

# INFLUENCE IN SYSTEMS WITH CONVEX DECISIONS

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**ABSTRACT.** Given a system where the real-valued states of the agents are aggregated by a function to a real-valued state of the entire system, we are interested in the influence of the different agents on that function. This generalizes the notion of power indices for binary voting systems to decisions over convex one-dimensional policy spaces and has applications in economics, engineering, security analysis, and other disciplines. Here, we provide a solid theoretical framework to study the question of influence in systems with convex decisions. Based on the classical Shapley-Shubik and Penrose-Banzhaf index, from binary voting, we develop two influence measures, whose properties then are analyzed. We present some results for parametric classes of aggregation functions.

**Keywords:** Influence; power; convex decisions; state aggregation; Shapley-Shubik index; Penrose-Banzhaf index.

**MSC:** 91B12; 94C10

## 1. INTRODUCTION

Consider a system where the agents (or components) each determine a real number  $x_i$ , which is then aggregated to a another real number  $f(\vec{x})$  representing the state of the entire system, where  $\vec{x} := (x_1, \dots, x_n)$ . This abstract setting occurs in several applications. The values  $x_i$  may encode the fault condition, normalized between zero and one, for several components of a complex system. Then, a suitable aggregation function might be simply given by  $f(\vec{x}) = \max\{x_1, \dots, x_n\}$ . For the estimation of unknown quantities, see [13], the “wisdom of the crowds”, see [33], can be applied.<sup>1</sup> Depending on the context the mean or the median might be a suitable aggregation function, see e.g. [2, 12] for some discussion. Things get more interesting if the components or agents are heterogeneous in terms of their impact on the aggregated value. An example is given by the weighted median, where each agent gets a non-negative integer weight  $w_i$  such that  $\sum_{i=1}^n w_i$  is odd.<sup>2</sup> Assume, to ease the notation, that the values  $x_i$  are pairwise different. Arranging the values in increasing order  $x_{i_1} < x_{i_2} < \dots < x_{i_n}$ , let  $1 \leq j \leq n$  be the smallest index such that  $\sum_{h=1}^j w_{i_h} > \sum_{h=1}^n w_h / 2$ . With this, the weighted median is given by  $x_{i_j}$ . Reasons for taking the weighted median instead of the median are manifold. When combining the judgment of multiple experts to a single value, different degrees of competence may be reflected by different weights, see e.g. Chapter 16 in [31]. In meta analysis, see e.g. [5], aggregated data of differently sized experiments are combined. More generally, citing [3]: “The aggregation problem can be defined as the information loss which occurs in the substitution of aggregate, or macrolevel, data for individual, or microlevel, data.”. Whenever datasets are heterogeneous this has to be reflected somehow in the aggregation, where the weighted median is just one possible method, that, however, is commonly applied. For the effects of data aggregation in wireless sensor networks we refer e.g. to [20]. Even if the micro level data is completely available, data aggregation makes sense due to the computational complexity, see e.g. [35]. Weighted median filters are also applied to sharpen images, see e.g. [11]. Due to the increasing share of solar and wind energy, transmission system operators are in need of accurate weather forecasts

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<sup>1</sup>The effect itself is a purely statistical phenomenon and is studied widely in the literature. It can also be simulated by a single individual, see [32]. For binary decisions similar effects are studied under the name “Condorcet Jury Theorem”, see e.g. [18].

<sup>2</sup>For the definition of the weighted median it is neither necessary to restrict to integer weights nor to restrict the possible weight sums of subsets of the agents. However, this way we can simplify the technical details cf. Section 7 for the more general version.

in order to economically regulate the stability of the power grid, see e.g. [7]. Typically, several such forecasts are combined with different weights in practice. Also fashion retailers invest quite some money to buy more accurate weather forecasts and combine them with freely available data, see e.g. [1] for the impact of temperature on sales. The median voter model in politics explains the output produced in the public sector by the preferences of the median voter, see e.g. [17]. While there is some criticism, it is nevertheless applied in several applications. Assuming a two-tier system, differently sized constituencies of an assembly call for the weighted median, see e.g. [28].

Using the weighted median or another aggregation function, whenever not all agents have an equal impact on the aggregation function, the question of the influence of an agent arises. For an example let us continue with the weighted median. Assume that we have four agents with weights  $w_1 = 5$ ,  $w_2 = 4$ ,  $w_3 = 3$ , and  $w_4 = 1$ . Here the threshold or quota is 7. Observe that in any ordering of the  $x_i$ , the value  $x_4$  is never the weighted median. So, it is justified to say that the fourth agent has no influence on the aggregation function. For any two of the other agents we observe that their weight sum meets or exceeds the quota. So, the second largest values of the  $x_i$  restricted to the first three agents determine the weighted median. Assuming equal distributions of the values  $x_i$ , we can say that the first three agents are symmetrical and have the same influence. Normalized to one, the influence vector of the four agents is given by  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ . As a by-product we get the information that the weights  $\vec{w} = (1, 1, 1, 0)$ , with a quota of 2, lead to the same aggregation function when using the weighted median. So, weights can be different from influence. While it was easy to determine the influence vector in our example, things get more complicated and even ambiguous in more intricate examples like e.g. for the weight vector  $\vec{w} = (3, 1, 1, 1)$ .

Nevertheless, the question of determining the influence of an agent or a component in a complex system is very important. Identifying a component without any influence may allow to remove that component and to reduce production costs. Due to security reasons or fault tolerance it might be beneficial if the influence of any component is not too large. Influential agents can be the goal of bribery or influential components be the target of technical attacks. For a firm it is important to be not too dependent from one of her external suppliers. From the other side, a supply firm is interested in knowing the impact of their contribution to the final product to potentially raise prices. There is also another point of view. In an application, the shape of the aggregation function may be defined besides some weights for the components, like in the case of the weighted median. Reliability, expertise, accuracy, or any other measure for a desired influence vector  $\sigma$  given, the question arises how to choose the weights such that the resulting influence meets  $\sigma$  as closely as possible. So, we face a problem of system design.

The aggregation problem can also be considered as the combination of probability distributions, see e.g. [4, 14] for an entry point into the related literature.

The question of the influence of agents is most studied in the context of binary voting systems. There, the agents vote either “yes” or “no”, encoded as 1 and 0, respectively, on a certain proposal. The aggregated group decision then is either to accept (and implement) or to reject the proposal. Von Neumann and Morgenstern introduced the notion of a simple game  $v$  in [34], which is an appropriate model in many applications. For any subset  $S$  of supporters  $v(S) \in \{0, 1\}$ , where  $v$  is surjective and monotone, i.e.,  $v(S) \leq v(T)$  for all  $S \subseteq T$ . The influence or power of an agent in a simple game is measured by so-called power indices like the Shapley-Shubik or the Penrose-Banzhaf index. The model is appropriate to model situations as complex as networks of companies, where several agents own shares of some companies that are owning shares of other companies themselves and so are indirectly controlling each other. However, the setting is binary, so that economic issues like e.g. monetary policy, tax rates, or spending on climate change mitigation does not fit and call for an interval of policy alternatives instead. In the context of voting the system design problem, from the previous paragraph, is called “inverse power index problem”, see e.g. [6, 19, 23]. For non-binary continuous decisions we refer to [26] and the references therein. Binary decisions with continuous signals are e.g. considered in [30]. Even in the binary case the influence or power vectors of a given simple game can differ for different power indices, so that the

question for the “right” index arises. Axiomatizations and comparative studies of the properties of the proposed power indices aid the practitioner in that task.

Having argued the importance of the problem, of influence in a complex system with states from a convex set, and highlighted its connection to voting, we aim to develop influence measures for this setting in the present paper. To this end, we interpret the classic Shapley-Shubik and Penrose-Banzhaf indices from a slightly different perspective and generalize the underlying definition to our setting. In the same vein the notion of a simple game is generalized. This lays the ground for a solid theoretical framework to study the question of influence in systems with convex decisions. We remark that some preliminary ideas in that direction have been presented in [24]. While the question is interesting in convex spaces of any dimension, see e.g. [29], we limit ourselves to intervals of real numbers. The introduction of two measurements of influence is not comprehensive at all and more suggestions are deserved. Evaluating the defined influence measures directly becomes computationally infeasible quickly if the number of agents increases, which is similar to the situation for power indices for simple games. For some classes of aggregation functions we are able to determine either improved algorithms or analytical formulas. Also the study of the mathematical properties of the two influence measures is initiated.

The remaining part of this paper is structured as follows. In Section 2 we briefly collect the basic definitions and facts for binary voting systems and power indices. A specific interpretation of the Shapley-Shubik and the Penrose-Banzhaf index is the topic of Section 3. In Section 4 we generalize simple games to simple aggregation functions and power indices to influence measures. Based on the stated interpretation, we generalize the Shapley-Shubik and the Penrose-Banzhaf index in Section 5. First mathematical properties of these two influence measures are studied in Section 6. Nevertheless, we did not completely succeed in revealing the properties of influence in weighted medians, we collect our findings in Section 7. We close with a conclusion and some open problems in Section 8.

## 2. BINARY VOTING SYSTEMS AND POWER INDICES

As mentioned in the introduction, we will go by the insights obtained in studying binary voting systems and corresponding power indices in order to develop more general influence measures. By  $N = \{1, \dots, n\}$  we denote the set of agents. A *simple game* is a surjective and monotone mapping  $v: 2^N \rightarrow \{0, 1\}$  from the set of subsets  $\{S \subseteq N\}$  of  $N$ , i.e., the power set  $2^N$  of  $N$ , into a binary output  $\{0, 1\}$ . *Monotone* means  $v(S) \leq v(T)$  for all  $\emptyset \subseteq S \subseteq T \subseteq N$ . The values of this mapping can be interpreted as follows. For each subset  $S$  of  $N$ , called *coalition*, we have  $v(S) = 1$  if the members of  $S$  can bring through a proposal nevertheless the members of  $N \setminus S$  are against it. If  $v(S) = 1$  we speak of a *winning coalition* and a *losing coalition* otherwise. The required monotonicity is quite natural in that context, i.e., if the members of a coalition  $S$  can bring through a proposal, then additional supporters should not harm. The technical condition of surjectivity, in conjunction with monotonicity, implies that  $\emptyset$  is a losing coalition and  $N$  a winning coalition. This is indeed also quite natural, i.e., if no one supports a proposal then it should not be accepted and if otherwise everybody is in favor of a proposal, then there is no reason to reject it. (Typically, surjectivity of  $v$  is replaced by the equivalent conditions  $v(\emptyset) = 0$  and  $v(N) = 1$ .) Simple majority for five agents can be modeled by a simple game whose winning coalitions are exactly those that have at least three members.

Each simple game is uniquely characterized by either listing all winning or losing coalitions. However, such a representation is not very compact. A slight reduction can be obtained by further exploiting monotonicity. To this end, a winning coalition  $S$  is called *minimal* if all of its proper subsets are losing. Similarly, a losing coalition  $T$  is called *maximal* if all of its proper supersets are winning. In our example of simple majority for five agents, the minimal winning coalitions are those with exactly three members and the maximal losing coalitions are those with exactly two members. In some cases an even more compact representation, based on weights, is possible. Therefore, we call a simple game  $v$  weighted if there exist weights  $w_1, \dots, w_n \in \mathbb{R}_{\geq 0}$  and a quota  $q \in \mathbb{R}_{> 0}$  such that  $v(S) = 1$  exactly if  $\bar{w}(S) := \sum_{i \in S} w_i \geq q$ . As notation we use  $[q; \bar{w}]$ , i.e.,  $[3; 1, 1, 1, 1, 1]$  describes simple majority for five agents. As

observed in the introduction, different weights can represent the same weighted game, e.g.,  $[7; 5, 4, 3, 1] = [7; 4, 4, 4, 1] = [2; 1, 1, 1, 0]$ . For any weighted game  $[q; \vec{w}]$  the difference between the minimum weight of a winning and the maximum weight of a losing coalition is some finite positive number, so that we can slightly modify weights and quota to rational numbers without changing the underlying simple game. Moreover, by multiplying with the least common multiple of the denominators we can assume that the quota and all weights are integers. We note that not every simple game is weighted. However, every simple game  $v$  can be written as the intersection of a finite number of weighted games  $[q_1; \vec{w}_1], \dots, [q_r; \vec{w}_r]$ , where

$$\left( \bigcap_{i=1}^r [q_i; \vec{w}_i] \right) (S) = \min \{ [q_i; \vec{w}_i](S) : 1 \leq i \leq r \} \quad (1)$$

for all coalitions  $S \subseteq N$ . For the description of the weighted median in terms of weighted games we need further subclasses of simple games. A simple game  $v$  is called *proper* if the complement  $N \setminus S$  of any winning coalition  $S \subseteq N$  is losing. If a simple game is not proper then it may happen that a coalition and its complement can change the status quo by turns, which leads to a very unpleasant and unstable situation, so that some researchers only consider proper simple games. Similarly, a simple game is called *strong* if the complement  $N \setminus T$  of any losing coalition  $T \subseteq N$  is winning. A simple game that is both proper and strong is called *constant-sum* (or self-dual or decisive). Weighted constant-sum games allow the definition of a corresponding aggregation function with a unique weighted median in all cases where the values  $x_i$  are pairwise different. Integer weights with an odd sum and a quota of half the weight sum (plus one half) are sufficient to guarantee the constant-sum property, see Section 7.

Several types of agents can be distinguished in a simple game  $v$ . Agent  $i \in N$  is called *null player* if  $v(S) = v(S \cup \{i\})$  for all  $\emptyset \subseteq S \subseteq N \setminus \{i\}$ , i.e., agent  $i$  is not contained in any minimal winning coalition. An agent that is contained in every minimal winning coalition is called a veto player. If  $\{i\}$  is a winning coalition (note that  $\emptyset$  is a losing coalition), then player  $i$  is called a *passer*. If additionally all other agents are null players, then we call agent  $i$  a *dictator*. Two agents  $i$  and  $j$  are called *symmetric*, if  $v(S \cup \{i\}) = v(S \cup \{j\})$  for all  $\emptyset \subseteq S \subseteq N \setminus \{i, j\}$ . In  $[7; 5, 4, 3, 1] = [2; 1, 1, 1, 0]$  the first three agents are symmetric and the fourth agent is a null player.

In order to measure the influence of agents in simple games several power indices were introduced in the literature. A power index  $p$  is a mapping from the set of simple (or weighted) games on  $n$  agents into  $\mathbb{R}_{\geq 0}^n$ . Typically power indices are defined for all positive integers  $n$ , so that we have a family of such mappings. By  $p_i(v)$  we denote the  $i$ th component of  $p(v)$ , i.e., the power of agent  $i$ . The *Shapley-Shubik index* is defined as

$$\text{SSI}_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|! \cdot (n - |S| - 1)!}{n!} \cdot (v(S \cup \{i\}) - v(S)). \quad (2)$$

We call  $S \subseteq N \setminus \{i\}$  a *swing* for agent  $i$  if  $v(S \cup \{i\}) - v(S) = 1$  in a given simple game  $v$ . In other words,  $S$  is a losing coalition and  $S \cup \{i\}$  a winning coalition. Counting the swings by

$$\sum_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S))$$

gives the *absolute Penrose-Banzhaf index*. Normalizing via the transformation  $p_i(v) / \sum_{j=1}^n p_j(v)$  then gives the *(relative) Penrose-Banzhaf index*. In general, we call a power index *efficient* if  $\sum_{i=1}^n p_i(v) = 1$  for all games  $v$ . We call a power index  $p$  *symmetric* if  $p_i(v) = p_j(v)$  for symmetric agents  $i, j$  in  $v$ . If  $p_i(v) = 0$  for every null player  $i$  of  $v$ , then we say that  $p$  satisfies the *null player property*. Both, the Shapley-Shubik and the Penrose-Banzhaf index, are efficient, symmetric, and satisfy the null player property. The Shapley-Shubik index additionally satisfies the *transfer axiom*

$$\varphi_i(u) + \varphi_i(v) = \varphi_i(u \vee v) + \varphi_i(u \wedge v) \quad (3)$$

for all  $1 \leq i \leq n$ , where  $(u \vee v)(S) = \max\{u(S), v(S)\}$  and  $(u \wedge v)(S) = \min\{u(S), v(S)\}$ . In the other direction, the Shapley-Shubik index is the unique power index that satisfies symmetry, efficiency, the null player property, and the transfer axiom, see [8]. An axiomatization of the Penrose-Banzhaf index was given in [9]. The absolute Penrose-Banzhaf index also satisfies the transfer axiom.

### 3. THE DEFINITION OF THE SHAPLEY-SHUBIK AND THE PENROSE-BANZHAF INDEX REVISITED

Based on precedent work, the following model was considered in [10]: Agents perform a roll-call. More precisely, all  $n!$  possible orders  $\pi: N \rightarrow N$  in which the agents are called are assumed to be equiprobable and the votes of each agent are independent with expectation  $0 \leq p \leq 1$  for voting 1, i.e., the probability for voting 1 is exactly  $p$ . For a given simple game  $v$  the *pivotal* agent  $i$  is determined by the unique index  $i$  such that  $\{j \in N : \pi(j) < \pi(i)\}$  is losing and  $\{j \in N : \pi(j) \leq \pi(i)\}$  is winning in  $v$ . Interestingly enough, the Shapley-Shubik index of agent  $i$  in  $v$  equals the probability that agent  $i$  is pivotal in the above roll-call model. Note that this statement is independent of  $p$ . The assumptions on the model can be even further weakened to correlated agents still maintaining the coincidence between the Shapley-Shubik index and pivot probabilities, see [25].

Let us take another perspective and consider the Shapley-Shubik index as a measurement for the reduction of uncertainty. To this end, note that if the votes of all  $n$  agents are known, then the aggregated decision, modeled by  $v(S)$  for the given simple game  $v$  and the coalition  $S \subseteq N$  of the agents voting “yes”, is uniquely determined. In the roll-call model we can consider our knowledge on the set of possible outcomes before and after an agent announces his or her vote. In the beginning an aggregated decision of both “yes” and “no” is possible, since  $v(\emptyset) = 0$ ,  $v(N) = 1$ , and we do not know how the agents will be voting. After the announcement of a certain agent the outcome is definitely determined. In other words, that agent reduces the uncertainty about the aggregated outcome by one. Let us consider a small example for the simple game  $v = [2; 1, 1, 1]$  and the ordering  $(1, 2, 3)$  of the agents. Moreover, let us assume that the agents will vote 0, 0, and 1, respectively. After agent 1 announces his or her vote both outcomes, 1 or 0, are possible since the other two agents may both vote 1 or 0. After the announcement of agent 2, the aggregated outcome is determined to be 0. Also for the ordering  $(1, 3, 2)$  agent 2 decides the final outcome. Averaging over all possible orderings and voting vectors again gives the Shapley-Shubik index, see [24, 25]. Note that the aggregated outcome can be determined to either 1 or 0, where both cases are symmetric in a certain sense, so that the notion of being pivotal applies. If the aggregated decision is a real number in  $[0, 1]$  instead of  $\{0, 1\}$  the uncertainty about the final outcome can be reduced by several agents at different points in time. Also the degree of reduction can be different within the same ordering and input vector  $\vec{x}$ . So, while there is not much a difference for the binary case, those things play a role in the continuous case, see Section 5.

Now let us look at the Penrose-Banzhaf index again. Assume, for a given agent  $i$ , that all other agents have announced their vote. To what degree can agent  $i$  move the aggregated outcome? In the binary setting the range is given by  $v(S \cup \{i\}) - v(S)$ , where  $v$  is the underlying simple game and  $S \subseteq N \setminus \{i\}$  is the set of agents voting “yes”. If  $v(S \cup \{i\}) - v(S) = 0$ , then agent  $i$  cannot change the aggregated outcome at all. If  $v(S \cup \{i\}) - v(S) = 1$ , then agent  $i$  can shift the aggregated outcome between 0 and 1. We may talk of a strategic point of view. In terms of orderings in the roll-call model we might say that the Shapley-Shubik index treats all possible orderings equally likely while the Penrose-Banzhaf index just considers orderings where the considered agent is last. Again the range of influence is more segmented in the continuous case, see Section 5.

### 4. THE GENERAL DECISION MODEL

Consider a system of  $n$  agents each described by some value  $x_i \in \mathcal{C}$ . As abbreviation we write  $\vec{x}$  for the vector  $(x_1, \dots, x_n)$ , i.e., the state vector of all agents. Further assume the existence of an aggregation function  $f: \mathcal{C}^n \rightarrow \mathcal{C}$ , where  $f(\vec{x})$  is a single *aggregated* state. As a crucial assumption we require that  $\mathcal{C}$  is *convex*, i.e., for any  $x, y \in \mathcal{C}$  and  $\lambda \in [0, 1]$  we have  $\lambda \cdot x + (1 - \lambda) \cdot y \in \mathcal{C}$ . The effect is that there is always

a possible state in between two different states  $x, y \in \mathcal{C}$ . In the introduction we have mentioned several applications calling for such state spaces. In general the subsequent problem is interesting for arbitrary convex spaces  $\mathcal{C}$ . Here, we specialize to convex one-dimensional spaces  $\mathcal{C} \subseteq \mathbb{R}$ , i.e., real intervals. To further ease the notation we consider bounded and closed intervals  $[a, b]$  only. Via  $x \mapsto (x - a)/(b - a)$  we can normalize any such interval of positive length to the interval  $[0, 1]$  that we are considering in the following.

We are interested in the *influence* of variable  $x_i$  on  $f$ . At a certain state vector  $\vec{x}$  the partial derivative with respect to variable  $x_i$  is an appropriate quantification (assuming differentiability). However, we are interested in a more global measure assigning a single non-negative real value to each index  $i$ , i.e., a mapping  $p$  from the set of (suitable) aggregation functions into  $\mathbb{R}_{\geq 0}^n$ . As outlined in the introduction this models the influence of a certain agent in a complex system on the decision outcome of the entire system, with the aim to distinguish heterogeneous agents according to their degree of influence. As a normalization we require that the entries of  $p(f)$  sum to one.

With this rather vague description of an influence measure  $p$  we remark that, again, this question is interesting for a huge variety of aggregation functions. However, in order to obtain stronger results we restrict ourselves on specific classes of aggregation functions. Conducted by the concept of simple games we define, cf. [24]:

**Definition 4.1.** For a positive integer  $n$  a *simple aggregation function*  $f$  is a mapping from  $[0, 1]^n$  to  $[0, 1]$  that is surjective, continuous, and weakly monotonic increasing, i.e.,  $f(\vec{x}) \leq f(\vec{y})$  for all  $\vec{x} \leq \vec{y}$ , i.e.,  $x_i \leq y_i$  for all  $1 \leq i \leq n$ .

For each  $\vec{w} \in \mathbb{R}_{\geq 0}^n$  with  $\sum_{i=1}^n w_i = 1$  the *weighted mean*  $f(\vec{x}) := \vec{w}^\top \vec{x} = \sum_{i=1}^n w_i x_i$  is a simple aggregation function. For  $\vec{w} \in \mathbb{N}^n$ , such that  $\sum_{i=1}^n w_i$  is odd, the *weighted median* is a simple aggregation function. Other examples are given e.g. by  $\hat{f}(x_1, x_2, x_3) = \frac{1x_1^2 + 2x_2^2 + 3x_3^2}{6}$  or  $\tilde{f}(x_1, x_2, x_3) = x_1 x_2^2 x_3^3$ . Of course there is no need for an explicit formula. As an example we consider the so-called *Hegselmann–Krause* or *bounded confidence model* [15]. Adjusted to our state space  $[0, 1]$  the model is as follows:  $n$  agents have initial opinions  $y_i^0 \in [0, 1]$ . For a given parameter  $\varepsilon \in (0, \frac{1}{2})$  opinions change in discrete time steps via the recursion

$$y_i^{t+1} := \sum_{j \in N_i(t)} y_j^t / |N_i(t)|, \quad (4)$$

where  $N_i(t) := \{1 \leq j \leq n : |y_i^t - y_j^t| \leq \varepsilon\}$ . Under the stated assumptions, and in several generalizations, the opinions converge to a steady state in finite time. Taking  $\vec{x}$  as the initial opinions we may define  $f(\vec{x})$  as the resulting steady state and easily check that this also gives a simple aggregation function. Since all agents (or variables) are homogeneous the question of influence is not very interesting. However, we can simply generalize the model and include some weights. Opinion dynamics based on further ways of averaging are studied in [16] and may serve for the definition of interesting simple aggregation functions.

Having the transfer axiom in mind, we easily observe:

**Lemma 4.2.** *Let  $f$  and  $g$  be two simple aggregation functions for the same number  $n \geq 1$  of agents. Then,  $f \vee g$  and  $f \wedge g$ , defined by  $(f \vee g)(\vec{x}) = \max\{f(\vec{x}), g(\vec{x})\}$  and  $(f \wedge g)(\vec{x}) = \min\{f(\vec{x}), g(\vec{x})\}$  for all  $\vec{x} \in [0, 1]^n$ , are simple aggregation functions.*

We also want to transfer the classification of types of agents in a simple aggregation function.

**Definition 4.3.** Let  $f$  be a simple aggregation function for  $n$  agents.

- (i) If  $f(\vec{x}) = f(\vec{y})$  for all  $\vec{x}, \vec{y} \in [0, 1]^n$  with  $x_j = y_j$  for all  $j \in \{1, \dots, n\} \setminus \{i\}$ , then we call agent  $i$  a *null*.
- (ii) If  $f(\vec{x}) = f(\vec{y})$  for all  $\vec{x}, \vec{y} \in [0, 1]^n$  with  $x_i = y_i$ , then we call agent  $i$  a *dictator*.
- (iii) If  $f(\vec{x}) = f(\vec{y})$  for all  $\vec{x}, \vec{y} \in [0, 1]^n$  with  $x_i = y_j$ ,  $x_j = y_i$ , and  $x_h = y_h$  for all  $h \in \{1, \dots, n\} \setminus \{i, j\}$ , then we call agent  $i$  and agent  $j$  *equivalent*.

It seems reasonable to require the following conditions for a measure of influence in a simple aggregation function:

**Definition 4.4.** An *influence measure* is a mapping  $p$  from the set of all simple aggregation functions for  $n$  agents into  $\mathbb{R}_{\geq 0}^n$  that is

- (i) *efficient*, i.e.,  $\sum_{i=1}^n p_i(f) = 1$ ;
- (ii) *symmetric*, i.e.,  $p_i(f) = p_j(f)$  for symmetric agents  $1 \leq i, j \leq n$ ; and
- (iii) has the *null property*, i.e.,  $p_i(f) = 0$  for every null  $1 \leq i \leq n$ .

Note that efficiency and the null property implies  $p_i(f) = 1$  for an agent  $i$  that is a dictator, since all other agents have to be nulls.

**Definition 4.5.** An *influence measure*  $p$  satisfies the *transfer axiom* if  $p_i(f) + p_i(g) = p_i(f \vee g) + p_i(f \wedge g)$  for all simple aggregation functions  $f$  and  $g$  on  $n \geq 1$  agents and all agents  $1 \leq i \leq n$ .

Depending on the application further properties might be desirable. In the following section we will introduce two reasonable influence measures.

## 5. TWO INFLUENCE MEASURES FOR SYSTEMS WITH CONVEX DECISIONS

Motivated by the interpretation of the Shapley-Shubik and the Penrose-Banzhaf index in Section 3, we introduce two influence measures for simple aggregation functions. Let us start with the generalization of the Shapley-Shubik index. We stick to the roll-call model and assume a given ordering  $\pi: N \rightarrow N$  of the agents and a given simple aggregation function  $f$ . For a given agent  $i$  we consider the case where all agents with  $\pi(j) < \pi(i)$  have already announced their state  $x_j$ . Given that information we are interested in the uncertainty of the possible value of  $f(\vec{x})$ , where  $\vec{x}$  is only partially specified. As an abbreviation we introduce:

**Definition 5.1.** For a positive integer  $n$  we set:

- $\bar{\tau}: [0, 1]^n \times \{1, \dots, n\} \rightarrow [0, 1]^n$ ,  $(\vec{x}, h) \mapsto (y_1, \dots, y_n)$ , where  $y_j = x_j$  for all  $1 \leq j \leq h$  and  $y_j = 1$  otherwise;
- $\underline{\tau}: [0, 1]^n \times \{1, \dots, n\} \rightarrow [0, 1]^n$ ,  $(\vec{x}, h) \mapsto (y_1, \dots, y_n)$ , where  $y_j = x_j$  for all  $1 \leq j \leq h$  and  $y_j = 0$  otherwise.

Since  $f$  is monotone,  $f(\bar{\tau}(\vec{x}, \pi^{-1}(i) - 1))$  is the maximal value that can be attained by  $f(\vec{x})$  if the  $x_j$  of all agents with  $\pi(j) < \pi(i)$  are fixed. Similarly,  $f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1))$  is the minimal value that can be attained by  $f(\vec{x})$  if the  $x_j$  of all agents with  $\pi(j) < \pi(i)$  are fixed. Since  $f$  is continuous, all values in the interval between  $f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1))$  and  $f(\bar{\tau}(\vec{x}, \pi^{-1}(i) - 1))$  can be attained by some vector  $\vec{x} \in [0, 1]^n$ , where the entries of all agents with  $\pi(j) < \pi(i)$  are fixed. The length

$$f(\bar{\tau}(\vec{x}, \pi^{-1}(i) - 1)) - f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1)) \quad (5)$$

of that interval is a suitable measure for the uncertainty of the simple aggregation function  $f$  before agent  $i$  announces his or her state  $x_i$ , with respect to the ordering  $\pi$  and the state vector  $\vec{x}$ . Similarly, the uncertainty after the announcement of agent  $i$  is given by

$$f(\bar{\tau}(\vec{x}, \pi^{-1}(i))) - f(\underline{\tau}(\vec{x}, \pi^{-1}(i))). \quad (6)$$

The difference between both values

$$\begin{aligned} & \left( f(\bar{\tau}(\vec{x}, \pi^{-1}(i) - 1)) - f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1)) \right) - \left( f(\bar{\tau}(\vec{x}, \pi^{-1}(i))) - f(\underline{\tau}(\vec{x}, \pi^{-1}(i))) \right) \\ &= \left( f(\bar{\tau}(\vec{x}, \pi^{-1}(i) - 1)) - f(\bar{\tau}(\vec{x}, \pi^{-1}(i))) \right) + \left( f(\underline{\tau}(\vec{x}, \pi^{-1}(i))) - f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1)) \right) \end{aligned}$$

summed over all possible orderings and averaged over all possible state vectors  $\vec{x}$  may serve as a suitable measurement of influence:

**Definition 5.2.** For a simple aggregation function  $f$  for  $n \geq 1$  agents we set

$$\begin{aligned} \varphi_i(f) &:= \frac{1}{n!} \cdot \sum_{\pi \in \mathcal{S}_n} \int_0^1 \cdots \int_0^1 \left( f(\overline{\tau}(\vec{x}, \pi^{-1}(i) - 1)) - f(\overline{\tau}(\vec{x}, \pi^{-1}(i))) \right) \\ &\quad + \left( f(\underline{\tau}(\vec{x}, \pi^{-1}(i))) - f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1)) \right) d\vec{x}_1 \dots d\vec{x}_n, \end{aligned} \quad (7)$$

for each agent  $i \in N$ , where  $\mathcal{S}_n$  denotes the set of permutations or bijections from  $N$  to  $N$ .

Here we assume that the states of all agents are independent and that all state vectors  $\vec{x}$  occur with equal probability. This assumption can of course be adjusted easily.

As an example we consider the two simple aggregation functions  $\hat{f}(x_1, x_2, x_3) = \frac{1x_1^2+2x_2^2+3x_3^2}{6}$  and  $\tilde{f}(x_1, x_2, x_3) = x_1x_2^2x_3^3$ .

| $\pi \in \mathcal{S}_3$ | 3-fold integral  |
|-------------------------|--|
| (1, 2, 3)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{6}{6} - \frac{x_1^2+5}{6} + \frac{x_2^2}{6} - \frac{0}{6} \right) d\vec{x} = \frac{1}{6}$   |
| (1, 3, 2)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{6}{6} - \frac{x_1^2+5}{6} + \frac{x_2^2}{6} - \frac{0}{6} \right) d\vec{x} = \frac{1}{6}$   |
| (2, 1, 3)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{2x_2^2+4}{6} - \frac{x_1^2+2x_2^2+3}{6} + \frac{x_1^2+2x_2^2}{6} - \frac{2x_2^2}{6} \right) d\vec{x} = \frac{1}{6}$                           |
| (2, 3, 1)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{2x_2^2+3x_3^2+1}{6} - \frac{x_1^2+2x_2^2+3x_3^2}{6} + \frac{x_1^2+2x_2^2+3x_3^2}{6} - \frac{2x_2^2+3x_3^2}{6} \right) d\vec{x} = \frac{1}{6}$ |
| (3, 1, 2)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{3x_3^2+3}{6} - \frac{x_1^2+3x_3^2+2}{6} + \frac{x_1^2+3x_3^2}{6} - \frac{3x_3^2}{6} \right) d\vec{x} = \frac{1}{6}$                           |
| (3, 2, 1)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{2x_2^2+3x_3^2+1}{6} - \frac{x_1^2+2x_2^2+3x_3^2}{6} + \frac{x_1^2+2x_2^2+3x_3^2}{6} - \frac{2x_2^2+3x_3^2}{6} \right) d\vec{x} = \frac{1}{6}$ |

TABLE 1. Details for  $\varphi_1(\hat{f})$ .

| $\pi \in \mathcal{S}_3$ | 3-fold integral  |
|-------------------------|--|
| (2, 1, 3)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{6}{6} - \frac{x_2^2+4}{6} + \frac{x_2^2}{6} - \frac{0}{6} \right) d\vec{x} = \frac{2}{6}$   |
| (2, 3, 1)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{6}{6} - \frac{x_2^2+4}{6} + \frac{x_2^2}{6} - \frac{0}{6} \right) d\vec{x} = \frac{2}{6}$   |
| (1, 2, 3)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{x_1^2+5}{6} - \frac{x_1^2+2x_2^2+3}{6} + \frac{x_1^2+2x_2^2}{6} - \frac{x_1^2}{6} \right) d\vec{x} = \frac{2}{6}$                           |
| (3, 2, 1)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{3x_3^2+3}{6} - \frac{2x_2^2+3x_3^2+1}{6} + \frac{2x_2^2+3x_3^2}{6} - \frac{3x_3^2}{6} \right) d\vec{x} = \frac{2}{6}$                       |
| (1, 3, 2)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{x_1^2+3x_3^2+2}{6} - \frac{x_1^2+2x_2^2+3x_3^2}{6} + \frac{x_1^2+2x_2^2+3x_3^2}{6} - \frac{x_1^2+3x_3^2}{6} \right) d\vec{x} = \frac{2}{6}$ |
| (3, 1, 2)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{x_1^2+3x_3^2+2}{6} - \frac{x_1^2+2x_2^2+3x_3^2}{6} + \frac{x_1^2+2x_2^2+3x_3^2}{6} - \frac{x_1^2+3x_3^2}{6} \right) d\vec{x} = \frac{2}{6}$ |

TABLE 2. Details for  $\varphi_2(\hat{f})$ .



| $\pi \in \mathcal{S}_3$ | 3-fold integral  |
|-------------------------|--|
| (3, 1, 2)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{6}{6} - \frac{3x_3^2+3}{6} + \frac{3x_3^2}{6} - \frac{0}{6} \right) d\vec{x} = \frac{3}{6}$   |
| (3, 2, 1)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{6}{6} - \frac{3x_3^2+3}{6} + \frac{3x_3^2}{6} - \frac{0}{6} \right) d\vec{x} = \frac{3}{6}$   |
| (1, 3, 2)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{x_1^2+5}{6} - \frac{x_1^2+3x_3^2+2}{6} + \frac{x_1^2+3x_3^2}{6} - \frac{x_1^2}{6} \right) d\vec{x} = \frac{3}{6}$                           |
| (2, 3, 1)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{2x_2^2+4}{6} - \frac{2x_2^2+3x_3^2+1}{6} + \frac{2x_2^2+3x_3^2}{6} - \frac{2x_2^2}{6} \right) d\vec{x} = \frac{3}{6}$                       |
| (1, 2, 3)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{x_1^2+2x_2^2+3}{6} - \frac{x_1^2+2x_2^2+3x_3^2}{6} + \frac{x_1^2+2x_2^2+3x_3^2}{6} - \frac{x_1^2+2x_2^2}{6} \right) d\vec{x} = \frac{3}{6}$ |
| (2, 1, 3)               | $\int_{\vec{x} \in [0,1]^3} \left( \frac{x_1^2+2x_2^2+3}{6} - \frac{x_1^2+2x_2^2+3x_3^2}{6} + \frac{x_1^2+2x_2^2+3x_3^2}{6} - \frac{x_1^2+2x_2^2}{6} \right) d\vec{x} = \frac{3}{6}$ |

 TABLE 3. Details for  $\varphi_3(\hat{f})$ .

In tables 1-3 we give the respective summands for each permutation  $\pi \in \mathcal{S}_3$  for  $\hat{f}$ . Summarizing the results we obtain

$$\varphi(\hat{f}) = \left( \frac{1}{6}, \frac{2}{6}, \frac{3}{6} \right). \quad (8)$$

Note that  $\varphi$  is efficient in our example, which we will prove in general in Lemma 6.1. Moreover, the entries of  $\varphi(\hat{f})$  coincide with the coefficients of the linear combination of functions defining  $\hat{f}$ . Again, this is a general phenomenon, see Theorem 6.3.

| $\pi \in \mathcal{S}_3$ | 3-fold integral   |
|-------------------------|---|
| (1, 2, 3)               | $\int_{\vec{x} \in [0,1]^3} (1 - x_1 + 0 - 0) d\vec{x} = \frac{1}{2}$                                 |
| (1, 3, 2)               | $\int_{\vec{x} \in [0,1]^3} (1 - x_1 + 0 - 0) d\vec{x} = \frac{1}{2}$                                 |
| (2, 1, 3)               | $\int_{\vec{x} \in [0,1]^3} (x_2^2 - x_1x_2^2 + 0 - 0) d\vec{x} = \frac{1}{6}$                        |
| (2, 3, 1)               | $\int_{\vec{x} \in [0,1]^3} (x_3^3 - x_1x_3^3 + 0 - 0) d\vec{x} = \frac{1}{8}$                        |
| (3, 1, 2)               | $\int_{\vec{x} \in [0,1]^3} (x_2^2x_3^3 - x_1x_2^2x_3^3 + x_1x_2^2x_3^3 - 0) d\vec{x} = \frac{1}{12}$ |
| (3, 2, 1)               | $\int_{\vec{x} \in [0,1]^3} (x_2^2x_3^3 - x_1x_2^2x_3^3 + x_1x_2^2x_3^3 - 0) d\vec{x} = \frac{1}{12}$ |

 TABLE 4. Details for  $\varphi_1(\tilde{f})$ .

In tables 4-6 we give the respective summands for each permutation  $\pi \in \mathcal{S}_3$  for  $\tilde{f}$ . Summarizing the results we obtain

$$\varphi(\tilde{f}) = \left( \frac{35}{144}, \frac{50}{144}, \frac{59}{144} \right) = (0.2430\bar{5}, 0.347\bar{2}, 0.4097\bar{2}). \quad (9)$$

In Theorem 6.4 we state a more general formula for  $\varphi(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n})$  where  $\alpha_i \in \mathbb{R}_{>0}$ .

For the generalization of the Penrose-Banzhaf index we stick to the strategic point of view outlined in Section 3. So, for a given simple aggregation function  $v$  and a state vector  $\vec{x}$  agent  $i$  increases the value of  $f(\vec{x})$  maximally by choosing  $x_i = 1$ , due to monotonicity of  $f$ . Similarly, the minimum is attained for  $x_i = 0$ .

| $\pi \in \mathcal{S}_3$ | 3-fold integral  |
|-------------------------|--|
| (1, 2, 3)               | $\int_{\vec{x} \in [0,1]^3} (x_1 - x_1 x_2^2 + 0 - 0) \, d\vec{x} = \frac{1}{3}$                           |
| (1, 3, 2)               | $\int_{\vec{x} \in [0,1]^3} (x_1 x_3^3 - x_1 x_2^2 x_3^3 + x_1 x_2^2 x_3^3 - 0) \, d\vec{x} = \frac{1}{8}$ |
| (2, 1, 3)               | $\int_{\vec{x} \in [0,1]^3} (1 - x_2^2 + 0 - 0) \, d\vec{x} = \frac{2}{3}$                                 |
| (2, 3, 1)               | $\int_{\vec{x} \in [0,1]^3} (1 - x_2^2 + 0 - 0) \, d\vec{x} = \frac{2}{3}$                                 |
| (3, 1, 2)               | $\int_{\vec{x} \in [0,1]^3} (x_1 x_3^3 - x_1 x_2^2 x_3^3 + x_1 x_2^2 x_3^3 - 0) \, d\vec{x} = \frac{1}{8}$ |
| (3, 2, 1)               | $\int_{\vec{x} \in [0,1]^3} (x_3^3 - x_2^2 x_3^3 + 0 - 0) \, d\vec{x} = \frac{1}{6}$                       |

TABLE 5. Details for  $\varphi_2(\tilde{f})$ .

| $\pi \in \mathcal{S}_3$ | 3-fold integral  |
|-------------------------|--|
| (1, 2, 3)               | $\int_{\vec{x} \in [0,1]^3} (x_1 x_2^2 - x_1 x_2^2 x_3^3 + x_1 x_2^2 x_3^3 - 0) \, d\vec{x} = \frac{1}{6}$ |
| (1, 3, 2)               | $\int_{\vec{x} \in [0,1]^3} (x_1 - x_1 x_3^3 + 0 - 0) \, d\vec{x} = \frac{3}{8}$                           |
| (2, 1, 3)               | $\int_{\vec{x} \in [0,1]^3} (x_1 x_2^2 - x_1 x_2^2 x_3^3 + x_1 x_2^2 x_3^3 - 0) \, d\vec{x} = \frac{1}{6}$ |
| (2, 3, 1)               | $\int_{\vec{x} \in [0,1]^3} (x_2^2 - x_2^2 x_3^3 + 0 - 0) \, d\vec{x} = \frac{1}{4}$                       |
| (3, 1, 2)               | $\int_{\vec{x} \in [0,1]^3} (1 - x_3^3 + 0 - 0) \, d\vec{x} = \frac{3}{4}$                                 |
| (3, 2, 1)               | $\int_{\vec{x} \in [0,1]^3} (1 - x_3^3 + 0 - 0) \, d\vec{x} = \frac{3}{4}$                                 |

TABLE 6. Details for  $\varphi_3(\tilde{f})$ .

**Definition 5.3.** For a simple aggregation function  $f$  for  $n \geq 2$  agents we set

$$\begin{aligned} \tilde{\psi}_i(f) &:= \int_0^1 \cdots \int_0^1 \left( f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) \right. \\ &\quad \left. - f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \right) \, dx_1 \dots dx_{i-1} \, dx_{i+1} \dots dx_n. \end{aligned} \quad (10)$$

for each agent  $i \in N$ . With this, we normalize to  $\psi_i(f) = \tilde{\psi}_i(f) / \sum_{j=1}^n \tilde{\psi}_j(f)$ .

Here we again assume that all state vectors  $\vec{x}$  occur with equal probability, which can of course be adjusted easily. As an example we consider the same two simple aggregation functions  $\hat{f}(x_1, x_2, x_3) = \frac{1x_1^2 + 2x_2^2 + 3x_3^2}{6}$  and  $\tilde{f}(x_1, x_2, x_3) = x_1 x_2^2 x_3^3$  as before. Here, we obtain:

$$\begin{aligned} \tilde{\psi}_1(\hat{f}) &= \int_0^1 \int_0^1 \left( \frac{1 + 2x_2^2 + 3x_3^2}{6} - \frac{0 + 2x_2^2 + 3x_3^2}{6} \right) \, dx_2 \, dx_3 = \frac{1}{6}, \\ \tilde{\psi}_2(\hat{f}) &= \int_0^1 \int_0^1 \left( \frac{1x_1 + 2 + 3x_3^2}{6} - \frac{1x_1^1 + 0 + 3x_3^2}{6} \right) \, dx_1 \, dx_3 = \frac{2}{6}, \\ \tilde{\psi}_3(\hat{f}) &= \int_0^1 \int_0^1 \left( \frac{1x_1^2 + 2x_2^2 + 3}{6} - \frac{1x_1^2 + 2x_2^2 + 0}{6} \right) \, dx_1 \, dx_2 = \frac{3}{6}. \end{aligned} \quad (11)$$

Since  $\tilde{\psi}_1(\hat{f}) + \tilde{\psi}_2(\hat{f}) + \tilde{\psi}_3(\hat{f}) = 1$  no normalization is necessary. Moreover, the example is covered by Theorem 6.3. For the other example we obtain:

$$\begin{aligned}\tilde{\psi}_1(\tilde{f}) &= \int_0^1 \int_0^1 (x_2^2 x_3^3 - 0) \, dx_2 \, dx_3 = \frac{1}{12}, \\ \tilde{\psi}_2(\tilde{f}) &= \int_0^1 \int_0^1 (x_1 x_3^3 - 0) \, dx_1 \, dx_3 = \frac{1}{8}, \\ \tilde{\psi}_3(\tilde{f}) &= \int_0^1 \int_0^1 (x_1 x_2^2 - 0) \, dx_1 \, dx_2 = \frac{1}{6}.\end{aligned}\tag{12}$$

After normalization we obtain  $\psi = \frac{1}{9} \cdot (2, 3, 4) = (0.\bar{2}, 0.\bar{3}, 0.\bar{4})$ . In Theorem 6.4 we capture this example as a special case.

## 6. PROPERTIES OF THE TWO INFLUENCE MEASURES

First, we have to verify that the two mappings  $\varphi$  and  $\psi$  (see Definition 5.2 and Definition 5.3) are indeed influence measures, i.e., that they satisfy the conditions of Definition 4.4.

**Lemma 6.1.** *For a positive integer  $n$  the mapping  $\varphi$  is a well-defined influence measure that satisfies the transfer axiom.*

*Proof.* Every argument  $f$  of  $\varphi$  is a simple aggregation function, which especially means that  $f$  is continuous over the compact domain  $[0, 1]^n$ . Thus, the integrals in the definition of  $\varphi(f)$  exist, so that the mapping  $\varphi$  is well-defined. Moreover,  $f(\bar{\tau}(\vec{x}, \pi^{-1}(i) - 1)) \geq f(\bar{\tau}(\vec{x}, \pi^{-1}(i)))$  and  $f(\underline{\tau}(\vec{x}, \pi^{-1}(i))) \geq f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1))$  for every  $\pi \in \mathcal{S}_n$ ,  $\vec{x} \in [0, 1]^n$ , and  $i \in N$ , so that  $\varphi_i(f) \geq 0$ .

For any permutation  $\pi \in \mathcal{S}_n$  and any state vector  $\vec{x} \in [0, 1]^n$ , we have

$$\begin{aligned}& \sum_{i=1}^n f(\bar{\tau}(\vec{x}, \pi^{-1}(i) - 1)) - f(\bar{\tau}(\vec{x}, \pi^{-1}(i))) + f(\underline{\tau}(\vec{x}, \pi^{-1}(i))) - f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1)) \\ &= \sum_{h=1}^n f(\bar{\tau}(\vec{x}, h - 1)) - f(\bar{\tau}(\vec{x}, h)) + \sum_{h=1}^n f(\underline{\tau}(\vec{x}, h)) - f(\underline{\tau}(\vec{x}, h - 1)) \\ &= f(\bar{\tau}(\vec{x}, 0)) - f(\bar{\tau}(\vec{x}, n)) + f(\underline{\tau}(\vec{x}, n)) - f(\underline{\tau}(\vec{x}, 0)) \\ &= f(\bar{\tau}(\vec{x}, 0)) - f(\underline{\tau}(\vec{x}, 0)) = 1 - 0 = 1,\end{aligned}\tag{13}$$

so that  $\sum_{i=1}^n \varphi_i(f) = 1$ , i.e.,  $\varphi$  is efficient.

For two distinct symmetric agents  $i, j \in N$  let  $\sigma \in \mathcal{S}_n$  be the transposition interchanging agent  $i$  and agent  $j$ . For a given state vector  $\vec{x} \in [0, 1]^n$  we define  $\vec{y} \in [0, 1]^n$  as the vector arising from interchanging the  $i$ th and the  $j$ th coordinate of  $\vec{x}$ . By  $\kappa \in \mathcal{S}_n$  we denote the concatenation of  $\sigma$  with  $\pi$ . With this, we have

$$\begin{aligned}& \sum_{\pi \in \mathcal{S}_n} f(\bar{\tau}(\vec{x}, \pi^{-1}(i) - 1)) - f(\bar{\tau}(\vec{x}, \pi^{-1}(i))) + f(\underline{\tau}(\vec{x}, \pi^{-1}(i))) - f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1)) \\ &= \sum_{\kappa \in \mathcal{S}_n} f(\bar{\tau}(\vec{x}, \kappa^{-1}(i) - 1)) - f(\bar{\tau}(\vec{x}, \kappa^{-1}(i))) + f(\underline{\tau}(\vec{x}, \kappa^{-1}(i))) - f(\underline{\tau}(\vec{x}, \kappa^{-1}(i) - 1)) \\ &= \sum_{\pi \in \mathcal{S}_n} f(\bar{\tau}(\vec{y}, \pi^{-1}(j) - 1)) - f(\bar{\tau}(\vec{y}, \pi^{-1}(j))) + f(\underline{\tau}(\vec{y}, \pi^{-1}(j))) - f(\underline{\tau}(\vec{y}, \pi^{-1}(j) - 1)),\end{aligned}$$

so that  $\varphi_i(f) = \varphi_j(f)$ , i.e.,  $\varphi$  is symmetric.

If agent  $i \in N$  is a null and  $\pi \in \mathcal{S}_n$  arbitrary, then  $f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1)) = f(\underline{\tau}(\vec{x}, \pi^{-1}(i)))$  and  $f(\bar{\tau}(\vec{x}, \pi^{-1}(i) - 1)) = f(\bar{\tau}(\vec{x}, \pi^{-1}(i)))$ , so that  $\varphi_i(f) = 0$ , i.e.,  $\varphi$  satisfies the null property.

Since  $x + y = \max\{x, y\} + \min\{x, y\}$  for all  $x, y \in \mathbb{R}$  and due to the linearity of finite sums and integrals,  $\varphi$  satisfies the transfer axiom.  $\square$

**Lemma 6.2.** *For a positive integer  $n \geq 2$  the mapping  $\psi$  is a well-defined influence measure. Moreover,  $\tilde{\psi}$  satisfies the transfer axiom.*

*Proof.* Every argument  $f$  of  $\tilde{\psi}$  and  $\psi$  is a simple aggregation function, which especially means that  $f$  is continuous over the compact domain  $[0, 1]^n$ . Thus, the integrals in the definition of  $\tilde{\psi}(f)$  exist, so that the mapping  $\tilde{\psi}$  is well-defined. Since  $f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) \geq f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$  for every  $i \in N$ , due to monotonicity, we have  $\tilde{\psi}_i(f) \geq 0$ .

Since  $f$  is weakly monotonic increasing,  $f(0, \dots, 0) = 0$ , and  $f(1, \dots, 1) = 1$  there exists a state vector  $\vec{x} \in [0, 1]^n$  and an agent  $i \in N$  such that

$$\varepsilon := f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) > 0.$$

Due to continuity, there exists a constant  $0 < \delta < \frac{1}{2}$  such that  $f(x'_1, \dots, x'_{i-1}, 1, x'_{i+1}, \dots, x'_n) - f(x'_1, \dots, x'_{i-1}, 0, x'_{i+1}, \dots, x'_n) \geq \varepsilon/2$  for all  $l_h := \max\{0, x_h - \delta\} \leq x'_h \leq \min\{1, x_h + \delta\} =: u_h$  and all  $h \in N \setminus \{i\}$ . Since  $u_h - l_h \geq \delta$  we have

$$\begin{aligned} \tilde{\psi}_i(f) &\geq \int_{l_0}^{u_1} \cdots \int_{l_{i-1}}^{u_{i-1}} \int_{l_{i+1}}^{u_{i+1}} \cdots \int_{l_n}^{u_n} \left( f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) - \right. \\ &\quad \left. f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \right) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n \geq \frac{\delta^{n-1} \varepsilon}{2} > 0, \end{aligned}$$

so that  $\sum_{h=1}^n \tilde{\psi}_h(f) > 0$  and  $\sum_{h=1}^n \psi_h(f) = 1$ , i.e.,  $\psi$  is efficient.

Let  $i, j \in N$  be distinct symmetric agents. For a given state vector  $\vec{x} \in [0, 1]^n$  with  $x_i = x_j$  and  $h \in N$  let  $\vec{x}_h = (x_1, \dots, x_{h-1}, 0, x_{h+1}, \dots, x_n)$  and  $\vec{x}_h = (x_1, \dots, x_{h-1}, 1, x_{h+1}, \dots, x_n)$ . Symmetry of  $\tilde{\psi}$  and  $\psi$  follows from  $f(\vec{x}_i) - f(\vec{x}_i) = f(\vec{x}_j) - f(\vec{x}_j)$ .

Since  $f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) = f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$  for every null  $i \in N$ ,  $\tilde{\psi}$  and  $\psi$  satisfy the null property.

Since  $x + y = \max\{x, y\} + \min\{x, y\}$  for all  $x, y \in \mathbb{R}$  and due to the linearity of finite sums and integrals,  $\tilde{\psi}$  satisfies the transfer axiom.  $\square$

Next, we show that the examples  $\hat{f}(x_1, x_2, x_3) = \frac{1x_1^2 + 2x_2^2 + 3x_3^2}{6}$  and  $\tilde{f}(x_1, x_2, x_3) = x_1 x_2^2 x_3^3$  from Section 5 both are part of more general families for which formulas for  $\varphi$  and  $\psi$  can be determined.

**Theorem 6.3.** *For a positive integer  $n$ ,  $\vec{w} \in \mathbb{R}_{\geq 0}^n$  with  $\sum_{i=1}^n w_i = 1$ , and surjective, continuous, weakly monotonic increasing functions  $f_i: [0, 1] \rightarrow [0, 1]$  for  $1 \leq i \leq n$ , let  $f: [0, 1]^n \rightarrow [0, 1]$  defined by  $\vec{x} \mapsto \sum_{i=1}^n w_i \cdot f_i(x_i)$ . With this,  $f$  is a simple aggregation function and  $\varphi_i(f) = \psi_i(f) = w_i$  for all  $1 \leq i \leq n$ .*

*Proof.* By construction,  $f$  is continuous. If  $\vec{x} \leq \vec{y}$  for two vectors  $x, y \in [0, 1]^n$ , then  $x_i \leq y_i$  for all  $1 \leq i \leq n$  so that  $f_i(x_i) \leq f_i(y_i)$  and  $w_i \cdot f_i(x_i) \leq w_i \cdot f_i(y_i)$ , which implies that  $f$  is weakly monotonic increasing. For  $1 \leq i \leq n$  we define  $\vec{x}_i = (1, \dots, 1, 0, \dots, 0) \in [0, 1]^n$  with  $i - 1$  leading ones and  $\vec{x}_i = (1, \dots, 1, 0, \dots, 0) \in [0, 1]^n$  with  $i$  leading ones. Since the  $f_i$  are surjective,  $f_i(0) = 0$  and  $f_i(1) = 1$ , the image of  $\lambda \cdot \vec{x}_i + (1 - \lambda) \cdot \vec{x}_i$  for  $\lambda \in [0, 1]$  under  $f$  is given by  $\left[ \sum_{j=1}^{i-1} w_j, \sum_{j=1}^i w_j \right]$  for all  $1 \leq i \leq n$ . Thus,  $f$  is surjective.

Since

$$\begin{aligned} &f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \\ &= w_i \cdot f_i(1) - w_i \cdot f_i(0) = w_i \end{aligned} \tag{14}$$

for all  $1 \leq i \leq n$ , we have  $\tilde{\psi}_i(f) = w_i$  and  $\psi_i(f) = w_i$  (due to  $\sum_{j=1}^n w_j = 1$ ).

For each  $\pi \in \mathcal{S}_n$  and each  $i \in N$  we similarly have

$$f(\bar{\tau}(\vec{x}, \pi^{-1}(i) - 1)) - f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1)) = \sum_{j=\pi^{-1}(i)}^n w_{\pi(j)} \quad (15)$$

and

$$f(\bar{\tau}(\vec{x}, \pi^{-1}(i))) - f(\underline{\tau}(\vec{x}, \pi^{-1}(i))) = \sum_{j=\pi^{-1}(i)+1}^n w_{\pi(j)}. \quad (16)$$

Thus, the difference equals  $w_i$ , so that  $\varphi_i(f) = w_i$ .  $\square$

**Theorem 6.4.** For a positive integer  $n$  and positive real numbers  $\alpha_1, \dots, \alpha_n$  let  $f: [0, 1]^n \rightarrow [0, 1]$  be defined by  $\vec{x} \mapsto \prod_{i=1}^n x_i^{\alpha_i}$ . Then,  $f$  is a simple aggregation function,

$$\varphi_i(f) = \frac{1}{n! \cdot \Lambda} \cdot \left( (n-1)! + \alpha_i \cdot \sum_{T \subseteq N \setminus \{i\}} |T|! \cdot (n-1-|T|)! \cdot \prod_{j \in T} (a_j + 1) \right) \quad (17)$$

for each agent  $i \in N$ , and  $\psi_i(f) = \frac{\alpha_i}{n + \sum_{j=1}^n \alpha_j}$  for each agent  $i \in N$  if  $n \geq 2$ .

*Proof.* We directly check the conditions from Definition 4.1, cf. the proof of Theorem 6.3. For each agent  $i \in N$  employing the definition of  $f$  gives

$$\tilde{\psi}_i(f) = \int_0^1 \cdots \int_0^1 \prod_{j=1}^{i-1} x_j^{\alpha_j} \cdot \prod_{j=i+1}^n x_j^{\alpha_j} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n. \quad (18)$$

Since  $\int_0^1 cx^\alpha dx = \frac{c}{\alpha+1}$  for each  $\alpha > 0$ , we recursively compute

$$\tilde{\psi}_i(f) = \frac{\alpha_i + 1}{\prod_{j=1}^n (\alpha_j + 1)}, \quad (19)$$

for each agent  $i \in N$ , so that the stated formula for  $\psi$  follows.

For the computation of  $\varphi(f)$  we first observe

$$\int_0^1 \cdots \int_0^1 \prod_{j=1}^r x_j^{\beta_j} dx_1 \cdots dx_r = \prod_{j=1}^r \frac{1}{\beta_j + 1} \quad (20)$$

for  $0 \leq r \leq n$  and  $\beta_j \in \mathbb{R}_{>0}$  for  $1 \leq j \leq r$ . Now let  $i \in N$  an arbitrary but fixed agent and  $\pi \in \mathcal{S}_n$  an arbitrary but fixed permutation. As abbreviation we set  $S = \{j \in N : \pi(j) < \pi(i)\} \subseteq N \setminus \{i\}$ . With this, we have

$$\begin{aligned} & \int_0^1 \cdots \int_0^1 \left( f(\bar{\tau}(\vec{x}, \pi^{-1}(i) - 1)) - f(\bar{\tau}(\vec{x}, \pi^{-1}(i))) \right) \\ & + \left( f(\underline{\tau}(\vec{x}, \pi^{-1}(i))) - f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1)) \right) dx_1 \cdots dx_n \\ & = \left( \prod_{j \in S} \frac{1}{a_j + 1} \right) \cdot \left( 1 - \frac{1}{\alpha_i + 1} \right) = \frac{\alpha_i}{\Lambda} \cdot \prod_{j \in N \setminus (S \cup \{i\})} (a_j + 1) \end{aligned} \quad (21)$$

for  $S \neq N \setminus \{i\}$ , where  $\Lambda = \prod_{j=1}^n (\alpha_j + 1)$ . For  $S = N \setminus \{i\}$  the first expression evaluates to  $\frac{\alpha_i + 1}{\Lambda} = \frac{\alpha_i}{\Lambda} + \frac{1}{\Lambda}$  instead of  $\frac{\alpha_i}{\Lambda}$ , so that we obtain the stated formula for  $\varphi_i(f)$ .  $\square$

In our second example in Section 5 we have  $n = 3$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 2$ , and  $\alpha_3 = 3$ , so that  $n! \cdot \Lambda = 144$ . For  $i = 1$  the expression  $|T|! \cdot (n - 1 - |T|)! \cdot \prod_{j \in T} (a_j + 1)$  evaluates to 2, 3, 4, and 24 for  $T = \emptyset$ ,  $T = \{2\}$ ,  $T = \{3\}$ , and  $T = \{2, 3\}$ , respectively, so that  $\varphi_1(\tilde{f}) = \frac{35}{144}$ . For  $i = 2$  we obtain the values 2, 2, 4, and 16, so that  $\varphi_2(\tilde{f}) = \frac{50}{144}$ .

## 7. WEIGHTED MEDIANS

Here we want to give a definition for the weighted median as a simple aggregation function which is more general than the description from the introduction in Section 1. For a positive integer  $n$  we consider real numbers  $x_i$  for  $1 \leq i \leq n$ . Arrange the values in weakly increasing order  $x_{i_1} \leq x_{i_2} \leq \dots \leq x_{i_n}$ . For  $\vec{w} \in \mathbb{R}_{\geq 0}^n \setminus \mathbf{0}$  let  $1 \leq j \leq n$  be the smallest index such that  $\sum_{h=1}^j w_{i_h} \geq \sum_{h=1}^n w_h / 2$ . If we have equality, then we set the weighed median, with respect to  $\vec{w}$ , to  $(x_{i_j} + x_{i_{j+1}}) / 2$  and to  $x_{i_j}$  otherwise. We can easily check that this definition gives a simple aggregation function, which we denote by  $m_{\vec{w}}$ .

In the introduction we have seen that different weight vectors can lead to the same function  $m_{\vec{w}}$ . Here we want to study the equivalence question and the correspondence to weighted games. We restrict ourselves onto the cases where  $\vec{w}(S) \neq \vec{w}(N \setminus S)$  for every coalition  $S \subseteq N$ , i.e., there is always a unique weighted median. So, modeling as a simple game  $v$ , for any  $S \subseteq N$  the two complementary sets  $S$  and  $N \setminus S$  form exactly one winning and exactly one losing coalition, i.e.,  $v$  is a weighted constant-sum game. Two simple aggregation functions  $m_{\vec{w}}$  and  $m_{\vec{w}'}$  coincide if and only if  $[q; \vec{w}] = [q'; \vec{w}']$ , where  $q = \vec{w}(N)/2$  and  $q' = \vec{w}'(N)/2$ .

**Lemma 7.1.** *If  $[q'; \vec{w}']$  is constant-sum game for  $n$  agents, then there exist  $q \in \mathbb{N}_{>0}$  and  $\vec{w} \in \mathbb{N}^n$  such that  $\sum_{i=1}^n w_i = 2q - 1$ , which is odd, and  $[q'; \vec{w}'] = [q; \vec{w}]$ .*

*Proof.* In Section 2 we have argued that for every weighted game there exists a representation with integer weights and integer quota. Choose an integer  $q$  and  $\vec{w} \in \mathbb{N}^n$  such that  $[q'; \vec{w}'] = [q; \vec{w}]$  and  $\sum_{i=1}^n w_i$  is minimized. Let  $l$  denote the maximum weight of a losing coalition and  $u$  the minimum weight of a winning coalition, so that  $l + 1 \leq u$ . If  $l + 2 \leq u$ , then we may decrease the weight of one agent, with a positive weight, by one and set the quota to  $l + 1$ , which contradicts the minimality of  $\sum_{i=1}^n w_i$ . Thus, we have  $l + 1 = u = q$ . Let  $S$  be a winning coalition with weight  $l + 1$ . Then  $N \setminus S$  is losing and has a weight of  $\sum_{i=1}^n w_i - l - 1 \leq l$ , so that  $\vec{w}(N) \leq 2q - 1$ . Let  $T$  be a losing coalition of weight  $l$ . Then  $N \setminus T$  is winning and has a weight of  $\sum_{i=1}^n w_i - l \geq l + 1$ , so that  $\vec{w}(N) \geq 2q - 1$ , which gives  $\vec{w}(N) = 2q - 1$ .  $\square$

We remark that for  $n \geq 8$  agents there may be several representations of a weighted game with integer weights, integer quota, and minimum weight sum, see e.g. [22].

**Proposition 7.2.** *Let  $\vec{w} \in \mathbb{N}^n$ , where  $n \geq 3$  and  $\sum_{i=1}^n w_i$  is odd. For  $q = (\vec{w}(N) + 1)/2$  let  $f = m_{\vec{w}}$  be the weighted median simple aggregation function corresponding to the weighted game  $v = [q; \vec{w}]$ . If all coalitions of size 1 and all coalitions of size  $n - 1$  are winning in  $v$ , then*

$$\begin{aligned} \tilde{\psi}_i(f) &= \sum_{S \subseteq N \setminus \{i\} : v(S)=1} |\{j \in S : v(S \setminus \{j\}) = 0\}| \cdot |S|! \cdot (n - 1 - |S|)! / n! \\ &\quad - \sum_{S \subseteq N \setminus \{i\} : v(S \cup \{i\})=1} |\{j \in S : v(S \setminus \{j\} \cup \{i\}) = 0\}| \cdot |S|! \cdot (n - 1 - |S|)! / n! \end{aligned} \quad (22)$$

for all  $1 \leq i \leq n$ .

*Proof.* We consider

$$\int_0^1 \dots \int_0^1 f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n \quad (23)$$

for an agent  $i \in N$ . In order to evaluate this expression we decompose the integration domain. For a given state vector  $\vec{x}$  with pairwise different coordinates let  $x_{i_1} < \dots < x_{i_n} = 1$  be the ordering of the coordinates and  $S \subseteq N \setminus \{i\}$  be a set of the form  $\{i_h : 1 \leq h \leq j\}$ , where  $j$  is chosen minimal such that  $S$  is winning in  $[q; \vec{w}]$ , i.e.,  $f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) = x_{i_j}$ . Note that  $\emptyset \neq S \subseteq N \setminus \{i\}$  is a winning coalition and  $S \setminus \{j\}$  is losing. If  $s$  denotes the cardinality of  $S$ , then

$$\begin{aligned} & c(S) \cdot \int_0^1 \int_0^{x_s} \dots \int_0^{x_s} \int_{x_s}^1 \dots \int_{x_s}^1 x_s \, dx_{s+1} \dots dx_{n-1} dx_1 \dots dx_s \\ &= c(S) \cdot \int_0^1 x^s (1-x)^{n-1-s} \, dx = c(S) \cdot \frac{s! \cdot (n-1-s)!}{n!} \end{aligned} \quad (24)$$

gives the value of the above integral over the integration domain that corresponds to  $S$ , where  $c(S) = |\{j \in S : v(S \setminus \{j\}) = 0\}|$  denotes the number of *critical* agents in  $S$ .

For the integral over  $f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$  only slight adjustments are necessary, so that we finally end up with the stated formula.  $\square$

For the influence measure  $\varphi$  we can also eliminate the integrals in the definition of  $\varphi$  for a simple aggregation function based on the weighted median. For each fixed agent  $i \in N$  and each fixed permutation  $\pi \in \mathcal{S}_n$  we define the set  $S = \{j \in N : \pi(j) < \pi(i)\} \subseteq N \setminus \{i\}$ . The value of

$$\left( f(\bar{\tau}(\vec{x}, \pi^{-1}(i) - 1)) - f(\bar{\tau}(\vec{x}, \pi^{-1}(i))) \right) + \left( f(\underline{\tau}(\vec{x}, \pi^{-1}(i))) - f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1)) \right)$$

is the same for any two permutations  $\pi$  and  $\pi'$ , which correspond to the same set and there correspond exactly  $|S|! \cdot (n-1-|S|)!$  permutations to a set  $S \subseteq N \setminus \{i\}$ . This allows to replace the sum over  $\mathcal{S}_n$  by a sum over the subsets of  $N \setminus \{i\}$ . In order to evaluate the above expression, we need to know the ordering of the  $x_j$  for all  $j \in S$  or all  $j \in S \cup \{i\}$ . The value  $x_h$  for an agent  $h$  outside of such a set is not relevant since it is replaced by either a 0 or a 1 in the formula for  $\varphi_i(f)$ .

For order  $y_1 \leq y_2 \leq \dots \leq y_r$  and weighted median  $y_i$  we have

$$\int_0^1 \int_0^{y_r} \int_0^{y_{r-1}} \dots \int_0^{y_2} y_i \, dy_1 \, dy_2 \dots dy_r = \frac{i}{(r+1)!}. \quad (25)$$

If the weighted median is 1, which can of course occur applying  $\bar{\tau}$ , then we have

$$\int_0^1 \int_0^{y_r} \int_0^{y_{r-1}} \dots \int_0^{y_2} 1 \, dy_1 \, dy_2 \dots dy_r = \frac{1}{r!}. \quad (26)$$

It may also happen that the weighted median is zero, where the corresponding integral of course is zero.

So, we have implemented the following algorithm: We loop over all  $S \subseteq N \setminus \{i\}$ . Then, we loop over all possible orderings of the  $x_j$ , where  $j \in S$ . The integral over

$$\left( f(\bar{\tau}(\vec{x}, \pi^{-1}(i) - 1)) - f(\underline{\tau}(\vec{x}, \pi^{-1}(i) - 1)) \right)$$

for the integration domain according to the fixed ordering within  $S$  can be evaluated by one of the above three cases - depending on the position of the weighted median in the ordering. Similarly, for

$$\left( f(\underline{\tau}(\vec{x}, \pi^{-1}(i))) - f(\bar{\tau}(\vec{x}, \pi^{-1}(i))) \right)$$

we loop over all possible orderings of the  $x_j$ , where  $j \in S \cup \{i\}$ , determine the position of the weighted median, and evaluate the integral.

We have applied the algorithm sketched above for all weighted constant-sum games with up to  $n = 9$  agents, which supports:

**Conjecture 7.1.** *Let  $\vec{w} \in \mathbb{N}^n$ , where  $n \geq 1$  and  $\sum_{i=1}^n w_i$  is odd. For  $q = (\vec{w}(N) + 1)/2$  let  $f = m_{\vec{w}}$  be the weighted median simple aggregation function corresponding to the weighted game  $v = [q; \vec{w}]$ . Then,  $\varphi(f)$  coincides with the Shapley-Shubik index of  $[q; \vec{w}]$ .*

We remark that the number of weighted constant-sum games with up to  $n = 9$  agents is given by 1, 1, 2, 3, 7, 21, 135, 2470, and 175 428, respectively, see also Table 1 in [21], where those objects were called “games in  $Z_n^r$ ”.

## 8. CONCLUSION

We have introduced simple aggregation functions which mimic simple games. Via influence functions we aim to measure the influence of an agent in a given simple aggregation function. As outlined in the introduction, there is a large variety of applications for this quantification. Exemplarily, we have introduced two influence measures, which mimic the Shapley-Shubik and the Penrose-Banzhaf index, respectively. Having proven several properties of these two influence measures, a suitable axiomatization remains an open problem. The evaluation of both influence measures is computationally involved. For two parametric classes of simple aggregation functions we have derived a more direct formula. For simple aggregation functions based on the weighted median, we have stated a reasonable simplification for  $\psi$ . It would be interesting to study whether the expression from Proposition 7.2 can serve as a reasonable power index for simple games. We also stated a first simplification for  $\varphi$ . It would be quite interesting to know if Conjecture 7.1 is true in general.

A further line of research might be to identify other interesting parametric classes of simple aggregation functions (arising from applications). In a second step, exact formulas for our two influence measures are beneficial. Also the generalization of further power indices for simple games to influence measures or the development of completely new influence measures is an interesting task for the future. To incorporate social influence between agents seems worthwhile to study, since social influence can, e.g., undermine the wisdom of the crowd effect, see [27].

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#### APPENDIX A. AN EXAMPLE FOR THE WEIGHTED MEDIAN FOR $\psi$

Consider the simple aggregation function given by the weighted median based on the weighted game  $[3; 2, 1, 1, 1]$  for four agents. We aim to evaluate  $\psi(f)$ . We start by computing  $\tilde{\psi}_1(f)$ . Assuming  $x_2 \leq x_3 \leq x_4$  we have  $f(1, x_2, x_3, x_4) = x_4$  and  $f(0, x_2, x_3, x_4) = x_2$ , so that

$$\int_0^1 \int_0^{x_4} \int_0^{x_3} f(1, x_2, x_3, x_4) \, dx_2 \, dx_3 \, dx_4 = \int_0^1 \int_0^{x_4} \int_0^{x_3} x_4 \, dx_2 \, dx_3 \, dx_4 = \frac{1}{8}$$

and

$$\int_0^1 \int_0^{x_4} \int_0^{x_3} f(0, x_2, x_3, x_4) \, dx_2 \, dx_3 \, dx_4 = \int_0^1 \int_0^{x_4} \int_0^{x_3} x_2 \, dx_2 \, dx_3 \, dx_4 = \frac{1}{24}.$$

We remark that Equation (25) gives a general formula for the above integrals. Since agents 2, 3, and 4 are symmetric, we gave the same result for all  $3!$  orderings of  $x_2, x_3$ , and  $x_4$ , so that  $\tilde{\psi}_1(f) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$ . Note that for the first sum in Equation (22) only  $S = \{2, 3, 4\}$  leads to a non-zero multiplier, which here is 3. In the second sum only the set  $\{2\}$ ,  $\{3\}$ , and  $\{4\}$  lead to a non-zero multiplier, which always is 1. Of course this gives the same result.

Next we compute  $\tilde{\psi}_2(f)$ . If  $x_1 \leq x_3 \leq x_4$ , then  $f(x_1, 1, x_3, x_4) = x_3$  and  $f(x_1, 0, x_3, x_4) = x_1$ , so that

$$\int_0^1 \int_0^{x_4} \int_0^{x_3} f(x_1, 1, x_3, x_4) \, dx_1 \, dx_3 \, dx_4 = \int_0^1 \int_0^{x_4} \int_0^{x_3} x_3 \, dx_1 \, dx_3 \, dx_4 = \frac{1}{12}$$

and

$$\int_0^1 \int_0^{x_4} \int_0^{x_3} f(x_1, 0, x_3, x_4) \, dx_1 \, dx_3 \, dx_4 = \int_0^1 \int_0^{x_4} \int_0^{x_3} x_1 \, dx_1 \, dx_3 \, dx_4 = \frac{1}{24}.$$

If  $x_3 \leq x_1 \leq x_4$ , then  $f(x_1, 1, x_3, x_4) = x_1$  and  $f(x_1, 0, x_3, x_4) = x_1$ , so that

$$\int_0^1 \int_0^{x_4} \int_0^{x_1} f(x_1, 1, x_3, x_4) \, dx_3 \, dx_1 \, dx_4 = \int_0^1 \int_0^{x_4} \int_0^{x_1} x_1 \, dx_3 \, dx_1 \, dx_4 = \frac{1}{12}$$

and

$$\int_0^1 \int_0^{x_4} \int_0^{x_1} f(x_1, 0, x_3, x_4) \, dx_3 \, dx_1 \, dx_4 = \int_0^1 \int_0^{x_4} \int_0^{x_1} x_1 \, dx_3 \, dx_1 \, dx_4 = \frac{1}{12}$$

If  $x_3 \leq x_4 \leq x_1$ , then  $f(x_1, 1, x_3, x_4) = x_1$  and  $f(x_1, 0, x_3, x_4) = x_4$ , so that

$$\int_0^1 \int_0^{x_1} \int_0^{x_4} f(x_1, 1, x_3, x_4) \, dx_3 \, dx_4 \, dx_1 = \int_0^1 \int_0^{x_1} \int_0^{x_4} x_1 \, dx_3 \, dx_4 \, dx_1 = \frac{1}{8}$$

and

$$\int_0^1 \int_0^{x_1} \int_0^{x_4} f(x_1, 0, x_3, x_4) \, dx_3 \, dx_4 \, dx_1 = \int_0^1 \int_0^{x_1} \int_0^{x_4} x_4 \, dx_3 \, dx_4 \, dx_1 = \frac{1}{12}.$$

Exchanging agent 3 with agent 4 gives the same number again, so that

$$\tilde{\psi}_2(f) = 2 \cdot \left( \frac{1}{12} + \frac{1}{12} + \frac{1}{8} \right) - 2 \cdot \left( \frac{1}{24} + \frac{1}{12} + \frac{1}{12} \right) = \frac{7}{12} - \frac{5}{12} = \frac{1}{6}.$$

Looking at the first sum of Equation (22) again, we have multipliers of two for  $S = \{1, 3\}$  or  $S = \{1, 4\}$ , a multiplier of one for  $S = \{1, 3, 4\}$ , and multipliers of zero in all other cases. I.e., the first sum equals  $\frac{2 \cdot 2 \cdot 1}{24} + \frac{2 \cdot 2 \cdot 1}{24} + \frac{1 \cdot 6 \cdot 1}{24} = \frac{7}{12}$ . For the second sum we have a multiplier of two for  $S = \{3, 4\}$ , multipliers of one for  $S = \{1\}$ ,  $S = \{1, 3\}$ ,  $S = \{1, 4\}$ , and multipliers of zero in all other cases. I.e., the second sum equals  $\frac{2 \cdot 2 \cdot 1}{24} + \frac{1 \cdot 1 \cdot 2}{24} + \frac{1 \cdot 2 \cdot 1}{24} + \frac{1 \cdot 2 \cdot 1}{24} = \frac{5}{12}$ .

The resulting power distribution  $\frac{1}{6} \cdot (3, 1, 1, 1)$  coincides with the Shapley-Shubik, the Penrose-Banzhaf, and the Public Good index of  $[3; 2, 1, 1, 1]$ . However, for  $[4; 3, 1, 1, 1, 1]$  we should get a power distribution of  $\frac{1}{32} \cdot (12, 5, 5, 5, 5)$ , which differs from the three power indices of the considered weighted game. If no error occurred during the computation, then for  $[4; 3, 2, 1, 1]$  we get  $\tilde{\psi}_1(f) = \frac{18}{120} - \frac{6}{120} = \frac{1}{10}$ ,  $\tilde{\psi}_2(f) = \frac{14}{120} - \frac{10}{120} = \frac{1}{30}$ , and  $\tilde{\psi}_3(f) = \tilde{\psi}_4(f) = \frac{14}{120} - \frac{6}{120} = \frac{1}{15}$ . Note that the power distribution is not monotone in the weights.

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