kj}^\gamma + (1 - q_{ik}^\gamma)(1 - q_{kj}^\gamma)}, \quad \text{(IV.191)}$$

hold for every $i,j,k = 1,\ldots,n, \ i < j, \ k \neq i,j$.

Proof. It is sufficient to demonstrate the equivalence of the expression (IV.190) and (IV.179). The demonstration of the equivalence of (IV.191) and (IV.180) is analogous.

First, let us demonstrate that when the inequalities (IV.190) hold for every $i,j,k = 1,\ldots,n, \ i < j, \ k \neq i,j$, then they hold for every $i,j,k = 1,\ldots,n$. The inequalities (IV.190) are always satisfied for $i,j,k = 1,\ldots,n$ such that

(i) $i = j \neq k$:

$$\frac{q_{ik}^\alpha q_{ki}^\alpha}{q_{ik}^\alpha q_{ki}^\alpha + (1 - q_{ik}^\alpha)(1 - q_{ki}^\alpha)} = \frac{q_{ik}^\alpha q_{ki}^\alpha}{q_{ik}^\alpha q_{ki}^\alpha + q_{ik}^\delta q_{ki}^\delta} \leq \frac{q_{ik}^\alpha q_{ki}^\alpha}{2q_{ik}^\alpha q_{ki}^\alpha} = 0.5 = q_{ii}^\alpha$$

$$\frac{q_{ik}^\delta q_{ki}^\delta}{q_{ik}^\delta q_{ki}^\delta + (1 - q_{ik}^\delta)(1 - q_{ki}^\delta)} = \frac{q_{ik}^\delta q_{ki}^\delta}{q_{ik}^\delta q_{ki}^\delta + q_{ik}^\alpha q_{ki}^\alpha} \geq \frac{q_{ik}^\delta q_{ki}^\delta}{2q_{ik}^\delta q_{ki}^\delta} = 0.5 = q_{ii}^\delta$$

(ii) $i \neq j = k$:

$$\frac{q_{ij}^\alpha q_{jj}^\alpha}{q_{ij}^\alpha q_{jj}^\alpha + (1 - q_{ij}^\alpha)(1 - q_{jj}^\alpha)} = \frac{0.5q_{ij}^\alpha}{0.5q_{ij}^\alpha + 0.5(1 - q_{ij}^\alpha)} = q_{ij}^\alpha$$

$$\frac{q_{ij}^\delta q_{jj}^\delta}{q_{ij}^\delta q_{jj}^\delta + (1 - q_{ij}^\delta)(1 - q_{jj}^\delta)} = \frac{0.5q_{ij}^\delta}{0.5q_{ij}^\delta + 0.5(1 - q_{ij}^\delta)} = q_{ij}^\delta$$

(iii) $j \neq k = i$:

$$\frac{q_{ii}^{\alpha} q_{ij}^{\alpha}}{q_{ii}^{\alpha} q_{ij}^{\alpha} + (1 - q_{ii}^{\alpha})(1 - q_{ij}^{\alpha})} = \frac{0.5 q_{ij}^{\alpha}}{0.5 q_{ij}^{\alpha} + 0.5(1 - q_{ij}^{\alpha})} = q_{ij}^{\alpha}$$

$$\frac{q_{ij}^{\delta} q_{jj}^{\delta}}{q_{ii}^{\delta} q_{ij}^{\delta} + (1 - q_{ii}^{\delta})(1 - q_{ij}^{\delta})} = \frac{0.5 q_{ij}^{\delta}}{0.5 q_{ij}^{\delta} + 0.5(1 - q_{ij}^{\delta})} = q_{ij}^{\delta}$$

(iv) $i = j = k$:

$$\frac{q_{ii}^{\alpha} q_{ii}^{\alpha}}{q_{ii}^{\alpha} q_{ii}^{\alpha} + (1 - q_{ii}^{\alpha})(1 - q_{ii}^{\alpha})} = \frac{0.5 \cdot 0.5}{0.5 \cdot 0.5 + 0.5 \cdot 0.5} = 0.5 = q_{ij}^{\alpha}$$

$$\frac{q_{ii}^{\delta} q_{ii}^{\delta}}{q_{ii}^{\delta} q_{ii}^{\delta} + (1 - q_{ii}^{\delta})(1 - q_{ii}^{\delta})} = \frac{0.5 \cdot 0.5}{0.5 \cdot 0.5 + 0.5 \cdot 0.5} = 0.5 = q_{ij}^{\delta}$$

Further, when the inequalities (IV.190) are satisfied for $i, j, k = 1, \ldots, n$, $i < j$, $k \neq i, j$, then they are satisfied also for $j, i, k = 1, \ldots, n$, $j > i$, $k \neq i, j$,

$$\frac{q_{jk}^{\alpha} q_{ki}^{\alpha}}{q_{jk}^{\alpha} q_{ki}^{\alpha} + (1 - q_{jk}^{\alpha})(1 - q_{ki}^{\alpha})} = \frac{q_{jk}^{\alpha} q_{ki}^{\alpha}}{q_{jk}^{\alpha} q_{ki}^{\alpha} + q_{jk}^{\delta} q_{ki}^{\delta}} = 1 - \frac{q_{ik}^{\delta} q_{kj}^{\delta}}{q_{jk}^{\alpha} q_{ki}^{\alpha} + q_{ik}^{\delta} q_{kj}^{\delta}}$$

$$= 1 - \frac{q_{ik}^{\delta} q_{kj}^{\delta}}{q_{ik}^{\delta} q_{kj}^{\delta} + (1 - q_{ik}^{\delta})(1 - q_{kj}^{\delta})} \leq 1 - q_{ij}^{\delta} = q_{ji}^{\alpha},$$

$$\frac{q_{jk}^{\delta} q_{ki}^{\delta}}{q_{jk}^{\delta} q_{ki}^{\delta} + (1 - q_{jk}^{\delta})(1 - q_{ki}^{\delta})} = \frac{q_{jk}^{\delta} q_{ki}^{\delta}}{q_{jk}^{\delta} q_{ki}^{\delta} + q_{jk}^{\alpha} q_{ki}^{\alpha}} = 1 - \frac{q_{ik}^{\alpha} q_{kj}^{\alpha}}{q_{jk}^{\delta} q_{ki}^{\delta} + q_{ik}^{\alpha} q_{kj}^{\alpha}}$$

$$= 1 - \frac{q_{ik}^{\alpha} q_{kj}^{\alpha}}{q_{ik}^{\alpha} q_{kj}^{\alpha} + (1 - q_{ik}^{\alpha})(1 - q_{kj}^{\alpha})} \geq 1 - q_{ij}^{\alpha} = q_{ji}^{\delta}.$$

To finalize the proof, it is sufficient to show that the inequalities (IV.190) are equivalent to the condition (IV.179) for every $i, j, k = 1, \ldots, n$.

First, let $\widetilde{Q}$ be a trapezoidal FAPCM-M multiplicatively consistent according to Definition 67. Then for $q_{ij} := q_{ij}^{\alpha}$ $\exists q_{ik} \in \left[q_{ik}^{\alpha}, q_{ik}^{\delta}\right] \wedge \exists q_{kj} \in \left[q_{kj}^{\alpha}, q_{kj}^{\delta}\right] : q_{ij}^{\alpha} = \frac{q_{ik} q_{kj}}{q_{ik} q_{kj} + (1 - q_{ik})(1 - q_{kj})}$. Since $q_{ik} \geq q_{ik}^{\alpha}, q_{kj} \geq q_{kj}^{\alpha}$, and since $\frac{q_{ik} q_{kj}}{q_{ik} q_{kj} + (1 - q_{ik})(1 - q_{kj})}$ is increasing in both variables on interval $]0, 1[$, then clearly (IV.190) is valid. Analogously, for $q_{ij} := q_{ij}^{\delta}$ $\exists q_{ik} \in \left[q_{ik}^{\alpha}, q_{ik}^{\delta}\right] \wedge \exists q_{kj} \in \left[q_{kj}^{\alpha}, q_{kj}^{\delta}\right] : q_{ij}^{\delta} = \frac{q_{ik} q_{kj}}{q_{ik} q_{kj} + (1 - q_{ik})(1 - q_{kj})}$. Since $q_{ik} \leq q_{ik}^{\delta}, q_{kj} \leq q_{kj}^{\delta}$, then clearly (IV.191) is valid.

Second, let the expression (IV.190) be valid for a trapezoidal FAPCM-M $\widetilde{Q}$. Because $\frac{q_{ik} q_{kj}}{q_{ik} q_{kj} + (1 - q_{ik})(1 - q_{kj})}$ is increasing in both variables q_{ik} and q_{kj} on intervals $[q_{ik}^{\alpha}, q_{ik}^{\delta}]$ and $[q_{kj}^{\alpha}, q_{kj}^{\delta}]$, respectively, then from the inequalities (IV.190) we get

$$\forall q_{ij} \in \left[q_{ij}^{\alpha}, q_{ij}^{\delta}\right] : \frac{q_{ik}^{\alpha} q_{kj}^{\alpha}}{q_{ik}^{\alpha} q_{kj}^{\alpha} + (1 - q_{ik}^{\alpha})(1 - q_{kj}^{\alpha})} \leq q_{ij} \leq \frac{q_{ik}^{\delta} q_{kj}^{\delta}}{q_{ik}^{\delta} q_{kj}^{\delta} + (1 - q_{ik}^{\delta})(1 - q_{kj}^{\delta})}$$

and, therefore, (IV.179) is satisfied. $\qquad\square$

Theorem 74. *A trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}, \widetilde{q}_{ij} = \left(q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta}\right)$, *is multiplicatively consistent according to Definition 67 if and only if*

$$q_{ij}^{\alpha} \geq \max_{\substack{k=1,\ldots,n \\ k \neq i,j}} \left\{ \frac{q_{ik}^{\alpha} q_{kj}^{\alpha}}{q_{ik}^{\alpha} q_{kj}^{\alpha} + (1 - q_{ik}^{\alpha})(1 - q_{kj}^{\alpha})} \right\}, \quad q_{ij}^{\delta} \leq \min_{\substack{k=1,\ldots,n \\ k \neq i,j}} \left\{ \frac{q_{ik}^{\delta} q_{kj}^{\delta}}{q_{ik}^{\delta} q_{kj}^{\delta} + (1 - q_{ik}^{\delta})(1 - q_{kj}^{\delta})} \right\}, \qquad \text{(IV.192)}$$

$$q_{ij}^{\beta} \geq \max_{\substack{k=1,\ldots,n \\ k \neq i,j}} \left\{ \frac{q_{ik}^{\beta} q_{kj}^{\beta}}{q_{ik}^{\beta} q_{kj}^{\beta} + (1 - q_{ik}^{\beta})(1 - q_{kj}^{\beta})} \right\}, \quad q_{ij}^{\gamma} \leq \min_{\substack{k=1,\ldots,n \\ k \neq i,j}} \left\{ \frac{q_{ik}^{\gamma} q_{kj}^{\gamma}}{q_{ik}^{\gamma} q_{kj}^{\gamma} + (1 - q_{ik}^{\gamma})(1 - q_{kj}^{\gamma})} \right\}, \qquad \text{(IV.193)}$$

hold for every $i, j = 1, \ldots, n$, $i < j$.

Proof. The inequalities (IV.192) and (IV.193) follow immediately from Theorem 73. $\qquad\square$

Theorem 75. *A trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}, \widetilde{q}_{ij} = \left(q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta}\right)$, *is multiplicatively consistent according to Definition 67 if and only if one of the following conditions holds for every* $i, j, k = 1, \ldots, n$, $i < j$, $k \neq i, j$:

$$q_{ij}^{\alpha} q_{jk}^{\delta} q_{ki}^{\delta} \geq q_{ik}^{\alpha} q_{kj}^{\alpha} q_{ji}^{\delta}, \qquad q_{ij}^{\delta} q_{jk}^{\alpha} q_{ki}^{\alpha} \leq q_{ik}^{\delta} q_{kj}^{\delta} q_{ji}^{\alpha} \qquad\qquad \text{(IV.194)}$$

$$q_{ij}^{\beta} q_{jk}^{\gamma} q_{ki}^{\gamma} \geq q_{ik}^{\beta} q_{kj}^{\beta} q_{ji}^{\gamma}, \qquad q_{ij}^{\gamma} q_{jk}^{\beta} q_{ki}^{\beta} \leq q_{ik}^{\gamma} q_{kj}^{\gamma} q_{ji}^{\beta} \qquad\qquad \text{(IV.195)}$$

(ii)

$$\frac{q_{ij}^{\alpha}}{q_{ji}^{\delta}} \frac{q_{jk}^{\delta}}{q_{kj}^{\alpha}} \frac{q_{ki}^{\delta}}{q_{ik}^{\alpha}} \geq 1, \qquad \frac{q_{ij}^{\delta}}{q_{ji}^{\alpha}} \frac{q_{jk}^{\alpha}}{q_{kj}^{\delta}} \frac{q_{ki}^{\alpha}}{q_{ik}^{\delta}} \leq 1 \qquad\qquad \text{(IV.196)}$$

$$\frac{q_{ij}^{\beta}}{q_{ji}^{\gamma}} \frac{q_{jk}^{\gamma}}{q_{kj}^{\beta}} \frac{q_{ki}^{\gamma}}{q_{ik}^{\beta}} \geq 1, \qquad \frac{q_{ij}^{\gamma}}{q_{ji}^{\beta}} \frac{q_{jk}^{\beta}}{q_{kj}^{\gamma}} \frac{q_{ki}^{\beta}}{q_{ik}^{\gamma}} \leq 1 \qquad\qquad \text{(IV.197)}$$

(iii)

$$\frac{q_{ij}^{\alpha}}{q_{ji}^{\delta}} \frac{q_{jk}^{\delta}}{q_{kj}^{\alpha}} \frac{q_{ki}^{\delta}}{q_{ik}^{\alpha}} \geq \frac{q_{ik}^{\alpha}}{q_{ki}^{\delta}} \frac{q_{kj}^{\alpha}}{q_{jk}^{\delta}} \frac{q_{ji}^{\delta}}{q_{ij}^{\alpha}}, \qquad \frac{q_{ij}^{\delta}}{q_{ji}^{\alpha}} \frac{q_{jk}^{\alpha}}{q_{kj}^{\delta}} \frac{q_{ki}^{\alpha}}{q_{ik}^{\delta}} \leq \frac{q_{ik}^{\delta}}{q_{ki}^{\alpha}} \frac{q_{kj}^{\delta}}{q_{jk}^{\alpha}} \frac{q_{ji}^{\alpha}}{q_{ij}^{\delta}} \qquad \text{(IV.198)}$$

$$\frac{q_{ij}^{\beta}}{q_{ji}^{\gamma}} \frac{q_{jk}^{\gamma}}{q_{kj}^{\beta}} \frac{q_{ki}^{\gamma}}{q_{ik}^{\beta}} \geq \frac{q_{ik}^{\beta}}{q_{ki}^{\gamma}} \frac{q_{kj}^{\beta}}{q_{jk}^{\gamma}} \frac{q_{ji}^{\gamma}}{q_{ij}^{\beta}}, \qquad \frac{q_{ij}^{\gamma}}{q_{ji}^{\beta}} \frac{q_{jk}^{\beta}}{q_{kj}^{\gamma}} \frac{q_{ki}^{\beta}}{q_{ik}^{\gamma}} \leq \frac{q_{ik}^{\gamma}}{q_{ki}^{\beta}} \frac{q_{kj}^{\gamma}}{q_{jk}^{\beta}} \frac{q_{ji}^{\beta}}{q_{ij}^{\gamma}} \qquad \text{(IV.199)}$$

(iv)

$$\frac{q_{ij}^{\alpha}}{q_{ji}^{\delta}} \geq \frac{q_{ik}^{\alpha}}{q_{ki}^{\delta}} \frac{q_{kj}^{\alpha}}{q_{jk}^{\delta}}, \qquad \frac{q_{ij}^{\delta}}{q_{ji}^{\alpha}} \leq \frac{q_{ik}^{\delta}}{q_{ki}^{\alpha}} \frac{q_{kj}^{\delta}}{q_{jk}^{\alpha}} \qquad\qquad \text{(IV.200)}$$

$$\frac{q_{ij}^{\beta}}{q_{ji}^{\gamma}} \geq \frac{q_{ik}^{\beta}}{q_{ki}^{\gamma}} \frac{q_{kj}^{\beta}}{q_{jk}^{\gamma}}, \qquad \frac{q_{ij}^{\gamma}}{q_{ji}^{\beta}} \leq \frac{q_{ik}^{\gamma}}{q_{ki}^{\beta}} \frac{q_{kj}^{\gamma}}{q_{jk}^{\beta}} \qquad\qquad \text{(IV.201)}$$

Proof. The inequalities (i)–(iv) are obtained directly from the inequalities (IV.190) and (IV.191) by applying the additive-reciprocity properties $q_{pq}^{\alpha} = 1 - q_{qp}^{\delta}, q_{pq}^{\beta} = 1 - q_{qp}^{\gamma}, q_{pq}^{\gamma} = 1 - q_{qp}^{\beta}, q_{pq}^{\delta} = 1 - q_{qp}^{\alpha}, p, q = i, j, k.$ $\square$

Remark 35. *Notice the similarities between the inequality conditions (i)–(iv) in Theorem 75 and the inequalities* (IV.190)*,* (IV.191) *in Theorem 73, and the definitions of multiplicatively consistent interval FAPCMs-M reviewed in Section 4.3.3.1.1. In particular, the inequalities* (IV.194) *are in some sense similar to the multiplicative-transitivity property* (IV.173) *in Wu and Chiclana's Definition 65. Condition* (IV.198) *has a form similar to the condition* (IV.168)*, which was derived from the condition* (IV.167)*. Finally, the inequalities* (IV.190) *are similar to the condition* (IV.172) *in Wu and Chiclana's Definition 64.*

However, there is a significant difference between the definitions reviewed in Section 4.3.3.1.1 and the new Definition 67 and the inequality properties in Theorem 75. Definitions 62–65 were obtained by extending the expressions from Theorem 4 that are mutually equivalent thanks to the additive reciprocity of PCs in APCMs-M. However, because the interval extension of the expressions from Theorem 4 was done by applying the standard interval arithmetic, the additive reciprocity of PCs is not preserved anymore in interval FAPCMs-M. This drawback leads to the fact that Definitions 62–65 are not mutually equivalent. Remember that Wang and Li (2012) even stated that "(IV.169) is not equivalent to (IV.167) as in the case of regular APCMs-M" (Wang and Li (2012), p. 183). This does not hold true for new Definition 67. The constrained fuzzy arithmetic was applied properly to the fuzzy extension of the condition (II.53) *in order to preserve the additive reciprocity of PCs in trapezoidal FAPCMs-M. As a result, it was possible to extend also the conditions from Theorem 4 so that their fuzzy extensions are mutually equivalent, see Theorem 72.*

In the following example, Definition 67 of multiplicative consistency is confronted with Definitions 62–66. In particular, it is demonstrated how the drawbacks regarding the dependence of Definitions 62 and 64 on permutation of objects and violation of the additive-reciprocity property in Definitions 63 and 65 are removed by Definition 67.

Example 57. Let us examine the interval FAPCM-M given by (IV.165). In Example 54, it was demonstrated that Definition 62 is not invariant under permutation of objects since the interval FAPCM-M (IV.165) is judged as multiplictively consistent while its permutation (IV.166) is judged as multiplicatively inconsistent. Analogously, in Example 55, it was demonstrated that Definition 64 is not invariant under permutation of objects since the interval FAPCM-M (IV.165) is judged as multiplictively consistent while its permutation (IV.166) is judged as multiplicatively inconsistent according to the definition.

Now, let us apply Definition 67 to the interval FAPCM-M (IV.165). By using Theorem 73, the interval FAPCM-A (IV.165) is judged multiplicatively consistent since it satisfies the inequalities (IV.190); see Tab. IV.12. Also the permuted interval FAPCM-A (IV.166) satisfies the inequalities (IV.190); see Tab. IV.13. Therefore, it is again

judged as multiplicatively consistent. Moreover, from Theorem 70 it follows that any permutation of the interval FAPCM-M (IV.165) is multiplicatively consistent.

In Example 56, it was shown that Definitions 63 and 65 violate additive reciprocity of PCs in the interval FAPCM-M (IV.165). According to Theorem 71, the additive-reciprocity property is preserved in new Definition 67. This basically means that by taking any value from any interval PC in the interval FAPCM-M (IV.165), there exist values in the remaining interval PCs such that they form a multiplicatively consistent APCM-M. Let us examine the triplet $i = 1, j = 2, k = 3$ of indices and let us consider the value $q_{12} = \frac{3}{5} \in [\frac{1}{2}, \frac{3}{5}]$. Then, according to (IV.179), there exist values $q_{13} \in [\frac{3}{5}, \frac{6}{7}]$ and $q_{32} \in [\frac{1}{5}, \frac{2}{5}]$ such that $\frac{3}{5} = \frac{q_{13}q_{32}}{q_{13}q_{32}+(1-q_{13})(1-q_{32})}$. It is, for example, $q_{13} = \frac{4}{5}, q_{32} = \frac{3}{11}$. The additive reciprocity is clearly not violated. More interestingly, let us consider the triplet $i = 1, j = 1, k = 2$. Then, according to (IV.179), there exist values $q_{12} \in [\frac{1}{2}, \frac{3}{5}]$ and $q_{21} \in [\frac{2}{5}, \frac{1}{2}]$ such that $\frac{1}{2} = \frac{q_{12}q_{21}}{q_{12}q_{21}+(1-q_{12})(1-q_{21})}$. This equality is satisfied by any value $q_{12} \in [\frac{1}{2}, \frac{3}{5}]$ and the corresponding value $q_{21} \in [\frac{1}{2}, \frac{2}{5}]$ such that $q_{21} = 1 - q_{12}$, which again preserves the additive reciprocity.

Notice that the interval FAPCM-M (IV.165) is also multiplicatively weakly consistent according to Definition 66. For example, the priority vector $\underline{u} = (0.5217, 0.3478, 0.1304)^T$ satisfies the condition (IV.175). △

In the following theorem, the relation between the multiplicative consistency and the multiplicative weak consistency given by Definitions 67 and 66, respectively, is formulated.

Theorem 76. *Let $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n, \widetilde{q}_{ij} = \left(q_{ij}^\alpha, q_{ij}^\beta, q_{ij}^\gamma, q_{ij}^\delta\right)$, be a trapezoidal FAPCM-M. If $\widetilde{Q}$ is multiplicatively consistent according to Definition 67, then it is also multiplicatively weakly consistent according to Definition 66.*

Proof. The statement follows immediately from Theorem 74. In particular, the inequality (IV.176) is obtained immediately from the inequalities (IV.192). □

Remark 36. *According to Theorem 76, when a trapezoidal FAPCM-M is multiplicatively consistent according to Definition 67, then it is also automatically multiplicatively weakly consistent according Definition 66. However, this does not hold the other way around. Clearly, the multiplicative weak consistency is much weaker than the multiplicative consistency; it only requires existence of one crisp multiplicatively consistent FAPCM-M obtainable by combining particular elements from the closures of the supports of trapezoidal FAPCM-M. Thus, the set of all trapezoidal FAPCMs-M multiplicatively consistent according to Definition 67 is a proper subset of the set of all trapezoidal FAPCMs-M multiplicatively weakly consistent according to Definition 66.*

Example 58. Let us examine multiplicative consistency and multiplicative weak consistency of the trapezoidal FAPCM-M

$$\widetilde{Q} = \begin{pmatrix} \frac{1}{2} & \left(\frac{8}{15}, \frac{9}{15}, \frac{10}{15}, \frac{11}{15}\right) & \left(\frac{16}{20}, \frac{16}{20}, \frac{17}{20}, \frac{18}{20}\right) & \left(\frac{16}{20}, \frac{17}{20}, \frac{18}{20}, \frac{19}{20}\right) \\ \left(\frac{4}{15}, \frac{5}{15}, \frac{6}{15}, \frac{7}{15}\right) & \frac{1}{2} & \left(\frac{13}{20}, \frac{14}{20}, \frac{15}{20}, \frac{15}{20}\right) & \left(\frac{15}{20}, \frac{16}{20}, \frac{17}{20}, \frac{17}{20}\right) \\ \left(\frac{2}{20}, \frac{3}{20}, \frac{4}{20}, \frac{4}{20}\right) & \left(\frac{5}{20}, \frac{6}{20}, \frac{6}{20}, \frac{7}{20}\right) & \frac{1}{2} & \left(\frac{9}{20}, \frac{10}{20}, \frac{12}{20}, \frac{13}{20}\right) \\ \left(\frac{1}{20}, \frac{2}{20}, \frac{3}{20}, \frac{4}{20}\right) & \left(\frac{3}{20}, \frac{3}{20}, \frac{4}{20}, \frac{5}{20}\right) & \left(\frac{7}{20}, \frac{8}{20}, \frac{10}{20}, \frac{11}{20}\right) & \frac{1}{2} \end{pmatrix}. \tag{IV.202}$$

By verifying the inequalities (IV.190) and (IV.191), we find out that $\widetilde{Q}$ is not multiplicatively consistent; e.g.

$$\frac{q_{12}^\delta q_{24}^\delta}{q_{12}^\delta q_{24}^\delta + (1-q_{12}^\delta)(1-q_{24}^\delta)} = \frac{\frac{11}{15}\frac{17}{20}}{\frac{11}{15}\frac{17}{20} + \frac{4}{15}\frac{3}{20}} = \frac{187}{199} \not\geq q_{14}^\delta = \frac{19}{20}.$$

However, $\widetilde{Q}$ can still be at least multiplicatively weakly consistent. Let us verify that by using Theorem 69. According to Tab. IV.14, the property (IV.177) is satisfied, and thus $\widetilde{Q}$ is multiplicatively weakly consistent.

Table IV.12: Inequality conditions (IV.190) for the interval FAPCM-M (IV.165).

$i < j :$	$q_{ij}^L \geq \frac{q_{ik}^L q_{kj}^L}{q_{ik}^L q_{kj}^L + (1-q_{ik}^L)(1-q_{kj}^L)}$	$q_{ij}^U \leq \frac{q_{ik}^U q_{kj}^U}{q_{ik}^U q_{kj}^U + (1-q_{ik}^U)(1-q_{kj}^U)}$
$1, 2 :$	$\frac{1}{2} \geq \frac{\frac{3}{5}\frac{1}{5}}{\frac{3}{5}\frac{1}{5} + \frac{2}{5}\frac{4}{5}}$	$\frac{3}{5} \leq \frac{\frac{6}{7}\frac{2}{5}}{\frac{6}{7}\frac{2}{5} + \frac{1}{7}\frac{3}{5}}$
$1, 3 :$	$\frac{3}{5} \geq \frac{\frac{1}{2}\frac{3}{5}}{\frac{1}{2}\frac{3}{5} + \frac{1}{2}\frac{2}{5}}$	$\frac{6}{7} \leq \frac{\frac{3}{5}\frac{4}{5}}{\frac{3}{5}\frac{4}{5} + \frac{2}{5}\frac{1}{5}}$
$2, 3 :$	$\frac{3}{5} \geq \frac{\frac{2}{5}\frac{3}{5}}{\frac{2}{5}\frac{3}{5} + \frac{3}{5}\frac{2}{5}}$	$\frac{4}{5} \leq \frac{\frac{1}{2}\frac{6}{7}}{\frac{1}{2}\frac{6}{7} + \frac{1}{2}\frac{1}{7}}$

Table IV.13: Inequality conditions (IV.190) for the permuted interval FAPCM-M (IV.166).

$i < j :$	$q_{ij}^L \geq \dfrac{q_{ik}^L q_{kj}^L}{q_{ik}^L q_{kj}^L + (1-q_{ik}^L)(1-q_{kj}^L)}$	$q_{ij}^U \leq \dfrac{q_{ik}^U q_{kj}^U}{q_{ik}^U q_{kj}^U + (1-q_{ik}^U)(1-q_{kj}^U)}$
$1,2 :$	$\dfrac{1}{7} \geq \dfrac{\frac{1}{5}\frac{2}{5}}{\frac{1}{5}\frac{2}{5} + \frac{4}{5}\frac{3}{5}}$	$\dfrac{2}{5} \leq \dfrac{\frac{2}{5}\frac{1}{2}}{\frac{2}{5}\frac{1}{2} + \frac{3}{5}\frac{1}{2}}$
$1,3 :$	$\dfrac{1}{5} \geq \dfrac{\frac{1}{7}\frac{1}{2}}{\frac{1}{7}\frac{1}{2} + \frac{6}{7}\frac{1}{2}}$	$\dfrac{2}{5} \leq \dfrac{\frac{2}{5}\frac{3}{5}}{\frac{2}{5}\frac{3}{5} + \frac{3}{5}\frac{2}{5}}$
$2,3 :$	$\dfrac{1}{2} \geq \dfrac{\frac{3}{5}\frac{1}{5}}{\frac{3}{5}\frac{1}{5} + \frac{2}{5}\frac{4}{5}}$	$\dfrac{3}{5} \leq \dfrac{\frac{6}{7}\frac{2}{5}}{\frac{6}{7}\frac{2}{5} + \frac{1}{7}\frac{3}{5}}$

Table IV.14: Condition (IV.177) for the trapezoidal FAPCM-M (IV.202).

$i < j :$	$\max\limits_{k=1,\ldots,4}\left\{\dfrac{q_{ik}^\alpha q_{kj}^\alpha}{q_{ik}^\alpha q_{kj}^\alpha + (1-q_{ik}^\alpha)(1-q_{kj}^\alpha)}\right\}$	$\leq$	$\min\limits_{k=1,\ldots,4}\left\{\dfrac{q_{ik}^\delta q_{kj}^\delta}{q_{ik}^\delta q_{kj}^\delta + (1-q_{ik}^\delta)(1-q_{kj}^\delta)}\right\}$
$1,2 :$	$\max\left\{\frac{8}{15}, \frac{8}{15}, \frac{4}{7}, \frac{12}{29}\right\}$	$\leq$	$\min\left\{\frac{11}{15}, \frac{11}{15}, \frac{63}{76}, \frac{19}{20}\right\}$
$1,3 :$	$\max\left\{\frac{16}{20}, \frac{104}{153}, \frac{16}{20}, \frac{28}{41}\right\}$	$\leq$	$\min\left\{\frac{18}{20}, \frac{35}{37}, \frac{18}{20}, \frac{209}{218}\right\}$
$1,4 :$	$\max\left\{\frac{16}{20}, \frac{24}{31}, \frac{36}{47}, \frac{16}{20}\right\}$	$\leq$	$\min\left\{\frac{19}{20}, \frac{187}{199}, \frac{117}{124}, \frac{19}{20}\right\}$
$2,3 :$	$\max\left\{\frac{16}{27}, \frac{13}{20}, \frac{13}{20}, \frac{21}{34}\right\}$	$\leq$	$\min\left\{\frac{63}{71}, \frac{15}{20}, \frac{15}{20}, \frac{187}{214}\right\}$
$2,4 :$	$\max\left\{\frac{16}{27}, \frac{15}{20}, \frac{117}{194}, \frac{15}{20}\right\}$	$\leq$	$\min\left\{\frac{133}{141}, \frac{19}{20}, \frac{39}{46}, \frac{19}{20}\right\}$
$3,4 :$	$\max\left\{\frac{8}{27}, \frac{1}{2}, \frac{9}{20}, \frac{9}{20}\right\}$	$\leq$	$\min\left\{\frac{19}{23}, \frac{119}{158}, \frac{13}{20}, \frac{13}{20}\right\}$

A vector satisfying the inequalities (IV.175) in Definition 66 is, for example, $\underline{u} = \left(\frac{9}{16}, \frac{9}{32}, \frac{3}{32}, \frac{1}{16}\right)^T$ with the corresponding APCM-M Q^* in the form

$$Q^* = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & \frac{6}{7} & \frac{9}{10} \\ \frac{1}{3} & \frac{1}{2} & \frac{3}{4} & \frac{9}{11} \\ \frac{1}{7} & \frac{1}{4} & \frac{1}{2} & \frac{3}{5} \\ \frac{1}{10} & \frac{2}{11} & \frac{2}{5} & \frac{9}{11} \end{pmatrix}. \tag{IV.203}$$

$\triangle$

In the rest of this section, some interesting properties of multiplicatively weakly consistent and multiplicatively consistent trapezoidal FAPCMs-M are examined.

Theorem 77. *Let $\widetilde{Q}$ be a trapezoidal FAPCM-M multiplicatively weakly consistent according to Definition 66. The interval FAPCM-M $\widetilde{Q}^*$ constructed from $\widetilde{Q}$ by eliminating the l-th row and the l-th column, $l \in \{1, \ldots, n\}$, is again multiplicatively weakly consistent.*

Proof. For $\widetilde{Q}$, (IV.175) is valid for every $i, j = 1, \ldots, n$. After eliminating the l-th row and the l-th column of $\widetilde{Q}$, (IV.175) is still valid for every remaining $i, j \in \{1, \ldots, n\} \setminus \{l\}$. Therefore, the new trapezoidal FAPCM-M $\widetilde{Q}^*$ is still multiplicatively weakly consistent. $\square$

The same holds also for multiplicatively consistent trapezoidal FAPCMs-M.

Theorem 78. *Let $\widetilde{Q}$ be a trapezoidal FAPCM-M multiplicatively consistent according to Definition 67. The trapezoidal FAPCM-M $\widetilde{Q}^*$ constructed from $\widetilde{Q}$ by eliminating the l-th row and the l-th column, $l \in \{1, \ldots, n\}$, is again multiplicatively consistent.*

Proof. For $\widetilde{Q}$, (IV.179) is valid for every $i, j, k = 1, \ldots, n$. After eliminating the l-th row and the l-th column of $\widetilde{Q}$, (IV.179) is still valid for every remaining $i, j, k \in \{1, \ldots, n\} \setminus \{l\}$. Therefore, the new interval FAPCM-M $\widetilde{Q}^*$ is multiplicatively consistent. $\square$

Remark 37. *Theorems 77 and 78 are useful in situations when the set of objects compared pairwisely is being reduced. According to the theorems, elimination of one or more objects has no impact on the multiplicative or multiplicative weak consistency of fuzzy PCs of remaining objects.*

The following theorems provide some results regarding aggregation of multiplicatively and multiplicatively weakly consistent trapezoidal FAPCMs-M into one trapezoidal FAPCM-M, which are particularly useful in group decision making.

Theorem 79. *Let* $\widetilde{Q}^1 = \{\widetilde{q}^1_{ij}\}^n_{i,j=1}$, $\widetilde{q}^1_{ij} = \left(q^{1\alpha}_{ij}, q^{1\beta}_{ij}, q^{1\gamma}_{ij}, q^{1\delta}_{ij}\right)$, *and* $\widetilde{Q}^2 = \{\widetilde{q}^2_{ij}\}^n_{i,j=1}$, $\widetilde{q}^2_{ij} = \left(q^{2\alpha}_{ij}, q^{2\beta}_{ij}, q^{2\gamma}_{ij}, q^{2\delta}_{ij}\right)$, *be trapezoidal FAPCMs-M multiplicatively weakly consistent according to Definition 66. Then* $\widetilde{Q} = \{\widetilde{q}_{ij}\}^n_{i,j=1}$, $\widetilde{q}_{ij} = \left(q^{\alpha}_{ij}, q^{\beta}_{ij}, q^{\gamma}_{ij}, q^{\delta}_{ij}\right)$, *such that*

$$q^{\alpha}_{ij} = \frac{\left(\frac{q^{1\alpha}_{ij}}{q^{1\delta}_{ji}}\right)^{\epsilon}\left(\frac{q^{2\alpha}_{ij}}{q^{2\delta}_{ji}}\right)^{1-\epsilon}}{\left(\frac{q^{1\alpha}_{ij}}{q^{1\delta}_{ji}}\right)^{\epsilon}\left(\frac{q^{2\alpha}_{ij}}{q^{2\delta}_{ji}}\right)^{1-\epsilon}+1}, \quad q^{\beta}_{ij} = \frac{\left(\frac{q^{1\beta}_{ij}}{q^{1\gamma}_{ji}}\right)^{\epsilon}\left(\frac{q^{2\beta}_{ij}}{q^{2\gamma}_{ji}}\right)^{1-\epsilon}}{\left(\frac{q^{1\beta}_{ij}}{q^{1\gamma}_{ji}}\right)^{\epsilon}\left(\frac{q^{2\beta}_{ij}}{q^{2\gamma}_{ji}}\right)^{1-\epsilon}+1},$$

$$q^{\gamma}_{ij} = \frac{\left(\frac{q^{1\gamma}_{ij}}{q^{1\beta}_{ji}}\right)^{\epsilon}\left(\frac{q^{2\gamma}_{ij}}{q^{2\beta}_{ji}}\right)^{1-\epsilon}}{\left(\frac{q^{1\gamma}_{ij}}{q^{1\beta}_{ji}}\right)^{\epsilon}\left(\frac{q^{2\gamma}_{ij}}{q^{2\beta}_{ji}}\right)^{1-\epsilon}+1}, \quad q^{\delta}_{ij} = \frac{\left(\frac{q^{1\delta}_{ij}}{q^{1\alpha}_{ji}}\right)^{\epsilon}\left(\frac{q^{2\delta}_{ij}}{q^{2\alpha}_{ji}}\right)^{1-\epsilon}}{\left(\frac{q^{1\delta}_{ij}}{q^{1\alpha}_{ji}}\right)^{\epsilon}\left(\frac{q^{2\delta}_{ij}}{q^{2\alpha}_{ji}}\right)^{1-\epsilon}+1},$$

is a multiplicatively weakly consistent trapezoidal FAPCM-M for any $\epsilon \in [0,1]$.

Proof. First, let us show that $\widetilde{Q}$ is a trapezoidal FAPCM-M. For $i = 1,\ldots,n$, we get

$$q^{\alpha}_{ii} = \frac{\left(\frac{q^{1\alpha}_{ii}}{q^{1\delta}_{ii}}\right)^{\epsilon}\left(\frac{q^{2\alpha}_{ii}}{q^{2\delta}_{ii}}\right)^{1-\epsilon}}{\left(\frac{q^{1\alpha}_{ii}}{q^{1\delta}_{ii}}\right)^{\epsilon}\left(\frac{q^{2\alpha}_{ii}}{q^{2\delta}_{ii}}\right)^{1-\epsilon}+1} = \frac{1^{\epsilon}1^{1-\epsilon}}{1^{\epsilon}1^{1-\epsilon}+1} = 0.5,$$

Similarly, $q^{\beta}_{ii} = 0.5$, $q^{\gamma}_{ii} = 0.5$, $q^{\delta}_{ii} = 0.5$, and thus, $\widetilde{q}_{ii} = 0.5$, $i = 1,\ldots,n$. Further, for $i \neq j$, we have

$$q^{\alpha}_{ij} = \frac{\left(\frac{q^{1\alpha}_{ij}}{q^{1\delta}_{ji}}\right)^{\epsilon}\left(\frac{q^{2\alpha}_{ij}}{q^{2\delta}_{ji}}\right)^{1-\epsilon}}{\left(\frac{q^{1\alpha}_{ij}}{q^{1\delta}_{ji}}\right)^{\epsilon}\left(\frac{q^{2\alpha}_{ij}}{q^{2\delta}_{ji}}\right)^{1-\epsilon}+1} = \frac{(q^{1\alpha}_{ij})^{\epsilon}(q^{2\alpha}_{ij})^{1-\epsilon}}{(q^{1\alpha}_{ij})^{\epsilon}(q^{2\alpha}_{ij})^{1-\epsilon}+(q^{1\delta}_{ji})^{\epsilon}(q^{2\delta}_{ji})^{1-\epsilon}}$$

$$= \frac{1}{\left(\frac{q^{1\delta}_{ji}}{q^{1\alpha}_{ij}}\right)^{\epsilon}\left(\frac{q^{2\delta}_{ji}}{q^{2\alpha}_{ij}}\right)^{1-\epsilon}+1} = 1 - \frac{\left(\frac{q^{1\delta}_{ji}}{q^{1\alpha}_{ij}}\right)^{\epsilon}\left(\frac{q^{2\delta}_{ji}}{q^{2\alpha}_{ij}}\right)^{1-\epsilon}}{\left(\frac{q^{1\delta}_{ji}}{q^{1\alpha}_{ij}}\right)^{\epsilon}\left(\frac{q^{2\delta}_{ji}}{q^{2\alpha}_{ij}}\right)^{1-\epsilon}+1} = 1 - q^{\delta}_{ji},$$

and, analogously, the equalities $q^{\beta}_{ij} = 1 - q^{\gamma}_{ji}$, $q^{\gamma}_{ij} = 1 - q^{\beta}_{ji}$, $q^{\delta}_{ij} = 1 - q^{\alpha}_{ji}$ are derived. Therefore, $\widetilde{q}_{ij} = 1 - \widetilde{q}_{ji}$, $i,j = 1,\ldots,n$. Finally, because $\forall i,j = 1,\ldots,n : \widetilde{q}^1_{ij} \subseteq]0,1[$, $\widetilde{q}^2_{ij} \subseteq]0,1[$, then also $\widetilde{q}_{ij} \subseteq]0,1[$.

Second, let us show that $\widetilde{Q}$ is multiplicatively weakly consistent. It is sufficient to prove inequalities (IV.175). Since (IV.175) is valid for $\widetilde{Q}^1$ and $\widetilde{Q}^2$, there exist non-negative vectors $\underline{u}^1 = \left(u^1_1,\ldots,u^1_n\right)^T$ and $\underline{u}^2 = \left(u^2_1,\ldots,u^2_n\right)^T$ such that $q^{1\alpha}_{ij} \leq \frac{u^1_i}{u^1_i+u^1_j} \leq q^{1\delta}_{ij}$, $q^{2\alpha}_{ij} \leq \frac{u^2_i}{u^2_i+u^2_j} \leq q^{2\delta}_{ij}$, $i,j,k = 1,\ldots,n$. From this, it follows $\forall i,j,k = 1,\ldots,n$:

$$q^{\alpha}_{ij} = \frac{\left(\frac{q^{1\alpha}_{ij}}{q^{1\delta}_{ji}}\right)^{\epsilon}\left(\frac{q^{2\alpha}_{ij}}{q^{2\delta}_{ji}}\right)^{1-\epsilon}}{\left(\frac{q^{1\alpha}_{ij}}{q^{1\delta}_{ji}}\right)^{\epsilon}\left(\frac{q^{2\alpha}_{ij}}{q^{2\delta}_{ji}}\right)^{1-\epsilon}+1} = \frac{\left(\frac{q^{1\alpha}_{ij}}{1-q^{1\alpha}_{ij}}\right)^{\epsilon}\left(\frac{q^{2\alpha}_{ij}}{1-q^{2\alpha}_{ij}}\right)^{1-\epsilon}}{\left(\frac{q^{1\alpha}_{ij}}{1-q^{1\alpha}_{ij}}\right)^{\epsilon}\left(\frac{q^{2\alpha}_{ij}}{1-q^{2\alpha}_{ij}}\right)^{1-\epsilon}+1} \leq$$

$$\frac{\left(\frac{\frac{u^1_i}{u^1_i+u^1_j}}{1-\frac{u^1_i}{u^1_i+u^1_j}}\right)^{\epsilon}\left(\frac{\frac{u^2_i}{u^2_i+u^2_j}}{1-\frac{u^2_i}{u^2_i+u^2_j}}\right)^{1-\epsilon}}{\left(\frac{\frac{u^1_i}{u^1_i+u^1_j}}{1-\frac{u^1_i}{u^1_i+u^1_j}}\right)^{\epsilon}\left(\frac{\frac{u^2_i}{u^2_i+u^2_j}}{1-\frac{u^2_i}{u^2_i+u^2_j}}\right)^{1-\epsilon}+1} = \frac{\left(\frac{u^1_i}{u^1_j}\right)^{\epsilon}\left(\frac{u^2_i}{u^2_j}\right)^{1-\epsilon}}{\left(\frac{u^1_i}{u^1_j}\right)^{\epsilon}\left(\frac{u^2_i}{u^2_j}\right)^{1-\epsilon}+1} =$$

$$\frac{(u^1_i)^{\epsilon}(u^2_i)^{1-\epsilon}}{(u^1_i)^{\epsilon}(u^2_i)^{1-\epsilon}+(u^1_j)^{\epsilon}(u^2_j)^{1-\epsilon}} = \cdots \leq q^{\delta}_{ij}.$$

Thus, by denoting $u_i := (u^1_i)^{\epsilon}(u^2_i)^{1-\epsilon}$, $i = 1,\ldots,n$, we get a non-negative vector $\underline{u} = (u_1,\ldots,u_n)^T$ satisfying the inequalities (IV.175), which means that $\widetilde{Q}$ is multiplicatively weakly consistent. $\square$

Theorem 79 can be further extended to the aggregation of $p \geq 2$ multiplicatively weakly consistent trapezoidal FAPCMs-M as follows.

Theorem 80. *Let* $\widetilde{Q}^\tau = \{\widetilde{q}_{ij}^\tau\}_{i,j=1}^n, \widetilde{q}_{ij}^\tau = \left(q_{ij}^{\tau\alpha}, q_{ij}^{\tau\beta}, q_{ij}^{\tau\gamma}, q_{ij}^{\tau\delta}\right), \tau = 1, \ldots, p,$ *be trapezoidal FAPCMs-M multiplicatively weakly consistent according to Definition 66. Then* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n$ *such that* $\widetilde{q}_{ij} = \left(q_{ij}^\alpha, q_{ij}^\beta, q_{ij}^\gamma, q_{ij}^\delta\right) =$

$$\left(\frac{\prod_{\tau=1}^p \left(\frac{q_{ij}^{\tau\alpha}}{q_{ji}^{\tau\delta}}\right)^{\epsilon_\tau}}{\prod_{\tau=1}^p \left(\frac{q_{ij}^{\tau\alpha}}{q_{ji}^{\tau\delta}}\right)^{\epsilon_\tau} + 1}, \frac{\prod_{\tau=1}^p \left(\frac{q_{ij}^{\tau\beta}}{q_{ji}^{\tau\gamma}}\right)^{\epsilon_\tau}}{\prod_{\tau=1}^p \left(\frac{q_{ij}^{\tau\beta}}{q_{ji}^{\tau\gamma}}\right)^{\epsilon_\tau} + 1}, \frac{\prod_{\tau=1}^p \left(\frac{q_{ij}^{\tau\gamma}}{q_{ji}^{\tau\beta}}\right)^{\epsilon_\tau}}{\prod_{\tau=1}^p \left(\frac{q_{ij}^{\tau\gamma}}{q_{ji}^{\tau\beta}}\right)^{\epsilon_\tau} + 1}, \frac{\prod_{\tau=1}^p \left(\frac{q_{ij}^{\tau\delta}}{q_{ji}^{\tau\alpha}}\right)^{\epsilon_\tau}}{\prod_{\tau=1}^p \left(\frac{q_{ij}^{\tau\delta}}{q_{ji}^{\tau\alpha}}\right)^{\epsilon_\tau} + 1}\right)$$

is a multiplicatively weakly consistent trapezoidal FAPCM-M for any $\epsilon_\tau \in [0, 1], \tau = 1, \ldots, p,$ *with* $\sum_{\tau=1}^p \epsilon_\tau = 1.$

Proof. The proof is analogous to the proof of Theorem 79. $\qquad\square$

Similar theorems are formulated also for multiplicatively consistent trapezoidal FAPCMs-M.

Theorem 81. *Let* $\widetilde{Q}^1 = \{\widetilde{q}_{ij}^1\}_{i,j=1}^n, \widetilde{q}_{ij}^1 = \left(q_{ij}^{1\alpha}, q_{ij}^{1\beta}, q_{ij}^{1\gamma}, q_{ij}^{1\delta}\right),$ *and* $\widetilde{Q}^2 = \{\widetilde{q}_{ij}^2\}_{i,j=1}^n, \widetilde{q}_{ij}^2 = \left(q_{ij}^{2\alpha}, q_{ij}^{2\beta}, q_{ij}^{2\gamma}, q_{ij}^{2\delta}\right),$ *be trapezoidal FAPCMs-M multiplicatively consistent according to Definition 67. Then* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^n, \widetilde{q}_{ij} = \left(q_{ij}^\alpha, q_{ij}^\beta, q_{ij}^\gamma, q_{ij}^\delta\right),$ *such that*

$$q_{ij}^\alpha = \frac{\left(\frac{q_{ij}^{1\alpha}}{q_{ji}^{1\delta}}\right)^\epsilon \left(\frac{q_{ij}^{2\alpha}}{q_{ji}^{2\delta}}\right)^{1-\epsilon}}{\left(\frac{q_{ij}^{1\alpha}}{q_{ji}^{1\delta}}\right)^\epsilon \left(\frac{q_{ij}^{2\alpha}}{q_{ji}^{2\delta}}\right)^{1-\epsilon} + 1}, \quad q_{ij}^\beta = \frac{\left(\frac{q_{ij}^{1\beta}}{q_{ji}^{1\gamma}}\right)^\epsilon \left(\frac{q_{ij}^{2\beta}}{q_{ji}^{2\gamma}}\right)^{1-\epsilon}}{\left(\frac{q_{ij}^{1\beta}}{q_{ji}^{1\gamma}}\right)^\epsilon \left(\frac{q_{ij}^{2\beta}}{q_{ji}^{2\gamma}}\right)^{1-\epsilon} + 1},$$

$$q_{ij}^\gamma = \frac{\left(\frac{q_{ij}^{1\gamma}}{q_{ji}^{1\beta}}\right)^\epsilon \left(\frac{q_{ij}^{2\gamma}}{q_{ji}^{2\beta}}\right)^{1-\epsilon}}{\left(\frac{q_{ij}^{1\gamma}}{q_{ji}^{1\beta}}\right)^\epsilon \left(\frac{q_{ij}^{2\gamma}}{q_{ji}^{2\beta}}\right)^{1-\epsilon} + 1}, \quad q_{ij}^\delta = \frac{\left(\frac{q_{ij}^{1\delta}}{q_{ji}^{1\alpha}}\right)^\epsilon \left(\frac{q_{ij}^{2\delta}}{q_{ji}^{2\alpha}}\right)^{1-\epsilon}}{\left(\frac{q_{ij}^{1\delta}}{q_{ji}^{1\alpha}}\right)^\epsilon \left(\frac{q_{ij}^{2\delta}}{q_{ji}^{2\alpha}}\right)^{1-\epsilon} + 1},$$

is a multiplicatively consistent trapezoidal FAPCM-M for any $\epsilon \in [0, 1].$

Proof. From the first part of the proof of Theorem 79 we know that $\widetilde{Q}$ is a trapezoidal FAPCM-M. Therefore, it is sufficient to show that $\widetilde{Q}$ satisfies the inequalities (IV.190) and (IV.191). Only the inequalities (IV.190) will be proven here; the proof of the inequalities (IV.191) is analogous.

$\widetilde{Q}^1$ and $\widetilde{Q}^2$ satisfy the inequalities (IV.190), i.e.

$$q_{ij}^{1\alpha} \geq \frac{q_{ik}^{1\alpha} q_{kj}^{1\alpha}}{q_{ik}^{1\alpha} q_{kj}^{1\alpha} + (1 - q_{ik}^{1\alpha})(1 - q_{kj}^{1\alpha})}, \qquad q_{ij}^{1\delta} \leq \frac{q_{ik}^{1\delta} q_{kj}^{1\delta}}{q_{ik}^{1\delta} q_{kj}^{1\delta} + (1 - q_{ik}^{1\delta})(1 - q_{kj}^{1\delta})}, \tag{IV.204}$$

$$q_{ij}^{2\alpha} \geq \frac{q_{ik}^{2\alpha} q_{kj}^{2\alpha}}{q_{ik}^{2\alpha} q_{kj}^{2\alpha} + (1 - q_{ik}^{2\alpha})(1 - q_{kj}^{2\alpha})}, \qquad q_{ij}^{2\delta} \leq \frac{q_{ik}^{2\delta} q_{kj}^{2\delta}}{1 q_{ik}^{2\delta} q_{kj}^{2\delta} + (1 - q_{ik}^{2\delta})(1 - q_{kj}^{2\delta})}. \tag{IV.205}$$

By applying the inequalities (IV.204) and (IV.205), we obtain

$$q_{ij}^\alpha = \frac{\left(\frac{q_{ij}^{1\alpha}}{q_{ji}^{1\delta}}\right)^\epsilon \left(\frac{q_{ij}^{2\alpha}}{q_{ji}^{2\delta}}\right)^{1-\epsilon}}{\left(\frac{q_{ij}^{1\alpha}}{q_{ji}^{1\delta}}\right)^\epsilon \left(\frac{q_{ij}^{2\alpha}}{q_{ji}^{2\delta}}\right)^{1-\epsilon} + 1} \geq \frac{\left(\frac{\frac{q_{ik}^{1\alpha} q_{kj}^{1\alpha}}{q_{ik}^{1\alpha} q_{kj}^{1\alpha} + q_{ki}^{1\delta} q_{jk}^{1\delta}}}{1 - \frac{q_{ik}^{1\alpha} q_{kj}^{1\alpha}}{q_{ik}^{1\alpha} q_{kj}^{1\alpha} + q_{ki}^{1\delta} q_{jk}^{1\delta}}}\right)^\epsilon \left(\frac{\frac{q_{ik}^{2\alpha} q_{kj}^{2\alpha}}{q_{ik}^{2\alpha} q_{kj}^{2\alpha} + q_{ki}^{2\delta} q_{jk}^{2\delta}}}{\frac{q_{ik}^{2\delta} q_{kj}^{2\delta}}{q_{ik}^{2\alpha} q_{kj}^{2\alpha} + q_{ki}^{2\delta} q_{jk}^{2\delta}}}\right)^{1-\epsilon}}{\left(\frac{q_{ik}^{1\alpha} q_{kj}^{1\alpha}}{q_{jk}^{1\delta} q_{ki}^{1\delta}}\right)^\epsilon \left(\frac{q_{ik}^{2\alpha} q_{kj}^{2\alpha}}{q_{jk}^{2\delta} q_{ki}^{2\delta}}\right)^{1-\epsilon} + 1}$$

$$= \frac{\left[\frac{\left(\frac{q_{ik}^{1\alpha}}{q_{ki}^{1\delta}}\right)^\epsilon \left(\frac{q_{ik}^{2\alpha}}{q_{ki}^{2\delta}}\right)^{1-\epsilon}}{\left(\frac{q_{ik}^{1\alpha}}{q_{ki}^{1\delta}}\right)^\epsilon \left(\frac{q_{ik}^{2\alpha}}{q_{ki}^{2\delta}}\right)^{1-\epsilon} + 1}\right]\left[\frac{\left(\frac{q_{kj}^{1\alpha}}{q_{jk}^{1\delta}}\right)^\epsilon \left(\frac{q_{kj}^{2\alpha}}{q_{jk}^{2\delta}}\right)^{1-\epsilon}}{\left(\frac{q_{kj}^{1\alpha}}{q_{jk}^{1\delta}}\right)^\epsilon \left(\frac{q_{kj}^{2\alpha}}{q_{jk}^{2\delta}}\right)^{1-\epsilon} + 1}\right]}{\left[\frac{\left(\frac{q_{ik}^{1\alpha}}{q_{ki}^{1\delta}}\right)^\epsilon \left(\frac{q_{ik}^{2\alpha}}{q_{ki}^{2\delta}}\right)^{1-\epsilon}}{\left(\frac{q_{ik}^{1\alpha}}{q_{ki}^{1\delta}}\right)^\epsilon \left(\frac{q_{ik}^{2\alpha}}{q_{ki}^{2\delta}}\right)^{1-\epsilon} + 1}\right]\left[\frac{\left(\frac{q_{kj}^{1\alpha}}{q_{jk}^{1\delta}}\right)^\epsilon \left(\frac{q_{kj}^{2\alpha}}{q_{jk}^{2\delta}}\right)^{1-\epsilon}}{\left(\frac{q_{kj}^{1\alpha}}{q_{jk}^{1\delta}}\right)^\epsilon \left(\frac{q_{kj}^{2\alpha}}{q_{jk}^{2\delta}}\right)^{1-\epsilon} + 1}\right] + \frac{1}{\left[\left(\frac{q_{ik}^{1\alpha}}{q_{ki}^{1\delta}}\right)^\epsilon \left(\frac{q_{ik}^{2\alpha}}{q_{ki}^{2\delta}}\right)^{1-\epsilon} + 1\right]\left[\left(\frac{q_{kj}^{1\alpha}}{q_{jk}^{1\delta}}\right)^\epsilon \left(\frac{q_{kj}^{2\alpha}}{q_{jk}^{2\delta}}\right)^{1-\epsilon} + 1\right]}}$$

$$= \frac{q_{ik}^\alpha q_{kj}^\alpha}{q_{ik}^\alpha q_{kj}^\alpha + \frac{(q_{ki}^{1\delta})^\epsilon (q_{ki}^{2\delta})^{1-\epsilon}}{(q_{ki}^{1\delta})^\epsilon (q_{ki}^{2\delta})^{1-\epsilon} + (q_{ik}^{1\alpha})^\epsilon (q_{ik}^{2\alpha})^{1-\epsilon}} \frac{(q_{jk}^{1\delta})^\epsilon (q_{jk}^{2\delta})^{1-\epsilon}}{(q_{jk}^{1\delta})^\epsilon (q_{jk}^{2\delta})^{1-\epsilon} + (q_{kj}^{1\alpha})^\epsilon (q_{kj}^{2\alpha})^{1-\epsilon}}}$$

$$= \frac{q_{ik}^\alpha q_{kj}^\alpha}{q_{ik}^\alpha q_{kj}^\alpha + \frac{\left(\frac{q_{ki}^{1\delta}}{q_{ik}^{1\alpha}}\right)^\epsilon \left(\frac{q_{ki}^{2\delta}}{q_{ik}^{2\alpha}}\right)^{1-\epsilon}}{\left(\frac{q_{ki}^{1\delta}}{q_{ik}^{1\alpha}}\right)^\epsilon \left(\frac{q_{ki}^{2\delta}}{q_{ik}^{2\alpha}}\right)^{1-\epsilon} + 1} \frac{\left(\frac{q_{jk}^{1\delta}}{q_{kj}^{1\alpha}}\right)^\epsilon \left(\frac{q_{jk}^{2\delta}}{q_{kj}^{2\alpha}}\right)^{1-\epsilon}}{\left(\frac{q_{jk}^{1\delta}}{q_{kj}^{1\alpha}}\right)^\epsilon \left(\frac{q_{jk}^{2\delta}}{q_{kj}^{2\alpha}}\right)^{1-\epsilon} + 1}} = \frac{q_{ik}^\alpha q_{kj}^\alpha}{q_{ik}^\alpha q_{kj}^\alpha + (1 - q_{ik}^\alpha)(1 - q_{kj}^\alpha)}.$$

Analogously, the inequality $q_{ij}^{\delta} \leq \frac{q_{ik}^{\delta}q_{kj}^{\delta}}{q_{ik}^{\delta}q_{kj}^{\delta}+(1-q_{ik}^{\delta})(1-q_{kj}^{\delta})}$ is proved. $\qquad \square$

Theorem 81 can be further extended to the aggregation of $p \geq 2$ multiplicatively consistent interval FAPCMs-M as follows.

Theorem 82. *Let* $\widetilde{Q}^{\tau} = \{\widetilde{q}_{ij}^{\tau}\}_{i,j=1}^{n}, \widetilde{q}_{ij}^{\tau} = \left(q_{ij}^{\tau\alpha}, q_{ij}^{\tau\beta}, q_{ij}^{\tau\gamma}, q_{ij}^{\tau\delta}\right), \tau = 1, \ldots, p,$ *be trapezoidal FAPCMs-M multiplicatively consistent according to Definition 67. Then* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$ *such that* $\widetilde{q}_{ij} = \left(q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta}\right) =$

$$\left(\frac{\prod_{\tau=1}^{p}\left(\frac{q_{ij}^{\tau\alpha}}{q_{ji}^{\tau\delta}}\right)^{\epsilon_{\tau}}}{\prod_{\tau=1}^{p}\left(\frac{q_{ij}^{\tau\alpha}}{q_{ji}^{\tau\delta}}\right)^{\epsilon_{\tau}}+1}, \frac{\prod_{\tau=1}^{p}\left(\frac{q_{ij}^{\tau\beta}}{q_{ji}^{\tau\gamma}}\right)^{\epsilon_{\tau}}}{\prod_{\tau=1}^{p}\left(\frac{q_{ij}^{\tau\beta}}{q_{ji}^{\tau\gamma}}\right)^{\epsilon_{\tau}}+1}, \frac{\prod_{\tau=1}^{p}\left(\frac{q_{ij}^{\tau\gamma}}{q_{ji}^{\tau\beta}}\right)^{\epsilon_{\tau}}}{\prod_{\tau=1}^{p}\left(\frac{q_{ij}^{\tau\gamma}}{q_{ji}^{\tau\beta}}\right)^{\epsilon_{\tau}}+1}, \frac{\prod_{\tau=1}^{p}\left(\frac{q_{ij}^{\tau\delta}}{q_{ji}^{\tau\alpha}}\right)^{\epsilon_{\tau}}}{\prod_{\tau=1}^{p}\left(\frac{q_{ij}^{\tau\delta}}{q_{ji}^{\tau\alpha}}\right)^{\epsilon_{\tau}}+1} \right)$$

is a multiplicatively consistent trapezoidal FAPCM-M for any $\epsilon_{\tau} \in [0,1], \tau = 1, \ldots, p,$ *with* $\sum_{\tau=1}^{p} \epsilon_{\tau} = 1.$

Proof. The proof is analogous to the proof of Theorem 81. $\qquad \square$

4.3.3.2 Deriving priorities from FAPCMs-M

In this section, the focus is put on methods for obtaining fuzzy priorities of objects from FAPCMs-M. The notation $\widetilde{\underline{u}} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)^T, \widetilde{u}_i = (u_i^{\alpha}, u_i^{\beta}, u_i^{\gamma}, u_i^{\delta}), i = 1, \ldots, n,$ will be used hereafter to represent exclusively a fuzzy priority vector associated with a FAPCM-M.

Various methods have been proposed to derive interval priorities of objects from interval FAPCMs-M. These methods are mostly based on linear programming models rather than on interval arithmetic. Xu and Chen (2008a), for example, proposed linear programming models for obtaining interval priorities of objects from interval FAPCMs-M. The models are based on satisfying the inequalities (IV.163) in the case when the interval FAPCM-M is multiplicatively consistent according to Definition 61 or on satisfying a relaxed version of the inequalities (IV.163) with additional deviation variables in the case when the interval FAPCM-M is not multiplicatively consistent. Genç et al. (2010) showed that in the case of multiplicative consistency, the interval priority vector can be calculated from the multiplicatively consistent interval FAPCM-M directly without the need to solve the linear programming models. Very similar linear programming models for obtaining interval priorities from interval FAPCMs-M based on Tanino's characterization were also proposed by Wang and Li (2012).

As far as I am aware, the only approach for obtaining interval priorities from interval FAPCMs-M not based on linear programming models is the approach presented by Xia and Xu (2011). Xia and Xu (2011) derived formulas for obtaining interval priorities from interval FAPCMs-M based on the extension of the formula (II.62). This approach is reviewed at the beginning of this section and it is shown that this approach is not invariant under permutation of objects. Afterwards, new formulas for obtaining fuzzy priorities from FAPCMs-M are proposed and their properties are discussed. In particular, it is proved that the new formulas preserve the two desired properties - invariance under permutation and additive reciprocity of PCs.

Xia and Xu (2011) proposed an extension of the formula (II.62) to interval FAPCMs-M. For an interval FAPCM-M $\overline{Q} = \{\overline{q}_{ij}\}_{i,j=1}^{n}, \overline{q}_{ij} = [q_{ij}^{L}, q_{ij}^{U}]$, they constructed two APCMs-M $C = \{c_{ij}\}_{i,j=1}^{n}$ and $D = \{d_{ij}\}_{i,j=1}^{n}$ by applying (IV.164). Afterwards, they derived priorities $u_i(C)$ and $u_i(D), i = 1, \ldots, n,$ of objects from these APCMs-M C and D, respectively, by using the formula (II.62). The interval priorities $\overline{u}_i = [u_i^{L}, u_i^{U}], i = 1, \ldots, n,$ are then determined as

$$u_i^{L} = \min\left\{u_i(C), u_i(D)\right\}, \qquad u_i^{U} = \max\left\{u_i(C), u_i(D)\right\}. \qquad \text{(IV.206)}$$

However, this method, similarly to Definition 62 of multiplicative consistency for interval FAPCMs-M proposed by Xia and Xu (2011) and reviewed already in Section 4.3.3.1.1, is not invariant under permutation of objects. This drawback is illustrated on the following example.

Example 59. Let us consider the interval FAPCM-M $\overline{Q}$ of three objects $o_1, o_2,$ and o_3 given by (IV.165). The interval priorities of the objects obtained by the formulas (IV.206) are in the form

$$\overline{u}_1 = [1.1447, 2.0801], \ \overline{u}_2 = [1.1447, 1.3867], \ \overline{u}_3 = [0.3467, 0.7631].$$

Now, let us consider the corresponding permuted interval FAPCM-M $\overline{Q}^{\pi}$ given as (IV.166). The interval priorities of objects obtained from the permuted interval FAPCM-M $\overline{Q}^{\pi}$ by the formulas (IV.206) are in the form

$$\overline{u}_{\pi(1)}^{\pi} = [1.3104, 1.8171], \ \overline{u}_{\pi(2)}^{\pi} = [1.0000, 1.5874], \ \overline{u}_{\pi(3)}^{\pi} = [0.3467, 0.7631].$$

As we can see, $\overline{u}_1 \neq \overline{u}_{\pi(1)}^{\pi}$ and $\overline{u}_2 \neq \overline{u}_{\pi(2)}^{\pi}$. $\qquad \triangle$

Since the method for deriving interval priorities from interval FAPCMs-M proposed by Xia and Xu (2011) is not invariant under permutation of objects, it is not suitable for deriving interval priorities. It is indispensable to obtain the interval priorities from interval FAPCMs-M in such a way that they do not change under the permutation of objects in the interval FAPCMs-M.

The formula (II.62) for obtaining non-normalized priorities from APCMs-M has to be again extended to FAPCMs-M by properly applying constrained fuzzy arithmetic. For a trapezoidal FAPCM-M $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, the non-normalized fuzzy priorities $\widetilde{u}_i = (u_i^{\alpha}, u_i^{\beta}, u_i^{\gamma}, u_i^{\delta})$, $i = 1, \ldots, n$, should be obtained by applying (III.45) as:

$$u_i^{\alpha} = \min \left\{ \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{q_{ji}}}; \begin{array}{l} q_{ij} \in \left[q_{ij}^{\alpha}, q_{ij}^{\delta}\right], \\ q_{ji} = 1 - q_{ij}, \\ j = 1, \ldots, n \end{array} \right\}, \tag{IV.207}$$

$$u_i^{\beta} = \min \left\{ \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{q_{ji}}}; \begin{array}{l} q_{ij} \in \left[q_{ij}^{\beta}, q_{ij}^{\gamma}\right], \\ q_{ji} = 1 - q_{ij}, \\ j = 1, \ldots, n \end{array} \right\}, \tag{IV.208}$$

$$u_i^{\gamma} = \max \left\{ \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{q_{ji}}}; \begin{array}{l} q_{ij} \in \left[q_{ij}^{\beta}, q_{ij}^{\gamma}\right], \\ q_{ji} = 1 - q_{ij}, \\ j = 1, \ldots, n \end{array} \right\}, \tag{IV.209}$$

$$u_i^{\delta} = \max \left\{ \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{q_{ji}}}; \begin{array}{l} q_{ij} \in \left[q_{ij}^{\alpha}, q_{ij}^{\delta}\right], \\ q_{ji} = 1 - q_{ij}, \\ j = 1, \ldots, n \end{array} \right\}. \tag{IV.210}$$

Unlike in the formulas (IV.143)–(IV.146), the additive-reciprocity constraints in the formulas (IV.207)–(IV.210) are indispensable since with every PC q_{ij} also the reciprocal PC q_{ji} appears in the optimized function. Nevertheless, also in this case the optima of the optimization problems (IV.207)–(IV.210) can be determined easily:

$$u_i^{\alpha} = \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\alpha}}{q_{ji}^{\delta}}} = \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\alpha}}{1 - q_{ij}^{\alpha}}}, \tag{IV.211}$$

$$u_i^{\beta} = \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\beta}}{q_{ji}^{\gamma}}} = \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\beta}}{1 - q_{ij}^{\beta}}}, \tag{IV.212}$$

$$u_i^{\gamma} = \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\gamma}}{q_{ji}^{\beta}}} = \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\gamma}}{1 - q_{ij}^{\gamma}}}, \tag{IV.213}$$

$$u_i^{\delta} = \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\delta}}{q_{ji}^{\alpha}}} = \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\delta}}{1 - q_{ij}^{\delta}}}. \tag{IV.214}$$

Note that the formulas (IV.211)–(IV.214) could be obtained also by simply applying simplified standard fuzzy arithmetic (III.36) to the fuzzy extension of the formula (II.62) since both constrained and standard fuzzy arithmetic give the same results in this particular case. As it will be shown later, such simplification is not possible when extending the formula (II.63) to FAPCMs-M.

Example 60. Let us consider the interval FAPCM-M $\overline{Q}$ given by (IV.165). The interval priorities of objects obtained by formulas (IV.211)–(IV.214) are

$$\overline{u}_1 = [1.1447, 2.0801], \quad \overline{u}_2 = [1, 1.15874], \quad \overline{u}_3 = [0.3467, 0.7631].$$

The same interval priorities are obtained also from the permuted interval FAPCM-M $\overline{Q}^{\pi}$ given as (IV.166), i.e. $\overline{u}_{\pi(1)}^{\pi} = \overline{u}_1, \overline{u}_{\pi(2)}^{\pi} = \overline{u}_2, \overline{u}_{\pi(3)}^{\pi} = \overline{u}_3$. Compare the resulting interval priorities with the interval priorities in Example 59 obtained by applying the method proposed by Xia and Xu (2011). △

Analogously as in the previous sections, formula (II.63) for obtaining normalized priorities from APCMs-M needs to be extended to FAPCMs-M by using the constrained fuzzy arithmetic. Only in this way the preservation of the additive reciprocity of PCs and of invariance under permutation of objects can be guaranteed.

By applying constrained fuzzy arithmetic (III.45) to the fuzzy extension of the formula (II.63) for obtaining normalized priorities of objects, the fuzzy priorities $\widetilde{u}_{Ci} = \left(u_{Ci}^{\alpha}, u_{Ci}^{\beta}, u_{Ci}^{\gamma}, u_{Ci}^{\delta} \right)$, $i = 1, \ldots, n$, (the lower index C stands for the applied concept of constrained fuzzy arithmetic) are obtained from a FAPCM-M $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, in this form:

$$u_{Ci}^{\alpha} = \min \left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{q_{ji}}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} \frac{q_{kj}}{q_{jk}}}} ; \begin{array}{l} q_{rs} \in \left[q_{rs}^{\alpha}, q_{rs}^{\delta} \right], \\ q_{sr} = 1 - q_{rs}, \\ r, s = 1, \ldots, n \end{array} \right\}, \tag{IV.215}$$

$$u_{Ci}^{\beta} = \min \left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{q_{ji}}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} \frac{q_{kj}}{q_{jk}}}} ; \begin{array}{l} q_{rs} \in \left[q_{rs}^{\beta}, q_{rs}^{\gamma} \right], \\ q_{sr} = 1 - q_{rs}, \\ r, s = 1, \ldots, n \end{array} \right\}, \tag{IV.216}$$

$$u_{Ci}^{\gamma} = \max \left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{q_{ji}}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} \frac{q_{kj}}{q_{jk}}}} ; \begin{array}{l} q_{rs} \in \left[q_{rs}^{\beta}, q_{rs}^{\gamma} \right], \\ q_{sr} = 1 - q_{rs}, \\ r, s = 1, \ldots, n \end{array} \right\}, \tag{IV.217}$$

$$u_{Ci}^{\delta} = \max \left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{q_{ji}}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} \frac{q_{kj}}{q_{jk}}}} ; \begin{array}{l} q_{rs} \in \left[q_{rs}^{\alpha}, q_{rs}^{\delta} \right], \\ q_{sr} = 1 - q_{rs}, \\ r, s = 1, \ldots, n \end{array} \right\}. \tag{IV.218}$$

Theorem 83. *The fuzzy priorities $\widetilde{u}_{Ci} = (u_{Ci}^{\alpha}, u_{Ci}^{\beta}, u_{Ci}^{\gamma}, u_{Ci}^{\delta})$, $i = 1, \ldots, n$, obtained from a FAPCM-M $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$ by the formulas* (IV.215)–(IV.218) *are normalized.*

Proof. It is sufficient to prove that the fuzzy priorities $\widetilde{u}_{Ci}, i = 1, \ldots, n$, obtained by the formulas (IV.215)–(IV.218) satisfy the inequalities (III.13). From the formula (IV.215), it follows that u_{Ci}^{α} was obtained by applying the formula (II.63) to one particular APCM-M $Q^{\alpha i} = \{q_{rs}\}_{r,s=1}^{n}$, $q_{rs} \in [q_{rs}^{\alpha}, q_{rs}^{\delta}], r, s = 1, \ldots, n$. Let us denote $u_k^{\alpha i}$ the priorities of objects $o_k, k \neq i$, obtainable by the formula (II.63) from the same APCM-M $Q^{\alpha i}$. Obviously, $u_{Ci}^{\alpha} + \sum_{k=1 \atop k \neq i}^{n} u_k^{\alpha i} = 1$, and $u_k^{\alpha i} \in [u_{Ck}^{\alpha}, u_{Ck}^{\delta}], k \neq i$. From this, it follows that $u_{Ci}^{\alpha} + \sum_{k=1 \atop k \neq i}^{n} u_{Ck}^{\delta} \geq 1$. The remaining inequalities in (III.13) are proved analogously. $\qquad\square$

Remark 38. *According to Theorem 83, the fuzzy priorities $\widetilde{u}_{Ci}, i = 1, \ldots, n$, obtained from a FAPCM-M by the formulas* (IV.215)–(IV.218) *are normalized in the sense of Definition 29. Notice that the normality of the fuzzy priorities was again reached naturally by just properly applying constrained fuzzy arithmetic to the fuzzy extension of the formula* (II.63) *for obtaining normalized priorities from an APCM-M, similarly as in the case of the fuzzy extension of the EVM and the GMM; no forced normalization was needed.*

Theorem 84. *The method for obtaining the normalized fuzzy priorities of objects from FAPCMs-M by using the formulas* (IV.215)–(IV.218) *is invariant under permutation of objects in FAPCMs-M.*

Proof. It is sufficient to show that for a given object $o_i, i \in \{1, \ldots, n\}$, its priority $\widetilde{u}_{Ci}$ obtained by the formulas (IV.215)–(IV.218) does not change under permutation of objects in a FAPCM-M $\widetilde{Q}$.

From the invariance of the formula (II.63) reviewed in Section 2.3.3.3, it follows that the priority u_i of object o_i determined by the formula (II.63) from the given APCM-M Q does not change under any permutation $Q^{\pi} = PQP^T$ of Q, it is just permuted accordingly. This means that the priority u_i obtained from Q is equal to the corresponding priority $u_{\pi(i)}^{\pi}$ obtained from Q^{π}.

Therefore, neither the minimum u_{Ci}^{α} nor the maximum u_{Ci}^{δ} of the priority u_i of object o_i obtained by (II.63) over all APCMs-M obtainable from the closures of the supports of the trapezoidal fuzzy numbers in the trapezoidal FAPCM-M $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, change. Analogously, also the minimum u_{Ci}^{β} and the

maximum u_{Ci}^{γ} of the priority u_i obtained by (II.63) over all APCMs-M obtainable from the cores of the trapezoidal fuzzy numbers in the trapezoidal FAPCM-M $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$ do not change. Therefore, the fuzzy priority $\widetilde{u}_{Ci} = (u_{Ci}^{\alpha}, u_{Ci}^{\beta}, u_{Ci}^{\gamma}, u_{Ci}^{\delta})$ obtained by the formulas (IV.215)–(IV.218) does not change under permutation (it is only permuted accordingly), which concludes the proof. $\qquad\square$

The optimization problems solved in (IV.215)–(IV.218) have $n^2 - n$ variables and $\frac{n^2-n}{2}$ additive-reciprocity constraints (the number of variables and additive-reciprocity constraints gets reduced when crisp numbers are present above and below the main diagonal of the FAPCM-M). Thus, the computational complexity of the optimization problems increases rapidly with an increasing dimension n. However, the following theorem shows that the optimization problems (IV.215)–(IV.218) can be simplified significantly. First, the additive-reciprocity constraints can be incorporated into the objective functions. Second, when u_{Ci}^{α} is computed, the variables q_{ij}, $j = 1, \ldots, n$, can be fixed as the lower boundary values of the trapezoidal fuzzy numbers in the i-th row of the FAPCM, i.e. as $q_{ij} := q_{ij}^{\alpha}$. Analogously, also for the representing values $u_{Ci}^{\beta}, u_{Ci}^{\gamma}$, and u_{Ci}^{δ}. In this way, the number of variables is reduced from $n^2 - n$ to $\frac{n^2-n}{2} - (n-1)$.

Theorem 85. *Let* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = \left(q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta}\right)$, *be a trapezoidal FAPCM-M. The optimization problems* (IV.215)–(IV.218) *can be simplified for* $i = 1, \ldots, n$ *in the following way:*

$$
u_{Ci}^{\alpha} = \frac{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\alpha}}{1-q_{ij}^{\alpha}}}}{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\alpha}}{1-q_{ij}^{\alpha}}} + \max\left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1-q_{ik}^{\alpha}}{q_{ik}^{\alpha}} \prod_{\substack{l=1 \\ l \neq i}}^{k-1} \frac{1-q_{lk}}{q_{lk}} \prod_{\substack{l=k+1 \\ l \neq i}}^{n} \frac{q_{kl}}{1-q_{kl}}} \ ; \ \begin{array}{l} q_{rs} \in \left[q_{rs}^{\alpha}, q_{rs}^{\delta}\right], \\ r = 1, \ldots, n-1, \\ s = r+1, \ldots, n, \\ r,s \neq i \end{array} \right\}}, \tag{IV.219}
$$

$$
u_{Ci}^{\beta} = \frac{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\beta}}{1-q_{ij}^{\beta}}}}{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\beta}}{1-q_{ij}^{\beta}}} + \max\left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1-q_{ik}^{\beta}}{q_{ik}^{\beta}} \prod_{\substack{l=1 \\ l \neq i}}^{k-1} \frac{1-q_{lk}}{q_{lk}} \prod_{\substack{l=k+1 \\ l \neq i}}^{n} \frac{q_{kl}}{1-q_{kl}}} \ ; \ \begin{array}{l} q_{rs} \in \left[q_{rs}^{\beta}, q_{rs}^{\gamma}\right], \\ r = 1, \ldots, n-1, \\ s = r+1, \ldots, n, \\ r,s \neq i \end{array} \right\}}, \tag{IV.220}
$$

$$
u_{Ci}^{\gamma} = \frac{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\gamma}}{1-q_{ij}^{\gamma}}}}{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\gamma}}{1-q_{ij}^{\gamma}}} + \min\left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1-q_{ik}^{\gamma}}{q_{ik}^{\gamma}} \prod_{\substack{l=1 \\ l \neq i}}^{k-1} \frac{1-q_{lk}}{q_{lk}} \prod_{\substack{l=k+1 \\ l \neq i}}^{n} \frac{q_{kl}}{1-q_{kl}}} \ ; \ \begin{array}{l} q_{rs} \in \left[q_{rs}^{\beta}, q_{rs}^{\gamma}\right], \\ r = 1, \ldots, n-1, \\ s = r+1, \ldots, n, \\ r,s \neq i \end{array} \right\}}, \tag{IV.221}
$$

$$
u_{Ci}^{\delta} = \frac{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\delta}}{1-q_{ij}^{\delta}}}}{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\delta}}{1-q_{ij}^{\delta}}} + \min\left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1-q_{ik}^{\delta}}{q_{ik}^{\delta}} \prod_{\substack{l=1 \\ l \neq i}}^{k-1} \frac{1-q_{lk}}{q_{lk}} \prod_{\substack{l=k+1 \\ l \neq i}}^{n} \frac{q_{kl}}{1-q_{kl}}} \ ; \ \begin{array}{l} q_{rs} \in \left[q_{rs}^{\alpha}, q_{rs}^{\delta}\right], \\ r = 1, \ldots, n-1, \\ s = r+1, \ldots, n, \\ r,s \neq i \end{array} \right\}}. \tag{IV.222}
$$

Proof. First, let us show that the formulas (IV.215) and (IV.219) are identical. For any $i \in \{1, \ldots, n\}$, the formula (IV.215) can be written in the following way:

$$
u_{Ci}^{\alpha} = \min\left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{1-q_{ij}}}}{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{1-q_{ij}}} + \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1-q_{ik}}{q_{ik}} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{q_{kj}}{1-q_{kj}}}} \ ; \ \begin{array}{l} q_{rs} \in \left[q_{rs}^{\alpha}, q_{rs}^{\delta}\right], \\ q_{sr} = 1 - q_{rs}, \\ r,s = 1, \ldots, n \end{array} \right\}.
$$

Let us denote $\quad x_i := \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{1-q_{ij}}}, \quad$ and $\quad y_i := \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1-q_{ik}}{q_{ik}} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{q_{kj}}{1-q_{kj}}}.$

Obviously, $x_i > 0$ for $q_{is} \in \left[q_{is}^{\alpha}, q_{is}^{\delta}\right]$, $s = 1, \ldots, n$, and $y_i > 0$ for $q_{rs} \in \left[q_{rs}^{\alpha}, q_{rs}^{\delta}\right]$, $q_{sr} = 1 - q_{rs}$, $r, s = 1, \ldots, n$. Further, x_i is increasing in all variables $q_{is}, s \neq i$:

$$\frac{\partial x_i}{\partial q_{is}} = \left[\prod_{\substack{j=1 \\ j \neq s}}^{n} \left(\frac{q_{ij}}{1-q_{ij}}\right)^{\frac{1}{n}}\right] \frac{1}{n} \left(\frac{q_{is}}{1-q_{is}}\right)^{\frac{1-n}{n}} \frac{1}{(1-q_{is})^2} > 0, \quad \begin{array}{l} q_{ik} \in [q_{ik}^{\alpha}, q_{ik}^{\delta}], \\ k = 1, \ldots, n, \end{array}$$

and y_i is decreasing in variables $q_{is}, s \neq i$:

$$\frac{\partial y_i}{\partial q_{is}} = \left[\prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{q_{sj}}{1-q_{sj}}\right)^{\frac{1}{n}}\right] \cdot \frac{1}{n} \cdot \left(\frac{1-q_{is}}{q_{is}}\right)^{\frac{1-n}{n}} \cdot \frac{-1}{q_{is}^2} < 0, \quad \begin{array}{l} q_{kj} \in [q_{kj}^{\alpha}, q_{kj}^{\delta}], \\ k, j = 1, \ldots, n. \end{array}$$

Further, let us denote $f_i := \frac{x_i}{x_i+y_i}$. Then $\frac{\partial f_i}{\partial x_i} = \frac{y_i}{(x_i+y_i)^2} > 0$, and $\frac{\partial f_i}{\partial y_i} = \frac{-x_i}{(x_i+y_i)^2} < 0$. Hence, f_i is an increasing function of x_i and a decreasing function of y_i. It means that for minimizing the function f_i, we have to minimize x_i and maximize y_i. Since the function x_i is increasing in all the variables, we obtain

$$x_i' := \min \left\{x_i ~;~ q_{ij} \in \left[q_{ij}^{\alpha}, q_{ij}^{\delta}\right], ~ j = 1, \ldots, n\right\} = \sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}^{\alpha}}{1-q_{ij}^{\alpha}}}.$$

Since the function y_i is decreasing in the variables $q_{i1}, \ldots, q_{in}$, we obtain

$$y_i' := \max \left\{ y_i ~;~ \begin{array}{l} q_{rs} \in \left[q_{rs}^{\alpha}, q_{rs}^{\delta}\right], q_{sr} = 1 - q_{rs}, \\ r, s = 1, \ldots, n \end{array} \right\} =$$

$$\left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \sqrt[n]{\frac{1-q_{ik}^{\alpha}}{q_{ik}^{\alpha}} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{q_{kj}}{1-q_{kj}}} ~;~ \begin{array}{l} q_{rs} \in \left[q_{rs}^{\alpha}, q_{rs}^{\delta}\right], q_{sr} = 1 - q_{rs}, \\ r, s = 1, \ldots, n \end{array} \right\}$$

Finally, thanks to the additive reciprocity of $\widetilde{Q}$, we can also replace all the elements q_{sr}, $r, s = 1, \ldots, n$, $r < s$, i.e. the elements below the main diagonal, by the reciprocals $1 - q_{rs}$ of the corresponding elements q_{rs} above the main diagonal. By that we obtain formula (IV.219).

Analogously, it can be demonstrated that (IV.216) is equivalent to (IV.220), (IV.217) is equivalent to (IV.221), and (IV.218) is equivalent (IV.222). $\qquad\square$

Example 61. Let us consider the trapezoidal FAPCM-M

$$\widetilde{Q} = \begin{pmatrix} \frac{1}{2} & \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}\right) & \left(\frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}\right) & \left(\frac{6}{9}, \frac{7}{9}, \frac{7.5}{9}, \frac{8}{9}\right) \\ \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}\right) & \frac{1}{2} & \left(\frac{1}{2}, \frac{1}{2}, \frac{3}{5}, \frac{3}{5}\right) & \left(\frac{4.5}{7}, \frac{5}{7}, \frac{5.5}{7}, \frac{6}{7}\right) \\ \left(\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}\right) & \left(\frac{2}{5}, \frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right) & \frac{1}{2} & \left(\frac{4}{6}, \frac{4.5}{6}, \frac{5}{6}, \frac{5}{6}\right) \\ \left(\frac{1}{9}, \frac{1.5}{9}, \frac{2}{9}, \frac{3}{9}\right) & \left(\frac{1}{7}, \frac{1.5}{7}, \frac{2}{7}, \frac{2.5}{7}\right) & \left(\frac{1}{6}, \frac{1}{6}, \frac{1.5}{6}, \frac{2}{6}\right) & 1 \end{pmatrix}. \tag{IV.223}$$

The fuzzy priorities of objects obtained from this FAPCM-M by the formulas (IV.219)–(IV.222) are given as

$$\begin{aligned} \widetilde{u}_1 &= (0.2096, 0.3358, 0.4251, 0.6036), \\ \widetilde{u}_2 &= (0.1774, 0.2666, 0.3378, 0.4493), \\ \widetilde{u}_3 &= (0.1161, 0.1894, 0.2914, 0.3455), \\ \widetilde{u}_4 &= (0.0438, 0.0666, 0.1001, 0.1472). \end{aligned} \tag{IV.224}$$

Let us now examine in detail how the upper boundary value $u_2^{\delta} = 0.4493$ of the fuzzy priority $\widetilde{u}_2$ was obtained. By applying the formula (IV.222) for the fixed $i = 2$, the optimum 0.4493 was obtained from an

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additively reciprocal matrix, in particular from the APCM-M

$$
Q^* = \begin{pmatrix}
\frac{1}{2} & \frac{1}{3} & 0.5359 & \frac{6}{9} \\
\frac{2}{3} & \frac{1}{2} & \frac{3}{5} & \frac{6}{7} \\
0.4641 & \frac{2}{5} & \frac{1}{2} & \frac{4}{6} \\
\frac{3}{9} & \frac{1}{7} & \frac{2}{6} & \frac{1}{2}
\end{pmatrix}.
\tag{IV.225}
$$

The elements of this APCM-M clearly belong to the closures of the supports of the respective trapezoidal fuzzy numbers in the trapezoidal FAPCM-M (IV.223). In the same way, it could be shown that all representing values of all four fuzzy priorities were obtained from APCMs-M by the formulas (IV.215)–(IV.218). $\triangle$

4.4 Transformations between FMPCMs and FAPCMs

In Section 2.4, transformations between MPCMs, APCMs-A, and APCMs-M, and between the related consistency conditions and the priority vectors obtainable from these PCMs were reviewed. In this section it will be proved that also FMPCMs, FAPCMs-A, and FAPCMs-M and the related methods proposed in Sections 4.2 and 4.3 are equivalent.

4.4.1 Transformations between FMPCMs and FAPCMs-A

In this section, transformation between FMPCMs and FAPCMs-A and between the related methods proposed in Sections 4.2 and 4.3.2, respectively, are examined.

Theorem 86. *A trapezoidal FMPCM* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$, $\widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, *can be transformed into a trapezoidal FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$, $\widetilde{r}_{ij} = (r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta})$, *by transformation formulas*

$$
\begin{aligned}
r_{ij}^{\alpha} &= \frac{1}{2}(1 + \log_9 m_{ij}^{\alpha}), & r_{ij}^{\beta} &= \frac{1}{2}(1 + \log_9 m_{ij}^{\beta}), \\
r_{ij}^{\gamma} &= \frac{1}{2}(1 + \log_9 m_{ij}^{\gamma}), & r_{ij}^{\delta} &= \frac{1}{2}(1 + \log_9 m_{ij}^{\delta}).
\end{aligned}
\tag{IV.226}
$$

Proof. From the transformation formula (II.64) for transforming a MPCM into an APCM-A it is obvious that $\widetilde{r}_{ij} \in [0, 1]$ and $\widetilde{r}_{ii} = 0.5$. It remains to show that $\widetilde{R}$ is additively reciprocal, i.e. $r_{ij}^{\alpha} = 1 - r_{ji}^{\delta}, r_{ij}^{\beta} = 1 - r_{ji}^{\gamma}, i, j = 1, \ldots, n$. Clearly

$$
r_{ij}^{\alpha} = \frac{1}{2}(1 + \log_9 m_{ij}^{\alpha}) = \frac{1}{2}(1 + \log_9 \frac{1}{m_{ji}^{\delta}}) = 1 - \frac{1}{2}(1 + \log_9 m_{ji}^{\delta}) = 1 - r_{ji}^{\delta}.
$$

Analogously, the validity of $r_{ij}^{\beta} = 1 - r_{ji}^{\gamma}$ is proved. $\square$

Corollary 7. *A trapezoidal FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$, $\widetilde{r}_{ij} = (r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta})$, *can be transformed into a trapezoidal FMPCM* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$, $\widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, *by transformation formulas*

$$
\begin{aligned}
m_{ij}^{\alpha} &= 9^{2r_{ij}^{\alpha}-1}, & m_{ij}^{\beta} &= 9^{2r_{ij}^{\beta}-1}, \\
m_{ij}^{\gamma} &= 9^{2r_{ij}^{\gamma}-1}, & m_{ij}^{\delta} &= 9^{2r_{ij}^{\delta}-1}.
\end{aligned}
\tag{IV.227}
$$

Remark 39. *The validity of Corollary 7 follows immediately from Theorem 86 by utilizing properties of an inverse function. Note that this form of representing the results is used in the whole section. This means that the transformation of a particular property is formulated in a theorem and proved only in one direction. Afterwards, each such theorem is followed by a corollary showing the transformation of the property in the opposite direction without providing the proof.*

In the following, it is proved that the transformation formulas (IV.226) and (IV.227) transform the multiplicative weak consistency into the additive weak consistency and vice versa.

Theorem 87. *Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$, $\widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, *be a trapezoidal FMPCM multiplicatively weakly consistent according to Definition 50. Then the trapezoidal FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$, $\widetilde{r}_{ij} = (r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta})$, *obtained from* $\widetilde{M}$ *by the transformations* (IV.226) *is additively weakly consistent according to Definition 59.*

Proof. It is sufficient to show that when the inequalities (IV.16) are valid for a FMPCM M, then the inequalities (IV.125) are valid for the FAPCM-A $\widetilde{R}$ obtained from $\widetilde{M}$ by the transformations (IV.226).

$$\max_{k=1,\ldots,n}\left\{r^{\alpha}_{ik}+r^{\alpha}_{kj}-0.5\right\}=\max_{k=1,\ldots,n}\left\{\frac{1}{2}\left(1+\log_9 m^{\alpha}_{ik}\right)+\frac{1}{2}\left(1+\log_9 m^{\alpha}_{kj}\right)-0.5\right\}=$$

$$\max_{k=1,\ldots,n}\left\{\frac{1}{2}\log_9 m^{\alpha}_{ik}+\frac{1}{2}\log_9 m^{\alpha}_{kj}+\frac{1}{2}\right\}=\max_{k=1,\ldots,n}\left\{\frac{1}{2}\log_9\left(m^{\alpha}_{ik}m^{\alpha}_{kj}\right)+\frac{1}{2}\right\}\overset{\text{(IV.16)}}{\leq}$$

$$\min_{k=1,\ldots,n}\left\{\frac{1}{2}\log_9\left(m^{\delta}_{ik}m^{\delta}_{kj}\right)+\frac{1}{2}\right\}=\min_{k=1,\ldots,n}\left\{\frac{1}{2}\log_9 m^{\delta}_{ik}+\frac{1}{2}\log_9 m^{\delta}_{kj}+\frac{1}{2}\right\}=$$

$$\min_{k=1,\ldots,n}\left\{\frac{1}{2}\left(1+\log_9 m^{\delta}_{ik}\right)+\frac{1}{2}\left(1+\log_9 m^{\delta}_{kj}\right)-0.5\right\}=\min_{k=1,\ldots,n}\left\{r^{\delta}_{ik}+r^{\delta}_{kj}-0.5\right\}.$$

□

Corollary 8. *Let $\widetilde{R}=\{\widetilde{r}_{ij}\}^n_{i,j=1}$, $\widetilde{r}_{ij}=(r^{\alpha}_{ij},r^{\beta}_{ij},r^{\gamma}_{ij},r^{\delta}_{ij})$, be a trapezoidal FAPCM-A additively weakly consistent according to Definition 59. Then the trapezoidal FMPCM $\widetilde{M}=\{\widetilde{m}_{ij}\}^n_{i,j=1}$, $\widetilde{m}_{ij}=(m^{\alpha}_{ij},m^{\beta}_{ij},m^{\gamma}_{ij},m^{\delta}_{ij})$, obtained from $\widetilde{R}$ by the transformations (IV.227) is multiplicatively weakly consistent according to Definition 50.*

Similarly, also multiplicative consistency is transformed into additive consistency and vice versa by the transformation formulas (IV.226) and (IV.227), respectively.

Theorem 88. *Let $\widetilde{M}=\{\widetilde{m}_{ij}\}^n_{i,j=1}$, $\widetilde{m}_{ij}=(m^{\alpha}_{ij},m^{\beta}_{ij},m^{\gamma}_{ij},m^{\delta}_{ij})$, be a trapezoidal FMPCM multiplicatively consistent according to Definition 51. Then the trapezoidal FAPCM-A $\widetilde{R}=\{\widetilde{r}_{ij}\}^n_{i,j=1}$, $\widetilde{r}_{ij}=(r^{\alpha}_{ij},r^{\beta}_{ij},r^{\gamma}_{ij},r^{\delta}_{ij})$, obtained from $\widetilde{M}$ by the transformations (IV.226) is additively consistent according to Definition 60.*

Proof. It is sufficient to show that when the inequalities (IV.25) and (IV.26) are valid for a FMPCM $\widetilde{M}$, then the inequalities (IV.133) and (IV.134) are valid for the FAPCM-A $\widetilde{R}$ obtained from $\widetilde{M}$ by the transformations (IV.226).

$$r^{\alpha}_{ik}+r^{\alpha}_{kj}-0.5=\frac{1}{2}\left(1+\log_9 m^{\alpha}_{ik}\right)+\frac{1}{2}\left(1+\log_9 m^{\alpha}_{kj}\right)-0.5=$$

$$\frac{1}{2}\left(1+\log_9\left(m^{\alpha}_{ik}m^{\alpha}_{kj}\right)\right)\overset{\text{(IV.25)}}{\leq}\frac{1}{2}\left(1+\log_9 m^{\alpha}_{ij}\right)=r^{\alpha}_{ij}.$$

The remaining inequalities are proved in the same way. □

Corollary 9. *Let $\widetilde{R}=\{\widetilde{r}_{ij}\}^n_{i,j=1}$, $\widetilde{r}_{ij}=(r^{\alpha}_{ij},r^{\beta}_{ij},r^{\gamma}_{ij},r^{\delta}_{ij})$, be a trapezoidal FAPCM-A additively consistent according to Definition 60. Then the trapezoidal FMPCM $\widetilde{M}=\{\widetilde{m}_{ij}\}^n_{i,j=1}$, $\widetilde{m}_{ij}=(m^{\alpha}_{ij},m^{\beta}_{ij},m^{\gamma}_{ij},m^{\delta}_{ij})$, obtained from $\widetilde{R}$ by the transformations (IV.227) is multiplicatively consistent according to Definition 51.*

In the following, a relation between fuzzy priorities obtained from FMPCMs and from FAPCMs-A is shown.

Theorem 89. *Let $\widetilde{M}=\{\widetilde{m}_{ij}\}^n_{i,j=1}$, $\widetilde{m}_{ij}=(m^{\alpha}_{ij},m^{\beta}_{ij},m^{\gamma}_{ij},m^{\delta}_{ij})$, be a trapezoidal FMPCM and let $\underline{\widetilde{w}}=(\widetilde{w}_1,\ldots,\widetilde{w}_n)^T$, $\widetilde{w}_i=(w^{\alpha}_i,w^{\beta}_i,w^{\gamma}_i,w^{\delta}_i)$, $i=1,\ldots,n$, be the fuzzy priority vector obtained from $\widetilde{M}$ by the formulas (IV.85). The fuzzy priority vector $\underline{\widetilde{w}}=(\widetilde{w}_1,\ldots,\widetilde{w}_n)^T$ can be transformed into a fuzzy priority vector $\underline{\widetilde{v}}=(\widetilde{v}_1,\ldots,\widetilde{v}_n)^T$, $\widetilde{v}_i=(v^{\alpha}_i,v^{\beta}_i,v^{\gamma}_i,v^{\delta}_i)$, $i=1,\ldots,n$, obtainable by formulas (IV.147) from the corresponding FAPCM-A $\widetilde{R}=\{\widetilde{r}_{ij}\}^n_{i,j=1}$, $\widetilde{r}_{ij}=(r^{\alpha}_{ij},r^{\beta}_{ij},r^{\gamma}_{ij},r^{\delta}_{ij})$, by using the transformation formulas*

$$v^{\alpha}_i=1+\log_9 w^{\alpha}_i,\qquad v^{\beta}_i=1+\log_9 w^{\beta}_i,$$

$$v^{\gamma}_i=1+\log_9 w^{\gamma}_i,\qquad v^{\delta}_i=1+\log_9 w^{\delta}_i.$$

(IV.228)

Proof. The validity of the transformation formulas follows immediately from the transformation formula (II.66) for the crisp case. □

Corollary 10. *Let $\widetilde{R}=\{\widetilde{r}_{ij}\}^n_{i,j=1}$, $\widetilde{r}_{ij}=(r^{\alpha}_{ij},r^{\beta}_{ij},r^{\gamma}_{ij},r^{\delta}_{ij})$, be a trapezoidal FAPCM-A and let $\underline{\widetilde{v}}=(\widetilde{v}_1,\ldots,\widetilde{v}_n)^T$, $\widetilde{v}_i=(v^{\alpha}_i,v^{\beta}_i,v^{\gamma}_i,v^{\delta}_i)$, $i=1,\ldots,n$, be the fuzzy priority vector obtained from $\widetilde{R}$ by the formulas (IV.147). The fuzzy priority vector $\underline{\widetilde{v}}=(\widetilde{v}_1,\ldots,\widetilde{v}_n)^T$ can be transformed into a fuzzy priority vector $\underline{\widetilde{w}}=(\widetilde{w}_1,\ldots,\widetilde{w}_n)^T$, $\widetilde{w}_i=(w^{\alpha}_i,w^{\beta}_i,w^{\gamma}_i,w^{\delta}_i)$, $i=1,\ldots,n$, obtainable by formulas (IV.85) from the corresponding FMPCM $\widetilde{M}=\{\widetilde{m}_{ij}\}^n_{i,j=1}$, $\widetilde{m}_{ij}=(m^{\alpha}_{ij},m^{\beta}_{ij},m^{\gamma}_{ij},m^{\delta}_{ij})$, by using the transformation formulas*

$$w^{\alpha}_i=9^{v^{\alpha}_i-1},\qquad w^{\beta}_i=9^{v^{\beta}_i-1},$$

$$w^{\gamma}_i=9^{v^{\gamma}_i-1},\qquad w^{\delta}_i=9^{v^{\delta}_i-1}.$$

(IV.229)

Similarly to the crisp case (see the discussion on p. 26), it is not possible to derive transformation formulas for transforming normalized fuzzy priorities (IV.91)–(IV.94) obtained from a FMPCM into the normalized fuzzy priorities (IV.158) obtained from the corresponding FAPCM-A and vice versa.

4.4.2 Transformations between FMPCMs and FAPCMs-M

In this section, transformation between FMPCMs and FAPCMs-M and between the related methods proposed in Sections 4.2 and 4.3.3, respectively, are examined.

Theorem 90. *A trapezoidal FMPCM* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$, $\widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, *can be transformed into a trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *by transformation formulas*

$$q_{ij}^{\alpha} = \frac{m_{ij}^{\alpha}}{1 + m_{ij}^{\alpha}}, \qquad q_{ij}^{\beta} = \frac{m_{ij}^{\beta}}{1 + m_{ij}^{\beta}},$$
$$q_{ij}^{\gamma} = \frac{m_{ij}^{\gamma}}{1 + m_{ij}^{\gamma}}, \qquad q_{ij}^{\delta} = \frac{m_{ij}^{\delta}}{1 + m_{ij}^{\delta}}. \tag{IV.230}$$

Proof. From the transformation formula (II.75) for transforming a MPCM into an APCM-M it is obvious that $\widetilde{q}_{ij} \in]0, 1[$ and $\widetilde{q}_{ii} = 0.5$. It remains to show that $\widetilde{Q}$ is additively reciprocal, i.e. $q_{ij}^{\alpha} = 1 - q_{ji}^{\delta}, q_{ij}^{\beta} = 1 - q_{ji}^{\gamma}, i, j = 1, \ldots, n$. Clearly

$$q_{ij}^{\alpha} = \frac{m_{ij}^{\alpha}}{1 + m_{ij}^{\alpha}} = \frac{\frac{1}{m_{ji}^{\delta}}}{1 + \frac{1}{m_{ji}^{\delta}}} = \frac{1}{1 + m_{ji}^{\delta}} = 1 - \frac{m_{ji}^{\delta}}{1 + m_{ji}^{\delta}} = 1 - q_{ji}^{\delta}.$$

Analogously, the validity of $q_{ij}^{\beta} = 1 - q_{ji}^{\gamma}$ is proved. $\square$

Corollary 11. *A trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *can be transformed into a trapezoidal FMPCM* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$, $\widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, *by transformation formulas*

$$m_{ij}^{\alpha} = \frac{q_{ij}^{\alpha}}{q_{ji}^{\delta}}, \qquad m_{ij}^{\beta} = \frac{q_{ij}^{\beta}}{q_{ji}^{\gamma}},$$
$$m_{ij}^{\gamma} = \frac{q_{ij}^{\gamma}}{q_{ji}^{\beta}}, \qquad m_{ij}^{\delta} = \frac{q_{ij}^{\delta}}{q_{ji}^{\alpha}}. \tag{IV.231}$$

In the following, it is proved that the transformation formulas (IV.230) and (IV.231) transform the multiplicative weak consistency for FMPCMs into the multiplicative weak consistency for FAPCMs-M and vice versa.

Theorem 91. *Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}$, $\widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, *be a trapezoidal FMPCM multiplicatively weakly consistent according to Definition 50. Then the trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *obtained from* $\widetilde{M}$ *by the transformations* (IV.230) *is multiplicatively weakly consistent according to Definition 66.*

Proof. It is sufficient to show that when the inequalities (IV.16) are valid for a FMPCM $\widetilde{M}$, then the inequalities (IV.177) are valid for the FAPCM-M $\widetilde{Q}$ obtained from $\widetilde{M}$ by the transformations (IV.230).

$$\max_{k=1,\ldots,n} \left\{ \frac{q_{ik}^{\alpha} q_{kj}^{\alpha}}{q_{ik}^{\alpha} q_{kj}^{\alpha} + (1 - q_{ik}^{\alpha})(1 - q_{kj}^{\alpha})} \right\} = \max_{k=1,\ldots,n} \left\{ \frac{\frac{m_{ik}^{\alpha}}{1 + m_{ik}^{\alpha}} \frac{m_{kj}^{\alpha}}{1 + m_{kj}^{\alpha}}}{\frac{m_{ik}^{\alpha}}{1 + m_{ik}^{\alpha}} \frac{m_{kj}^{\alpha}}{1 + m_{kj}^{\alpha}} + \frac{1}{1 + m_{ik}^{\alpha}} \frac{1}{1 + m_{kj}^{\alpha}}} \right\} =$$

$$\max_{k=1,\ldots,n} \left\{ \frac{m_{ik}^{\alpha} m_{kj}^{\alpha}}{1 + m_{ik}^{\alpha} m_{kj}^{\alpha}} \right\} = \frac{1}{\min_{k=1,\ldots,n} \left\{ \frac{1 + m_{ik}^{\alpha} m_{kj}^{\alpha}}{m_{ik}^{\alpha} m_{kj}^{\alpha}} \right\}} = \frac{1}{1 + \frac{1}{\max_{k=1,\ldots,n} \{m_{ik}^{\alpha} m_{kj}^{\alpha}\}}} \overset{\text{(IV.16)}}{\leq}$$

$$\frac{1}{1 + \frac{1}{\min_{k=1,\ldots,n} \{m_{ik}^{\delta} m_{kj}^{\delta}\}}} = \frac{1}{\max_{k=1,\ldots,n} \left\{ \frac{1 + m_{ik}^{\delta} m_{kj}^{\delta}}{m_{ik}^{\delta} m_{kj}^{\delta}} \right\}} = \min_{k=1,\ldots,n} \left\{ \frac{m_{ik}^{\delta} m_{kj}^{\delta}}{1 + m_{ik}^{\delta} m_{kj}^{\delta}} \right\} =$$

$$\min_{k=1,\ldots,n} \left\{ \frac{\frac{q_{ik}^{\delta}}{q_{ki}^{\alpha}} \frac{q_{kj}^{\delta}}{q_{jk}^{\alpha}}}{1 + \frac{q_{ik}^{\delta}}{q_{ki}^{\alpha}} \frac{q_{kj}^{\delta}}{q_{jk}^{\alpha}}} \right\} = \min_{k=1,\ldots,n} \left\{ \frac{q_{ik}^{\delta} q_{kj}^{\delta}}{q_{ik}^{\delta} q_{kj}^{\delta} + (1 - q_{ik}^{\delta})(1 - q_{kj}^{\delta})} \right\}.$$

$\square$

Corollary 12. *Let* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}, \widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *be a trapezoidal FAPCM-M multiplicatively weakly consistent according to Definition 66. Then the trapezoidal FMPCM* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, *obtained from* $\widetilde{Q}$ *by the transformations* (IV.231) *is multiplicatively weakly consistent according to Definition 50.*

Similarly, also the multiplicative consistency is transformed into the multiplicative consistency and vice versa by the transformation formulas (IV.230) and (IV.231), respectively.

Theorem 92. *Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, *be a trapezoidal FMPCM multiplicatively consistent according to Definition 51. Then the trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}, \widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *obtained from* $\widetilde{M}$ *by the transformations* (IV.230) *is multiplicatively consistent according to Definition 67.*

Proof. It is sufficient to show that when the inequalities (IV.25) and (IV.26) are valid for a FMPCM M, then the inequalities (IV.190) and (IV.191) are valid for the FAPCM-M $\widetilde{Q}$ obtained from $\widetilde{M}$ by the transformations (IV.230).

$$\frac{q_{ik}^{\alpha} q_{kj}^{\alpha}}{q_{ik}^{\alpha} q_{kj}^{\alpha} + (1 - q_{ik}^{\alpha})(1 - q_{kj}^{\alpha})} = \frac{m_{ik}^{\alpha} m_{kj}^{\alpha}}{1 + m_{ik}^{\alpha} m_{kj}^{\alpha}} \overset{(IV.25)}{\leq} \frac{m_{ij}^{\alpha}}{1 + m_{ij}^{\alpha}} = \frac{\frac{q_{ij}^{\alpha}}{q_{ji}^{\delta}}}{1 + \frac{q_{ij}^{\alpha}}{q_{ji}^{\delta}}} = \frac{q_{ij}^{\alpha}}{q_{ij}^{\alpha} + q_{ji}^{\delta}} = q_{ij}^{\alpha}.$$

The remaining inequalities are proved in the same way. $\qquad\square$

Corollary 13. *Let* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}, \widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *be a trapezoidal FAPCM-M multiplicatively consistent according to Definition 67. Then the trapezoidal FMPCM* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, *obtained from* $\widetilde{Q}$ *by the transformations* (IV.231) *is multiplicatively consistent according to Definition 51.*

In the following, a relation between fuzzy priorities obtained from FMPCMs and from FAPCMs-M is shown.

Theorem 93. *Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, *be a trapezoidal FMPCM and let* $\underline{\widetilde{w}} = (\widetilde{w}_1, \ldots, \widetilde{w}_n)^T$, $\widetilde{w}_i = (w_i^{\alpha}, w_i^{\beta}, w_i^{\gamma}, w_i^{\delta}), i = 1, \ldots, n$, *be the fuzzy priority vector obtained from* $\widetilde{M}$ *by the formulas* (IV.85). *The fuzzy priority vector* $\underline{\widetilde{w}} = (\widetilde{w}_1, \ldots, \widetilde{w}_n)^T$ *is identical to the fuzzy priority vector* $\underline{\widetilde{u}} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)^T$, $\widetilde{u}_i = (u_i^{\alpha}, u_i^{\beta}, u_i^{\gamma}, u_i^{\delta}), i = 1, \ldots, n$, *obtainable by the formulas* (IV.211)–(IV.214) *from the corresponding trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}, \widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *i.e.*

$$\widetilde{w} = \widetilde{u}. \tag{IV.232}$$

Proof. The validity of the transformation formulas follows immediately from the transformation formula (II.77) for the crisp case. $\qquad\square$

Moreover, similarly to the crisp case, also the normalized fuzzy priorities obtained from a FMPCM and from the corresponding FAPCM-M are identical.

Theorem 94. *Let* $\widetilde{M} = \{\widetilde{m}_{ij}\}_{i,j=1}^{n}, \widetilde{m}_{ij} = (m_{ij}^{\alpha}, m_{ij}^{\beta}, m_{ij}^{\gamma}, m_{ij}^{\delta})$, *be a trapezoidal FMPCM and let* $\underline{\widetilde{w}}_C = (\widetilde{w}_{C1}, \ldots, \widetilde{w}_{Cn})^T, \widetilde{w}_{Ci} = (w_{Ci}^{\alpha}, w_{Ci}^{\beta}, w_{Ci}^{\gamma}, w_{Ci}^{\delta}), i = 1, \ldots, n$, *be the normalized fuzzy priority vector obtained from* $\widetilde{M}$ *by the formulas* (IV.91)–(IV.94). *The normalized fuzzy priority vector* $\underline{\widetilde{w}}_C = (\widetilde{w}_{C1}, \ldots, \widetilde{w}_{Cn})^T$ *is identical to the normalized fuzzy priority vector* $\underline{\widetilde{u}}_C = (\widetilde{u}_{C1}, \ldots, \widetilde{u}_{Cn})^T, \widetilde{u}_{Ci} = (u_{Ci}^{\alpha}, u_{Ci}^{\beta}, u_{Ci}^{\gamma}, u_{Ci}^{\delta}), i = 1, \ldots, n$, *obtainable by the formulas* (IV.215)–(IV.218) *from the corresponding trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}, \widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *i.e.*

$$\widetilde{w}_C = \widetilde{u}_C. \tag{IV.233}$$

Proof. Let us demonstrate the equality of the lower boundary values of the fuzzy priorities $\widetilde{w}_{Ci}$ and $\widetilde{u}_{Ci}$ obtained by (IV.91) and (IV.215), respectively. The equality of the remaining representing values would be demonstrated in the same way.

By applying (IV.231), and $m_{rs} = \frac{q_{rs}}{q_{sr}}, r, s = 1, \ldots, n$, to the optimization problem (IV.91), we obtain

$$w_{Ci}^{\alpha} = \min \left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} m_{ij}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} m_{kj}}} \; ; \; \begin{array}{l} m_{rs} \in \left[m_{rs}^{\alpha}, m_{rs}^{\delta} \right], \\ m_{sr} = \frac{1}{m_{rs}}, \\ r, s = 1, \ldots, n \end{array} \right\} =$$

$$\min \left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{q_{ji}}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} \frac{q_{kj}}{q_{jk}}}} \; ; \; \begin{array}{l} \frac{q_{rs}}{1-q_{rs}} \in \left[\frac{q_{rs}^{\alpha}}{q_{sr}^{\delta}}, \frac{q_{rs}^{\delta}}{q_{sr}^{\alpha}} \right], \\ q_{sr} = 1 - q_{rs}, \\ r, s = 1, \ldots, n \end{array} \right\} =$$

$$\min \left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} \frac{q_{ij}}{q_{ji}}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} \frac{q_{kj}}{q_{jk}}}} \; ; \; \begin{array}{l} q_{rs} \in \left[q_{rs}^{\alpha}, q_{rs}^{\delta} \right], \\ q_{sr} = 1 - q_{rs}, \\ r, s = 1, \ldots, n \end{array} \right\} = u_{Ci}^{\alpha}.$$

$\square$

4.4.3 Transformations between FAPCMs-A and FAPCMs-M

In this section, transformations between FAPCMs-A and FAPCMs-M and between the related methods proposed in Sections 4.3.2 and 4.3.3, respectively, are examined. Analogously as for the transformations between APCMs-A and APCMs-M, the transformation formulas can be derived directly by composing the corresponding formulas from the previous two sections as specified in the following theorems.

Theorem 95. *A trapezoidal FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$, $\widetilde{r}_{ij} = (r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta})$, *can be transformed into a trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *by transformation formulas*

$$q_{ij}^{\alpha} = \frac{9^{2r_{ij}^{\alpha}-1}}{1 + 9^{2r_{ij}^{\alpha}-1}}, \qquad q_{ij}^{\beta} = \frac{9^{2r_{ij}^{\beta}-1}}{1 + 9^{2r_{ij}^{\beta}-1}},$$
$$q_{ij}^{\gamma} = \frac{9^{2r_{ij}^{\gamma}-1}}{1 + 9^{2r_{ij}^{\gamma}-1}}, \qquad q_{ij}^{\delta} = \frac{9^{2r_{ij}^{\delta}-1}}{1 + 9^{2r_{ij}^{\delta}-1}}. \tag{IV.234}$$

Proof. Because the transformation formulas (IV.227) transform a FAPCM-A into a FMPCM, and the transformation formulas (IV.230) transform a FMPCM into a FAPCM-M, then the composition of these formulas transforms a FAPCM-A into a FAPCM-M. By composing (IV.227) and (IV.230) we immediately obtain (IV.234). $\square$

Corollary 14. *A trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *can be transformed into a trapezoidal FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$, $\widetilde{r}_{ij} = (r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta})$, *by transformation formulas*

$$r_{ij}^{\alpha} = \frac{1}{2} \left(1 + \log_9 \frac{q_{ij}^{\alpha}}{q_{ji}^{\delta}} \right), \qquad r_{ij}^{\beta} = \frac{1}{2} \left(1 + \log_9 \frac{q_{ij}^{\beta}}{q_{ji}^{\gamma}} \right),$$
$$r_{ij}^{\gamma} = \frac{1}{2} \left(1 + \log_9 \frac{q_{ij}^{\gamma}}{q_{ji}^{\beta}} \right), \qquad r_{ij}^{\delta} = \frac{1}{2} \left(1 + \log_9 \frac{q_{ij}^{\delta}}{q_{ji}^{\alpha}} \right). \tag{IV.235}$$

In the following, it is proved that additive weak consistency is transformed into multiplicative weak consistency and vice versa by the transformation formulas (IV.234) and (IV.235), respectively.

Theorem 96. *Let* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$, $\widetilde{r}_{ij} = (r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta})$, *be a trapezoidal FAPCM-A additively weakly consistent according to Definition 59. Then the trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *obtained from* $\widetilde{R}$ *by the transformations* (IV.234) *is multiplicatively weakly consistent according to Definition 66.*

Proof. Because the transformation formulas (IV.227) transform the additive weak consistency (IV.123) of a FAPCM-A into the multiplicative weak consistency (IV.14) of the corresponding FMPCM, and the transformation formulas (IV.230) transform multiplicative weak consistency (IV.14) of a FMPCM into multiplicative weak consistency (IV.175) of the corresponding FAPCM-M, then the composition of these formulas transforms additive weak consistency of a FAPCM-A into multiplicative weak consistency of the corresponding FAPCM-M. By composing (IV.227) and (IV.230) we immediately obtain (IV.234). $\square$

Corollary 15. *Let* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *be a trapezoidal FAPCM-M multiplicatively weakly consistent according to Definition 66. Then the trapezoidal FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$, $\widetilde{r}_{ij} = (r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta})$, *obtained from* $\widetilde{Q}$ *by the transformations* (IV.235) *is additively weakly consistent according to Definition 59.*

Similarly, also the additive consistency is transformed into the multiplicative consistency and vice versa by the transformation formulas (IV.234) and (IV.235), respectively.

Theorem 97. *Let* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$, $\widetilde{r}_{ij} = (r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta})$, *be a trapezoidal FAPCM-A additively consistent according to Definition 60. Then the trapezoidal FAPCM-M* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *obtained from* $\widetilde{R}$ *by the transformations* (IV.234) *is multiplicatively consistent according to Definition 67.*

Proof. Because the transformation formulas (IV.227) transform the additive consistency (IV.127)–(IV.128) of a FAPCM-A into the multiplicative consistency (IV.18)–(IV.19) of the corresponding FMPCM, and the transformation formulas (IV.230) transform multiplicative consistency (IV.18)–(IV.19) of a FMPCM into multiplicative consistency (IV.179)–(IV.180) of the corresponding FAPCM-M, then the composition of these formulas transforms additive consistency of a FAPCM-A into multiplicative consistency of the corresponding FAPCM-M. By composing (IV.227) and (IV.230) we immediately obtain (IV.234). □

Corollary 16. *Let* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *be a trapezoidal FAPCM-M multiplicatively consistent according to Definition 67. Then the trapezoidal FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$, $\widetilde{r}_{ij} = (r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta})$, *obtained from* $\widetilde{Q}$ *by the transformations* (IV.235) *is addititively consistent according to Definition 60.*

In the following, a relation between fuzzy priorities obtained from FAPCMs-A and from FAPCMs-M is shown.

Theorem 98. *Let* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$, $\widetilde{r}_{ij} = (r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta})$, *be a trapezoidal FAPCM-A and let* $\underline{\widetilde{v}} = (\widetilde{v}_1, \ldots, \widetilde{v}_n)^T$, $\widetilde{v}_i = (v_i^{\alpha}, v_i^{\beta}, v_i^{\gamma}, v_i^{\delta})$, $i = 1, \ldots, n$, *be the fuzzy priority vector obtained from* $\widetilde{R}$ *by the formulas* (IV.147). *The fuzzy priority vector* $\underline{\widetilde{v}} = (\widetilde{v}_1, \ldots, \widetilde{v}_n)^T$ *can be transformed into the fuzzy priority vector* $\underline{\widetilde{u}} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)^T$, $\widetilde{u}_i = (u_i^{\alpha}, u_i^{\beta}, u_i^{\gamma}, u_i^{\delta})$, $i = 1, \ldots, n$, *obtainable by formulas* (IV.211)–(IV.214) *from the corresponding FAPCM-M by using the transformation formulas*

$$u_i^{\alpha} = 9^{v_i^{\alpha}-1}, \qquad u_i^{\beta} = 9^{v_i^{\beta}-1},$$
$$u_i^{\gamma} = 9^{v_i^{\gamma}-1}, \qquad u_i^{\delta} = 9^{v_i^{\delta}-1}. \tag{IV.236}$$

Proof. Because the transformation formulas (IV.229) transform the fuzzy priority vector (IV.147) of a FAPCM-A into the fuzzy priority vector (IV.85) of the corresponding FMPCM, and the transformation formula (IV.232) transforms the fuzzy priority vector (IV.85) of a FMPCM into the fuzzy priority vector (IV.211)–(IV.214) of the corresponding FAPCM-M, then the composition of these transformation formulas transforms the fuzzy priority vector (IV.147) of a FAPCM-A into the fuzzy priority vector (IV.211)–(IV.214) of the corresponding FAPCM-M. By composing (IV.229) and (IV.232) we immediately obtain (IV.236). □

Corollary 17. *Let* $\widetilde{Q} = \{\widetilde{q}_{ij}\}_{i,j=1}^{n}$, $\widetilde{q}_{ij} = (q_{ij}^{\alpha}, q_{ij}^{\beta}, q_{ij}^{\gamma}, q_{ij}^{\delta})$, *be a trapezoidal FAPCM-M and let* $\underline{\widetilde{u}} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)^T$, $\widetilde{u}_i = (u_i^{\alpha}, u_i^{\beta}, u_i^{\gamma}, u_i^{\delta})$, $i = 1, \ldots, n$, *be the fuzzy priority vector obtainable from* $\widetilde{Q}$ *by the formulas* (IV.211)–(IV.214). *The fuzzy priority vector* $\underline{\widetilde{u}} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)^T$ *can be transformed into the fuzzy priority vector* $\underline{\widetilde{v}} = (\widetilde{v}_1, \ldots, \widetilde{v}_n)^T$, $\widetilde{v}_i = (v_i^{\alpha}, v_i^{\beta}, v_i^{\gamma}, v_i^{\delta})$, $i = 1, \ldots, n$, *obtainable by formulas* (IV.147) *from the corresponding FAPCM-A* $\widetilde{R} = \{\widetilde{r}_{ij}\}_{i,j=1}^{n}$, $\widetilde{r}_{ij} = (r_{ij}^{\alpha}, r_{ij}^{\beta}, r_{ij}^{\gamma}, r_{ij}^{\delta})$, *by using the transformation formulas*

$$v_i^{\alpha} = 1 + \log_9 u_i^{\alpha}, \qquad v_i^{\beta} = 1 + \log_9 u_i^{\beta},$$
$$v_i^{\gamma} = 1 + \log_9 u_i^{\gamma}, \qquad v_i^{\delta} = 1 + \log_9 u_i^{\delta}. \tag{IV.237}$$

Similarly to the crisp case, it is not possible to derive transformation formulas for transforming normalized fuzzy priorities (IV.158) obtained from a FAPCM-A into normalized fuzzy priorities (IV.215)–(IV.218) obtained from the corresponding FAPCM-M and vice versa; see the discussion on p. 26 and p. 30.

4.5 Conclusion

In this Chapter, the first research question, *"Based on a FPCM of objects, how should fuzzy priorities of these objects be determined so that they reflect properly all preference information available in the FPCM?"*, was answered. Three types of FPCMs were examined in this chapter - FMPCMs, FAPCMs-A, and FAPCMs-M. Construction of FPCMs, defining and verifying their consistency, and deriving fuzzy priorities of objects from them have been studied in detail for each of the three types of FPCMs.

First, the relevant methods proposed in the literature based on the fuzzy extension of methods originally proposed for PCMs were reviewed and their major drawbacks were identified (task (1.b) formulated in Section

1.3). In particular, it was find out that "equal preference" of two compared objects is very often modeled inappropriately in FPCMs which results in misinterpretation of the preference information provided by the DM and leads to false results. Further, it was find out that most of the definitions of consistency reviewed in this chapter violate the reciprocity of PCs in FPCMs or are not invariant under permutation of objects. These are sever drawbacks that lead to false conclusions about consistency/inconsistency of the FPCMs. Similarly, also the reviewed approaches for obtaining fuzzy maximal eigenvalues of FMPCMs violate the reciprocity of the related PCs or the invariance under permutation of objects. This again leads to the distortion of the preference information contained in FMPCMs. In particular, the fuzzy maximal eigenvalue obtained by these approaches is not necessarily greater or equal to the dimension of the FMPCM, which is an inherent property of the maximal eigenvalues of MPCMs. Analogously, also the reviewed methods for deriving fuzzy priorities of objects from FPCMs violate the reciprocity of the related PCs or the invariance under permutation of objects. The consequences in this case are even more critical. The fuzzy priorities of objects obtained by applying these defective methods do not reflect properly the preference information contained in the FPCM. Such fuzzy priorities are not only excessively uncertain, but often completely distorted, which may also lead to a completely different ranking of the compared objects and thus to a decision that is not optimal. Second, it was shown that in order to reflect appropriately the preference information contained in the FPCM, in particular the reciprocity of the related PCs, it is necessary to apply constrained fuzzy arithmetic to the fuzzy extension of the methods instead of standard fuzzy arithmetic (task (1.c) formulated in Section 1.3). From the multiplicative reciprocity $m_{ji} = \frac{1}{m_{ij}}$, $i,j = 1, \ldots, n$, for MPCMs, the equality $m_{ij}m_{ji} = 1$, $i,j = 1, \ldots, n$, automatically follows. Similarly, from the additive reciprocity $a_{ji} = 1 - a_{ij}$, $i,j = 1, \ldots, n$, for APCMs, the equality $a_{ij} + a_{ji} = 1$, $i,j = 1, \ldots, n$, follows. These properties are inherent to every MPCM and every APCM, respectively. However, these properties are not preserved for FMPCMs and FAPCMs when standard fuzzy arithmetic is applied to the computations. In particular, the multiplicative reciprocity $\widetilde{m}_{ji} = \frac{1}{\widetilde{m}_{ij}}$, $i,j = 1, \ldots, n$, holds for FMPCMs but $\widetilde{m}_{ij}\widetilde{m}_{ji} \neq 1$, $i,j = 1, \ldots, n$, $i \neq j$. Similarly, the additive reciprocity $\widetilde{a}_{ji} = 1 - \widetilde{a}_{ij}$, $i,j = 1, \ldots, n$, holds for FAPCMs but $\widetilde{a}_{ij} + \widetilde{a}_{ji} \neq 1$, $i,j = 1, \ldots, n$, $i \neq j$. This s a serious drawback. It was shown that validity of the equalities $\widetilde{m}_{ij}\widetilde{m}_{ji} = 1$ and $\widetilde{a}_{ij} + \widetilde{a}_{ji} = 1$, $i,j = 1, \ldots, n$, can be guaranteed by appropriately applying constrained fuzzy arithmetic.

Third, a complete approach based on constrained fuzzy arithmetic was proposed to deal with all three types of FPCMs (task (1.d) formulated in Section 1.3). Namely, it was shown how to appropriately model the meaning of the linguistic term "equal preference" used for PCs in FPCMs. Further, definitions of consistency for FMPCMs, FAPCMs-A, and FAPCMs-M were proposed in such a way that they are invariant under permutation of objects and do not violate the reciprocity of the related PCs. Two definitions of consistency, weak version and strong version, were proposed for each type of FPCMs, and useful tools for verifying the consistency according to each definition were provided. Moreover, by using constrained fuzzy arithmetic, it was also possible to properly extend to FPCMs the properties equivalent to the corresponding consistency conditions for PCMs in such a way that they are still equivalent. This was not possible with standard fuzzy arithmetic. Further, a method for obtaining the fuzzy maximal eigenvalue of a FMPCM was proposed in such a way that it is invariant under permutation of objects and does not violate the reciprocity of the related PCs in a FMPCM. By preserving the reciprocity of the related PCs, the fuzzy maximal eigenvalue is always greater than (or equal to) the dimension of the FMPCM, which is an inherent property of the maximal eigenvalues of MPCMs. Afterwards, methods for obtaining fuzzy priorities of objects from FMPCMs, FAPCMs-A, and FAPCMs-M were proposed based on constrained fuzzy arithmetic so that they preserve the reciprocity of the related PCs and are invariant under permutation of objects. The fuzzy priorities obtained by these methods thus properly represent the preference information contained in the FPCMs. Moreover, applying constrained fuzzy arithmetic to the fuzzy extension of the formulas for obtaining normalized priorities preserves the normality property, i.e. the fuzzy priorities obtained by these methods form a normalized fuzzy vector. This was again not possible by using standard fuzzy arithmetic. The approaches defined for each type of the FPCMs are mutually equivalent; each type of the FPCMs can be transformed into another together with the respective consistency properties. In the same way, fuzzy priorities obtained from each type of the FPCMs can be transformed one into another.

Chapter V

Incomplete large-dimensional pairwise comparison matrices

5.1   Introduction to large-dimensional pairwise comparison problems

In Chapter II, methods for dealing with PC problems were reviewed. These methods are based on constructing PCMs where every to objects from a given set of n objects are compared pairwisely. As we know from Chapter II, $n(n-1)/2$ PCs are required to compare n objects pairwisely. To compare 5 objects, for example, 10 PCs need to be provided by the DM. Now imagine that 20 objects need to be compared. In this case, 190 PCs are required from the DM. This number is clearly very high. Such a high number of PCs is not easy to obtain in sufficient quality. In fact, the more PCs need to be made, the less reliable the information expressed by the DMs might be (due to fatigue, due to time constraints and similar factors). In these cases, Saaty (1977, 2008) suggests to split a large-dimensional problem into several subproblems of smaller dimensions. This implies creating supercategories of objects. The objects are then compared pairwisely only within the defined supercategories. Additionally, PCs of the supercategories have to be also provided. This results in the reduction of the complexity of the problem and in making the preference information requirements (the number of PCs needed) feasible. However, this procedure also results in a slight loss of information; the objects from different supercategories are not directly compared pairwisely. This loss of information can be to some extent compensated for by introducing a strong enough consistency condition on the preferences expressed by the experts, which provides means of calculating the missing values in the PCM.

For some real-life problems, the above described approach works well. There are, however, situations when splitting the problem into several smaller ones renders parts of the problem too abstract and hence intractable for the experts providing the information on the preferences among objects. Stoklasa et al. (2013) provided a real-life example of such a problem in the area of arts evaluation. Stoklasa et al. (2013) refers to the development of the evaluation model for the Registry of Artistic Performances that has been used in the Czech Republic within the principles and rules of financing public universities; see Ministry of Education of the Czech Republic (2011). This model has been used since 2012 to provide a basis for the distribution of a part of the subsidy from the state budget among public universities in the Czech Republic. The mathematical model presented by Stoklasa et al. (2013) is designed to compute evaluations (priorities) for different categories of works of art (currently 27 categories) based on combination of expert assessment of the significance of the respective work of art and two more objective criteria (extent and institutional reception). The authors dealt with a 27x27 PCM that represents a problem that could not be split into several smaller ones due to partial dependencies among the evaluation criteria and due to the necessity of providing real-life examples to all the compared categories for the experts to be able to express their intensities of preference.

In large-dimensional PC problems which cannot be split into smaller subproblems, the required priorities of objects may be obtained from incomplete PCMs. In that case, focusing on an appropriate reduction of the number of PCs which have to be provided by the DM and obtaining enough preference information in the incomplete PCM to be able to compute the priorities of objects are of paramount importance. When using

incomplete PCMs, we have to deal adequately with two key tasks: (i) finding a method for efficiently selecting the subset of the $n(n-1)/2$ PCs that should be provided by the DM, and (ii) finding an appropriate method for deriving the priority vector from the incomplete PCM.

Harker (1987a,b,c) and later Harker and Millet (1990) were the first to deal with the problem of reducing the number of PCs. They proposed to perform only a part of the $n(n-1)/2$ PCs by means of an algorithm which iteratively selects the PCs to be submitted to the DM. This selection is made according to the largest modification in the priority vector. The process of inputting PCs is then stopped when the provided PC changes the priority vector by less than a fixed threshold. Wedley et al. (1993) focused on the choice of only $n-1$ PCs, which is the minimum number required for comparing n objects, and they compared and discussed several methods of entering them. Sanchez and Soyer (1998) proposed to use entropy-based measures of the information content to evaluate judgment accuracy and to state a stopping rule of the process of inputting PCs. Ra (1999) worked with n PCs which form a closed chain. Fedrizzi and Giove (2013) proposed a method for selecting PCs in an incomplete PCM which takes into account both the robustness of the collected data and the consistency of the expressed preferences.

For what concerns the methods for deriving the priority vector from incomplete PCMs, several different approaches have been proposed; see e.g. Alonso et al. (2008); Chen and Triantaphillou (2001); Fedrizzi and Giove (2007, 2013); Harker (1987a,b); Harker and Millet (1990); Kwiesielewicz (1996); Kwiesielewicz and van Uden (2003); Ramík (2016); Shiraishi et al. (1998); Xu (2004, 2005). Some of these methods are aimed at automatically determining the missing PCs in order to complete the incomplete PCM. Once the PCM is filled in, one of known methods for deriving the priorities from a complete PCM can be used. Conversely, other methods compute the priorities from the incomplete PCM directly. Clearly, having first computed the priorities, every missing PCs in the incomplete PCM can then be determined accordingly, thus completing the PCM.

In this chapter, we aim to propose a method for obtaining priorities of objects from large-dimensional incomplete PCMs where the consistency preservation plays a crucial role. In particular, the weak consistency defined in Section 2.2.2.2 for MPCMs and in Section 2.3.3.2 for APCMs is employed in the method as a minimum requirement of consistency that has to be satisfied. The method differs from the other PC methods mentioned above since the weak consistency of the incomplete PCM is preserved in every step of the method. A similar property is not required in any other known method.

The objective of the method proposed in this chapter is not simply reducing the number of PCs required from the DM. It is known that this number could be radically reduced to $n-1$, as proposed by Wedley et al. (1993); Herrera-Viedma et al. (2004) and others. Such choice completely fulfills the requirement of maximally reducing the number of PCs required from the DM. However, it gives up the fundamental characterizing property of the PC methods - the ability to use the redundancy of information contained in a PCM in order to suitably manage the unavoidable inconsistency of human judgments. In the numerical example in Section 5.3.2, it will be demonstrated that the methods requiring only $n-1$ PCs do not always result in reliable outcomes.

The main objective of the method proposed in this chapter is to find an ideal compromise between requiring as little preference information from the DM as possible and still obtaining enough information to calculate priorities of objects that are close to the hypothetical full-information case (i.e. the case when the DM provides all PCs in the PCM). Moreover, the final priorities of objects provided by the new method are computed in such a way that they contain information concerning the uncertainty which stems from the fact that some PCs are not provided by the DM, nor are they entered automatically by the proposed algorithm. The priorities of objects are computed as intervals in order to reflect the missing information in the incomplete PCM and to provide ranges for the values of the crisp priorities of objects obtainable from any weakly consistent completion of the incomplete PCM. The range of the interval priorities depends on the amount of preference information that is missing in the incomplete PCM. The formulas for calculating fuzzy priorities introduced in Sections 4.2, 4.3.2, and 4.3.3 are applied to the method depending on the type of the PCM used for expressing DM's preferences.

In the following section, preliminaries indispensable for introducing the new method for large-dimensional PC problems and for demonstrating its performance are given.

5.2 Background

In this section the large-dimensional evaluation model for the registry of artistic performances proposed by Stoklasa et al. (2013) is described in more detail as its results are later confronted with the results obtained by the new method. Further, an overview of the algorithm for the optimal choice of PCs in incomplete large-dimensional PCMs proposed by Fedrizzi and Giove (2013) is given here since this algorithm is partially utilized in the new method.

5.2.1 Case study: Evaluation model for the Registry of Artistic Performances

The evaluation model for the Registry of Artistic Performances as mentioned in Section 5.1 was a motivation for developing the novel method for large-dimensional pairwise-comparison problems. This evaluation model will be also used later in this chapter to validate the performance of the new method. Therefore, it is necessary to introduce the original evaluation model in more detail. For a more detailed description of the model, the readers can refer to Stoklasa et al. (2013).

The outputs of artistic performance are currently evaluated in the Czech Republic based on the following three criteria, for each of which there are three levels distinguished:

Criterion 1 - Relevance or significance of the piece of art

      A - a new piece of art or a performance of crucial significance

      B - a new piece of art or a performance containing numerous important innovations

      C - a new piece of art or a performance pushing forward modern trends

Criterion 2 - Extent of the piece of art

      K - a piece of art or a performance of large extent

      L - a piece of art or a performance of medium extent

      M - a piece of art or a performance of limited extent

Criterion 3 - Institutional and media reception/impact of the piece of art

      X - international reception/impact

      Y - national reception/impact

      Z - regional reception/impact

Criterion 1 is an expertly assessed criterion that brings a peer-review element into the evaluation. Each segment of art provided a general linguistic specification for each level of this criterion to be made available for the expert evaluators, real-life (historical) examples for levels A, B, and C are also available. Also the levels of Criterion 2 are specified linguistically. This criterion was, however, intended to be measurable for each segment on such a level of accuracy that most of the ambiguity in categorizing works of art according to this criterion is removed. For Criterion 3, lists of institutions corresponding to level X, Y and Z are provided. Hence, there is no subjectivity in evaluation against this criterion in the process.

By combining the various levels of the three criteria, 27 categories of works of art can be defined. These categories are represented in the model by triplets of the capital letters identifying the levels (e.g. AKY, BLZ, or CMZ). Each of these 27 categories needs to be assigned a score (priority). The original idea was to obtain all PCs of the 27 well-represented (that is represented by real-life examples) categories of works of art (351 PCs in total) using Saaty's scale given in Tab. II.1, and afterwards, to compute the score for each category using the GMM (II.24).

Because the MPCMs and Saaty's scale were not intended for large-dimensional problems, Saaty (1977) proposed to approach these problems by splitting them into subproblems of lower dimensions. However, this approach was not applicable to the problem in question for the following reasons: a) there are some dependencies among the criteria which are not easy to describe or capture, b) to compare various levels of one criterion (e.g. big, medium and small) without any real-life representatives (good representatives for such broad categories proved to be difficult to find) is not easy for the experts, c) the experts were not able to express their preferences between the criteria (these too proved to be too abstract to provide enough representation for the experts to be able to express their preferences). For these reasons, all 27 categories were compared pairwisely. Since the multiplicative-consistency condition (II.4) is almost impossible to achieve for large-dimensional MPCMs, the much more relaxed weak-consistency condition (II.11) was used to control the consistency of the PCs of the categories provided by the experts. The weak consistency was considered as a minimum requirement on the consistency of the MPCM.

As the weak-consistency condition (II.11) is easy to check during the process of inputting preferences, and even more so when the rows and columns of the MPCM are ordered in accordance with the preference ordering of the categories (from the most preferred to the least preferred one), the 27 categories were first ordered according to their preference using the PC method; see Stoklasa et al. (2013) for more details. Afterwards, the experts provided 351 PCs of the categories using the elements from Saaty's scale given in Tab. II.1 and the normalized priorities of the categories were obtained from the complete MPCM by using the GMM (II.24). The MPCM and the derived priorities of all 27 categories are shown later in Section 5.3.3 where these results are compared with the results obtained by applying the novel method that is going to be proposed in Section 5.3.1.

After two years of using the described model and the computed evaluations, minor adjustments to the evaluation methodology proved to be necessary. Adding one more level to one of the criteria and changing the initial preference ordering of the categories were considered; see Stoklasa et al. (2016). Changing the preference ordering of the categories would result in the need of inputting the large MPCM again. In the case of adding one level of one of the criteria the dimension of the MPCM would increase, thus dramatically

increasing the number of PCs required. Generally, it has to be expected that these changes might occur in the model in the future in order to meet new requirements. If such an adaptation results in the need of providing all the PCs again (or even in providing more of them), an algorithm capable of reducing the number of PCs that need to be provided would be most needed in order to reduce the strain and the time consumption for the experts without substantial loss of information.

5.2.2 Overview of the algorithm of Fedrizzi and Giove (2013) for optimal sequencing in incomplete large-dimensional PCMs

In this section, the algorithm for optimal sequencing (i.e. the optimal choice of PCs) in incomplete large-dimensional PCMs proposed by Fedrizzi and Giove (2013) is briefly summarized as its part is utilized in the method for dealing with incomplete large-dimensional PCMs proposed in the following section. For a more detailed description of the algorithm, interested readers can refer to Fedrizzi and Giove (2013).

Fedrizzi and Giove (2013) proposed an algorithm for iteratively selecting PCs that should be provided by the DM in an incomplete PCM. The algorithm was presented in the form for APCMs-A. Nevertheless, the authors themselves emphasized that the approaches based on APCMs and MPCMs are equivalent (this was also shown in detail in Section 2.4).

The algorithm uses a selection rule based on two criteria. The first criterion, quantified by y_{ij}, is used to achieve enough indirect PCs (r_{ik}, r_{kj}) for every missing PC r_{ij} of the APCM-A $R = \{r_{ij}\}_{i,j=1}^{n}$. The second criterion, quantified by z_{ij}, is used to reduce possible inconsistency of judgments. A scoring function F is defined to determine the usefulness of selecting a particular pair of not yet mutually compared objects o_i and o_j, $i, j \in \{1, \ldots, n\}$. A high value of the scoring function indicates high necessity to compare o_i with o_j, $i, j \in \{1, \ldots, n\}$. Thus, at each step of the algorithm, the pair of objects with the maximal value of F is selected. The scoring function is defined as

$$F(y_{ij}, z_{ij}) = \lambda y_{ij} + (1 - \lambda)z_{ij}, \tag{V.1}$$

where $\lambda \in [0, 1]$ is the parameter quantifying the importance of the criterion y_{ij} over the criterion z_{ij}. Using the simplified notation $f(o_i, o_j) := F(y_{ij}, z_{ij})$ to refer directly to the pair of objects, the selection rule is defined as

$$(o_i, o_j) = \arg \max_{(o_k, o_l) \in \Omega \setminus Q} f(o_k, o_l), \tag{V.2}$$

where Q is the set of PCs that were already provided by the DM during the questioning process and $\Omega = \{(o_i, o_j); i, j = 1, \ldots, n, \ i < j\}$ is the set of all PCs between the n objects. The criteria used in the scoring function (V.1) are defined by the following formulas:

$$y_{ij} = 1 - \frac{|s_i| + |s_j|}{2(n-2)}, \tag{V.3}$$

$$z_{ij} = \frac{\varphi_{ij}}{|s_i \cap s_j| + 1} \frac{1}{3} = \frac{3}{|s_i \cap s_j| + 1} \varphi_{ij}. \tag{V.4}$$

First, let us analyze the expression (V.3), where $s_i = \{k; (o_i, o_k) \in Q \vee (o_k, o_i) \in Q\}$ and $|s_i|$ is the cardinality[1] of the set s_i. Then $|s_i| + |s_j|$ is the number of PCs involving object o_i or object o_j. The maximum value of $|s_i|$ is $n - 2$ since (o_i, o_j) was not yet provided and (o_i, o_i) is excluded. Thus, the maximum value of $|s_i| + |s_j|$ is $2(n-2)$, and $\frac{|s_i| + |s_j|}{2(n-2)}$ represents the normalized number of PCs involving objects o_i or o_j. Criterion y_{ij} is defined by (V.3) in order to have the scoring function F increasing in both variables. Criterion y_{ij} determines the lack of PCs suffered by objects o_i and o_j.

Now, let us analyze the expression (V.4), where φ_{ij} is the mean inconsistency of the indirect PCs of objects o_i and o_j. First, let us define the variable μ_{ij} representing the mean value of all indirect PCs of o_i and o_j, based on the additive-consistency condition (II.28):

$$\mu_{ij} = \begin{cases} 0 & \text{if } s_i \cap s_j = \varnothing, \\ \sum_{k \in s_i \cap s_j} \frac{r_{ik} + r_{kj} - 0.5}{|s_i \cap s_j|} & \text{if } s_i \cap s_j \neq \varnothing. \end{cases} \tag{V.5}$$

Because indirect PCs of objects o_i and o_j are usually not completely consistent, the mean inconsistency φ_{ij} of indirect PCs of o_i and o_j is defined as

$$\varphi_{ij} = \begin{cases} 0 & \text{if } s_i \cap s_j = \varnothing, \\ \sum_{k \in s_i \cap s_j} \frac{(r_{ik} + r_{kj} - 0.5 - \mu_{ij})^2}{|s_i \cap s_j|} & \text{if } s_i \cap s_j \neq \varnothing. \end{cases} \tag{V.6}$$

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[1] cardinality $|s|$ of the set s is the number of its elements; e.g. $|\{2, 4, 5\}| = 3$

Note that for $s_i \cap s_j \neq \emptyset$, φ_{ij} is the variance of $(r_{ih} + r_{hj} - 0.5)$, and it holds that $\varphi_{ij} = 0$ if and only if all the indirect PCs of o_i and o_j are additively consistent according to Definition 9.

The maximum achievable reduction $\Delta\varphi_{ij}$ of φ_{ij} is obtained if the direct PC is $r_{ij} = \mu_{ij}$ and, in such a case, $\Delta\varphi_{ij} = \frac{\varphi_{ij}}{|s_i \cap s_j| + 1}$. In the formula (V.4), $\Delta\varphi_{ij}$ is normalized, i.e. it is divided by $\frac{1}{3}$ as it is the maximum achievable value of $\Delta\varphi_{ij}$; see Fedrizzi and Giove (2013). The criterion z_{ij} expresses the normalized maximum achievable reduction of the inconsistency φ_{ij} which can be reached by means of the direct PCs of o_i and o_j.

The algorithm for selecting the PCs that should be provided by the DM in an incomplete APCM-A given by Fedrizzi and Giove (2013) consists of the following steps:

1. At the beginning, no PCs are performed and $Q = \emptyset$. Thus, $y_{ij} = 1$, $z_{ij} = 0$, and $f(o_i, o_j) = \lambda$ for all $i, j = 1, \ldots, n$. Instead of a random selection, recommended initial PCs are $\{(o_{2i-1}, o_{2i}); i = 1, \ldots, \frac{n}{2}\}$ if n is even and $\{(o_{2i-1}, o_{2i}); i = 1, \ldots, \frac{n-1}{2}\}$ if n is odd.

2. In each step of the selection process, the value of the scoring function f is quantified for each missing PC (o_i, o_j) by using the formula (V.1). According to (V.2), the suitable PC (o_i, o_j) is selected. In the case of equal values of $f(o_i, o_j)$, the pair of objects o_{i^*}, o_{j^*} such that $i^* + j^*$ minimizes $i + j$ is selected. In the case of equal values of $i + j$, the pair containing the minimum index is selected.

3. The selection is stopped when the value of the scoring function becomes lower than the threshold $\delta \in [0, 1]$ which is subjectively defined by the DM, i.e.

$$\max_{(o_i, o_j) \in \Omega \setminus Q} f(o_i, o_j) \leq \delta. \tag{V.7}$$

5.3 New method for incomplete large-dimensional PCMs

In this section a novel method for large-dimensional PCMs is proposed. In particular, Section 5.3.1 provides a detailed description of the method. In Section 5.3.2, the application of the method is demonstrated on an illustrative example and compared with another well-known method for large-dimensional PCMs. In Section 5.3.3, the method is applied to the evaluation model for the Registry of Artistic Performances and the results are confronted with the results obtained by the original model proposed by Stoklasa et al. (2013). In Section 5.3.4, the results of numerical simulations are provided in order to analyze the performance of the method.

5.3.1 Description of the method

In this section, a novel method for inputting PCs in large-dimensional PCMs and for computing interval priorities from incomplete PCMs is proposed. The method combines the concept of weak consistency with the PC-selection process proposed by Fedrizzi and Giove (2013). The proposed interactive algorithm guides the DM through the PC-input phase by identifying which pair of objects should be compared next. This way, the increase of preference information in the incomplete PCM is maximized and the compliance with the weak-consistency condition is ensured in each step of the algorithm. This results in a weakly consistent incomplete PCM after each input. Moreover, information on all feasible preference intensities of each missing PC of an incomplete PCM (that is such values that would not violate the weak consistency when put in the PCM) is available in each step of the algorithm. Values that are unambiguous are input automatically into the PCM and the DM is not bothered to provide these. This way, the amount of information contained in the incomplete PCM can increase after each step without the effort of the DM. When enough information is provided by the DM, the algorithm stops asking the DM for inputs and determines the preference ordering of the objects and their priorities, which are in this case in the form of intervals.

Let us consider objects $o_1, o_2, \ldots, o_n$ to which priorities need to be assigned. The PC of a pair of objects o_i and o_j will be denoted as (o_i, o_j). Considering that the MPCM approach, the APCM-A approach, and the APCM-M approach are equivalent (transformation of one representation into the other can be done using the formulas reviewed in Section 2.4), the DM can express the preference intensities in any of these forms. For the sake of the algorithm presentation and without any loss of generality by presenting the algorithm only for one PCM approach, the MPCM approach is chosen in this section to present the algorithm. This approach is used also because the practical application of large-dimensional PCMs presented in Section 5.2.1 was actually done using a MPCM. In this way, it will be possible to confront the outputs of the algorithm proposed in this section with the practical result using the full-information MPCM approach directly.

Saaty's scale given in Tab. II.1 is used here for expressing the PCs in the MPCM $M = \{m_{ij}\}_{i,j=1}^{n}$, i.e. $m_{ij} \in \{\frac{1}{9}, \frac{1}{8}, \ldots, \frac{1}{2}, 1, 2, \ldots, 8, 9\}$, $i, j = 1, \ldots, n$, with the meanings described in Tab. II.1. Since MPCM M is multiplicatively reciprocal, it is sufficient to enter only the PCs above the main diagonal of M or alternatively only the PCs below the main diagonal of M. In this algorithm, without any loss of information, the PCs above the main diagonal of M are required from the DM. Hence, the set Ω of all PCs required to complete the PCM is

$\Omega = \{(o_i, o_j); i, j = 1, \ldots, n, \ i < j\}$, the cardinality of Ω being $|\Omega| = n(n-1)/2$. The objective of this algorithm is twofold: (i) finding such a set $\bar{\Omega} \subset \Omega$ that its cardinality (i.e. the number of the PCs required from the DM) allows for the computation of all the priorities of objects, and (ii) proposing a way of generating the elements of this set in such order that minimizes the cardinality of $\bar{\Omega}$.

The set of all PCs already performed will be denoted by Q, and the set of PCs not yet entered into the MPCM will be denoted $\Omega \setminus Q$. For each $(o_i, o_j) \in \Omega \setminus Q$, the set $FV_{ij} \subseteq \{\frac{1}{9}, \frac{1}{8}, \ldots, \frac{1}{2}, 1, 2, \ldots, 8, 9\}$ of all feasible values that are in compliance with the weak-consistency condition (II.11) will be always given. For simplicity and in the figures, the notation $[\min FV_{ij}, \max FV_{ij}]$ will be used where there is no risk of ambiguity. The notation $[\min FV_{ij}, \max FV_{ij}]$ represents a range of the values from Saaty's scale from $\min FV_{ij}$ to $\max FV_{ij}$ for a given $(o_i, o_j) \in \Omega \setminus Q$. For example, the set $\{6, 7, 8, 9\}$ will be denoted as $[6, 9]$ and interpreted as a range of values of Saaty's scale from 6 to 9. An incomplete MPCM will be denoted $\widehat{M} = \{\widehat{m}_{ij}\}_{i,j=1}^n$, where

$$\widehat{m}_{ij} = \begin{cases} [m_{ij}^L, m_{ij}^U] & \text{for } (o_i, o_j) \in \Omega \setminus Q, \\ m_{ij} & \text{for } (o_i, o_j) \in Q. \end{cases}$$

It is obvious that $[m_{ij}^L, m_{ij}^U] = [\min FV_{ij}, \max FV_{ij}]$ for each $(o_i, o_j) \in \Omega \setminus Q$.

The process of guided input of the preference information and computation of the priorities of n compared objects can be summarized in the following steps:

1. The DM chooses which PCM will be used to express the preference intensities (MPCM is considered for the purpose of the description of the algorithm). The diagonal elements (o_i, o_i) of MPCM $\widehat{M} = \{\widehat{m}_{ij}\}_{i,j=1}^n$ are set, i.e. $\widehat{m}_{ii} = 1$ for all $i = 1, \ldots, n$. The sets of feasible values (FV sets) FV_{ij} are established for $(o_i, o_j) \in \Omega \setminus Q$. At the beginning, $FV_{ij} = [\frac{1}{9}, 9]$ for $(o_i, o_j) \in \Omega$.

2. The DM provides initial PCs. In this algorithm, the setting proposed by Fedrizzi and Giove (2013) is used, i.e. the set of initial PCs $\{(o_{2i-1}, o_{2i}), \ i = 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, where $\lfloor \frac{n}{2} \rfloor$ is the floor[2] of $\frac{n}{2}$, is required from the DM. However, also a different set of initial PCs can be selected. The only restriction is that these initial PCs do not violate the weak-consistency condition (II.11).

The following Steps 3–5 are repeated until the stopping criterion is met:

3. Based on the algorithm of Fedrizzi and Giove (2013), we determine iteratively which PC $(o_i, o_j) \in \Omega \setminus Q$ is to be provided next by the DM. The PC (o_i, o_j) that maximizes the scoring function (V.1) is selected, and the DM is asked to provide the corresponding preference intensity into the incomplete PCM $\widehat{M}$. The DM selects the value of the PC (o_i, o_j) from its FV set FV_{ij}.

4. Based on the weak-consistency requirement, the FV set FV_{ij} is recalculated for each missing PC $(o_i, o_j) \in \Omega \setminus Q$. The weak-consistency rules (II.11)–(II.14) for MPCMs are used in this step of the algorithm to determine $[\min FV_{ij}, \max FV_{ij}]$.

   Obviously, the FV set is restricted only when an indirect PC exists. That is when for a PC (o_i, o_j) not yet entered into the MPCM there exists at least one object with index $k, k \neq i, j$, such that the PCs (o_i, o_k) and (o_k, o_j) are already entered into the incomplete MPCM $\widehat{M}$ or restricted FV sets are determined for them.

5. Missing PCs $(o_i, o_j) \in \Omega \setminus Q$, for which FV_{ij} contains just a single element, are entered into the incomplete MPCM $\widehat{M}$ automatically. Obviously, the occurrence of such single-element FV_{ij} sets is far more frequent when a discrete scale is used for making PCs of objects. In real-life applications, the requirement of a discrete scale rather than a continuous scale is not a constraint of the decision-making problem. That is because in real-life applications discrete scales of numbers (either crisp of fuzzy) with assigned linguistic terms expressing the intensities of preference are used far more frequently than continuous scales. Discrete scales are more natural for DMs as they provide the required simplifying granularity for continuous universes similar to the common language. The algorithm, however, remains valid also for continuous scales. Since the choice of the scale is out of the scope of this thesis, discrete Saaty's scale as given in Tab. II.1 is assumed for the description of the algorithm. The sets FV_{ij} are recalculated (Step 4 is performed) after each such input and Step 5 is performed again. Steps 4 and 5 are repeated until there are no elements of the incomplete MPCM $\widehat{M}$ that could be entered automatically this way.

6. Stopping criterion: For every missing PC in the incomplete MPCM $\widehat{M}$, there exists at least one indirect PC.

   This condition requires us to be able to determine for each missing PC (o_i, o_j) a restricted set of feasible intensities of preference which could be entered in order to preserve weak consistency of the MPCM.

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[2]floor $\lfloor x \rfloor$ of $x \in \mathbb{R}$ is the largest integer lower or equal to x; e.g. $\lfloor 5.7 \rfloor = 5$

Once the stopping criterion is met, we know for each PC (o_i, o_j) of the incomplete MPCM $\widehat{M}$ either its value or its FV set FV_{ij} restricted by the weak consistency if the PC (o_i, o_j) was not entered yet.

This stopping criterion varies from the stopping criterion proposed by Fedrizzi and Giove (2013). Since the scope of this method is to be able to compute the interval priorities of objects, it is required that for each missing PC in the incomplete MPCM $\widehat{M}$ there exists at least one indirect PC. This means that, for each missing PC $\widehat{m}_{ij}$, we are able to determine a (restricted) set FV_{ij} of feasible intensities of preference which can be entered in order to preserve the weak-consistency condition (II.11).

7. The so-called reciprocal FV sets are identified, i.e. such FV_{ij}, $(o_i, o_j) \in V \subseteq \Omega \setminus Q$, that contain at least one of the values of the respective scale along with its reciprocal value. As an example, a set containing the two numbers 3 and $\frac{1}{3}$ is a reciprocal FV set. From a reciprocal FV set FV_{ij}, it is not possible to derive which object from the pair (o_i, o_j) is preferred to the other. This ambiguity is not desired. Thus, all reciprocal FV sets need to be replaced by a specific value provided by the DM or by a non-reciprocal FV set (as a consequence of filling in a value from another reciprocal FV set), so that $V = \emptyset$. The DM is asked to provide a PC $(o_k, o_l) \in V$ such that $(o_k, o_l) = \arg \max\limits_{(o_i, o_j) \in V} |FV_{ij}|$. In the case that there are more pairs of objects with the same maximal cardinality of their reciprocal FV sets, one of them is chosen randomly. Alternatively, to make the algorithm more user friendly, the DM can be asked to provide the PC of one pair of objects of his/her choice. After such PC is provided, i.e. after the DM chooses one value from the given reciprocal FV set, FV_{ij}, $(o_i, o_j) \in \Omega \setminus Q$, are recalculated using Steps 4 and 5. This step is repeated until there are no reciprocal FV sets left.

Described technique enables us to reduce the amount of information required from the DM as much as possible since providing the PC of the pair of objects with the maximal cardinality of the problematic set adds the most information to the MPCM.

8. The preference ordering of objects is derived from the incomplete MPCM $\widehat{M}$. For each object (represented by the corresponding row of the MPCM), we determine the number of elements in the given row of the MPCM that are greater than or equal to the indifference value or for which the elements of the FV set are all greater than or equal to the indifference value, which is 1 for MPCMs. Based on this information, the objects $o_1, o_2, \ldots, o_n$ can be reordered from the most preferred one to the least preferred one, i.e. $o_{(1)} \succeq o_{(2)} \succeq \cdots \succeq o_{(n)}$. The respectively permuted MPCM with rows and columns ordered from the most preferred object to the least preferred one will be denoted $\widehat{M}^o$.

9. In order to obtain the priorities of objects from the incomplete MPCM $\widehat{M}$, the sets FV_{ij} of feasible intensities of preference for all missing PCs are considered to be intervals given by the minimal and the maximal value in the set (for example, the set $\{3, 4, 5\}$ is now considered to be the interval $[3, 5]$). This allows us to obtain the priorities of objects in the form of intervals. The interval priorities can be obtained either from the preference-ordered MPCM $\widehat{M}^o$ or from the non-preference-ordered MPCM $\widehat{M}$. It is obvious that in both cases we would obtain the same interval priorities as the matrices are the same up to a permutation.

To obtain the interval priorities $\overline{w}_1, \ldots, \overline{w}_n$ of objects, the fuzzy extension of the GMM proposed in Section 4.2.3.2 is used here. Specifically, either the formulas (IV.91)–(IV.94) or the formulas (IV.97)–(IV.100) are applied to the incomplete MPCM $\widehat{M} = \{\widehat{m}_{ij}\}_{i,j=1}^n$. Realize that the incomplete MPCM $\widehat{M}$ is in fact an interval FMPCM; the filled-in PCs are crisp numbers, which are a special case of intervals, and we have intervals of feasible values for all missing PCs. All the formulas for obtaining fuzzy priorities provided in Chapter IV are explicitly written for trapezoidal FPCMs. Nevertheless, recall that interval FPCMs are a particular case of trapezoidal FPCMs. Therefore, keeping this in mind, we can easily apply the formulas (IV.91)–(IV.94) or (IV.97)–(IV.100) to the incomplete MPCM $\widehat{M} = \{\widehat{m}_{ij}\}_{i,j=1}^n$.

Note that in the case when the DM provides preference information utilizing APCMs-A or APCMs-M, the formulas (IV.158) and the formulas (IV.219)–(IV.222), respectively, are used for deriving the interval priorities.

From the formulas (IV.91)–(IV.94) and from the argumentation preceding their construction (see p. 84) it is obvious that the resulting interval priorities contain all the priorities that would be computed for any particular selection of real values from the sets FV_{ij} corresponding to the missing PCs in $\widehat{M}$ (that is if $\widehat{M}$ was completed) preserving the weak-consistency condition. This means that if the DM provided all the missing PCs preserving the weak consistency, the crisp priorities computed from such a MPCM would lie within the computed interval priorities.

Furthermore, because the interval priorities $\overline{w}_i = [w_i^L, w_i^U]$, $i = 1, \ldots, n$, obtained by the formulas (IV.97)–(IV.100) from an incomplete weakly consistent MPCM are normalized according to Definition 28, i.e.

$\overline{w}_i \subseteq [0,1]$ and

$$w_i^L + \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j^U \geq 1, \qquad w_i^U + \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j^L \leq 1 \qquad i = 1, \ldots, n,$$

the interval priorities get very narrow with an increasing dimension n of a MPCM.

Further, the interval priorities obtained from the preference-ordered MPCM $\widehat{M}^o$ by the formulas (IV.97)–(IV.100) have the following property. From the weak consistency and particularly from the property of a non-decreasing sequence of elements in each row and a non-increasing sequence of elements in each column of an ordered weakly consistent MPCM, it follows that any two interval priorities $\overline{w}_i, \overline{w}_j, i, j \in \{1, \ldots, n\}$, obtained by formulas (IV.97)–(IV.100) can be ordered according to the standard partial order $\leq$ on intervals; $[a, b] \leq [c, d]$ if $a \leq c, b \leq d$. Therefore, $\leq$ is a total order on the set of all interval priorities $\overline{w}_i, i = 1, \ldots, n$. Recall that, according to Step 8, the preference ordering of objects is derived immediately from the preference information in $\widehat{M}^o$ without the need of computing the interval priorities. Moreover, for any two objects o_i, o_j such that $o_i \succ o_j$, it holds that $\overline{w}_i > \overline{w}_j$; for the case when $o_i \succeq o_j$ and $o_j \succeq o_i$ it holds that $\overline{w}_i = \overline{w}_j$.

5.3.2 Illustrative example and comparison study

For better understanding, the novel method is demonstrated step-by-step on a simple illustrative example of a weakly consistent MPCM of seven objects. In addition, the performance of the method is compared with the well-known method for incomplete MPCMs proposed by Herrera-Viedma et al. (2004).

Obviously, applying the proposed method to a PCM of just several (in this case seven) objects has only limited significance in practice as such a PCM does not require many PCs from the DM in the first place. However, for better visual illustration of each step of the proposed algorithm, an example with just several objects is more convenient.

Let $o_1, o_2, \ldots, o_7$ be objects which need to be compared and whose priorities need to be determined by the DM. In order to compare seven objects pairwisely, the DM would have to provide 21 PCs in the full-information case. By applying the new algorithm, only a part of these 21 PCs will be required from the DM. In order to evaluate the performance of the new algorithm we need to confront its results with the results obtainable in the hypothetical full-information case. Therefore, let us consider the MPCM given in Fig. V.1 as the full-information MPCM M that would be obtained if the DM provided all 21 PCs. For better illustration, easier understanding and an easy check of the compliance with the weak-consistency condition (II.11), the objects in the MPCM M given in Fig. V.1 are ordered from the most preferred one to the least preferred one. For the sake of simplicity, only the PCs above the main diagonal are given since the PCs below the main diagonal are the reciprocals of the corresponding PCs above the main diagonal. The priorities $w_1, \ldots, w_7$ of objects $o_1, \ldots, o_7$ obtainable from the full-information MPCM M by the GMM (II.24) are given in the second column of Tab. V.1.

Figure V.1: MPCM with all PCs provided by the DM.

	o_1	o_2	o_3	o_4	o_5	o_6	o_7
o_1	1	9	9	9	9	9	9
o_2		1	2	2	9	9	9
o_3			1	1	5	8	8
o_4				1	5	8	8
o_5					1	7	8
o_6						1	7
o_7							1

The method proposed in this chapter is designed to be applicable to general PC problems with no information about the preference ordering of the objects which are to be compared pairwisely, i.e. we suppose that the ordering of the objects from the most preferred one to the least preferred one is not known at the beginning of the decision-making process. This means that the method can be applied to any random initial ordering of objects in the MPCM. Let us therefore assume that the preference ordering of objects is not known in advance and, instead, the objects are ordered randomly. Let us assume the random initial order of the objects as given in Fig. V.2. The empty MPCM $\widehat{M} = \{\widehat{m}_{ij}\}_{i,j=1}^{n}$ in Fig. V.2 is the starting matrix where the PCs identified by the algorithm are going to be provided by the DM or entered automatically based on the weak-consistency condition.

Notice that the labeling of objects in the incomplete MPCM $\widehat{M} = \{\widehat{m}_{ij}\}_{i,j=1}^{n}$ in Fig. V.2 does not correspond to the numbering of rows and columns of the MPCM anymore. For example, object o_1 is not in the first row

Figure V.2: Starting empty MPCM $\widehat{M}$.

	o_3	o_4	o_6	o_1	o_2	o_5	o_7
o_3	1						
o_4		1					
o_6			1				
o_1				1			
o_2					1		
o_5						1	
o_7							1

of the MPCM, but, instead, it is in the fourth row now. Therefore, it is important to realize that from now on when we refer to a PC (o_i, o_j), this does not necessarily correspond to the PC $\widehat{m}_{ij}$ in the $i-$th row and the $j-$th column of $\widehat{M}$.

At the beginning of the algorithm, the diagonal elements are set to the value 1 and the DM is asked to provide initial PCs $(o_3, o_4), (o_6, o_1)$, and (o_2, o_5) as it is required in Step 2 of the algorithm. Any value from Saaty's scale can be chosen in this step. This is because the FV sets for all missing PCs are $[\frac{1}{9}, 9]$. For easier orientation in the figures the initial FV sets $[\frac{1}{9}, 9]$ are replaced by empty fields. Only the FV sets calculated from indirect PCs in the following steps of the algorithm will be entered into the incomplete MPCM $\widehat{M}$.

Steps 3 to 5 are repeated until the stopping criterion is met. In Step 3, we apply the algorithm based on searching for a missing PC (o_i, o_j) with the maximum value of the scoring function (V.1), i.e. the missing PC that should be provided by the DM. In this illustrative example, both criteria of the scoring function (V.1) are considered to have the same importance, therefore the parameter $\lambda = 0.5$ is set.

As already mentioned in the previous section, in contrast to the method proposed by Fedrizzi and Giove (2013), we require the incomplete MPCM $\widehat{M}$ to be weakly consistent; it has to satisfy the properties (II.11)–(II.14). According to this requirement, in Step 4, we are able to restrict the sets FV_{ij} of feasible intensities of preference for some missing PCs. If any set FV_{ij} contains only one value, this value is entered automatically into the incomplete MPCM $\widehat{M}$ as suggested in Step 5.

Fig. V.3 demonstrates the incomplete MPCM $\widehat{M}$ after the initial PCs $(o_3, o_4) = 1$, $(o_6, o_1) = \frac{1}{9}$, and $(o_1, o_5) = 9$ and after the first iteration of the algorithm. The first PC chosen in the first iteration and provided by the DM is $(o_3, o_7) = 8$. As it can be seen from the incomplete MPCM $\widehat{M}$, the PC $(o_4, o_7) = 8$ was filled in automatically according to the weak consistency since $(o_3, o_4) = 1$ and $(o_3, o_7) = 8$. The legend explaining the notation used in the figures in this section is provided in Fig. V.4.

Figure V.3: Incomplete MPCM $\widehat{M}$ after the first iteration.

	o_3	o_4	o_6	o_1	o_2	o_5	o_7
o_3	1	1					8
o_4		1					8
o_6			1	1/9			
o_1				1			
o_2					1	9	
o_5						1	
o_7							1

Figure V.4: Legend.

1/9	initial PC provided by the DM
8	PC provided by the DM during the algorithm
8	PC filled in automatically according to the weak consistency
[1/9,1/2]	FV set
[1/9,9]	reciprocal FV set
1/9	value from the reciprocal FV set provided by the DM

Fig. V.5 shows the incomplete MPCM $\widehat{M}$ after the second iteration of the algorithm. The PC $(o_4, o_6) = 8$ was provided by the DM, and according to the weak consistency, one missing PC and ranges for other three missing PCs were added automatically. For example, the range $[1/9, 1/2]$ for the missing PC (o_3, o_1) was derived from

the PCs $(o_3, o_6) = 8$ and $(o_6, o_1) = 1/9$ according to the first rule of the weak-consistency property (II.13). The DM continues providing the missing PCs until the stopping criterion is met. The incomplete MPCM obtained at the moment of meeting the stopping criterion is given in Fig. V.6.

Figure V.5: Incomplete MPCM $\widehat{M}$ after the second iteration.

	o_3	o_4	o_6	o_1	o_2	o_5	o_7
o_3	1	1	8	[1/9,1/2]			8
o_4		1	8	[1/9,1/2]			8
o_6			1	1/9			[1/8,8]
o_1				1			
o_2					1	9	
o_5						1	
o_7							1

Figure V.6: Incomplete MPCM $\widehat{M}$ after the stopping criterion is met.

	o_3	o_4	o_6	o_1	o_2	o_5	o_7
o_3	1	1	8	1/9	[1/9,1/2]	5	8
o_4		1	8	1/9	[1/9,1/2]	5	8
o_6			1	1/9	[1/9,9]	[1/8,1/2]	[1/8,8]
o_1				1	9	9	9
o_2					1	9	9
o_5						1	[2,8]
o_7							1

Two reciprocal FV sets are present in the incomplete MPCM $\widehat{M}$ in Fig. V.6; see the PCs $(o_6, o_2) = [\frac{1}{9}, 9]$ and $(o_6, o_7) = [\frac{1}{8}, 8]$. This means that we are not even able to decide which object is preferred to the other one for these pairs of objects; the information obtained from the indirect PCs is too vague. Therefore, according to Step 7, we have to ask the DM to determine the intensities of preference for these pairs of objects.

First the DM is asked to provide the PC (o_6, o_2) as its reciprocal FV set has the biggest cardinality ($|FV_{62}| = |\{\frac{1}{9}, \frac{1}{8}, \ldots, \frac{1}{2}, 1, 2, \ldots, 9\}| = 17$). In this particular case no restriction of the other FV sets occurs. Afterwards, the DM provides the PC (o_6, o_7) and, as a consequence, the FV set of (o_5, o_7) is reduced from $[2, 8]$ to $[7, 8]$. Fig. V.7 shows the incomplete MPCM $\widehat{M}$ after Step 7.

Figure V.7: Incomplete MPCM $\widehat{M}$ after removing the reciprocal FV sets.

	o_3	o_4	o_6	o_1	o_2	o_5	o_7
o_3	1	1	8	1/9	[1/9,1/2]	5	8
o_4		1	8	1/9	[1/9,1/2]	5	8
o_6			1	1/9	1/9	[1/8,1/2]	7
o_1				1	9	9	9
o_2					1	9	9
o_5						1	[7,8]
o_7							1

Once the reciprocal FV sets are removed, we are able to order the compared objects from the most preferred one to the least preferred one according to Step 8 and to reorder the whole incomplete MPCM $\widehat{M}$ accordingly. Fig. V.8 demonstrates the preference-ordered incomplete MPCM $\widehat{M}^o$ with FV sets for all missing PCs. The reader can verify that by choosing any value from any of the FV sets the weak consistency of the incomplete MPCM $\widehat{M}^o$ is not violated.

According to Step 9 of the algorithm, the interval priorities of objects are obtained from the incomplete MPCM $\widehat{M}^o$. The interval priorities are summarized in Tab. V.1 along with the crisp priorities computed from the full-information MPCM M given in Fig. V.1.

Let us summarize the results of this illustrative example. In order to have complete preference information and to compute crisp priorities of objects, the DM would have to provide 21 PCs. Using the algorithm for incomplete PCMs proposed in Section 5.3.1, the DM had to provide only 10 PCs (approx. 48%). Other 7 PCs (approx. 33%) were added automatically based on the weak-consistency condition and 4 PCs (approx. 19%)

Figure V.8: Final preference-ordered incomplete MPCM $\widehat{M}^o$.

	o_1	o_2	o_3	o_4	o_5	o_6	o_7
o_1	1	9	9	9	9	9	9
o_2		1	[2,9]	[2,9]	9	9	9
o_3			1	1	5	8	8
o_4				1	5	8	8
o_5					1	[2,8]	[7,8]
o_6						1	7
o_7							1

were missing with FV sets the elements of which do not violate the weak consistency of the incomplete MPCM $\widehat{M}^o$. The proposed algorithm did not only spare the DM more than half of the PCs, but it also provided very good results. The calculated interval priorities are quite narrow and contain the original priorities; see Tab. V.1. Recall that it was mentioned in Step 9 of the algorithm that this is a general property that always holds.

Table V.1: Priorities of objects.

Objects	Crisp priorities obtained by the GMM	Interval priorities obtained by the new method	Priorities obtained according to Herrera-Viedma et al. (2004)
o_1	0.5083	[0.4840, 0.5102]	0.2377
o_2	0.1765	[0.1765, 0.2594]	0.1128
o_3	0.1166	[0.0896, 0.1170]	0.2284
o_4	0.1166	[0.0896, 0.1170]	0.2284
o_5	0.0463	[0.0363, 0.0472]	0.0535
o_6	0.0228	[0.0213, 0.0273]	0.1128
o_7	0.0128	[0.0122, 0.0131]	0.0264

To emphasize the advantage and the significant contribution of this method to the decision-making theory, it is also compared here with another well-known method for incomplete MPCMs. Particularly, the method proposed by Herrera-Viedma et al. (2004) will be applied to this illustrative example for the comparison. Notice that the paper of Herrera-Viedma et al. (2004) has been cited over 560-times which suggests wide recognition of the method). In the method proposed by Herrera-Viedma et al. (2004), only $n-1$ PCs above the main diagonal, i.e. $\{(o_i, o_{i+1}); i = 1, \ldots, n-1\}$, are required from the DM. The remaining PCs are completed automatically so that the resulting MPCM $M = \{m_{ij}\}_{i,j=1}^n$ is multiplicatively consistent according to (II.4). Clearly, in most of the cases, the missing PCs completed by this automatic procedure exceed Saaty's scale $\left[\frac{1}{9}, 9\right]$. That is why Herrera-Viedma et al. (2004) suggest to transform the obtained MPCM M given on scale $\left[\frac{1}{c}, c\right], c > 9$, into the MPCM $M' = \{m'_{ij}\}_{i,j=1}^n$ given on scale $\left[\frac{1}{9}, 9\right]$ by using transformation formula

$$m'_{ij} = m_{ij}^{1/\log_9 c}, \qquad i, j = 1, \ldots, n. \tag{V.8}$$

In Fig. V.9, the completed, transformed and ordered MPCM M' after providing the 6 initial PCs above the main diagonal is given. The 6 PCs provided by the DM are highlighted in bold. Obviously, unlike the incomplete MPCM $\widehat{M}^o$ in Fig. V.8, the MPCM M' in Fig. V.9 differs substantially from the original MPCM in Fig. V.1. Thus, also the priorities obtained from this MPCM given in the last column of Tab. V.1 vary essentially from the original priorities given in the second column. Even the ranking of the objects based on these priorities varies from the ranking obtained in the full-information case.

In order too demonstrate how far the MPCM M' in Fig. V.9 obtained by the method proposed by Herrera-Viedma et al. (2004) is from the original MPCM M in comparison to the incomplete MPCM $\widehat{M}^o$ obtained by the method proposed in the previous section, their distances will be measured. In particular, the distance for MPCMs defined by Cook and Kress (1988) is utilized for this scope. Since the incomplete MPCM $\widehat{M}^o$ in Fig. V.8 contains intervals, it is necessary to generalize the distance of Cook and Kress (1988) to interval FMPCMs first. For two interval FMPCMs $\overline{M}^1 = \{\overline{m}_{ij}^1\}_{i,j=1}^n, \overline{m}^1 = \left[m_{ij}^{1L}, m_{ij}^{1U}\right], \overline{M}^2 = \{\overline{m}_{ij}^2\}_{i,j=1}^n, \overline{m}^2 = \left[m_{ij}^{2L}, m_{ij}^{2U}\right]$, the

	o_1	o_2	o_3	o_4	o_5	o_6	o_7
o_1	1	**2.1080**	1.0408	1.0408	4.4436	**2.1080**	9
o_2		1	0.4937	0.4937	**2.1080**	1	4.2695
o_3			1	1	4.2695	2.0254	8.6473
o_4				1	4.2695	**2.0254**	8.6473
o_5					1	0.4742	**2.0254**
o_6						1	4.2695
o_7							1

interval distance based on the distance defined by Cook and Kress (1988) is given as $\overline{D}(\overline{M}^1, \overline{M}^2) = [d^L, d^U]$

$$
d^L = \min_{\substack{m_{ij}^1 \in [m_{ij}^{1L}, m_{ij}^{1U}] \\ m_{ij}^2 \in [m_{ij}^{2L}, m_{ij}^{2U}]}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |\ln(m_{ij}^1 / m_{ij}^2)|
$$

$$
= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \min_{\substack{m_{ij}^1 \in [m_{ij}^{1L}, m_{ij}^{1U}] \\ m_{ij}^2 \in [m_{ij}^{2L}, m_{ij}^{2U}]}} |\ln(m_{ij}^1 / m_{ij}^2)|,
$$

(V.9)

$$
d^U = \max_{\substack{m_{ij}^1 \in [m_{ij}^{1L}, m_{ij}^{1U}] \\ m_{ij}^2 \in [m_{ij}^{2L}, m_{ij}^{2U}]}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |\ln(m_{ij}^1 / m_{ij}^2)|
$$

$$
= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \max_{\substack{m_{ij}^1 \in [m_{ij}^{1L}, m_{ij}^{1U}] \\ m_{ij}^2 \in [m_{ij}^{2L}, m_{ij}^{2U}]}} |\ln(m_{ij}^1 / m_{ij}^2)|.
$$

(V.10)

For crisp MPCMs, the interval distance given by (V.9), (V.10) is identical to the distance originally defined by Cook and Kress (1988). By applying the formulas (V.9) and (V.10), the distance of the MPCM obtained by the method proposed by Herrera-Viedma et al. (2004) given in Fig. V.9 and the original MPCM given in Tab. V.1 is $D = 22.8941$. The interval distance of the interval FMPCM in Fig. V.8 from the original MPCM in Fig. V.1 is $\overline{D} = [0, 4.3934]$. Clearly, $\overline{D} = [0, 4.3934]$ is significantly smaller than $D = 22.8941$, which demonstrates better performance of the method proposed in this thesis.

Notice that the lower boundary value d^L of the distance of any incomplete MPCM with intervals of feasible values for all missing PCs obtained by the new method from the hypothetical full-information MPCM is always 0. This follows from the fact that the incomplete MPCM always contains the hypothetical full-information MPCM obtainable if all PCs were provided by the DM, which is the substance and the main advantage of the proposed method.

5.3.3 Application of the method to the Evaluation model for the Registry of Artistic Performances

In this section, the large-dimensional problem of evaluating outcomes of artistic performance in the Czech Republic that was solved by Stoklasa et al. (2013) and briefly introduced in Section 5.2.1 is approached. The method for optimal construction of an incomplete MPCM and obtaining interval priorities of the categories of artistic production introduced in Section 5.3.1 is applied. The outcome of the new method is compared with the outcome given by Stoklasa et al. (2013). Again, we draw from the knowledge of the complete MPCM and conduct a numerical experiment. In particular, we start with an empty MPCM of randomly ordered 27 categories of works of art, we utilize the novel method proposed in Section 5.3.1, and, whenever a PC is required from the DM, we find the appropriate value in the complete MPCM given in Fig. V.10.

We assumed the randomly generated initial order of the categories (i.e. categories are not ordered according to their preference but randomly) given in the heading of Fig. V.11. For better orientation and simpler notation, the number of the corresponding row was assigned to each category. First, the DM was asked to provide 13 initial PCs $\{(2i-1, 2i)\,;\ i = 1, \dots, 13\}$. Subsequently, the algorithm for selecting the missing PCs $(i, j), i, j \in \{1, 2, \dots, 27\}, i < j$, which should be provided by the DM was applied. The parameter $\lambda = 0.5$ was used in the scoring function (V.1) as both its criteria were considered to have the same importance. The algorithm was stopped after just 109 PCs provided by the DM.

Because missing PCs with reciprocal FV sets were present in the incomplete MPCM at that stage, it was not possible to order the categories from the most preferred one to the least preferred one immediately. First,

Figure V.10: Complete weakly consistent MPCM obtained by Stoklasa et al. (2013).

	1 AKX	2 AKY	3 AKZ	4 ALX	5 AMX	6 ALY	7 ALZ	8 BKX	9 AMY	10 AMZ	11 BKY	12 BKZ	13 BLX	14 BMX	15 BLY	16 BLZ	17 BMY	18 BMZ	19 CKX	20 CLX	21 CKY	22 CKZ	23 CMX	24 CLY	25 CLZ	26 CMY	27 CMZ	
AKX	1																											1
AKY	5	1																										2
AKZ	5	5	1																									3
ALX	5	5	3	1																								4
AMX	5	5	3	3	1																							5
ALY	5	5	5	5	5	1																						6
ALZ	5	5	5	5	5	3	1																					7
BKX	5	5	5	5	5	3	3	1																				8
AMY	5	5	5	5	5	5	5	1																				9
AMZ	5	5	5	5	5	5	5	3	1																			10
BKY	5	5	5	5	5	5	5	5	1																			11
BKZ	5	5	5	5	5	5	5	5	5	1																		12
BLX	5	5	5	5	5	5	5	5	5	3	1																	13
BMX	5	5	5	5	5	5	5	5	5	5	1																	14
BLY	5	5	5	5	5	5	5	5	5	3	1																	15
BLZ	7	5	5	5	5	5	5	5	5	3	3	1																16
BMY	7	5	5	5	5	5	5	5	5	3	3	3	1															17
BMZ	9	7	5	5	5	5	5	5	5	5	5	1																18
CKX	9	7	7	5	5	5	5	5	5	5	5	1																19
CLX	9	7	7	7	5	5	5	5	5	5	3	1																20
CKY	9	7	7	7	5	5	5	5	5	5	5	1																21
CKZ	9	9	7	7	5	5	5	5	5	5	3	1																22
CMX	9	9	9	7	7	5	5	5	5	5	5	3	3	1														23
CLY	9	9	9	7	7	5	5	5	5	5	5	5	1															24
CLZ	9	9	9	9	7	7	7	7	7	5	5	5	5	5	3	1												25
CMY	9	9	9	9	9	7	7	7	7	7	7	5	5	5	3	3	1											26
CMZ	9	9	9	9	9	7	7	7	7	7	7	7	5	5	5	3	3	3	1									27

Figure V.11: Incomplete weakly consistent MPCM after filling in the PCs by the DM.

Table V.2: Interval and crisp priorities of the categories.

| Categories		Crisp priorities	Interval priorities
1	AKX	0.1357	$[0.1314, 0.1370]$
2	AKY	0.1132	$[0.1126, 0.1166]$
3	AKZ	0.0967	$[0.0917, 0.0995]$
4	ALX	0.0862	$[0.0829, 0.0895]$
5	ALY	0.0761	$[0.0687, 0.0799]$
6	ALZ	0.0612	$[0.0593, 0.0660]$
7	AMX	0.0552	$[0.0542, 0.0573]$
8	AMY	0.0498	$[0.0495, 0.0509]$
9	AMZ	0.0418	$[0.0415, 0.0423]$
10	BKX	0.0385	$[0.0382, 0.0390]$
11	BKY	0.0335	$[0.0333, 0.0340]$
12	BKZ	0.0292	$[0.0280, 0.0296]$
13	BLX	0.0269	$[0.0258, 0.0273]$
14	BLY	0.0222	$[0.0211, 0.0249]$
15	BLZ	0.0204	$[0.0194, 0.0215]$
16	BMX	0.0184	$[0.0176, 0.0192]$
17	BMY	0.0167	$[0.0160, 0.0175]$
18	BMZ	0.0134	$[0.0133, 0.0140]$
19	CKX	0.0117	$[0.0114, 0.0125]$
20	CKY	0.0106	$[0.0102, 0.0112]$
21	CKZ	0.0088	$[0.0088, 0.0092]$
22	CLX	0.0080	$[0.0077, 0.0082]$
23	CLY	0.0072	$[0.0067, 0.0074]$
24	CLZ	0.0057	$[0.0053, 0.0066]$
25	CMX	0.0047	$[0.0045, 0.0051]$
26	CMY	0.0042	$[0.0040, 0.0045]$
27	CMZ	0.0038	$[0.0035, 0.0040]$

it was necessary to remove all reciprocal FV sets FV_{ij}. This was done iteratively and after the replacement of every single reciprocal FV set FV_{ij} either by a PC provided by the DM or by a non-reciprocal FV set, all the remaining missing elements were recalculated. In order to eliminate all the reciprocal FV sets, 23 PCs were required from the DM overall. The incomplete MPCM obtained after this step is shown in Fig. V.11. Finally, the categories compared in the incomplete MPCM were ordered from the most preferred one to the least preferred one. The preference-ordered weakly consistent incomplete MPCM is given in Fig. V.12.

In the original method proposed by Stoklasa et al. (2013), the experts had to provide all 351 PCs. When the new method was applied to the problem, only 145 PCs (approx. 41%) were required. Other 153 PCs (approx. 44%) were added automatically according to the weak consistency and, for the remaining 53 PCs (approx. 15%), sets of feasible intensities of preference were derived from the weak-consistency properties. These FV sets are relatively narrow containing at most 4 values. Furthermore, the incomplete MPCM contains the original complete MPCM, i.e. all the filled-in PCs are the same and the FV set provided for each missing PC in the incomplete MPCM always contains the preference intensity of the corresponding PC in the complete MPCM; compare Fig. V.10 and Fig. V.12.

Interval priorities of the categories were obtained from the incomplete MPCM in Fig. V.12 by using the formulas (IV.97)–(IV.100). The interval priorities together with the crisp priorities obtained from the complete MPCM in Fig. V.10 by the GMM (II.24) are given in Tab. V.2. Obviously, the crisp priorities of the categories lie within the intervals delimited by the interval priorities. This result is natural since the FV sets for missing PCs in the incomplete PCM obtain all feasible intensities of preference that preserve the weak consistency. Therefore, the complete MPCM given in Fig. V.10 provided by Stoklasa et al. (2013) can be obtained from the weakly consistent incomplete MPCM in Fig. V.12 by a particular combination of values from the FV sets.

Using the new method, we obtained the interval priorities of the categories which represent very well the actual priorities obtained from the complete MPCM (compare the results in Tab. V.2). In contrast to the original method, however, only 145 PCs were required from the DMs instead of 351. This means that the amount of the information required from the DM was reduced to only 41% of the information required by Stoklasa et al. (2013). This is a very significant reduction of the information required from the DM that reduces considerably the strain and time demands and raises the quality of the information provided.

Figure V.12: Preference-ordered incomplete weakly consistent MPCM.

5.3.4 Simulations and numerical results

Simulations were performed to evaluate the benefit of the proposed method from the point of view of sparing a part of PCs required from the DM. MPCMs of dimensions $n = 5, 10, \ldots, 30$ were considered.

The proposed method was applied to 600 randomly generated weakly consistent MPCMs, 100 of each dimension. For each such a MPCM, an empty initial MPCM of the given dimension n was considered and the new method was applied to the empty MPCM in order to identify iteratively the missing PCs that should be provided by the DM. Whenever such a missing PC was identified, the value of the PC was taken from the complete MPCM and entered into the incomplete MPCM. The number x of PCs required in the iterative algorithm from the DM was computed. Consequently, also the number of spared PCs was computed as $n(n-1)/2 - x$. After applying the method to all 100 MPCMs of the given dimension n, an average number of spared PCs was computed as well as an average % of spared PCs. The numerical results are presented in Tab. V.3.

According to the results presented in Tab. V.3, the average percentage of the spared PCs increases with the increasing dimension of the MPCM. For MPCMs of dimension 15 and greater, more than 60% of the PCs are spared on average. However, it is necessary to point out that, despite this huge reduction in the number of PCs required from the DM, the resulting interval priorities obtained by the formulas (IV.97)–(IV.100) from the final incomplete weakly consistent MPCM always contain the crisp priorities obtainable from the original randomly generated weakly consistent MPCM. Moreover, as discussed on p. 145, the interval priorities get very narrow with the increasing dimension of the MPCM.

Table V.3: Average number of spared PCs required from the DM.

dimension of PCMs	$n = 5$	$n = 10$	$n = 15$	$n = 20$	$n = 25$	$n = 30$
number of PCs required in the full-information case	10	45	105	190	300	435
average number of spared PCs	4	24	64	123	207	312
average % of spared PCs	42%	53%	61%	65%	69%	72%

5.4 Conclusion

An answer to the second research question, *"How can the amount of preference information required from the DM in a large-dimensional PCM be reduced while still obtaining comparable priorities of objects"*, was provided in this chapter. In particular, a novel approach for dealing with incomplete large-dimensional PCMs was introduced. The approach is applicable to all three types of PCMs examined in this thesis, i.e. MPCMs, APCMs-A, as well as APCMs-M.

The proposed method strives to identify the tradeoff between decreasing the number of PCs required from the DM and obtaining a sufficient amount of information to compute relevant priorities of objects. The method is suggested as a possible solution to large-dimensional problems where the complete information (i.e. providing all PCs) is either costly, too time consuming, or infeasible to obtain or where the preference intensities in the large-dimensional PCMs require (frequent) revisions.

In the first part of the method, an iterative algorithm for optimal choice of PCs that should be provided by the DM in an incomplete large-dimensional PCM is applied. The algorithm is based on the combination of the weak-consistency condition introduced by Jandová and Talašová (2013) for MPCMs and by Jandová et al. (2017) for APCMs, with the modified version of the optimal PC-selection algorithm proposed by Fedrizzi and Giove (2013). The weak consistency is imposed as a minimum consistency requirement on the incomplete PCM and it is required during the whole process of entering PCs into the incomplete PCM. As a consequence, certain PCs are added automatically on the base of PCs previously provided by the DM and, for some PCs, intervals of feasible values that could be entered without violating the weak consistency are provided. The algorithm is stopped when for every missing PC in the incomplete PCM there exists an interval of feasible values. Afterward, interval priorities of objects are derived from such an incomplete PCM by means of formulas proposed in Chapter IV.

The whole method is designed in such a way that the interval priorities derived from the incomplete large-dimensional PCM include the priorities of objects obtainable from any weakly consistent completion of the incomplete PCM. This means that the interval priorities contain the crisp priorities that would be obtained from the hypothetical complete PCM obtainable if the DM provided all PCs in the PCM.

The numerical results of performed simulations demonstrate that the application of the proposed algorithm can significantly reduce the number of PCs required from the DM and thus results in significant resource sav-

ings. At the same time, a high accuracy of the output is guaranteed by the algorithm as the resulting interval priorities contain the priorities that would be obtained from the hypothetical complete PCM. For randomly generated MPCMs of dimension $n \geq 15$, even more than 60% of PCs required (with respect to the full-information case) were spared on average. The numerical example of a small 7x7 PCM exhibited a reduction of ca. 50% in the number of PCs required (from 21 to 10). In the real-life case study of the works-of-art evaluation model utilizing a 27x27 PCM, the number of PCs required from the experts was reduced by ca. 60% (from 351 PCs only 145 were required). The obtained interval priorities of the 27 categories of works of art contain the crisp priorities obtained from the original complete PCM and they are very narrow.

Chapter VI

Discussion and future research

6.1 Discussion

"Traditional" methods based on PCMs were not designed to cope with MCDM problems under uncertainty. However, uncertainty is integral to human mind and, thus, it is necessarily closely related to decision making. In order to properly handle uncertainty, methods based on PCMs were extended to fuzzy numbers that allow for better modeling of uncertain PCs of objects. When extending PCMs to FPCMs, it is of paramount importance to extend appropriately the key properties of PCMs and of the related methods in order to reflect properly the preference information contained in FPCMs.

Beside the inability to capture uncertainty, the "traditional" methods based on PCMs are also unable to deal with incompleteness of preference information. The problem of incomplete preference information concerns especially large-dimensional PCMs where it is not possible or reasonable to obtain complete preference information from the DM, e.g. due to time or cost limitations. When dealing with incomplete large-dimensional PCMs, compromise between maximally reducing the number of PCs required from the DM and obtaining reasonable priorities of objects from the incomplete PCM is of crucial importance.

Thus, the thesis was aiming at answering two research questions:

(1) *Based on a FPCM of objects, how should fuzzy priorities of these objects be determined so that they reflect properly all preference information available in the FPCM?*

(2) *How can the amount of preference information required from the DM in a large-dimensional PCM be reduced while still obtaining comparable priorities of objects?*

The first research question was answered by pursuing four tasks identified in Section 1.3. Each task and the related findings are summarized as follows:

(1.a) Well-known and in practice most often applied methods based on PCMs were critically reviewed in Chapter II. In particular, three types of PCMs were examined - MPCMs, APCMs-A, and APCMs-M.

Two key properties of PCMs and of the related methods were identified - reciprocity of the related PCs and invariance under permutation of objects. Reciprocity of the related PCs is an inherent property of every PCM that results from the meaning of PCs in the PCM (multiplicative reciprocity for MPCMs and additive reciprocity for APCMs-A and APCMs-M). Invariance under permutation of objects had been introduced as a property that every "good" method should satisfy. is

Therefore, it is necessary to extended properly both these properties also to FPCMs.

(1.b) In Chapter IV, critical review of the approaches to the fuzzy extension of the methods reviewed in Chapter II within task (1.a) was done and two main drawbacks were identified.

The reviewed approaches are mostly based on applying standard fuzzy arithmetic to the fuzzy extension of the methods and they violate the reciprocity of the related PCs or the invariance under permutation of objects. This leads to false results (resulting fuzzy priorities of objects in particular) that distort the preference information contained in the FPCM.

(1.c) Necessity of applying constrained fuzzy arithmetic to the fuzzy extension of the methods based on PCMs in order to reflect properly the preference information contained in the FPCM was demonstrated in Chapter IV.

Constrained fuzzy arithmetic allows for imposing constraints on operands of arithmetic operations with fuzzy numbers. Therefore, reciprocity of the related PCs, which is an inherent property of PCMs, is

introduced as a constraint in the computations with PCs in a FPCM. Applying constrained fuzzy arithmetic with reciprocity constraints to the fuzzy extension of the methods reviewed in Chapter II also automatically ensures invariance of the methods under permutation of objects.

(1.d) The fuzzy extension of the methods critically reviewed within task (1.a) was proposed in Chapter IV in such a way that it reflects properly all preference information contained in the FPCM.

Specifically, a whole set of methods based on constrained fuzzy arithmetic was proposed in the thesis to deal with three types of FPCMs - FMPCMs, FAPCMs-A, and FAPCMs-M. FPCMs were defined properly and two definitions of consistency were given for each type of FPCMs. Formulas for obtaining the fuzzy maximal eigenvalue of a FMPCM were proposed and properties of the fuzzy maximal eigenvalues were identified. The fuzzy maximal eigenvalue is indispensable in order to define fuzzy extension of Consistency Index and Consistency Ratio for verifying acceptable multiplicative consistency of FMPCMs and to define a fuzzy extension of the EVM. Finally, methods for deriving fuzzy priorities of objects from FPCMs were proposed. The methods proposed for each type of FPCMs are mutually equivalent. FM-PCMs, FAPCMs-A, and FAPCMs-M can be transformed one into another together with the respective consistency properties. Similarly, fuzzy priorities obtained from FMPCMs, FAPCMs-A, and FAPCMs-M can be transformed one into another. The proposed methods were compared with the methods critically reviewed within task (1.c). Further, it was proved that all new methods based on constrained fuzzy arithmetic preserve the reciprocity of PCs and are invariant under permutation of objects. By preserving these two key properties, the fuzzy priorities obtained by the new methods reflect better the preference information contained in FPCMs in comparison to the fuzzy priorities obtained by the methods reviewed within task (1.c).

The methods based on constrained fuzzy arithmetic introduced within the answer to the research question (1) require the same amount of preference information from the DM as the reviewed methods based on standard fuzzy arithmetic. However, unlike them, they are invariant under permutation of objects and they preserve the reciprocity of the related PCs. This means that, based on the same amount of preference information from the DM, the new methods provide results that better reflect the preference information contained in the FPCMs, which leads to a better quality of decisions.

The second research question was answered by pursuing two tasks identified in Section 1.3. The tasks were

(2.a) to propose an efficient method for partially filling an incomplete large-dimensional PCM that minimizes the number of PCs required from the DM but provides a sufficient amount of preference information;

(2.b) to propose a suitable method for deriving priorities from an incomplete large-dimen- sional PCM that reflect the incompleteness of preference information and that are "close" to the priorities obtainable from the hypothetical complete PCM.

The tasks resulted to be highly interconnected. In particular, development of the method in task (2.a) was substantially influenced by the requirement to obtain priorities that are "close" to the priorities obtainable from the hypothetical complete PCM. Thus, it is difficult to draw a clear line between the two tasks and, consequently, it is not possible to represent the findings separately for each task.

Tasks (2.a) and (2.b) were carried out by proposing an iterative algorithm for optimal choice of PCs that should be provided by the DM in an incomplete large-dimensional PCM. The algorithm is based on the concept of weak consistency. The weak-consistency condition is a minimum requirement of consistency that has to be satisfied in each step of the algorithm. Based on the weak-consistency condition, some missing PCs are entered into the PCM automatically and, for some, intervals of feasible values are provided. The whole process is based on searching for a compromise between minimizing the number of PCs provided by the DM and maximizing the amount of preference information contained in the incomplete PCM. At the end of the process, interval priorities of objects are computed using the formulas proposed in Chapter IV. The interval priorities include the priorities of objects obtainable from any weakly consistent completion of the incomplete PCM. This means that the interval priorities contain the priorities that would be obtained from the hypothetical complete PCM obtainable if the DM provided all PCs in the PCM. The average percentage of spared PCs in an incomplete PCM increases with the increasing dimension of the PCM; for PCMs of 15 or more objects, more than 60% of PCs are spared on average. Despite this great reduction of PCs required from the DM, the resulting interval priories are very narrow for large-dimensional PCMs.

The novel method is particularly useful for real-life decision-making problems where providing all PCs is either costly, too time consuming, or infeasible to obtain. It is also very effective in dealing with large-dimensional problems where PCs provided by the DM require frequent revisions. By applying the method, the preference information required from the DM is significantly reduced which leads to cost reduction and time saving. Despite this reduction, the method provides results (resulting interval priorities) that are very close to the hypothetical results obtainable from the complete preference information.

Naturally, the methods developed in this thesis have some limitations. In Chapter IV, new MCDM methods based on FPCMs were introduced. The methods were developed by applying constrained fuzzy arithmetic to the fuzzy extension of well-known and in practice most often applied methods based on PCMs (that were critically reviewed in Chapter II). Unlike the methods based on standard fuzzy arithmetic, the new methods preserve both the reciprocity of the related PCs and the invariance under permutation of objects, which are two key properties identified for PCMs and for the related methods. Thus, it is justifiable to claim that the fuzzy priorities of objects obtained by the new methods reflect the preference information contained in FPCMs better in comparison to the fuzzy priorities obtained by the methods based on standard fuzzy arithmetic. However, the whole idea of "properly reflecting" the preference information contained in FPCMs by means of the fuzzy extension of methods based on PCMs in this thesis is based on the assumption that the original methods are a suitable means of representing the preference information contained in PCMs.

The limitation of the method for dealing with large-dimensional PCMs proposed in Chapter V is that the method is based on the assumption that the preference system of the DM is in compliance with the weak-consistency condition. The weak-consistency condition is imposed as a minimal and very natural requirement of consistency in the method, and the DM is expected to provide weakly consistent preference information. Nevertheless, it is not guaranteed that every DM is able or willing to keep weak consistency during the process of providing PCs. If the DM refuses weak consistency as not reflecting properly his or her preference system, the method proposed in Chapter V cannot be used.

Another limitation might be that the formulas proposed in the thesis are based on highly non-linear optimization problems and should be, therefore, carefully managed by numerical computation. In this thesis, some optimization methods predefined in Matlab were used.

6.2 Future research

Despite the effort, the thesis could not cover all issues related to the fuzzy extension of MCDM methods based on PCMs. Therefore, there is still a lot of space for future research. In the following, some ideas are presented.

Calibration: In Section 2.2.1, Saaty's scale of linguistic terms with assigned integers for expressing intensities of preference in MPCMs was reviewed, and its fuzzy extension was studied in Section 4.2.1. However, as mentioned in Section 2.2.1, the linguistic terms do not correspond very well to the respective numerical values that are distributed uniformly in the interval $[1, 9]$. This problem naturally concerns also the fuzzy extension of Saaty's scale. Thus, as mentioned in Section 4.2.1, it would be appropriate to customize Saaty's scale for each DM with respect to the given decision problem. The first attempt of customizing the scale by using fuzzy numbers was done by Ishizaka and Nguyen (2013). However, as mentioned in Section 4.2.1, the process for customizing the scale is not designed well, which results in an inappropriate calibration. Therefore, this area still needs to be explored more thoroughly in order to design an appropriate calibration process.

Consistency: In Chapter IV, two consistency conditions were proposed for each type of FPCMs, one very weak and easy to reach and one very strong and difficult to reach. For real-life applications, a compromise between these two definitions of consistency might be useful. Therefore, searching for such a definition of consistency that is again invariant under permutation and preserves the reciprocity property of PCs is a subject for future research.

Weak consistency: As discussed in Chapter II, weak-consistency conditions for MPCMs and APCMs provide an intuitive minimum consistency requirement. These definitions of consistency are less restrictive than traditional definitions of consistency reviewed in Chapter II and fuzzified in Chapter IV, and they provide DMs with some space for expressing their preferences. Further, weak consistency is much easier to reach and to control during the process of entering PCs into a PCM. This is especially convenient for real-life applications. Therefore, it would be useful to have such definitions of weak consistency also for FPCMs. The first step towards the fuzzy extension of the weak-consistency condition was done by Krejčí and Stoklasa (2016) who applied a fuzzy extension of the weak-consistency condition for MPCMs in the evaluation of scientific monographs.

Aggregation: Because of the excessive extent of the topic, the fuzzy extension of aggregation methods for obtaining final priorities of alternatives representing their final multi-criteria evaluations was not dealt with in this thesis. However, it is a very important part of the methods for dealing with FPCMs. Similarly as for definitions of consistency and methods for deriving fuzzy priorities of objects from FPCMs, also aggregation methods have to be extended properly to FPCMs by applying constrained fuzzy arithmetic in order to preserve the reciprocity property of the related PCs. Such fuzzy extension becomes considerably more complex in comparison to the fuzzy extension of the consistency conditions and of the methods for deriving priorities of objects since fuzzy priorities of criteria and alternatives obtained from several FPCMs are involved in the aggregation process. Nevertheless, for the completeness of the MCDM methods based on FPCMs proposed in this thesis, it is necessary to deal also with this issue. The fuzzy extension of the weighted average for aggregating fuzzy priorities obtained from FMPCMs was already introduced by Krejčí et al. (2017). The fuzzy extension of the

aggregation methods for FAPCMs-A and FAPCMs-M are still left for future research.

Multiple DMs: Another issue that was not addressed in this thesis is considering multiple DMs. A large number of methods based on PCMs have been proposed in the literature to deal with MCDM problems involving multiple DMs and some of the methods have been extended also to FPCMs. With the fuzzy extension of these methods very same challenges that have been approached in this thesis arise. In particular, it is again necessary to preserve the reciprocity of PCs in FPCMs as well as the invariance of the methods in order to reflect appropriately the preference information provided by multiple DMs.

Incomplete large-dimensional FPCMs: The last but not least topic for future research is the fuzzy extension of the method for dealing with incomplete large-dimensional PCMs that was proposed in Chapter V. The method proposed in Chapter V is designed for large-dimensional problems where the DM provides PCs by means of crisp numbers. However, as discussed in the thesis, crisp numbers cannot model properly uncertainty stemming from subjectivity of human thinking and from vagueness of information about the problem that are very often related to MCDM problems. Therefore, a fuzzy extension of the method proposed in Chapter V is needed in order to handle properly large-dimensional problems with uncertainty as well as with incompleteness of preference information provided by the DM. Thus, the last but not least topic for future research is the fuzzy extension of the method for dealing with incomplete large-dimensional PCMs proposed in Chapter V. The fuzzy extension of the method requires a fuzzy extension of the weak-consistency condition, on which the method is based. Besides that, a large number of rules derived from the weak-consistency condition has to be fuzzified accordingly and employed in the iterative process of identifying PCs that should be provided by the DM. At the end of the process of entering PCs into the incomplete large-dimensional FPCM, we would obtain a FPCM instead of an interval PCM, which is the output of the current method. Afterwards, in order to derive fuzzy priorities from such a FPCM, it would be sufficient to apply one of the methods proposed in Chapter IV.

List of abbreviations

AHP	Analytic Hierarchy Process
APCM	additive pairwise comparison matrix
APCM-A	additive pairwise comparison matrix with additive representation
APCM-M	additive pairwise comparison matrix with multiplicative representation
DM	decision maker
EVM	eigenvector method
FAPCM	fuzzy additive pairwise comparison matrix
FAPCM-A	fuzzy additive pairwise comparison matrix with additive representation
FAPCM-M	fuzzy additive pairwise comparison matrix with multiplicative representation
FMPCM	fuzzy multiplicative pairwise comparison matrix
FPCM	fuzzy pairwise comparison matrix
GMM	geometric mean method
LLSM	logarithmic least squares method
MCDM	multi-criteria decision making
MPCM	multiplicative pairwise comparison matrix
PC	pairwise comparison
PCM	pairwise comparison matrix

List of mathematical symbols

$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of positive real numbers greater than 0
$U \times V$	cartesian product of two sets U and V
$\mathbb{R}^n$	$n-$ary cartesian power of set $\mathbb{R}$
$\mathcal{F}(\mathbb{R})$	set of all fuzzy sets defined on $\mathbb{R}$
$\mathcal{F}_N(\mathbb{R})$	set of all fuzzy numbers defined on $\mathbb{R}$
$\mathcal{F}_N(\mathbb{R}^+)$	set of all positive fuzzy numbers
$\mathcal{F}_N(\mathbb{R})^n$	$n-$ary cartesian power of set $\mathcal{F}_N(\mathbb{R})$
$x \in \Omega$	element belonging to set Ω
$|\Omega|$	cardinality of set Ω
$\Omega \backslash Q$	difference of sets Ω and Q
$\Omega \cap Q$	intersection of sets Ω and Q
$\Omega \cup Q$	union of sets Ω and Q
$(x_1, x_2, \ldots, x_n)$	$n-$tuple, i.e. an ordered list of n elements, $n \in \mathbb{N}$
$[a, b]$, $\bar{c} = [c^L, c^U]$	closed interval
$]a, b[$	open interval
$\widetilde{c}$	fuzzy set

$Supp\,\widetilde{c}$	support of $\widetilde{c}$
$Core\,\widetilde{c}$	core of $\widetilde{c}$
$\widetilde{c}_{(\alpha)}$	$\alpha-$cut of $\widetilde{c}$
$\widetilde{c}_{(0)} = Cl(Supp\,\widetilde{c})$	closure of the support of $\widetilde{c}$
$c \in \widetilde{c}$	element belonging to the closure of the support of $\widetilde{c}$
$\widetilde{c} = (c^L, c^M, c^U)$	triangular fuzzy number
$\widetilde{c} = (c^\alpha, c^\beta, c^\gamma, c^\delta)$	trapezoidal fuzzy number
$A = \{a_{ij}\}_{i,j=1}^n$	square matrix
$\widetilde{A} = \{\widetilde{a}_{ij}\}_{i,j=1}^n$	square fuzzy matrix
$|A|$	determinant of matrix A
A^T	transpose of matrix A
$\widetilde{A}^T$	transpose of fuzzy matrix A
$\lambda = EVM_\lambda(A)$	maximal eigenvalue of matrix A
$\underline{w} = EVM_{\underline{w}}(A)$	normalized maximal eigenvector of matrix A
$\underline{w} = (w_1, \dots, w_n)^T$	column vector
$\underline{w}^T = (w_1, \dots, w_n)$	row vector
$\underline{\widetilde{w}} = (\widetilde{w}_1, \dots, \widetilde{w}_n)^T$	column fuzzy vector
$\underline{\overline{w}} = (\overline{w}_1, \dots, \overline{w}_n)^T$	column interval vector
$\wedge$	logical conjunction
$\vee$	logical disjunction
$\ln$	natural logarithm
$\log_9$	logarithm of base 9
f^{-1}	inverse of function f
$\lfloor x \rfloor$	floor of $x \in \mathbb{R}$
$k!$	factorial of number $k \in \mathbb{N}$

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