BOUNDS FOR THE NAKAMURA NUMBER

JOSEP FREIXAS AND SASCHA KURZ

ABSTRACT. The Nakamura number is an appropriate invariant of a simple game in order to study the existence of social equilibria and the possibility of cycles. For symmetric quota games its number can be obtained by an easy formula. For some subclasses of simple games the corresponding Nakamura number has also been characterized. However, in general, not much is known about lower and upper bounds depending of invariants of simple, complete or weighted games. Here, we present several results in that direction.

1. INTRODUCTION

Consider a committee with a finite set N of committee members. Suppose that a subset S of the committee members is in favor of variant A of a certain proposal, while all others, i.e., those in $N\backslash S$, are in favor of variant B. If the committee's decision rule is such that both S and $N\backslash S$ can change the status quo, then we may end up in an infinite chain of status quo changes between variant A and variant B – a very unpleasant and unstable situation. In the context of simple games the described situation can be prevented easily. One just has to restrict the allowed class of voting systems to *proper simple games*, i.e., each two *winning coalitions* have at least one common player. As a generalization the Nakamura number of a simple game is the smallest number k such that there exist k winning coalitions with empty intersection. So, a simple game is proper if and only if its Nakamura number is at least 3. Indeed, the *Nakamura number* turned out to be an appropriate invariant of a simple game in order to study the existence of social equilibria and the possibility of cycles in a more general setting, see [Schofield, 1984]. As the author coins it, individual convex preferences are insufficient to guarantee convex social preference. If, however, the Nakamura number of the used decision rule is large enough with respect to the dimension of the involved policy space, then convex individual preference guarantees convex social preferences. Having this relation at hand, a stability result of [Greenberg, 1979] on q-majority games boils down to the computation of the Nakamura number for these games. The original result of [Nakamura, 1979] gives a necessary and sufficient condition for the non-emptiness of the *core* of a simple game obtained from individual preferences. Further stability results in terms of the Nakamura number are e.g. given by [Le Breton and Salles, 1990]. A generalization to coalition structures can be found in [Deb et al., 1996]. For other notions of stability and acyclicity we refer e.g. to [Martin, 1998, Schwartz, 2001, Truchon, 1996]. Unifications of related theorems have been presented in [Saari, 2014].

Here we study lower and upper bounds for the Nakamura number of different types of voting games. For the mentioned q-majority games with n players the Nakamura could be analytically determined to be $\left\lceil\frac{n}{n-q}\right\rceil$ by [Ferejohn and Grether, 1974] and [Peleg, 1978]. For general weighted games with normalized weights, i.e., with weight sum one, we prove that the corresponding expression $\left\lceil\frac{1}{1-q}\right\rceil$ is a lower bound for the Nakamura number. While relatively tight bounds for the Nakamura number of weighted games can be obtained, the natural invariants of simple and complete simple games allow only weaker bounds. The excess minimization problem in the first stage of a nucleolus computation allows an adequate counterpart for weights in the case on non-weighted cases and partially allows to improve bounds for weighted games. Additionally we show up a relation to the one-dimensional cutting stock problem.

[Kumabe and Mihara, 2008] studied the 32 combinations of five properties of simple games. In each of the cases the authors determined the generic Nakamura number or the best possible lower bound if several values can be attained. As a generalization of simple games with more than two alternatives, so-called (j,k)-simple games have been introduced, see e.g. [Freixas and Zwicker, 2009]. The notion of the Nakamura number and a first set of stability results for $(j,2)$-simple games have been transfered by [Tchantcho et al., 2010].

The remaining part of the paper is organized in follows. In Section 2 we define simple games, the Nakamura number for simple games and state related lower and upper bounds. Special subclasses of simple games, including weighted games, are studied in Section 3. The maximum possible Nakamura number within special subclasses of

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simple games is the topic of Section 4. Further relation of the Nakamura number to other concepts of cooperative game theory are discussed in Section 5. In this context the one-dimensional cutting stock problem is treated in Subsection 5.1. Some enumeration results for special subclasses of complete and weighted simple games and their corresponding Nakamura numbers are given in Section 6. We close with a conclusion in Section 7.

2. Preliminaries and bounds for simple games

A pair (N, v) is called simple game if N is a finite set, $v : 2^N \to \{0,1\}$ satisfies $v(\emptyset) = 0$, $v(N) = 1$, and $v(S) \le v(T)$ for all $S \subseteq T \subseteq N$. In this paper we limit ourselves to non-trivial monotonic simple games. The subsets of N are called coalitions and N is called the grand coalition. By $n = |N|$ we denote the number of players. If $v(S) = 1$, we call S a winning coalition and a losing coalition otherwise. By $\mathcal{W}$ we denote the set of winning coalitions and by $\mathcal{L}$ the set of losing coalitions. If S is a winning coalition such that each proper subset is losing we call S a minimal winning coalition. Similarly, if T is a losing coalition such that each proper superset is winning, we call T a maximal losing coalition. By $\mathcal{W}^m$ we denote the set of minimal winning coalitions and by $\mathcal{L}^M$ we denote the set of maximal losing coalitions. We remark that each of the sets $\mathcal{W}$, $\mathcal{L}$, $\mathcal{W}^m$ and $\mathcal{L}^M$ uniquely characterizes a simple game. Instead of (N, v) we also write $(N, \mathcal{W})$ for a simple game. Next we introduce special kinds of players in a simple game.

Definition 1. *Let (N, v) be a simple game. A player $i \in N$ such that $i \in S$ for all winning coalitions S is called a vetoer. Each player $i \in N$ that is not contained in any minimal winning coalition is called a null player. If $\{i\}$ is a winning coalition, we call player i a passer. If $\{i\}$ is the unique minimal winning coalition, then we call player i a dictator.*

Note that a dictator is the strongest form of being both a passer and a vetoer. Obviously, there can be at most one dictator.

Definition 2. *Given a simple game $(N, \mathcal{W})$ its Nakamura number, cf.* [Nakamura, 1979]*, $\nu(N, \mathcal{W})$ is given by the minimum number of winning coalitions whose intersection is empty. If the intersection of all winning coalitions is non-empty we set $\nu(N, \mathcal{W}) = \infty$.*

Lemma 1. *For each simple game $(N, \mathcal{W})$ the Nakamura number $\nu(N, \mathcal{W})$ equals the minimum number of minimal winning coalitions whose intersection is empty.*

Proof. Since each minimal winning coalition is also a winning coalition, the Nakamura number is a lower bound. For the other direction we consider r winning coalitions S_i for $1 \le i \le r$, where $\nu(N, \mathcal{W}) = r$ and $\cap_{1 \le i \le r} S_i = \emptyset$. Now let $T_i \subseteq S_i$ be an arbitrary minimal winning coalition for all $1 \le i \le r$. Clearly, we also have $\cap_{1 \le i \le r} T_i = \emptyset$. □

We easily observe:

Lemma 2. *For each simple game $(N, \mathcal{W})$ we have $\nu(N, \mathcal{W}) = \infty$ if and only if $(N, \mathcal{W})$ contains at least one vetoer.*

Proof. If $\nu(N, \mathcal{W}) = \infty$ then $U := \cap_{S \in \mathcal{W}} \neq \emptyset$, i.e., all players in U are vetoers. If player i is a vetoer, then i is contained in the intersection of all winning coalitions, which then has to be non-empty. □

Since dictatorship is the strongest form of having a veto we conclude:

Corollary 1. *For a simple game $(N, \mathcal{W})$ containing a dictator, we have $\nu(N, \mathcal{W}) = \infty$.*

Lemma 3. *Let $(N, \mathcal{W})$ be a simple game with at least one passer and $|N| \ge 2$ then*

$$\nu(N, \mathcal{W}) = \begin{cases} \infty & \text{if } \mathcal{W} \text{ contains a dictator,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Let i be a passer in $(N, \mathcal{W})$. If i is a dictator, then Corollary 1 applies. Otherwise $(N, \mathcal{W})$ contains another non-null player $j \in N \backslash \{i\}$. Let S be a minimal winning coalition containing j. Since S is a minimal winning coalition, we have $i \notin S$ and the intersection of the winning coalitions $\{i\}$ and S is empty. □

Lemma 4. *For each simple game $(N, \mathcal{W})$ without vetoers we have $2 \le \nu(N, \mathcal{W}) \le n$.*

Proof. Obviously we have $\nu(N, \mathcal{W}) \in \mathbb{N}_{>0} \cup \{\infty\}$. Since $\emptyset$ is a losing coalition, at least two winning coalitions are needed to get an empty intersection. Thus, we have $\nu(N, \mathcal{W}) \ge 2$. For each player $i \in N$ let S_i be a winning coalition without player i, which needs to exist since player i is not a vetoer. With this we have $\cap_{1 \le i \le n} S_i = \emptyset$, so that $\nu(N, \mathcal{W}) \le n$. □

We can easily state an integer linear programming formulation for the determination of $\nu(N,\mathcal{W})$:

Lemma 5. *For each simple game $(N,\mathcal{W})$ without vetoers the corresponding Nakamura number $\nu(N,\mathcal{W})$ is given as the optimal target value of:*

$$
\begin{aligned}
\min r & \\
\sum_{S\in\mathcal{W}} x_S &= r \\
\sum_{S\in\mathcal{W}\,:\,i\in S} x_S &\le r-1 \quad \forall i\in N \\
x_S &\in \{0,1\} \quad \forall S\in\mathcal{W}
\end{aligned}
$$

Corollary 2. *For each simple game $(N,\mathcal{W})$ without vetoers the corresponding Nakamura number $\nu(N,\mathcal{W})$ is given as the optimal target value of:*

$$
\begin{aligned}
\min r & \\
\sum_{S\in\mathcal{W}^m} x_S &= r \\
\sum_{S\in\mathcal{W}^m\,:\,i\in S} x_S &\le r-1 \quad \forall i\in N \\
x_S &\in \{0,1\} \quad \forall S\in\mathcal{W}^m
\end{aligned}
$$

Lemma 6. *Let m be the minimum cardinality of a (minimal) winning coalition*[1] *of a simple game $(N,\mathcal{W})$. With this we have $\nu(N,\mathcal{W})\ge\left\lceil\frac{n}{n-m}\right\rceil$. If $(N,\mathcal{W})$ has no vetoers, then $\nu(N,\mathcal{W})\le m+1$.*

Proof. We set $r=\nu(N,\mathcal{W})$ and choose r winning coalitions $S_1,\dots,S_r$ with empty intersection. Starting with $I_0:=N$, we recursively set $I_i:=I_{i-1}\cap S_i$ for $1\le i\le r$. By induction we prove $|I_i|\ge n-i\cdot(n-m)$ for all $0\le i\le r$. The statement is true for I_0 by definition. For $i\ge 1$ we have $|I_{i-1}|\ge n-(i-1)\cdot(n-m)$. Since $|S_i|\ge m$ we have $|I_{i-1}\cap S_i|\ge|I_{i-1}|-(n-m)\ge n-i\cdot(n-m)$. Thus we have $\nu(N,\mathcal{W})\ge\left\lceil\frac{n}{n-m}\right\rceil$, where we set $\frac{n}{0}=\infty$ and remark that this can happen only, if N is the unique winning coalition, i.e., all players are vetoers. This proves the lower bound.

For the upper bound let S be a minimal winning coalition with minimum cardinality m. Let $T_i=N\backslash\{i\}$ for all $i\in S$. With this, we have $(\cap_{i\in S}T_i)\cap S=\emptyset$, where the T_i are winning coalitions since $(N,\mathcal{W})$ has no vetoers. □

We remark that the lower bound of Lemma 6 is tight for all values of $n,m\in\mathbb{N}_{>0}$, where $n\ge m$. An example is given by the simple game uniquely characterized by the minimal winning coalitions

$$\{1,\dots,m\}\,,\quad\{m+1,\dots,2m\}\,,\quad\dots,\quad\{(j-1)m+1,\dots,jm\}$$

for $j=\left\lceil\frac{n}{n-m}\right\rceil-1$ and $\{n-m+1,\dots,n\}$. The upper bound of Lemma 6 is also tight for all positive integers $n>m$. An example is given by the simple game uniquely characterized by the minimal winning coalitions S and T_i for $i\in S$, as described in the proof of Lemma 6.

We can turn the packing-type proof of Lemma 6 into a constructive heuristic to obtain another upper bound for the Nakamura number of a simple game.

Lemma 7. *Let M be the maximum cardinality of a minimal winning coalition in a simple game $(N,\mathcal{W})$. Then, we have $\nu(N,\mathcal{W})\le\left\lceil\frac{n}{n-M}\right\rceil$.*

Proof. If $M=n$, we obtain the trivial bound $\nu(N,\mathcal{W})\le\infty$ so that we assume $M\le n-1$. As in the proof of Lemma 6 we recursively define $I_i:=I_{i-1}\cap S_i$ for $1\le i\le r$ and set $I_0=N$. In order to construct a winning coalition S_i we determine $U=N\backslash\{I_{i-1}\}$ and choose a $\max(0,M-|U|)$-element subset V of I_{i-1}. With this we set $S_i=U\cup V$. If $|S_i|>M$, we remove some arbitrary elements so that $|S_i|=M$, i.e. all coalitions S_i have cardinality exactly M and thus are winning for all $i\ge 1$. By induction we prove $|I_i|\ge\max(0,n-i\cdot(n-M))$, so that the stated upper bound follows. □

[1] A winning coalition of minimum cardinality is clearly a minimal winning coalition.

We remark that Lemma 7 is tight for e.g. $m = M$, where $\nu(N, \mathcal{W}) = \left\lceil \frac{n}{n-M} \right\rceil$ due to Lemma 6. By slightly modifying the proof strategy we obtain a greedy type heuristic. Instead of choosing V arbitrarily completing S_i to cardinality M, we may choose a minimum cardinality V such that $S_i = U \cup V$ becomes a winning coalition. Especially we may start with a minimal winning coalition of minimum size. Note that $\left\lceil \frac{n}{n-M} \right\rceil \le m + 1$, i.e., the upper bound in Lemma 7 is more accurate than the upper bound in Lemma 6.

Corollary 3. *For a given simple game $(N, \mathcal{W})$ let m be the cardinality of its smallest minimal winning coalition and M be the cardinality of its largest minimal winning coalition. Assume $M \le n - 1$. With this we have*

$$\nu(N, \mathcal{W}) \le 1 + \left\lceil \frac{m}{n - M} \right\rceil$$

Let S be a winning coalition of a simple game (N, v). Then, $(S, v_{|S})$, where $v_{|S}(T) = v(T)$ for all $T \subseteq S$ is a simple game. Removing null players from a simple game does not change its Nakamura number.

Lemma 8. *Let (N, v) be a simple game and $D \subseteq N$ be its set of null players, then we have $\nu(N, v) = \nu(N \backslash D, v')$, where $v' = v_{|N \backslash D}$.*

Proof. First we note that $N \backslash D$ is a winning coalition, so that $(N \backslash D, v')$ is a simple game. If player i is a vetoer in (N, v) then it remains a vetoer in $(N \backslash D, v')$. So let $r = \nu(N, v)$ and $S_1, \ldots, S_r$ be winning coalitions of (N, v) with empty intersection. We set $T_i = S_i \cap (N \backslash D)$ for all $1 \le i \le r$. In $(N \backslash D, v')$ the T_i are winning coalitions with empty intersection so that $\nu(N, v) \ge \nu(N \backslash D, v')$. For the other direction let T_i be winning coalitions of $(N \backslash D, v')$ with empty intersection. In (N, v) the coalitions T_i remain winning coalitions with empty intersection. □

Lemma 9. *Let $(N, \mathcal{W})$ be a simple game without vetoers and d be the number of its null players. We have $\nu(N, \mathcal{W}) \le \min\left(|\mathcal{W}^m|, n - d\right)$.*

Proof. Since $\nu(N, \mathcal{W}) < \infty$ we have $\nu(N, \mathcal{W}) \le n$. Using Lemma 8 we obtain $\nu(N, \mathcal{W}) \le n - d$. Due to $\nu(N, \mathcal{W}) < \infty$ and Lemma 1 the intersection of all minimal winning coalitions is empty, so that we have $\nu(N, \mathcal{W}) \le |\mathcal{W}^m|$. □

We remark that the proposed bound is tight: For each integer $2 \le k \le n$ we consider the weighted game $[k - 1; 1, \ldots, 1, 0, \ldots, 0]$, see Definition 4 in Section 3, with k players of weight 1. It has $n - k$ null players, k minimal winning coalitions and a Nakamura number of k. We further remark that $|\mathcal{W}^m| = 1$ implies that all players of the unique minimal winning coalition are veto players, i.e., the Nakamura number is infinite.

Lemma 10. *For two simple games $(N, \mathcal{W}_1)$ and $(N, \mathcal{W}_2$, with $\mathcal{W}_1 \subseteq \mathcal{W}_2$ we have $\nu(N, \mathcal{W}_1) \ge \nu(N, \mathcal{W}_2)$.*

Proof. W.l.o.g. we assume $r := \nu(N, \mathcal{W}_1) < \infty$ and choose r winning coalitions S_i of $(N, \mathcal{W}_1)$ with empty intersection. Since $\mathcal{W}_1 \subseteq \mathcal{W}_2$ the S_i are winning in $(N, \mathcal{W}_2)$ too and we have $\nu(N, \mathcal{W}_2) \le r$. □

Remark 1. *For a simple game $(N, \mathcal{W})$ let m be the cardinality of its smallest minimal winning coalition and M be the cardinality of its largest minimal winning coalition. By D we denote the set of null players of N, where we assume that only the players with the largest indices are null players, if any at all. All coalitions of cardinality less than m in $N \backslash D$ are losing and all coalitions of cardinality at least M in $N \backslash D$ are winning so that we have*

$$\mathcal{W}_1 \subseteq \mathcal{W} \subseteq \mathcal{W}_2,$$

where $(N, \mathcal{W}_1) = [m; 1, \ldots, 1, 0, \ldots, 0]$ and $(N, \mathcal{W}_2) = [M; 1, \ldots, 1, 0, \ldots, 0]$ with $|D|$ players of weight 1 each. I.e., we can deduce Lemma 6 and Lemma 7 from Lemma 10 and Equation (1) in Section 3 about weighted games.

Lemma 11. *Let $(N, \mathcal{W})$ be a simple game without veto players and $S_1, \ldots, S_k$ be winning coalitions with $\left|\cap_{i=1}^k S_i\right| = t$, then we have $\nu(N, \mathcal{W}) \le t + k$.*

Proof. Since the game does not contain vetoers we can complement the list of winning coalitions $S_1, \ldots, S_k$ by the winning coalitions $N \backslash \{j\}$ for all $j \in \cap_{i=1}^k S_i$. □

Definition 3. *A simple game $(N, \mathcal{W})$ is called proper if the complement $N \backslash S$ of any winning coalition $S \in \mathcal{W}$ is losing. It is called strong if the complement $N \backslash T$ of any losing coalition T is winning. A simple game that is both proper and strong is called constant-sum (or self-dual or decisive).*

Lemma 12. [Kumabe and Mihara, 2008, Lemma 6] *Let $(N, \mathcal{W})$ be a simple game without vetoers. We have $\nu(N, \mathcal{W}) = 2$ if and only if $(N, \mathcal{W})$ is non-proper.*

Proof. Assume that $(N, \mathcal{W})$ is non-proper, i.e., there exist two winning coalitions S and $N\backslash S$. Thus we have $\nu(N, \mathcal{W}) = 2$ due to $S \cap N\backslash S = \emptyset$ and $\nu(N, \mathcal{W}) \geq 2$. If otherwise $\nu(N, \mathcal{W}) = 2$, then there exist two winning coalitions S and T with $S \cap T = \emptyset$. Since $T \subseteq N\backslash S$ also $N\backslash S$ is a winning coalition so that the game is non-proper. □

[Kumabe and Mihara, 2008, Lemma 7] states that the Nakamura number of each strong simple game without vetoers is either 2 or 3. As the combination of both results we obtain:

Lemma 13. *For each constant-sum simple game $(N, \mathcal{W})$ without veto players we have $\nu(N, \mathcal{W}) = 3$.*

Proof. Since the game is proper we have $\nu(N, \mathcal{W}) \geq 3$. Now let S be an arbitrary minimal winning coalition. Since the game is strong, $N\backslash S$ is a maximal losing coalition. Thus $R := N\backslash S \cup \{i\}$, where $i \in S$ is arbitrary, is a winning coalition. Since player i is not a veto player $N\backslash\{i\}$ is a winning coalition. We conclude the proof by noting that the intersection of the winning coalitions S, R, and $N\backslash\{i\}$ is empty. □

From lemmas 6 and 7 of [Kumabe and Mihara, 2008] we conclude that $\nu(N, \mathcal{W}) > 3$ implies that $(N, \mathcal{W})$ is proper and non-strong.

3. Bounds for weighted, α-weighted, and complete simple games

Definition 4. *A simple game (N, v) is weighted if there exists a quota $q > 0$ and weighs $w_i \geq 0$ for all $1 \leq i \leq n$ such that $v(S) = 1$ if and only if $w(S) = \sum_{i \in S} w_i \geq q$ for all $S \subseteq N$.*

As notation we use $[q; w_1, \ldots, w_n]$ for a weighted game. We remark that the weighted representations are far from being unique. In any case there exist some special weighted representations. By $[\hat{q}; \hat{w}_1, \ldots, \hat{w}_n]$ we denote a weighted representation, where all weights and the quota are integers. Instead of specializing to integers we can also normalize the weights to sum to one. By $[q'; w'_1, \ldots, w'_n]$ we denote a weighted representation with $q' \in (0, 1]$ and $w'(N) = 1$. For the existence of a normalized representation we remark that not all weights can be equal to zero, since $\emptyset$ is a losing coalition.

A special class of simple games are so-called symmetric games, where all players have equivalent capabilities. All these games are weighted and can be parametrized as $[\hat{q}; 1, \ldots, 1]$, where $\hat{q} \in \{1, 2, \ldots, n\}$. The Nakamura number for these games is well known, see e.g. [Ferejohn and Grether, 1974, Nakamura, 1979, Peleg, 1978]:

$$\nu([\hat{q}, 1, \ldots, 1]) = \left\lceil \frac{n}{n - \hat{q}} \right\rceil, \tag{1}$$

where we set $\frac{n}{0} = \infty$.

For the connection between the properties proper, strong or constant-sum and the quota of a weighted voting game we refer the interested reader to e.g. [Kurz, 2014]:

(1) A weighted game is proper if and only if it admits a normalized weighted representation with $q' \in (\frac{1}{2}, 1]$.
(2) A weighted game is strong if and only if it admits a normalized weighted representation with $q' \in (0, \frac{1}{2}]$.
(3) A weighted game is constant-sum if and only if there exists a $\varepsilon > 0$ such that for all $q' \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$, there exists a normalized weighted representation with quota q'.

Lemma 14. *If $\nu(N, \mathcal{W}) = n$ for a simple game, then $(N, \mathcal{W}) = [n - 1; 1, \ldots, 1]$.*

Proof. Assume that $(N, \mathcal{W})$ contains a minimal winning coalition S of cardinality at most $n - 2$. Then, we can apply Corollary 3 and deduce $\nu(N, \mathcal{W}) \leq 1 + |S| \leq n - 1$. Thus, all minimal winning coalitions have cardinality at least $n - 1$. If N is a minimal winning coalition, then all all players would be vetoers and $\nu(N, \mathcal{W}) = \infty$. Thus all minimal winning coalitions have cardinality exactly $n - 1$. If $N\backslash\{i\}$ is not a minimal winning coalition for an arbitrary player $i \in N$, then player i would be a vetoer, which is impossible for $\nu(N, \mathcal{W}) = n < \infty$. Thus all n coalitions of the form $N\backslash\{i\}$ are minimal winning coalitions and the statement follows. □

Each simple game $(N, \mathcal{W})$ can be written as the intersection of a finite minimum number r, called its dimension, of weighted voting games. Directly from Lemma 10 we conclude[2]:

Corollary 4. *Let $(N, \mathcal{W}_i)$ be simple games for $1 \leq i \leq r$ and $(N, \mathcal{W})$ be the simple game arising as the intersection of the $(N, \mathcal{W}_i)$, i.e., $\mathcal{W} = \cap_{1 \leq i \leq r} \mathcal{W}_i$. Then we have $\nu(N, \mathcal{W}) \geq \nu(N, \mathcal{W}_i)$ for all $1 \leq i \leq r$.*

[2] Since the intersection of simple games with the same grand coalition is a simple game too, we can state the result slightly more general.

Similarly each simple game can be written as a union of a finite number r of weighted voting games. The smallest possible value r is called co-dimension of the game by some authors, see e.g. [Freixas and Marciniak, 2009]. Again the union of simple game with the same grand coalition is a simple game.

Corollary 5. *Let $(N,\mathcal{W}_i)$ be simple games for $1\le i\le r$ and $(N,\mathcal{W})$ be the simple game arising as the union of the $(N,\mathcal{W}_i)$, i.e., $\mathcal{W}=\cup_{1\le i\le r}\mathcal{W}_i$. Then we have $\nu(N,\mathcal{W})\le\nu(N,\mathcal{W}_i)$ for all $1\le i\le r$.*

For weighted games we can improve on the lower bound of Lemma 6:

Lemma 15. *For each weighted game we have $\nu([q;w_1,\dots,w_n],N)\ge\left\lceil\frac{w(N)}{w(N)-q}\right\rceil$.*

Proof. Similarly as in the proof of Lemma 6 we set $r=\nu(N,\mathcal{W})$ and choose r winning coalitions $S_1,\dots,S_r$ with empty intersection. With $I_0:=N$ we recursively set $I_i:=I_{i-1}\cap S_i$ for $1\le i\le r$. By induction we prove $w(I_i)\ge w(N)-i\cdot(w(N)-q)$ for all $0\le i\le r$. The statement is true for I_0 by definition. For $i\ge 1$ we have $w(I_{i-1})\ge w(N)-(i-1)\cdot(w(N)-q)$. Since $w(S_i)\ge q$ we have $w(I_{i-1}\cap S_i)\ge w(I_{i-1})-(w(N)-q)=w(N)-i\cdot(w(N)-q)$. Thus we have $\nu([q;w_1,\dots,w_n])\ge\left\lceil\frac{w(N)}{w(N)-q}\right\rceil$. □

Corollary 6. $\nu([q';w'_1,\dots,w'_n],N)\ge\left\lceil\frac{1}{1-q'}\right\rceil$

We remark that one can use the freedom in choosing the representation of a weighted game to eventually improve the lower bound obtained by Lemma 15. For the representation $[2;1,1,1]$ we obtain $\nu([2;1,1,1])\ge\left\lceil\frac{3}{3-2}\right\rceil=3$. Since the same game is also represented by $[1+\varepsilon;1,1,1]$ for all $0<\varepsilon\le\frac{1}{2}$, we could also deduce $\nu([2;1,1,1])\ge\left\lceil\frac{3}{3-1-\varepsilon}\right\rceil=2$, which is a worser bound. The tightest possible bound is attained if the relative quota is maximized.

Using the greedy type approach, based on weights instead of cardinality, to construct a list winning coalitions with empty intersection we obtain:

Lemma 16. $\nu([\hat{q};\hat{w}_1,\dots,\hat{w}_n],N)\le\frac{\hat{w}(N)}{\hat{w}(N)-\hat{q}-\hat{w}_1+1}$*, where we assume $\hat{w}_1\ge\hat{w}_i$ for all $1\le i\le n$.*

Proof. Starting with $R_0=\emptyset$ we recursively construct winning coalitions S_i by setting $S_i=R_{i-1}$ and adding players from $N\backslash R_{i-1}$ to S_i until $\hat{w}(S_i)\ge\hat{q}$. By construction we cannot guarantee that S_i is a minimal winning coalition, but that S_i is a winning coalition with $\hat{w}(S_i)\le\hat{q}+\hat{w}_1-1$. With this we set $R_i=R_{i-1}\cap S_i$ and prove $\hat{w}(R_i)\le\max(0,\hat{w}(N)-i\cdot(\hat{w}(N)-\hat{q}-\hat{w}_1+1))$. Due to Lemma 8 we can assume that our weighted game does not contain null players, i.e., we especially have $\hat{w}_i\ge 1$ so that $\hat{w}(R_i)=0$ implies $R_i=\emptyset$. Thus, we obtain the stated upper bound. □

Corollary 7. $\nu([q';w'_1,\dots,w'_n],N)\le\left\lceil\frac{1}{1-q'-\omega}\right\rceil$*, where $w'_i\le\omega$ for all $1\le i\le n$.*

We remark for the special case of $\hat{w}_i\le 1$, i.e. $\hat{w}_i\in\{0,1\}$, for all $1\le i\le n$, the lower bound of Lemma 15 and the upper bound of Lemma 16 coincide, which is the null player extension of Equation (1) again.

While Lemma 3 characterizes several cases where the Nakamura number is equal to 2, in general it is an NP-hard problem to decide whether this is the case for weighted games.

Lemma 17. *The computational problem to decide whether $\nu([q;w_1,\dots,w_n],N)=2$ is NP-hard.*

Proof. We will provide a reduction to the NP-hard partition problem. So for integers $w_1,\dots,w_n$ we have to decide whether there exists a subset $S\subseteq N$ such that $\sum_{i\in S}w_i=\sum_{i\in N\backslash S}w_i$, where we use the abbreviation $N=\{1,\dots,n\}$. Consider the weighted game $[w(N)/2;w_1,\dots,w_n]$. It has Nakamura number 2 if and only if a subset S with $w(S)=w(N\backslash S)$ exists. □

There exists a relaxation of the notion of a weighted game.

Definition 5. *A simple game $(N,\mathcal{W})$ is α-roughly weighted, where $\alpha\ge 1$, if there exist non-negative weights $w_1,\dots,w_n$ such that each winning coalition S has a weight $w(S)$ of at least 1 and each losing coalition T has a weight of at most α.*

1-roughly weighted games are also called roughly weighted games in the literature.

Lemma 18. *Let $(N,\mathcal{W})$ be a simple game with α-roughly representation $(w_1,\dots,w_n)$. Then we have $\nu(N,\mathcal{W})\ge\left\lceil\frac{w(N)}{w(N)-1}\right\rceil$.*

Proof. We can repeat the argumentation of the proof of Lemma 6 in this situation. □

Note that for $w(N) \geq 2$ Lemma 18 is equivalent to $\nu(N,\mathcal{W}) \geq 2$. For $w(N)=1$ Lemma 18 states $\nu(N,\mathcal{W}) \geq \infty$, i.e., this can happen if and only if there exists a dictator in $(N,\mathcal{W})$. So, the only interesting case, for which $\left\lceil\frac{w(N)}{w(N)-1}\right\rceil \geq 3$, is achieved for $1 < w(N) < 2$. Assuming $\alpha < w(N)$, which is always possible by eventually decreasing α to the maximum weight of a losing coalition, we have $\frac{w(N)}{w(N)-1} < \frac{\alpha}{\alpha-1}$.

Lemma 19. *Let $(N,\mathcal{W})$ be a simple game with α-roughly representation $(w_1,\dots,w_n)$ and $\omega = \max\{w_i \mid i \in N\}$. If $\alpha+\omega > w(N)$, then then we have $\nu(N,\mathcal{W}) \leq \left\lceil\frac{w(N)}{w(N)-\alpha-\omega}\right\rceil$.*

Proof. We can proceed similarly as in the proof of Lemma 16. In order to construct winning coalitions S_i with empty intersection we set $S_i = R_{i-1}$ and add players from $N\backslash R_{i-1}$ until S_i becomes a winning coalition. We remark $w(S_i) \leq \alpha+\omega$ so that we can conclude the proposed statement. □

Of course an α-roughly weighted game is α'-roughly weighted for all $\alpha' \geq \alpha$. The minimum possible value of α, such that a given simple game, is α-roughly weighted is called critical threshold value in [Freixas and Kurz, 2014a]. Taking the critical threshold value gives the tightest upper bound.

The proofs of Lemma 15, Lemma 16 and the corresponding greedy-type heuristic suggest that weighted representations where all minimal winning coalitions have the same weight, equaling the quota, might have a good chance to meet the lower bound from Lemma 15. Those representations are called homogeneous representations and the corresponding games are called, whenever such a representation exists, homogeneous games.

Example 1. *The weighted voting game $(N,\mathcal{W}) = [90; 9^{10}, 2^4, 1^2]$ is homogeneous since all minimal winning coalitions have weight 90. The lower bound of Lemma 15 gives $\nu(N,\mathcal{W}) \geq \left\lceil\frac{100}{100-90}\right\rceil = 10$. In order to determine the exact Nakamura number of this game we study its minimal winning coalitions. To this end let S be a minimal winning coalition. If S contains a player of weight 2, then it has to contain all players of weight 2, one player of weight 1, and nine players of weight 9. If S contains a player of weight 1, then the other player of weight 1 is not contained and S has to contain all players of weight 2 and nine players of weight 9. If S contains neither a player of weight 1 not a player of weight 2, then S consists of all players of weight 9. Now we are ready to prove that the Nakamura number of $(N,\mathcal{W})$ equals 11. Let $S_1,\dots,S_r$ be a minimal collection of minimal winning coalitions whose intersection is empty. Clearly all coalitions are pairwise different. Since there has to be a coalition where not all players of weight 2 are present, say S_1, one coalition has to consist of all players of weight 9. Since each minimal winning coalition contains at least nine players of weight 9, we need 10 coalitions, where each of the players of weight 9 is missing once. Thus $\nu(N,\mathcal{W}) \geq 11$ and indeed one can easily state a collection of 11 minimal winning coalitions with empty intersection.*

The previous example can be generalized by choosing an integer $k \geq 3$ and considering the weighted game $[k(k+1); k^{k+1}, 2^l, 1^{k+1-2l}]$, where $1 \leq l \leq \lfloor k/2 \rfloor$ is arbitrary. The lower bound from Lemma 15 gives $\nu(N,\mathcal{W}) \geq k+1$, while $\nu(N,\mathcal{W}) = k+2$.

Next we want to study a another subclass of simple games and superclass of weighted games.

Definition 6. *Let (N,v) be a simple game. We write $i \sqsupset j$ (or $j \sqsubset i$) for two agents $i,j \in N$ if we have $v\Big(\{i\} \cup S\backslash\{j\}\Big) \geq v(S)$ for all $\{j\} \subseteq S \subseteq N\backslash\{i\}$ and we abbreviate $i \sqsupset j$, $j \sqsupset i$ by $i\square j$.*

The relation $\square$ partitions the set of players N into equivalence classes $N_1,\dots,N_t$.

Example 2. *For the weighted game $[4; 5, 4, 2, 2, 0]$ we have $N_1 = \{1,2\}$, $N_2 = \{3,4\}$, and $N_3 = \{5\}$.*

Obviously, players having the same weight are contained in the same equivalence class, while the converse is not necessarily true. But there always exists a different weighted representation of the same game such that the players of each equivalence class have the same weight. For Example 2, such a representation is e.g. given by $[2; 2, 2, 1, 1, 0]$.

Definition 7. *A simple game $(N,\mathcal{W})$ is called complete if the binary relation $\sqsupset$ is a total preorder, i.e.,*

(1) *$i \sqsupset i$ for all $i \in N$,*
(2) *$i \sqsupset j$ or $j \sqsupset i$ for all $i,j \in N$, and*
(3) *$i \sqsupset j$, $j \sqsupset h$ implies $i \sqsupset h$ for all $i,j,h \in N$.*

All weighted games are obviously complete since $w_i \geq w_j$ implies $i \sqsupset j$. For the weighted game $[7; 3, 3, 3, 1, 1, 1]$ the minimal winning coalitions are given by $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,2,6\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{1,3,6\}$, $\{2,3,4\}$, $\{2,3,5\}$, and $\{2,3,6\}$. Based on the equivalence classes of players one can state a more compact description.

Definition 8. *Let $(N, \mathcal{W})$ be a simple game with equivalence classes $N_1, \dots, N_t$. A coalition vector is a vector $c = (c_1, \dots, c_t) \in \mathbb{N}_{\geq 0}^t$ with $0 \leq c_i \leq |N_i|$ for all $1 \leq i \leq t$. The coalition vector of a coalition S is given by $(|S \cap N_1|, \dots, |S \cap N_t|)$. A coalition vector is called winning if the corresponding coalitions are winning and losing otherwise. If the corresponding coalitions are minimal winning or maximal losing the coalition vector itself is called minimal winning or maximal losing.*

In our previous example the minimal winning (coalition) vectors are given by $(3, 0)$ and $(2, 1)$, where $N_1 = \{1, 2, 3\}$ and $N_2 = \{4, 5, 6\}$.

Using the concept of coalition vectors the ILP from Corollary 2 can be simplified:

Lemma 20. *Let $(N, \mathcal{W})$ be a simple game without vetoers and $N_1, \dots, N_t$ be its decomposition into equivalence classes. Using the abbreviations $n_j = |N_j|$ for all $1 \leq j \leq t$ and $V \subseteq \mathbb{N}_{\geq 0}^t$ for the set of minimal winning coalition vectors, the Nakamura number of $(N, \mathcal{W})$ is given as the optimal target value of:*

$$\min \sum_{v \in V} x_v$$

$$\begin{array}{rcll} \sum\limits_{v=(v_1,\dots,v_t) \in V} (n_j - v_j) \cdot x_v & \geq & n_j & \forall 1 \leq j \leq t \\ x_v & \in & \mathbb{Z}_{\geq 0} & \forall v \in V \end{array}$$

Proof. At first we show that each collection $S_1, \dots, S_r$ of minimal winning coalitions with empty intersection can be mapped onto a feasible, not necessarily optimal, solution of the above ILP with target value r.

Each minimal winning coalition S_i has a minimal winning coalition vector v_i. We set x_v to the number of times vector v is the corresponding winning coalition vector. So the x_v are non-negative integers and the target value clearly coincides with r. The term

$$|N_j| - |S_i - N_j|$$

counts the number of players of type j which are missing in coalition S_i. Since every player has to be dropped at least once from a winning coalition, we have

$$\sum_{i=1}^{r} n_j - |S_i - N_j| \geq n_j$$

for all $1 \leq j \leq t$. The number on the left hand side is also counted by

$$\sum_{v=(v_1,\dots,v_t) \in V} (n_j - v_j) \cdot x_v,$$

so that all inequalities are satisfied.

For the other direction we chose r vectors $v^1, \dots, v^r \in V$ such that $\sum_{i=1}^{r} v^i = \sum_{v \in V} x_v \cdot v$, i.e. we take x_v copies of vector v for each $v \in V$, where $r = \sum_{v \in V} x_v$. In order to construct corresponding minimal winning coalitions $S_1, \dots, S_r$, we decompose those desired coalitions according to the equivalence classes of players: $S^i = \cup S_j^i$ with $S_j^i \subseteq N_j$ for all $1 \leq j \leq t$.

For an arbitrary by fix index $1 \leq j \leq t$ we start with $R_0 = N_j$ and recursively construct the sets S_j^i as follows: Starting from $i = 1$ we set $S_j^i = N_j \backslash R_{i-1}$ and $R_i = \emptyset$ if $|R_{i-1}| < n_j - v_j^i$. Otherwise we chose a subset $U \subseteq R_{i-1}$ of cardinality $n_j - v_j^i$ and set $S_j^i = N_j \backslash U$ and $R_i = R_{i-1} \backslash U$. For each $1 \leq i \leq r$ we have $N_j \backslash \cap_{1 \leq h \leq i} S_j^i = N_j \backslash R_i$.

By construction, the coalition vector of S_i is component-wise larger or equal to v^i, i.e. the S^i are winning coalitions. Since $\sum_{i=1}^{r} \left(n_j - v_j^i\right) \geq n_j$, we have $R_i = \emptyset$ in all cases, i.e. the intersection of the S_i is empty. □

As an example, we consider the weighted game $[4; 2, 2, 1, 1, 1, 1]$ with equivalence classes $N_1 = \{1, 2\}$, $N_2 = \{3, 4, 5, 6\}$ and minimal winning coalition vectors $(2, 0)$, $(1, 2)$, and $(0, 4)$. The corresponding ILP reads:

$$\begin{array}{rcl} \min x_{(2,0)} + x_{(1,2)} + x_{(0,4)} & & \\ 0 \cdot x_{(2,0)} + 1 \cdot x_{(1,2)} + 2 \cdot x_{(0,4)} & \geq & 2 \\ 4 \cdot x_{(2,0)} + 2 \cdot x_{(1,2)} + 0 \cdot x_{(0,4)} & \geq & 4 \\ x_{(2,0)}, x_{(1,2)}, x_{(0,4)} & \in & \mathbb{Z}_{\geq 0} \end{array}$$

Solutions with the optimal target value of 2 are given by $x_{(2,0)} = 1$, $x_{(1,2)} = 0$, $x_{(0,4)} = 1$ and $x_{(2,0)} = 0$, $x_{(1,2)} = 2$, $x_{(0,4)} = 0$. For the first solution we have $v^1 = (2, 0)$ and $v^2 = (0, 4)$ so that $S_1^1 = \{1, 2\}$, $S_1^2 = \emptyset$,

$S_2^1 = \emptyset$ and $S_2^2 = \{3,4,5,6\}$, where we have always chosen the players with the smallest index. For the second solution we have $v^1 = (1,2)$ and $v^2 = (1,2)$ so that $S_1^1 = \{1\}$, $S_1^2 = \{2\}$, $S_2^1 = \{3,4\}$ and $S_2^2 = \{5,6\}$.

Definition 9. *For two vectors $u, v \in \mathbb{N}_{\ge 0}^t$ we write $u \preceq v$ if $\sum_{j=1}^{i} u_j \le \sum_{j=1}^{i} v_j$ for all $1 \le i \le t$. If neither $u \preceq v$ nor $v \preceq u$, we write $u \bowtie v$. We call a winning coalition vector u shift-minimal winning if all coalition vectors $v \preceq u$, $v \ne u$ ($v \prec u$ for short) are losing. Similarly, we call a losing vector u shift-maximal losing if all coalition vectors $v \succ u$ are winning.*

In our previous example $(2,1)$ is shift-minimal winning and $(3,0)$ is not shift-minimal winning, since one player of type 1 can be shifted to be of type 2 without losing the property of being a winning vector. Complete simple games are uniquely characterized by their count vectors $\tilde{n} = (|N_1|, \ldots, |N_t|)$ and their matrix $\tilde{M}$ of shift-minimal winning vectors. In our example we have $\tilde{n} = (3,3)$, $\tilde{M} = \begin{pmatrix} 2 & 1 \end{pmatrix}$. The corresponding matrix of shift-maximal losing vectors is given by $\tilde{L} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$. By $\tilde{m}_1, \ldots, \tilde{m}_r$ we denote the shift-minimal winning vectors, i.e. the rows of $\tilde{M}$.

The crucial characterization theorem for complete simple games using vectors as coalitions and the partial order $\preceq$ was given in [Carreras and Freixas, 1996]:

Theorem 1.

(a) *Given are a vector*

$$\widetilde{n} = \begin{pmatrix} n_1 & \ldots & n_t \end{pmatrix} \in \mathbb{N}_{>0}^t$$

and a matrix

$$\mathcal{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & \ldots & m_{1,t} \\ m_{2,1} & m_{2,2} & \ldots & m_{2,t} \\ \vdots & \ddots & \ddots & \vdots \\ m_{r,1} & m_{r,2} & \ldots & m_{r,t} \end{pmatrix} = \begin{pmatrix} \widetilde{m}_1 \\ \widetilde{m}_2 \\ \vdots \\ \widetilde{m}_r \end{pmatrix}$$

satisfying the following properties

(i) $0 \le m_{i,j} \le n_j$, $m_{i,j} \in \mathbb{N}$ *for* $1 \le i \le r$, $1 \le j \le t$,

(ii) $\widetilde{m}_i \bowtie \widetilde{m}_j$ *for all* $1 \le i < j \le r$,

(iii) *for each* $1 \le j < t$ *there is at least one row-index* i *such that* $m_{i,j} > 0$, $m_{i,j+1} < n_{j+1}$ *if* $t > 1$ *and* $m_{1,1} > 0$ *if* $t = 1$, *and*

(iv) $\widetilde{m}_i \gtrdot \widetilde{m}_{i+1}$ *for* $1 \le i < r$, *where* $\gtrdot$ *denotes the lexicographical order.*

Then, there exists a complete simple game (N, χ) *associated to* $(\widetilde{n}, \mathcal{M})$.

(b) *Two complete simple games* $(\widetilde{n}_1, \mathcal{M}_1)$ *and* $(\widetilde{n}_2, \mathcal{M}_2)$ *are isomorphic if and only if* $\widetilde{n}_1 = \widetilde{n}_2$ *and* $\mathcal{M}_1 = \mathcal{M}_2$.

Shift-minimal winning coalitions are coalitions whose coalition vector is shift-minimal winning. For shift-minimal winning coalitions an analogue lemma like Lemma 1 for minimal winning coalitions does not exist in general. As an example consider the complete simple game uniquely characterized by $\tilde{n} = (5,5)$ and $\tilde{M} = \begin{pmatrix} 2 & 3 \end{pmatrix}$. Here we need three copies of the coalition vector $(2,3)$ since $2 \cdot (\tilde{n} - (2,3)) = (6,4) \not\ge (5,5) = \tilde{n}$ but $3 \cdot (\tilde{n} - (2,3)) \ge \tilde{n}$. On the other hand the Nakamura number is indeed 2 as one can choose the two minimal winning vectors $(2,3)$ and $(3,2)$, where the later is a shifted version of $(2,3)$.

Definition 10. *Given a vector $u \in \mathbb{N}_{\ge 0}^t$ the vector $v = \sum(u) \in \mathbb{N}_{\ge 0}^t$ is given by $v_i = \sum_{j=1}^{i} u_j$ for all $1 \le i \le t$.*

Directly from Definition 9 and Definition 10 we conclude:

Lemma 21. *Let $v \in \mathbb{N}_{\ge 0}^t$ be a minimal winning vector of a complete simple game $(N, \mathcal{W})$. If $v \preceq u$, then u is also a winning vector and $\sum(v) \le \sum(u)$.*

Lemma 22. *For each complete simple game, uniquely characterized by $\tilde{n}$ and $\tilde{M}$, without vetoers and equivalence classes $N_1, \ldots, N_t$ the corresponding Nakamura number $\nu(N, \mathcal{W})$ is given as the optimal target value of*

$$\min \sum_{i=1}^{r} x_i$$

$$\begin{aligned} \sum_{i=1}^{r} \left(o_j - p_j^i\right) \cdot x_i \quad &\ge \quad o_j \quad \forall 1 \le j \le t \\ x_i \quad &\in \quad \mathbb{Z}_{\ge 0} \quad \forall 1 \le i \le r, \end{aligned}$$

where $o := (o_1, \dots, o_t) = \sum(\tilde{n})$, $p^i := \left(p_1^i, \dots, p_t^i\right) = \sum(\tilde{m}_i)$, *and* $n_j = |N_j|$.

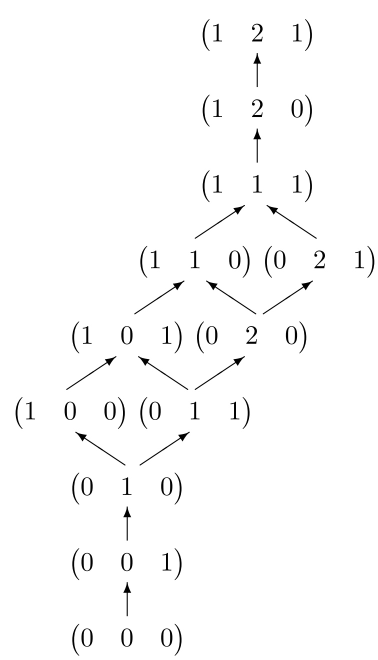
Proof. Consider a list of minimal winning vectors $v^1, \dots, v^r$ corresponding to an optimal solution of Lemma 20. We aim to construct a solution of the present ILP. To this end consider an arbitrary mapping τ from the set of minimal winning vectors into the set of shift-minimal winning vectors, such that $\tau(u) \preceq u$ for all minimal winning vectors u. We choose the x_i's as the number of occurrences of $\tilde{m}_i = \tau(v^j)$ for all $1 \le j \le j$. Thus, the x_i are non-negative numbers, which sum to the Nakamura number of the given complete game. Since $\tau(v^i) \preceq v^i$ we have $\sum(\tau(v^i)) \le \sum(v^i)$ due to Lemma 21. Thus $\sum(\tilde{n}) - \sum(\tau(v^i)) \ge \sum(\tilde{n}) - \sum(v^i)$ so that all inequalities are satisfied.

For the other direction let x_i be a solution of the present ILP. Choosing x_i copies of shift-minimal winning vector $\tilde{m}_i$ we obtain a list of shift-minimal winning vectors $v_0^1, \dots, v_0^r$ satisfying $\sum_{i=1}^r \sum(\tilde{n}) - \sum(v_0^i) \ge \sum(\tilde{n})$. Starting with $j = 1$ we iterate: As long we do not have $\sum_{i=1}^r \tilde{n} - v_j^i \ge \tilde{n}$, we choose an index $1 \le h \le t$ where the hth component of $\sum_{i=1}^r \tilde{n} - v_j^i$ is smaller then $\tilde{n}_h$. Since $\sum_{i=1}^r \sum(\tilde{n}) - \sum(v_j^i) \ge \sum(\tilde{n})$ we have $h \ge 2$ and the $(h-1)$th component of $\sum_{i=1}^r \sum(\tilde{n}) - \sum(v_j^i)$ is at least one larger than the $(h-1)$th component of $\sum(\tilde{n})$. Thus there exists a vector $v_j^{i'}$ where we can shift one player from class h to a class with index lower or equal than $h-1$ to obtain a new minimal winning vector $v_{j+1}^{i'}$. All other vectors remain unchanged. We can easily check, that the new list of minimal winning vectors also satisfies $\sum_{i=1}^r \sum(\tilde{n}) - \sum(v_{j+1}^i) \ge \sum(\tilde{n})$. Since $\sum_{i=1}^r \sum(\tilde{n}) - \sum(v_j^i)$ decreases one unit in a component in each iteration the process must terminate. Thus finally we end up with a list of minimal winning vectors satisfying $\sum_{i=1}^r \tilde{n} - v_j^i \ge \tilde{n}$. $\square$

In Figure 1 we have depicted the Hasse diagram of the shift-relation for coalition vectors for $\tilde{n} = (1, 2, 1)$. If we consider the complete simple game with shift-minimal winning vectors $(1, 0, 1)$ and $(0, 2, 0)$, then for the minimal winning vector $(1, 1, 0)$ we have two possibilities for τ.

As an example we consider the complete simple game uniquely characterized by $\tilde{n} = (10, 10)$ and $\tilde{M} = \begin{pmatrix} 7 & 8 \end{pmatrix}$. An optimal solution of the corresponding ILP is given by $x_1 = 4$. I.e. initially we have $v_0^1 = (7, 8)$, $v_0^2 = (7, 8)$, $v_0^3 = (7, 8)$, and $v_0^4 = (7, 8)$. We have $\sum_{i=1}^r \sum(\tilde{n}) - \sum(v_0^i) = (12, 20) \ge (10, 20) = \sum(\tilde{n})$ and $\sum_{i=1}^r \tilde{n} - v_0^i = (12, 8) \not\ge (10, 10) = \tilde{n}$. Here the second component, with value 8, is too small. Thus the first component must be at least 1 too large, and indeed $12 > 10$. We can shift one player from class 2 to class 1. We may choose $v_1^1 = (8, 7)$, $v_1^2 = (7, 8)$, $v_1^3 = (7, 8)$, and $v_1^4 = (7, 8)$, so that $\sum_{i=1}^r \sum(\tilde{n}) - \sum(v_0^1) = (11, 20) \ge (10, 20) = \sum(\tilde{n})$ and $\sum_{i=1}^r \tilde{n} - v_0^i = (11, 9) \not\ge (10, 10) = \tilde{n}$. Finally we may shift one player in v_1^1 again or in any of the three other vectors to obtain $v_2' = v_2^2 = (7, 8)$ and $v_2^3 = v_2^4 = (8, 7)$.

FIGURE 1. The Hasse diagram of the vectors with counting vector $(1, 2, 1)$.



Lemma 23. *A complete simple game $(N,\mathcal{W})$ uniquely characterized by its count vector $\tilde{n}$ and its matrix $\tilde{M} = (\tilde{m}^1,\dots,\tilde{m}^r)^T$ contains vetoers if and only if $\tilde{m}_1^i = \tilde{n}_1$ for all $1 \le i \le r$.*

The next two lemmas and Lemma 27 concern complete simple games with minimum, i.e., with a unique minimal winning vector in $\tilde{M}$.

Lemma 24. *The Nakamura number of a complete simple game uniquely characterized by $\tilde{n} = (n_1,\dots,n_t)$ and $\tilde{M} = \begin{pmatrix} m_1^1 & \dots & m_t^1 \end{pmatrix}$ is given by*

$$\max_{1\le i\le t} \left\lceil \frac{\sum_{j=1}^i n_j}{\sum_{j=1}^i n_j - m_j^1} \right\rceil.$$

Proof. We utilize the ILP in Lemma 22. In our situation it has only one variable x_1. The minimal integer satisfying the inequality number i is given by $\left\lceil \frac{\sum_{j=1}^i n_j}{\sum_{j=1}^i n_j - m_j^1} \right\rceil$. □

We remark that the above Lemma remains true in the case of vetoers due to $m_1^1 = n_1$ according to Lemma 23. Just knowing the sizes of the equivalence classes of players gives the following upper bound.

Lemma 25. *The Nakamura number of a complete simple game without vetoers uniquely characterized by $\tilde{n} = (n_1,\dots,n_t)$ and $\tilde{M} = \begin{pmatrix} m_1^1 & \dots & m_t^1 \end{pmatrix}$ is upper bounded by*

$$\max_{1\le i\le t} \left\lceil \frac{\sum_{j=1}^i n_j}{i} \right\rceil.$$

Proof. Since the complete simple game has no vetoers we have $m_1^1 \le n_1 - 1$. Due to the type conditions in the parameterization theorem of complete simple games, we have $1 \le m_j^1 \le n_j - 1$ and $n_j \ge 2$ for all $2 \le j \le t-1$. If $t \ge 2$ then we additionally have $0 \le m_t^1 \le n_t - 1$ and $n_t \ge 1$. Thus we have $\sum_{j=1}^i n_j - m_j^1 \ge i$ and conclude the proposed upper bound from Lemma 24. □

Observe that the previous upper bound is tight when $n_i = 2$ for each $1 \le i \le t$ since the only game with such $\tilde{n}$ is improper. For the remaining $\tilde{n}$ vectors both the upper bound and the trivial lower bound 2 are tight.

Lemma 26. *Let $(N,\mathcal{W})$ be a complete simple game with t types of players. If $(n_1-1,\dots,n_t-1)$ is a winning vector, then we have*

$$\nu(N,\mathcal{W}) \le \max_{1\le i\le t} \left\lceil \frac{\sum_{j=1}^i n_j}{i} \right\rceil \le n-t+1$$

Proof. Proceeding as in the proof of Lemma 24 yields the first bound. The second bound follows from

$$\frac{n_1+\dots+n_i}{i} \le \frac{n-t+i}{i} = \frac{n-t}{i} + 1 \le n-t+1.$$

□

Lemma 27. *The Nakamura number of a complete simple game without vetoers uniquely characterized by $\tilde{n} = (n_1,\dots,n_t)$ and $\tilde{M} = \begin{pmatrix} m_1^1 & \dots & m_t^1 \end{pmatrix}$ is upper bounded by* $\max(2, n-2t+3)$.

Proof. By shifting one player from N_i to N_{i-1} the upper bound from Lemma 25 does not decrease. Thus the minimum is attained at $n_t = 1$, and $n_i = 2$ for all $2 \le i \le t-1$. □

The example with the unique shift-minimal winning vector $(n_1-1, 1, \dots, 1, 0)$ shows that the stated bound is sharp.

Using the argument of Lemma 24 as a heuristic, i.e. using just a single shift-minimal winning vector, we obtain:

Lemma 28. *The Nakamura number of a complete simple game uniquely characterized by $\tilde{n} = (n_1,\dots,n_t)$ and $\tilde{M} = (m^1,\dots,m^r)^T$, where $m^i = (m_1^i,\dots,m_t^i)$, is upper bounded by*

$$\max_{1\le i\le t} \left\lceil \frac{\sum_{j=1}^i n_j}{\sum_{j=1}^i n_j - m_j^i} \right\rceil.$$

for all $1 \le i \le r$.

I.e., we may take the minimal upper bound over all $1 \le i \le r$, cf. Corollary 5.

4. Maximum Nakamura numbers within subclasses of simple games

By $\mathcal{S}$ we denote the set of simple games, by $\mathcal{C}$ we denote the set of complete simple games, and by $\mathcal{T}$ we denote the set of weighted games.

Definition 11. $\operatorname{Nak}^{\mathcal{X}}(n,t)$ *is the maximum Nakamura number of a game with* $n \geq 2$ *player and* $t \leq n$ *equivalence classes in* $\mathcal{X}$*, where* $\mathcal{X} \in \{\mathcal{S}, \mathcal{C}, \mathcal{T}\}$*, without vetoers.*

Clearly, we have

$$2 \leq \operatorname{Nak}^{\mathcal{T}}(n,t) \leq \operatorname{Nak}^{\mathcal{C}}(n,t) \leq \operatorname{Nak}^{\mathcal{S}}(n,t) \leq n,$$

if the corresponding set of games is non-empty.

Lemma 29. *For* $n \geq 2$ *we have* $\operatorname{Nak}^{\mathcal{T}}(n,1) = \operatorname{Nak}^{\mathcal{C}}(n,1) = \operatorname{Nak}^{\mathcal{S}}(n,1) = n$.

Proof. Consider the example $[n-1; 1, \dots, 1]$ with n players of weight 1. □

Lemma 30. *For* $n \geq 3$ *we have* $\operatorname{Nak}^{\mathcal{T}}(n,2) = \operatorname{Nak}^{\mathcal{C}}(n,2) = \operatorname{Nak}^{\mathcal{S}}(n,2) = n-1$.

Proof. Consider the example $[n-1; 1, \dots, 1, 0]$ with $n-1$ players of weight 1. Thus $\operatorname{Nak}^{\mathcal{T}}(n,2) \geq n-1$. By using Lemma 14 we conclude $\operatorname{Nak}^{\mathcal{S}}(n,2) \leq n-1$, so that we obtain the stated result. □

We remark that each simple game with two non-equivalent players achieving the maximum Nakamura number contains a vetoer and a null player.

Lemma 31. *For* $n \geq 4$ *we have* $\operatorname{Nak}^{\mathcal{T}}(n,3) = \operatorname{Nak}^{\mathcal{C}}(n,3) = \operatorname{Nak}^{\mathcal{S}}(n,3) = n-1$.

Proof. Consider the example $[5n-2k-9; 5^{n-k-1}, 3^k, 1^1]$, where $k \geq 2$ and $n-k-1 \geq 1$, i.e. $n \geq k+2$ and $n \geq 4$, with $n-k-1$ players of weight 5, k players of weight 3, and one player of weight 1 – this is indeed the minimum integer representation, so that we really have 3 types of players (this may also be checked directly).

Let S be a minimal winning coalition. If a player of weight 5 is missing in S, then all players of weight 3 and the player of weight 1 belong to S. Thus, we need $n-k-1$ such versions in order to get an empty intersection of winning coalitions. If a player of weight 3 is missing, then all of the remaining players of weight 3 and all players of weight 5 have to be present, so that we need k such versions. Thus the game has Nakamura number $n-1$ for all $n \geq 4$ (if k is chosen properly). Using Lemma 14 we conclude $\operatorname{Nak}^{\mathcal{S}}(n,3) \leq n-1$, so that we obtain the stated result.

□

We remark that for $n \leq 3$ there exists no weighted (or simple) game with $t = 3$ types. Similarly for $n \leq 4$ there exists no weighted game with $t = 4$ types. In our example we have freedom to distribute the players almost arbitrarily between the first two types of players, i.e., there are examples where two classes of types of players are *large*, i.e. there is little hope to get too many restrictions on the set of extremal examples.

As a follow-up to Lemma 14 we now want to classify all simple games with Nakamura number $n-1$.

Lemma 32. *Let* $(N, \mathcal{W})$ *be a simple game with* $\nu(N, \mathcal{W}) = n-1$*, then* $(N, \mathcal{W})$ *is of one of the following types:*

(1) $(N, \mathcal{W}) = [2n-4); 2^{n-2}, ^1 2]$, $t = 2$*, for all* $n \geq 3$*;*
(2) $(N, \mathcal{W}) = [1; 1^3]$, $t = 1$*, for all* $n = 3$*;*
(3) $(N, \mathcal{W}) = [2n-5; 2^{n-3}, 1^3]$, $t = 2$*, for all* $n \geq 4$*;*
(4) $(N, \mathcal{W}) = [n-1; 1^{n-1}, 0]$, $t = 2$*, for all* $n \geq 3$*;*
(5) $(N, \mathcal{W}) = [5n-2k-9; 5^{n-k-1}, 3^k, 1^1]$, $t = 3$*, for all* $n \geq 4$ *(*$2 \leq k \leq n-2$*);*

Proof. Since $\nu(N, \mathcal{W}) = n-1$ the game does not contain veto players and all coalitions without one player, i.e. $N \backslash \{i\}$, are winning. Due to Lemma 11 all coalitions missing at least three players have to be losing, since otherwise $\nu(N, \mathcal{W}) \leq n-2$. So, we can describe the game as a graph by taking N as the set of vertices and by taking edge $\{i,j\}$ if and only if $N \backslash \{i,j\}$ is a winning coalition. Again by using Lemma 11 we conclude that each two edges need to have a vertex in common. Thus our graph consists of isolated vertices and either a triangle or a star. To be more precise, we consider the following cases:

- only isolated vertices, which is the case of Lemma 14 and thus excluded;
- a single edge: this does not correspond to a simple game since the empty coalition has to be losing;
- a single edge and at least one isolated vertex: this is case (1);
- a triangle: this is case (2);
- a triangle and at least one isolated vertex: this is case (3);

- a star (with at least three vertices) and no isolated vertex: this is case (4);
- a star (with at least three vertices) and at least one isolated vertex: this is case (5).

□

Lemma 33. *For $n \geq 5$ we have* $\mathrm{Nak}^{\mathcal{T}}(n,4) = \mathrm{Nak}^{\mathcal{C}}(n,4) = \mathrm{Nak}^{\mathcal{S}}(n,4) = n-2$.

Proof. We append a null player to the stated extremal example for three types of players considered in the proof of Lemma 31, i.e., we consider the weighted game $[5n-2k-9; 5^{n-k-2}, 3^k, 1^1, 0^1]$, where $k \geq 2$ and $n-k-2 \geq 1$, which is possible for $n \geq 5$ players. Due to Lemma 8 it has a Nakamura number of $n-2$, so that $\mathrm{Nak}^{\mathcal{T}}(n,4) \geq n-2$. From the classification in Lemma 32 we conclude $\mathrm{Nak}^{\mathcal{S}}(n,4) \leq n-2$, so that the proposed equations follow. □

Conjecture 1. *If n is sufficiently large, then we have $n-t+1 \leq \mathrm{Nak}^{\mathcal{T}}(n,t) \leq n-t+2$, where $t \in \mathbb{N}_{>0}$.*

For simple games we can obtain tighter bounds.

Lemma 34. *For $n \geq t$ and $t \geq 6$ we have* $\mathrm{Nak}^{\mathcal{S}}(n,t) \geq n - \left\lfloor \frac{t-1}{2} \right\rfloor$.

Proof. Consider a simple game with t types of players given by the following list of minimal winning vectors:

$$\begin{aligned}
(n_1-1, n_2, \dots, n_t)\\
(n_1, n_2-1, n_3-1, n_4, \dots, n_t)\\
(n_1, n_2, n_3-1, n_4-1, n_5, \dots, n_t)\\
\vdots\\
(n_1, n_2, \dots, n_{t-2}, n_{t-1}-1, n_t-1)\\
(n_1, n_2-1, n_3, \dots, n_{t-1}, n_t-1),
\end{aligned}$$

i.e., if a player of class 1 is missing, then all other players have to be present in a winning coalition, no two players of the same type can be missing in a winning coalition, and at most two players can be missing in a winning vector, if they come from neighbored classes (where the classes $2, 3, \dots, t$ are arranged on a circle).

At first we check that this game has in fact t types. Obviously class 1 is different from the other ones. Let i, j be two different indices in $\{2, 3, \dots, t\}$. Since the circle has length at least five, on one side there are at least two vertices, say a and b, between i and j. Assume further that a is neighbored to i, but not to j, and b is neighbored to j, but not to i. Then exchanging i and j turns the type of the coalition with two players missing from $\{i, a\}$ and $\{j, b\}$.

With respect to the Nakamura number we remark that we have to choose n_1 coalitions of the form $(n_1 - 1, n_2, \dots, n_t)$. All other coalitions exclude 2 players, so that we need $\left\lceil \frac{n_2+\dots+n_t}{2} \right\rceil$ of these. Taking $n_2 = \dots = n_t = 1$ gives the proposed bound. □

Lemma 35. *Let $k \geq 3$ be an integer. For $2k+1 \leq t \leq k+2^k$ and $n \geq t$ we have* $\mathrm{Nak}^{\mathcal{S}}(n,t) \geq n-k$.

Proof. Let V be an arbitrary k-element subset of N. Let $U_1, \dots, U_{t-k-1}$ be disjoint subsets of V containing all k one-element subsets and the empty subset. For each $1 \leq i \leq t-l-1$ we choose a distinct vertex v_i in $N \backslash V$. We define the game by specifying the set of winning coalitions as follows: The grand coalition and all coalitions $N \backslash \{j\}$ of cardinality $n-1$ are winning. Coalition $N \backslash V$ and all of its supersets are winning. Additionally the following coalitions of cardinality $n-2$ are winning: For all $1 \leq i \leq t-l-1$ and all $u \in U_i$ the coalition $N \backslash \{v, u\}$ is winning.

We can now check that the k players in V are of k different types, where each equivalence class contains exactly one player (this is due to the one element subsets U_i of V). Vertices v_i also form their own equivalence class, consisting of exactly one player - except for the case of $U_i = \emptyset$, here all remaining players are pooled. Thus we have $2k+1 \leq t \leq k+2^k$ types of players.

Suppose we are given a list $S_1, \dots, S_l$ of winning coalitions with empty intersection, then $|N \backslash (S_i \backslash V)| = 1$, i.e. every winning coalition can miss at most one player from $N \backslash V$. Thus the Nakamura number is at least $n-k$. □

5. Further relations for the Nakamura number

As we have already remarked, the lower bound of Lemma 15 can be strengthened if we maximize the quota, i.e. solve

$$\begin{array}{rcll} \max q & & & \\ w(S) & \ge & q & \forall S \in \mathcal{W} \\ w(T) & < & q & \forall T \in \mathcal{L} \\ w(N) & = & 1 & \\ w_i & \ge & 0 & \forall 1 \le i \le n \end{array}$$

Looking at the proof of Lemma 15 again, we observe that our lower bound can eventually by further strengthened if we drop the condition on the losing coalitions, which are not used in the proof of the lower bound. So we consider the linear program

$$\begin{array}{rcll} \max q & & & \\ w(S) & \ge & q & \forall S \in \mathcal{W} \\ w(N) & = & 1 & \\ w_i & \ge & 0 & \forall 1 \le i \le n, \end{array}$$

which has the same set of optimal solutions, except for the target value, as

$$\begin{array}{rcll} \min 1-q & & & \\ w(S) & \ge & q & \forall S \in \mathcal{W} \\ w(N) & = & 1 & \\ w_i & \ge & 0 & \forall 1 \le i \le n, \end{array}$$

Here the optimal value $1-q$ is also called the minimum maximum excess $e^\star$, which arises in the determination of the nucleolus. We remark that, for an arbitrary simple game, $e^\star$ equals 1 if and only if the game contains a vetoer.

Corollary 8. *Let $e^\star$ be the minimum maximum excess of a simple game $(N,\mathcal{W})$, then we have $\nu(N,\mathcal{W}) \ge \left\lceil \frac{1}{e^\star} \right\rceil$.*

Dividing the target function by $q>0$ and replacing $w_i = w_i' q$, which is a monotone transform, we obtain that the set of the optimal solutions of the previous LP is the same as the one of:

$$\begin{array}{rcll} \min \frac{1-q}{q} = \frac{1}{q}-1 & & & \\ w'(S) & \ge & 1 & \forall S \in \mathcal{W} \\ w'(N) & = & \frac{1}{q} & \\ w_i' & \ge & 0 & \forall 1 \le i \le n, \end{array}$$

If we now set $\Delta := \frac{1}{q}-1$ and add $\Delta \ge 0$, we obtain the definition of the price of stability for games where the grand coalition is winning, see e.g. [Bachrach et al., 2009].

Corollary 9. *Let Δ be the price of stability of a simple game $(N,\mathcal{W})$, then we have $\nu(N,\mathcal{W}) \ge \left\lceil \frac{1+\Delta}{\Delta} \right\rceil$.*

We remark that we have $\Delta = \frac{e^*}{1-e^\star} = 0$ if and only if the game contains a vetoer. So one can say, that the Nakamura number is large if the price of stability is low.

It seems that Corollary 8 is the tightest and most applicable lower bound that we have at hand for the Nakamura number of a simple game. An interesting question is to study under what conditions it attains the exact value.

Here we want to focus on the easier subcase of weighted games, where the lower bound of Lemma 15 and Corollary 8 coincides. This happens e.g. for homogeneous representations. It is well known that one can homogenize each weighted game with integer weights by adding a sufficiently large number of players of weight 1.

Lemma 36. *Let $w_1 \ge \cdots \ge w_n \ge 2$ be integer weights with sum $\Omega = \sum_{i=1}^n w_i$ and $\overline{q} \in (0,1)$ be a rational number. For each positive integer r we consider the game*

$$\chi = \left[\overline{q} \cdot (\Omega + r); w_1, \dots, w_n, 1^r\right],$$

with r players of weight 1. If $r \ge \max\left(\Omega, \frac{2+w_1}{1-\overline{q}}\right)$ we have $\nu(\chi, N) = \left\lceil \frac{1}{1-q^r} \right\rceil$, where $q^r = \frac{\lceil \overline{q}(\Omega+r) \rceil}{\Omega+r}$.

Proof. At first we remark that $\chi = [\lceil \overline{q} \cdot (\Omega + r) \rceil ; w_1, \ldots, w_n, 1^r]$, so that the proposed exact value coincides with the lower bound from Lemma 15. Next we observe

$$q^r = \frac{\lceil \overline{q}(\Omega + r) \rceil}{\Omega + r} \le \frac{1 + \overline{q}(\Omega + r)}{\Omega + r} = \overline{q} + \frac{1}{\Omega + r} \le \overline{q} + \frac{1}{r}.$$

Consider the following greedy way of constructing the list $S_1, \ldots, S_k$ of winning coalitions with empty intersection. Starting with $i = 1$ and $h = 1$ we choose an index $h \le g \le n$ such that $U_i = \{h, h+1, \ldots, g\}$ has a weight of a most $(1 - q^r)(\Omega + r)$ and either $g = n$ or $U_i \cup \{g + 1\}$ has a weight larger then $(1 - q^r)(\Omega + r)$. Given U_i we set $S_i = \{1, \ldots, n + r\} \backslash U_i$, $h = g + 1$, and increase i by one. If $(1 - q^r)(\Omega + r) \ge w_i$ for all $1 \le i \le n$, then no player in $\{1, \ldots, n\}$ has a too large weight to be dropped in this manner. Since we assume the weights to be ordered, it suffices to check the proposed inequality for w_1. To this end we consider

$$(1 - q^r)(\Omega + r) \ge \left(1 - \overline{q} - \frac{1}{r}\right) \cdot (\Omega + r) = (1 - \overline{q})\Omega - 1 - \frac{\Omega}{r} + (1 - \overline{q})r \ge (1 - \overline{q})r - 2,$$

where we have used $r \ge \Omega$. Since $r \ge \frac{2+w_1}{1-\overline{q}} \ge \frac{2+w_i}{1-\overline{q}}$ the requested inequality is satisfied.

So far the winning coalitions S_i can have weights larger then $q^r(\Omega + r)$ and their intersection is given by the players of weight 1, i.e. by $\{n + 1, \ldots, n + r\}$. For all $1 \le i < k$ let h_i be the player with the smallest index in U_i, which is indeed one of the heaviest players in this subset. With this we conclude $w(S_i) \le q^r(\Omega+r) + w_{h_i} - 1$ since otherwise another player from U_{i+1} could have been added. In order to lower the weights of the S_i to $q^r(\Omega + r)$ we remove $w(S_i) - (q^r(\Omega + r)))$ players of S_i for all $1 \le i \le k$, starting from player $n + 1$ and removing each player exactly once. Since $\sum_{i=1}^{k-1} w_{h_i} \le \Omega \le r$ this is indeed possible. Now we remove the remaining, if any, players of weight 1 from S_k until they reach weight $q^r(\Omega + r)$ and eventually start new coalitions $S_i = \{1, \ldots, n + r\}$ removing players of weight 1. Finally we end up with $r + l$ winning coalitions with empty intersection, where the coalitions $1 \le i \le k + l - 1$ have weight exactly $q^r(\Omega + r)$ and the sets $\{1, \ldots, n + r\} \backslash S_i$ do contain only players of weight 1 for $i \ge r + 1$. Since each player is dropped exactly once the Nakamura number of the game equals $k + l = \left\lceil \frac{1}{1-q^r} \right\rceil$. □

Other possibilities are to consider replicas, i.e., each of the initial players is divided into k equal players all having the initial weight, where we assume a relative quota as in the previous lemma. If no players of weight 1 are present, then the game eventually does not become homogeneous, even if the replication factor k is large. But indeed the authors of [Kurz et al., 2014] have recently shown that for the case of a suitably large replication factor k the nucleolus coincides with the relative weights of the players, i.e., the lower bound of Corollary 8 and Lemma 15 coincide. Here we show that for sufficiently large replication factors k the lower bound of Lemma 15 is attained with equality.

Lemma 37. *Let $w_1 \ge \cdots \ge w_n \ge 1$ be (not necessarily pairwise) coprime integer weights with sum $\Omega = \sum_{i=1}^{n} w_i$ and $\overline{q} \in (0, 1)$ be a rational number. For each positive integer r we consider the game*

$$\chi = [\overline{q} \cdot (\Omega \cdot r); w_1^r, \ldots, w_n^r],$$

where each player is replicated r times. If r is sufficiently large, we have $\nu(\chi, N) = \left\lceil \frac{1}{1-q^r} \right\rceil$, where $q^r = \frac{\lceil \overline{q}(\Omega \cdot r) \rceil}{\Omega \cdot r}$.

Proof. We write $\overline{q} = \frac{p}{q}$ with positive comprime integers p, q. If $p \neq q - 1$, then

$$\left\lceil \frac{1}{1 - \overline{q}} \right\rceil \left\lceil \frac{q}{q - p} \right\rceil > \frac{1}{1 - \overline{q}},$$

i.e., we always round up. Obviously $\lim_{r\to\infty} q^r = \overline{q}$ (and $q^r \ge \overline{q}$). Since also

$$\lim_{r\to\infty} \frac{w(N^r)}{w(N^r) - q^r w(N^r) - w_1 + 1} = \lim_{r\to\infty} \frac{w(N^r)}{w(N^r) - q^r w(N^r)} = \frac{1}{1 - \overline{q}},$$

we can apply the upper bound of Lemma 16 to deduce that the lower bound is attained with equality for sufficiently large replication factors r.

In the remaining part we assume $p = q - 1$, i.e., $1 - \overline{q} = \frac{1}{q}$. If $\Omega \cdot r$ is not divisible by q, i.e. $q^r > \overline{q}$, we can apply a similar argument as before, so that we restrict ourselves to the case $q | \Omega \cdot r$, i.e. $\overline{q} = q^r$. Here we have to show that the Nakamura number exactly equals q (in the previous case it equals $q + 1$). This is possible if we can partition the grand coalition N into q subsets $U_1, \ldots, U_q$ all having a weight of exactly $\frac{\Omega \cdot r}{q}$. (The list of winning coalitions with empty intersection is then given by $S_i = N \backslash U_i$ for $1 \le i \le q$.) This boils down to a purely number theoretic question, which is solved in the subsequent lemma. □

Lemma 38. *Let $w_1,\dots,w_n$ be positive integers with $\sum\limits_{i=1}^{n} w_i=\Omega$ and greatest common divisor* $\gcd(w_1,\dots,w_n)=1$. *Let further an integer $q\ge 2$ be given. There exists an integer K such that for all $k\ge K$, where $\frac{k\cdot\Omega}{q}\in\mathbb{N}$ there exist non-negative integers u_j^i with*

$$\sum_{j=1}^{n} u_j^i\cdot w_j=\frac{k\cdot\Omega}{q},$$

for all $1\le i\le q$, and

$$\sum_{i=1}^{q} u_j^i=k,$$

for all $1\le j\le n$.

Proof. For $k=1$, setting $u_j^i=\frac{1}{q}$ is an inner point of the polyhedron

$$P=\left\{u_j^i\in\mathbb{R}_{\ge 0}\mid \sum_{j=1}^{n} u_j^i\cdot w_j=\frac{\Omega}{q}\;\forall 1\le i\le q \text{ and } \sum_{i=1}^{q} u_j^i=1\;\forall 1\le j\le n\right\},$$

so that is has non-zero volume.

For general $k\in\mathbb{N}_{>0}$ we are looking for lattice points in the dilation $k\cdot P$. If q is a divisor of $k\cdot\Omega$, then $\mathbb{Z}^{nq}\cap k\cdot P$ is a lattice of maximal rank in the affine space spanned by $k\cdot P$. Let k_0 the minimal positive integer such that q divides $k_0\cdot\Omega$. Using Erhart theory one can count the number of lattice points in the parametric rational polytope in $m\cdot k_0\cdot P$, where $m\in\mathbb{N}_{>0}$, see e.g. [Beck and Robins, 2007]. To be more precise, the number of (integer) lattice points in $m\cdot k_0\cdot P$ grows asymptotically as $m^d\operatorname{vol}_d(k_0P)$, where d is the dimension of the affine space A spanned by $k_0\cdot P$ and $\operatorname{vol}_d(k_0P)$ is the (normalized) volume of $k_0\cdot P$ within A. Due to the existence of an inner point we have $\operatorname{vol}_d(k_0P)>0$, so that the number of integer solutions is at least 1 for $m\gg 0$. □

5.1. **The Nakamura number and the one-dimensional cutting stock problem.** There is a relation between the problem of Lemma 38 and the Frobenius number, which asks for the largest integer which can not be expressed as a non-negative integer linear combination of the w_i, and which recently occurs in the context on minimum sum integer representations, see [Freixas and Kurz, 2014b]. According to the Frobenius theorem every sufficiently large number can be expressed as such a sum. Here we ask for several such representations which are balanced, i.e., each *coin* is taken equally often.

Finally we would like to mention another relation between the Nakamura number of a weighted game and a famous optimization problem – the one-dimensional cutting stock problem. Here, one-dimensional objects like e.g. paper reels or wooden rods, all having length $L\in\mathbb{R}_{>0}$ should be cut into pieces of lengths $l_1,\dots,l_m$ in order to satisfy the corresponding order demands $b_1,\dots,b_m\in\mathbb{Z}_{>0}$. The minimization of waste is the famous 1CSP. By possible duplicating some lengths l_i, we can assume $b_i=1$ for all $1\le i\le m$, while this transformation can increases the value of m. Using the abbreviations $l=(l_1,\dots,l_m)^T$ we denote an instance of 1CSP by $E=(m,L,l)$. The classical ILP formulation for the cutting stock problem by Gilmore and Gomory is based on so-called cutting patterns, see [Gilmore and Gomory, 1961]. We call a pattern $a\in\{0,1\}^m$ feasible (for E) if $l^\top a\le L$. By $P(E)$ we denote the set of all patterns that are feasible for E. Given a set of patterns $P=\{a^1,\dots,a^r\}$ (of E), let $A(P)$ denote the concatenation of the pattern vectors a^i. With this we can define

$$z_B(P,m):=\sum_{i=1}^{r} x_i\to\min\quad\text{subject to}\quad A(P)x=\mathbf{1},\;x\in\{0,1\}^r\quad\text{and}$$

$$z_C(P,m):=\sum_{i=1}^{r} x_i\to\min\quad\text{subject to}\quad A(P)x=\mathbf{1},\;x\in[0,1]^r.$$

Choosing $P=P(E)$ we obtain the mentioned ILP formulation for 1CSP of [Gilmore and Gomory, 1961] and its continuous relaxation. Obviously we have $z_B(P(E),m)\ge\lceil z_C(P(E),m)\rceil$. In cases of equality one speaks of an IRUP (integer round-up property) instance – a concept introduced for general linear minimization problems in [Baum and Trotter, 1981]. In practice almost all instances have the IRUP. Indeed, the authors of [Scheithauer and Terno, 1995] have conjectured that $z_B(P(E),m)\ge\lceil z_C(P(E),m)\rceil+1$ – called the MIRUP property, which is one of the most important theoretical issues about 1CSP, see also [Eisenbrand et al., 2013].

Lemma 39. *Let $(N,\mathcal{W})$ be a strong simple game on n players, then $\nu(N,\mathcal{W})\le z_B(\overline{\mathcal{L}},n)$, where $\overline{\mathcal{L}}$ denotes the incidence vectors corresponding to the losing coalitions $\mathcal{L}=2^N\backslash\mathcal{W}\subseteq 2^N$.*

Proof. The value $z_B(\overline{\mathcal{L}}, n)$ corresponds to the minimal number of losing coalitions that partition the set N, which is the same as the minimum number of (maximal) losing coalitions that cover the grand coalition N. Let $L_1, \dots, L_r$ denote a list of losing coalitions of minimum size. Since $(N, \mathcal{W})$ is strong the coalitions $N \backslash L_1, \dots, N \backslash L_r$ are winning and have an empty intersection, so that $\nu(N, \mathcal{W}) \le z_B(\overline{\mathcal{L}}, n)$. □

There is a strong relation between the 1CSP instances and weighted games, see [Kartak et al., 2015]. For each weighted games there exists an 1CSP instance where the feasible patterns correspond to the losing coalitions. For the other direction the feasible patterns of a 1CSP instance correspond to the losing coalitions of a weighted game if the all-one vector is non-feasible.

6. Enumeration results

In order to get a first idea of the distribution of the distribution of the attained Nakamura numbers we consider the class of complete simple games with a unique shift-minimal winning coalition, see Table 1 and Table 2, as well as their subclass of weighted games, see Table 3.

n	∞	2	3	4	5	6	7	8	9
1	1								
2	2	1							
3	4	2	1						
4	8	5	1	1					
5	16	9	4	1	1				
6	32	19	8	2	1	1			
7	64	34	18	7	2	1	1		
8	128	69	36	14	4	2	1	1	
9	256	125	86	24	12	4	2	1	1
10	512	251	160	60	24	8	4	2	1
11	1024	461	362	120	43	21	8	4	2
12	2048	923	724	240	86	42	16	8	4
13	4096	1715	1525	513	194	78	38	16	8
14	8192	3431	3050	1026	388	156	76	32	16
15	16384	6434	6529	2052	776	312	145	71	32
16	32768	12869	12785	4377	1517	659	290	142	64
17	65536	24309	27000	8614	3174	1318	580	276	136
18	131072	48619	54000	17228	6348	2636	1160	552	272
19	262144	92377	111434	35884	12696	5221	2371	1104	535
20	524288	184755	222868	71768	25392	10442	4742	2208	1070
21	1048576	352715	462532	142567	51468	21169	9484	4416	2140
22	2097152	705431	917312	292886	102936	42338	18898	8902	4280
23	4194304	1352077	1893410	585772	205872	84676	37796	17804	8560
24	8388608	2704155	3786820	1171544	411744	169352	75592	35608	17120
25	16777216	5200299	7738389	2379267	830572	338198	151690	71124	34332

TABLE 1. Complete simple games with minimum ($r = 1$) per Nakamura number – part 1

We have chosen these subclasses since they allow to exhaustively generate all corresponding games for moderate sizes of the number of players n, which is not the case for many other subclasses of simple games. Additionally, the corresponding Nakamura numbers can be evaluated easily applying Lemma 24.

One might say that being non-weighted increases the probability for a complete simple game with a unique shift-minimal winning vector to have a low Nakamura number. In Table 3 the last entries of each row seem to coincide with the sequence of natural numbers, where the number of entries increases every two rows.

7. Conclusion

The Nakamura number measures the degree of rationality of preference aggregation rules such as simple games in the voting context. It indicates the extent to which the aggregation rule can yield well defined choices. If the number of alternatives to choose from is less than this number, then the rule in question will identify "best"

n	10	11	12	13	14	15	16	17
10	1							
11	1	1						
12	2	1	1					
13	4	2	1	1				
14	8	4	2	1	1			
15	16	8	4	2	1	1		
16	32	16	8	4	2	1	1	
17	64	32	16	8	4	2	1	1
18	128	64	32	16	8	4	2	1
19	265	128	64	32	16	8	4	2
20	530	256	128	64	32	16	8	4
21	1050	522	256	128	64	32	16	8
22	2100	1044	512	256	128	64	32	16
23	4200	2077	1035	512	256	128	64	32
24	8400	4154	2070	1024	512	256	128	64
25	16800	8308	4128	2060	1024	512	256	128

TABLE 2. Complete simple games with minimum ($r = 1$) per Nakamura number – part 2

n	∞	2	3	4	5	6	7	8	9	10	11	12	13	14
			15	16	17									
1	1													
2	2	1												
3	4	2	1											
4	8	5	1	1										
5	16	8	4	1	1									
6	31	14	7	2	1	1								
7	57	20	11	6	2	1	1							
8	99	30	16	10	3	2	1	1						
9	163	40	26	11	8	3	2	1	1					
10	256	55	32	18	13	4	3	2	1	1				
11	386	70	45	25	14	10	4	3	2	1	1			
12	562	91	59	33	16	16	5	4	3	2	1	1		
13	794	112	74	42	25	17	12	5	4	3	2	1	1	
14	1093	140	91	52	34	19	19	6	5	4	3	2	1	1
15	1471	168	117	63	44	21	20	14	6	5	4	3	2	1
16	1941	204	136	84	46	32	22	22	7	6	5	4	3	2
17	2517	240	166	96	59	43	24	23	16	7	6	5	4	3
18	3214	285	198	110	72	55	26	25	25	8	7	6	5	4
19	4048	330	231	136	86	57	39	27	26	18	8	7	6	5
20	5036	385	267	163	101	60	52	29	28	28	9	8	7	6
21	6196	440	316	179	117	76	66	31	30	29	20	9	8	7
22	7547	506	355	210	134	92	68	46	32	31	31	10	9	8
23	9109	572	409	242	152	109	71	61	34	33	32	22	10	9
24	10903	650	466	276	171	127	74	77	36	35	34	34	11	10
25	12951	728	524	311	207	130	93	79	53	37	36	35	24	11

TABLE 3. Weighted games with minimum ($r = 1$) per Nakamura number

alternatives. The larger the Nakamura number of a rule, the greater the number of alternatives the rule can rationally deal with. This paper provides new results on: the computation of the Nakamura number, lower and upper bounds for it or the maximum achievable Nakamura number for subclasses of simple games and parameters as the number of players and the number of equivalent types of them. We highlight the results found in the classes of weighted, complete, and α-roughly weighted simple games. In addition, some enumerations for some classes of games with a given Nakamura number are obtained.

Further relations of the Nakamura number to other concepts of cooperative game theory like the price of stability of a simple game or the one-dimensional cutting stock problem are provided.

As future research, it would be interesting to study the truth of Conjecture 1 or finding new results on the Nakamura number for other interesting subclasses simple games, as for example, weakly complete simple games.

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JOSEP FREIXAS, DEPARTMENT OF MATHEMATICS AND ENGINEERING SCHOOL OF MANRESA, UNIVERSITAT POLYTÈCNICA DE CATALUNYA, SPAIN.[3]

E-mail address: josep.freixas@upc.edu

SASCHA KURZ, DEPARTMENT FOR MATHEMATICS, PHYSICS AND INFORMATICS, UNIVERSITY BAYREUTH, GERMANY

E-mail address: sascha.kurz@uni-bayreuth.de

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