On (the existence of) Control Lyapunov Barrier Functions

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Abstract

Recently, [17] proposed control Lyapunov barrier functions (CLBFs) for the stabilization of nonlinear dynamical systems with state constraints. Under the assumption of the existence of a smooth CLBF, Sontag's universal formula was used to define a continuous feedback ensuring that the constraints are satisfied. However, as we demonstrate in this paper, the above approach neglects the necessity of discontinuous controllers in the case of state constraints. Additionally, we show that the assumption of a smooth CLBF limits the state constraints to those defined by unbounded sets. Consequently, we introduce the notion of nonsmooth complete control Lyapunov functions (CCLFs) and indicate how they may be used for stabilization in the presence of state constraints.

Key words: (nonsmooth) control Lyapunov functions; control of constrained systems; asymptotic stabilization

1 Introduction & motivation

The problem of uniting local and global controllers formally traces back to [22], where the initial problem of interest was designing a global anti-windup controller to complement a (pre-designed) local controller. This work continued in [13] where it was shown that merging local and global stabilizing controllers

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generally requires the use of time-varying or dynamic hybrid controllers. Subsequently, the uniting output feedback stabilization problem was solved in [14] by means of a dynamic hybrid stabilizer.

Of particular note is that the strategies in [13,14] are robust to various small perturbations such as measurement noise or external disturbances. This stems from the use of Lyapunov-based techniques and the associated robustness that comes with such approaches [4,6,8,20,23].

Originally introduced for dynamical systems without inputs, the concept of Lyapunov functions [11] was extended to control Lyapunov functions (CLFs) by Artstein [2] and Sontag [18] for systems with additional degrees of freedom, in the form of inputs, providing the possibility to design feedback laws using CLFs. For controllable, control affine dynamical systems Sontag developed a universal formula [19], providing a (smooth except at the origin) asymptotically stabilizing feedback law based on the knowledge of a smooth CLF. However, assuming that a smooth CLF exists is, in general, overly restrictive, which can for example be illustrated on the dynamics of the nonholonomic integrator [3], showing the necessity of discontinuous feedback laws and nonsmooth CLFs.

Nonsmooth CLFs using the Dini derivative for the decrease condition were introduced by Sontag in [18]. With this definition Rifford [15,16] and Kellett and Teel [9,10] demonstrated existence of nonsmooth CLFs in the Dini sense assuming asymptotic controllability, and additionally provided feedback stabilizers. See [5] for a comprehensive survey.

Alternatively, control barrier functions (CBFs) were introduced in [24] where, given a set of "unsafe" states, a feedback law similar to Sontag's universal formula [19] was presented to guarantee "safety". Here, safety referred to a form of forward invariance where trajectories starting in the set of safe states will not enter the set of unsafe sets. Subsequent work looked to simultaneously address the safety and stabilization problems using integrator backstepping [12,21] and quadratic programming [1] to jointly construct a CLF and a CBF. A crucial assumption in these works is that the set of unsafe sets is unbounded.

Recently, [17] formalized the uniting controller problem for safety and stabilization. In other words, in contrast to the work of [22,13,14] looking to unite local and global stabilizing feedbacks, [17] proposed to unite a stabilizing feedback with a safety guaranteed feedback. To this end, smooth CLFs are combined with CBFs to guarantee satisfaction of state constraints and stabilization of the origin. These united functions are called Control Lyapunov Barrier Functions (CLBFs). Under the strong assumption of the existence of a smooth CLBF, [17] uses Sontag's universal formula [19] to define a continuous feedback law stabilizing the dynamical system while respecting the state constraints.

Critically, unboundedness of the state constraints is not assumed in [17] and, furthermore, numerical examples are presented that claim global (on the safe set domain) asymptotic stability of the origin. However, as we demonstrate below, the existence of a continuously differentiable CLBF necessarily requires that the constraint sets are unbounded. As an alternative, we propose (nonsmooth) complete control Lyapunov functions (CCLFs) as an extension of complete Lyapunov functions [7], to simultaneously solve the safety and asymptotic feedback stabilization problems.

The paper is structured as follows. Section 2 reviews results and definitions on CLBFs from [17]. Section 3 discusses the assumption on existence of CLBFs and shows that continuously differentiable CLBFs can only exist if the set of state constraints is unbounded. In Section 4 we illustrate the necessity of discontinuous feedback laws. CCLFs, using the Dini derivative, are described in Section 5 and examples are provided. The paper concludes in Section 6.

Throughout the paper, the following notation is used. For $x \in \mathbb{R}^n$ and $\varepsilon > 0$, $B_\varepsilon(x) = \{y \in \mathbb{R}^n | \|x - y\| < \varepsilon\}$ denotes an open ball with radius ε centered around x. Moreover, $\overline{\mathcal{D}}$ and $\partial\mathcal{D}$ denote the closure and the boundary of a set $\mathcal{D} \subset \mathbb{R}^n$. For a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^{n\times p} \to \mathbb{R}^n$ we denote the Lie derivative by

$$L_g V(x) = \frac{\partial V}{\partial x}(x) g(x).$$

A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is said to belong to class $\mathcal{K}_\infty$ if $\alpha(0) = 0$, α is strictly increasing and $\alpha(t) \to \infty$ for $t \to \infty$.

2 Definitions and results on control Lyapunov barrier functions

To be able to give results and definitions on control Lyapunov barrier functions derived in [17] we first provide necessary definitions on Lyapunov stability here. Throughout this paper we follow the notation used in [17] where possible.

2.1 Lyapunov stability

We consider nonlinear dynamical systems

$$\dot{x} = F(x, u), \qquad x(0) = x_0 \tag{1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$ and a Lipschitz continuous right hand side $F : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$. A special form of nonlinear systems, particularly considered in [17], are affine systems

$$F(x,u) = f(x) + g(x)u, \qquad x(0) = x_0 \tag{2}$$

with smooth right hand side $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times p}$. Without loss of generality we assume that $F(0) = 0$ and $f(0) = 0$, i.e., $0 \in \mathbb{R}^n$ is an equilibrium of the dynamical system.

For a given control function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ and a given initial value $x_0 \in \mathbb{R}^n$ the solution of the dynamical system (1) or (2), respectively, is denoted by $\phi(\cdot, u(\cdot), x_0)$. In the case that $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ is fixed, we rewrite (1) as a autonomous ordinary differential equation

$$\dot{x} = F_u(x) = F(x, u(\cdot)), \qquad x(0) = x_0, \tag{3}$$

$F_u : \mathbb{R}^n \to \mathbb{R}^n$, with solution $\phi_u(\cdot, x_0)$. Stability properties of the equilibrium can be characterized by Lyapunov and control Lyapunov functions (CLFs).

Definition 1 *A continuously differentiable proper*[1] *and positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the ordinary differential* (3) *if*

$$L_{F_u} V(x) < 0$$

for all $x \in \mathbb{R}^n \backslash \{0\}$.

Definition 2 *A continuously differentiable proper and positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ is a CLF for the nonlinear system* (1) *if for all $x \in \mathbb{R}^n \backslash \{0\}$ there exists $u \in \mathbb{R}^p$ such that*

$$\frac{\partial V}{\partial x}(x) F(x,u) < 0.$$

For affine dynamical systems, Definition 2 can be equivalently written as follows.

Definition 3 *A continuously differentiable proper and positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ is a CLF for the nonlinear system* (2) *if*

$$L_f V(x) < 0 \qquad \forall x \in \{z \in \mathbb{R}^n \backslash \{0\} | L_g V(z) = 0\}.$$

To ensure the existence of a continuous feedback particularly around the origin, the small control property needs to be satisfied.

[1] A function $V : \mathbb{R}^n \to \mathbb{R}$ is called proper if $\{x | V(x) \leq c\}$ is compact for all $c \in \mathbb{R}$.

Definition 4 *Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable CLF for the nonlinear system* (1)*. We say that* (1) *has the small control property with respect to* V *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for every* $x \in B_\delta(0)$ *there exists* $u \in B_\varepsilon(0)$ *with*

$$\frac{\partial V}{\partial x}(x)F(x,u) < 0.$$

If a CLF for an affine system (2) is known, a stabilizing feedback can be constructed by using Sontag's universal control law, $k : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p$,

$$k(\gamma, a, b) = \begin{cases} -\frac{a+\sqrt{a^2+\gamma\|b\|^4}}{b^T b} b, & \text{if } b \neq 0 \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Theorem 5 ([19]) *Let* V *be a continuously differentiable CLF for the nonlinear system* (2) *satisfying the small control property. Then for fixed* $\gamma > 0$ *the feedback law*

$$u(x) = k(\gamma, L_f V(x), (L_g V(x))^T)$$

is continuous and the closed loop system

$$\dot{x} = f(x) + g(x)k(\gamma, L_f V(x), (L_g V(x))^T)$$

is globally asymptotically stable.

Theorem 5 reduces nonlinear systems to autonomous ordinary differential equations

$$F_k(x) = f(x) + g(x)k(\gamma, L_f V(x), (L_g V(x))^T).$$

Thus, if V is a CLF for system (2) then V is a Lyapunov function for the ordinary differential equation (3) defined through Sontag's universal formula.

2.2 Barrier functions and state constraints

In [17] state constraints are incorporated in the consideration of stability for dynamical systems by introducing the notion of safety and by using the results derived in [24] on barrier functions. We here review key results from [17].

Definition 6 *Let* $\mathcal{D} \subset \mathbb{R}^n$*,* $\mathcal{X}_0 \subset \mathbb{R}^n$*, and* $\mathcal{X}_0 \cap \mathcal{D} = \emptyset$*. The autonomous ordinary differential equation* (3) *is called safe for all* $x_0 \in \mathcal{X}_0$ *with respect to* $\mathcal{D}$ *if* $\phi(t, x_0) \notin \mathcal{D}$ *for all* $t \geq 0$ *and for all* $x_0 \in \mathcal{X}_0$*.*

Definition 7 *Let $\mathcal{D} \subset \mathbb{R}^n$. A continuously differentiable function $B : \mathbb{R}^n \to \mathbb{R}$ is called control barrier function (CBF) for the system* (2) *with respect to $\mathcal{D}$ if it satisfies the following conditions:*

$$
\begin{aligned}
&B(x) > 0 && \forall x \ \in \mathcal{D}, \\
&L_f B(x) \leq 0 && \forall x \ \in \{z \in \mathbb{R}^n \backslash \mathcal{D} | L_g B(z) = 0\}, \\
&\mathcal{U} := \{x \in \mathbb{R}^n | B(x) \leq 0\} \neq \emptyset.
\end{aligned}
$$

Theorem 8 ([17, Thm. 2]) *Let $B : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable CBF of system* (2) *with respect to $\mathcal{D} \subset \mathbb{R}^n$. The autonomous system*

$$\dot{x} = f(x) + g(x)k(\gamma, L_f B(x), (L_g B(x))^T),$$

$\gamma > 0$*, is safe for all $x_0 \in \mathcal{X}_0 = \mathcal{U}$ according to Definition 6. If*

$$\overline{\mathbb{R}^n \backslash \mathcal{D} \cup \mathcal{U}} \cap \overline{\mathcal{U}} = \emptyset$$

then the autonomous system is safe with $\mathcal{X}_0 = \mathbb{R}^n \backslash \mathcal{D}$.

2.3 Control Lyapunov barrier functions

To achieve asymptotic stability and simultaneously satisfy state constraints (i.e., safety), [17] proposed combining CLFs and CBFs.

Definition 9 ([17, Def. 2]) *Given a set of unsafe states $\mathcal{D}$, a proper and lower-bounded continuously differentiable function $W : \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$
\begin{aligned}
&W(x) > 0 && \forall x \in \mathcal{D}, \\
&L_f W < 0 && \forall x \in \{z \in \mathbb{R}^n \backslash (\mathcal{D} \cup \{0\}) | L_g W(z) = 0\}, \\
&\mathcal{U} = \{x \in \mathbb{R}^n | W(x) \ \leq 0\} \neq \emptyset, \\
&\overline{\mathbb{R}^n \backslash \mathcal{D} \cup \mathcal{U}} \cap \overline{\mathcal{D}} = \emptyset,
\end{aligned}
$$

is called a control Lyapunov barrier function (CLBF).

With the definition of a CLBF the following result is presented in [17].

Theorem 10 ([17, Thm. 3]) *Assume that the system* (2) *admits a continuously differentiable CLBF $W : \mathbb{R}^n \to \mathbb{R}$ with a given set of unsafe states $\mathcal{D}$ and* (2) *satisfies the small control property with respect to W. Then the feedback law*

$$u = k(\gamma, L_f W(x), (L_g W(x))^T),$$

$\gamma > 0$, *satisfies*

$$\lim_{t\to\infty} \phi_u(t, x_0) = 0 \qquad \forall\, x_0 \in \mathbb{R}^n \backslash \mathcal{D}, \qquad \textit{and}$$
$$\phi_u(t, x_0) \notin \mathcal{D} \qquad \forall\, x_0 \in \mathbb{R}^n \backslash \mathcal{D}, \qquad \forall\, t \geq 0.$$

3 Discussion on the existence of continuously differentiable CLBFs

While, in principle, Theorem 10 provides an elegant method to accommodate state constraints in a Lyapunov-based stabilizing feedback, the assumption on the existence of CLBFs is very restrictive. Moreover, the construction of a CLBF based on the knowledge of a CLF and a CBF is not as simple as described in [17, Prop. 3] and applied in [17, Sec. 6] to two numerical examples. These two points are discussed in this and the following section.

The general assumption on the existence of a continuously differentiable CLBF is very strong. Here, we will show that a CLBF can only exist if the set of unsafe states $\mathcal{D}$ is unbounded.

Theorem 11 *Assume that* $W : \mathbb{R}^n \to \mathbb{R}$ *is a continuously differentiable CLBF of system* (2) *for a given set of unsafe states* $\mathcal{D}$*. Additionally, assume that* (2) *satisfies the small control property with respect to* W*. Then Theorem 10 can only hold if* $\mathcal{D}$ *is unbounded.*

PROOF. We consider the ordinary differential equation

$$\dot{x} = F_k(x) = f(x) + g(x)k(\gamma, L_f V(x), (L_g V(x))^T)$$

for $\gamma > 0$ with solution $\phi_k(\cdot, x_0)$, $x_0 \in \mathbb{R}^n \backslash \mathcal{D}$, and assume that the set $\mathcal{D}$ is bounded. Additionally, without loss of generality, we assume that the set $\mathcal{D}$ is connected. (If $\mathcal{D}$ is the union of disjoint sets, the following arguments can be used for each connected component.) According to Theorem 10 the feedback k asymptotically stabilizes every initial state $x_0 \in \mathbb{R}^n \backslash \mathcal{D}$, i.e.,

$$\lim_{t\to\infty} \phi_k(t, x_0) \to 0.$$

Let $\mathcal{X}_c$ be defined as the level set $\mathcal{X}_c = \{x \in \mathbb{R}^n | W(x) \leq c\}$ and let $c > 0$ be chosen such that $\mathcal{D} \cup \{0\} \subset X_c \backslash \partial X_c$. (This is possible since $\mathcal{D}$ is bounded and W is continuous on $\mathbb{R}^n$.) Since $L_{F_k}(x) \leq 0$ for all $x \in \partial \mathcal{X}_c$ the set is forward invariant, i.e.,

$$\phi_k(\cdot, x_0) \subset \mathcal{X}_c$$

for all $x_0 \in \partial X_c$.

Since the solutions $\phi_k(\cdot, x_0)$ are continuous with respect to the initial state x_0, there need to exist initial values $x_0 \in \partial\mathcal{X}_c$ such that $\phi_k(\cdot, x_0)$ passes $\mathcal{D}$ from all possible directions. This means the trajectories $\cup_{x_0 \in \partial\mathcal{X}_c} \phi_k(\cdot, x_0)$ surround the set $\mathcal{D}$. Moreover, again due to continuity with respect to the initial state, there needs to exist an $\tilde{x}_0 \in \partial\mathcal{X}_c$ such that for all $\varepsilon > 0$ the solutions

$$\bigcup_{x_0 \in \partial\mathcal{X}_c \cap B_\varepsilon(\tilde{x}_0)} \phi_k(\cdot, x_0)$$

surround the set $\mathcal{D}$ which implies the existence of a heteroclinic orbit around the set $\mathcal{D}$. In particular it implies that

$$\lim_{t\to\infty} \phi_k(t, \tilde{x}_0) \neq 0$$

which contradicts the global asymptotic stability. Thus, the assumption on the boundedness of $\mathcal{D}$ or the asymptotic stability of Theorem 10 was wrong. In the case that $\mathcal{D}$ is not connected, but is the union of disjoint subsets, with the same arguments a heteroclinic orbit around every connected subset of $\mathcal{D}$ can be concluded. This contradiction completes the proof.

□

The idea of the proof in the two dimensional case is visualized in Figure 1. The figure particularly shows the necessity of discontinuous feedback laws in the case of bounded sets $\mathcal{D}$ in order to select to pass the obstacle to the left or the right. This discontinuity cannot be achieved by the use of Sontag's formula (4).

4 Construction of CLBFs

In [17] two numerical two-dimensional examples for the construction of CLBFs and bounded sets $\mathcal{D}$ are provided. However, as we have just seen, a continuously differentiable CLBF W cannot exist if Sontag's formula is used for the feedback design. In this section, by considering a simplified dynamical system, we illustrate the missing condition on the CLBF in the derivation in [17, Sec. 6]. For completeness we provide the result given in [17] for the construction of CLBFs.

Claim 12 ([17, Prop. 3]) *Suppose that for system* (2)*, with a given open set of unsafe states* $\mathcal{D} \subset \mathbb{R}^n$*, there exist a continuously differentiable CLF* $V : \mathbb{R}^n \to \mathbb{R}$ *and a continuously differentiable CBF* $B : \mathbb{R}^n \to \mathbb{R}$ *which satisfy*

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2 \qquad \forall\, x \in \mathbb{R}^n, \tag{5}$$

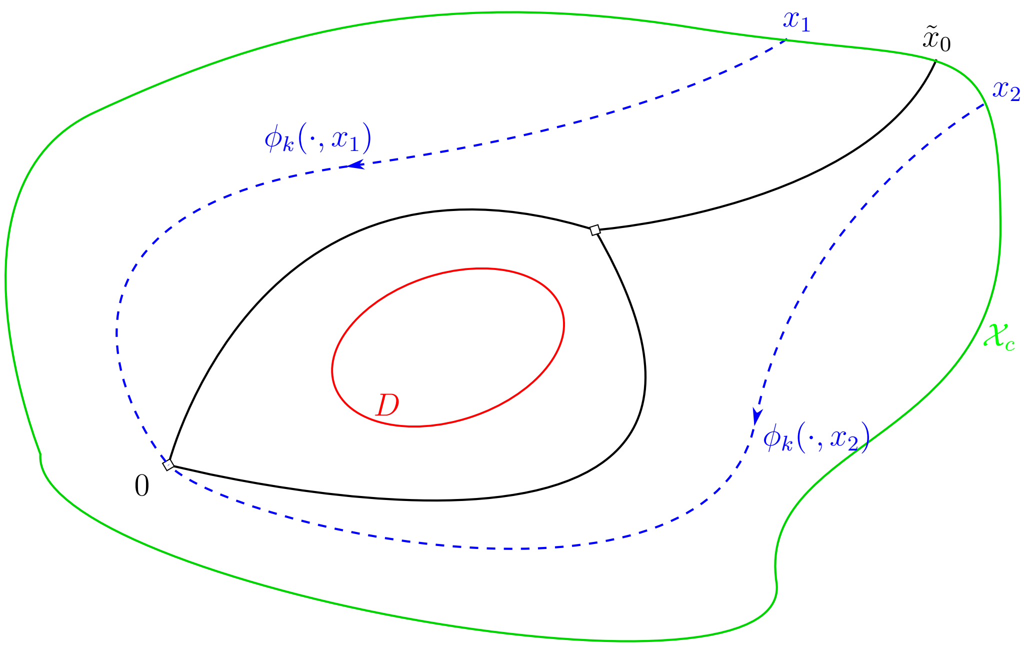

Fig. 1. Visualization of the idea of the proof of Theorem 11. The black lines indicate limit cycles and ⋄'s indicate equilibria of the ordinary differential equation. The existence of solutions $\phi_k(\cdot, x_1)$ and $\phi_k(\cdot, x_2)$ passing the set $\mathcal{D}$ from the left and the right implies the existence of an initial state $\tilde{x}_0$ such that $\phi_k(t, \tilde{x}_0)$ does not converge to the origin for $t \to \infty$.

$c_1 > 0$, $c_2 > 0$, *and a compact and connected set* $\mathcal{X}$ *such that*

$$\mathcal{D} \subset \mathcal{X},\ 0 \notin \mathcal{X},\ \textit{and } B(x) = -\varepsilon < 0\ \forall x \in \mathbb{R}^n \backslash \mathcal{X}.$$

If

$$L_f W(x) < 0 \ \ \forall x \in \{z \in \mathbb{R}^n \backslash (\mathcal{D} \cup \{0\}) | L_g W(z) = 0\} \tag{6}$$

where

$$W(x) = V(x) + \lambda B(x) + \kappa,$$

with $\lambda > \frac{c_2 c_3 - c_1 c_4}{\varepsilon}$, $\kappa = -c_1 c_4$, $c_3 = \max_{x \in \partial X} \|x\|^2$, $c_4 = \min_{x \in \partial D} \|x\|^2$, *then the feedback law*

$$u(x) = k(\gamma, L_f W(x), (L_g W(x))^T) \tag{7}$$

satisfies

$$\begin{aligned} &\lim_{t\to\infty} \phi_u(t, x_0) = 0 && \forall\, x_0 \in \mathbb{R}^n \backslash \mathcal{D}_{\text{relaxed}}, && \textit{and} \\ &\phi_u(t, x_0) \notin \mathcal{D}_{\text{relaxed}} && \forall\, x_0 \in \mathbb{R}^n \backslash \mathcal{D}_{\text{relaxed}}, && \forall\, t \geq 0, \end{aligned}$$

where $\mathcal{D}_{relaxed} = \{x \in \mathcal{X} | W(x) > 0\} \supset \mathcal{D}$*. Moreover if* (2) *satisfies the small control property with respect to* V *then it satisfies the small control property with respect to* W*.*

Remark 13 *Observe that due to the compactness of* $\mathcal{X}$ *in Claim 12 the set* $\mathcal{D} \subset \mathcal{X}$ *needs to be bounded.*

To illustrate a key point in the numerical examples in [17, Sec. 6], we consider the linear dynamical system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{8}$$

The smooth function

$$V(x) = \|x\|^2 \tag{9}$$

is a global CLF for the dynamical system and satisfies the technical constraints (5) trivially for the constants $c_1 = c_2 = 1$. For the construction of a CLBF we discuss two examples following the arguments given in [17, Sec. 6].

Example 14 *For the construction of the CBF, consider* $\mathcal{X} = (1, 3) \times (1, 3)$ *and*

$$\mathcal{D} = \left\{ x \in \mathcal{X} \,\middle|\, \frac{1}{1-(x_1-2)^2} + \frac{1}{1-(x_2-2)^2} < 4 \right\}$$

and define the CBF

$$B(x) = \begin{cases} e^{-\left(\frac{1}{1-(x_1-2)^2} + \frac{1}{1-(x_2-2)^2}\right)} - e^{-4}, & \text{for } x \in \mathcal{X} \\ -e^{-4}, & \text{for } x \in \mathbb{R}^2 \backslash \mathcal{X} \end{cases}$$

The constants c_3 *and* c_4 *can be defined as* $c_3 = 18$ *and* $c_4 = 3.34$*. Thus, according to Claim 12, with* $\varepsilon = 10^{-4}$ *the constants* λ *and* κ *can be defined as* $\lambda = 146600$ *and* $\kappa = -3.34$ *to obtain the CLBF*

$$W(x) = V(x) + \lambda B(x) + \kappa. \tag{10}$$

For the derivative, it holds that

$$L_f V(x) = -2\|x\|^2 < 0 \qquad \text{and} \qquad L_g V(x) = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix}.$$

For the function B it holds that

$$L_fB(x) = e^{\frac{1}{x_1^2-4x_1+3}+\frac{1}{x_2^2-4x_2+3}}\left(\frac{2x_1(x_1-2)}{(x_1^2-4x_1+3)^2}+\frac{2x_2(x_2-2)}{(x_2^2-4x_2+3)^2}\right)$$
$$L_gB(x) = -e^{\frac{1}{x_1^2-4x_1+3}+\frac{1}{x_2^2-4x_2+3}}\begin{bmatrix}\frac{2(x_1-2)}{(x_1^2-4x_1+3)^2} & \frac{2(x_2-2)}{(x_2^2-4x_2+3)^2}\end{bmatrix}$$

for all $x \in \mathcal{X}$ and $L_fB(x) = 0$ for all $x \in \mathbb{R}^n \backslash \mathcal{X}$ with $L_gB(x) = 0$. Thus it follows that B is a CBF, and following the arguments in [17], this implies that W defined in (10) *is a CLBF.*

However, even though V is a CLF and B is a CBF the decrease condition cannot be satisfied for all $x \in \mathcal{X}_0$ as visualized in Figure 2. For $x \in \mathbb{R}^2$ with $x_1 = x_2 > 3$ the trajectory $\phi_k(\cdot, x)$ does not converge to the origin but converges to a different equilibrium of the closed loop system, as indicated by the saddle point shown in Figure 3. Here, a discontinuous feedback would be necessary

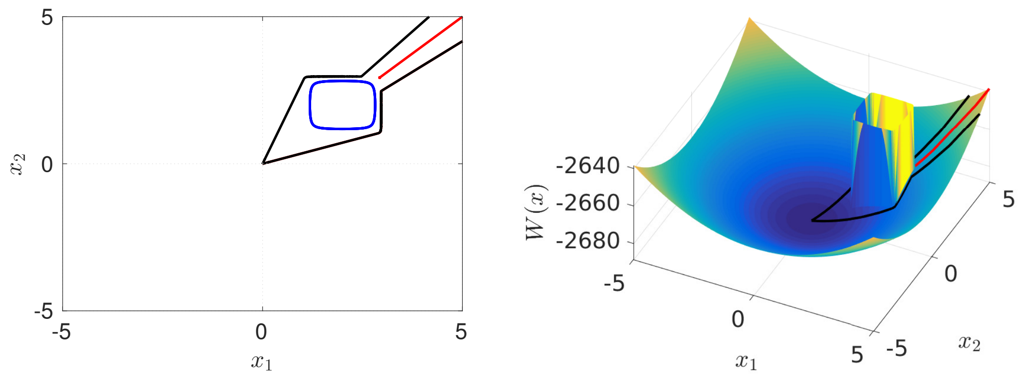

Fig. 2. Visualization of closed loop trajectories of the dynamical system for different initial states on the left and the CLBF on the right. On the left, the blue set denotes the set $\mathcal{D}$. For initial states on the axis $x_1 = x_2$ and x_1 large enough, asymptotic stability does not hold.

to make a decision to pass the equilibrium from the left or from the right. This situation exactly captures the situation illustrated in Figure 1.

The numerical examples in [17, Sec. 6] necessarily face the same issue, but the critical trajectory is not visualized and is harder to identify due to a more complex geometry.

Example 14 might lead to the conjecture that a form of almost everywhere stabilizability might be achievable; i.e., that the above issue is restricted to a set of measure zero. However, this is not the case as demonstrated in the next example where the set of unsafe states $\mathcal{D}$ is shifted to the x_1-axis.

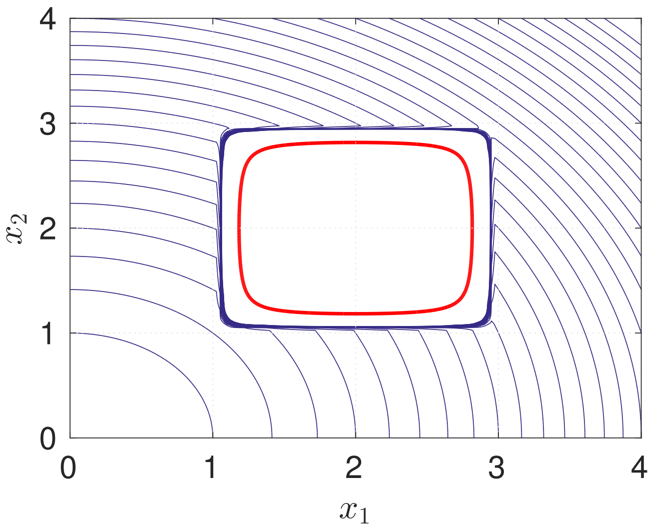

Fig. 3. Contour plot of the CLBF. The CLBF has a saddle point in the positive orthant. Trajectories $\phi_k(\cdot, x_0)$ converging to the saddle point are not asymptotically stabilized by the feedback law. The red line indicates the boundary of the set of unsafe states.

Example 15 *Consider the set* $\mathcal{X} = (1,3) \times (-1,1)$ *and*

$$\mathcal{D} = \left\{ x \in \mathcal{X} \,\middle|\, \frac{1}{1-(x_1-2)^2} + \frac{1}{1-x_2^2} < 4 \right\}$$

Define

$$B(x) = \begin{cases} e^{-\left(\frac{1}{1-(x_1-2)^2}+\frac{1}{1-x_2^2}\right)} - e^{-4} & \text{for } x \in \mathcal{X} \\ -e^{-4} & \text{for } x \in \mathbb{R}^2 \backslash \mathcal{X} \end{cases}$$

and the constants $c_3 = 10$ *and* $c_4 = 1.4$*. Thus, again according to Claim 12, we set* $\varepsilon = 10^{-4}$, $\lambda = 86000$ *and* $\kappa = -1.4$ *to obtain the CLBF*

$$W(x) = V(x) + \lambda B(x) + \kappa.$$

where $V(x) = \|x\|^2$ *as before.*

The function B *satisfies*

$$L_f B(x) = e^{\frac{1}{x_1^2-4x_1+3}+\frac{1}{x_2^2-1}} \left(\frac{2x_1(x_1-2)}{(x_1^2-4x_1+3)^2} + \frac{2x_2^2}{(1-x_2^2)^2} \right)$$

$$L_g B(x) = -e^{\frac{1}{x_1^2-4x_1+3}+\frac{1}{x_2^2-1}} \begin{bmatrix} \frac{2(x_1-2)}{(x_1^2-4x_1+3)^2} & \frac{2x_2}{(1-x_2^2)^2} \end{bmatrix}$$

for all $x \in \mathcal{X}$*, and* $L_f B(x) = 0$ *for all* $x \in \mathbb{R}^n \backslash \mathcal{X}$ *with* $L_g B(x) = 0$*. Thus it follows that* B *is a CBF.*

Similar to Example 14 asymptotic stability is not guaranteed for all $x \in \mathcal{X}_0$. In addition to the positive x_1-axis with $x_1 > 3$, trajectories corresponding to initial values close to the x_1-axis are not asymptotically stabilized as visualized in Figure 4. Hence, not only a set of measure zero is problematic in this example. [2]

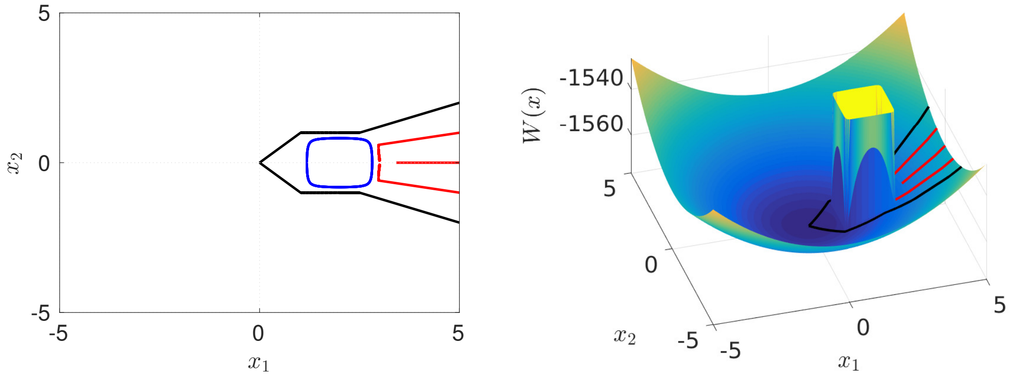

Fig. 4. Visualization of closed loop trajectories of the dynamical system for different initial states on the left and the CLBF on the right. On the left, the blue set denotes the set $\mathcal{D}$. Red lines indicate trajectories which do not converge to the origin. Here, not only a set of measure zero is not asymptotically stabilized by the feedback law.

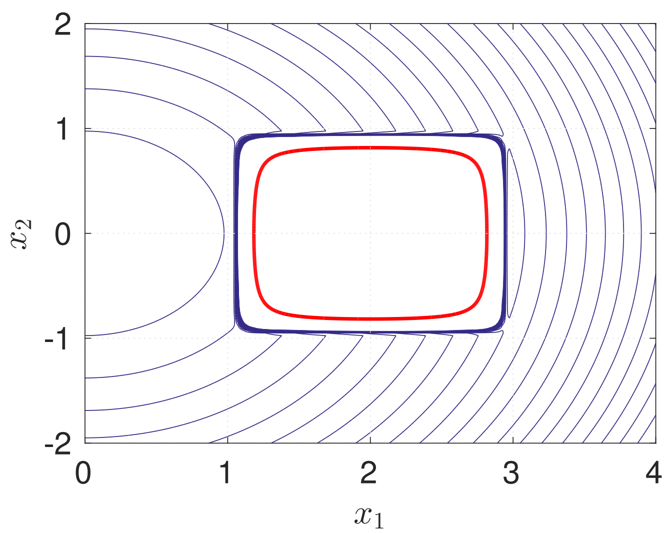

Fig. 5. Contour plot of the CLBF. The CLBF has a local minimum on the x_1-axis which explains why the feedback law does not asymptotically stabilize certain initial values x_0. The red line indicates the boundary of the set of unsafe states.

[2] The numerical simulations are performed in Matlab. Due to the steep slope of the function W, (which is used in the feedback law,) the numerical solution is very sensitive to the state x.

5 Complete Lyapunov functions using the Dini derivative

The previous sections show that a continuous feedback cannot be used in general to asymptotically stabilize a dynamical system with state constraints. As a consequence, the use of continuously differentiable CLBFs and Sontag's Formula (4) are insufficient for the consideration of this problem. Instead, based on existence results for nonsmooth CLFs using the Dini derivative or proximal gradients for systems which do not admit a smooth CLF, we propose an alternative idea to CLBFs which we refer to as complete control Lyapunov functions (CCLFs).

The lower right Dini derivative of a Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$D_+V(x;w) = \liminf_{t \searrow 0} \frac{V(x+tw) - V(x)}{t}.$$

If V is continuously differentiable, then the Dini derivative coincides with the Lie derivative, i.e.,

$$L_F V(x) = D_+V(x; F(x,u))$$

where $F(x,u)$ denotes the right hand side of the dynamical system (1). With this definition we can define CCLFs incorporating state constraints.

Definition 16 (CCLF) *For $i = 1, \dots, N$, $N \in \mathbb{N}$, let $\mathcal{D}_i \subset \mathbb{R}^n$ be compact and connected. A Lipschitz continuous function $V_C : \mathbb{R}^n \to \mathbb{R}$ is called complete control Lyapunov function (CCLF) for the dynamical system* (1) *with constraints $\mathcal{D}_i$, $i = 1, \dots, N$, if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and open neighborhoods $\mathcal{O}_i \subset \mathbb{R}^n$, $\mathcal{D}_i \subset \mathcal{O}_i$, $0 \notin \cup_{i=1}^N \overline{\mathcal{O}}_i$, such that*

$$V_C(x) < \min_{y \in \mathcal{D}_i} V_C(y), \quad V_C(x) = V_C(\tilde{x}) \qquad \forall\, x, \tilde{x} \in \partial \mathcal{O}_i, \tag{11}$$

$$\alpha_1(\|x\|) \le V_C(x) \le \alpha_2(\|x\|), \tag{12}$$

$$\min_{\|u\| \le \alpha_3(\|x\|)} D_+V_C(x, F(x,u)) < 0 \qquad \forall\, x \in \mathbb{R}^n \setminus \left(\cup_{i=1}^N \mathcal{O}_i \cup \{0\} \right). \tag{13}$$

Conditions (12) and (13) are the usual conditions for control Lyapunov functions in the Dini sense. Here, the $\mathcal{K}_\infty$-function α_3 ensures that the small control property is satisfied with respect to the function V_C. Condition (11) ensures that $\overline{\mathcal{O}}_i$ defines a local level set of the function V_C with a local maximum in $\mathcal{D}_i$. From the definition of a CCLF it is clear that for all $x \notin \cup_{i=1}^N \mathcal{O}_i$ there exists an input $u(\cdot)$ such that $V_C(\phi(t, u(t), x))$ is strictly decreasing for all t with $\phi(t, u(t), x)) \neq 0$. Thus, there exists an asymptotically stabilizing feedback (possibly discontinuous) and $\phi(t, u(t), x)) \notin \mathcal{D}_i$ for all $t \geq 0$.

Theorem 17 *Assume there exists a Lipschitz continuous CCLF* $V_C : \mathbb{R}^n \to \mathbb{R}$ *for a given dynamical system* (1) *and compact and disjoint state constraints* $\mathcal{D}_i$, $i = 1, \ldots, N$, $N \in \mathbb{N}$, *where each* $\mathcal{D}_i$ *is connected. Then for all* $x \in \mathbb{R}^n \backslash \left(\cup_{i=1}^N \mathcal{O}_i \right)$ *there exists a (possibly discontinuous) feedback law* $k : \mathbb{R}^n \to \mathbb{R}^p$, *such that*

$$\begin{aligned}
&\|k(x)\| \leq \alpha_3(\|x\|), \\
&\lim_{t\to\infty} \phi(t, k(x(t)), x) = 0, \\
&\phi(t_1, k(x(t_1)), x) < \phi(t_2, k(x(t_2)), x) \qquad && \forall\, t_1 \leq t_2, \\
&\phi(t, k(x(t)), x) \notin \cup_{i=1}^N \overline{\mathcal{O}}_i && \forall\, t > 0.
\end{aligned}$$

In Theorem 17 the feedback can for example be defined as

$$k(x) \in \operatorname*{argmin}_{\|u\| \leq \alpha_3(\|x\|)} D_+ V_C(x, F(x,u)), \tag{14}$$

even though it might in general be impossible to give a closed-form expression for $k : \mathbb{R}^n \to \mathbb{R}^p$.

Similar to the approach in [17], existence and the construction of CCLFs are crucial questions in this context. Here we discuss two simple examples particularly pointing out the necessity of nonsmooth CCLFs.

Example 18 *We start with the simple two dimensional fully actuated dynamical system* $\dot{x}_1 = u_1$, $\dot{x}_2 = u_2$. *For the state constraints we consider the set*

$$\mathcal{D} = \{x \in \mathbb{R}^2 | x_1^2 + (x_2 - 2)^2 \leq 0.25\} \tag{15}$$

and we define the function

$$\widehat{V}_C(x) = \max\{x_1^2 + x_2^2, 40 - 25(x_1^2 + (x_2 - 2)^2)\}.$$

The function $\widehat{V}_C$ *is visualized in Figure 6. The constraints are indicated by the red circle. The neighborhood* $\mathcal{O}$ *can for example be defined as*

$$\mathcal{O} = \{x \in \mathbb{R}^2 | x_1^2 + (x_2 - 2)^2 < 0.3\}.$$

The decrease condition

$$\min_{\|u\| \leq \|x\|} D_+ \widehat{V}_C(x, F(x,u)) < 0$$

holds for allmost all $x \in \mathbb{R}^n \backslash \left(\cup_{i=1}^N \mathcal{O}_i \cup \{0\} \right)$. *Moreover, functions* α_1 *and* α_2 *can for example be defined as* $\alpha_1(x) = \|x\|^2$ *and* $\alpha_2(\|x\|) = \max_{\|y\| \leq \|x\|} \widehat{V}_C(y) + \|x\|^2$.

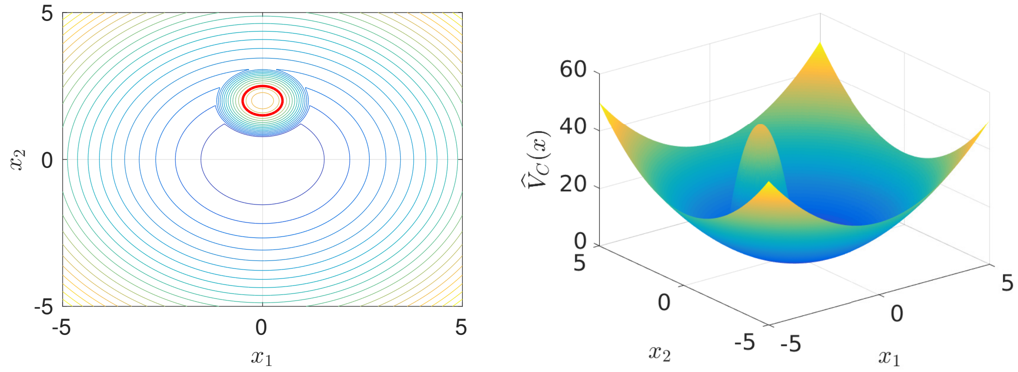

Fig. 6. Visualization of the function $\widehat{V}_C(x) = \max\{x_1^2 + x_2^2, 40 - 25(x_1^2 + (x_2 - 2)^2)\}$. The red circle (left) shows the boundary of the constraints $\partial\mathcal{D}$.

However, the function $\widehat{V}_C$ has a local minimum on the x_2-axis where

$$x_2^2 = 40 - 25(x_2 - 2)^2, \qquad x_2 > 0$$

holds. Since here the tangent of the inner circles and outer circles point in opposite directions (see Figure 6, left), we have a local minimum and not a saddle point. Thus, $\widehat{V}_C$ is not a CCLF.

The function

$$V_C(x) = \max\{x_1^2 + x_2^2, 40 - 25(|x_1| + |x_2 - 2|)\} \tag{16}$$

visualized in Figure 7 is a CCLF with respect to the constraints $\mathcal{D}$. Here the

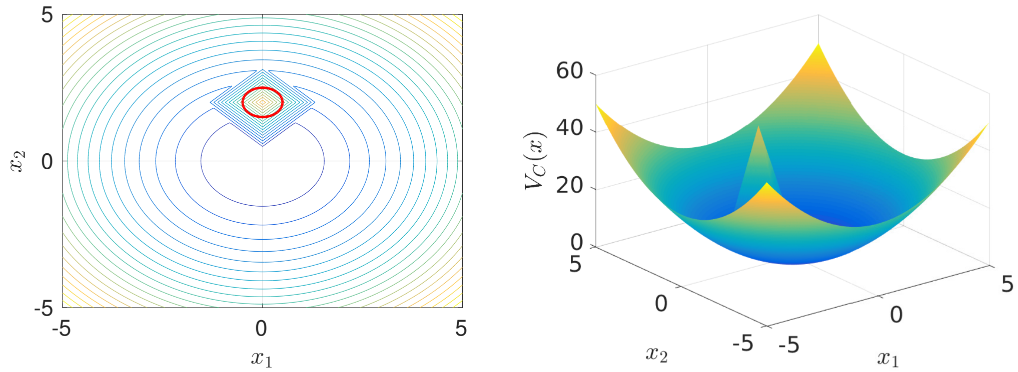

Fig. 7. Visualization of the function $V_C(x) = \max\{x_1^2 + x_2^2, 40 - 25(|x_1| + |x_2 - 2|)\}$. The red circle (left) shows the boundary of the constraints $\partial\mathcal{D}$.

same observations as in the case of $\widehat{V}_C$ hold, but additionally, the critical point on the x_2-axis is not a local minimum and there exist decreasing directions passing the diamond shaped peak on the left or on the right.

Example 18 indicates that even for fully actuated linear systems smooth CCLFs fail to exist. An ansatz of a smooth CCLF naturally creates a point $x \in$

$\mathbb{R}^n \backslash (\mathcal{O} \cup \{0\})$ behind the constraint set, where $\min_{u\in\mathbb{R}^n} D_+V_C(x, F(x, u)) = 0$.

In a second example we highlight the connection between the shape of the set $\mathcal{O}$ and the possible directions of the flow of the dynamical system.

Example 19 *Consider the linear dynamical system*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + u \\ -x_2 + u \end{pmatrix}$$

with input constraints $|u| \leq 5\|x\|$. Again, we consider the set of constraints $\mathcal{D}$ defined in (15)*. Here, the function* (16) *is not a CCLF. Since u is one dimensional it is not possible to move in an arbitrary direction. For a neighborhood around the set $\mathcal{D}$, V_C is visualized in Figure 8 (left). Possible directions of the dynamical system are indicated by blue ($u = 0$), red ($u = 5\|x\|$) and yellow ($u = -5\|x\|$) arrows. All possible directions at a state x are given by convex combinations of the red and the yellow arrows. One can see that for some states in the north-west of the constraint set $\mathcal{D}$ there does not exist a feasible decreasing direction since all possible directions point into higher level sets of V_C. By changing the shape of the function around the constraint set $\mathcal{D}$, for*

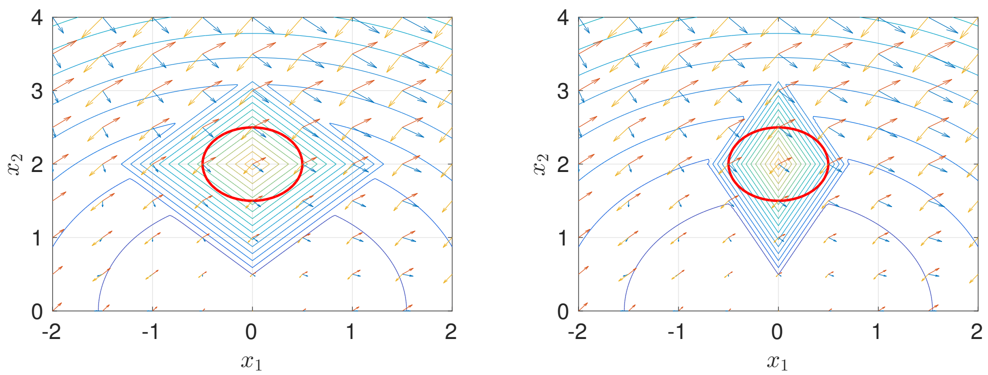

Fig. 8. Contour plots of the function $V_C(x) = \max\{x_1^2 + x_2^2, 40 - 25(|x_1| + |x_2 - 2|)\}$ (left) and $V_C(x) = \max\{x_1^2+x_2^2, 40-25(2|x_1|+|x_2-2|)\}$ (right). The red circle shows the boundary of the constraints $\partial\mathcal{D}$. The arrows indicate the possible directions for $u = 0$ (blue), $u = 5\|x\|$ (red) and $u = -5\|x\|$ (yellow).

example by using the function

$$V_C(x) = \max\{x_1^2 + x_2^2, 40 - 25(2|x_1| + |x_2 - 2|)\},$$

a decreasing direction can be guaranteed (Figure 8, right). Here, for example the input $u = -5\|x\|$ provides a decrease for the critical states in the north-west of the set $\mathcal{D}$.

6 Conclusion

In this paper we discussed the results on smooth CLBFs derived in [17]. In particular, we described the necessity of nonsmooth functions and discontinuous feedback laws in the context of control problems with state constraints. This led us to introduce nonsmooth CCLFs in the Dini sense. The definition of CCLFs naturally extends the definition of CLFs for nonlinear dynamical systems and complete Lyapunov functions for systems without inputs. The existence of a CCLF for a system with an asymptotically controllable origin with (compact) state constraints, and an associated robust feedback stabilizer, remains an open question.

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