Permission in Non-Monotonic Normative Reasoning

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謹致予我無盡關愛的家人

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Chapter 1

Introduction

「不以規矩，不能成方圓」
"Nothing can be accomplished without norms or standards."

–《孟子 · 離婁上》
Mèng Zǐ · Lí Lóu Shàng

Normative reasoning is a general theoretical topic that studies normative concepts involving obligation, prohibition, and *permission* in various systematic methods. It attracted a great deal of attention from experts in various fields including philosophy, linguistics, ethics, law, and artificial intelligence. A number of traditional formal methods including deontic logic (e.g. standard deontic logic [73], minimal deontic logic [101], dynamic deontic logic [74], and deontic action logic [20, 100]), the logic of agency (e.g. STIT-logics [12, 50]), Input/Output logic [82], logic and games [107, 48, 113, 96], and law and logic [56, 57, 63, 68, 54, 94, 64] have been used in researches into normative reasoning. Normative reasoning is often viewed as *non-monotonic*, because it has an important *common sense* aspect. This is reflected in many recent approaches, for instance deontic preference logic [42, 109], non-monotonic logic [76, 70, 95], argumentation and logic programming [14], and default theory [52].

For a long time, the dominant normative concept in normative reasoning, either in theoretical fields or in formal tools, has been that of obligation rather than permission. For instance, in standard deontic logic (SDL), permission is *reducible* to obligation. It is the dual of obligation. For the sake of *consistency* in this reductive account, it has to be that *obligation implies permission*. This so-called *consistency* principle in SDL seems to be a sufficient *prescriptive* criterion for normative reasoning [3, p.283]. It is now a widely-accepted principle in most of the normative systems in the reductive account.

Should we take for granted that permission must be viewed as a second class citizen? Indeed, from the perspective of rationality, obligation guides human actions, and hence it should be at the center of normative reasoning. However, the reducibility of permission to obligation is not universally accepted. Von Wright's classic paper [116] is the first attempt to study permission as a primitive concept, and reduce obligation to the dual of permission. Though "this choice was more or less an accident" [43], it shows a branching point in normative reasoning: the reducibility of permission to obligation should not be taken for granted [1].

For one thing, the reduction just mentioned entails several difficulties in capturing the common sense notion of permission expressed by the free choice principle [118, p.21-22]. "Permission here means freedom to choose between all the alternatives, if any, covered by the permitted thing." (See [118, p.32].) This idea approximates the concept *permission as at liberty* in philosophy [84], which is also based on common use in ordinary language. "When saying that an action is permitted we mean that one is at liberty to perform it, that one may either perform the action or refrain from performing it." (See [84].) If both concepts share the same logical structure as "permission as the dual of obligation" in the reductive approach [47, 73], then either *free choice permission* causes the famous puzzle, the so-called *Ross's Paradox* [2] [47, p.61-62] [73], or *permission as at liberty* fails the principle of *consistency*. I will come back to this in Section 1.1.3. In conclusion, permission as freedom to act is at odds with the standard notion of permission that is reducible to obligation.

In the face of this, one natural solution is to study normative systems where permission is not reducible to obligation. A number of normative systems of that kind have been developed [101, 21, 99, 89, 11, 10], in which the reducible tie between obligation and permission is broken, while two principles are maintained. The first is the *consistency* principle mentioned above. The second is the *free choice* principle regarding permission: *all performances of the alternatives* are allowed. These non-reductive approaches also seem to fit many uses of "permissions" in natural language.

A number of researches, especially in legal theory, have pointed out the many faces of permission related to obligation. Rather than being reduced to the dual of obligation, and implied by obligation, certain types of permission can also *generate obligation*. Hohfeldian *legal rights* involve two types of permission, one is called *privilege*, the dual of obligation, the other is called *power*, which is one's legal ability to *change* others' normative states constructed by obligation [49]. Various formal methods have been developed to model this classification emphasizing the *dynamic* aspect of permission [56, 57, 63, 68, 54, 94]. On the other hand, considering the *defeasible functions* of rights in a legal context [45, p.275], there are clearly many notions of permissions. One is the *protected right* proposed by Raz [85], which can defeat the others' reasons for interference, and so it is a claim that "the other has a duty not to interfere." These

[1] Von Wright mentioned once in his later paper [120, p.17] that "the analogy between modal and deontic ideas" is "untenable and that the identities which are extracted from it depend on a confusion between genuine norms and norm-propositions or statements about the existence or non-existence of norms." In deontic logic a lot of discussions concern the question of what are genuine norms. See the Jørgensen's dilemma related to the *prescriptive/descriptive* distinction [47, p.58-59]. In this thesis I just note this issue, however.

[2] Sometimes it is called *free choice permissions puzzle*, but this is different from the free choice permission paradox introduced later, because they have different causes, and result in different conclusions.

Raz-protections are, however, *defeasible*, as it frequently appears in legal texts and in discussion among legal scholars [121, 85]. However, few formal works have touched on this issue of defeasible permissions and rights.

The main research questions in this thesis are: How to capture the four principles *consistency*, *free choice*, *change*, and *defeasibility* for permissions? What are the appropriate *non-monotonic* logics for normative reasoning, in which one can characterize various types of permission? Though several formal works for free choice permission and for permissions as rights have been developed, few have touched the many facets of permission in non-monotonic normative reasoning. This is the goal of this thesis.

1.1 An Overview on Permission

I start with an overview on permission in normative reasoning, which focuses on the development of deontic logic [73, 47], a field which perhaps is the first to offer a systematic analysis of normative concepts in a formal way.

1.1.1 The Traditional Scheme: A Reductive Approach

The theoretical foundation of *Standard Deontic Logic* (SDL) has two origins. One can be dated it back to Gottfried Wilhelm Leibniz's *reductive* "modalities of law" (*iuris modalia*), which includes three categories: obligation (*debitum*), prohibition (*illicitum*), and permission (*licitum*) [47]. SDL follows this reductive method, and is developed based on such modal analogies. Along the path of its ancestor in the 14th century, SDL takes obligation as the primitive concept in the scheme, and thus prohibition and permission can be defined accordingly. Prohibition is something it is obligatory not to do, and permission is something it is not obligatory not to do. The core of the reductive approach is that: permission is the dual of obligation. This kind of permission is called *negative permission* or *weak permission* [118]. The other key feature of SDL, its second origin, is the so-called *consistency* principle, which can be traced back to von Wright's earliest work [116, 73, 47]. In its usual form it states that obligation implies permission, which together with the definition of permission in terms of obligation excludes inconsistent obligations. These two key points together constitute the basic scheme of SDL.

SDL is a branch of modern modal logics [13], and axiomatically specified by the system **D** [73]. As a member of the normal modal logics [13], the language of SDL is constructed by a set of atomic propositions, taken together with classical negation, conjunction, and the modality O for obligation. Disjunction and the material conditional are defined as usual. Permission, such as negative permission and weak permission, is defined by obligation: $P\varphi := \neg O \neg \varphi$. All well-formed formulas of SDL can be interpreted in a *serial* Kripke model, which includes a binary accessible relation R over a non-empty set of possible worlds, indicating an *ideal* relation between two possible worlds. The truth conditions for propositions are standard, and those for obligation and permission are of a universal/existential character:

- $O\varphi$ is true at w iff for all ideal-world u accessible from w, φ is true at u
- $P\varphi$ is true at w iff for some ideal-world u accessible from w, φ is true at u

The sound and complete axiomatization for SDL is presented in Table 1.1.

(K-O)	$O(\varphi \to \psi) \to (O\varphi \to O\psi)$	(NEC-O)	$\dfrac{\varphi}{O\varphi}$
(D-O)	$O\varphi \to \neg O \neg \varphi$		

Table 1.1: All propositional tautologies are taken as axioms, and Modus Ponens is taken as a rule.

As usual in normal modal logics, the logical characterization of negative permission is the following theorem in SDL:

$$P(\varphi \vee \psi) \leftrightarrow P\varphi \vee P\psi$$

This property for negative permission will sometime be called the *weak permission* property. Similarly, the following is the standard property for obligation in SDL:

$$O(\varphi \wedge \psi) \leftrightarrow O\varphi \wedge O\psi$$

For many purposes it is useful to extend the language of SDL with an *action* or *agency* operator, capturing what the agent does or "sees to it that." One recent prominent logic of agency is the so-called "STIT" theory, for "Seeing To It That" [50]. The main component in constructing the language of STIT is the non-normal modality [13] Do_i over propositions, which indicates that agent i "sees to it that." This modality is, however, often assumed to be a normal, **S5** modality [50, p.17]. One popular formulation of "seeing to it that" in legal theory is a deontic logic with the T axiom and the E rule for "seeing to it that," analyzed by Kanger and Kanger [57, 56], Lindahl [63], Makinson [68] and Sergot [93]. See the details in Table 1.2.

(T-Do)	$Do_i\varphi \to \varphi$	(E-Do)	$\dfrac{\varphi \leftrightarrow \psi}{Do_i\varphi \leftrightarrow Do_i\psi}$

Table 1.2: One sound and complete deontic logic for STIT. All propositional tautologies, Modus Ponens and all axioms and rules for O in SDL are also taken into account.

1.1.2 A Brief History of the Free Choice Principle

In what follows, I give a brief history about the development of the notion of *free choice permission*. The basic idea is that when a permission is given for a number of alternatives, one is *at liberty to choose* any of them.

The expression "free choice permission" is first noted in von Wright's later work [118, p.21], which addressed permission with a free distribution of one's choices:

> "On an ordinary understanding of the phrase 'it is permitted that,' the formula '$P(p \vee q)$' seems to entail '$Pp \wedge Pq$.' If I say to somebody 'you may work or relax' I normally mean that the person addressed has my permission to work and also my permission to relax. It is up to him to choose between the two alternatives."

I denote this logical principle by FCP. Free choice permission should be distinguished from the two alternative forms of permission that have been discussed. First, in parallel to free choice permission von Wright also introduced the notion of *strong permission* [117]. This appears to be stronger than free choice permission. An action is strongly permitted in this sense if "the authority has considered its normative status and decided to permit it." This might be seen as requiring that $P(\varphi \vee \psi)$ is *equivalent* to $P\varphi \wedge P\psi$. Second, the notion of *bilateral permission* is based on a similar intuition but involves a different logic structure. Bilateral permission is introduced by Raz [84, p.161], as a notion of permission under which one is *at liberty* to perform the permitted action: "that one may either perform the action or refrain from performing it." Free choice permission is often associated with strong permission, but not with bilateral permission.

The free choice permission is close to the later developed notion in accord with the principle *open specification* [25] (or *open interpretation* [18]) in the logic of actions, and to the *open reading* principle studied in [6] recently. The open interpretation of an action expression is first mentioned in [25]. Roughly speaking, an action expression is open whenever "the action denoted by that action expression occurs, possibly in combination with other actions" [26]. Under such an "open" specification for actions, a strong permission comes down to saying that an action is permitted if none of the ways of performing this action leads to a violation state, which takes us back to free choice permission in the sense of [118].

1.1.3 Problems for Free Choice Permission and Weak Permission

The so-called *free choice permission paradox* shows that the standard, weak reading of permission is incompatible with free choice permission. The free choice permission paradox was first discussed by von Wright [118, p.33], and later extensively discussed by logicians. It is generally taken to show an incompatibility between "our intuitive understanding of such statements of ordinary language, our usual procedure for symbolizing them, and the formal powers of the usual systems of deontic logic within which they are symbolized" [67].

I will come back in detail to the problem with FCP in Chapter 3. For now I just highlight one of the most acute forms of the paradox. As already pointed out by Hilpinen [46, p.176-177], if (classically) logically equivalent formulas can be substituted in the scope of the permission operator, then the following is a direct consequence of FCP: $P\varphi \rightarrow P(\varphi \wedge \psi)$ for arbitrary ψ. Hansson's well-known *vegetarian free lunch* [43, p.218] is one instance of this: if you are permitted to order a vegetarian lunch, then you are permitted to order a vegetarian lunch and not pay for this meal.

1.2 The Many Faces of Permission

So far I have encountered three types of permissions: weak, strong and bilateral. These have been extensively studied by philosophical logicians and computer scientists in the tradition that started with von Wright. Permission has also been studied in natural language semantics and in formal theories of rights. In what follows I give a short overview of the main insights arising from these different traditions.

1.2.1 Permissions in Natural Language

Permissions in Formal Linguistics

In linguistics, utterances with "may," "can" in a deontic context are ambiguous. An ordinary utterance of choices (1a) taken in a deontic context would normally be understood as the conjunction of its choices (1b) and (1c), in the sense of free choice permission:

(1a) Detectives may go by bus or boat.

(1b) Detectives may go by bus.

(1c) Detectives may go by boat.

However, some formal linguists have noticed [123] that, in ordinary language use, this kind of inference is not always warranted.

The following is an example, which seems to imply neither (1b) nor (1c). It rather behaves like a weak permission:

(1d) Detectives may go by bus or boat – but I forget which.

How to account for this phenomenon? Two general forms of reply are proposed in formal linguistics: the pragmatic account is rooted in the Gricean view [38], which treats the deontic utterances about permission as a *conversational implicature*. Nowadays, this investigation of permission is well-analysed through pragmatic concepts including *quantity implicature* [32] and *scalar implicature* [5, 31]. For the semantic account, one promising answer for the free choice phenomenon in the deontic context uses the *alternative semantics* [4].

Permissions in Dynamics: Reshape the Normative States

Another important insight from the formal semantics literature is the *dynamic* aspect of permission. This was explicitly mentioned by David Lewis [61] in the so-called *games between Master and Slave*. In this example, "the point of the game, as regards commanding and permitting, is to enable the Master to control the actions of the Slave [61, p.22]," and "his purpose is to control the Slave's actions by *changing* the sphere of permissibility [61, p.24]." In other words, the function of permission is to *influence the normative states* of others, in order to *guide* the others' actions.

In von Wright's work this dynamic aspect goes hand in hand with his notion of *strong permission* [117, p.86]: "An act is permitted in the strong sense if the authority has considered its normative status and decided to permit it. [. . .] Strong permission only is a norm-character." Various formal works have addressed the *norm change* function of permission in modeling legal rights [63, 68, 54]. I will come back to this in Chapter 4.

The formal theories of "changes" have been well-studied in dynamic epistemic logics [103, 9] and dynamic deontic logics [65, 109, 110]. Semantically, the logics of dynamics involve two kinds of model transformations. So do information dynamics. One is the so-called *update*, which concerns the change in the level of the worlds. After the announcement of new "hard information," it generates a model only containing the worlds that satisfy the announced information. The other, the so-called *upgrade* considers the change in the level of the preference structure. After the announcement of new "soft information," it generates a model with the same worlds but a new preference order. Recently, van Benthem *et al.* [110] developed a dynamic deontic logic for conditional obligation based on the second type of change, in which permission is modeled as a dynamic modality, and can be viewed as free choice permission in certain conditions.

1.2.2 Permissions and Rights in Legal Theory

Hohfeld's influential classification of fundamental legal rights [49] distinguishes between static and dynamic rights. Static rights include *claim* and *privilege*, and their correlatives of *duty* and *no-claim*; dynamic rights are *power* and *immunity* together with the correlatives of *liability* and *no-power*. Given that agent i has a claim against agent j to stay off i's land, this claim correlates to j's duty toward i to stay off her land. The classical duality and correlation between the different Hohfeldian types of right are presented in Table 1.3. Observe that the opposite relation in static rights is the dual relation between obligation and permission in deontic logic. If i has a privilege against j to enter her land, this is reduced to j's absence of a claim against i not to

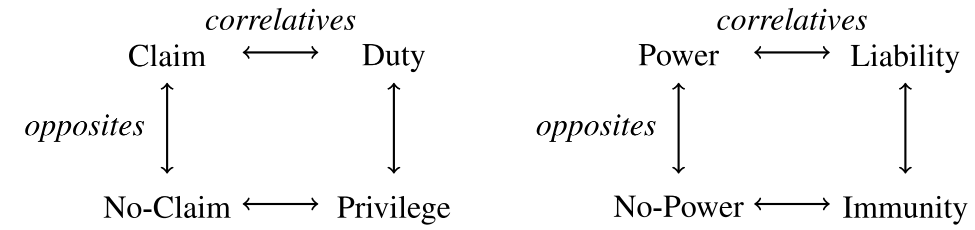


Table 1.3: Legal Rights [94]

enter her land. A number of formal theories [56, 57, 63, 68, 54, 94] have developed their frameworks of static rights based on this observation.

One important part of Hohfeld's legal theory is his emphasis on the dynamic character of power and immunity. On powers, he writes:

> A *change* in a given legal relation may result [. . .] from some *superadded fact* or group of facts which are *under the volitional control* of one or more human beings. [$\cdots$] the person (or persons) whose volitional control is paramount may be said to have the (legal) power to effect the particular change of legal relations that is involved in the problem.

This dynamic character can be spelled out in terms of *normative positions* [94]. The dynamic rights are the *legal ability* to influence the normative positions, which are the maximally consistent conjunctions consisting of the static rights.

There are two notable formal theories on Hohfeld's "power as change" [49, p.44-45]: reductive and non-reductive. I will argue in Chapter 4 that the first [56, 57, 63] does not distinguish deontic from non-deontic actions. The language of the reductive account is constructed by atomic propositions, negation, conjunction, a deontic modality O for obligation, and a Do_i-modality for agent i's "seeing to it that." The logic for O behaves essentially as in SDL, and the logic for Do_i is the non-normal modal logic satisfying the T axiom and the E rule [22, 50]. Then, $ODo_i\varphi$ indicates a claim against agent i to see to it that φ, and $\neg PDo_i\varphi$ indicates an absence of agent i's privilege to see to it that φ. Power and immunity are reduced as combinations of claim, privilege, and action performance. For instance, in [63], agent i has a power to see to it that j has a duty to see to it that φ can be defined as: $PDo_iODo_j\varphi$. See [68, 94] for summaries of the other formulations.

1.3 Outline of the Thesis

The aim of this thesis is to develop the formal theories of the various permissions that I have just introduced. The main running thread will be the *non-monotonic* character of permissions, such that inferences involving permissions, and in particular free choice permission, can be *defeated* as more information comes in. So this thesis can be viewed as a contribution to the formal theories of non-monotonic normative reasoning.

I start with an investigation of the non-reducible approach in Chapter 2, by honing in on a special case of permissions: rational permissions in games. Setting the free choice permission paradox aside, I argue there that such permission should satisfy free choice permission. In particular, I argue that, in games, obligations and permissions should be viewed, respectively, as the necessary and the sufficient conditions for rationality. This gives rise to a specific deontic logic where, for instance, O and P are not dual notions and P becomes a free choice permission operator. This feature is emphasized in the logic of obligation as the weakest permission in [90, 7], and is shared with three similar deontic logics proposed in the literature, as early as the minimal deontic logic [101], and more recently the deontic action logics in [58, 20]. This chapter studies the relationships between these deontic logics for rational agency in games. I compare their deductive power, provide the translation results, and emphasize the different views they take on what players ought to, or may do.

Chapter 3 is a reflection on the *free choice* principle imposed on actions as well as the well-known free choice permission paradox in natural language. This chapter proposes a new solution to the free choice permission paradox in three notable aspects [11, 43, 122], combining ideas from substructural logics and non-monotonic reasoning. The challenge for a logic of free choice permission is to exclude the counterintuitive consequences while not giving up too much deductive power. I suggest that the right way to do so is using a family of substructural logics augmented with principles from non-monotonic reasoning. This follows up on a proposal made in [6].

I then turn to the dynamic aspect of permission applied to the Hohfeldian notions of power and immunities in Chapter 4. I develop a dynamic logic to model and study legal competences, and in particular the Hohlfeldian categories of power and immunity. The logic improves the existing formalizations by taking explicitly into account the genuinely dynamic character of legal competences. It does so, while keeping a tight connection between both, with, however, the former being ultimately reducible to the latter. The logic is shown to be completely axiomatizable, and I provide an analysis of dynamic *normative positions*. And then it is applied to a concrete case in German contract law, as well as to diverse forms of permissions in the legal context.

Chapter 5 explores various types of permission in the light of defeasibility. I suggest augmenting current default theories along two dimensions: a distinction between obligatory and permissive defaults, and a multi-agent character. In doing so, a prioritized default theory is developed, which encompasses permissive and obligatory norms as *prima facie norms*. In this augmented default theory I can develop a rich typology of permissions, including weak permission, explicit permission, tacit permission, and protected right. I address two running examples as applications of analyzing legislation, one from Chinese tax law, and one from German driving regulations.

Chapter 2

Permission and Obligation for Rational Agency in Games

This chapter studies a family of deontic logics that diverge from *standard deontic logic* (SDL) [73] in that O and P are not dual, and P validates the notorious *free choice* principle:

$$P(\varphi \vee \psi) \rightarrow P\varphi \wedge P\psi \qquad \text{(FCP)}$$

In [90, 7], the authors argued that such deontic logics are well-suited to capturing rational obligations and permissions in games, i.e. what the players ought to, and may do according to particular solution concepts. The similarity between the logic proposed by the authors and a number of other deontic systems has been observed in [7]. But the precise comparison remained to be made. This is the main contribution of the present chapter.

This contribution should be of interest to philosophical logicians working on game theory for two reasons. It shows, first, that four independent proposals in deontic logic are well-suited to describe the rational obligations and permissions that bear on players in games, even though these systems might not have been originally devised for that purpose. This is a conceptual contribution. Second, on the formal side, it provides a systematic comparison of the deductive power of the first two systems studied here, and shows that the third can be embedded in the first, while the fourth can be partially embedded in the second.

Section 2.1 reviews the normative interpretation of solution concepts in game theory, and the argument given in [90, 7] for the particular structure of obligations and permissions to which they give rise. Section 2.2 provides the first comparison, between van Benthem's "Minimal Deontic Logic" [101] and Anglberger *et al.* "Obligations as Weakest Permissions" [89]. Section 2.3 compares van Benthem's system with Trupuz and Kulicki's "Deontic Boolean Action Logic" [99], and also compares Anglberger *et al.*'s system with Castro and Maibaum's "Propositional Boolean Dynamic Logic" [21]. Section 2.4 concludes.

2.1 The Deontic Logic of Rational Recommendations in Games

By rational recommendations in games I mean obligations and permissions stemming from classical game-theoretic solution concepts. The goal of this section is to argue that such recommendations give rise to a specific kind of deontic logics, one that differs from SDL. On the way, I give a brief, informal introduction to the game-theoretic solution concepts I have in mind (Section 2.1.1).

2.1.1 Normative Interpretation of Solution Concepts

In a game a number of self-interested players interact in what Schelling called "interdependent decisions" (see [92]). The result of each player's decision depends on what all the other players do. In this chapter I look only at the so-called *games in strategic form*. The formal definition is presented as follows:

2.1.1. DEFINITION. A game in strategic form $\mathbb{G}$ is a tuple $\langle I, \{S_i\}_{i\in I}, \pi\rangle$ where

- I is a finite set of agents or players.
- S_i is a finite set of actions or strategies for each player i.
- The payoff function $\pi : S \to \mathbb{R}^I$ assigns to each strategy profile a vector of real-valued payoffs for the players, such that S is the set of all strategy profiles σ that are combinations of strategies, one for each player.

I write σ_i for agent i's strategy in σ, and σ_{-i} for the strategies of all agents except i in σ. I use π_i to denote i's component in that vector. Let me consider a concrete example: the game "Guess 2/3 of the Average", a classic in introductory game-theory courses and experiments [1]. A number n of players have to choose a natural number between 0 and 100. They do so simultaneously, without knowing what the others do. The winner is the player whose choice is closest to 2/3 of the average number chosen. In the event there is more than one "winner", the players split the prize. Players prefer having more of the prize than less. So each prefers to be the unique winner.

Consider a very simple, two-player version of this game, say between Ann and Bob. So I = {Ann, Bob}, and $S_{Ann} = \{0, 1, ..., 100\}$, and similarly for Bob. The payoff function is first defined player-wise, as follows, with a pair (k, l) representing Ann's choice k and Bob's choice l, and $a = 2/3\frac{k+l}{2}$.

$$\pi_{Ann}(k,l) = \begin{cases} 1 & |k-a| < |l-a| \\ 1/2 & |k-a| = |l-a| \\ 0 & otherwise \end{cases}$$

[1]See `https://en.wikipedia.org/wiki/Guess_2/3_of_the_average` for an overview.

Bob's payoff function $\pi_{Bob}(k, l)$ is just defined as $1 - \pi_{Ann}(k, l)$, while $\pi(k, l)$ as $(\pi_{Ann}(k, l), \pi_{Bob}(k, l))$.

What should Ann and Bob do in this game? The standard solution concept to be applied here is that of *Nash equilibrium*, which is computed using the *best response dynamics*. Put yourself in Ann's position, and consider the case in which you and Bob choose 100. In that case you split the prize. Both of you are equally near to 2/3 of 100. But then, given that Bob chooses 100, you (Ann) could have done better by choosing any lower number, claiming the prize for yourself. In game-theoretic terms, choosing 100 is not a *best response* for Ann when Bob chooses 100. Technically, a best response function outputs an action that, *given the choice of the other players*, yields an outcome that is at least as good as the outcome yielded by any other action. Formally, for each $l \in S_{Bob}$, Ann's best response $br(Ann, l)$ is defined as $\{k : \pi_{Ann}(k, l) \geq \pi_{Ann}(k', l)$, for all $k' \in S_{Ann}\}$.[2] Observe that for Ann this set is not a singleton. The best response need not be unique. Playing *anything* lower than 100 will make her the unique winner, given that Bob plays 100. But playing 100 is *not* a best response. The situation is entirely symmetric for Bob, of course. Given that Ann plays 100, his best response is to play something lower.

Suppose Ann and Bob then play different numbers, each lower than 100, but higher than or equal to 1. Say Ann plays the highest number. This is not a best response for her. She should play a lower number, either slightly above or slightly below Bob's, depending on how far he is from 2/3 of the average point. But doing so will lower that point, which now makes even lower choices best responses for Bob.

This dynamic will continue until both Ann and Bob have chosen 0. There they play a *mutual best response*, a.k.a. a *Nash equilibrium*. Given that you choose 0, Bob has no incentive to choose anything else. In all other cases he forfeits the prize entirely to Ann, and vice-versa for her. The formal definition of a Nash equilibrium is as follows:

2.1.2. DEFINITION. Let $\mathbb{G}$ be a game in strategic form and *br* be a best response function for that game. Then σ is a Nash equilibrium iff $\sigma_i = br(i, \sigma_{-i})$ for all players $i \in I$.

Best response and equilibrium play are two *solution concepts* for games. In their normative interpretations, they are intended to capture the idea of a rational action or a rational play. In this chapter I use best response as my running example. Ann and Bob should not, on pain of irrationality, play actions that are not a best response to one another.[3] I call *rational recommendations* the normative prescriptions that one obtains from such solution concepts in games, such as the recommendation that Ann should

[2]The general definition of the best response set $br(i, \sigma_{-i})$ for player i to the choice σ_{-i} of the others is $\{s_i \in S_i : \pi_i(s_i, \sigma_{-i}) \geq \pi_{Ann}(s'_i, \sigma_{-i})$, for all $s'_i \in S_i\}$.

[3]Why? One way to answer is to go back to decision theory. There the standard of rationality for decisions under risk is the maximization of expected utility. Simply put, a player should choose actions for which the player has the strongest belief that the action will lead to a good outcome. Choosing otherwise can lead to practical incoherence. See [55] for an overview of the normative interpretation of decision theory, and [80] for an overview of its application to games.

not play 100 if Bob plays that too. Different solution concepts will of course yield different (but not unrelated!) rational recommendations in games. Moreover, Nash equilibrium and best response are surely not the only solution concepts available. In recent years, for instance, iterated elimination procedures, using either strict or weak dominance, have attracted much attention from epistemic game theorists and logicians (see [104, 78, 79, 24]). Here, however, I do not look at the structures of specific solution concepts. There is already an extensive literature on logical characterizations of, say, iterated strict dominance or equilibrium play [113, 96]. Rather, my aim is that the abstract logics of rational recommendations in games, *whatever the underlying solution concept*, should have a particular structure. This is what I argue now.

2.1.2 The Logical Structure of Rational Recommendations

I now review the argument given in [90, 7] for the following claim: rational obligations and permissions in games should be seen, respectively, as giving necessary and sufficient conditions for rational plays, and as a result the two notions should not be seen as the dual of each other. This is a philosophical argument. If the argument is correct, then this has important implications for the deontic logics of such rational obligations and permissions.

Rationality is the key normative notion underlying solution concepts. As we have seen, solution concepts pinpoint a subset of profiles that are intuitively deemed rational in a game, sometimes given additional information about the strategies that are in play or the beliefs of the players. Consider again Guess 2/3 of the Average. Here, the best response prescribes that both Ann and Bob play a lower number, given that the other plays 100. A Nash equilibrium profile in that game is one where Ann and Bob play a mutual best response to what the other is doing. The profile $(0, 0)$ is the unique Nash equilibrium in pure strategies of that game.[4]

Solution concepts, interpreted normatively, give recommendations to the players. But of what kind? My first claim is that they provide rational *permissions*, as opposed to obligations.

> **Rational Permissions** Solution concepts in games pinpoint rational *permissions*, not necessarily rational obligations.

The argument for this claim starts with the basic observation that there is in general no unique solution to a given game. Consider again the recommendation of best response given that the other plays 100. Any number from 0 to 99 is a best response. The only non-best response is playing 100 oneself. In the face of such a plurality of solutions it does not make sense to say that the players *ought* to play *all* of these numbers. They simply cannot do that. These are mutually exclusive actions. So, if "ought implies can"

[4]The situation is more complicated for more than four players. There everyone playing 1 can be an equilibrium in pure strategies. A unique deviation to 0 might not lower the average sufficiently to ensure a win.

then it is not the case that players are under a rational obligation to play every solution. What remains is that playing any solution of a game is rationally permissible. In Guess 2/3 of the Average, given that the other is playing 100, any number between 0 and 99 is rationally permissible. The situation is of course not a particular best response in this particular instance of that game. Non-unique solutions are ubiquitous in game theory. In the face of this, the appropriate way to understand rational recommendations from solution concepts is in terms of rational permissions.[5]

Our next claim is that rational permissions provide sufficient conditions for best response. Here I only illustrate this by using our running example. The argument is developed in detail in [90, 7]. Given that Bob plays 100, it is rationally permissible for Ann to play any number, as long as it is lower than 100. Let me introduce some action-theoretic terminology, which I will formalize later on. Call an *action* or a *strategy type* just a set of actions/strategies for one player, and similarly for strategy profiles. In Guess 2/3 of the Average, the type "playing a number lower than 100" is rationally permitted for Ann by best response, given that Bob plays 100. *If* she plays any strategy of that type *then* she plays a best response strategy. Playing less than 100 is sufficient for rationality. Observe, furthermore, that playing any number which is an instance of a logically stronger action type will also be best response for Ann against Bob's playing 100.[6] So picking among the set of *even* numbers lower than 100, or just picking 0 for that matter, will imply playing a best response. From the perspective of best response to 100 alone, these are all on a par.

So if "playing a number lower than 100" is a rationally permitted type, and a rational permission for the action type provides sufficient conditions for rationality, then any of these logically stronger types should be seen as also permitted. Best response cannot distinguish between them any further. In [90, 7] it is argued that this holds more generally, for any rational permission in games. In a nutshell, this gives me the following principle:

> **Strong Rational Permissions (SRP)** An action type φ is rationally permitted in game $\mathbb{G}$ if and only if playing a strategy of type φ implies playing a rational strategy.

If SRP is correct, this has important consequences for the logical analysis of rational recommendations. The most important is that one should take *free choice permission* on board:[7]

$$P(\varphi \lor \psi) \to P\varphi \land P\psi \tag{FCP}$$

[5] Of course, when there is a unique profile that is rationally permitted, that profile becomes rationally obligatory. This is the case, for instance, for the profile $(0, 0)$, using rational recommendations from Nash equilibrium in the game above. This is not only what the players are rationally permitted to do. They ought to play $(0, 0)$. This will correspond to a logical principle connecting obligations and permissions, which I will encounter later on.

[6] A type φ is logically stronger than type ψ when playing a strategy of type φ implies playing strategy of type ψ. In this case, I say φ is a sub-type of ψ.

[7] This principle has a reputation for being misconstructed in combination with SDL. If permissions are normal modalities in the technical sense, then $P\varphi \to P\psi$ becomes easy to derive by using FCP.

Indeed, if $\varphi \vee \psi$ is viewed as a non-deterministic choice between playing φ or ψ, then if playing a strategy of either type is sufficient for rationality, then playing both any strategy of type φ and any strategy of type ψ is sufficient for rational play. So both are permitted by SRP.

If permissions provide sufficient conditions for rationality, the natural counterpart is the view of obligations as necessary conditions. In other words:

> **Weak Rational Obligations (WRO)** An action type φ is rationally obligatory, or *rationally required* in game $\mathbb{G}$ if and only if not playing a strategy of type φ implies not playing a rational strategy.

I call this principle "weak" because it suggests a form of closure of obligations under logically weaker types. If it is rationally required to do φ, then playing rationally implies playing a strategy of type φ. But then this also implies playing any weaker type of strategy, and in particular the trivial type $\varphi \vee \neg\varphi$. So logically very weak types of action will turn out to be obligatory. I shall see this in the concrete case of Guess 2/3 of the Average in Section 2.2.

An important consequence of accepting both SRP and WRO is that obligations and permissions are no longer necessarily duals. In our example it is not the case that playing a number higher than 50 is permitted for Ann as best response to 100. This is not sufficient for best response to 100, because playing 100 herself is a strategy of that type. So by SRP this is not permitted. But not playing a number higher than 50 is just the same as playing a number lower than or equal to 49. But this cannot be obligatory either, because not playing this does not entail not playing a best response. So rational permissions and rational requirements are not dual here.

I take these as the central features of rational recommendations in games: SRP, WRO — viz. obligations and permissions provide, respectively, necessary and sufficient conditions for rationality — and these two normative categories are not dual. In this section I have sketched the philosophical arguments for these claims. They are developed in more detail in [90, 7], where the authors present a deontic logic of "obligation as weakest permission" that has all these features. In the next sections we compare this logic with two very congenial proposals, an earlier one by van Benthem and a more recent one by Trypuz and Kulicki. As will be seen, they all share the three central features, and as such can be viewed as logical for rational recommendations in games, although they differ either in their philosophical commitment or their expressive power.

An example of this is given in Section 2.2. So rational permissions should not be normal, and indeed they are not in any of the logical systems presented below. FCP can also cause problems for non-normal modalities, as long as they are extensional. This is the now familiar "vegetarian free lunch" example [43]. If ordering a vegetarian meal is permitted, then by FCP the logically stronger action type "ordering a vegetarian meal and not paying for it" must also be permitted, at least if the Boolean constructors on action types are classical. See [6] for an answer to that criticism.

2.2 Minimal Deontic Logic and Weakest Permissions

I start by comparing the logic of "obligations as weakest permissions" [89] with van Benthem's "minimal deontic logic" [101]. They have much in common, both formally and in their analysis of rational recommendations. They differ, however, in their view of obligation. Applied to rational recommendations, van Benthem's deontic logic makes obligatory every necessary condition for rational play. In other words, playing any action type that rules out being rational is forbidden. This is not the case in the logic of obligations as weakest permissions. There the unique obligation bearing on the players is to play a rational strategy. The crux of this difference turns out to be the relation between obligation and permission in these two systems.

2.2.1 Common Language

The two logics that I study now share the same language. They take both O and P as primitive, and use a universal modality $\Box$.

2.2.1. DEFINITION. Let p be any element of a given countable set $Prop_0$ of atomic propositions. The language $\mathcal{L}$ is defined as follows:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid P\varphi \mid O\varphi$$

The existential modality $\Diamond$ can be defined as $\neg \Box \neg$ as usual.

2.2.2 Propositions and Action Types

First a note on the interpretation of the structures used in the semantics of my first two systems. These structures are familiar to modal logicians: binary relations or neighborhood functions defined on a set of objects. These objects, however, are here viewed from an action-theoretic perspective. Instead of thinking of them as possible worlds, I take them to be atomic actions. When discussing concrete games, these are either strategy profiles or strategies for individual players. While sets of states in standard Kripke semantics are propositions, here they are taken as action types. In my running example, for instance, "playing an even number", "playing a number less than 100", or "playing 0" are all action types, with the latter just happening to be an atomic one. So the standard Boolean connectives on propositions correspond here to action type constructors, pretty much as in Propositional Dynamic Logic (PDL) [98] or Boolean Modal Logic (BML) [13], and the resulting deontic logic is one of "ought to do", as opposed to "ought to be."

2.2.3 Minimal Deontic Logic

In van Benthem's Minimal Deontic Logic (MDL) the obligation operator O is a normal modality, and the permission operator P is a so-called *window modality* [13] defined on the set of normatively ideal action tokens. The presentation here is slightly different than in [101], to bring it more into line with what comes later.

2.2.2. DEFINITION. A MDL model M is a tuple $\langle W, R_D, V\rangle$ where:

- $W \neq \emptyset$ is a set of atomic action (tokens).
- $R_D \subseteq W \times W$.
- $V : Prop_0 \to \wp(W)$ is a valuation function

This is just a standard Kripke model for deontic logic. The relation R_D pinpoints the normatively ideal action type, from the perspective of each atomic action (token) or profile. In games, the normatively ideal actions will be those recommended by a specific solution concept. In principle what is rational or ideal in an MDL model may vary from action to action. This can be used to represent typical cases of interdependence between what one player does and what is rational for the others to do, as observed for instance in the best response dynamic that leads to the $(0,0)$ equilibrium of Guess 2/3 of the average. This section and the next, however, considers only *uniform* models, where $R_D[w] = R_D[w']$ for all w, w', with $R_D[w] = \{v : R_D(w,v)\}$. The set of rational atomic actions is the same throughout the model.

The difference from SDL shows in the truth conditions for P.

$$\begin{array}{lll} M,w \models \Box\varphi & \text{iff} & \forall v \in W.\ M,v \models \varphi \\ M,w \models O\varphi & \text{iff} & \forall v \in W.\ (R_Dwv \Rightarrow M,v \models \varphi) \\ M,w \models P\varphi & \text{iff} & \forall v \in W.\ (M,v \models \varphi \Rightarrow R_Dwv) \end{array}$$

P is thus a "window modality" [13]. $P\varphi$ is true iff all action types φ are the sub-types of the ideal type specified by R_D. In less technical terms, $P\varphi$ is true whenever playing a strategy of type φ ensures a rational play. So in this logical system permissions provide sufficient conditions for an action type to be "legal" or "licensed" by a given normative theory. The normative theory I consider now, of course, is the rational recommendations stemming from a given solution concept in games. An action type is permitted, in this view, if playing that action type implies playing a rational strategy. So the logic embodies SRP.

Obligations, on the other hand, can be seen as providing necessary conditions for rationality in that system, and hence also to capture WRO. The core interaction principle behind this is the following, which I shall often encounter later on in the chapter:

$$O\varphi \wedge P\psi \to \Box(\psi \to \varphi)$$

In the context of games, this principle states that if one is not rational unless one plays a strategy of type φ, while playing a strategy of type ψ guarantees a rational play, then it must be the case that all strategies of type ψ are the strategies of type φ. Let Ra be the type of all rational strategies. Then combining SRP and WRO, I have:

$$\psi \Rightarrow Ra \Rightarrow \varphi$$

(K-$\Box$)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	(Incl)	$\Box\varphi \to O\varphi$
(K-O)	$O(\varphi \to \psi) \to (O\varphi \to O\psi)$	(CoIncl)	$\Box\neg\varphi \to P\varphi$
(ConV)	$P\varphi \wedge P\psi \to P(\varphi \vee \psi)$	(WP)	$O\varphi \wedge P\psi \to \Box(\psi \to \varphi)$
(NEC)	$\dfrac{\varphi}{\Box\varphi}$	(Flip)	$\dfrac{\varphi \to \psi}{P\psi \to P\varphi}$

Table 2.1: The sound and complete axiom system for van Benthem's Minimal Deontic Logic. All propositional tautologies are taken as axioms, and Modus Ponens is a rule.

This connection between ψ and φ is expressed in the consequent of the previous formula, crucially using the universal modality. Observe that this logic also captures the third core feature of logics for rational recommendations in games: O and P are not dual here.

Let me illustrate how this system would handle rational recommendations in Guess 2/3 of the Average, with Ann and Bob playing the game, and Bob playing 100. Then best response recommends Ann to play any number lower than 100. A natural way to represent this situation as a minimal deontic model is to take W to be the set of all pairs $(n, 100)$, with $0 \leq n \leq 100$, i.e. all possible choices for Ann given that Bob plays 100. The set of plays where Ann plays a best response to Bob playing 100, i.e. $br(Ann, 100)$, is then simply W minus the pair $(100, 100)$. Defining $R_D[w]$ as $br(Ann, 100)$, for all w, then I get the expected recommendations. Ann ought not to play 100, and playing anything below that is permitted, because it implies playing a best response.

So with van Benthem's minimal deontic logic I have the first logic for rational recommendations in games. It is worth considering its axiomatization before moving to my second logic. The full system is presented in Table 2.1.
Since O is given in normal Kripke semantics, it validates K and the necessitation rule. The latter is readily derivable using NEC and Incl. So O is a normal modality. Note in passing that this implies that in this logic some of the classical deontic paradoxes, for instance contrary to duty [23] or Ross's paradox, will be valid.

2.2.3. PROPOSITION. *In* MDL*, the necessitation rule holds for O, i.e.,* $\dfrac{\varphi}{O\varphi}$.

Proof:
Suppose φ is a theorem of MDL. Then by NEC I have $\Box\varphi$. So $O\varphi$ is derivable from $\Box\varphi$ using Incl and Modus Ponens. □

P, on the other hand, is not a normal modality. Necessitation fails for permissions in that logic. Permissions do validate FCP, which is directly derivable by Flip. Note also that from this rule one can both derive the K axiom for P and the extensionality rule E, familiar to axiomatization of non-normal modal logics, stating that if $\varphi \leftrightarrow \psi$ is a theorem of the logic, then one can substitute φ and ψ within the scope of a permission operator.

2.2.4. PROPOSITION. *In* MDL*, the principle* FCP*, the K axiom as well as the E rule hold for P, i.e.,*

1. $P(\varphi \vee \psi) \to P\varphi \wedge P\psi$,
2. $P(\varphi \to \psi) \to (P\varphi \to P\psi)$, *and*
3. $\dfrac{\varphi \leftrightarrow \psi}{P\varphi \leftrightarrow P\psi}$.

Proof:

1. Because $\varphi \vee \psi$ is derivable from both φ and ψ, I have $P(\varphi \vee \psi) \to P\varphi$ and $P(\varphi \vee \psi) \to P\psi$ by applying Flip twice. So $P(\varphi \vee \psi) \to P\varphi \wedge P\psi$.
2. Suppose $P(\varphi \to \psi)$ and $P\varphi$ are theorems of MDL. Then $P((\varphi \to \psi) \vee \varphi)$ is followed by ConV. On the other hand, I know that $\psi \to (\varphi \to \psi) \vee \varphi$ is a tautology. Then $P((\varphi \to \psi) \vee \varphi) \to P\psi$ is derivable by using Flip. Thus, I have $P\psi$ by Modus Ponens.
3. Suppose $\varphi \leftrightarrow \psi$ is a theorem of MDL. So $\varphi \to \psi$ and $\psi \to \varphi$ are both theorems. Then, by using Flip twice, I have $P\psi \to P\varphi$ and $P\varphi \to P\psi$. So $P\varphi \leftrightarrow P\psi$ is derivable.

□

2.2.4 Obligations as Weakest Permissions

The second logic I consider is the logic of obligations as weakest permissions. The main difference from van Benthem's system is that obligations are no longer normal modalities. An action type φ is obligatory, in this logic, whenever it is *exactly* the normatively ideal action type, i.e. a type of rational strategy. To put it bluntly, in this logic agents ought to do only one thing : be rational!

2.2.5. DEFINITION. A model for the logic of obligations as weakest permissions, or OWP-model for short, is a tuple $M = \langle H, n_P, n_O, ||\cdot||\rangle$ where

- H is a set of atomic actions.
- $n_P : H \to \wp\wp(H)$ is a neighborhood function assigning a set of subsets of H to each $h \in H$ such that
  - If $X \cup Y \in N_P(h)$ then $X \in n_P(h)$ & $Y \in n_P(h)$
- $n_O : H \to \wp\wp(H)$ is a neighborhood function assigning a set of subsets of H to each $h \in H$ such that

  - $\varnothing \notin n_O(h)$.
  - (Ought-Perm) If $X \in n_O(h)$ then $X \in n_P(h)$
  - (Weakest-Perm) If $X \in n_O(h)$ then $Y \subseteq X$ for all $Y \in n_P(h)$.

- $||\cdot|| : Prop_0 \to \wp(H)$ is a valuation function.

As usual, an OWP-frame is an OWP-model minus the valuation function $||\cdot||$. I postpone the discussion of the frame conditions for a moment, observing only that putting (Ought-Perm) and (Weakest-Perm) together I obtain the following:

2.2.6. PROPOSITION. *If* $n_O(h) \neq \varnothing$ *then*[8]

$$n_O(h) = \{\bigcup n_P(h)\} \tag{C1}$$

Proof:
I need to show $X \in n_O(h)$ implies $X = \bigcup n_P(h)$. Suppose $X \in n_O(h)$. First, by using the (Ought-Perm) condition, $X \in n_P(h)$ and then $X \subseteq \bigcup n_P(h)$. Second, let $x \in \bigcup n_P(h)$. Then $\exists Y \in n_P(h)$ s.t. $x \in Y$. By using the (Weakest-Perm) condition, $x \in Y \subseteq X$. Hence $\bigcup n_P(h) \subseteq X$. In conclusion, $X = \bigcup n_P(h)$. □
In other words, if an action type is obligatory, then it is the *unique* action type that is obligatory, up to logical equivalence, and this is the logically weakest permission that the agent has.

The truth conditions for O and P in OWP-models are standard for neighborhood semantics, and validity is defined as usual. Let me abuse my notation and write $||\varphi||$ for $\{h : M, h \models \varphi\}$.

$$\begin{array}{lll} M,h \models \Box\varphi & \text{iff} & \forall h' \in H.\ M,h' \models \varphi \\ M,h \models O\varphi & \text{iff} & ||\varphi|| \in n_o(h) \\ M,h \models P\varphi & \text{iff} & ||\varphi|| \in n_p(h) \end{array}$$

(Univ)	$\Box(\varphi \leftrightarrow \psi) \to (D\varphi \leftrightarrow D\psi)$	(WP)	$(O\varphi \land P\psi) \to \Box(\psi \to \varphi)$
(O-P)	$O\varphi \to P\varphi$	(Flip)	$\dfrac{\varphi \to \psi}{P\psi \to P\varphi}$
(O-Can)	$O\varphi \to \Diamond\varphi$		

Table 2.2: The sound and complete axiom system for OWP. All propositional tautologies, Modus Ponens, as well as the S5 axioms and rules for $\Diamond$, are also assumed here. In (Univ) D is either O or P.

The set of valid formulas is completely axiomatizable by the system in Table 2.2. Some of its features are worth highlighting. First, the E rule mentioned earlier for P in

[8] Keep in mind that $\varnothing \in n_O(h)$ and $n_O(h) = \varnothing$ are two very different conditions. Here the first would come down to a violation of the "ought implies can" principle. It makes the impossible action type obligatory. In the second case *nothing* is obligatory.

MDL is derivable here as well, both for P, in exactly the same way as above, and for O. The latter follows from necessitation for $\Box$ and (Univ). Maybe more surprisingly, K for O is derivable in this logic! Model-theoretically, this is a consequence of the uniqueness of obligations in that system. If both $\varphi \to \psi$ and φ are obligatory, then by Proposition 2.2.6 they must have the same extension. Axiomatically, two applications of WP make $\Box((\varphi \to \psi) \leftrightarrow \varphi)$ derivable. But then a few steps of normal modal and propositional reasoning with $\Box$ gives me $\Box(\varphi \leftrightarrow \psi)$, from which one application of the E rule for O together with Univ outputs $O\psi$. Now deontic logicians might fear that paradoxes loom again in the presence of the K axiom. But the undesirable consequences of that theorem are limited by the absence of necessitation for O in that logic. Neither the Ross nor the Contrary to Duty paradox holds here. See [89] for the details.

2.2.7. PROPOSITION. *In* OWP*, the E rule holds for P and O, i.e.,*

1. $\dfrac{\varphi \leftrightarrow \psi}{P\varphi \leftrightarrow P\psi}$, *and*

2. $\dfrac{\varphi \leftrightarrow \psi}{O\varphi \leftrightarrow O\psi}$.

Proof:

1. Suppose $\varphi \leftrightarrow \psi$ is a theorem of OWP. So, by using Flip twice, I have $P\varphi \leftrightarrow P\psi$.
2. Suppose $\varphi \leftrightarrow \psi$ is a theorem of OWP. By necessitation for $\Box$, I have $\Box(\varphi \leftrightarrow \psi)$. So, by using Univ, it follows $O\varphi \leftrightarrow O\psi$.

$\Box$

2.2.8. PROPOSITION. *In* OWP*, the K axiom holds for O, i.e.,*

$$O(\varphi \to \psi) \to (O\varphi \to O\psi).$$

Proof:
By applying WP twice, $O(\varphi \to \psi) \land P\varphi \to \Box(\varphi \to (\varphi \to \psi))$ and $O\varphi \land P(\varphi \to \psi) \to \Box((\varphi \to \psi) \to \psi)$ are both theorems of OWP. In accordance with O-P, $O\varphi \to P\varphi$ and $O(\varphi \to \psi) \to P(\varphi \to \psi)$ are theorems of OWP too. Taking these four theorems of OWP together, it then implies that $O(\varphi \to \psi) \land O\varphi \to \Box((\varphi \to \psi) \leftrightarrow \varphi)$. In addition, $(\varphi \to \psi) \leftrightarrow \varphi$ and $\varphi \leftrightarrow \psi$ are logically equivalent. So I have that $\Box((\varphi \to \psi) \leftrightarrow \varphi) \leftrightarrow \Box(\varphi \leftrightarrow \psi)$ is derived by using the E rule for $\Box$. By Modus Ponens, $O(\varphi \to \psi) \land O\varphi \to \Box(\varphi \leftrightarrow \psi)$ is derived in OWP. As what Univ shows, I have $\Box(\varphi \leftrightarrow \psi) \to (O\varphi \leftrightarrow O\psi)$. So $O(\varphi \to \psi) \land O\varphi \to (O\varphi \leftrightarrow O\psi)$. According

to propositional reasoning, $O(\varphi \to \psi) \wedge O\varphi \to O\psi$ is derived. □

Rational recommendations in games work in OWP-models essentially as in van Benthem's minimal deontic logic. Take my running example again. Ann's obligations are defined exactly as before. Taking H to be the set of all pairs $(n, 100)$, $n_O(h)$ just contains the set $br(Ann, 100)$, for all h, and each $n_P(h)$ is defined by taking the closure under subsets of $br(Ann, 100)$. This gives me essentially the same result as before: Ann ought to play something less that 100, and any sub-type of that, for instance playing less than 10, or just 1, is permitted.

2.2.5 Comparison

Our first two logics for rational recommendations in games thus have many things in common. They share the main substantive principles for which I argued in Section 2.1.2: O and P are not dual, P validates FCP by Flip, and I have WP in both logics. Furthermore, obligations validate K in both systems.

Some of the axiomatic divergences between the two systems reflect minute frame-theoretic differences that can easily be accommodated. There is for instance no "ought implies can" principle in MDL, because the semantics allow for "blind" atomic actions, i.e. actions from which no normatively ideal actions can be reached. But the principle can be added, forcing us into the class of serial MDL frames. On the other hand, both the obligation and the permission neighborhoods can be empty in OWP models, which explains the invalidity of $\Box\neg\varphi \to P\varphi$ in that logic. If, however, the permission neighborhood is not empty, then the principle holds. This is reflected by the following theorem of OWP:

$$P\psi \to (\Box\neg\varphi \to P\varphi)$$

The derivation starts by factoring ψ into its logically equivalent $(\psi \wedge \varphi) \vee (\psi \wedge \neg\varphi)$. Using the derivable E rule and Flip, one gets $P(\psi \wedge \varphi)$. But then since $\neg\varphi$ propositionally implies $(\psi \wedge \varphi) \leftrightarrow \varphi$, a standard bit of normal modal reasoning with $\Box$, and one last application of E with Univ gives me $P\varphi$, as required.

2.2.9. PROPOSITION. $P\psi \to (\Box\neg\varphi \to P\varphi)$ *is a theorem of* OWP.

Proof:
I know ψ is logically equivalent to $(\psi \wedge \varphi) \vee (\psi \wedge \neg\varphi)$. By using the E rule for P, Flip and Modus Ponens, $P\psi \to P(\varphi \wedge \psi)$ is a theorem of OWP. On the other hand, $\neg\varphi$ logically implies $(\psi \wedge \varphi) \leftrightarrow \varphi$. By applying the E rule for $\Box$, Univ and Modus Ponens, $\Box\neg\varphi \to (P(\varphi \wedge \psi) \leftrightarrow P\varphi)$ is a theorem in OWP too. Taking these two theorems together, I have $P\psi \to (\Box\neg\varphi \to P\varphi)$ as a theorem of OWP. □

A similar argument explains the absence of ConV in OWP. The principle is valid only in the class of OWP frames where the *obligation* neighborhood is not empty. Note that in OWP-frames this might happen if the permission neighborhood is not empty.

This is mirrored, once again, by the following prefixed version of Univ, which is also a theorem of OWP.[9]

$$O\chi \to (P\varphi \wedge P\psi \to P(\varphi \vee \psi))$$

This time the derivation starts with applying (WP) to $O\chi$ and $P\varphi$, to deliver $\Box(\varphi \to \chi)$, and similarly for ψ. Then after some steps of propositional and normal modal reasoning we get $\Box((\chi \wedge (\varphi \vee \psi)) \leftrightarrow (\varphi \vee \psi))$. The proof finishes by using O-P on $O\chi$ again, and working my way to $P(\varphi \vee \psi)$ using the same factorization and flipping routine as above.

2.2.10. PROPOSITION. $O\chi \to ((P\varphi \wedge P\psi) \to P(\varphi \vee \psi)$ *is a theorem of* OWP.

Proof:
In accordance with WP, I have $O\chi \wedge P\varphi \to \Box(\varphi \to \chi)$ and $O\chi \wedge P\psi \to \Box(\psi \to \chi)$ as theorems in OWP. They imply that $O\chi \wedge P\varphi \wedge P\psi \to \Box(\varphi \vee \psi \to \chi)$ is a theorem. On the other hand, by using normal modal logic, I have $\Box(\varphi \vee \psi \to \chi) \to \Box((\chi \wedge (\varphi \vee \psi)) \leftrightarrow (\varphi \vee \psi))$. By using Univ, its consequent implies $P(\chi \wedge (\varphi \vee \psi)) \leftrightarrow P(\varphi \vee \psi)$. In addition, by WP, Flip, and propositional logic, I have $O\chi \to P(\chi \wedge (\varphi \vee \psi))$. Taking all these together, I have $O\chi \to ((P\varphi \wedge P\psi) \to P(\varphi \vee \psi)$ as a theorem of OWP. □

The main point of divergence between the two systems, how they handle obligations, rests on the apparently innocuous O-P principle:

$$O\varphi \to P\varphi$$

This principle is not valid in van Benthem's MDL, while in OWP it nails down the uniqueness of obligations.

The absence of O-P in MDL requires us to abandon some old thinking habits from Standard Deontic Logic. Consider again the example of Ann's best response to Bob's playing 100 in Guess 2/3 of the Average. The result of the construction sketched in Section 2.1.1 is that Ann ought not only to "play any number lower than 100", she ought also to play any action type that is logically weaker than "playing any number lower than 100". So in particular the trivial action type $\top$ is rationally required of her. She ought to play *a* number, whatever that number is, simply because if she does not play any number then she will not play any best response number. But playing any number whatever is *not* permitted for Ann, despite the fact that she is rationally required to do this. This is a particular case where obligation does not imply permission in MDL. This might feel counterintuitive to the reader, probably because of the ease with which we have learned to derive permissions from obligations in Standard Deontic Logic. Against this one should keep in mind the interpretation of obligations and permissions in MDL as necessary and sufficient conditions for rationality. Necessary conditions need not be sufficient, of course, so O should not imply P in that interpretation.

[9] I am grateful to Frederik van de Putte for drawing our attention to this fact.

In fact adding this principle to MDL results in the same deontic trivialization as when FCP is added to SDL (c.f. footnote 7 on p. 15). Everything becomes permitted. Necessitation for O gives $O\top$, which then with O-P and Flip yields $P\varphi$ for any φ whatsoever, as anything implies the tautology. OWP avoids this trivialization because obligations are not closed under logical consequences. Although it satisfies K, this logic invalidates the so-called inheritance rule:

$$\frac{\varphi \rightarrow \psi}{O\varphi \rightarrow O\psi}$$

This can be illustrated again in my running example. Ann's best response to Bob playing 100 is to play any number lower than his. This logically implies that she plays *a* number. But, in contrast to MDL, here it does not follow that she ought to play any number as well. What Ann ought to do, here, is to play *only* a best response. This is a direct consequence of the interplay between O-P and WP.

2.2.11. PROPOSITION. *For arbitrary φ, $P\varphi$ is derivable after adding O-P into* MDL.

Proof:
By the necessitation rule for O in MDL, I have $O\top$. Then $P\top$ is followed by O-P. On the other hand, by using Flip, $P\top \rightarrow P\varphi$ for any φ. Thus, $P\varphi$ is derivable, after adding O-P into MDL. □

More generally, by accepting that obligation implies permission, OWP is committed to the view that obligations pinpoint necessary *and* sufficient conditions for rationality. Hence the uniqueness of obligations, up to coextensionality. As mentioned earlier, the only types of strategy agents ought to play in that logic are rational strategies.

MDL and OWP do overlap, but precisely in the trivial cases where nothing but the trivial action type $\top$ is obligatory, and hence everything is permitted.[10] Indeed, any MDL-model where $R_D = W \times W$ can be turned into an OWP-model, by taking $n_O(w) = \{W\}$ for all w, and $n_P(w)$ the full power set of W, and conversely for starting from such an OWP-model. It should be clear that the two will satisfy exactly the same formulas. The converse is also true. For any MDL-frame where R_D is not the universal relation, taking the set of accessible atomic actions at each w to construct $n_O(w)$ will yield divergent obligations in OWP and MDL, at some w.

Let me summarize the findings of this first comparison. The most important point of agreement between MDL and OWP, the way in which they differ most from Standard Deontic Logic, is that rational obligation and permission provide necessary and sufficient conditions for rational play in games. This is witnessed by their acceptance of the Flip rule and the WP axiom. Accepting this view, however, raises a dilemma.

[10] Note that with Flip or FCP in the system, $\top$ being the only obligatory action and everything being permitted are just two sides of the same coin.

On the one hand one can stay as close as possible to SRP and WRO. Then one is forced, on pain of trivialization, to abandon the familiar "O implies P" principle. This leads to the situations like Ann's described above, where a player ought to play an action type that is not rationally permitted. On the other hand, if one chooses to retain the implication from obligation to permission, then one restricts the former to one particular necessary condition for rationality, namely the necessary *and* sufficient one, abandoning the "only if" direction of WRO. In short, the main axiomatic difference between MDL and OWP reflects a difference in philosophical commitment, to the familiar "obligation implies permission" principle and to the main features of logics of rational recommendations in games.

2.3 Deontic Action Logics

I now look at a richer language to describe rational recommendations in games, Deontic Boolean Action Logic (DBAL), proposed in [99]. This logic differs from MDL and OWP in that it draws a sharper distinction between the Boolean construction of action types and the Boolean connectives applied to obligations and permissions. In contrast to the previous section, where we mainly looked at axiomatic differences between MDL and OWP, here the main contributions are translation results, showing that DBAL is embeddable in MDL.

2.3.1 Deontic Boolean Action Logic

By interpreting the points in MDL and OWP models as actions, atomic and complex formulas in $\mathcal{L}$ were naturally interpreted as action types, constructed much as in Propositional Dynamic Logic (PDL) [98]. The logic I consider now explicitly uses a language similar to Boolean Modal Logic (BML), and in doing so it distinguishes action type constructors and standard Boolean connectives for formulas. The result is the following two-sorted language:

2.3.1. DEFINITION. The language $\mathcal{L}^*$ for Deontic Boolean Action Logic (DBAL) is defined as follows:

$$\varphi := \alpha \doteq \alpha \mid \neg\varphi \mid \varphi \rightarrow \varphi \mid P\alpha \mid F\alpha$$
$$\alpha := a \mid \mathbf{1} \mid \overline{\alpha} \mid \alpha \cup \alpha \mid \alpha \cap \alpha$$

where a is one element of a finite set Act_0 of action generators.

Action types $\alpha \in Act$ are thus constructed out of primitive action types, taken from a given set of action generators Act_0, the trivial **1** action, and the usual BML connectives of complementation $\overline{\alpha}$, nondeterministic choice $\cup$ and parallel execution $\cap$. The impossible action **0** is defined as $\overline{\mathbf{1}}$.

Strictly speaking, the language itself has three types of atomic sentence. First, the language can describe the equivalence of action types, using the $\doteq$ connective. Second come the deontic statements, stating that certain action types are (rationally) permitted or forbidden. An obligation to do α, in that language, is defined as the complement $\overline{\alpha}$ of that action type being prohibited, namely $O\alpha := F\overline{\alpha}$.

This syntax reveals the main difference between DBAL and the two systems presented in the previous section. In DBAL only action types are allowed within the scope of the deontic modalities. So I am still in the realm of logics for "ought to do". Here, however, no deontic modality, or Boolean connective for that matter, can occur in the scope of an F or a P. Formulas of the form $PF\alpha$ or $F\neg\alpha$ are not well-formed here. The deontic modalities function as properties of action types, and deontic statements that such-and-such a type is forbidden or permitted are (structured) atomic sentences in this language. The Boolean connectives $\neg$, $\rightarrow$, etc., are here to form complex statements about equivalence and the deontic properties of action types.

The semantics offer tools to interpret the construction of complex action types and their deontic properties. Although there is a syntactic distinction between the action-theoretic and the rest of the language here, in the model I only work with atomic actions and deontic properties.

2.3.2. DEFINITION. A DBAL model M is a tuple $\langle E, LEG, ILL, I\rangle$ where:

- E is a non-empty set of atomic actions
- LEG and ILL are subsets of $\wp(E)$ such that
  (a) If $X \in \mathcal{K}$ and $Y \subseteq X$ then $Y \in \mathcal{K}$, where $\mathcal{K} \in \{LEG, ILL\}$
  (b) If $X \in \mathcal{K}$ and $Y \in \mathcal{K}$ then $X \cup Y \in \mathcal{K}$, where $\mathcal{K} \in \{LEG, ILL\}$
  (c) $LEG \cap ILL = \{\varnothing\}$
- $I : Act_0 \rightarrow \wp(E)$ is an interpretation function assigning each action generator to a subset of actions, i.e., $I(a) \subseteq E$ where $a \in Act_0$

Reflecting the syntactic primitives, DBAL models come equipped with two sets of action types, the legal and the illegal. In its LEG instance, condition (a) gives us the by now familiar downward closure of the permissions, corresponding to FCP. For ILL this condition gives us, given the definition of obligation, the supersets closure condition that I have already encountered in MDL. To spell this out in detail I need first to extend the interpretation function to arbitrary action types.

$$
\begin{aligned}
I(\mathbf{1}) &:= E \\
I(\overline{\alpha}) &:= E - I(\alpha) \\
I(\alpha \cup \beta) &:= I(\alpha) \cup I(\beta) \\
I(\alpha \cap \beta) &:= I(\alpha) \cap I(\beta)
\end{aligned}
$$

(OR-P)	$P(\alpha \cup \beta) \leftrightarrow P\alpha \wedge P\beta$	(Incl)	$P\alpha \wedge F\beta \rightarrow \alpha \cap \beta \doteq \mathbf{0}$
(OR-F)	$F(\alpha \cup \beta) \leftrightarrow F\alpha \wedge F\beta$	(BA)	$\varphi \wedge \alpha \doteq \beta \rightarrow \varphi[\alpha/\beta]$
(**0**-P/F)	$\alpha \doteq \mathbf{0} \leftrightarrow F\alpha \wedge P\alpha$		

Table 2.3: The sound and complete axiom system for Deontic Boolean Action Logic. All propositional tautologies are also taken as axioms, as well as the usual axioms for Boolean algebras for Boolean action terms α. Modus Ponens is taken as a rule. Above $\varphi[\alpha/\beta]$ indicates that all occurrences of α in φ are replaced by β.

With this in hand the truth conditions for formulas of $\mathcal{L}^*$ become the following. Observe that the Boolean connectives for formulas are interpreted globally, rather than locally.

$$\begin{array}{lll} M \models \alpha \doteq \beta & \text{iff} & I(\alpha) = I(\beta) \\ M \models \neg\varphi & \text{iff} & M \not\models \varphi \\ M \models \varphi \wedge \psi & \text{iff} & M \models \varphi \ \& \ M \models \psi \\ M \models P\alpha & \text{iff} & I(\alpha) \in LEG \\ M \models F\alpha & \text{iff} & I(\alpha) \in ILL \end{array}$$

The sound and complete axiomatization of DBAL is very close to that for van Benthem's MDL, modulo the additional apparatus to describe the equivalence of action types, and the fact that prohibitions are primitive, rather than obligations. See Table 2.3. One indeed recognizes FCP and its converse (ConV) in the biconditional (OR-P). Observe that the following valid condition, analogous to (WP) is expressible in this system:

2.3.3. PROPOSITION. $O\alpha \wedge P\beta \rightarrow (\beta \cap \alpha) \doteq \beta$ *is a theorem of* MDL.

Proof:
By using Incl and the definition of obligation, $O\alpha \wedge P\beta \rightarrow \overline{\alpha} \cap \beta \doteq \mathbf{0}$ is a theorem of DBAL. In addition, as the axioms for Boolean action terms, $\overline{\alpha} \cap \beta \doteq \mathbf{0} \rightarrow (\beta \cap \alpha) \doteq \beta$. Taking them together, it is clear that the result holds. □

2.3.2 Equivalence of Level-1: MDL and DBAL

I now show that a DBAL corresponds to a simple fragment of MDL. They are intertranslatable. I start with the syntactic translation, from DBAL to MDL. Let Act_0 be my given set of action generators. I first create one atomic proposition per generator. So set $Prop_{Act_0} = \{p_a : a \in Act_0\}$. Call $\mathcal{L}_{Act_0}$ the language $\mathcal{L}$ generated by the rule on page 17 using $Prop_{Act_0}$. The translation function $T : \mathcal{L}^* \rightarrow \mathcal{L}_{Act_0}$ is then defined as follows:

$$
\begin{aligned}
T(a) &:= p_a \\
T(\mathbf{1}) &:= \top \\
T(\overline{\alpha}) &:= \neg T(\alpha) \\
T(\alpha \cup \beta) &:= T(\alpha) \vee T(\beta) \\
T(\alpha \cap \beta) &:= T(\alpha) \wedge T(\beta) \\
T(\alpha \doteq \beta) &:= \Box(T(\alpha) \leftrightarrow T(\beta)) \\
T(\neg\varphi) &:= \Box\neg T(\varphi) \\
T(\varphi \to \psi) &:= \Box(T(\varphi) \to T(\psi)) \\
T(P\alpha) &:= PT(\alpha) \\
T(F\alpha) &:= O\neg T(\alpha)
\end{aligned}
$$

Now let $M = \langle E, LEG, ILL, I\rangle$ be a DBAL model. I construct a uniform MDL model $M^T = \langle W, R_D, V\rangle$ as follows. The only subtle matter lies in the construction of R_D, as Act_0 is finite but E need not be, not all subsets of E need to be definable. I use definable ones here:

- $W = E$
- For all $w \in W$, $R_D[w] = I(\alpha)$, where $M \vDash P\alpha$ and for all β that $M \vDash P\beta$ and $I(\beta) \subseteq I(\alpha)$
- $V(p_a) = I(a)$

2.3.4. PROPOSITION. *Let* $M = \langle E, LEG, ILL, I\rangle$ *be a* DBAL *model. Then* M^T *is an* MDL *model.*

Proof:
The only non-trivial condition is that for R_D. I need to check that the set $R_D[w]$ is well defined. This follows directly from the finiteness of Act_0 and the fact that if $P\alpha$ and $P\beta$ are true, then $P(\alpha \cup \beta)$ is too. □
With this in hand I can show the main result of this section, namely that the translation T is truth-preserving.

2.3.5. THEOREM. *For any formula* φ *of* $\mathcal{L}^*$, DBAL *model* M, *and state* $w \in W$ *for* M^T*:*

$$M \vDash \varphi \textit{ iff } M^T, w \vDash T(\varphi)$$

Proof:
The proof is by induction on the complexity of $\varphi \in \mathcal{L}^*$, which in turns requires sub-inductions on action types for atomic formulas. The former is straightforward. I focus on the latter cases.

There are three cases to consider. First $\varphi = \alpha \doteq \beta$, where $\alpha, \beta \in Act$. Then $M \vDash \alpha \doteq \beta$ iff $I(\alpha) = I(\beta)$ iff $V(p_\alpha) = V(p_\beta)$, by construction. But the latter happens iff $M^T, w \vDash \Box(T(\alpha) \leftrightarrow T(\beta))$. The inductive steps follow similarly, using the inductive hypothesis to go from $I(\alpha)$ and $I(\beta)$, for complex α and β, to the truth sets $||T(\alpha)||$ and $||T(\beta)||$, respectively.

Now consider the case where $\varphi = P\alpha$ for any $\alpha \in Act$. Then $M \vDash P\alpha$ iff $I(\alpha) \in LEG$. Now the key observation, which follows directly from our construction, is that the latter happens iff $V(p_a) \subseteq R_D(w)$ in M^T, for any w, and similarly for arbitrary action type α.

Finally, suppose that $\varphi = F\alpha$ for any $\alpha \in Act$. I have that $M \vDash F\alpha$ iff $I(\alpha) \in ILL$. Now I know that $I(\alpha) \cap X = \varnothing$ for all $X \in LEG$. But by my construction of M^T this means that $R_D[w] \in LEG$, so $R_D[w] \cap V(p_\alpha) = \varnothing$, so $R_D[w] \subseteq W - V(p_\alpha) = ||\neg T(\alpha)||$, so $M, w \vDash O\neg T(\alpha)$. The inductive step proceeds similarly.

□

Observe that the syntactic translation maps formulas of $\mathcal{L}^*$ into a simple fragment of MDL. First of all, the fragment without embedded deontic operators, simply by the syntactic restrictions on DBAL. No □ or ◇ occur in the scope of a deontic operator either. Furthermore, no atomic proposition occurs "free" in the translated formulas. They are always in the scope of either a universal modality, when translated from atoms $\alpha \doteq \beta$, or a deontic operator. Finally, since Act_0 is finite, this translation only uses finitely many atomic proposition. Call this simple fragment $\mathcal{L}_1$. I show now that $\mathcal{L}_1$ can be translated back into DBAL. The translation goes in two steps, first for action types and then for arbitrary formulas.

Let $\mathcal{L}_0$ be a fragment of MDL where no modal operator (□, O or P) occurs, defined over a given finite set of atomic propositions $Prop_0$. Define $Act_{Prop} = \{a_p : p \in Prop_0\}$. Then the action translation $\tau : \mathcal{L}_0 \to \mathcal{L}$ is defined as follows:

$$\begin{aligned} \tau(p) &:= a_p \\ \tau(\neg\varphi) &:= \overline{\tau(\varphi)} \\ \tau(\varphi \wedge \psi) &:= \tau(\varphi) \cap \tau(\psi) \end{aligned}$$

Now I am ready to define my translation from $\mathcal{L}_1$ to $\mathcal{L}^*$. Recall that no atomic proposition occurs "free" in $\mathcal{L}_1$. The translation $\rho : \mathcal{L}_1 \to \mathcal{L}^*$ is thus only defined for complex formula, and ultimately relies on the action-translation τ just defined.

$$\begin{aligned} \rho(\neg\varphi) &:= \neg\rho(\varphi) \\ \rho(\varphi \wedge \psi) &:= \rho(\varphi) \wedge \rho(\psi) \\ \rho(\Diamond\varphi) &:= \neg\tau(\varphi) \doteq \mathbf{0} \\ \rho(P\varphi) &:= P(\tau(\varphi)) \\ \rho(O\varphi) &:= F(\overline{\tau(\varphi)}) \end{aligned}$$

I now show how to transform MDL models into DBAL ones. The construction works locally, generating one DBAL model for each point w in the original deontic model. Of course, in the special case of uniform MDL models, where the set of normatively ideal or rational atomic actions is the same at all w, this is not necessary.

Let $M = \langle W, R_D, V \rangle$ be a MDL model. Given a point $w^* \in W$, I can construct $M^{w^*} = \langle E, LEG, ILL, I \rangle$ w.r.t. w^* as follows:

- $E = W$
- $LEG = \{X : X \subseteq R_D[w^*]\}$
- $ILL = \{X : R_D[w^*] \subseteq \overline{X}\}$
- $I(\tau(p)) = ||p||$ for each $p \in Prop_0$ in DAL-language

The resulting model is indeed a DBAL model.

2.3.6. Proposition. *The model $M^{w^*} = \langle E, LEG, ILL, I \rangle$ w.r.t. $w^* \in W$ constructed before is a* DBAL *model.*

Proof:
The verification of LEG is easy to see, so I only verify those three conditions for ILL.

1. Suppose $X \in ILL$ and $Y \subseteq X$. Then I have $R_D[w^*] \subseteq \overline{X}$ and $\overline{X} \subseteq \overline{Y}$. So $R_D[w^*] \subseteq \overline{Y}$. So $Y \in ILL$.
2. Suppose $X \in ILL$ and $Y \in ILL$. Then $R_D[w^*] \subseteq \overline{X}$ and $R_D[w^*] \subseteq \overline{Y}$. This then implies $R_D[w^*] \subseteq \overline{X} \cap \overline{Y}$. That is, $R_D[w^*] \subseteq \overline{X \cup Y}$. Thus $X \cup Y \in ILL$.
3. First I show that LEG and ILL are not empty. Otherwise, either $X \nsubseteq R_D[w^*]$ for any $X \subseteq W$, or $R_D[w^*] \nsubseteq \overline{X}$ for any $X \subseteq W$. However, none of them happen. So LEG and ILL are not empty. I will show that $LEG \cap ILL = \{\varnothing\}$.

   If not, then there is some $X \in LEG$ and $X \in ILL$ such that $X \neq \varnothing$. So $X \subseteq R_D[w^*]$ and $R_D[w^*] \subseteq \overline{X}$. It follows that $X \subseteq \overline{X}$. But that is not possible because $X \neq \varnothing$. So, $LEG \cap ILL = \{\varnothing\}$.

□

With this in hand I can now show the main result of this section:

2.3.7. Theorem. *Let $M = \langle W, R_D, V \rangle$ be a* MDL *model. Then for any $w^* \in W$, any formulas $\psi \in \mathcal{L}_0$ and $\varphi \in \mathcal{L}_1$,*

1. *$M, w \vDash \psi$ iff $w \in I(\tau(\psi))$ for every $w \in W$, i.e., $||\psi|| = I(\tau(\psi))$*
2. *$M, w^* \vDash \varphi$ iff $M^{w^*} \vDash \rho(\varphi)$*

Proof:
Statement 1 follows directly from the construction of M^{w^*} and the definition of the translation function τ. For statement 2, I only show the cases of $\Diamond\varphi, P\varphi$ and $O\varphi$. The cases for the Boolean connectives follow directly.

1. Case $\Diamond\varphi$: Suppose $M, w^* \models \Diamond\varphi$. Then there is some $w' \in W$ such that $M, w' \models \varphi$. By using 1 and the construction, $w' \in I(\tau(\varphi))$ where $w' \in E$. This means that $M^{w^*} \models \neg(\tau(\varphi) \doteq \mathbf{0})$. For the other directoin, suppose $M^{w^*} \models \neg(\tau(\varphi) \doteq \mathbf{0})$, so there is some $w' \in I(\tau(\varphi))$. By using 1, $M, w' \models \varphi$. That is $M, w \models \Diamond\varphi$.

2. Case $P\varphi$: Suppose $M, w^* \models P\varphi$. That is, $||\varphi|| \subseteq R_D[w^*]$. Then $I(\tau(\varphi)) \in LEG$ by (1) and the construction. Thus $M^{w^*} \models P(\tau(\varphi))$. For the other directoin, suppose $M^{w^*} \models P(\tau(\varphi))$. That is to say, $I(\tau(\varphi)) \in LEG$. By the construction, it implies $||\varphi|| \subseteq R_D[w^*]$. So I have $M, w^* \models P\varphi$.

3. Case $O\varphi$: Suppose $M, w^* \models O\varphi$. That is $R_D[w^*] \subseteq ||\varphi||$. Then $\overline{I(\tau(\varphi))} \in ILL$ by (1) and the construction. Namely $I(\overline{\tau(\varphi)}) \in ILL$. So I have $M, w^* \models F\overline{\tau(\varphi)}$. For the other directoin, suppose $M, w^* \models F\overline{\tau(\varphi)}$. This implies that $\overline{I(\tau(\varphi))} \in ILL$. So $R_D[w^*] \subseteq ||\varphi||$ by the first result and the construction. Thus $M, w^* \models O\varphi$.

□

Taken together, these two translation results show that DBAL can be embedded in MDL. The upshot of the comparison between MDL and OWP was that the two logics embodied different philosophical commitments regarding the structure of rational obligations and permissions. This is not the case here. The main difference between MDL and DBAL is that the latter does not allow deontic modalities to be embedded.

2.3.3 Alternative: Propositional Boolean Dynamic Logic

An alternative to Deontic Boolean Action Logic is the Propositional Boolean Dynamic Logic (PBDL) introduced by Castro and Maibaum [21], which goes further than DBAL in that it makes a distinction between propositions and action types. In doing so the language for PBDL contains both atomic propositions and action modalities. In addition, PBDL replaces the atomic construction for obligation with "weak permission." which is not contained in the three logics just presented. The dual of this weak permission is the standard notion of obligation, which here expresses necessary conditions for rationality. These two differences prevent a complete embedding of PBDL into OWP, or vice versa. I show, however, two translation results concerning interesting fragments of each of these logics.

2.3.8. DEFINITION. The language L^+ for PBDL is defined as follows:

$$\varphi := p \mid \alpha \doteq \alpha \mid \neg\varphi \mid \varphi \to \varphi \mid \langle\alpha\rangle\varphi \mid P\alpha \mid P_W\alpha$$
$$\alpha := a \mid \mathbf{0} \mid \mathbf{1} \mid \overline{\alpha} \mid \alpha \cup \alpha \mid \alpha \cap \alpha$$

where p is an element of the countable set of atomic propositions $Prop_0$, and a is an element of the finite set of action generators Act_0. I denote the set of all actions as Act.

The syntax of PBDL is richer than DBAL. Action types in PBDL are constructed like DBAL, but formulas are constructed as in PDL. The language contains two types of atomic sentence and two deontic modalities, P_W for weak and P for strong permissions. Obligation is then defined using a combination of the two types of permissions: $O\alpha := P\alpha \wedge \neg P_W\overline{\alpha}$. The second new type in the language is the standard PDL-style action modality $\langle\cdot\rangle$. $[\cdot]$ is defined in the form of $\neg\langle\cdot\rangle\neg$. Formulas of the form $(\alpha \doteq \alpha) \vee \varphi$ or $[\alpha \cup \beta](P\alpha \wedge P\beta)$ are well-formed here.

The semantics of PBDL is also richer than DBAL. The PBDL models use the standard Kripke structures to interpret action types, their deontic properties, and their dynamic effects on propositions as well. Like the syntactic difference between propositions and actions, the models for PBDL are two-sorted: possible worlds for interpreting propositions, and possible events for interpreting action types. The model contains one Kripke relation R over worlds for giving the dynamic functions, and one Kripke relation R_P connecting worlds and events for evaluating the deontic properties. PBDL models also have to contain two interpretations, one $||\cdot||$ for primitive propositions, and one I for primitive actions.

2.3.9. DEFINITION. A PBDL model is a tuple $M = \langle W, E, R, I, R_P, ||\cdot||\rangle$ where

- W is a non-empty set of possible worlds
- E is a non-empty set of possible events
- $R \subseteq W \times W \times E$ is a functional relation requiring that: if $R(w,u,e)$ and $R(w,v,e)$ then $u = v$
- $I : Act_0 \to \wp(E)$ is an interpretation function on actions such that $I(a) \subseteq E$ for each $a \in Act_0$, which satisfies the following three conditions:

  (I.1) For each $a \in Act_0$, $|I(a) - \bigcup\{I(b) : b \in (Act_0 - \{a\})\}| \leq 1$

  (I.2) For each $e \in E$, if $e \in I(a) \cap I(b)$ such that $a \neq b \in Act_0$, then

  $$\bigcap\{I(a) : a \in Act_0 \wedge e \in I(a)\} = \{e\}$$

  (I.3) $E = \bigcup_{a \in Act_0} I(a)$
- $R_P \subseteq W \times E$ is a relation between worlds and events

- $||\cdot|| : Prop_0 \to \wp(W)$ is an interpretation function such that $||p|| \subseteq W$ for each $p \in Prop_0$

Since R is functional, we write $R_e(w) = u$ for $R(w, u, e)$. The extension of the interpretation function I on actions from the set of actions to the set of all possible events is defined in the same way as in DBAL. The conditions I.1 to I.3 are there to ensure that the interpretation of action types is well constructed. I leave out the details here. See [21] for a discussion.

The truth conditions for formulas of L^+ are defined in a standard method for Kripke models. Observe that the Boolean connectives and the equivalence between action types are interpreted locally as usual, while the dynamic sentences, strong permission and weak permission are interpreted in the following way:

$M, w \vDash \langle\alpha\rangle\varphi$	iff	$\exists u \in W \exists e \in I(\alpha)$ s.t. $R_e(w) = u$ & $M, u \vDash \varphi$
$M, w \vDash P\alpha$	iff	$\forall e \in I(\alpha)$ s.t. $R_P we$
$M, w \vDash P_W\alpha$	iff	$\exists e \in I(\alpha)$ s.t. $R_P we$

Strong permission and weak permission together capture rationality. They are interpreted in a universal/existential character by R_P, in which strong permission plays as the sufficient condition for rationality, while the dual of weak permission as the necessary condition for rationality. See Table 2.4 for the axiom system for PBDL. Strong permission satisfies FCP and its converse ConV by one biconditional axiom. Observe that a validity similar to WP is contained in this system:

2.3.10. PROPOSITION. $O\alpha \wedge P\beta \wedge \neg(\beta \doteq \mathbf{0}) \to (\beta \cap \alpha) \doteq \beta$ *is a theorem of* PBDL.

Proof:
By using OSW, I have $O\alpha \to \neg P_W\overline{\alpha}$. By using **0**-P/W, I have $P\beta \wedge \neg(\beta \doteq \mathbf{0}) \to P_W\beta$. Taking these two results with CW, I infer (1): $O\alpha \wedge P\beta \wedge \neg(\beta \doteq \mathbf{0}) \to \neg P_W(\overline{\alpha} \cap \beta)$. One the other hand, from ConV, it has (2): $P\beta \to P(\overline{\alpha} \cap \beta)$. Taking (1) and (2) together with **0**-P/W, it concludes $O\alpha \wedge P\beta \wedge \neg(\beta \doteq \mathbf{0}) \to (\beta \cap \alpha) \doteq \beta$. □

2.3.4 Equivalence of Level-0: PBDL and OWP

Now I provide a correspondence from PBDL and OWP, and this result shows that PBDL does partially overlap OWP. In other words, a fragment of the language for PBDL is a translation of a fragment for OWP, and vice versa. The corresponding construction is similar to the method introduced in [77].

I start with a syntactic translation from a fragment of PBDL to a fragment of OWP in two steps. First I define a fragment of PBDL in which no dynamic sentences and weak permission occur. Let L^- be the fragment of language for PBDL, which involves formulas containing atomic propositions, action generators, and the formulas constructed with Boolean connectives, dynamic operator, and strong permission. To

Basic axioms and rule:		The other axioms:
(OR-P)	$P(\alpha \cup \beta) \leftrightarrow P\alpha \wedge P\beta$	$[\mathbf{0}]\varphi$
(ConV)	$P\alpha \vee P\beta \rightarrow P(\alpha \cap \beta)$	$[\alpha \cup \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$
(CW)	$P_W(\alpha \cap \beta) \rightarrow P_W\alpha \wedge P_W\beta$	$\alpha \doteq \beta \rightarrow [\gamma](\alpha \doteq \beta)$
(**0**-P/W)	$P\alpha \wedge \neg P_W\alpha \rightarrow \alpha \doteq \mathbf{0}$	$\langle\beta\rangle(\alpha \doteq \alpha') \rightarrow \alpha \doteq \alpha'$
(OSW)	$O\alpha \leftrightarrow P\alpha \wedge \neg P_W(\overline{\alpha})$	$\langle\gamma\rangle\varphi \rightarrow [\gamma]\varphi$
(UN)	$(a_1 \cup \cdots \cup a_n) \doteq \mathbf{1}$	$P\mathbf{0}$
(BA)	$\varphi \wedge \alpha \doteq \beta \rightarrow \varphi[\alpha/\beta]$	$\neg P_W\mathbf{0}$
(K)	$[\alpha](\varphi \rightarrow \psi) \rightarrow ([\alpha]\varphi \rightarrow [\alpha]\psi)$	$P_W(\alpha \cup \beta) \leftrightarrow P_W\alpha \vee P_W\beta$
(NEC)	$\dfrac{\varphi}{[\alpha]\varphi}$	$P_W\gamma \rightarrow P\gamma$

Table 2.4: The sound and complete axiom system for Castro and Maibaum's Propositional Boolean Dynamic Logic. All propositional tautologies, Modus Ponens, and the axioms for Boolean Algebras for action terms are also taken into account. Above $\gamma \in Act_0$, and $\varphi[\alpha/\beta]$ indicates that all occurrences of α in φ are replaced by β.

construct the translation fragment of PBDL, I allow for dynamic operators and permissions to scope over formulas of L^-. I simply call this fragment $L^\times$. Let L be the language for OWP. The translation proceeds in two steps. Define $\sigma : L^- \rightarrow L$ be a translation as follows:

$$\begin{aligned}
\sigma(a) &\in Prop_0, \text{ for each } a \in Act_0\\
\sigma(\mathbf{0}) &:= \bot\\
\sigma(\mathbf{1}) &:= \top\\
\sigma(\overline{\alpha}) &:= \neg\sigma(\alpha)\\
\sigma(\alpha \cup \beta) &:= \sigma(\alpha) \vee \sigma(\beta)\\
\sigma(\alpha \cap \beta) &:= \sigma(\alpha) \wedge \sigma(\beta)\\
\sigma(p) &\in Prop_0, \text{ for each } p \in Prop_0\\
\sigma(\alpha \doteq \beta) &:= \Box(\sigma(\alpha) \leftrightarrow \sigma(\beta))\\
\sigma(\neg\varphi) &:= \neg\sigma(\varphi)\\
\sigma(\varphi \wedge \psi) &:= \sigma(\varphi) \wedge \sigma(\psi)\\
\sigma(P\alpha) &:= P\sigma(\alpha)
\end{aligned}$$

And then I can define $\delta : L^\times \rightarrow L$ be a translation, for each of the formulas $\alpha \in L^-$, $\psi \in L^-$, and $\varphi \in L^\times$,

$$\begin{aligned}
\delta(\psi) &:= \sigma(\psi)\\
\delta(\langle\alpha\rangle\varphi) &:= \Diamond(\sigma(\alpha) \wedge \Diamond\delta(\varphi))\\
\delta(\neg P_W\overline{\alpha}) &:= \bigwedge_{\beta \in Act} [P\sigma(\beta) \rightarrow \Box(\sigma(\beta) \rightarrow \sigma(\alpha))]
\end{aligned}$$

The resulting $\delta(\neg P_W\overline{\alpha})$ embodies WRO. So α is a type of strategy required to be played, iff all rationally permissible strategies are α-type strategies.

Let $M = \langle W, E, R, I, R_P, ||\cdot||\rangle$ be a PBDL model. I construct a model $M^* = \langle H, n_P, n_O, ||\cdot||^*\rangle$ as follows:

- $H = W \cup E$.
- $n_P(h) = \begin{cases} \{X \subseteq E : \forall e \in X.\ R_P he\} & \text{if } h \in W \\ \varnothing & \text{if } h \in E \end{cases}$
- $n_O = \varnothing$
- $||\sigma(a)||^* = I(a)$
- $||\sigma(p)||^* = ||p||$

I denote M^* as the unique model generated from a PBDL model M. It should be noticed that M^* is an OWP model.

2.3.11. PROPOSITION. *Given a* PBDL *model* $M = \langle W, E, R, I, R_P, ||\cdot||\rangle$, M^* *is an* OWP *model.*

Proof:
The condition for n_P is easy to verify by the construction. As n_O is empty, the three conditions required for OWP models are automatically satisfied. □

I now can show one of the main results in this section.

2.3.12. THEOREM. *Given a* PBDL *model* $M = \langle W, E, R, I, R_P, ||\cdot||\rangle$. *Then, for each* $\alpha, \varphi \in L^-$, *and* $\psi \in L^\times$,

$$\begin{array}{lcl} e \in I(\alpha) & \textit{if and only if} & e \in ||\sigma(\alpha)||^* \\ M, w \models \varphi & \textit{if and only if} & M^*, w \models \sigma(\varphi) \\ M, w \models \psi & \textit{only if} & M^*, w \models \delta(\psi) \end{array}$$

Proof:
It is not difficult to prove $e \in ||\sigma(\alpha)||^*$ iff $e \in I(\alpha)$. Now I am going to prove that $M, w \models \varphi$ iff $M^*, w \models \sigma(\varphi)$ for each $\varphi \in L^-$. I do so by induction on the complexity of $\varphi \in L^-$.

1. The cases of atomic propositions, negation, implication are easy to verify by the construction.
2. The case of $\alpha \doteq \beta$:

$$\begin{aligned} M, w \models \alpha \doteq \beta \text{ iff } & I(\alpha) = I(\beta) \\ \text{iff } & e \in ||\sigma(\alpha)||^* \Leftrightarrow e \in ||\sigma(\beta)||^* \\ \text{iff } & M^*, w \models \Box(\sigma(\alpha) \leftrightarrow \sigma(\beta)) \end{aligned}$$

3. The case of $P\alpha$: According to the aforementioned construction, then

$$\begin{aligned} M, w \vDash P\alpha & \text{ iff } \forall e \in I(\alpha).\ R_P we \\ & \text{ iff } ||\sigma(\alpha)||^* \in n_P(w) \end{aligned}$$

Next I will verify that $M, w \vDash \psi \Rightarrow M^*, w \vDash \delta(\psi)$ for each $\psi \in L^\times$. I show this by induction on the complexity of $\varphi \in L^\times$.

1. The case of $\langle\alpha\rangle\varphi$:

$$\begin{aligned} M, w \vDash \langle\alpha\rangle\varphi & \text{ iff } \exists(w, u, e) \in R \text{ and } e \in I(\alpha) \text{ s.t. } M, u \vDash \varphi \\ & \text{ only if } \exists w, e, u \in W \text{ s.t. } e \in ||\sigma(\alpha)||^* \text{ and } u \in ||\delta(\varphi)||^* \\ & \text{ iff } M^*, w \vDash \Diamond(\sigma(\alpha) \wedge \Diamond\delta(\varphi)) \end{aligned}$$

2. The case of $\neg P_W\alpha$:

$$\begin{aligned} M, w \vDash \neg P_W\alpha & \text{ iff } \neg\exists e \in I(\alpha) \text{ s.t. } R_P we \\ & \text{ iff } \forall e \in E.\ (R_P we \Rightarrow e \in I(\overline{\alpha})) \\ & \text{ only if } \forall\beta \in Act. \forall e \in E.\ ((e \in I(\beta) \Rightarrow R_P we) \\ & \qquad\qquad \Rightarrow (e \in I(\beta) \Rightarrow e \in I(\overline{\alpha}))) \\ & \text{ iff } \forall\beta \in Act.[||\sigma(\beta)||^* \in n_P(w) \Rightarrow ||\sigma(\beta)||^* \subseteq ||\sigma(\overline{\alpha})||^*] \\ & \text{ iff } M^*, w \vDash P\sigma(\beta) \rightarrow \Box(\sigma(\beta) \rightarrow \sigma(\overline{\alpha})), \text{ for all } \beta \in Act \\ & \text{ iff } M^*, w \vDash \bigwedge_{\beta \in Act} [P\sigma(\beta) \rightarrow \Box(\sigma(\beta) \rightarrow \sigma(\overline{\alpha}))] \end{aligned}$$

□

Now I can start with the second contribution in this section, which is a correspondence result from a fragment of OWP to a fragment of PBDL. Recall that OWP and MDL share the common language. So given the language L for PBDL, similarly, I simply call the fragment of OWP where no modal operator (□, O or P) occurs L_0, and the fragment, the so-called L_1 of OWP, is defined within the classical conjunction, negation, and modal operators based on L_0. Again, since Act_0 is finite, my translation only considers finitely many atomic propositions. I show that L_1 can be translated back to PBDL in a non-trivial way. Similarly, the translation runs in two steps.

Let L_0 be a fragment of OWP in which neither □, O, nor P occur. Define a function $\mu : L_0 \rightarrow L^+$ as an action-translation as follows:

$$\begin{aligned} \mu(p) & \in Act_0 \text{ for each } p \in Prop_0 \\ \mu(\neg\varphi) & := \overline{\mu(\varphi)} \\ \mu(\varphi \wedge \psi) & := \mu(\varphi) \cap \mu(\psi) \end{aligned}$$

I can then define a translation from L_1 to L^+. The translation $\epsilon : L_1 \to L^+$ is defined for complex formulas, which relies on the action-translation μ defined above.

$$\begin{aligned}
\epsilon(p) &\in Prop_0 \text{ for each } p \in Prop_0\\
\epsilon(\neg\varphi) &:= \neg\epsilon(\varphi)\\
\epsilon(\varphi \wedge \psi) &:= \epsilon(\varphi) \wedge \epsilon(\psi)\\
\epsilon(\Diamond\varphi) &:= \langle\mu(\varphi)\rangle\top\\
\epsilon(P\varphi) &:= P\mu(\varphi)\\
\epsilon(O\varphi) &:= P\mu(\varphi) \wedge \neg P_W\mu(\neg\varphi)
\end{aligned}$$

Notice that obligation in OWP can be translated by a combination of strong permission and weak permission in PBDL via the translation ϵ.

The transformation from the OWP models to PBDL models need certain restrictions. Not every OWP model can be transformed into a PBDL model. To do so, the transformation needs to satisfy two conditions. The first is the valuation of atomic propositions in the transformed models should satisfy the required conditions (I.1)-(I.3). Second, the domain of the transformed model should be restricted to the valuation of atomic propositions in OWP.

To satisfy the first restriction, I define the following "functional OWP models." An OWP model $M_{\text{OWP}} = \langle H, n_P, n_O, ||\cdot||\rangle$ is functional, if and only if it satisfies the following three conditions:

- For each $p \in Prop_0$, $|||p|| - \bigcup\{||q|| : q \in (Prop_0 - \{p\})\}| \leq 1$
- For each $w \in W$, if $w \in ||p|| \cap ||q||$ such that $p \neq q \in Prop_0$, then

$$\bigcap\{||p|| : p \in Prop_0 \text{ and } w \in ||p||\} = \{w\}$$

- $W = \bigcup_{p\in Prop_0} ||p||$

The three conditions correspond to the conditions (I.1)-(I.3) in PBDL models.

I now can construct a model transformed from a functional OWP model. Let $M_{\text{OWP}} = \langle H, n_P, n_O, ||\cdot||\rangle$ be a functional OWP model, I then can construct a model $M^* = \langle W, E, R, I, R_P, ||\cdot||^*\rangle$ as follows:

- $W = E = H$
- $R = \{(x, y, y) \mid (x, y) \in Alt\}$
- $R_P = \{(x, y) \mid \exists X \in n_P(x) \text{ s.t. } y \in X\}$
- $I(\mu(p)) = ||p||$
- $||\epsilon(p)||^* = ||p||$ for each $p \in Prop_0$ in OWP-language

I denote M^* as the unique model generated from a functional OWP model M_{OWP}.

2.3.13. PROPOSITION. *Given a functional* OWP *model* $M_{\text{OWP}} = \langle H, n_P, n_O, ||\cdot||\rangle$, *then the model* M^* *defined above is a* PBDL *model.*

Proof:
I only need to check whether R is functional. Because M_{OWP} is functional, it is thus easy to check that R in M^* is also functional. □
With this result in hand, I now can turn to the final step of my transformation.

2.3.14. THEOREM. *Given a functional* OWP *model* $M = \langle H, n_P, n_O, ||\cdot||\rangle$, *I construct the model* $M^* = \langle W, E, R, I, R_P, ||\cdot||^*\rangle$ *generated from* M. *Then, given any OWP sentence* φ *and* $h \in \bigcup_{p \in Propo_0} ||p||$,

1. *If* $\varphi \in L_0$, *then* $M, h \vDash \varphi$ *iff* $h \in I(\mu(\varphi))$
2. *If* $\varphi \in L_1$, *then* $M, h \vDash \varphi$ *iff* $M^*, h \vDash \epsilon(\varphi)$

Proof:

1. Suppose $\varphi \in L_0$. By induction on the complexity of φ.

   (a) For the case of $p \in Prop_0$, then

   $$M, h \vDash p \text{ iff } h \in ||p|| \text{ iff } h \in I(\mu(p))$$

   (b) For the case of $\varphi = \psi \wedge \varphi \in L_0$, then also $\psi, \varphi \in L_0$. Now I have

   $$\begin{aligned} M, h \vDash \psi \wedge \varphi &\text{ iff } M, h \vDash \psi \text{ and } M, h \vDash \varphi \\ &\text{ iff } h \in I(\mu(\psi)) \text{ and } h \in I(\mu(\varphi)) \\ &\text{ iff } h \in I(\mu(\psi)) \cap I(\mu(\varphi)) \\ &\text{ iff } h \in I(\mu(\psi) \cap \mu(\varphi)) \\ &\text{ iff } h \in I(\mu(\psi \wedge \varphi)) \end{aligned}$$

   (c) For the case of $\neg\varphi \in L_0$, then also $\varphi \in L_0$. We can see that

   $$\begin{aligned} M, h \vDash \neg\varphi &\text{ iff } M, h \nvDash \varphi \\ &\text{ iff } h \notin I(\mu(\varphi)) \\ &\text{ iff } h \in I(\overline{\mu(\varphi)}) \\ &\text{ iff } h \in I(\mu(\neg\varphi)) \end{aligned}$$

2. Suppose $\varphi \in L_1$. By induction on the complexity of φ.

(a) For the case of $p \in Prop_0$, then

$$M, h \vDash p \text{ iff } h \in ||p|| \text{ iff } h \in ||\epsilon(p)||^* \text{ iff } M^*, h \vDash \epsilon(p)$$

(b) For the case of $\psi \wedge \varphi \in L_1$, then also $\psi, \varphi \in L_1$. I can then infer

$$\begin{aligned} M, h \vDash \psi \wedge \varphi &\text{ iff } M, h \vDash \psi \text{ and } M, h \vDash \varphi \\ &\text{ iff } M^*, h \vDash \epsilon(\psi) \text{ and } M^*, h \vDash \epsilon(\varphi) \\ &\text{ iff } M^*, h \vDash \epsilon(\psi) \wedge \epsilon(\varphi) \\ &\text{ iff } M^*, h \vDash \epsilon(\psi \wedge \varphi) \end{aligned}$$

(c) For the case of $\neg\psi \in L_1$, then also $\psi \in L_1$. So I infer

$$\begin{aligned} M, h \vDash \neg\psi &\text{ iff } M, h \nvDash \psi \\ &\text{ iff } M^*, h \nvDash \epsilon(\psi) \\ &\text{ iff } M^*, h \vDash \epsilon(\neg\psi) \end{aligned}$$

(d) For the case of $\Diamond\psi \in L_1$, then also $\psi \in L_0$ by the definition. So

$$\begin{aligned} M, h \vDash \Diamond\psi &\text{ iff } \exists h' \in H \text{ s.t. } Alt(h, h') \text{ and } M, h' \vDash \psi \\ &\text{ iff } \exists h' \in I(\mu(\psi)) \text{ s.t. } R_{h'}(h) = h' \text{ and } M^*, h' \vDash \top \\ &\text{ iff } M^*, h \vDash \langle \mu(\psi) \rangle \top \\ &\text{ iff } M^*, h \vDash \epsilon(\Diamond\psi) \end{aligned}$$

(e) Similar to the case of $P\psi \in L_1$, then $\psi \in L_0$. So

$$\begin{aligned} M, h \vDash P\psi &\text{ iff } ||\psi|| \in n_P(h) \\ &\text{ iff } \forall h' \in ||\psi|| \text{ that } R_P h h' \qquad \text{by the Construction of } R_P \\ &\text{ iff } \forall h' \in I(\mu(\psi)) \text{ that } R_P h h' \\ &\text{ iff } M^*, h \vDash P\mu(\psi) \\ &\text{ iff } M^*, h \vDash \epsilon(P\psi) \end{aligned}$$

(f) For the case of $O\psi \in L_1$, then I infer $\psi \in L_0$. So

$$
\begin{aligned}
M, h \vDash O\psi \text{ iff } & ||\psi|| \in n_O(h) \\
\text{iff } & ||\psi|| = \bigcup n_P(h), \text{ by } n_O(h) \neq \varnothing \\
\text{iff } & ||\psi|| \subseteq \bigcup n_P(h) \text{ and } \bigcup n_P(h) \subseteq ||\psi|| \\
\text{iff } & ||\psi|| \subseteq \bigcup n_P(h) \in n_P(h) \text{ and } \bigcup n_P(h) \subseteq ||\psi|| \\
& \quad \text{by the condition for } n_P \\
\text{iff } & \forall h' \in I(\mu(\psi)).R_P hh', \text{ and} \\
& \quad \forall h' \in \bigcup n_P(h).[h' \in I(\mu(\psi))] \\
\text{iff } & \forall h' \in I(\mu(\psi)).R_P hh', \text{ and} \\
& \quad \forall h'.[R_P(h, h') \Rightarrow h' \in I(\mu(\psi)] \\
\text{iff } & M^*, h \vDash P\mu(\psi) \text{ and } M^*, h \vDash \neg P_W \overline{\mu(\psi)} \\
\text{iff } & M^*, h \vDash P\mu(\psi) \wedge \neg P_W \mu(\neg\psi) \\
\text{iff } & M^*, h \vDash \epsilon(O\psi)
\end{aligned}
$$

□

As these two translation results show, PBDL and OWP are partially overlap. One of their differences is similar to that between DBAL and MDL: the former does not allow deontic modalities be iterated. The second difference is that the language in PBDL is more fine-grained than that in OWP. Because the effect of the dynamic sentences and weak permissions, only part of the language of PBDL can be translated into the language of OWP.

2.4 Conclusion

The goal of this chapter was to provide an explicit comparison between four related logics which, I argued, are well-suited to studying rational recommendations in games: van Benthem's "Minimal Deontic Logic", Anglberger et al.'s "Obligation as Weakest Permission", Trypuz and Kulicki's "Deontic Boolean Action Logic," and Castro and Maibaum's "Propositional Boolean Dynamic Logic." All four systems can be seen as endorsing the idea that obligations and permissions in games provide necessary and sufficient conditions for rationality. The scope of O and P in the first two logics are propositions, and then modalities can be iterated; while the latter two distinguish propositions and action types, and no deontic modalities can be iterated. I have argued that the first two differ in their view of the relation between obligations and permissions: in OWP the former imply the latter, but not in MDL. As a result of this, I argued, MDL stays closer to the core philosophical principles for rational recommendations in games, while OWP stays closer to the intuitive idea that an action type cannot be obligatory while at the same time not being permitted. I then showed that DBAL can be embedded in MDL, and PBAL be partially embedded in OWP. The upshot of the first translation result is that even though the former two logics draw a sharper syntactic distinction between propositional connectives and constructors for complex action types, this distinction is blurred again at the semantic level. For instance, DBAL can be seen as a syntactically restricted MDL, but this need not be conceptually implausible. Given the action-theoretic interpretation I used for the semantics of MDL, formulas of the form $OP\varphi$ means that the “action type” $P\varphi$ is obligatory . I leave it to the reader to decide whether $P\varphi$ can plausibly support this action-theoretic reading.

The comparisons in this chapter have shown that in the context of rational recommendations in games, it is natural to assume that permissions satisfy FCP. This is so because they are viewed as providing sufficient conditions for rationality. Of course the resulting logics remain prone to the free choice permission paradox discussed in the introduction. This is due to the fact that the underlying logic of actions is classical in all the systems that I studied in this chapter. I address this issue in the next chapter, where I study the logics where the action-theoretic notions are *not* classical.

Chapter 3

Free Choice Permission in Open Reading

A Perspective in Substructural Logics

This chapter proposes a new solution to the Free Choice Permission Paradox [11, 43, 122], combining ideas from substructural logics and non-monotonic reasoning. Recall that free choice permission is intuitively understood as "if it is permitted to do α or β then it is permitted to do α and it is permitted to do β." This is usually formalized as follows:

$$P(\alpha \vee \beta) \to P\alpha \wedge P\beta, \tag{FCP}$$

where $\to$ represents the material conditional "if ... then ...." There are many well-known problems associated with FCP. I have already alluded to one in the Introduction. In this chapter I focus on three of them. First, in many deontic systems adding FCP allows for a form of conjunctive inference which seems clearly unacceptable: if it is permitted to order a vegetarian lunch then it is permitted to order a lunch and not pay for it. This is the so called "vegetarian free lunch" example [43]. Second, many deontic logics become resource-insensitive in the presence of FCP. They validate inferences of the form "if the patient with stomach trouble is allowed to eat one cookie then he is allowed to eat more than one," which are also counter-intuitive. Third, in its classical form FCP entails that the classically equivalent formulas can be substituted to the scope of a permission operator. This is also implausible: It is permitted to eat an apple or not iff it is permitted to sell a house or not. The challenge for a logic of free choice permission is to exclude such counter-intuitive consequences while not giving up too much deductive power. In other words, I need a suitable non-classical calculus for free choice permission. I suggest one way of doing this is using a family of substructural logics augmented with a principle borrowed from non-monotonic reasoning.

The solution that I put forward in this chapter is to build a family of logics including FCP on top of a plausible calculus for action types. This calculus must, in my view, be at the same time relevant, non-monotonic and resource-sensitive. So the present chapter can be seen as a follow-up to a suggestion made in [6]. It is structured as follows. I first clarify various background concepts related to free choice permission, and apply the semantic strategy suggested in [6] to analyze those three problems. In the next two sections I emphasize that one of the main goals of this chapter is to balance the counter-intuitive consequences of free choice permission with the deductive power of its logic. My solution is to augment substructural logics with a principle from non-monotonic reasoning, as a plausible characterization of a conditional over action types. I argue that this preserves enough deductive power while keeping counter-intuitive FCP inferences at bay. And these logics are presented in the fourth section, where it is shown that they are sound and complete, and the basic logic and one of its extensions satisfy cut elimination.

3.1 Background Concepts

This section will circumscribe the understanding of FCP that I will study in this chapter by correlating it with a number of different views of permissions: strong permission, weak permission, explicit permission/implicit permission, the "open reading." Furthermore, I propose an intuitive reading of conditionals on actions, which is proved to be suitable for analyzing FCP in the next section.

3.1.1 Strong Permission

Here I review some of the notions already presented in the Introduction. Free choice permission is often argued as going hand in hand with the idea of strong permission [117, 8]. The notion of strong permission goes back at least to von Wright who stated that an action is strongly permitted "if the authority has considered its normative status and decided to permit it [117]." [1] Recall that in contrast to strong permission, weak permission is defined as the absence of prohibition, i.e. the dual of obligation in standard deontic logic [73, 72]. Due to these presence/absence features, von Wright roughly classified strong permission as explicit permission and weak permission as tacit permission [119]:

> *I think we are well advised to distinguish between things being permitted in the weak sense of simply not being forbidden and things being permitted in some stronger sense. Exactly in what this stronger sense 'consists' may be difficult to tell. That which is in the strong sense permitted is, somehow, expressly permitted, subject to norm and not just void of deontic status altogether.*

[1] Subsequent literature has not always used "strong permissions" in the same sense as von Wright. Asher and Bovenac [8] use the term in a way which is closer to what I call the "open reading" in Section 1.2. In [71] two different senses of explicit permissions, static and dynamic, are distinguished and studied. In this chapter I will use "strong permission" in von Wright's sense, unless otherwise specified.

It has been defended that strong permission satisfies free choice whereas weak permission does not [118, 26]. More precisely, it is strongly permitted to do α or β, iff it is strongly permitted to do α *and* it is strongly permitted to do β; while it is weakly permitted to do α or β, iff it is weakly permitted to do α *or* it is weakly permitted to do β. In [118, p.31], von Wright claimed that strong permission is closely associated with free choice permission. In this chapter I do not take a stand on whether strong permission as illustrated above is free choice permission or the other way around. I do observe, however, that the permissions that I study here share a number of important features with strong permissions: they are explicit, they are not the dual of obligations, and they of course satisfy a restricted form of FCP.

3.1.2 Open Reading

The so-called open reading of permission is defined as follows:

An action type α is permitted iff each instantiation of action type α is normatively okay. (OR)

The open reading provides a "sufficiency" reading [112]: instantiating a permitted action type is *sufficient* for being okay [101, 100, 58, 7], which I have already studied in detail in Chapter 2. The open reading thus becomes: if instantiating action type α is instantiating action type β, I conclude that if it is permitted to do β then it is also permitted to do α.

$$\alpha \multimap \beta \vdash P\beta \rightarrow P\alpha \qquad (\text{OR}^+)$$

Here $\vdash$ is a logical consequence relation about deontic propositions and $\multimap$ a conditional over action types. A classical reading of $\alpha \multimap \beta$ is "an instantiation of action type α is an instantiation of action type β." In addition, $P\beta \rightarrow P\alpha$ is read as "If type β is permitted then so is type α." The representation OR^+ captures the idea of OR: whether α is a permitted action type, depends on whether action type α is *sufficient* for action type β, and whether type β is permitted. In Section 3.2.1 I will argue that the conditional $\multimap$ should be non-monotonic, resource sensitive, and relevant.

FCP follows from OR in a classical reading of the conditional $\multimap$ [6, 7, 27]. Given that an instantiating type α is an instantiation of this disjunctive type (α or β), if type (α or β) is permitted then type α is also permitted. The idea of open reading goes back to open interpretation in dynamic deontic logic [18] and disjunctive permissions in conditional logic [46, 67]. The open reading underlies the analysis of rational permission in games in Chapter 2.

One of the driving ideas of this chapter is to take a controlled version of OR, in the form of OR^+, as the core of free choice permission. In this controlled version, the conditional $\multimap$ on actions should not be classical. So is the consequence relation $\vdash$ for permission.

3.1.3 The "Licensed Instance" Relation on Actions

The key to my solution of the free choice permission paradox is to interpret the $\multimap$ connective used above in such a way that its logical behavior becomes non-classical. Recall that this connective takes two action types as arguments. I view it as stating a particular relation between those types: "being a licensed instance of." More precisely, I say that α is a licensed instance of β in state s just in case:

(b1) executing α in s is an instance of β, and

(b2) the execution of α is otherwise licensed in s.

For α to be a licensed instance of β the first must of course constitute a way of doing the second. So this relation between action types can be seen as a special case of the "by means of" relation introduced by Goldman [36] and later applied in legal theory by Lindahl [63]. Since the "by means of" relation is already non-classical, this means that my connective $\multimap$ will be as well. But not all instances are licensed. Ordering a lunch and not paying for it is not a licensed instance of ordering a lunch.

The relation "being a licensed instance of" is non-monotonic, resource sensitive and relevant. Non-monotonicity and relevance are direct consequences of (b2). If α is licensed it doesn't mean that α together with β will be. My cookie example above shows that this relation is also resource sensitive. Eating one cookie might be licensed, while eating more might not be.

On the positive side, this relation seems to satisfy a property that I will call "rational monotony" below. It states that if α is a licensed instance of β, and that it is not the case that something else than γ is a licensed instance of β, than doing α together with γ is a licensed instance of β. I will argue below that the close cousin called "cautious monotony" is, however, not acceptable for the "being a licensed instance of" relation.

3.2 The Basic Inferences of Free Choice Permission

In this section I will differentiate different types of inferences on FCP (i.e. disjunctive and conjunctive) by adopting the so-called open reading as the semantic core.

3.2.1 Disjunctive and Conjunctive Free Choice Inferences

Using OR$^+$, the three types of undesired counter-intuitive consequences of free choice permission can be illustrated as special cases of two general patterns of free choice permission inferences: disjunctive and conjunctive free choice inferences.

Disjunctive inferences are the canonical forms of inferences using FCP. The premise in a disjunctive inference contains "or" inside the scope of permission. Here is one instance of the typical disjunctive inferences:

$\because$ (1a) It is permitted to eat an apple *or* eat a pear.
$\therefore$ (1b) It is permitted to eat an apple, and (1c) it is permitted to eat a pear.

This instance of disjunctive FCP inference is that from the premise (1a) I can infer the consequents (1b) and (1c). FCP will be valid if the conditionals $\alpha \multimap (\alpha \vee \beta)$ and $\beta \multimap (\alpha \vee \beta)$ are valid.

OR^+ leads to several problems if $\multimap$ is classical. The first one I notice is that it entails unrestricted substitutions of classically equivalent formulas [122]. An instance derived from OR^+ is the following:

$$\alpha \multimapboth \beta \vdash P\beta \leftrightarrow P\alpha \tag{P-E}$$

where $\alpha \multimapboth \beta$ means that $\alpha \multimap \beta$ and $\beta \multimap \alpha$, and $\alpha \leftrightarrow \beta$ means that $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$. This principle P-E is a form of the substitution principle. If $\multimapboth$ is strong enough to make classical tautologies equivalent, then "eating an apple or not is permitted iff selling a house or not is permitted." Yet these permissions seem different. The permission to sell a house or not involves a different authority than the permission to eat an apple or not. Eating an apple or not is, in my view, *irrelevant* to selling a house [2]. In my view a plausible calculus of actions should give up the substitution of classically equivalent formulas.

By conjunctive inference I mean the inference of the following form:

∵ (2a) It is permitted to order a vegetarian lunch.
∴ (2b) It is permitted to order a vegetarian lunch *and* not pay for it.

whose abstract form is shown as follows:

$$P\alpha \rightarrow P(\alpha \wedge \beta), \tag{CI}$$

where β is an arbitrary action type. CI is valid under OR^+, if $(\alpha \wedge \beta) \multimap \alpha$ is valid. Yet the classical "vegetarian free lunch" example is one instance of CI. Moreover, CI is logically equivalent to FCP under OR^+ if $\multimap$ is classical [3]. This is not the case that in the logic that I develop below. Otherwise CI will bring at least two "abnormal" cases into the permissions. The first case is the unrestricted "vegetarian free lunch" example just mentioned. The implied permission (2b) is not a *normal* case to the explicit permitted type (2a), because the composition of ordering a lunch and not paying for this order is not a *licensed* instance of ordering a lunch. The *monotonic* $(\alpha \wedge \beta) \multimap \alpha$ should be restricted by the notion of *licensed instance*.

CI also encounters problems from *resource-sensitivity*. If eating one cookie is permitted, for a patient with stomach trouble, it is unfortunate to imply that eating more than one is also permitted. In this case, the classical conditional $(\alpha \wedge \cdots \wedge \alpha) \multimap \alpha$ indicates an unrestricted resource composition. The above two conjunctive cases are

[2] I take "irrelevance" in a different sense, but still very close to relevant logic [88]: only $\alpha \multimap (\beta \vee \sim \beta)$ is not valid in my logics. My logics validate $(\alpha \circ \sim \alpha) \multimap \beta$, which is rejected in relevant logic. In this chapter $\sim$ and $\neg$ are different negations, and $\sim$ will be seen as the negation for actions.

[3] By taking $\alpha \multimapboth ((\alpha \wedge \beta) \vee (\alpha \wedge \sim \beta))$ and $\alpha \multimapboth ((\alpha \vee \beta) \wedge (\alpha \vee \sim \beta))$ as classical validities, the logical equivalence between CI and FCP holds after using the substitution of logical equivalences.

brought up by the unrestrained conjunctions appearing in the conditional over action types, in which the conjunctions lead to *resource-insensitive* action compositions and so *abnormal* cases of permission.

My solution to the aforementioned problems related to FCP and CI is a semantic one. I insist on OR as the semantic core of permissions, and propose a non-classical interpretation for the conditional $\multimap$ as well as for OR^+, in which the conditional is required to be *relevant, non-monotonic*, and *resource sensitive*. This interpretation provides a mechanism to select *licensed instances* on action types, especially action compositions.

3.2.2 Goal: Balance between Cautiousness and Deductive Power

Unrestricted classical conditional on actions is thus problematic in the presence of FCP. To avoid this I follow the solution proposed by Barker [11] and move to a substructural interpretation of the conditional and the conjunction. Yet in doing so one should be careful not to put the deductive barrier too high. In the words of van Benthem [102, p.95]: “This is like turning down the volume on your radio so as not to hear the bad news. You will not hear much good news either.” Indeed, as we have seen earlier there are still plausible cases of FCP inferences. Combining the idea of “licensed instance” on action types, my strategy to do so is to adopt this action types relation, and borrow the principle shared in “licensed instance” from non-monotonic logic. This provides just enough deductive power to keep control of conjunctive FCP inferences. This is what I do in the next section.

3.3 Rational Monotonicity in a Substructural Framework

In this section I argue for a combination of ideas from substructural logics [87, 33] and non-monotonic reasoning to achieve a good balance in deductive power. From now on the intuitive reading of the conditional $\multimap$ in OR^+ is “be a licensed instance of” adopted from Section 3.1.3.

3.3.1 From Classical to Non-Classical: Give Up Left-Hand Weakening and Mingle

A plausible calculus of action types should give up left-hand weakening and mingle as rules of inference. The absence of these rules will exclude the two unwelcome classical properties in OR^+ mentioned above: unrestricted monotonicity and resource-insensitivity.

The following left-hand weakening should be excluded as one of the rules of inference on licensed action types. For arbitrary γ,

$$\alpha \multimap \beta \vdash (\alpha \wedge \gamma) \multimap \beta$$

The reason for giving up this rule under the "licensed instance" interpretation has already been alluded to. This principle is a form of monotonicity. But it should be clear that even though α is a licensed instance of β, it doesn't mean that α together with γ is. The unrestricted left-hand weakening can allow for $(\alpha \wedge \beta) \multimap \alpha$ and $(\alpha \wedge \cdots \wedge \alpha) \multimap \alpha$, which brings back the foregoing examples of monotonic and resource insensitive conditionals on action types.

A similar reasoning forces me to reject the following mingle rule for the "licensed instance" interpretation:

$$(\alpha \multimap \beta) \wedge (\gamma \multimap \beta) \vdash (\alpha \wedge \gamma) \multimap \beta$$

Indeed, $(\alpha \wedge \beta) \multimap \alpha$ and $(\alpha \wedge \cdots \wedge \alpha) \multimap \alpha$ are two instances of mingle. The mingle rule can even bring the following additional *unlicensed* instance into actions as well as the *abnormality* on permissions. (3a) selling the house to Ann is a licensed instance of selling the house, and (3b) selling the house to Bob is a licensed instance of selling the house. Applying mingle to these two conditionals on action types, I can conclude the counter-intuitive (3c) that selling the house to Ann and selling the house to Bob is a licensed instance of selling the house. Yet the conclusion (3c) selling one house to two individuals is not a licensed instance of a selling contract. Though action type α and action type γ are both licensed instances of action type β, if they are mutually exclusive, then their composition as a contradiction is not a licensed instance of the action type β any longer. The mingle rule therefore is not a desired rule for the "licensed instance."

3.3.2 Getting Back Some Deductive Power: Cautious Monotony or Rational Monotony?

To recover the plausible free choice permission inferences, I will examine two principles from non-monotonic reasoning. I want to restore certain plausible examples such as ordering a lunch and paying for it is a licensed instance of ordering a lunch.

The first option is the following non-monotonic principle:

$$(\alpha \multimap \gamma) \wedge (\alpha \multimap \beta) \vdash (\alpha \circ \beta) \multimap \gamma \qquad \text{(CaM)}$$

where fusion $\circ$ is a non-classical counterpart of the classical conjunction $\wedge$. This principle is called cautious monotony. It means that, if action type α is a licensed instance of action type γ, and it is also a licensed instance of action type β, then the composite action type α and β is still a licensed instance of action type γ. By using CaM, the selling example can be accommodated as follows. Let (3b') be the case that selling the house to Ann is a licensed instance of not selling the house to Bob. Applying CaM to (3a) and (3b'), I conclude (3c') that selling the house to Ann and not selling the house to Bob is a licensed instance of selling the house.

Yet accepting CaM re-introduces the kind of resource-insensitivity that I want to avoid in this chapter. I can infer $(\alpha \circ \cdots \circ \alpha) \multimap \alpha$ by applying $(\alpha \multimap \alpha)$ on CaM. Therefore this non-monotonic principle does not help to solve the problems I want to address here. Instead, I consider a closely related principle: rational monotony.

Rational monotony states that: If action type α is a licensed instance of action type γ, and it is not the case that action type α is a licensed instance of any actions except β, then the composition of action type α and action type β is a licensed instance of action type γ. I read $\sim \beta$ as the action type "any actions except β," and it is equal to saying $\beta \multimap \mathbf{0}$, where $\mathbf{0}$ is an impossible action. It captures the intuition of any actions except β: this action type β fails to be licensed. Formally, rational monotony is presented in the following form:

$$(\alpha \multimap \gamma) \wedge \neg(\alpha \multimap \sim \beta) \vdash (\alpha \circ \beta) \multimap \gamma \qquad \text{(RaM)}$$

Under a classical reading of conjunction and implication, rational monotony implies cautious monotony. This is not the case here.

3.4 A Substructural Calculus of Actions and Permissions

The goal of this section is to develop a family of sound and complete substructural logics for actions and permissions. I argued earlier that, in the context of free choice permission, the calculus of actions should not include the classically valid left-hand weakening and mingle, which are the undesired logical assumptions emphasized on action compositions, and the substitution under (classical) equivalence, which is the other unwanted assumption applied on conditional over actions. The calculus I develop is a family of substructural logics. All logics in this family exclude the structural rules for left-hand weakening, mingle and cautious monotony. They can thus avoid the unrestricted monotonic "vegetarian free lunch" and the resource-insensitive cases. Because the negation on actions is defined by the non-classical conditional $\multimap$ together with the impossible action type $\mathbf{0}$, the irrelevant $(\alpha \uplus \sim \alpha) \multimapboth (\beta \uplus \sim \beta)$ fails too [4]. Furthermore, to restore the deductive power, one of these logics will include RaM. I adopt OR as my semantic core, and develop a family of logics that contains OR^{+} as the main proof theoretical principle.

[4] To mark the disjunction for action types, here I use $\uplus$ to replace the classical disjunction $\vee$.

3.4.1 Substructural Logics for Actions and Permissions

The presentation in this section is close to the standard systems and the standard semantics in [87]. Notice that the following is a calculus of actions as types and permissions as propositions. The language is then of two sorts: one is well-formed types for actions, and the other is well-formed formulas for permissions.

3.4.1. DEFINITION. [Types and Formulas] The set $\mathcal{L}_A$ of well-formed types of actions and the set $\mathcal{L}_n$ of well-formed formulas of norms are defined as follows:

$$\mathcal{L}_A : \alpha ::= \mathbf{0} \mid a \mid \alpha \uplus \alpha \mid \alpha \circ \alpha \mid \sim \alpha$$
$$\mathcal{L}_n : \varphi ::= \alpha \multimap \alpha \mid P\alpha \mid \bot \mid \varphi \rightarrow \varphi$$

where $a \in Act_0$ the (countable) set of action generators, and $\alpha \in \mathcal{L}_A$. Let $\mathcal{L} = \mathcal{L}_A \cup \mathcal{L}_n$.

The set $\mathcal{L}_A$ of well-formed action types are constructed out of the set Act_0 of primitive action types and the impossible action $\mathbf{0}$, by taken together with $\uplus$ the Boolean disjunction, the fusion $\circ$ the non-classical conjunction, and $\sim$ the action negation. So $\alpha \uplus \beta$ is a choice between actions α and β, read as "doing α or doing β." And $\alpha \circ \beta$ is an *action composition*, read as "doing α together with β." [5] Then the action negation $\sim \alpha$ is read as "doing anything else than α." Moreover, $\sim$ is not a De Morgen negation, in the sense that, although $\alpha \multimap \sim\sim \alpha$ holds, its converse $\sim\sim \alpha \multimap \alpha$ does not.

The set $\mathcal{L}_n$ of well-formed formulas includes three kinds of atomic formula. The first kind of atomic formula is constructed by the binary conditional $\multimap$ over action types, which is understood as the "licensed instance" relation discussed in Section 3.1.3. So the "licensed instance" formula $\alpha \multimap \beta$ is read as "doing α is a licensed instance of doing β." Paying for a lunch is a licensed instance of ordering a lunch. In contrast, not-paying for a lunch is not a licensed instance of ordering a lunch in commercial circumstances. I simply construct a set $\mathcal{L}_B$ of "action-formulas" by taking all action types and "licensed instance" formulas. The second kind of primitive well-formed formulas capture permissions of actions, which is constructed by taking the P-modality for (free choice) permission, in the scope of which only action types occur. Then $P\alpha$ indicates that doing α is permitted. The third kind of primitive well-formed formula is the false constant $\bot$. Taking all three kinds of primitive well-formed formula together with the material conditional $\rightarrow$, I can construct the whole set $\mathcal{L}_n$ of well-formed formulas. Observe that the classical negation $\neg$ over formulas can be defined as $\neg\varphi := \varphi \rightarrow \bot$. So the true constant $\top := \neg\bot$. The classical disjunction $\vee$ over propositions is defined as usual: $\varphi \vee \psi := \neg\varphi \rightarrow \psi$. I use the Greek letters $\alpha, \beta, \gamma, \dots$ to represent action types, and $\varphi, \psi, \chi, \dots$ to represent formulas. Sometimes, in certain particular situations, I

[5] The composition operator $\circ$ is not the sequent composition operator in propositional dynamic logic (PDL) [44, p.168]. This operator $\circ$, sometimes, may be understood as a non-standard concurrency operator of actions [44, p.268, p.276]. For example, $Listen \circ WriteNote$. I suggest reading $\alpha \circ \beta$ as "doing action α and action β (together)." Though it seems impossible to excute $EatCookies \circ DrinkCoffee$ at the same time, I can still think of eating cookies and drinking coffee as a *licensed* action.

use the capital letters $A, B, C, \dots$ to encompass either action types or formulas, if no confusion arises.

I can then introduce another important syntactic component in substructural logic: structures. In my logics, structures are categorized into two sorts, corresponding to action-formulas in $\mathcal{L}_\mathcal{B}$ and deontic formulas in $\mathcal{L}_n$.

3.4.2. DEFINITION. [Structures] The set $\mathcal{S} = \mathcal{S}_\mathcal{B} \cup \mathcal{S}_d$ of structures is defined as follows:

$$\mathcal{S}_\mathcal{B} : X ::= \psi \mid X; X \text{ where } \psi \in \mathcal{L}_\mathcal{B}$$
$$\mathcal{S}_d : X ::= \varphi \mid X, X \text{ where } \varphi \in \mathcal{L}_n$$

Observe that $\mathcal{L} \subseteq \mathcal{S}$, and the set $\mathcal{S}$ of structures is the union of the set $\mathcal{S}_\mathcal{B}$ of action-structures and the set $\mathcal{S}_d$ of deontic-structures. The semicolon ; is a binary punctuation mark over action-formulas $\psi \in \mathcal{L}_\mathcal{B}$. Usually $X; Y$ is read as "a structure X combines with a structure Y." The comma , is a binary punctuation mark over well-formed formulas $\varphi \in \mathcal{L}_n$. Then X, Y is read as "a structure X and a structure Y." Here I use $X, Y, Z, H, \cdots$ to represent structures in $\mathcal{S}$. I write $X[Y]$ as a structure X with a substructure Y, while writing $X[Z/Y]$ as a structure X replacing the occurrences of the substructure Y by Z. For instance, $\alpha\circ \sim \alpha$ is a substructure for the structure $(\alpha\circ \sim \alpha; \alpha; \alpha\circ \sim \alpha)$, and then $(\alpha\circ \sim \alpha; \alpha; \alpha\circ \sim \alpha)[\mathbf{0}/\alpha\circ \sim \alpha] = (\mathbf{0}; \alpha; \mathbf{0})$.

The models for my logics are the standard models in substructural logics [87]. The basic ontological entities in these models are states, which, as I will show in Section 3.4.4, are incomplete. Here states have two characters in accordance with the valuation function V: to instantiate action types, and to give well-formed formulas truth value. In other words, action types can be instantiated or not at states, while formulas can be true or false at states. Moreover, two different types of relations should be contained in the substructural models. One is the ternary relation M, which is understood as a semantic characterization of the licensed instance relation in Section 3.1.3, in the sense that $Mxyz$ holds, iff it satisfies: (b1) x and y together is an instance of z; and (b2) the third state z gives rise to a reason for the circumstance to consist of x and y. $Mxyz$ is read as " the circumstance consisting of state x together with state y is licensed by state z." I simply call M "state-license," and use it to interpret the "licensed instance" relation on action types. See Figure 3.1. The frame conditions for M-relation will be discussed in Section 3.4.2. The second main relation is the binary relation OK for interpreting (free choice) permission. $OK(x, y)$ is read as "a state x is normatively okay w.r.t. a state y."

3.4.3. DEFINITION. [Models] A model is a tuple $\mathcal{M} = \langle W, M, OK, V\rangle$ where

- W is a non-empty set of possible states
- $M \subseteq W \times W \times W$ is a ternary relation on W
- $OK \subseteq W \times W$ is a binary relation on W

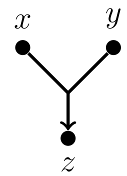


Figure 3.1: A state-license $Mxyz$: (1) a state consisting of x and y is an instance of z, and (2) the circumstance together with x and y is licensed by the third state z.

- $V : Act_0 \cup \{\mathbf{0}, \bot\} \to \wp(W)$ is a valuation function

The OK-relation is a standard to evaluate the deontic facts at each state, while the M-relation, as a characterization of "licensed instance" between actions, governs how to regulate and license one action in accordance with a given deontic standard.

As usual, a frame for my substructural logics is a model minus the valuation V.

Observe that the language $\mathcal{L}$ is two-sorted, and so are the truth conditions. Now I can interpret action types and formulas in $\mathcal{L}$ and structures in $\mathcal{S}$ by taking these models as a standard method in substructural logics [87].

3.4.4. DEFINITION. [Truth Conditions] Let $\mathcal{M} = \langle W, M, OK, V\rangle$ be a model. A well-formed type $\alpha \in \mathcal{L}_A$ is *instantiated* at state w in model $\mathcal{M}$, written $\mathcal{M}, w \vDash \alpha$, iff

$\mathcal{M}, w \vDash a$	iff	$w \in V(a)$
$\mathcal{M}, w \nvDash \mathbf{0}$	for all	$w \in W$
$\mathcal{M}, w \vDash \alpha \uplus \beta$	iff	$\mathcal{M}, w \vDash \alpha$ or $\mathcal{M}, w \vDash \beta$
$\mathcal{M}, w \vDash \alpha \circ \beta$	iff	$\exists y, z \in W.\ (Myzw, \mathcal{M}, y \vDash \alpha\ \&\ \mathcal{M}, z \vDash \beta)$
$\mathcal{M}, w \vDash \sim \alpha$	iff	$\forall y, z \in W.(Mwyz \Rightarrow \mathcal{M}, y \nvDash \alpha)$

Now I construct the truth conditions for well-formed formulas: $\varphi \in \mathcal{L}_n$ is *true* at state w in model $\mathcal{M}$, written $\mathcal{M}, w \vDash \varphi$, iff

$\mathcal{M}, w \vDash \alpha \multimap \beta$	iff	$\forall y, z \in W.(Mwyz\ \&\ \mathcal{M}, y \vDash \alpha \Rightarrow \mathcal{M}, z \vDash \beta)$
$\mathcal{M}, w \vDash P\alpha$	iff	$\forall y \in W.\ (\mathcal{M}, y \vDash \alpha \Rightarrow OK(y, w))$
$\mathcal{M}, w \nvDash \bot$	for all	$w \in W$
$\mathcal{M}, w \vDash \varphi \to \psi$	iff	$\mathcal{M}, w \vDash \varphi \Rightarrow \mathcal{M}, w \vDash \psi$

So I can extend the truth conditions to all structures such that

$\mathcal{M}, w \vDash X; Y$	iff	$\exists y, z \in W.\ (Myzw, \mathcal{M}, y \vDash X\ \&\ \mathcal{M}, z \vDash Y)$
$\mathcal{M}, w \vDash X, Y$	iff	$\mathcal{M}, w \vDash X\ \&\ \mathcal{M}, w \vDash Y$

The truth conditions for action types are interpreted via the concept "instantiation." $V(a)$ is a set of states that can instantiate action generator $a \in Act_0$, and then $\mathcal{M}, w \vDash a$ indicates that action generator a is instantiated at state w. $\alpha \uplus \beta$ is instantiated at state w iff either α or β is instantiated at this state. The non-classical interpretations over actions are for the fusion $\circ$ and the action negation $\sim$, which need to be interpreted via state-license M. $\alpha \circ \beta$ is instantiated at w, iff there exist y and z s.t. (b1') the state consisting of y, the instantiation of α, and z, the instantiation of β, is an instance of w which instantiates $\alpha \circ \beta$, and (b2') the circumstance consisting of y and z is licensed by w. For instance, ordering a lunch together with paying for it would be instantiated at w, iff there are two states y and z s.t. y instantiates ordering a lunch, z instantiates paying for the meal, and (b1") y and z together is an instance of w as well as (b2") the circumstance consisting of y and z is licensed by w. We can see that the state-license in an existential manner can offer a good interpretation for $\alpha \circ \beta$: there exists a licensed instantiation of doing α and β together. Similarly, the action negation $\sim \alpha$ is instantiated at w, iff, that if w together with arbitrary state y is a licensed instance, then this state y does not instantiate α.

The truth conditions for well-formed formulas are interpreted via the usual concept "truth." An action-formula $\alpha \multimap \beta$ is true at w iff the state consisting of w and arbitrary state instantiating α is always a licensed instance of β. With this in hand, for instance, $(\alpha \circ \beta) \multimap \alpha$, where α is ordering a lunch and β is paying for the meal, is true at state w, iff an instantiation of α ordering a lunch and β paying for the meal together is a licensed instance of α ordering a lunch. A deontic formula $P\alpha$ is true at w iff for arbitrary state instantiating α, it is normatively okay w.r.t. the state w. The truth conditions for $\perp$ and $\rightarrow$ are defined as usual.

The truth conditions for structures are similar as for formulas. Validity is defined as usual. A structure X is valid in a model $\mathcal{M}$, denoted as $\mathcal{M} \models X$, iff $\mathcal{M}, w \models X$ for all $w \in W$. $\mathcal{M} \in \mathcal{F}$ indicates that $\mathcal{M}$ is a model based on a frame $\mathcal{F}$ with a valuation function. A structure X is valid in a frame $\mathcal{F}$, denoted as $\mathcal{F} \models X$, iff $\mathcal{M} \models X$ for all $\mathcal{M} \in \mathcal{F}$. I define $X \models_{\mathcal{F}} A$ iff if $\mathcal{M}, w \models X$ then $\mathcal{M}, w \models A$ where $\mathcal{M} \in \mathcal{F}$.

I define the turnstile $\vdash \subseteq \mathcal{S} \times \mathcal{L}$ as the basic syntactic consequences of actions and permissions, iff it is closed under the axioms and rules in Table 3.1, which is presented in the sequent calculus style. I read $X; B \vdash A$ as "if X combines with B, normally, then A", where $X; B$ is a combination in the premise and A is a consequent. I write the equivalent consequences $A \vdash B$ and $B \vdash A$ as $A \dashv\vdash B$. In accordance with Table 3.1, Table 3.2, Table 3.3, and Table 3.4, a system in sequent calculus is defined in a standard way [75, 81]. Here (Id) and (**0**) are called axioms, and the others are rules in a given sequent calculus N. I define derivations and their heights in Definition 3.4.5 and Definition 3.4.6, which are inspired by the notions in [75, 81]. They are the key notions for cut elimination.

3.4.5. DEFINITION. [Derivations] I say $X \vdash A$ is a derivation in the system N, iff:

1. either $X \vdash A$ is an instance of axioms in N, or
2. there are derivations $Y \vdash B$ and $Z \vdash C$ s.t. $X \vdash A$ is concluded by the application of a rule in N on $Y \vdash B$ and $Z \vdash C$ as its premises.

3.4.6. DEFINITION. [Heights of Derivations] Given a system N, I say $X \vdash A$ is a derivation with the height n, denoted as $X \vdash_n A$, iff

(Id)	$a \vdash a$ where $a \in Act_0$	(**0**)	$X[\mathbf{0}] \vdash \gamma$ for all $\gamma \in \mathcal{L}_A$
$(\sim L)$	$\dfrac{X \vdash \alpha \qquad Y[\mathbf{0}] \vdash \gamma}{Y[\sim \alpha; X] \vdash \gamma}$	$(\sim R)$	$\dfrac{X; \alpha \vdash \mathbf{0}}{X \vdash \sim \alpha}$
$(\circ L)$	$\dfrac{X[\alpha; \beta] \vdash \gamma}{X[\alpha \circ \beta] \vdash \gamma}$	$(\circ R)$	$\dfrac{X \vdash \alpha \qquad Y \vdash \beta}{X; Y \vdash \alpha \circ \beta}$
$(\uplus L)$	$\dfrac{X[\alpha] \vdash \gamma \qquad X[\beta] \vdash \gamma}{X[\alpha \uplus \beta] \vdash \gamma}$	$(\uplus R_1)$	$\dfrac{X \vdash \alpha}{X \vdash \alpha \uplus \beta}$
		$(\uplus R_2)$	$\dfrac{X \vdash \beta}{X \vdash \alpha \uplus \beta}$

Table 3.1: The fragment of the sequent calculus N^0 for action types $\mathcal{L}_A$.

$(\multimap L)$	$\dfrac{X \vdash \alpha \qquad Y[\beta] \vdash \gamma}{Y[(\alpha \multimap \beta); X] \vdash \gamma}$	$(\multimap R)$	$\dfrac{X; \alpha \vdash \beta}{X \vdash \alpha \multimap \beta}$
$(\bot)$	$X[\bot] \vdash \varphi$ for all $\varphi \in \mathcal{L}_n$	(OR)	$\dfrac{X \vdash \alpha \multimap \beta}{X, P\beta \vdash P\alpha}$
$(\rightarrow L)$	$\dfrac{X \vdash \varphi \qquad Y[\psi] \vdash \chi}{Y[\varphi \rightarrow \psi, X] \vdash \chi}$	$(\rightarrow R)$	$\dfrac{X, \varphi \vdash \psi}{X \vdash \varphi \rightarrow \psi}$

Table 3.2: The fragment of the sequent calculus N^0 for norms $\mathcal{L}_n$.

1. either $X \vdash_0 A$ is an instance of axioms in N, or
2. $\exists Y \vdash_{n-k} B, Z \vdash_k C$ s.t. $X \vdash_{n+1} A$ is a derivation concluded by the application of a rule in N with premises $Y \vdash_{n-k} B$ and $Z \vdash_k C$, where $0 \leq k \leq n$.

A derivation $X \vdash_0 A$ with the height 0 is an instance of an axiom. In Table 3.1, Table 3.2, Table 3.3, and Table 3.4, if the upper consequences are at height n then the lower consequences are at height $n + 1$. The left rules and right rules are to introduce each operator from the upper consequences to its derived consequence. The left rules show how to introduce the operator in the left side of $\vdash$ and the right rules show how to introduce the operator in the right side of $\vdash$. For instance, $(\circ L)$ and $(\circ R)$ are the left introduction and the right introduction for the fusion $\circ$.

Given any sequent calculus system N, an extension N^R of N with a set R of axioms or rules means that N^R is closed under all the axioms and rules from N in addition with those in R. $X \vdash A$ is a theorem of the system N iff it is derivable in N. I use $Th(N)$ to denote all theorems of the sequent calculus N. Then N^R as an extension of N with R can be denoted as $Th(N) \subseteq Th(N^R)$, sometimes is simplified as $N \subseteq N^R$. Notice that the system N^0 in Table 3.1, Table 3.2, and Table 3.3 is the basic substructural logic in which left-hand weakening, mingle, and cautious monotony are absent. Based

(Cut) $\dfrac{X \vdash A \qquad Y[A] \vdash B}{Y[X/A] \vdash B}$ (Tra) $\dfrac{X \vdash \alpha \multimap \beta \qquad Y \vdash \beta \multimap \gamma}{X, Y \vdash \alpha \multimap \gamma}$

where either $A, B \in L_A$ or $A, B \in L_n$.

Table 3.3: The fragment of the sequent calculus N^0 for actions $\mathcal{L}_A$ and norms $\mathcal{L}_n$.

(BI) $\dfrac{Z[X;Y] \vdash A}{Z[Y;X] \vdash A}$ (CaM) $\dfrac{X \vdash \alpha \multimap \beta \qquad Y \vdash \alpha \multimap \gamma}{X;Y \vdash (\alpha \circ \beta) \multimap \gamma}$

(B) $\dfrac{H[(X;Y);Z] \vdash A}{H[X;(Y;Z)] \vdash A}$ (RaM) $\dfrac{X \vdash \alpha \multimap \gamma}{X, \neg(\alpha \multimap\sim \beta) \vdash (\alpha \circ \beta) \multimap \gamma}$

(M) $\dfrac{H[X] \vdash A}{H[X;X] \vdash A}$

Table 3.4: Additional Rules, where either $A \in L_A$ or $A \in L_n$.

on the additional rules in Table 3.4, I can extend the basic system N^0 of actions and permissions as follows. N^E is an extension of N^0 together with BI and B (which are called exchange rules). N^M is an extension of N^E with M, while N^{RaM} is an extension of N^E with RaM. And so N^{CaM} is an extension of N^M with CaM. I therefore have two families of substructural logics of actions and permissions, and their relations are presented in the following branching chains:

$$N^0 \subseteq N^E \begin{array}{l} \subseteq N^{RaM} \\ \subseteq N^M \subseteq N^{CaM} \end{array}$$

All substructural logics in the upper branch exclude left-hand weakening, mingle, and cautious monotony. The lower branch shows how cautious monotony correlates with mingle, and why it is resource insensitive. In addition, the upper family of substructural logics has the following useful properties.

3.4.7. THEOREM. *1. ID is a theorem of N^0: $\alpha \vdash \alpha$ for all $\alpha \in \mathcal{L}_A$.*

2. (OR^+) is a theorem of N^0.

3. Action composition is associative and commutative in N^E:

$$\dfrac{A;B \vdash C}{\overline{B;A \vdash C}}\ BI \qquad \dfrac{(A;B);C \vdash D}{\overline{A;(B;C) \vdash D}}\ B, BI$$

4. The following two rules are theorems of N^0:

$$\dfrac{X \vdash \varphi}{Y[\neg\varphi, X] \vdash \psi}\ (\neg L) \qquad \dfrac{X, \varphi \vdash \bot}{X \vdash \neg\varphi}\ (\neg R)$$

Proof:

1. To show that $\alpha \vdash \alpha$ for all $\alpha \in \mathcal{L}_A$.
   - If $\alpha = a \in Act_0$, by (Id), I have $a \vdash a$.
   - If $\alpha = \mathbf{0}$, by ($\mathbf{0}$), I have $\mathbf{0} \vdash \mathbf{0}$.
   - If $\alpha = \beta \uplus \gamma$, then, by ($\uplus L$), ($\uplus R_1$), and ($\uplus R_2$), the proof is:
   $$\dfrac{\dfrac{\beta \vdash \beta}{\beta \vdash \beta \uplus \gamma}(\uplus R_1) \qquad \dfrac{\gamma \vdash \gamma}{\gamma \vdash \beta \uplus \gamma}(\uplus R_2)}{\beta \uplus \gamma \vdash \beta \uplus \gamma}(\uplus L)$$
   - If $\alpha = \beta \circ \gamma$, then, by ($\circ L$) and ($\circ R$), the proof is:
   $$\dfrac{\dfrac{\beta \vdash \beta \qquad \gamma \vdash \gamma}{\beta ; \gamma \vdash \beta \circ \gamma}(\circ R)}{\beta \circ \gamma \vdash \beta \circ \gamma}(\circ L)$$
   - If $\alpha = \sim \beta$, then, by ($\sim L$) and ($\sim R$), the proof is:
   $$\dfrac{\dfrac{\beta \vdash \beta \qquad \mathbf{0} \vdash \mathbf{0}}{\sim \beta ; \beta \vdash \mathbf{0}}(\sim L)}{\sim \beta \vdash \sim \beta}(\sim R)$$
2. The first derivation goes by the rule ($\multimap R$).
3. The derivation goes as follows:
$$\dfrac{\dfrac{\alpha \multimap \beta \vdash \alpha \multimap \beta}{\alpha \multimap \beta, P\beta \vdash P\alpha}\text{OR}}{\alpha \multimap \beta \vdash P\beta \to P\alpha} \to R$$
4. These two instances are derived in the following:
$$\dfrac{\dfrac{X \vdash \varphi \qquad Y[\bot] \vdash \psi}{Y[\varphi \to \bot, X] \vdash \psi}(\to L)}{Y[\neg\varphi, X] \vdash \psi}\text{(Definition)} \qquad \dfrac{\dfrac{X, \varphi \vdash \bot}{X \vdash \varphi \to \bot}(\neg R)}{X \vdash \neg\varphi}\text{(Definition)}$$

□

3.4.8. THEOREM. *The "licensed instance" relation $\multimap$ is irreflexive, asymmetric, and transitive in the following sense:*

1. $\top \vdash \alpha \multimap \alpha$ *is invalid.*
2. $\alpha \multimap \beta \vdash \beta \multimap \alpha$ *is invalid.*

3. $(\alpha \multimap \beta), (\beta \multimap \gamma) \vdash \alpha \multimap \gamma$ *is a theorem of* N^0.

And $\multimap$ *satisfies rational monotony:*

1. $(\alpha \multimap \gamma), \neg(\alpha \multimap\sim \beta) \vdash (\alpha \circ \beta) \multimap \gamma$ *is a theorem of* N^{RaM}.

Proof:

1. Construct a model $\mathcal{M} = \langle W, M, OK, V\rangle$ s.t. $W = \{x, y, z\}$, $Mxyz$, $OK = \varnothing$, and $V(a) = \{y\}$. Obviously $\mathcal{M}, x \nVdash a \multimap a$.
2. Construct a model $\mathcal{M} = \langle W, M, OK, V\rangle$ s.t. $W = \{x, y, z\}$, $Mxyz$, $OK = \varnothing$, $V(a) = \{y\}$ and $V(b) = \{y, z\}$ where $a, b \in Act_0$. I see that $\mathcal{M}, x \nVdash a \multimap b$ but $\mathcal{M}, x \nVdash b \multimap a$.
3. This theorem is ensured by (Tra) in N^0.
4. This is an instance of (RaM).

□

3.4.2 Standard Translation and Frame Correspondence

I apply the standard first-order translation ST for the ternary frames suggested in [59, p.29] for (OR)'s modal correspondence, as well as for rules (Tra), (BI), (B), (M), (CaM), and (RaM)'s correspondences. First, the definition of the standard first-order translation is given as follows:

3.4.9. DEFINITION. [Standard Translation]

$$
\begin{array}{lcl}
ST_x(a) & := & A(x) \\
ST_x(\mathbf{0}) & := & false \\
ST_x(\alpha \uplus \beta) & := & ST_x(\alpha) \vee ST_x(\beta) \\
ST_x(\alpha \circ \alpha) & := & \exists y \exists z.(Myzx \wedge ST_x(\alpha)[x := y] \wedge ST_x(\beta)[x := z]) \\
ST_x(\sim \alpha) & := & \forall y \forall z.(Mxyz \rightarrow \neg ST_x(\alpha)[x := y]) \\
ST_x(\alpha \multimap \beta) & := & \forall y \forall z.(Mxyz \wedge ST_x(\alpha)[x := y] \rightarrow ST_x(\beta)[x := z]) \\
ST_x(P\alpha) & := & \forall y.(ST_x(\alpha)[x := y] \rightarrow OK(y, x)) \\
ST_x(\bot) & := & false \\
ST_x(\varphi \rightarrow \psi) & := & ST_y(\varphi) \rightarrow ST_y(\psi) \\
ST_x(X; Y) & := & \exists y \exists z.(Myzx \wedge ST_x(X)[x := y] \wedge ST_x(Y)[x := z]) \\
ST_x(X, Y) & := & ST_x(X) \wedge ST_x(Y)
\end{array}
$$

It is straightforward to check that this standard translation is adequate [13]. Now I turn to the frame correspondence. I will present in turn the correspondents of (OR), (Tra), (BI), (B), (M), (CaM), and (RaM).

3.4.10. DEFINITION. [Open Reading] Let a tuple $\mathcal{F} = \langle W, M, OK \rangle$ be a frame. I say $\mathcal{F}$ is an open-reading frame iff $\mathcal{F}$ satisfies the following (or) condition:

$$\forall w \forall x \forall y' [\forall y (Mwy'y \to OK(y,x)) \to OK(y',x)].$$

The open reading frame condition can be taken as a closure. Given a state w, if for any y, which licenses to the circumstance consisting of w and y', is normatively okay, then y' is also normatively okay. This frame condition can be seen as a closure of normatively okay states closed under the state-license relation w.r.t. a given w:

$$\mathbf{OR}(w) = \{(y',x) \in OK \mid \forall y.[Mwy'y \Rightarrow (y,x) \in OK]\}.$$

3.4.11. DEFINITION. [License-Transitivity] Let a tuple $\mathcal{F} = \langle W, M, OK \rangle$ be a frame. I say $\mathcal{F}$ is a license-transitive frame iff $\mathcal{F}$ satisfies the following (tra) condition:

$$\forall x \forall y \forall z \exists u [Mxyz \to Mxyu \wedge Mxuz].$$

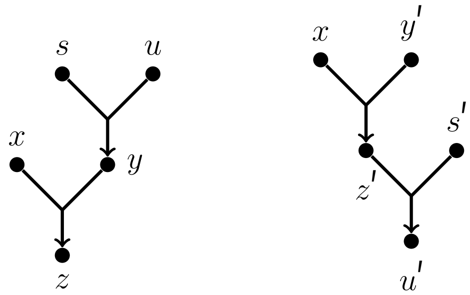


Figure 3.2: Two kinds of combination of state-licenses starting at x.

3.4.12. DEFINITION. [Cautious Monotonicity] Let a tuple $\mathcal{F} = \langle W, M, OK \rangle$ be a frame. I say $\mathcal{F}$ is a cautiously monotonic frame iff $\mathcal{F}$ satisfies the following (cam) condition:

$$\forall x \forall y \forall z \forall s \forall u \exists y' \exists z' \exists m \exists n [(Mxyz \wedge Msuy) \wedge (s = y' \vee s = m) \wedge (u = z' \to Mxy'z') \to (z = n \to Mxmn)].$$

By use of the intuitive structure depicted in Figure 3.2 I want to shed light on the formal semantics, i.e. given a combination of state-licenses $Mxyz$ and $Msuy$. In condition (cam), the conjunct $(u = z' \to Mxy'z')$ indicates that it is possible that the circumstance consisting of state x together with state y' is licensed by state u, and, similarly, the conclusion $(z = n \to Mxmn)$ indicates that it is possible that the circumstance consisting of state x together with state m is licensed by state z. Hence, the condition (cam) indicates that the given combination of two state-licenses can be shortened: if there exists a state-license of x by u, then there exists a state-license of x by z. However, this *existential* character is not adequate for capturing the licensed instance on action types.

3.4.13. DEFINITION. [Rational Monotonicity] Let a tuple $\mathcal{F} = \langle W, M, OK \rangle$ be a frame. I say $\mathcal{F}$ is a rationally monotonic frame iff $\mathcal{F}$ satisfies the following (ram) condition:

$$\forall x\forall y\forall z\forall s\forall u\forall y'\forall z'\forall s'\forall u'[(Mxyz \wedge Msuy) \wedge (Mxy'z' \wedge Mz's'u') \\ \rightarrow (Mxsz \vee Mxy'z)].$$

Similar to condition (cam), the condition (ram) also provides a method to shorten the same combination of state-licenses starting at x: If there are two kinds of combination of state-licenses starting at x, then the circumstances of state x together with two kinds of "closest licensed" state (s or y' in Figure 3.2) are licensed the "closest licensing" state z. In contrast to (cam), here the shortening strategy given by (ram) can provide a standard to select suitable state-licenses, which fits into the *universal* character of licensed instance.

Let $\mathcal{F} = \langle W, M, OK \rangle$ be a frame. Then [6]

- $\mathcal{F}$ is a (bi)-frame iff it satisfies that $\forall xyz.(Mxyz \rightarrow Myxz)$.
- $\mathcal{F}$ is a (b)-frame iff it satisfies that $\forall xyzw.[Mx(yx)w \rightarrow M(xy)zw]$.
- $\mathcal{F}$ is a (mig)-frame iff it satisfies that $\forall x\forall y\forall z[Mxyz \rightarrow x = z \vee y = z]$.

3.4.14. DEFINITION. [Correspondences] A frame condition (r) that corresponds to the rule (R) is defined as follows: For any frame $\mathcal{F}$, $\mathcal{F}$ is a (r)-frame iff (R) is valid in $\mathcal{F}$.

The frame conditions (bi), (b), and (mig) are suggested in [87, p.250], and corresponds to (BI), (B), and (M) respectively.[7] I propose the conditions (or), (tra), and (ram) in this chapter. I show that (or) corresponds to (OR), (tra) corresponds to (Tra), and (ram) corresponds to (RaM).

3.4.15. THEOREM. *The (or) condition corresponds to (OR).*

$$\alpha \multimap \beta \vDash_{\mathcal{F}} P\beta \rightarrow P\alpha \textit{ iff } \mathcal{F} \vDash \forall w\forall x\forall y'[\forall y(Mwy'y \rightarrow OK(y,x)) \rightarrow OK(y',x)].$$

Proof:
For the Left-to-Right direction, I apply the contrapositive method in [59, p.43]. Suppose $a \multimap b \vDash_{\mathcal{F}} Pb \rightarrow Pa$. That is, for any valuation V and any state w, if $w \notin V(Pb \rightarrow Pa)$ then $w \notin V(a \multimap b)$. Let A, B be predicates denoting the sets of

[6]The following shorthands are used to define frame conditions for the rules indicated in brackets:

- $M(xy)zw := \exists u.(Mxyu \wedge Muzw)$.
- $Mz(xy)w := \exists u.(Mxyu \wedge Mzuw)$.

[7]My proofs are similar but different than [87]: It has to contain an inclusion relation in the given frames.

possible states where a, b hold, respectively. This is reflected in the corresponding second-order formula

$$\forall B \forall A \forall w[\exists x(\forall y(By \to OK(y,x)) \land \exists y'(Ay' \land \neg OK(y',x))) \to \exists s \exists u(Mwsu \land As \land \neg Bu)]$$

This formula is equal to

$$\forall A \forall B \forall w \forall x \forall y' \exists y \exists s \exists u[(By \to OK(y,x)) \land (Ay' \land \neg OK(y',x)) \to (Mwsu \land As \land \neg Bu)]$$

Then I define a *minimal valuation* V^* as follows:

- $V^*(a) = \{v \mid v = y'\}$
- $V^*(b) = \{v \mid v = y \to OK(y,x)\}$

This corresponds to the syntactic substitution

- $A^*(v) := v = y'$
- $B^*(v) := v = y \to OK(y,x)$

The required frame condition is obtained by instantiation:

$$\forall w \forall x \forall y' \exists y \exists s \exists u[(B^* y \to OK(y,x)) \land (A^* y' \land \neg OK(y',x)) \to (Mwsu \land A^* s \land \neg B^* u)]$$

After the instantiation, the first-order form is presented as follows

$$\forall w \forall x \forall y' \exists y[\neg OK(y',x) \to Mwy'y \land \neg OK(y,x)]$$

This can be simplified as

$$\forall w \forall x \forall y'[\forall y(Mwy'y \to OK(y,x)) \to OK(y',x)].$$

As universal second-order formulas imply all their instantiations, this shows that the original (OR) rule implies this frame property.

But also conversely, the Right-to-Left direction, I can prove that (OR) is true as each frame satisfies condition (or). Let $\mathcal{M}, w \vDash \alpha \multimap \beta$. I need to show that $\mathcal{M}, w \vDash P\beta \to P\alpha$. Suppose $\mathcal{M}, x \vDash P\beta$. Let $\mathcal{M}, y' \vDash \alpha$. Assume that it is not the case that $OK(y',x)$. Applying this to (or), it follows that $\exists y$ s.t. $Mwy'y$ and $\neg OK(y,x)$. From $Mwy'y, \mathcal{M}, y' \vDash \alpha$ and $\mathcal{M}, w \vDash \alpha \multimap \beta$, I have $\mathcal{M}, y \vDash \beta$. But I already know that $\neg OK(y,x)$. It then implies that $\mathcal{M}, x \nvDash P\beta$, which contradicts my assumption $\mathcal{M}, x \vDash P\beta$. □

3.4.16. THEOREM. *The (tan) condition corresponds to (Tra).*

$$(\alpha \multimap \beta), (\beta \multimap \gamma) \vDash_{\mathcal{F}} \alpha \multimap \gamma \text{ iff } \mathcal{F} \vDash \forall x \forall y \forall z \exists u[Mxyz \to Mxyu \wedge Mxuz].$$

Proof:
The proof is similar to the previous one. □

3.4.17. THEOREM. *The (ram) condition corresponds to RaM. In other words,*

$$(\alpha \multimap \gamma), \neg(\alpha \multimap\!\sim \beta) \vDash_{\mathcal{F}} (\alpha \circ \beta) \multimap \gamma$$
$$\text{iff}$$
$$\mathcal{F} \vDash \forall x \forall y \forall z \forall s \forall u \forall y' \forall z' \forall s' \forall u'[(Mxyz \wedge Msuy) \wedge (Mxy'z' \wedge Mz's'u') \to (Mxsz \vee Mxy'z)].$$

Proof:
From the Left-to-Right direction of the proof, I apply the contrapositive method in [59, p.43] again. Suppose $(a \multimap c), \neg(a \multimap\!\sim b) \vDash_{\mathcal{F}} (a \circ b) \multimap c$. That is, for any valuation V and any state x, if $x \notin V(a \circ b \multimap c)$ and $x \notin V(a \multimap\!\sim b)$, then $x \notin V(a \multimap c)$. Let A, B, C be the predicates denoting the sets of possible states where a, b, c hold, respectively. This is reflected in the corresponding second-order formula

$$\forall A \forall B \forall C \forall x[\exists y \exists z.(Mxyz \wedge \exists s \exists u.(Msuy \wedge As \wedge Bu) \wedge \neg Cz) \wedge \exists y' \exists z'.(Mxy'z' \wedge Ay' \wedge \exists s' \exists u'.(Mz's'u' \wedge Bs')) \to \exists y'' \exists z''.(Mxy''z'' \wedge Ay'' \wedge \neg Cz'')].$$

This formula is equal to

$$\forall A \forall B \forall C \forall x \forall y \forall z \forall s \forall u \forall y' \forall z' \forall s' \forall u'[(Mxyz \wedge Msuy \wedge As \wedge Bu \wedge \neg Cz) \wedge (Mxy'z' \wedge Ay' \wedge Mz's'u' \wedge Bs') \to \exists y'' \exists z''(Mxy''z'' \wedge Ay'' \wedge \neg Cz'')].$$

Next I define a *minimal valuation* V^* as follows:

- $V^*(a) = \{s, y'\}$
- $V^*(b) = \{u, s'\}$
- $V^*(\neg c) = \{z\}$

This corresponds to the syntactic substitution

- $A^*(w) := (w = s) \vee (w = y')$
- $B^*(w) := (w = u) \vee (w = s')$
- $C^*(w) := (w \neq z)$

The required frame condition is obtained by instantiation:

$$\forall x \forall y \forall z \forall s \forall u \forall y' \forall z' \forall s' \forall u' [(Mxyz \wedge Msuy \wedge A^* s \wedge B^* u \wedge \neg C^* z) \wedge$$
$$(Mxy'z' \wedge A^* y' \wedge Mz's'u' \wedge B^* s') \rightarrow \exists y'' \exists z'' (Mxy''z'' \wedge A^* y'' \wedge \neg C^* z'')].$$

The first-order form is presented as

$$\forall x \forall y \forall z \forall s \forall u \forall y' \forall z' \forall s' \forall u' [(Mxyz \wedge Msuy) \wedge (Mxy'z' \wedge Mz's'u')$$
$$\rightarrow \exists y'' \exists z'' (Mxy''z'' \wedge ((y'' = s) \vee (y'' = y')) \wedge (z'' = z)].$$

which is equal to

$$\forall x \forall y \forall z \forall s \forall u \forall y' \forall z' \forall s' \forall u' [(Mxyz \wedge Msuy) \wedge (Mxy'z' \wedge Mz's'u')$$
$$\rightarrow (Mxsz \vee Mxy'z)].$$

As universal second-order formulas imply all their instantiations, this shows that the original RaM axiom implies this frame condition. Even so, to illustrate my general proof method, I provide some explicit steps leading "backwards." Consider any valuation. Assume that

$$(Mxyz \wedge Msuy \wedge As \wedge Bu \wedge \neg Cz) \wedge (Mxy'z' \wedge Ay' \wedge Mz's'u' \wedge Bs')$$

and show the consequent that

$$\exists y'' \exists z'' (Mxy''z'' \wedge Ay'' \wedge \neg Cz'')$$

First, I define the special valuation V^* as above. This follows

$$(Mxyz \wedge Msuy \wedge A^* s \wedge B^* u \wedge \neg C^* z) \wedge (Mxy'z' \wedge A^* y' \wedge Mz's'u' \wedge B^* s')$$
$$\text{for any } x, \text{ if } A^*(x), \text{ then } A(x)$$
$$\text{for any } x, \text{ if } B^*(x), \text{ then } B(x)$$
$$\text{for any } x, \text{ if } \neg C^*(x), \text{ then } \neg C(x)$$

Moreover, $A, \neg C$ occur monotone positively in the consequent. Therefore, I see that

$$\exists y'' \exists z'' (Mxy''z'' \wedge A^* y'' \wedge \neg C^* z'')$$

and together with semantic monotonicity, this implies

$$\exists y'' \exists z'' (Mxy''z'' \wedge Ay'' \wedge \neg Cz'').$$

Now it is the Right-to-Left direction of the proof: Assume that $\mathcal{F}$ is a (ram)-frame. I will show that $\mathcal{F}$ validates (RaM). Let $Mwxy$, $\mathcal{M}, x \vDash \alpha \circ \beta$, $\mathcal{M}, w \nvDash \alpha \multimap\!\sim \beta$ and

$\mathcal{M}, w \vDash \alpha \multimap \gamma$. From $\mathcal{M}, w \nvDash \alpha \multimap\!\!\sim \beta$, I have $\exists y', s' \in W$ s.t. $Mwy'z', \mathcal{M}, y' \vDash \alpha$ and $\mathcal{M}, z' \nvDash\sim \beta$. It follows that $\exists y', z', s', u' \in W$ s.t. $Mwy'z', Mz's'u', \mathcal{M}, y' \vDash \alpha$, and $\mathcal{M}, s' \vDash \beta$. From $\mathcal{M}, x \vDash \alpha \circ \beta$ I have $\exists s, u \in W$ s.t. $\mathcal{M}, s \vDash \alpha, \mathcal{M}, u \vDash \beta$, and $Msuy$. Applying all these results on (ram), I have $Mwsz$ or $Mwy'z$. If $Mwsz$, from $\mathcal{M}, s \vDash \alpha$ and $\mathcal{M}, w \vDash \alpha \multimap \gamma$, I have $\mathcal{M}, z \vDash \gamma$. If $Mwy'z$, similarly, I have $\mathcal{M}, z \vDash \gamma$. Hence, I know that $\mathcal{M}, w \vDash (\alpha \circ \beta) \multimap \gamma$. □

3.4.18. THEOREM. *(or) corresponds to (OR), (tra) corresponds to (Tra), (bi) corresponds to (BI), (b) corresponds to (B), (mig) corresponds to (M), (cam) corresponds to (CaM), and (ram) corresponds to (RaM).*

3.4.3 Soundness and Completeness

Now I move onto the proof of soundness and completeness of the logics N^0, N^E, and N^{RaM} with respect to the class of frames just defined. I define the normatively non-empty, non-trivial prime theories which later are used as canonical states, which are based on structures rather than formulas (as $\mathcal{L} \subseteq \mathcal{S}$).

3.4.19. DEFINITION. [Theories, Prime Theories, Non-Trivial, Normatively Non-Empty, and Consequents]

- A set w of structures $\mathcal{S}$ is a theory for logic N iff $X \in w$ and $X \vdash_N B$ imply $B \in w$, where $X \in \mathcal{S}$ and $B \in \mathcal{L}$.
- A theory w is prime iff $\alpha \uplus \beta \in w$ implies $\alpha \in w$ or $\beta \in w$.
- A theory w is trivial iff either **0** or $\bot$ is contained in w; otherwise it is non-trivial.
- A theory w is normatively empty iff $P\mathbf{0} \notin w$; otherwise it is normatively non-empty.
- Given a theory w. I say that $A \in \mathcal{L}$ is a consequent of a set w, denoted as $w \vdash A$, iff $\exists X \in w$ s.t. $X \vdash A$ where $X \in \mathcal{S}$.

Definition 3.4.19 is close but different than the definitions in [87, p.89-90]. I can now construct the canonical model as follows.

3.4.20. DEFINITION. [Canonical Model] Given a logic N, the canonical model $\mathcal{M}^C$ w.r.t. N is a structure $\langle W^C, M^C_\circ, M^C_\multimap, OK^C, V^C \rangle$ defined as follows:

- W^C be the set of all normatively non-empty, non-trivial, and prime theories over the structure $\mathcal{S}$ for logic N
- $M^C_\circ suw$ iff for all $\alpha \in s$ and $\beta \in u$ I have that $\alpha \circ \beta \in w$
- $M^C_\multimap wsu$ iff for all $\alpha \multimap \beta \in w$ and $\alpha \in s$ I have that $\beta \in u$

- $OK^C(s, w)$ iff for all $\alpha \in s$ I have that $P\alpha \in w$
- $V^C : ACT_0 \cup \{\mathbf{0}, \bot\} \to \wp(W^C)$ s.t. $w \in V^C(a)$ iff $a \in w$, where $a \in Act_0 \cup \{\mathbf{0}, \bot\}$

3.4.21. THEOREM. *Let $\mathcal{M}^C = \langle W^C, M^C_\circ, M^C_{\multimap}, OK^C, V^C \rangle$ be the canonical model. I then have that $M^C_\circ$ is equal to $M^C_{\multimap}$.*

This theorem is similar to the lemma in [87, p.254]. According to this theorem, I can simplify the canonical model into $\mathcal{M}^C = \langle W, M, OK, V \rangle$ without the superscript C, if no confusion arises.

The strategy that I use to prove this witness lemma is different from that in [87, p.255], because, without the inclusion relation in the frame, the key theorem Pair Extension Theorem [87, p.94] is not applicable in my logics. My general strategy is to apply consistency as in standard Existence Lemma in normal modal logic [13, p.200-201], i.e., (1) it is not the case that α and $\sim \alpha$ are consequents in a set, and (2) it is not the case that either $\mathbf{0}$ or $\bot$ are consequents in a set.

3.4.22. LEMMA (WITNESS LEMMA). *Let $\mathcal{M}^C = \langle W, M, OK, V \rangle$ be the canonical model, I have*

1. *If $\alpha \circ \beta \in w$, then exist $s, u \in W$ s.t. $Msuw$, $\alpha \in s$ and $\beta \in u$.*
2. *If $\alpha \multimap \beta \notin w$, then exist $s, u \in W$ s.t. $Mwsu$, $\alpha \in s$ and $\beta \notin u$.*
3. *If $P\alpha \notin w$, then exists $s \in W$ s.t. $\neg OK(s, w)$ and $\alpha \in s$.*

Proof:

I prove the third case here as an example. The proofs of the first two cases are similar.

Suppose $P\alpha \notin w$. Enumerate $\alpha_1, \cdots, \alpha_n, \cdots \in \mathcal{L}_A$. Construct u_n as follows:

1. $u_0 = \{\alpha\}$
2. $u_{i+1} = \begin{cases} u_i \cup \{\alpha_i\} & \text{if } P\alpha_i \notin w \\ u_i & \text{otherwise.} \end{cases}$

I need to show that $u_n \nvdash \mathbf{0}$ for each $n \in \omega$. I do so by induction on n.

1. If $n = 0$. If $u_0 \vdash \mathbf{0}$, then it implies $\alpha \vdash \mathbf{0}$. This follows $\alpha = \mathbf{0}$ because applying ($\mathbf{0}$) is the only way to introduce $\mathbf{0}$. So the assumption $P\alpha \notin w$ means that $P\mathbf{0} \notin w$. But it contradicts w is normatively non-empty.
2. Assume that $u_i \nvdash \mathbf{0}$. To show that $u_{i+1} \nvdash \mathbf{0}$. If not, then $u_i \cup \{\alpha_i\} \vdash \mathbf{0}$ where $P\alpha_i \notin w$. Since $u_i \nvdash \mathbf{0}$, it follows that $\alpha_i \vdash \mathbf{0}$ according to Definition 3.4.19 about consequents. This result implies that $\alpha_i = \mathbf{0}$ and $P\mathbf{0} \notin w$. Similar to the previous argument, this conclusion will lead to a contradiction. Therefore it concludes that $n_{i+1} \nvdash \mathbf{0}$.

Hence, $u_n \nvdash \mathbf{0}$ for each $n \in \omega$. Let $u^* = \bigcup_{i\in\omega} u_i$. So $u^* \nvdash \mathbf{0}$. Otherwise $\exists i \in \omega$ such that $u_i \vdash \mathbf{0}$, which contradicts $u_n \nvdash \mathbf{0}$ for each $n \in \omega$.

And then I enumerate $\varphi_1, \cdots, \varphi_n, \cdots \in \mathcal{L}_n$. Construct $u_n^{'}$ as follows:

1. $u_0^{'} = u^*$
2. $u_{i+1}^{'} = \begin{cases} u_i^{'} \cup \{\varphi_i\} & \text{if } \bot \notin u_i^{'} \cup \{\varphi_i\} \\ u_i^{'} & \text{otherwise.} \end{cases}$

I need to show that $u_n^{'} \nvdash \bot$ for each $n \in \omega$. Observe that its proof is easy to verify in the same method. Hence, $u_n^{'} \nvdash \bot$ for each $n \in \omega$. Let $u = \bigcup_{i\in\omega} u_i^{'}$. So $u \nvdash \bot$. Otherwise $\exists i \in \omega$ such that $u_i^{'} \vdash \bot$, which contradicts $u_n^{'} \nvdash \bot$ for each $n \in \omega$. In conclusion: $u \nvdash \mathbf{0}$ and $u \nvdash \bot$.

It remains to be shown that u is a normatively non-empty non-trivial prime theory. First show that u is a theory. Assume that $X \in u$ and $X \vdash D$ where $X \in \mathcal{S}$ and $D \in \mathcal{L}$. If u is not a theory, then $D \notin u$. By construction of u, if $D \in \mathcal{L}_A$, it follows that $\exists u_i$ s.t. $u_i \cup \{D\} \vdash \mathbf{0}$. Because $u_i \nvdash \bot$, by Definition 3.4.19, it then follows that $D \vdash \mathbf{0}$. According to (Cut), it implies that $X \vdash \mathbf{0}$. Yet it contradicts $u \nvdash \mathbf{0}$. If $D \in \mathcal{L}_n$, then the proof is similar. In sum, $D \in u$ where $D \in \mathcal{L}$, and thus u is a theory. Secondly I show that the theory u is prime. Assume that $\beta \uplus \gamma \in u$. Suppose $\beta, \gamma \notin u$. So $\exists u_i, u_j$ s.t. $u_i \cup \{\beta\} \vdash \mathbf{0}$ and $u_j \cup \{\gamma\} \vdash \mathbf{0}$. Let $u_i \subseteq u_j$. Then $u_j \cup \{\beta\} \vdash \mathbf{0}$ and $u_j \cup \{\gamma\} \vdash \mathbf{0}$. Because $u_i \nvdash \mathbf{0}$. Similar to the previous argument, I have that $\beta \vdash \mathbf{0}$ and $\gamma \vdash \mathbf{0}$. By ($\uplus L$) it follows that $\beta \uplus \gamma \vdash \mathbf{0}$. Yet it contradicts $u \vdash \mathbf{0}$. So u is a prime theory. The third is to show that u is non-trivial. If not, then $\mathbf{0}, \bot \in u$. By ($\mathbf{0}$), ($\bot$) and u is a theory, I then infer $u \vdash \mathbf{0}$ and $u \vdash \bot$, which contradicts my previous results. As the construction of u, $P\mathbf{0} \in u$. So u is normatively non-empty.

I need to construct a s as follows. Enumerate $\alpha_1, \cdots, \alpha_n, \cdots \in \mathcal{L}_A$. Construct s_n as follows: For any $n \in \omega$,

1. $s_0 = \{\alpha\}$
2. $s_{i+1} = \begin{cases} s_i \cup \{\alpha_i\} & \text{if } s_i \cup \{\alpha_i\} \nvdash \mathbf{0} \\ s_i & \text{otherwise.} \end{cases}$

I need to show that $s_n \nvdash \mathbf{0}$ for each $n \in \omega$. The proof is similar to the previous one. Let $s^* = \bigcup_{i\in\omega} s_i$. Notice that $s^* \nvdash \mathbf{0}$. Its verification follows from $s_n \nvdash \mathbf{0}$ for each $n \in \omega$. And then, I enumerate $\varphi_1, \cdots, \varphi_n, \cdots \in \mathcal{L}_n$. Construct $s_n^{'}$ as follows:

1. $s_0^{'} = u^*$
2. $s_{i+1}^{'} = \begin{cases} s_i^{'} \cup \{\varphi_i\} & \text{if } \bot \notin s_i^{'} \cup \{\varphi_i\} \\ s_i^{'} & \text{otherwise.} \end{cases}$

Observe that $s_n^{'} \nvdash \bot$ for each $n \in \omega$. Let $s = \bigcup_{i\in\omega} s_i^{'}$. So $s \nvdash \bot$. Similarly, I have $s \nvdash \mathbf{0}$ and $s \nvdash \bot$. The verification of s as a normatively non-empty non-trivial prime theory is similar to the previous proof for u.

Because $\alpha \in u$ and $P\alpha \notin w$, I conclude that $OK(s, w)$ does not hold. In summary, there exists a non-empty non-trivial prime theory s such that $\neg OK(s, w)$ and $\alpha \in s$. □

3.4.23. LEMMA (TRUTH LEMMA). *Given the canonical model $\mathcal{M}^C = \langle W, M, OK, V \rangle$, I have*

$$w \models A \textit{ iff } A \in w,$$

where $A \in \mathcal{L}$.

The truth lemma can be proved by applying the witness lemma in the standard way [13].

To prove the completeness, I need the Lindenbaum Lemma. Its proof strategy is similar to Lemma 3.4.22.

3.4.24. LEMMA (LINDENBAUM LEMMA). *Let $\mathcal{M}^C = \langle W, M, OK, V \rangle$ be the canonical model. If $X \nVdash A$, then $\exists w \in W$ s.t. $X \in w$ and $A \notin w$, where $A \in \mathcal{L}$.*

Proof:
Let $X \nVdash A$. Enumerate $C_1, \cdots, C_n, \cdots \in \mathcal{L}$. Construct w_n as follows:

1. $w_0 = \{X\}$
2. $w_{i+1} = \begin{cases} w_i \cup \{C_i\} & \text{if } w_i \cup \{C_i\} \nVdash A; \\ w_i & \text{otherwise.} \end{cases}$

I need to show that $w_n \nVdash A$ for all $n \in \omega$. I prove this by induction on $n \in \omega$.

1. When $n = 0$. If not, then $X \vdash A$. This contradicts the assumption.
2. When $n = i + 1$. Assume that $w_i \nVdash A$. I need to show that $w_{i+1} \nVdash A$. By the construction of w_{i+1}, I have $w_{i+1} = w_i \cup \{C_i\}$ such that $w_{i+1} \nVdash A$. Otherwise $w_{i+1} = w_i$. Then obviously $w_{i+1} \nVdash A$ by $w_i \nVdash A$.

Let $w = \bigcup_{i\in\omega} w_i$. I need to show that $w \nVdash A$. If not, then $\exists w_i \subseteq w \exists Y \in w_i$ s.t. $Y \vdash A$. This directly contradicts $w_i \nVdash A$.

Now I show that w is a non-empty non-trivial prime theory. The argument is similar to the previous one in Lemma 3.4.22. By the construction of w, $X \in w$ but $A \notin w$. Hence I get the result. □

3.4.25. THEOREM (SOUNDNESS AND COMPLETENESS). *The logics N^0, N^E and N^{RaM} are sound and complete w.r.t. the class of (or), (b) and (bi), and (ram) frames.*

Proof:
The soundness and completeness can be proved in a standard way within witness lemma, truth lemma, and Lindenbaum lemma, as well as correspondence theorems 3.4.18.

Here I prove the soundness of N^0.

- For (Id) it is clear.

- For ($\mathbf{0}$). Let $\mathcal{M}, w \vDash X[\mathbf{0}]$. Induction on the complexity of X. The first case is that $X \in \mathcal{S}_{\mathcal{B}}$. The first subcase is that $X \in \mathcal{L}_{\mathcal{B}}$. Assume that $X =: \alpha \in \mathcal{L}_A$, then $X[\mathbf{0}] = \alpha[\mathbf{0}] = \mathbf{0}$. However, this implies $\mathcal{M}, w \vDash \mathbf{0}$ for all $w \in W$. I then of course have $\mathcal{M}, w \vDash \gamma$ for arbitrary $\gamma \in \mathcal{L}_A$. Assume that $X =: \alpha \multimap \beta \in \mathcal{L}_{\mathcal{B}}$, the proof is similar. The second subcase is $X = (Y; Z)$. I assume that $X[\mathbf{0}] = Y[\mathbf{0}]; Z$. So there exist $y, z \in W$ s.t. $Mwyz$ and $\mathcal{M}, y \vDash Y[\mathbf{0}]$. By the inductive hypothesis, this implies $\mathcal{M}, y \vDash \mathbf{0}$ by the validity of ($\mathbf{0}$). Yet it is impossible, as y is non-trivial. Then neither does $\mathcal{M}, w \vDash X[\mathbf{0}]$. So I have $\mathcal{M}, w \vDash \gamma$ for arbitrary $\gamma \in \mathcal{L}$ in a trivial way. This is similar to the second case $X \in \mathcal{S}_d$.

- The other cases are standard, except for (OR) and (Tra). Luckily, these two cases are ensure by theorem 3.4.18. Similar to the soundness of N^E and N^{RaM}.

The completeness is as standard in [13]. □

The soundness and completeness can be proved in a standard way within witness lemma, truth lemma, and Lindenbaum lemma, as well as correspondences theorem 3.4.18.

3.4.4 Applications to Free Choice Inferences

The substructural logics I have just studied provide a good balance between cautiousness and deductive power. They avoid the unwelcome free choice inferences, while still allowing plausible ones. Here I call the frame class of all frames satisfying (or), (bi), (b), and (ram) the open-reading frame class.

3.4.26. THEOREM. *The following consequences are* **not valid** *in the open reading frame class: given $\alpha, \beta, \gamma \in \mathcal{L}_A - \{\mathbf{0}\}$, and $X \in \mathcal{S}$ where $\mathbf{0}$ is not a substructure in X,*

1. $(\alpha \circ \beta) \vdash \alpha$

2. $\alpha \multimap \beta \vdash (\alpha \circ \gamma) \multimap \beta$

3. $(\alpha \circ \alpha) \vdash \alpha$

4. $\alpha \multimap \alpha \vdash (\alpha \circ \alpha) \multimap \alpha$

5. $(\alpha \multimap \beta), (\gamma \multimap \beta) \vdash (\alpha \circ \gamma) \multimap \beta$

6. $X \vdash (\alpha \uplus \sim \alpha)$

7. $X \vdash (\alpha \uplus \sim \alpha) \multimapboth (\beta \uplus \sim \beta)$

8. $X \vdash \alpha \multimap (\beta \uplus \sim \beta)$

The above invalidities are part of my solution to the three counter-intuitive free choice permission inferences discussed earlier. First, since the monotonic $\alpha \multimap \beta \vdash (\alpha \circ \gamma) \multimap \beta$ is not valid in the open reading frame class, $\alpha \multimap \beta \vdash P\beta \rightarrow P(\alpha \circ \gamma)$ is not valid either. So the systems N^0, N^E, and N^{RaM} can exclude the unrestricted monotonic cases like "if it is permitted to order a vegetarian lunch, then it is permitted to order a vegetarian lunch and not pay for it." Second, because the resource-insensitive $\alpha \multimap \alpha \vdash (\alpha \circ \alpha) \multimap \alpha$ is invalid in the open reading frame class, $\alpha \multimap \alpha \vdash P\alpha \rightarrow P(\alpha \circ \alpha)$ is also not valid. I then can exclude resource insensitive cases like "if it is permitted to eat one cookie then it is permitted to eat more than one" in the sequent calculus systems N^0, N^E, and N^{RaM} too. Third, the irrelevant cases are also excluded in the sequent calculus N^0, because $X \vdash (\alpha \uplus \sim \alpha) \multimapboth (\beta \uplus \sim \beta)$ and $X \vdash \alpha \multimap (\beta \uplus \sim \beta)$ are not valid in the open reading frame class. Thus I cannot conclude the following two consequences $X \vdash P(\alpha \uplus \sim \alpha) \subset\!\!\rightarrow P(\beta \uplus \sim \beta)$ and $X \vdash P(\beta \uplus \sim \beta) \rightarrow P\alpha$, which contain cases like "it is permitted to eat an apple or not iff it is permitted to sell a house or not" and "if it is permitted to eat an apple or not then it is permitted to sell a house." The family of substructural logics N^0, N^E, and N^{RaM} studies in the previous section avoid the aforementioned unrestricted monotonic, resource insensitive, and irrelevant free choice permission inferences.

The family of substructural logics N^0, N^E, and N^{RaM} can also exclude the following conflict cases by excluding the mingle rule:

$$(\alpha \multimap \beta), (\gamma \multimap \beta) \vdash P\beta \rightarrow P(\alpha \circ \gamma),$$

by giving up $(\alpha \multimap \beta), (\gamma \multimap \beta) \vdash (\alpha \circ \gamma) \multimap \beta$. So, though in a normal situation given "selling the house to Ann is a licensed instance of selling the house" and "selling the house to Bob is a licensed instance of selling the house," I will not conclude the conflict case "if it is permitted to sell the house then it is permitted to sell the house to Ann and sell the house to Bob" in N^0, N^E, and N^{RaM}.

On the other hand, the logic N^{RaM} in this family can derive some restricted monotonic cases in the following form:

$$(\alpha \multimap \beta), \neg(\alpha \multimap\sim \gamma) \vdash P\beta \rightarrow P(\alpha \circ \gamma).$$

For instance, the good variant of the vegetarian lunch example can be derived in logic N^{RaM}. From a normal situation "it is not the case that ordering a lunch is a licensed instance of doing any other action except paying for the lunch," this logic can derive the consequence that "if it is permitted to order a lunch then it is permitted to order the lunch and pay for it."

3.4.27. Theorem (Incomplete and Consistent). *1. The consequence $\alpha \uplus \sim \alpha$ is not valid in (or)-frames, so the states to instantiate types are not complete. In fact, this property ensures that the free choice permission inferences are relevant.*

2. *The (bi)-frame class of all frames,* $(\alpha \circ \sim \alpha) \vdash \mathbf{0}$ *is valid.*[8] *So the states to instantiate types are consistent.*

3. *The double negation introduction* $\alpha \vdash \sim\sim \alpha$ *is valid in the (bi)-frame class, but this is not the case for the double negation elimination* $\sim\sim \alpha \vdash \alpha$.

The family of N^0 and its extensions N^E and N^{RaM} not only excludes the undesired properties discussed in Section 3.2 and saves the deductive power of free choice permission inferences. It also satisfies some interesting properties: the states are incomplete but consistent.[9]

3.4.5 Proof Theory

Cut elimination states that, if $X \vdash A$ is a derivation by the application of the cut rule, then it can be a derivation without using the cut rule. Cut elimination is important because it implies the subformula property. The definition of cut-height that I use is inspired by [75, p.35], and the definition of principal formulas is inspired by [75, p.29].

3.4.28. DEFINITION. [Cut-Height] The cut-height of a derivation with application of the rule of cut in a proof is the sum of heights of the two premises of cut plus 1.

3.4.29. DEFINITION. [Principal Formulas] A formula is principal in a derivation iff it is introduced in a conclusion derived from a rule in the system.

The cut elimination can be formalized in the following way, by using cut-height.

3.4.30. DEFINITION. [Cut Elimination] A system satisfies the cut elimination iff for arbitrary conclusion $Y[X] \vdash B$ concluded from the cut rule with premises $X \vdash_n A$ and $Y[A] \vdash_m B$ in a cut-height $n + m + 1$, the system can derive $Y[X] \vdash B$ in a cut-height at most $n + m$, where $n, m \in \omega$.

3.4.31. THEOREM. *The systems* N^0 *and* N^E *satisfy cut elimination.*

Proof:
Given a derivation $X \vdash A$ which is derived from the cut rule. The proof of cut elimination is proved in the standard way by inducing on the cut-height of $X \vdash A$. Here I only present the interesting cases in below.

[8] So the action negation $\sim$ still satisfies the *ex contradictione quodlibet* rule (ECQ) $(\alpha \circ \sim \alpha) \vdash \beta$ in the (bi)-frame class, which is rejected in relevant logics.

[9] Given arbitrary frame $\mathcal{F}$ for open reading, all states in $\mathcal{F}$ are consistent and not complete, can be understood in this way: $(\alpha \circ \sim \alpha) \multimap \beta$ is valid in $\mathcal{F}$, but not $\alpha \multimap (\beta \uplus \sim \beta)$.

- $X \vdash A$ is a derivation by applying Cut with at least one premise such that its height of derivation is 0. Consider two subcases: either (1) the left premise $X \vdash_0 A$ is an instance of either (Id), ($\mathbf{0}$) or ($\bot$); or (2) the right premise $Y[A] \vdash_0 B$ is an instance of either (Id), ($\mathbf{0}$) or ($\bot$).

  For the first subcase, I only present the derivation by ($\mathbf{0}$) as an example for the first subcase:

$$\dfrac{X[\mathbf{0}] \vdash_0 \gamma \qquad Y[\gamma] \vdash_n \beta}{Y[X[\mathbf{0}]] \vdash_{n+1} \beta}\ (\text{Cut})$$

  It's clear that $Y[X[\mathbf{0}]] \vdash_0 \beta$ according to ($\mathbf{0}$).

  For the second subcase, I have

$$\dfrac{Y \vdash_n \mathbf{0} \qquad X[\mathbf{0}] \vdash_0 \gamma}{X[Y] \vdash_{n+1} \gamma}\ (\text{Cut})$$

  Two possibilities of this proof:

  1. If $\mathbf{0}$ in $Y \vdash_n \mathbf{0}$ is principal. The non-trivial case is that $Y \vdash_n \mathbf{0}$ is derived by ($\sim L$). Then $Y \vdash_n \mathbf{0}$ is in the form $Z[\sim \alpha; Z'] \vdash_n \mathbf{0}$, and the above proof is in the following form:

$$\dfrac{\dfrac{Z' \vdash_{n-k-1} \alpha \qquad Z[\mathbf{0}] \vdash_k \mathbf{0}}{Z[\sim \alpha; Z'] \vdash_n \mathbf{0}}\ (\sim L) \qquad X[\mathbf{0}] \vdash_0 \gamma}{X[Z[\sim \alpha; Z']] \vdash_{n+1} \gamma}\ (\text{Cut})$$

     where $0 \leq k \leq n-1$. This proof can be transferred into:

$$\dfrac{Z' \vdash_{n-k-1} \alpha \qquad \dfrac{Z[\mathbf{0}] \vdash_k \mathbf{0} \qquad X[\mathbf{0}] \vdash_0 \gamma}{X[Z[\mathbf{0}]] \vdash_{k+1} \gamma}\ (\text{Cut})}{X[Z[\sim \alpha; Z']] \vdash_{n+1} \gamma}\ (\sim L)$$

     whose cut-height is at most n.

  2. If $\mathbf{0}$ in $Y \vdash_n \mathbf{0}$ is not principal. For example, $Y \vdash_n \mathbf{0}$ is derived by ($\circ L$), and the proof concerned is presented as follows:

$$\dfrac{\dfrac{Y[\alpha; \beta] \vdash_{n-1} \mathbf{0}}{Y[\alpha \circ \beta] \vdash_n \mathbf{0}}\ (\circ L) \qquad X[\mathbf{0}] \vdash_0 \gamma}{X[Y[\alpha \circ \beta]] \vdash_{n+1} \gamma}\ (\text{Cut})$$

     Obviously this proof can be transferred into:

$$\dfrac{\dfrac{Y[\alpha; \beta] \vdash_{n-1} \mathbf{0} \qquad X[\mathbf{0}] \vdash_0 \gamma}{X[Y[\alpha; \beta]] \vdash_n \gamma}\ (\text{Cut})}{X[Y[\alpha \circ \beta]] \vdash_{n+1} \gamma}\ (\circ L)$$

     whose cut-height is at most n.

- Cut with premises which are derived with height is at least 1. So its premises cannot be derived from (Id), ($\mathbf{0}$) nor ($\perp$). In this case, there are three subcases: Cut formula A is not principal in the left premise; Cut formula A is principal in the left premise only; and Cut formula A is principal in both premises. Given $n, m \geq 1$:

  1. For the first subcase, I only show the application of ($\multimap L$) as an example. Suppose that the concluded derivation is in the cut-height $n + m + 1$ where $n \geq k \geq 1$:

$$\dfrac{\dfrac{Z \vdash_{n-k} \gamma \qquad X[\beta] \vdash_{k-1} \alpha}{X[\gamma \multimap \beta; Z] \vdash_n \alpha}(\multimap L) \qquad Y[\alpha] \vdash_m \delta}{Y[X[\gamma \multimap \beta; Z]] \vdash_{n+m+1} \delta}(\text{Cut})$$

     This proof can be transformed into the following proof such that the concluded derivation is in a cut-height $m+k$, which is of course at most $n+m$:

$$\dfrac{Z \vdash_{n-k} \gamma \qquad \dfrac{X[\beta] \vdash_{k-1} \alpha \qquad Y[\alpha] \vdash_m \delta}{Y[X[\beta]] \vdash_{m+k} \delta}(\text{Cut})}{Y[X[\gamma \multimap \beta; Z]] \vdash_{n+m+1} \delta}(\multimap L)$$

  2. For the second subcase, I show two applications as examples.

     – Application of ($\multimap R$). Suppose that the concluded derivation is in a cut-height $n + m + 1$:

$$\dfrac{X \vdash_n \alpha \qquad \dfrac{Y[\alpha]; \beta \vdash_{m-1} \gamma}{Y[\alpha] \vdash_m \beta \multimap \gamma}(\multimap R)}{Y[X] \vdash_{n+m+1} \beta \multimap \gamma}(\text{Cut})$$

       This proof can be transformed into the following proof such that the concluded derivation in a cut-height is $n + m$, which is at most $n + m$:

$$\dfrac{\dfrac{X \vdash_n \alpha \qquad Y[\alpha]; \beta \vdash_{m-1} \gamma}{Y[X]; \beta \vdash_{n+m} \gamma}(\text{Cut})}{Y[X] \vdash_{n+m+1} \beta \multimap \gamma}(\multimap R)$$

  3. For the third subcase, I show two applications:

     – Application of ($\multimap R$) and ($\multimap L$): Suppose that the concluded derivation is in a cut height $n + m + 1$ where $m \geq k \geq 1$:

$$\dfrac{\dfrac{X; \alpha \vdash_{n-1} \beta}{X \vdash_n \alpha \multimap \beta}(\multimap R) \qquad \dfrac{Z \vdash_{m-k} \alpha \qquad Y[\beta] \vdash_{k-1} \gamma}{Y[\alpha \multimap \beta; Z] \vdash_m \gamma}(\multimap L)}{Y[X; Z] \vdash_{n+m+1} \gamma}(\text{Cut})$$

       It can be transformed into the proof which concluded the original derivation in cut-heights $n + m - k$ and $n + m$, which are both at most $n + m$:

$$\dfrac{\dfrac{Z \vdash_{m-k} \alpha \qquad X; \alpha \vdash_{n-1} \beta}{X; Z \vdash_{n+m-k} \beta}(\text{Cut}) \qquad Y[\beta] \vdash_{k-1} \gamma}{Y[X; Z] \vdash_{n+m} \gamma}(\text{Cut})$$

– Application of (OR): Suppose that the concluded derivation is in a cut-height $n + m + 1$:

$$\dfrac{\dfrac{X;\beta \vdash_{n-1} \alpha}{X, P\alpha \vdash_n P\beta}\text{(OR)} \qquad \dfrac{Y;\gamma \vdash_{m-1} \beta}{Y, P\beta \vdash_m P\gamma}\text{(OR)}}{Y,(X, P\alpha) \vdash_{n+m+1} P\gamma}\text{(Cut)}$$

It can be transformed into the proof which concluded the original derivation in the cut height at most $n + m$:

$$\dfrac{\dfrac{Y;\gamma \vdash_{m-1} \beta \qquad X;\beta \vdash_{n-1} \alpha}{(Y,X);\gamma \vdash_{n+m-1} \alpha}\text{(Tra)}}{Y, X, P\alpha \vdash_{n+m} P\gamma}\text{(OR)}$$

– Application of (Tra): Suppose that the concluded derivation is in a cut-height $n + m + 1$:

$$\dfrac{\dfrac{X;\alpha \vdash_{n-k-1} \beta \qquad Z'[Y;\beta] \vdash_k P\gamma}{Z'[(X,Y);\alpha] \vdash_n P\gamma}\text{(Tra)} \qquad \dfrac{Z;\delta \vdash_{m-1} \gamma}{Z, P\gamma \vdash_m P\delta}\text{(OR)}}{Z, Z'[(X,Y);\alpha] \vdash_{n+m+1} P\delta}\text{(Cut)}$$

It can be transformed into the proof concluding the original derivation in the cut height at most $n + m$:

$$\dfrac{X;\alpha \vdash_{n-k-1} \beta \qquad \dfrac{Z'[Y;\beta] \vdash_k P\gamma \qquad Z, P\gamma \vdash_m P\delta}{Z, Z'[Y;\beta] \vdash_{k+m+1} P\delta}\text{(Cut)}}{Z, Z'[(X,Y);\alpha] \vdash_{n+m+1} P\delta}\text{(Tra)}$$

□

In conclusion, this section provides a family of sound and complete substructural logics with cut-elimination that characterize a calculus for one kind of conditional over action types and permissions, such that it can avoid deriving the undesired consequences in free choice permission, which are irrelevant, unrestricted monotonic, and resource insensitive.

3.5 Conclusion and Discussion of Related Work

This chapter suggests a family of substructural logics for excluding the unrestricted monotonic, resource insensitive, and irrelevant free choice permission inferences. One logic among this family can derive the consistent monotonic cases of free choice permission by accepting my version of rational monotony (RaM). The driving idea of this family of substructural logics comes from the open reading of free choice permission. Following this semantic core, these logics offer various sound and complete calculus on action types and permissions, such that the calculus not only is monotonic, resource sensitive, and relevant, but also consistent and incomplete.

The logic I develop in this chapter is close to, but different from the one developed by Barker in [11]. First, my substructural logics are weaker than Barker's linear logic [11], though similar in that Barker's and my proposal both exclude contraction and left-hand weakening.[10] Statements like "if it is permitted to do A and B then it is permitted to do B and A" are derivable by using the multiplicative "and" in Barker's linear logic. These statements are not derivable in my substructural logic N^0. In addition my logic N^{RaM} gains deductive power in comparison to Barker's by using the reformed rational monotony. My logic N^{RaM} is thus able to derive more intuitive free choice permission statements like "if it is permitted to order a lunch then it is permitted to order a lunch and pay for it," which are not derivable in Barker's logic.

In this chapter I have not considered conditional permissions [71, 37]. Instead I focus on the analysis of *prima facie* permissions through the study of the consequences on action types. Even though my logics are inspired by non-monotonic reasoning, both in their interpretations (by the reading "*licensed instances*") and its proof system (by (RaM)), they are substantially weaker because of the resource-sensitive and relevant aspects of them.

Obligations are missing in this chapter. This is the natural step for future work. For instance, I can define obligations as the dual of permissions, viewing permissions as weak permissions. This logic would be still weaker than standard deontic logic, since negation is not classical in my logics. Another option is to view permissions as strong permissions.[11] In this case, it is not clear whether obligations should be defined as the dual of permissions. And so the relation between obligations and permissions might become weaker.

The logics developed in this chapter provide a view of *action guiding* by taking the logical property RaM into normative reasoning. What RaM gives is a simple evaluation of how plausible a composition of two action types is, if one would not undermine the licensed aspect of the other. However, in a more concrete case, such an evaluation could be more complicated. In the next two chapters, I will see two other different properties in non-monotonic reasoning for normative concepts, in order to deal with concrete cases, especially in a number of legal situations.

[10] Removing (OR) and (Tra) from the logic N^0, obviously, this system is a basic substructural logic for full Lambek calculus **FL** [33]. It is weaker than Barker's linear logic, because my fusion is neither associative nor commutative in N^0.

[11] As I discussed earlier, strong permissions are not the dual of obligations.

Chapter 4

Permission in Dynamics: Power and Immunity

In this chapter I look at permission from the perspective of legal theory. More precisely, I am interested in the Hohfeldian typology of rights. The main contribution is a new theory of what I call dynamic rights: Power and Immunity. This theory uses modern tools from dynamic epistemic logic, but gives it a deontic interpretation.

The Hohfeldian [49] typology of rights distinguishes what one might call *static* and *dynamic* rights. Static basic rights encompass claims and privileges, as well as their respective correlatives of duties and no-claim. On the dynamic side one finds power and immunity together with the correlatives of liability and no-power. See Figure 1.3 for the classic presentation.

What I call here static and dynamic rights have been labelled in various ways in the formal literature. Kanger called static rights the "type of the states of affairs" and dynamic ones the "type of influence" [56]. Makinson instead used the "deontic family" and the "legally capacitative family" for static and dynamic rights, respectively [68]. Bentham, von Wright and Hart on the other hand used "legal validity" and "norm-creating action" [63], while Lindahl [63] followed this action viewpoint, and called it "the range of action."

Although logical approaches to legal competences are scarcer than for static normative positions, existing theories can be divided into two broad families. The first formalizes power and immunity as (legal) permissibility, or absence thereof, to see to it that a certain normative position obtains [57, 63]. Lindahl [63], for instance, captured j's power to make it the case that i ought to see to it that φ using a combination of action and embedded deontic modalities:

$$P\ Do_j O(Do_i \varphi)$$

I call such an approach reductive because it takes power and immunity as definable in the language of obligations, permissions, and action, where claims and privileges are also defined. Non-reductive approaches, on the other hand, view power and immunity as position-changing actions that are not reducible to static normative positions [68, 54]. A typical example of this is Jones and Sergot [54], who captured legal power through "counts as" conditionals.

Each of these families has assets and drawbacks. Reductive approaches come with a rich logical theory of the relationship between static normative conditions and legal competences, with the latter inheriting its logic from the former. Defining power as above, however, obfuscates the dynamic character of legal competence by reducing it to permissibility, a simple static legal relation. This dynamic aspect was arguably crucial for Hohfeld who defined power as the ability to "*change* legal relations" [49, p.44-45]. The formalization above furthermore conflates legal ability (*rechtliches Können*) with legal permissibility (*rechtliches Dürfen*), although these two concepts are distinct [68, 54]. Non-reductive approaches, on the other hand, do better justice to the dynamic character of legal competences by taking norm-changing actions as first class citizens in the logic. This allows us, by the same token, to distinguish legal ability and legal permissibility. The cost of this is a relatively weak logic of legal competences, which is at least at the outset completely independent from the logic of the static normative position.

The dynamic logic that I present in this chapter provides a plausible middle ground between these two types of approaches. It is reductive, and as such comes with a rich set of principles of interaction between static and dynamic rights. It does so, however, while retaining both the dynamic character of legal competences and the distinction between legal ability and legal permissibility.

The reader familiar with dynamic *epistemic* logic [103] will recognize both the modelling methodology and many of the canonical results (axiomatization, bisimulation invariance) that I present here. What I propose is a deontic re-interpretation of this framework. I show that this yields interesting insights for the theories of legal competence, and can be applied to the concrete question of the distinction between legal ability and legal permissibility.

The rest of the chapter is structured as follows. My main contribution being the dynamic part of the model, for the static part I use a fairly standard model for conditional obligations. I present it briefly in Section 4.1, and then move to dynamic modalities and legal competence in Section 4.2. I show how the two put together capture the four Hohfeldian basic types of right, present a complete axiomatization, and study its model theory. I then turn to the combinatorics of conditional normative positions both in statics and dynamics in Section 4.3. Finally, I apply it to a concrete case in the German civil code in order to show that legal ability and legal permissibility can be naturally distinguished in this logic.

4.1 Static Rights

My starting point is a conditional version of the Kangerian model of claims and privileges [56, 63, 68]. The latter is the standard in current theories of the normative position [94], and hence comes with well-studied models of claims and privilege. The conditional version I propose follows the one developed in [110], but goes back at least to [41]. Little, however, rests on this modelling choice in the sense that the dynamic methodology that I present later is fairly modular. In other contexts it has been successfully used to extend very different static logical systems [105, 106]. The same exercise could be done here.

On the surface, the language I use differs from classical Kangerian approaches in that it contains two Kripke modalities on two underlying independent preference orders, along with the usual "seeing to it that" modality.[1] I use this language instead of the classical deontic one for technical reasons. It facilitates the axiomatization of the dynamic modalities. It is well known, however, the language with one preferential modality, which lies on a converse well-founded pre-order,[2] can define the conditional obligations and permissions [15, 111]. We will come back to this at the end of the section. The logic of conditional obligation can be completely axiomatized, both in the current language [103] or by taking just conditional obligations as primitive [9]. Several works have done this for the reducible characterization of the conditional within the converse well-founded condition [108, 35], but its complete axiomatization with Löb axiom is left open [9, 35]. Here, instead, I follow a suggestion made in [9], and introduce two independent Kripke modalities $[<]$ and $[\cong]$. The standard preferential $[\leq]$ modality is definable in this language.

4.1.1. DEFINITION. Let $Prop$ be a countable set of propositions and $\mathcal{I}$ a set of agents. The language $\mathcal{L}$ is defined as follows:

$$\varphi := p \in Prop \mid \neg\varphi \mid \varphi \wedge \varphi \mid [<]\varphi \mid [\cong]\varphi \mid A\varphi \mid Do_i\varphi$$

where $i \in \mathcal{I}$.

I write $\langle<\rangle\varphi$ for $\neg[<]\neg\varphi$, $\langle\cong\rangle\varphi$ for $\neg[\cong]\neg\varphi$, and $E\varphi$ for $\neg A\neg\varphi$. A formula $A\varphi$ is read as "it is necessary that φ." $Do_i\varphi$ indicates a *non-deontic* or *ontic* [114] action of agent i, and should be read in the usual sense of "i sees to it that φ." In addition, the modality $[<]$ is the Kripke modality for the "strictly better" relation, and $[\cong]$ for the "as good as" relation. I define a formula $[\leq]\varphi$ as $([<]\varphi \wedge [\cong]\varphi)$ for the standard preferential concept "at least as good as," and write $\langle\leq\rangle\varphi$ for $\neg[\leq]\neg\varphi$.

The semantics of this language is provided in the models including two preferential orders $<$ and $\cong$, augmented with a Kripke relation for the Do_i operators. Because of

[1] In the literature of formal theories of rights, there are two main accounts to define "seeing to it that." One formulates "seeing to it that" into an unary normal Kripke modality, which, for instance in the Kangerian tradition [56, 63, 68], usually satisfies *T* axiom as well as the *E* rule the substitution of logical equivalence, but in STIT it can also be a **S5** modality [22, 50]. The other dominant account defines "seeing to it that" in the form of binary conditionals, as the Jones and Sergot framework [54] presented, in which the logic of "seeing to it that" is quite weak.

[2] An order is a pre-order iff it is reflexive and transitive. A relation R is converse well-founded (CWF) iff for each non-empty set X, there exists a R-greatest element of X, an element w of X s.t. $\neg wRx$ for all $x \in X$. This is equal to saying there is no infinite path of $x_0R\cdots Rx_nR\cdots$. In the literature of dynamic update on preference [9, 110, 35], a pre-order $\leq$ is CWF iff for each non-empty set X, there exists a $<$-greatest element of X. In fact, the CWF condition is stronger than David Lewis's limit assumption [60], which restricts those sets merely representing formulas.

this, the standard preferential models [109, 110] can be embedded into my construction, by defining the standard pre-order as the union of these two preference orderings. I do not assume that the two preference orderings are connected, and nor does the pre-order. This will give rise to slight differences from, e.g. [109, 110].

4.1.2. DEFINITION. Let $Prop$ and $\mathcal{I}$ be as above. A *preference-action model* $\mathcal{M}$ is a tuple $\langle W, <, \cong, \{\sim_i\}_{i \in \mathcal{I}}, V \rangle$ where:

- W is a non-empty set of states
- $<$ is a converse well-founded and transitive relation on W, and $\cong$ is an equivalence relation on W, such that they satisfy this property:

$$\forall xy \in W.(z \cong y \wedge y < x \rightarrow z < x) \qquad \text{(Interaction}_1\text{)}$$

- for each $i \in \mathcal{I}$, $\sim_i$ is an equivalence relation
- $V : Prop \rightarrow \mathcal{P}(W)$ is a valuation function

Observe that Interaction$_1$ corresponds to $[<]\varphi \rightarrow [\cong][<]\varphi$. I can then define a relation $\leq$ according to this Inclusion property: $\leq = < \cup \cong$. Indeed, $\leq$ is transitive and reflexive. And this model satisfies two further interaction principles, which will be useful later on:

$$\text{(Interaction}_2\text{)} \qquad \forall xy \in W.(z \leq y \wedge y < x \rightarrow z < x)$$
$$\text{(Interaction}_3\text{)} \qquad \forall xy \in W.(z \cong y \wedge y \leq x \rightarrow z \leq x)$$

Preference-action frames are preference-action models minus the valuation. The assumption that the relations $\sim_i$ are equivalence relations is present only to simplify the treatment of static rights and thus put the emphasis on my dynamic extension. As above, this assumption could be lifted. I can then interpret the sentences in language $\mathcal{L}$ according to a preference-action model as follows.

4.1.3. DEFINITION. The truth conditions for sentence $\varphi \in \mathcal{L}$ are defined in the following:

$\mathcal{M}, w \vDash p$	iff	$w \in V(p)$
$\mathcal{M}, w \vDash \neg\varphi$	iff	$\mathcal{M}, w \nvDash \varphi$
$\mathcal{M}, w \vDash \varphi \wedge \psi$	iff	$\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, w \vDash \psi$
$\mathcal{M}, w \vDash [<]\varphi$	iff	$\mathcal{M}, w' \vDash \varphi$ for all $w' > w$
$\mathcal{M}, w \vDash [\cong]\varphi$	iff	$\mathcal{M}, w' \vDash \varphi$ for all $w' \cong w$
$\mathcal{M}, w \vDash A\varphi$	iff	$\mathcal{M}, w' \vDash \varphi$ for all $w' \in W$
$\mathcal{M}, w \vDash Do_i\varphi$	iff	$\mathcal{M}, w' \vDash \varphi$ for all $w' \sim_i w$

Validity on models and frames, and the classes thereof, is defined as usual. I define $||\varphi||$, the truth set of φ, as $\{w : \mathcal{M}, w \vDash \varphi\}$. Notice that the truth condition for $[\leq]$ turns out to be the standard one:

$$\mathcal{M}, w \models [\leq]\varphi \text{ iff } \mathcal{M}, w' \models \varphi \text{ for all } w' \geq w$$

As mentioned, conditional obligation, understood in terms of "truth in all the most preferred worlds," is definable in this language. The argument is standard [15], I nonetheless give the proof for the sake of completeness. Let $O(\psi/\varphi)$ be defined as follows:

$$\mathcal{M}, w \models O(\psi/\varphi) \text{ iff } \mathcal{M}, w' \models \psi \text{ for all } w' \in max_{\geq}(||\varphi||)$$

with $max_{\geq}(X) = \{w \in X : \neg\exists w' \in X \; s.t. \; w' > w\}$. Then conditional obligation is definable as:

$$O(\psi/\varphi) \leftrightarrow A(\varphi \to \langle\leq\rangle(\varphi \wedge [\leq](\varphi \to \psi)))$$

Proof:
I need to show that $\mathcal{M}, w \models O(\psi/\varphi) \Leftrightarrow \mathcal{M}, w \models A(\varphi \to \langle\leq\rangle(\varphi \wedge [\leq](\varphi \to \psi)))$.

(LtR) Assume that $\mathcal{M}, w \models O(\psi/\varphi)$, which means that $max_{\geq}(||\varphi||) \subseteq ||\psi||$. Let $\mathcal{M}, u \models \varphi$. By reflexivity of $\leq$ it follows that $u \leq u$, which indicates that $\{w' \in ||\varphi|| \mid u \leq w'\} \neq \emptyset$. By applying that $\leq$ is converse well-founded, I have $(*)$ $\exists w \in \{w' \in ||\varphi|| \mid u \leq w'\}$ s.t. if $s \in \{w' \in ||\varphi|| \mid u \leq w'\}$ then $w \nless s$. Given arbitrary $t \geq w$ that $t \in ||\varphi||$. I need to show $t \in max_{\geq}(||\varphi||)$. By $w \geq u$ and transitivity, it has $t \geq u$. It means that $t \ngtr w$ by applying $(*)$, and then $t \cong w$ by the Inclusion condition. Given $v > t$, it implies that $v \geq u$ by $t \geq u$ and Interaction$_2$. It also implies that $v > w$ by $t \cong w$ and Interaction$_1$. Together with $(*)$ this implies $v \notin \{w' \in ||\varphi|| \mid u \leq w'\}$. Since $v \geq u$, it then concludes that $v \notin ||\varphi||$. I therefore conclude that $t \in max_{\geq}(||\varphi||) \subseteq ||\psi||$. I then have $\mathcal{M}, w \models \Box(\varphi \to \langle\leq\rangle(\varphi \wedge [\leq](\varphi \to \psi)))$.

(RtL) Assume that $\mathcal{M}, w \models \Box(\varphi \to \langle\leq\rangle(\varphi \wedge [\leq](\varphi \to \psi)))$. Let $u \in max_{\geq}(||\varphi||)$. It means that $u \in ||\varphi||$ and $\forall s \in ||\varphi||$ that $u \nless s$. It follows that $\mathcal{M}, u \models \langle\leq\rangle(\varphi \wedge [\leq](\varphi \to \psi))$ by the assumption, which implies that $\exists u' \geq u$ s.t. $\mathcal{M}, u' \models \varphi$ and $\mathcal{M}, u' \models [\leq](\varphi \to \psi)$. Yet I know that $u' \ngtr u$ by the first conclusion and the previous result, it then has $u' \cong u$ by Inclusion. On the other hand, according to $\mathcal{M}, u' \models [\leq](\varphi \to \psi)$, it implies that $\forall s \geq u'$ that $\mathcal{M}, s \models \varphi \to \psi$. By $u' \cong u$ and Interaction$_3$, it follows that $\forall s \geq u$ that $\mathcal{M}, s \models \varphi \to \psi$. Because $u \geq u$ by reflexivity and $\mathcal{M}, u \models \varphi$, it thus follows that $u \in ||\psi||$. Hence I can conclude that $max_{\geq}(||\varphi||) \subseteq ||\psi||$.

□

Unconditional obligations $O\varphi$ are defined as $O(\varphi/\top)$, and permission as "weak permissions", i.e. $P(\varphi/\psi)$ iff $\neg O(\neg\varphi/\psi)$. With this in hand I have the machinery required to define claims and privileges, which I do again using the standard Kangerian approach:

- Given ψ, agent i has a claim against j regarding φ: $O(Do_j\varphi/\psi)$

- Given ψ, agent i has a privilege against j regarding φ: $\neg O(Do_i \neg\varphi/\psi)$ iff $P(\neg Do_i \neg\varphi/\psi)$

I close this section with a short example, to which I will return later.

4.1.4. EXAMPLE. Ivy has parked her car but she forgot to put the mandatory parking permit in her windshield. Parking with a parking permit ($\neg p$) is strictly better than parking without (p). It can of course have deontic consequences, but only if a city clerk with the *power* to *issue* parking tickets passes by. Absent this deontic action of issuing a ticket, the city has no claim against Ivy regarding the payment of a fine ($\neg O(Do_{Ivy} f/p)$). Not possessing a permit being mandatory for parking, Ivy is forbidden to park. In other words, the city has a claim against her not to park her car where she did ($O(Do_{Ivy} \neg p)$). This is illustrated in Figure 4.1.

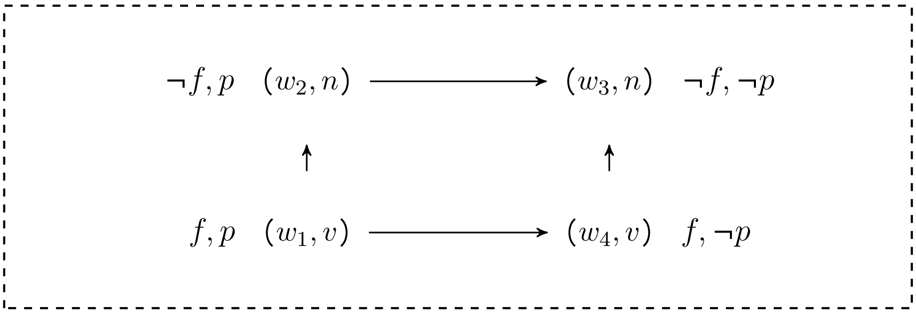


Figure 4.1: A static model of Ivy's example. The arrows → represent the preference order < between states.

The theory of normative positions for unconditional static rights is well studied [94], but less so for conditional ones. I will come back to it in Section 4.3.2.

4.2 Dynamic Rights

4.2.1 Core Model

My modeling of legal competence follows the so-called "event models" methodology developed in [9] for epistemic modalities. The key idea there is to model the *structure* of a particular learning event using the same tools as for an agent's static information state, that is Kripke models. The result of updating one's knowledge or belief in the light of new information is then computed using some form of restricted product of these models. See [103] for details. Transposed into my deontic context, the proposal is to model explicitly the structure of deontic action or legal competences using what I call *deontic action models*. These are agent-indexed to capture the fact that different agents will have different legal competences. The deontic action models I define include two preference ordering over acts as Definition 4.1.2 does.

4.2.1. DEFINITION. A *deontic action model $\mathcal{A}_i$ for agent i* is a tuple $\langle A, >^{\mathcal{A}_i}, \cong^{\mathcal{A}_i}, Pre\rangle$ where:

- A is a non-empty finite set of acts.
- $>^{\mathcal{A}_i}$ is a converse well-founded and transitive relation on A
- $\cong^{\mathcal{A}_i}$ is an equivalence relation on A
- $Pre : A \to \mathcal{L}$ is a precondition function.

Each act $a \in A$ should be seen as a deontic action or a legal ability. It encodes an action that agent i can take in order to bring about changes in obligations and permissions or, in more Hohfeldian terminology, changes in underlying legal relations. These acts also come in different levels of ideality, which is encoded by the preference orders $>^{\mathcal{A}_i}$, $\cong^{\mathcal{A}_i}$, and $\geq^{\mathcal{A}_i}$, among which $\geq^{\mathcal{A}_i}$ is a standard reflexive and transitive relation [9, 103] defined using $>^{\mathcal{A}_i}$ and $\cong^{\mathcal{A}_i}$, as in the previous section. Finally, the preconditions function Pre specifies for each act a the conditions in the underlying static models that need to obtain for a to be executable in the first place.

4.2.2. EXAMPLE. John is city clerk. He can confirm a violation of the parking regulations (v), or not (n). He confirms a violation if a fine applies (f), otherwise ($\neg f$) not. Given that Ivy's car has no permit in the windshield, the preferred situation is one where the fine indeed applies. This is illustrated in Figure 4.2.

$$\neg f \quad n \longmapsto v \quad f$$

Figure 4.2: The deontic action model $\mathcal{A}_{John}$ for John the city clerk. The arrow $\mapsto$ represents the preference order $>^{\mathbf{A}_j}$. The precondition of n and v are written down to the left and the right, respectively.

The effect of executing a deontic action in a particular situation is computed by the so-called lexicographic update.

4.2.3. DEFINITION. Let $\mathcal{M}$ be a preference-action model and $\mathcal{A}_i$ be a deontic action model. The preference-action model $\mathcal{M} \otimes \mathcal{A}_i = \langle W', >', \cong', \{\sim'_i\}_{i \in \mathcal{I}}, V'\rangle$ is defined as follows:

- $W' = \{(w, a) \mid \mathcal{M}, w \models Pre(a)$, where $a \in A\}$.
- $(w, a) >' (w', a')$ iff either $a >^{\mathcal{A}_i} a'$ or $a \cong^{\mathcal{A}_i} a'$ and $w > w'$.
- $(w, a) \cong' (w', a')$ iff $a \cong^{\mathcal{A}_i} a'$ and $w \cong w'$.
- $(w, a) \sim' (w', a')$ iff $w \sim_i w'$

- $(w, a) \in V'(p) \Leftrightarrow w \in V(p)$.

The lexicographic update takes pairs of preference-action models and deontic action models and returns an updated model $\mathcal{M} \otimes \mathcal{A}_i$. The adjective "lexicographic" comes from the update rule for the preference orders $>'$ and $\cong'$, which give priority to the deontic action. The domain of that new model is the set of pairs (w, a) such that $\mathcal{M}, w$ satisfies the pre-condition of a, written $\mathcal{M}, w \vDash Pre(a)$. Instead, combining the two update rules for $>'$ and $\cong'$, I get the following rule for the pre-order $\geq'$:

$$(w, a) \geq' (w', a') \text{ iff either } a >^{\mathcal{A}_i} a' \text{ or } a \cong^{\mathcal{A}_i} a' \text{ and } w \geq w'.$$

Lexicographic updates capture what I call *pure* deontic actions. These are actions that *only* change legal relations. This is encoded in the condition defining the valuation V' in the updated model: $(w, a) \in V'(p) \Leftrightarrow w \in V(p)$. One can take pure deontic action to be acts that are explicitly defined by the legislator, for instance entering into a contract or getting married. Of course non-deontic action might change the legal relation too. By breaking your neighbor's window you create a claim for her against you to cover the repair costs. Such mixed deontic and non-deontic actions are the object of [54]. A full comparison between their and my models of deontic actions and legal competences is left for future work.

4.2.4. LEMMA. *The lexicographic update models are preference-action models.*

Proof:
Clearly that $\geq' = >' \cup \cong'$ and $>' \cap \cong' = \varnothing$. In this case, I need to show that $\leq'$ is reflexive and transitive, $<'$ is CWF, and $\cong'$ is an equivalence relation. I only prove the interesting case that $<'$ is CWF, and the other cases are easily checked. Now I need to show that $\forall X \neq \varnothing, \exists x \in X$ s.t. $\forall y \in X$ that $y \not>' x$.

Let $\varnothing \neq X = \{(w, a) \mid w \in Y \subseteq W \text{ and } a \in B \subseteq A\}$ where $Y \neq \varnothing$ and $B \neq \varnothing$. As $>$ and $>^{\mathcal{A}_i}$ are CWF, there are w^* be a $>$-maximal element on Y and a^* be a $>^{\mathcal{A}_i}$-maximal element on B. Namely that $\forall y \in Y$ that $y \not> w^*$ and $\forall b \in B$ that $b \not>^{\mathcal{A}_i} a^*$. Given $(y, b) \in X$, then $b \not>^{\mathcal{A}_i} a^*$, and if $b \cong^{\mathcal{A}_i} a^*$, then obviously $y \not> w^*$. So $(y, b) \not>' (w^*, a^*)$. This means that (w^*, a^*) is the $>'$-maximal element on X. □

4.2.5. EXAMPLE. John notices that Ivy's car doesn't have a permit. He issues a parking ticket, which results in the city having a claim against Ivy regarding the payment of a fine. This is represented by updating the model in Figure 4.1 with the one in Figure 4.2. The result is in Figure 4.3. After the ticket has been issued, all states a fine applies to (f) are strictly better than those in which they do not $(\neg f)$. Now Ivy still ought not to park there, but she ought to pay a fine.

Of course, executing different deontic actions will have different effects on the same initial legal relations. This notion of "different deontic action" can be made precise using the standard notions of bisimulation [13] and action emulation [115], but I leave it until Section 4.2.3. For now it is sufficient to illustrate this with an intuitive example.

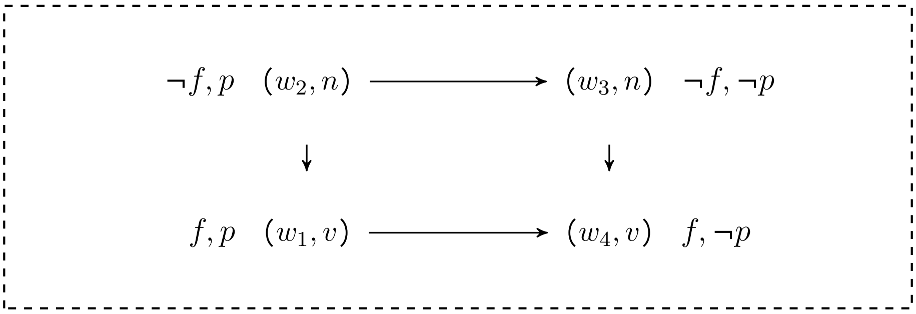


Figure 4.3: The model $\mathcal{M} \otimes \mathcal{E}_{John}$ resulting from John's execution of a deontic action to issue a parking ticket.

4.2.6. EXAMPLE. Suppose that Mary has the authority to grant Ivy an exception that allows her to park without a permit. Such a deontic action is represented in Figure 4.4, and the result of updating Ivy's initial situation (Figure 4.1) is in Figure 4.5, where Ivy still ought not to pay a fine but now enjoys a privilege to park her car. Notice that this update crucially uses a non-connected preference relation in the action model.

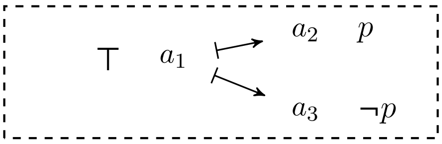


Figure 4.4: The deontic action model $\mathcal{A}_{Mary}$ for Mary.

To express the effect of deontic action the language $\mathcal{L}$ is extended with a dynamic, unary operator $[\mathcal{A}_i, a]$, with the following semantics:

- $\mathcal{M}, w \vDash [\mathcal{A}_i, a]\varphi$ iff if $\mathcal{M}, w \vDash Pre(a)$ then $\mathcal{M} \otimes \mathcal{A}_i, (w, a) \vDash \varphi$.

A formula $[\mathcal{A}_i, a]\varphi$ thus reads "if i's deontic action a is executable, then doing so results in φ." Dynamic modalities allow me to introduce my key notions, powers and immunity. Let $T(i, j, \psi/\varphi)$ denote an arbitrary (conditional) normative position definable in the static language $\mathcal{L}$. Then:

- i has a *power* against j regarding $T(i, j, \psi/\varphi)$:

$$\bigvee_{a \in \mathcal{A}_i} [\mathcal{A}_i, a]T(i, j, \psi/\varphi)$$

- i has an *immunity* against j regarding $T(i, j, \psi/\varphi)$:

$$\neg \bigvee_{a \in \mathcal{A}_j} [\mathcal{A}_j, a]T(i, j, \psi/\varphi)$$

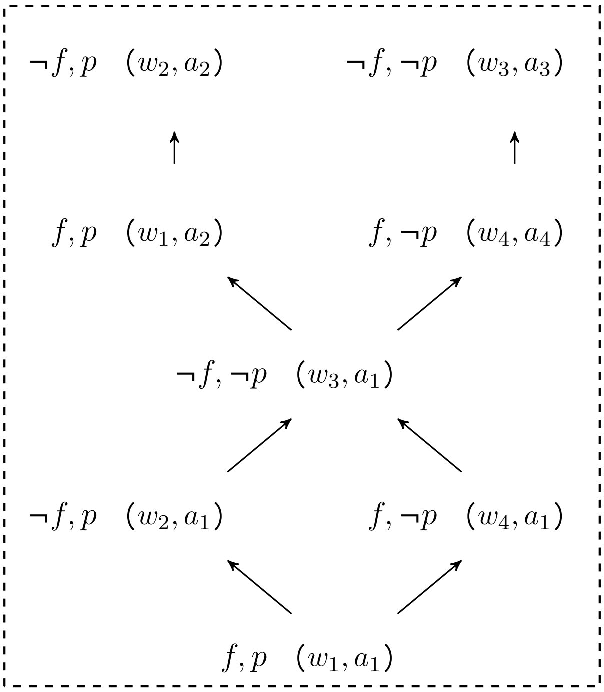


Figure 4.5: The model $\mathcal{M} \otimes \mathcal{A}_{John}$ resulting from Mary's granting Ivy an exception to park without a permit.

In other words, i has a power against j regarding the normative position $T(i, j, \psi/\varphi)$ whenever there is a deontic action that i can be executed which results in $T(i, j, \psi/\varphi)$. Similarly, i has an immunity against j regarding $T(i, j, \psi/\varphi)$ if j doesn't have a power against i regarding that position. A quick check of the example above reveals that, as expected, John has a power against Ivy regarding her paying a fine.

This formalization of dynamic rights has two assets in comparison with classical, reductive approaches. First, it explicitly captures, both semantically and syntactically, the dynamic character of power and immunity. Second, as I will see below, this clear static-dynamic distinction allows for a natural distinction between legal ability and legal permissibility. This analysis of power and immunity does so, however, while staying reductive. This is what I show now.

4.2.2 Axiomatization and Reduction to Static Positions

Axiomatizing the set of validities for the frames, models and updates just defined proceeds in two modules: one for the static modalities of $\mathcal{L}$ and one for the dynamic extension. For the static part the axiomatization proceeds close to a standard manner. I use the K axiom for $<$, together with the Löb axiom for well-foundedness and transitivity. $[\cong]$, A and Do_i are **S5** modalities. Interaction between $[<]$, $[\cong]$ and A can be captured by two standard inclusion axioms and one interaction axiom. Of course, each modality satisfies Modus Ponens and the necessitation rule. See Table 4.1 for details.

Axioms for $[<]$:

K	$\vdash [<](\varphi \rightarrow \psi) \rightarrow ([<]\varphi \rightarrow [<]\psi)$
Löb	$\vdash [<]([<]\varphi \rightarrow \varphi) \rightarrow [<]\varphi$

Axioms for $[\cong]$:

K	$\vdash [\cong](\varphi \rightarrow \psi) \rightarrow ([\cong]\varphi \rightarrow [\cong]\psi)$
T	$\vdash [\cong]\varphi \rightarrow \varphi$
4	$\vdash [\cong]\varphi \rightarrow [\cong][\cong]\varphi$
5	$\vdash \neg[\cong]\varphi \rightarrow [\cong]\neg[\cong]\varphi$

The same axioms are used *mutatis mutandis* for A and Do_i.

Inclusion axioms:

Incl_1	$\vdash A\varphi \rightarrow [<]\varphi$
Incl_2	$\vdash A\varphi \rightarrow [\cong]\varphi$

Interaction axiom:

Int	$\vdash [<]\varphi \rightarrow [\cong][<]\varphi$

All modalities satisfy:

NEC	If $\vdash \varphi$ then $\vdash B\varphi$ where $B \in \{[<], [\cong], A, Do_i\}$.

Table 4.1: Sound and complete axiomatization of preference-action frames

Axiomatizing the dynamic part uses the well-known "reduction axioms" methodology [9, 110]. Formulas containing dynamic modalities are shown to be semantically equivalent to formulas of $\mathcal{L}$, that is without dynamic modalities. The formulas in Table 4.2 indeed show how to "push" dynamic modalities inside the various connectives and modal operators of the static language, until they range over atomic propositions where they can be eliminated. These formulas are sound with respect to the lexicographic update over preference-action models. Taking them as axioms thus makes formulas containing dynamic modalities provably equivalent to formulas of $\mathcal{L}$. Completeness for the extended language then follows from completeness of the static part with respect to the class of preference-action models.

Through the soundness of the reduction axioms, together with the fact that conditional obligations are definable in $\mathcal{L}$, I obtain that powers and abilities are also reducible to static normative positions. So the approach presented here is reductive. This reduction, however, is more complex than the simple reduction proposed for instance in [63].

$\vdash [\mathcal{A}_i, a]p \leftrightarrow (Pre(a) \rightarrow p)$
$\vdash [\mathcal{A}_i, a]\neg\varphi \leftrightarrow (Pre(a) \rightarrow \neg[\mathcal{A}_i, a]\varphi)$
$\vdash [\mathcal{A}_i, a](\varphi \land \psi) \leftrightarrow [\mathcal{A}_i, a]\varphi \land [\mathcal{A}_i, a]\psi$
$\vdash [\mathcal{A}_i, a]A\varphi \leftrightarrow (Pre(a) \rightarrow A[\mathcal{A}_i, a]\varphi)$
$\vdash [\mathcal{A}_i, a][<]\varphi \leftrightarrow (Pre(a) \rightarrow \bigwedge_{c >^{\mathcal{A}_i} a} A[\mathcal{A}_i, c]\varphi)$
$\vdash [\mathcal{A}_i, a][\cong]\varphi \leftrightarrow (Pre(a) \rightarrow \bigwedge_{c \cong^{\mathcal{A}_i} a}[\cong][\mathcal{A}_i, c]\varphi)$
$\vdash [\mathcal{A}_i, a]Do_j\varphi \leftrightarrow (Pre(a) \rightarrow Do_j[\mathcal{A}_i, a]\varphi)$

Table 4.2: Reduction Axioms for Lexicographic Update

Here is the valid reduction validity for conditional obligation:

$$[\mathcal{A}_i, a]O(\psi/\varphi) \leftrightarrow [Pre(a) \rightarrow A \bigwedge_{d \in A}((\langle\mathcal{A}_i, d\rangle\varphi \land A \bigwedge_{c >^{\mathcal{A}_i} d} [\mathcal{A}_i, c]\neg\varphi) \rightarrow O([\mathcal{A}_i, d]\psi / \bigvee_{c \cong^{\mathcal{A}_i} d} \langle\mathcal{A}_i, c\rangle\varphi))]$$

with $\langle\mathcal{A}_i, d\rangle\varphi$ being the dual of $[\mathcal{A}_i, a]$. The complexity of this formula results from it essentially encoding syntactically the lexicographic update rule in combination with the specific semantic definition of obligations as truth in all the *most* ideal worlds. For unconditional obligations, however, it simplifies to the following:

$$[\mathcal{A}_i, a]O(\psi) \leftrightarrow [Pre(a) \rightarrow A \bigwedge_{d \in A}((pre(d) \land A \bigwedge_{c >^{\mathcal{A}_i} d} \neg pre(c)) \rightarrow O([\mathcal{A}_i, d]\psi / \bigvee_{c \cong^{\mathcal{A}_i} d} pre(c)))]$$

The effect of changes in legal relations are thus reducible to statements describing legal relations holding *before* the deontic action takes place. In particular, the latter formula states that executing an action a would result in an obligation to ψ exactly when, if a is executable in the first place, for any maximally ideal and executable action d, it ought to be the case before executing d that ψ would hold after d.

The reduction allows to distinguish the legal permissibility of a deontic action a and its legal ability. The latter boils down to a being executable in a particular situation, which in turn reduces to the preconditions of a obtaining. This, however, is *not* equivalent to the execution of a being permitted. Defining permissibility of deontic action requires some additional machinery. I return to this question in Section 4.4.

4.2.7. THEOREM. *All reduction axioms for lexicographic update including the conditional obligation are valid.*

Proof:
Here I am only interested in the reductions of $[\mathcal{A}_i, a][<]\varphi$, $[\mathcal{A}_i, a][\cong]\varphi$, as well as $[\mathcal{A}_i, a]O(\psi/\varphi)$. The other cases are similar to those in the literature [103, p.162-164] [9].

First, I will prove the validity of reduction axiom of $[\mathcal{A}_i, a][<]\varphi$.

(LtR) Let $M, w \models \langle \mathcal{A}_i, a\rangle\langle<\rangle\varphi$. It follows that $\mathcal{M}, w \models Pre(a)$ and $\mathcal{M}\otimes\mathcal{A}_i, (w, a) \models \langle<\rangle\varphi$. By the later case, $\exists(w', a') \in \mathcal{M}\otimes\mathcal{A}_i$ s.t. $(w', a') >' (w, a)$ and $\mathcal{M}\otimes\mathcal{A}_i, (w', a') \models \varphi$. I thus have that $\exists w' \in W$ and $\exists a' \in A$ s.t. $\mathcal{M}, w' \models Pre(a')$, and $a' >^{\mathcal{A}_i} a$ and $\mathcal{M}\otimes\mathcal{A}_i, (w', a') \models \varphi$. This follows that $\mathcal{M}, w' \models \langle\mathcal{A}_i, a'\rangle\varphi$ where $a' >^{\mathcal{A}_i} a$. Then $\mathcal{M}, w \models E\langle\mathcal{A}_i, a'\rangle\varphi$ where $a' >^{\mathcal{A}_i} a$, which indicates that $\mathcal{M}, w \models \bigvee_{a' >^{\mathcal{A}_i} a} E\langle\mathcal{A}_i, a'\rangle\varphi$.

(RtL) Let $\mathcal{M}, w \models Pre(a) \wedge \bigvee_{a' >^{\mathcal{A}_i} a} E\langle\mathcal{A}_i, a'\rangle\varphi$. It means that $\mathcal{M}, w \models Pre(a)$ and $\exists w' \in W$ s.t. $\exists a' >^{\mathcal{A}_i} a$ and $\mathcal{M}, w' \models \langle\mathcal{A}_i, a'\rangle\varphi$. According to this result, $(w', a') \in \mathcal{M}\otimes\mathcal{A}_i$, $\mathcal{M}, w' \models Pre(a')$, and $\mathcal{M}\otimes\mathcal{A}_i, (w', a') \models \varphi$. By $a' >^{\mathcal{A}_i} a$, it follows $(w', a') >' (w, a)$. So $\mathcal{M}\otimes\mathcal{A}_i, (w, a) \models \langle<\rangle\varphi$

Next I prove that $[\mathcal{A}_i, a][\cong]\varphi \leftrightarrow (Pre(a) \to \bigwedge_{c \cong^{\mathcal{A}_i} a}[\leq][\mathcal{A}_i, c]\varphi)$.

(LtR) Let $M, w \models \langle\mathcal{A}_i, a\rangle\langle\cong\rangle\varphi$. It follows that $\mathcal{M}, w \models Pre(a)$ and $\mathcal{M}\otimes\mathcal{A}_i, (w, a) \models \langle\leq\rangle\varphi$. By the latter case, $\exists(w', a') \in \mathcal{M}\otimes\mathcal{A}_i$ s.t. $(w', a') \cong' (w, a)$ and $\mathcal{M}\otimes\mathcal{A}_i, (w', a') \models \varphi$. I thus have that $\exists w' \in W$ and $\exists a' \in A$ s.t. $\mathcal{M}, w' \models Pre(a')$, and $a' \cong^{\mathcal{A}_i} a, w' \cong w$, and $\mathcal{M}\otimes\mathcal{A}_i, (w', a') \models \varphi$. It follows that $\mathcal{M}, w' \models \langle\mathcal{A}_i, a'\rangle\varphi$ where $a' \cong^{\mathcal{A}_i} a$. Because $w' \cong w$, I have $\mathcal{M}, w \models \langle\cong\rangle\langle\mathcal{A}_i, a'\rangle\varphi$ where $a' \cong^{\mathcal{A}_i} a$. It is $\mathcal{M}, w \models \bigvee_{a' \cong^{\mathcal{A}_i} a}\langle\cong\rangle\langle\mathcal{A}_i, a'\rangle\varphi$.

(RtL) Let $\mathcal{M}, w \models Pre(a) \wedge \bigvee_{a' \cong^{\mathcal{A}_i} a}\langle\cong\rangle\langle\mathcal{A}_i, a'\rangle\varphi$. It means that $\mathcal{M}, w \models Pre(a)$ and $\mathcal{M}, w \models \bigvee_{a' \cong^{\mathcal{A}_i} a}\langle\cong\rangle\langle\mathcal{A}_i, a'\rangle\varphi$. So $\exists w' \cong w$ s.t. $\exists a' \cong^{\mathcal{A}_i} a$ and $\mathcal{M}, w' \models \langle\mathcal{A}_i, a'\rangle\varphi$. According to this result, it follows that $(w', a') \in \mathcal{M}\otimes\mathcal{A}_i$, $\mathcal{M}, w' \models Pre(a')$, and $\mathcal{M}\otimes\mathcal{A}_i, (w', a') \models \varphi$. By $w' \cong w$ and $a' \cong^{\mathcal{A}_i} a$, it follows $(w', a') \cong' (w, a)$. I thus have $\mathcal{M}\otimes\mathcal{A}_i, (w, a) \models \langle\cong\rangle\varphi$. Because $\mathcal{M}, w \models Pre(a)$, it follows that $\mathcal{M}, w \models \langle\mathcal{A}_i, a\rangle\langle\cong\rangle\varphi$.

Finally I prove the validity of the reduction axiom of $[\mathcal{A}_i, a]O(\psi/\varphi)$. I need to show that

$$\forall(x, c) \in \mathcal{M}\otimes\mathcal{A}_i.\ (x, c) \in max_{\geq}(||\varphi||) \Rightarrow (x, c) \in ||\psi|| \tag{6}$$

iff

$$\forall y \in W \forall d \in A. y \in (||\langle\mathcal{A}_i, d\rangle\varphi \wedge A \bigwedge_{c >^{\mathcal{A}_i} d}[\mathcal{A}_i, c]\neg\varphi||) \cap max_{\geq}(|| \bigvee_{c \cong^{\mathcal{A}_i} d}\langle\mathcal{A}_i, c\rangle\varphi||)$$
$$\Rightarrow y \in ||[\mathcal{A}_i, d]\psi||. \tag{7}$$

(LtR) Given arbitrary $y \in W$ and $d \in A$ that $y \in (||\langle\mathcal{A}_i, d\rangle\varphi \wedge A \bigwedge_{c >^{\mathcal{A}_i} d}[\mathcal{A}_i, c]\neg\varphi||) \cap max_{\geq}(|| \bigvee_{c \cong^{\mathcal{A}_i} d}\langle\mathcal{A}_i, c\rangle\varphi||)$ and $\mathcal{M}, y \models Pre(d)$, then

1. $\mathcal{M}, y \models \langle\mathcal{A}_i, d\rangle\varphi$,
2. $\forall z \in W\ \forall c >^{\mathcal{A}_i} d$ that $\mathcal{M}, z \models [\mathcal{A}_i, c]\neg\varphi$, and
3. $y \in max_{\geq}(|| \bigvee_{c \cong^{\mathcal{A}_i} d}\langle\mathcal{A}_i, c\rangle\varphi||)$.

It follows that

1. $\mathcal{M} \otimes \mathcal{A}_i, (y, d) \vDash \varphi$,
2. $\forall z \in W$ that if $\mathcal{M}, z \vDash \neg[\mathcal{A}_i, c]\neg\varphi$ then $c \not>^{\mathcal{A}_i} d$,
3. $\mathcal{M}, y \vDash \bigvee_{c \cong^{\mathcal{A}_i} d} \langle \mathcal{A}_i, c \rangle \varphi$, and
4. $\forall z \in W$ that if $\mathcal{M}, z \vDash \bigvee_{c \cong^{\mathcal{A}_i} d} \langle \mathcal{A}_i, c \rangle \varphi$ then $z \not> y$.

From cases 1, 2 and 4 it follows that

1. $\mathcal{M} \otimes \mathcal{A}_i, (y, d) \vDash \varphi$,
2. $\forall z \in W \ \forall c \in A$ that if $\mathcal{M}, z \vDash \langle \mathcal{A}_i, c \rangle \varphi$ then $c \not>^{\mathcal{A}_i} d$, and
3. $\forall z \in W \ \forall c \cong^{\mathcal{A}_i} d$ that if $\mathcal{M}, z \vDash \langle \mathcal{A}_i, c \rangle \varphi$ then $z \not> y$.

which implies

1. $\mathcal{M} \otimes \mathcal{A}_i, (y, d) \vDash \varphi$,
2. $\forall (z, c) \in \mathcal{M} \otimes \mathcal{A}_i$ that if $\mathcal{M} \otimes \mathcal{A}_i, (z, c) \vDash \varphi$ then $c \not>^{\mathcal{A}_i} d$, and
3. $\forall (z, c) \in \mathcal{M} \otimes \mathcal{A}_i$ that if $\mathcal{M} \otimes \mathcal{A}_i, (z, c) \vDash \varphi$ then ($c \cong^{\mathcal{A}_i} d \Rightarrow z \not> y$).

It means that

1. $\mathcal{M} \otimes \mathcal{A}_i, (y, d) \vDash \varphi$,
2. $\forall (z, c) \in \mathcal{M} \otimes \mathcal{A}_i$ that if $\mathcal{M} \otimes \mathcal{A}_i, (z, c) \vDash \varphi$ then $(z, c) \not> (y, d)$.

I therefore conclude that $(y, d) \in max_{\geq}(||\varphi||)$. Applying this result into statement (6) it follows that $(y, d) \in ||\psi||$. It thus concludes that $\mathcal{M}, y \vDash [\mathcal{A}_i, d]\psi$.

(RtL) Given arbitrary $(x, d) \in \mathcal{M} \otimes \mathcal{A}_i$ that $(x, d) \in max_{\geq}(||\varphi||)$. It follows that given arbitrary $x \in W$ and arbitrary $d \in A$,

1. $\mathcal{M}, x \vDash Pre(d)$,
2. $\mathcal{M} \otimes \mathcal{A}_i, (x, d) \vDash \varphi$, and
3. $\forall (z, c) \in \mathcal{M} \otimes \mathcal{A}_i$ that if $\mathcal{M} \otimes \mathcal{A}_i, (z, c) \vDash \varphi$ then $(z, c) \not> (x, d)$.

It follows that given any $x \in W$ and for all $d \in A$,

1. $\mathcal{M}, x \vDash \langle \mathcal{A}_i, d \rangle \varphi$
2. $\forall (z, c) \in \mathcal{M} \otimes \mathcal{A}_i$ that if $\mathcal{M} \otimes \mathcal{A}_i, (z, c) \vDash \varphi$ then $c \not>^{\mathcal{A}_i} d$, and
3. $\forall (z, c) \in \mathcal{M} \otimes \mathcal{A}_i$ that if $\mathcal{M} \otimes \mathcal{A}_i, (z, c) \vDash \varphi$ then ($c \cong^{\mathcal{A}_i} d$ implies $z \not> x$).

From case 2 and case 3 it implies that

1. $\forall z \in W \ \forall c >^{\mathcal{A}_i} d$ that $\mathcal{M}, z \vDash [\mathcal{A}_i, c]\neg\varphi$, and

2. $\forall z \in W\ \forall c \cong^{\mathcal{A}_i} d$ that if $\mathcal{M}, z \models \langle \mathcal{A}_i, c \rangle \varphi$ then $z \not> x$.

In sum, given any $x \in W$ and for all $d \in A$,

1. $\mathcal{M}, x \models \langle \mathcal{A}_i, d \rangle \varphi$
2. $\forall z \in W$ that $\mathcal{M}, z \models \bigwedge_{c >^{\mathcal{A}_i} d} [\mathcal{A}_i, c] \neg \varphi$, and
3. $\forall z \in W$ that if $\mathcal{M}, z \models \bigvee_{c \cong^{\mathcal{A}_i} d} \langle \mathcal{A}_i, c \rangle \varphi$ then $z \not> x$.

It implies that given any $x \in W$ and for all $d \in A$,

1. $\mathcal{M}, x \models \langle \mathcal{A}_i, d \rangle \varphi$
2. $\forall z \in W$ that $\mathcal{M}, z \models \bigwedge_{c >^{\mathcal{A}_i} d} [\mathcal{A}_i, c] \neg \varphi$, and
3. $x \in max_{\geq}(|| \bigvee_{c \cong^{\mathcal{A}_i} d} \langle \mathcal{A}_i, c \rangle \varphi ||)$

It means $x \in ||\langle \mathcal{A}_i, d \rangle \varphi \wedge A \bigwedge_{c >^{\mathcal{A}_i} d} [\mathcal{A}_i, c] \neg \varphi|| \cap max_{\geq}(|| \bigvee_{c \cong^{\mathcal{A}_i} d} \langle \mathcal{A}_i, c \rangle \varphi ||)$ for arbitrary $x \in W$ and $d \in A$. Applying this result to statement (7) it implies that $x \in ||[\mathcal{A}_i, d]\psi||$. Since $\mathcal{M}, x \models Pre(d)$, I conclude that $(x, d) \in ||\psi||$.

□

4.2.3 Model Theory

At the end of Section 4.2.1 I showed that "different" deontic actions can have different effects on a given set of initial legal relations. Now I make this notion of sameness of deontic actions precise using the standard notion of bisimulation.

4.2.8. DEFINITION. A relation Z between the domains of two preference-action models $\mathcal{M}$ and $\mathcal{M}'$ is a bisimulation whenever, for all $w \in W$ and $w' \in W'$, if wZw' then:

- For all $p \in Prop$, $\mathcal{M}, w \models p$ iff $\mathcal{M}', w' \models p$
- (>-Forward-Condition) If $w > v$ then there exists $v' \in W'$ s.t. $w' > v'$ and vZv'
- (>-Backward-Condition) If $w' > v'$ then there exists $v \in W$ s.t. $w > v$ and vZv'
- and similarly for $\cong$ and $\sim_i$

A textbook argument shows that two bisimilarities imply modal invariance for $\mathcal{L}$ [13, 105]. In particular two bisimilar preference-action models support exactly the same static rights or legal relations. These are "the same" as far as static rights are concerned. Accordingly, I also can expect a similar bisimulation of two preference-action models based on $\geq$.

Bisimulation is straightforwardly adapted to deontic action models. I do not, however, have an explicit language to describe such actions. Recall that the dynamic modalities only describe their *effects*. So instead I use a notion of behavioral equivalence. I say that two deontic actions are "the same" when executing them in indistinguishable legal circumstances guarantees indistinguishable results.

4.2.9. DEFINITION. A relation Z between two deontic action models $\mathcal{A}_i$ and $\mathcal{A}_j'$ is a bisimulation whenever, for all $a \in \mathcal{A}_i$ and $a' \in \mathcal{A}_j'$ such that aZa':

- $Pre(a) = Pre(a')$
- (>-Forward-Condition) If $b >^{\mathcal{A}_i} a$ then there exists $b' \in \mathcal{A}_i'$ s.t. $b' >^{\mathcal{A}_i} a'$ and bZb'
- (≅-Forward-Condition) If $b \cong^{\mathcal{A}_i} a$ then there exists $b' \in \mathcal{A}_i'$ s.t. $b' \cong^{\mathcal{A}_i} a'$ and bZb'
- Similarly in the converse direction

A bisimulation Z between two deontic action models is *total* whenever

- for each $a \in \mathcal{A}_i$ there exists $a' \in \mathcal{A}_i'$ s.t. aZa', and
- for each $a' \in \mathcal{A}_i'$ there exists $a \in \mathcal{A}_i$ s.t. aZa'

I are now in position to show that bisimulation for deontic-action models implies behavioral equivalence.

4.2.10. FACT. Given two deontic action models $\mathcal{A}_i$ and $\mathcal{A}_i'$ for agent i, and two preference-action models $\mathcal{M}$ and $\mathcal{M}'$. If $\mathcal{A}_i$ and $\mathcal{A}_i'$ are bisimilar, and $\mathcal{M}$ and $\mathcal{M}'$ are bisimilar, then $\mathcal{M} \otimes \mathcal{A}_i$ and $\mathcal{M}' \otimes \mathcal{A}_i'$ are bisimilar.

Proof:
I construct a bisimulation between $\mathcal{M} \otimes \mathcal{A}_i$ and $\mathcal{M}' \otimes \mathcal{A}_i'$ as follows. Suppose that $\mathcal{M}Z_1\mathcal{M}'$ and $\mathcal{A}_iZ_2\mathcal{A}_i'$. I define $Z_1 \otimes Z_2$ as the set of all bisimilar pairs $\langle (w,a),(w',a')\rangle$ s.t. wZ_1w' and aZ_2a', where $(w,a) \in \mathcal{M} \otimes \mathcal{A}_i$ and $(w',a') \in \mathcal{M}' \otimes \mathcal{A}_i'$. I prove that $Z_1 \otimes Z_2$ is a bisimulation between $\mathcal{M} \otimes \mathcal{A}_i$ and $\mathcal{M}' \otimes \mathcal{A}_i'$ in the following.

- For $\mathcal{M}Z_1\mathcal{M}'$, I know the standard result that $w \in ||\varphi||$ iff $w' \in ||\varphi||$ where $\varphi \in \mathcal{L}$. For $\mathcal{A}_iZ_2\mathcal{A}_i'$, I have $\psi \in Pre(a)$ iff $\psi \in Pre(a')$. Combining them together, (w,a) and (w',a') verify the same preconditions for actions, in the sense that $\mathcal{M}, w \vDash Pre(a)$ iff $\mathcal{M}', w' \vDash Pre(a')$.
- For the forward condition of >: If $(w,a) > (u,b)$ in $\mathcal{M} \otimes \mathcal{A}_i$ and $(w,a)Z_1 \otimes Z_2(w',a')$. Then

  - either $a >^{\mathcal{A}_i} b$ or ($a \cong^{\mathcal{A}_i} b$ and $w > u$), and
  - wZ_1w' and aZ_2a'.

  If $a >^{\mathcal{A}_i} b$. Since aZ_2a', this implies $\exists b' \in \mathcal{A}_i'$ s.t. bZ_2b' and $a' >^{\mathcal{A}_i} b'$. If $a \cong b$ and $w > u$. Since wZ_1w', then $\exists u' \in \mathcal{M}'$ s.t. uZ_1u' and $w' > u'$. Both imply that $(w', a') > (u', b')$. For the backward condition of $\geq$, the proof is similar to the forward condition.

- For the forward condition of $\cong$, the proof is similar to the case of $>$.

- For the forward condition of $\sim_i$: If $(w, a) \sim_i (u, b)$ in $\mathcal{M} \otimes \mathcal{A}_i$, then $w \sim_i u$. From wZ_1w', then $\exists u' \in \mathcal{M}'$ s.t. uZ_1u' and $w' \sim_i u'$. The proof of the backward condition of $\sim_i$ is similar.

Hence, I conclude that $Z_1 \otimes Z_2$ is a bisimulation between $\mathcal{M} \otimes \mathcal{A}_i$ and $\mathcal{M}' \otimes \mathcal{A}_i'$. □

The following is a direct corollary:

4.2.11. COROLLARY. *Given $a \in \mathcal{A}_i$ and $b \in \mathcal{A}_i'$. If Z is a total bisimulation between $\mathcal{A}_i$ and $\mathcal{A}_i'$ and aZb, then*

$$[\mathcal{A}_i, a]\chi \leftrightarrow [\mathcal{A}_i', b]\chi$$

Proof:
I need to prove that $\mathcal{M} \otimes \mathcal{A}_i, (w, a) \vDash \chi \Leftrightarrow \mathcal{M} \otimes \mathcal{A}_i', (w, b) \vDash \chi$. By induction on the complexity of χ. Here I only focus on the most interesting cases the atomic case, the case of when $\chi = A\varphi$, and the case of when $\chi = [<]\varphi$.

- The atomic case $\chi = p \in Prop$. $\mathcal{M} \otimes \mathcal{A}_i, (w, a) \vDash p$ iff $(w, a) \in V(p)$ iff $w \in V(p)$ iff $(w, b) \in V(p)$ iff $\mathcal{M} \otimes \mathcal{A}_i', (w, b) \vDash p$.

- If $\chi = A\varphi$. For the Left-to-Right direction, I assume that $\mathcal{M} \otimes \mathcal{A}_i, (w, a) \vDash A\varphi$ iff $\forall w' \in W \forall a' \in \mathcal{A}_i . \mathcal{M} \otimes \mathcal{A}_i, (w', a') \vDash \varphi$. If given an arbitrary $b' \in \mathcal{A}_i'$. In accordance with $\mathcal{A}_i Z \mathcal{A}_i'$ where Z is total, there exists $a'' \in \mathcal{A}_i$ s.t. $a''Zb'$. By inductive hypothesis and the assumption, I have $\mathcal{M} \otimes \mathcal{A}_i', (w', b') \vDash \varphi$. Thus I can conclude the result in this direction: $\mathcal{M} \otimes \mathcal{A}_i', (w, b) \vDash A\varphi$. The other direction can be proved in a similar way.

- If $\chi = [<]\varphi$. NTS: $\mathcal{M} \otimes \mathcal{A}_i, (w, a) \vDash [<]\varphi \Leftrightarrow \mathcal{M} \otimes \mathcal{A}_i', (w, b) \vDash [<]\varphi$.

  For the Left-to-Right direction, I assume that (*) $\mathcal{M} \otimes \mathcal{A}_i, (w, a) \vDash [<]\varphi$. Assume $(w', b') > (w, b)$ where $w' \in W$ and $b' \in \mathcal{A}_i'$. This means that either $b' >^{\mathcal{A}_i} b$, or ($b' \cong^{\mathcal{A}_i} b$ and $w' > w$). For the subcase (1): $b' >^{\mathcal{A}_i} b$. Now I need to show $\mathcal{M} \otimes \mathcal{A}_i', (w', b') \vDash \varphi$. By $b' >^{\mathcal{A}_i} b$, aZb and the $>$-backward-condition, there exists $a' \in \mathcal{A}_i$ s.t. $a'Zb'$ and $a' >^{\mathcal{A}_i} a$. It then implies $\mathcal{M} \otimes \mathcal{A}_i, (w', a') \vDash \varphi$ by $a' >^{\mathcal{A}_i} a$ and the assumption (*). According to $a'Zb'$ and the inductive hypothesis, this implies $\mathcal{M} \otimes \mathcal{A}_i', (w', b') \vDash \varphi$. For the subcase (2): ($b' \cong^{\mathcal{A}_i} b$

and $w' > w$). From $b' \cong^{\mathcal{A}_i} b$, aZb and the $\cong$-backward-condition, there exists $a' \in \mathcal{A}_i$ s.t. $a \cong^{\mathcal{A}_i} a'$ and $a'Zb'$. Together with $w' > w$ and the assumption, I have $(w', a) > (w, a)$, which implies $\mathcal{M} \otimes \mathcal{A}_i, (w', a') \vDash \varphi$. According to $a'Zb'$ and the inductive hypothesis, it implies that $\mathcal{M} \otimes \mathcal{A}'_i, (w', b') \vDash \varphi$. In the sum of (1) and (2), I get $\mathcal{M} \otimes \mathcal{A}'_i, (w', b') \vDash \varphi$. Hence I can conclude $\mathcal{M} \otimes \mathcal{A}'_i, (w, b) \vDash [<]\varphi$, and thus $[\mathcal{A}_i, a][<]\varphi \rightarrow [\mathcal{A}'_i, b][<]\varphi$. I can prove $[\mathcal{A}'_i, b][<]\varphi \rightarrow [\mathcal{A}_i, a][<]\varphi$ by applying the forward conditions and the inductive hypothesis in a similar method.

The proof for $[\cong]$ is similar to the case for $[<]$. □

This implies that executing two bisimilar deontic actions in the same model will give rise to exactly the same legal relations. So the dynamic modelling that I propose here provides a natural identity criterion for deontic actions.

4.3 Conditional Normative Positions

The formal theories of normative positions [94] examine all possible complete and logically consistent configurations of the Hohfedian four fundamental rights. Kanger and Kanger were the first to formulate this theory, and show that there are 26 "atomic types of rights" [57] based on a sound and complete logic presented in Table 1.2 in Section 1.1. Utilizing the same logic, Lindahl provides a refinement of 35 complete "individualistic types" w.r.t. one-agent, and 127 "collectivistic types" w.r.t. two-agent in his fine-grained "range of legal action" [63]. A general and refined computational theory of normative positions is introduced by Sergot [93]. See [68] for a recent survey of normative positions.

4.3.1 Normative Positions in Conditionals

One significant advantage of conditional normative positions is the gain in expressive power they provide. As Sergot pointed out [93], the simple theory of *monadic* normative positions is too weak to capture the following situations:

if *conditions* then *normative-position*

"If you have paid the ticket, then you have a privilege to park here" and "if someone attempts to murder you, then you have a privilege to defend yourself" cannot be expressed by monadic normative positions. The representation of conditional normative positions is "far from straightforward" [93], though Alchourrón and Bulygin [1] have provided its first systematic study.

4.3.2 Static Normative Positions

This section studies the theory of conditional static normative positions, based on the models introduced in Section 4.1 for a formal theory of static normative positions. My computational result is obtained in three steps in accordance with the method proposed in [68, 93], but adapted to my theory of conditional static normative positions.

The *simple types of conditional static rights* for i, j regarding ψ on the condition φ are defined in the following scheme, with expressions of *claim* and *privilege* in the language defined in Section 4.1:

$$\pm O(\pm\binom{Do_i}{Do_j} \pm \psi / \pm \varphi)$$

This notation is Makinson's *choice-scheme* [68], in which, for instance, $\binom{Do_i}{Do_j}$ here indicates the (two) alternatives Do_i and Do_j. Obviously the simple types of conditional static rights are more general and expressive than the *monadic* normative positions of Kanger and Kange's "simple types of rights" [57]. According to the four possible basic types of non-deontic actions Do in Def. 4.1.1, I unfold the static normative positions $O(\pm Do_i \pm \psi/\varphi)$ into the following four types:

(o1) $O(Do_i\psi/\varphi)$ indicates a *claim*: it is a claim against i to do ψ conditional φ

(o2) $O(\neg Do_i\psi/\varphi)$ indicates an *omission-claim*: it is a claim against i not to do ψ conditional φ, or it is a claim against i to omit ψ conditional φ

(o3) $O(Do_i\neg\psi/\varphi)$ indicates a *prevention-claim*: it is a claim against i to prevent ψ conditional φ

(o4) $O(\neg Do_i\neg\psi/\varphi)$ indicates a particular case of (o2)-claim: it is a claim against i to omit to prevent ψ conditional φ

Similarly, I can enumerate the four static normative positions in $P(\pm Do_i \pm \psi/\varphi)$ (which is $\neg O(\pm Do_i \pm \psi/\varphi)$) as follows:

(p1) $P(Do_i\psi/\varphi)$: *privilege*

(p2) $P(\neg Do_i\psi/\varphi)$: *omission-privilege*

(p3) $P(Do_i\neg\psi/\varphi)$: *prevention-privilege*

(p4) $P(\neg Do_i\neg\psi/\varphi)$: a particular case of (p2)-privilege

As the standard opposite relation between claim and privilege, (oi) and (pi) are dual to each other, where $1 \leq i \leq 4$. These eight simple types of conditional static rights exhaust all possibilities of the conditional static rights.

The *atomic types of conditional static rights* for i, j regarding ψ conditional on φ are generated from the simple types of conditional static rights in the following schema:

$$[\![\pm O(\pm\binom{Do_i}{Do_j} \pm \psi / \pm \varphi)]\!]$$

where the brackets denote the *max-conjunctions* defined by Makinson [68]: Given a choice-scheme Σ, $[\![\Sigma]\!]$ are the set of all maximal consistent conjunctions of expressions belonging to Σ. I call the components in $[\![\pm O(\pm\binom{Do_i}{Do_j} \pm \psi / \pm \varphi)]\!]$ the *atomic types of conditional static rights*.

There are three steps to figure out the total numbers of the atomic types of conditional static rights:

1. What are the theorems regarding $O(\psi/\varphi)$ and Do-operator?
2. How many components are in $[\![\pm O(\pm Do_i \pm \psi/\varphi)]\!]$?
3. How many components are in $[\![\pm O(\pm\binom{Do_i}{Do_j} \pm \psi/\varphi)]\!]$?

For the first step here is my answer: The relevant theorems for O and Do, which can be used to remove the redundant conjunctions in the max-conjunctions in a given choice-scheme, are the following:

(O.E) $\dfrac{\psi \leftrightarrow \psi'}{O(\psi/\varphi) \leftrightarrow O(\psi'/\varphi)}$

(O.C) $O(\psi/\varphi) \wedge O(\psi'/\varphi) \to O(\psi \wedge \psi'/\varphi)$

(O.M) $O(\psi \wedge \psi'/\varphi) \to O(\psi/\varphi) \wedge O(\psi'/\varphi)$

(O.D) $O(\psi/\varphi) \to \neg O(\neg\psi/\varphi)$ [3]

(O.N) $\dfrac{\psi}{O(\psi/\varphi)}$

(E.E) $\dfrac{\psi \leftrightarrow \psi'}{Do_i\psi \leftrightarrow Do_i\psi'}$

(E.T) $Do_i\varphi \to \varphi$

(E.D) $Do_i\varphi \to \neg Do_i\neg\varphi$

By applying the above theorems, the second step is to calculate the maximally consistent elements in $[\![\pm O(\pm Do_i \pm \psi/\varphi)]\!]$. They are the six *max-conjunctions* presented as follows:

[3] Ensured by that > is converse well-founded.

(C_1) $O(Do_i\psi/\varphi)$

(C_2) $O(Do_i\neg\psi/\varphi)$

(C_3) $O(\neg Do_i\psi/\varphi) \wedge O(\neg Do_i\neg\psi/\varphi) \wedge \neg O(Do_i\neg\psi/\varphi)$

(C_4) $O(\neg Do_i\psi/\varphi) \wedge \neg O(Do_i\neg\psi/\varphi) \wedge \neg O(\neg Do_i\neg\psi/\varphi)$

(C_5) $O(\neg Do_i\neg\psi/\varphi) \wedge \neg O(Do_i\psi/\varphi) \wedge \neg O(\neg Do_i\psi/\varphi)$

(C_6) $\neg O(Do_i\psi/\varphi) \wedge \neg O(Do_i\neg\psi/\varphi) \wedge \neg O(\neg Do_i\psi/\varphi) \wedge \neg O(\neg Do_i\neg\psi/\varphi)$

These six *max-conjunctions* are called the *simple types of conditional static rights* for i. They are consistent and logically independent, and the disjunction of them is a tautology. Moreover, they are more fine-grained than Kanger and Kanger's six "simple types of rights" for one agent. In more detail, if the conditions in these six conjunctions are tautologies, then C_1 becomes K_3, C_2 becomes K_5, C_4 becomes K_6, and C_5 becomes K_4. Moreover, given the same condition, C_3 implies K_2, and C_6 implies K_1.

I now can move to the third step: the calculation of the *max-conjunctions* in the schema of the two-agent case $[\![\pm O(\pm Do_i \pm \psi/\varphi)]\!] \cdot [\![\pm O(\pm Do_j \pm \psi/\varphi)]\!]$:

(B_1) $O(Do_i\psi/\varphi) \wedge O(Do_j\psi/\varphi)$

(B_2) $O(Do_i\psi/\varphi) \wedge O(\neg Do_j\psi/\varphi) \wedge O(\neg Do_j\neg\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi)$

(B_3) $O(Do_i\psi/\varphi) \wedge O(\neg Do_j\neg\psi/\varphi) \wedge \neg O(Do_j\psi/\varphi) \wedge \neg O(\neg Do_j\psi/\varphi)$

(B_4) $O(Do_i\neg\psi/\varphi) \wedge O(Do_j\neg\psi/\varphi)$

(B_5) $O(Do_i\neg\psi/\varphi) \wedge O(\neg Do_j\psi/\varphi) \wedge O(\neg Do_j\neg\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi)$

(B_6) $O(Do_i\neg\psi/\varphi) \wedge O(\neg Do_j\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi) \wedge \neg O(\neg Do_j\neg\psi/\varphi)$

(B_7) $O(\neg Do_i\psi/\varphi) \wedge O(\neg Do_i\neg\psi/\varphi) \wedge \neg O(Do_i\neg\psi/\varphi) \wedge O(\neg Do_j\psi/\varphi) \wedge O(\neg Do_j\neg\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi)$

(B_8) $O(\neg Do_i\psi/\varphi) \wedge O(\neg Do_i\neg\psi/\varphi) \wedge \neg O(Do_i\neg\psi/\varphi) \wedge O(\neg Do_j\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi) \wedge \neg O(\neg Do_j\neg\psi/\varphi)$

(B_9) $O(\neg Do_i\psi/\varphi) \wedge O(\neg Do_i\neg\psi/\varphi) \wedge \neg O(Do_i\neg\psi/\varphi) \wedge O(\neg Do_j\neg\psi/\varphi) \wedge \neg O(Do_j\psi/\varphi) \wedge \neg O(\neg Do_j\psi/\varphi)$

(B_{10}) $O(\neg Do_i\psi/\varphi) \wedge O(\neg Do_i\neg\psi/\varphi) \wedge \neg O(Do_i\neg\psi/\varphi) \wedge \neg O(Do_j\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi) \wedge \neg O(\neg Do_j\psi/\varphi) \wedge \neg O(\neg Do_j\neg\psi/\varphi)$

(B_{11}) $O(\neg Do_i\psi/\varphi) \wedge \neg O(Do_i\neg\psi/\varphi) \wedge \neg O(\neg Do_i\neg\psi/\varphi) \wedge O(\neg Do_j\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi) \wedge \neg O(\neg Do_j\neg\psi/\varphi)$

(B_{12}) $O(\neg Do_i\psi/\varphi) \wedge \neg O(Do_i\neg\psi/\varphi) \wedge \neg O(\neg Do_i\neg\psi/\varphi) \wedge O(\neg Do_j\neg\psi/\varphi) \wedge \neg O(Do_j\psi/\varphi) \wedge \neg O(\neg Do_j\psi/\varphi)$

(B_{13}) $O(\neg Do_i\psi/\varphi)\wedge\neg O(Do_i\neg\psi/\varphi)\wedge\neg O(\neg Do_i\neg\psi/\varphi)\wedge\neg O(Do_j\psi/\varphi)\wedge\neg O(Do_j\neg\psi/\varphi)\wedge \neg O(\neg Do_j\psi/\varphi)\wedge\neg O(\neg Do_j\neg\psi/\varphi)$

(B_{14}) $O(\neg Do_i\neg\psi/\varphi)\wedge\neg O(Do_i\psi/\varphi)\wedge\neg O(\neg Do_i\psi/\varphi)\wedge O(\neg Do_j\neg\psi/\varphi)\wedge\neg O(Do_j\psi/\varphi)\wedge \neg O(\neg Do_j\psi/\varphi)$

(B_{15}) $O(\neg Do_i\neg\psi/\varphi)\wedge\neg O(Do_i\psi/\varphi)\wedge\neg O(\neg Do_i\psi/\varphi)\wedge\neg O(Do_j\psi/\varphi)\wedge\neg O(Do_j\neg\psi/\varphi)\wedge \neg O(\neg Do_j\psi/\varphi)\wedge\neg O(\neg Do_j\neg\psi/\varphi)$

(B_{16}) $\neg O(Do_i\psi/\varphi)\wedge\neg O(Do_i\neg\psi/\varphi)\wedge\neg O(\neg Do_i\psi/\varphi)\wedge\neg O(\neg Do_i\neg\psi/\varphi)\wedge\neg O(Do_j\psi/\varphi)\wedge \neg O(Do_j\neg\psi/\varphi)\wedge\neg O(\neg Do_j\psi/\varphi)\wedge\neg O(\neg Do_j\neg\psi/\varphi)$

Given Σ, Δ as sets of sentences, then $\Sigma\cdot\Delta$ is the set of all the consistent conjunctions that can be formed by conjoining an expression from set Σ with an expression from set Δ.

These sixteen elements in $[\![\pm O(\pm Do_i\pm\psi/\varphi)]\!]\cdot[\![\pm O(\pm Do_j\pm\psi/\varphi)]\!]$ are consistent and logically independent, and the disjunction of them is a tautology. Similarly, there are sixteen consistent and logically independent elements of $[\![\pm O(\pm Do_i\pm\psi/\neg\varphi)]\!]\cdot[\![\pm O(\pm Do_j\pm\psi/\neg\varphi)]\!]$, which are also consistent and logically independent with those sixteen elements in $[\![\pm O(\pm Do_i\pm\psi/\varphi)]\!]\cdot[\![\pm O(\pm Do_j\pm\psi/\varphi)]\!]$. Hence, there are 16^2 consistent and logically independent elements of $[\![\pm O(\pm\binom{Do_i}{Do_j}\pm\psi/\pm\varphi)]\!]$, and the disjunction of them is a tautology. All these results are formulated in the following two theorems.

4.3.1. THEOREM. *For the two-agent case, we know that:*

$$\begin{aligned} &|\ [\![\pm O(\pm\binom{Do_i}{Do_j}\pm\psi/\varphi)]\!]\ |\\ =&|\ [\![\pm O(\pm Do_i\pm\psi/\varphi)]\!]\cdot[\![\pm O(\pm Do_j\pm\psi/\varphi)]\!]\ |\\ =&16 \end{aligned}$$

Using the elements from (B_1) to (B_{16}), I then can calculate the total number of elements in the schema $[\![\pm O(\pm\binom{Do_i}{Do_j}\pm\psi/\varphi)]\!]$, as shown in Theorem 4.3.1.

4.3.2. THEOREM.

$$\begin{aligned} &|\ [\![\pm O(\pm\binom{Do_i}{Do_j}\pm\psi/\pm\varphi)]\!]\ |\\ =&|\ [\![\pm O(\pm\binom{Do_i}{Do_j}\pm\psi/\varphi)]\!]\cdot[\![\pm O(\pm\binom{Do_i}{Do_j}\pm\psi/\neg\varphi)]\!]\ |\\ =&16^2 \end{aligned}$$

Theorem 4.3.2 provides a calculus for $|\ [\![\pm O(\pm\binom{Do_i}{Do_j}\pm\psi/\pm\varphi)]\!]\ |$. I therefore have $16\times 16 = 256$ elements of $[\![\pm O(\pm\binom{Do_i}{Do_j}\pm\psi/\pm\varphi)]\!]$, and they are consistent and logically independent. Compared to Kanger and Kanger's 26 "atomic types of rights" and Lindahl's 127 "collectivistic types" w.r.t. two-agent, my "atomic types of conditional static rights" are thus combinatorially richter.

I denote an *atomic type of conditional static rights*, involving i and j regarding ψ, conditional on φ, as $T_n(i, j, \psi/\varphi)$, where $1 \leq n \leq 256$, for the n-th element in $[\![\pm O(\pm\binom{Do_i}{Do_j} \pm \psi / \pm \varphi)]\!]$. And $\mathcal{T}(i, j, \psi/\varphi)$ denotes the set of the 256 atomic types of conditional static rights, involving i and j regarding ψ, conditional on φ. Each element in it is independent, in the sense that they are all consistent and logically independent, and the disjunction of all the elements is a tautology.

4.3.3 Dynamic Normative Positions

I now turn to the study of the dynamic normative positions that are generated by my theory of legal competences. Just as I have done for static normative positions, we answer now the question of how many distinct, atomic legal competences there are. I have seen that for static normative positions this number increases substantially with the number of agents. This is also the case here, but there is an additional complication. As hinted at by the valid reduction law for conditional obligation, the number of dynamic normative positions will also grow with the size of the deontic action model.

Let me define the set of *atomic legal competences* for two agents i, j and a given static normative position $T(i, j, \psi/\varphi))$ using again Makinson's [68] notation:

$$[\![\pm \bigvee_{i \in \mathcal{I}} \bigvee_{a \in \mathcal{A}_i} [\mathcal{A}_i, a] \pm T(i, j, \psi/\varphi)]\!]$$

The question I ask now is thus how large is that set for a given action model of size at most n? In this chapter I restrict ourselves to the two-agents cases. My base result establishes the number of combinations of claims and no-claims rights which can result from a given deontic action.

4.3.3. THEOREM. *The total maximally consistent elements in* $[\mathcal{A}_i, a][\![\pm O(\pm\binom{Do_i}{Do_j} \pm \psi/\varphi)]\!]$ *are:*

(T_1) $[\mathcal{A}_i, a](O(Do_i\psi/\varphi) \wedge O(Do_j\psi/\varphi))$

(T_2) $[\mathcal{A}_i, a](O(Do_i\psi/\varphi) \wedge O(\neg Do_j\psi/\varphi) \wedge O(\neg Do_j\neg\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi))$

(T_3) $[\mathcal{A}_i, a](O(Do_i\psi/\varphi) \wedge O(\neg Do_j\neg\psi/\varphi) \wedge \neg O(Do_j\psi/\varphi) \wedge \neg O(\neg Do_j\psi/\varphi))$

(T_4) $[\mathcal{A}_i, a](O(Do_i\neg\psi/\varphi) \wedge O(Do_j\neg\psi/\varphi))$

(T_5) $[\mathcal{A}_i, a](O(Do_i\neg\psi/\varphi) \wedge O(\neg Do_j\psi/\varphi) \wedge O(\neg Do_j\neg\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi))$

(T_6) $[\mathcal{A}_i, a](O(Do_i\neg\psi/\varphi) \wedge O(\neg Do_j\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi) \wedge \neg O(\neg Do_j\neg\psi/\varphi))$

(T₇) $[\mathcal{A}_i, a](O(\neg Do_i\psi/\varphi) \wedge O(\neg Do_i\neg\psi/\varphi) \wedge \neg O(Do_i\neg\psi/\varphi)\wedge$
$O(\neg Do_j\psi/\varphi) \wedge O(\neg Do_j\neg\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi))$

(T_8) $[\mathcal{A}_i, a](O(\neg Do_i\psi/\varphi) \wedge O(\neg Do_i\neg\psi/\varphi) \wedge \neg O(Do_i\neg\psi/\varphi)\wedge$
$O(\neg Do_j\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi) \wedge \neg O(\neg Do_j\neg\psi/\varphi))$

(T_9) $[\mathcal{A}_i, a](O(\neg Do_i\psi/\varphi) \wedge O(\neg Do_i\neg\psi/\varphi) \wedge \neg O(Do_i\neg\psi/\varphi)\wedge$
$O(\neg Do_j\neg\psi/\varphi) \wedge \neg O(Do_j\psi/\varphi) \wedge \neg O(\neg Do_j\psi/\varphi))$

(T_{10}) $[\mathcal{A}_i, a](O(\neg Do_i\psi/\varphi) \wedge O(\neg Do_i\neg\psi/\varphi) \wedge \neg O(Do_i\neg\psi/\varphi)\wedge$
$\neg O(Do_j\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi) \wedge \neg O(\neg Do_j\psi/\varphi)\wedge$
$\neg O(\neg Do_j\neg\psi/\varphi))$

(T_{11}) $[\mathcal{A}_i, a](O(\neg Do_i\psi/\varphi)\wedge\neg O(Do_i\neg\psi/\varphi)\wedge\neg O(\neg Do_i\neg\psi/\varphi)\wedge O(\neg Do_j\psi/\varphi)\wedge$
$\neg O(Do_j\neg\psi/\varphi) \wedge \neg O(\neg Do_j\neg\psi/\varphi))$

(T_{12}) $[\mathcal{A}_i, a](O(\neg Do_i\psi/\varphi)\wedge\neg O(Do_i\neg\psi/\varphi)\wedge\neg O(\neg Do_i\neg\psi/\varphi)\wedge O(\neg Do_j\neg\psi/\varphi)\wedge$
$\neg O(Do_j\psi/\varphi) \wedge \neg O(\neg Do_j\psi/\varphi))$

(T_{13}) $[\mathcal{A}_i, a](O(\neg Do_i\psi/\varphi)\wedge\neg O(Do_i\neg\psi/\varphi)\wedge\neg O(\neg Do_i\neg\psi/\varphi)\wedge\neg O(Do_j\psi/\varphi)\wedge$
$\neg O(Do_j\neg\psi/\varphi) \wedge \neg O(\neg Do_j\psi/\varphi)\wedge$
$\neg O(\neg Do_j\neg\psi/\varphi))$

(T_{14}) $[\mathcal{A}_i, a](O(\neg Do_i\neg\psi/\varphi) \wedge \neg O(Do_i\psi/\varphi) \wedge \neg O(\neg Do_i\psi/\varphi)\wedge$
$O(\neg Do_j\neg\psi/\varphi) \wedge \neg O(Do_j\psi/\varphi) \wedge \neg O(\neg Do_j\psi/\varphi))$

(T_{15}) $[\mathcal{A}_i, a](O(\neg Do_i\neg\psi/\varphi) \wedge \neg O(Do_i\psi/\varphi) \wedge \neg O(\neg Do_i\psi/\varphi)\wedge$
$\neg O(Do_j\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi) \wedge \neg O(\neg Do_j\psi/\varphi) \wedge \neg O(\neg Do_j\neg\psi/\varphi))$

(T_{16}) $[\mathcal{A}_i, a](\neg O(Do_i\psi/\varphi)\wedge\neg O(Do_i\neg\psi/\varphi)\wedge\neg O(\neg Do_i\psi/\varphi)\wedge\neg O(\neg Do_i\neg\psi/\varphi)\wedge$
$\neg O(Do_j\psi/\varphi) \wedge \neg O(Do_j\neg\psi/\varphi) \wedge \neg O(\neg Do_j\psi/\varphi) \wedge \neg O(\neg Do_j\neg\psi/\varphi))$

With this in hand I can get my calculation going. There are 256 maximally consistent elements in

$$\bigvee_{a\in\mathcal{A}_i}[\mathcal{A}_i, a][\![\pm O(\pm\binom{Do_i}{Do_j} \pm \psi/ \pm \varphi)]\!]$$

when $|\mathcal{A}_i| = 1$. When $|\mathcal{A}_i| = 2$, the total number of the maximally consistent elements of that set is 2^{256}. This number happens to grow linearly. With $|\mathcal{A}_i| = n$, we get n^{256} possibilities. On the other hand, when $|\mathcal{A}_i| = 2$, the total number of the maximally consistent elements in

$$\neg\bigvee_{a\in\mathcal{A}_i}[\mathcal{A}_i, a][\![\pm O(\pm\binom{Do_i}{Do_j} \pm \psi/ \pm \varphi)]\!]$$

is 2^{256} This number is also n^{256} for $|\mathcal{A}_i| = n$. Furthermore, when $|\mathcal{A}_i| = 1$, for each element in the first set above there are 255 elements in the second. Accordingly, hence the total number of elements in

$$[\![\pm \bigvee_{a\in\mathcal{A}_i}[\mathcal{A}_i, a][\![\pm O(\pm\binom{Do_i}{Do_j} \pm \psi / \pm \varphi)]\!]]\!]$$

is $256 \times 255 = 65280$, when $|\mathcal{A}_i| = 1$, and the total number is $n^{256} \times (n^{255})$, when $|\mathcal{A}_i| = n$.

4.3.4 Bisimulation for Dynamic Normative Positions

We already know how large the size of the set of atomic legal competences is, given a static normative position. Now I turn to the question how much an action model may affect the atomic legal competences. I first provide an example to show the effect of a given action model on atomic legal competences, and then I present a general result based on the notion of bisimulation defined in Section 4.2.3.

4.3.4. EXAMPLE. Given a model $\mathcal{M} = \langle W, <, \cong, \{\sim_i\}_{i\in\mathcal{I}}, V\rangle$ presented in Table 4.3, such that $W = \{w_1, w_2, w_3, w_4\}$, $<=\cong=\varnothing$, $\sim_i = \{w_1 \sim_i w_2, w_1 \sim_i w_1, w_2 \sim_i w_2, w_3 \sim_i w_3, w_4 \sim_i w_4\}$, and $V(p) = \{w_1, w_3\}$. Let $A = \{a_1, a_2, a_3, a_4\}$, where $Pre(a_1) = Do_i p$, $Pre(a_2) = Do_i\neg p$, $Pre(a_3) = do_i\neg p$, $Pre(a_4) = do_i p$.

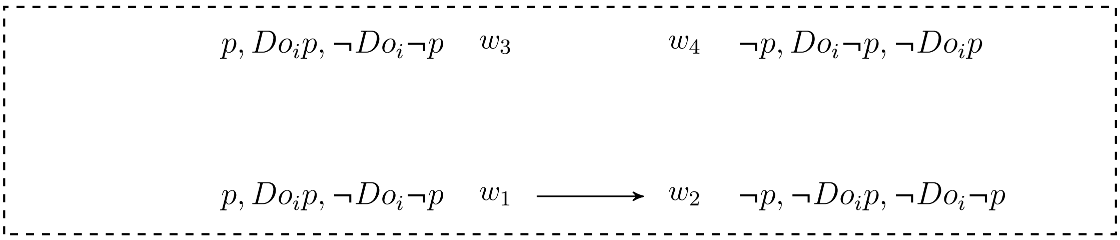


Table 4.3: The Atomic Types of Legal Competences

Now I can consider how much difference action models may bring to the given model $\mathcal{M}$. Let $\mathcal{A}_i$ be an action model $\langle A, >^{\mathcal{A}_i}, \cong^{\mathcal{A}_i}, Pre\rangle$ for agent i, where $>^{\mathcal{A}_i} = \cong^{\mathcal{A}_i} = \varnothing$. For every $a \in \mathcal{A}_i$,

- $\mathcal{M} \nvDash [\mathcal{A}_i, a]O(Do_ip/\top)$
- $\mathcal{M} \nvDash [\mathcal{A}_i, a]O(Do_i\neg p/\top)$
- $\mathcal{M} \nvDash [\mathcal{A}_i, a]O(\neg Do_ip/\top) \wedge [\mathcal{A}_i, a]O(\neg Do_i\neg p/\top) \wedge [\mathcal{A}_i, a]\neg O(Do_i\neg p/\top)$
- $\mathcal{M} \nvDash [\mathcal{A}_i, a]O(\neg Do_ip/\top) \wedge [\mathcal{A}_i, a]\neg O(Do_i\neg p/\top) \wedge [\mathcal{A}_i, a]\neg O(\neg Do_i\neg p/\top)$
- $\mathcal{M} \nvDash [\mathcal{A}_i, a]O(\neg Do_i\neg p/\top) \wedge [\mathcal{A}_i, a]\neg O(Do_ip/\top) \wedge [\mathcal{A}_i, a]\neg O(\neg Do_ip/\top)$

- $\mathcal{M} \models [\mathcal{A}_i, a]\neg O(Do_i p/\top) \wedge [\mathcal{A}_i, a]\neg O(Do_i \neg p/\top) \wedge [\mathcal{A}_i, a]\neg O(\neg Do_i p/\top) \wedge$ $[\mathcal{A}_i, a]\neg O(\neg Do_i \neg p/\top)$

Given a different action model $\mathcal{A}_i'$ as follows, I have a different result. Let $\mathcal{A}_i'$ be another action model $\langle A, >^{\mathcal{A}_i'}, \cong^{\mathcal{A}_i'}, Pre\rangle$ for agent i, so $\geq^{\mathcal{A}_i'} = >^{\mathcal{A}_i'} \cup \cong^{\mathcal{A}_i'}$. In this action model, $a_1 \geq^{\mathcal{A}_i'} a_2, a_1 \geq^{\mathcal{A}_i'} a_3, a_1 \geq^{\mathcal{A}_i'} a_4$. For every $a \in \mathcal{A}_i'$,

- $\mathcal{M} \models [\mathcal{A}_i', a]O(Do_i p/\top)$
- $\mathcal{M} \not\models [\mathcal{A}_i', a]O(Do_i \neg p/\top)$
- $\mathcal{M} \not\models [\mathcal{A}_i', a]O(\neg Do_i p/\top) \wedge [\mathcal{A}_i', a]O(\neg Do_i \neg p/\top) \wedge [\mathcal{A}_i', a]\neg O(Do_i \neg p/\top)$
- $\mathcal{M} \not\models [\mathcal{A}_i', a]O(\neg Do_i p/\top) \wedge [\mathcal{A}_i', a]\neg O(Do_i \neg p/\top) \wedge [\mathcal{A}_i', a]\neg O(\neg Do_i \neg p/\top)$
- $\mathcal{M} \not\models [\mathcal{A}_i', a]O(\neg Do_i \neg p/\top) \wedge [\mathcal{A}_i', a]\neg O(Do_i p/\top) \wedge [\mathcal{A}_i', a]\neg O(\neg Do_i p/\top)$
- $\mathcal{M} \not\models [\mathcal{A}_i', a]\neg O(Do_i p/\top) \wedge [\mathcal{A}_i', a]\neg O(Do_i \neg p/\top) \wedge [\mathcal{A}_i', a]\neg O(\neg Do_i p/\top) \wedge$ $[\mathcal{A}_i', a]\neg O(\neg Do_i \neg p/\top)$

In fact, there is a close link between action models and atomic legal competences. In order to show this I need first to establish the following Lemma:

4.3.5. LEMMA. *Given two action models $\mathcal{A}_i$ and $\mathcal{A}_i'$ for agent i, and two models $\mathcal{M}$ and $\mathcal{M}'$. If $\mathcal{M} \otimes \mathcal{A}_i$ and $\mathcal{M}' \otimes \mathcal{A}_i'$ are bisimilar, then* $\max_{\mathcal{M}\otimes\mathcal{A}_i}(||\pm\varphi||) \subseteq ||\pm\psi||$ *iff* $\max_{\mathcal{M}'\otimes\mathcal{A}_i'}(||\pm\varphi||) \subseteq ||\pm\psi||$.

Proof:
I consider $\max_{\mathcal{M}\otimes\mathcal{A}_i}(||\varphi||) \subseteq ||\psi||$ iff $\max_{\mathcal{M}'\otimes\mathcal{A}_i'}(||\varphi||) \subseteq ||\psi||$ as an example for all proofs of the four cases.

($\Rightarrow$) Let $(w', a') \in \max_{\mathcal{M}'\otimes\mathcal{A}_i'}(||\varphi||)$ where $w' \in \mathcal{M}'$ and $a' \in \mathcal{A}_i'$. It implies that $(w', a') \in ||\varphi||$ and $\neg\exists(u', b') \in ||\varphi||$ s.t. $(u', b') > (w', a')$ where $u' \in \mathcal{M}$ and $b' \in \mathcal{A}_i'$. From the latter, I have (*): $\forall(u', b') \in \mathcal{M}' \otimes \mathcal{A}_i'$ that if $(u', b') > (w', a')$ then $(u', b') \notin ||\varphi||$.

I assume that E is a bisimulation between $\mathcal{M} \otimes \mathcal{A}_i$ and $\mathcal{M}' \otimes \mathcal{A}_i'$, and then $\exists(w, a) \in \mathcal{M} \otimes \mathcal{A}_i$ s.t. $(w', a')E(w, a)$. From the invariance lemma in [105], I have (!): $\mathcal{M} \otimes \mathcal{A}_i \models \chi$ iff $\mathcal{M}' \otimes \mathcal{A}_i' \models$. So $(w, a) \in ||\varphi||$. Given any $(u, b) \in \mathcal{M} \otimes \mathcal{A}_i$ that $(u, b) > (w, a)$. As $(\mathcal{M} \otimes \mathcal{A}_i)E(\mathcal{M}' \otimes \mathcal{A}_i')$, $\exists(u', b') \in \mathcal{M}' \otimes \mathcal{A}_i'$ s.t. $(u', b') > (w', a')$, and $(u, b) \in ||\varphi||$ iff $(u', b') \in ||\varphi||$. According to (*), I have $(u', b') \notin ||\varphi||$. And then $(u, b) \notin ||\varphi||$. Thus, I conclude that $(w, a) \in \max_{\mathcal{M}\otimes\mathcal{A}_i}(||\varphi||)$, and so $(w, a) \in ||\psi||$. From (!), I can infer that $(w', a') \in ||\psi||$. Hence, $\max_{\mathcal{M}'\otimes\mathcal{A}_i'}(||\varphi||) \subseteq ||\psi||$.

($\Leftarrow$) Its proof is similar to the case ($\Rightarrow$).

The proofs of the other three cases are very similar to the previous case. □

4.3.6. THEOREM. *Given two action models $\mathcal{A}_i$ and $\mathcal{A}_i'$ for agent i, and two models $\mathcal{M}$ and $\mathcal{M}'$. Let $D_k(i, j, \varphi, \psi)$ be the k-th atomic types of legal competence for $\mathcal{A}_i$, and $D_l'(i, j, \varphi, \psi)$ the l-th atomic types of legal competence for $\mathcal{A}_i'$. If $\mathcal{A}_i$ and $\mathcal{A}_i'$ are bisimilar, and $\mathcal{M}$ and $\mathcal{M}'$ are bisimilar, then $\mathcal{M} \vDash D_k(i, j, \varphi, \psi)$ iff $\mathcal{M}' \vDash D_k'(i, j, \varphi, \psi)$.*

Proof:
From $\mathcal{A}_i$ and $\mathcal{A}_i'$ being bisimilar, and $\mathcal{M}$ and $\mathcal{M}'$ being bisimilar, by Lemma 4.2.10, I can infer that $\mathcal{M} \otimes \mathcal{A}_i$ and $\mathcal{M}' \otimes \mathcal{A}_i'$ are bisimilar. This may imply that $\max_{\mathcal{M} \otimes \mathcal{A}_i}(|| \pm \varphi||) \subseteq || \pm \psi||$ iff $\max_{\mathcal{M}' \otimes \mathcal{A}_i'}(|| \pm \varphi||) \subseteq || \pm \psi||$ by Lemma 4.3.5. Obviously, I can then conclude that $\mathcal{M} \vDash D_k(i, j, \varphi, \psi)$ iff $\mathcal{M}' \vDash D_k'(i, j, \varphi, \psi)$. □

Crucially, I also have a converse type of result. Indistinguishibility in terms of legal competence entails bisimilarity after update.

4.3.7. THEOREM. *Given two models $\mathcal{M}$ and $\mathcal{M}'$, and two action models $\mathcal{A}_i$ and $\mathcal{A}_i'$ for agent i. Let $D_k(i, j, \varphi, \psi)$ be the k-th atomic types of legal competence for $\mathcal{A}_i$, and $D_l'(i, j, \varphi, \psi)$ the l-th atomic types of legal competence for $\mathcal{A}_i'$. If $\mathcal{M} \vDash D_k(i, j, \varphi, \psi)$ iff $\mathcal{M}' \vDash D_k'(i, j, \varphi, \psi)$, then $\mathcal{M} \otimes \mathcal{A}_i$ and $\mathcal{M}' \otimes \mathcal{A}_i'$ are bisimilar.*

Proof:
The assumption that $\mathcal{M} \vDash D_k(i, j, \varphi, \psi)$ iff $\mathcal{M}' \vDash D_k'(i, j, \varphi, \psi)$ is simply equivalent to state: $\max_{\mathcal{M} \otimes \mathcal{A}_i}(|| \pm \psi||) \subseteq || \pm Do_i \pm \varphi||$ iff $\max_{\mathcal{M}' \otimes \mathcal{A}_i'}(|| \pm \psi||) \subseteq || \pm Do_i \pm \varphi||$. To prove that $\mathcal{M} \otimes \mathcal{A}_i$ and $\mathcal{M}' \otimes \mathcal{A}_i'$ are bisimilar. Otherwise, suppose that $(w, a)E(w', a')$, and $(w, a) > (u, b)$ in $\mathcal{M} \otimes \mathcal{A}_i$. And suppose that there is no $(u', b') \in \mathcal{M}' \otimes \mathcal{A}_i'$ s.t. $(w', a') > (u', b')$ and $(u, b)E(u', b')$. That is to say, there is a basic action type $D_k(i, j, \varphi, \psi)$ true at $\mathcal{M}, u$, and a different basic action type $D_l(i, j, \varphi, \psi)$ true at $\mathcal{M}', u'$, where $k \neq l$. This implies that, for example, $\max_{\mathcal{M} \otimes \mathcal{A}_i}(||\psi||) \subseteq ||Do_i\varphi||$ and $\max_{\mathcal{M}' \otimes \mathcal{A}_i'}(||\psi||) \subseteq ||Do_i\neg\varphi||$. Obviously, this contradicts my assumption. □

Together with these results, I can conclude that there is a strong link between atomic types of legal competences and lexicographic updates. This close link reflects that my dynamic logic is a natural characterization of legal competences.

4.4 Legal Ability and Legal Permissibility

Although legal ability and legal permissibility often go together, they are conceptually distinct notions [68, 54]. The German Civil Code (*Bürgerliches Gesetzbuch*) offers a concrete case where they can come apart.

Article 179 of this code regulates the contractual delegation of the right to auction one's land to a third party. The article allows for the following case. Suppose that i contracts j to auction her land. j sells to k, but k is not the highest bidder. The sale being a deontic action, it is still considered valid. It transfers property rights from i to k. j selling to k is, however, not legally permissible, because k is not the highest bidder. j can consequently be asked to compensate i for the difference between the selling price and the highest bid.

In this example j has the legal ability to sell the land to k. She is legally capable of executing a deontic action which transfers the set of static and dynamic property rights with regard to the land from i to k. This action, however, is impermissible. It cannot be executed without j incurring a sanction. I now show that my logical model of legal competences can capture this case in a simple manner.

4.4.1 Legal Ability

Let κ be the fact that k is in possession of the land's property titles, and S be that j compensate i for the price difference. Before the sale, i owns her land, and i contracts j to auction her land. j, so entrusted by i, has a privilege to transfer the land's property titles, which I represent here simply as transferring them to κ or to someone else $\neg\kappa$. Hence, even on the condition S, the states with $Do_j\neg\kappa$ are the most ideal, the states with $Do_j\kappa$ are the least ideal, and the others are as ideal as each other. This situation is illustrated in Figure 4.6. There, as before, the arrows $\rightarrow$ representing the preference ordering $\leq$ between distinct states, and the dashed arc labelled j represents the relations $\sim_j$. The reflexive loops for both relations are everywhere omitted. In this model the following simple type of conditional static right holds for j: $O(Do_j\neg\kappa)$. Also, $\neg O(\kappa)$. Here do_j is the dual of Do_j.

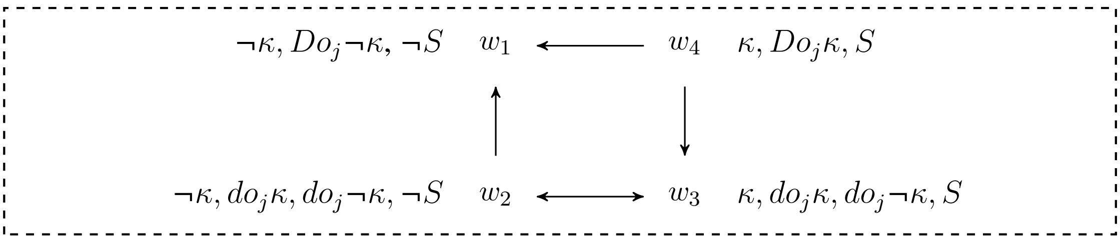


Figure 4.6: The preference-action model **C** for the normative positions before the sale.

j's legal ability can be modelled in the deontic action model of Figure 4.7. As before, the arrow $\mapsto$ represents the preference order $\leq^{\mathbf{S}_j}$ with the reflexive loops omitted. In this particular model, the precondition of action s is $Do_j\kappa$, and that of action n is $\neg Do_j\kappa$.

The result of the lexicographic update $\mathbf{C} \otimes \mathbf{S}_j$ is presented in Figure 4.8. In this model the normative position of B has changed. Now I have $O(Do_j\kappa)$, and $O(\kappa)$ as

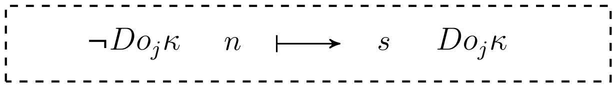


Figure 4.7: The deontic action model $\mathbf{S}_j$ for j's sale to k.

well. The action j selling the land to k of course changes i's normative position as well, but I omit those here.

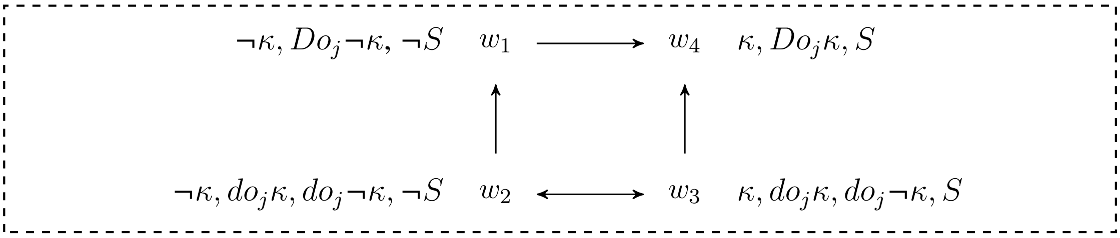


Figure 4.8: The preference-action model $\mathbf{C}$ for the normative positions before the sale.

This simple example shows how to model j's legal ability to sell i's land to k. Since k is not the highest bidder, however, j's action is not legally permissible. This can also be easily expressed here.

4.4.2 Legal Permissibility

As mentioned, the language $\mathcal{L}$, even extended with the dynamic modalities, cannot directly express the notion of legal permissibility. This language is designed to describe the effects of deontic action or, by having deontic modalities scoping over dynamic expressions, the normative status of those effects. But this is still different from saying that a certain deontic action is obligatory, permitted or forbidden. In my example it is arguably the case that it ought to be that k owns the land after j has sold it to her, even though this sale is not legally permitted.

To express legal permissibility I rather use a type of Anderson-Kanger reduction that is also present in dynamic deontic logic [74]. Let S, as above, represent the constant that a sanction will occur, viz. in my particular case that j must compensate i. Then we define "action a is legally permitted" as follows.

$$P(a) := [\mathcal{A}_l, a]\neg S$$

This is the standard definition of strong permission in dynamic deontic logic introduced by Dignum et. al. [26]. In my case $Do_j\kappa \rightarrow S$. j incurs a sanction upon selling the land to k. Since $\mathbf{C}, w_4 \models \langle \mathbf{S}_j, s\rangle S$, I get that $\mathbf{C}, w_4 \models \neg P(s)$ and, together with my analysis of legal ability $\mathbf{C}w_4 \models \langle \mathbf{S}_j, s\rangle \top \wedge \neg P(s)$. Selling to k is legally possible but not permissible.

4.5 Conclusion

This chapter can be seen as a test drive for a new way of representing the structure of legal competences, and of deontic action more generally: preference-action models, (deontic) action models and the related update mechanism. This methodology is well-established in epistemic and doxastic logic, and many of the results presented here closely follow that literature. My main contribution is to bring it to bear on the theory of dynamic rights.

Indeed, I have argued that the model of Hohfeldian power and immunity developed here improves on both the classical reductive and non-reductive approaches. In comparison with, for instance Lindahl's [63] approach, my model explicitly captures the norm-changing or dynamic character of legal competences. It does so both at the semantic level, through the explicit update mechanism, and at the semantic level, by using an explicit dynamic modality to express the effects of deontic actions. The approach I propose here is however still reductive in the sense that formulas with dynamic modalities are semantically and provably reducible to formulas without. As a result it comes with a rich set of interaction principles between static and dynamic rights. From that point of view it improves on Jones and Sergot's non-reductive approach to dynamic rights through "count as" conditionals [54]. Finally, I have shown that this system can capture the distinction between legal ability and legal permissibility in a more auspicious way than reductive approaches, without paying the price of full-blown non-reductionism.

I take this to be a promising starting point for the methodology we propose, but of course it also raises a number of questions that could not be addressed in this chapter. Probably the most important next step will be to enrich the model to cover not only pure actions but also the combination of deontic and non-deontic actions, for instance breaking someone's window or crossing someone's property. This would allow for a closer comparison between my and the Jones and Sergot approach just mentioned. Equally important in my view is to study the theory of legal competences that would result from extending a static base that is different from Standard Deontic Logic. In the epistemic context a wide variety of static logics of knowledge and belief have been "dynamified" using the action model methodology. The proposal here already slightly deviates from the classical Kangerian approach in that it includes conditional obligations. Studying the normative positions stemming from that addition gains me a deeper insight into conditional normative positions. Of course more radical departures from SDL have been proposed in this chapter to capture actual legal reasoning, and the question remains whether they would yield a plausible theory of power and immunity once augmented with a dynamic module as I have done here.

Chapter 5

The Defeasible Characters of Permission

A Prioritized Default Theory Perspective

This chapter studies permissions in Horty's prioritized default theory [52]. There are two main motivations for doing this. The first is to extend prioritized default theory with a richer typology of permissions, in order to study the relationship between permissions and defeasible norms. Only one type of permission has been studied in prioritized default theory: the one that I so far have called "weak permission," that is the dual of obligation [52]. Recall that this is the same type of permission as in SDL [47]. Weak permission does not, however, exhaust all types of permission [71, 43]. We have already encountered strong permission, free choice permission and bilateral permission. With the notable exception of [71], however, little work has been done on the relation between a richer typology of permissions and defeasible obligations. This gap should be filled, and Horty's prioritized default theory, being a prominent, recent theory of defeasible obligation, is a natural framework to do so. So by studying permissions in default theory, I am taking another step towards reaching the main goal of this thesis, that is understanding how permission goes together with non-monotonic normative reasoning.

The second motivation for studying permissions in prioritized default theory is to continue with the thread that I started in the previous chapter: application to legal reasoning. A number of salient pieces of legislation involve both permissions and defeasible norms. Horty's theory has already been applied to Common Law reasoning [51, 53]. I show in this chapter that, equipped with a richer typology of permissions, it can capture and analyze two running examples, one from Chinese Tax Law, and one from the German Driving Regulations.

5.0.1. EXAMPLE. [China Tax Incentive] In the season for businesses filing their tax returns, any foreign company c established in China should file the corporate income tax (CIT) returns with the local tax authority a. If, however, c is a solar energy company, and then it can benefit from a 15% preferential tax rate from the government, if it submits valid documentation.

5.0.2. EXAMPLE. [German Right of Way] Germany uses the "priority to the right" rule at intersections with no further priority signs. A driver on a priority road making a left turn is permitted to do so, unless another vehicle coming from the opposite direction on the same road drives straight on or turns right.

The main contribution of this chapter is thus to extend prioritized default theory with various notions of permission (Section 5.2), and to apply it to concrete cases in Chinese and German Law (Section 5.2). One additional contribution of this chapter is to connect the resulting theory to defeasible permissions in Horty's theory to other versions of default theories and to argumentation theory. The first issue concerns the relation between my formal default theory and Dung's stable semantics in argumentation theory [28]. This provides a theoretical foundation to my framework (Section 5.3). The second requires me to make a comparison of the representation of my *rebutting* defeater with the so-called *undercutting* defeater [52] in certain legal cases, which gives rise to different kinds of permission (Section 5.4). Here my contribution gives rise to a new typology of permissions, different from the one in [86] and Alchourrón and Bulygin [2].

This chapter is organized as follows. For the first task I draw from existing accounts of explicit and tacit permissions [71, 82]. The second requires developing a multi-agent version of Horty's theory. The third task needs a comparison with the existing theories. Section 5.1 introduces Horty's prioritized default theory and its agent-relative extension, Section 5.2 presents the various permissions and their applications, Section 5.3 argues that my formal theory is developed on a firm foundation for *defeasibility*, Section 5.4 demonstrates the legitimacy of my taxonomy of permissions, and Section 5.5 concludes.

5.1 Multi-Agent Prioritized Default Theory

I start by introducing the basics of Horty's prioritized default theory, generalized such that it allows for agent-relative duties and permissions. Prioritized default theory is a syntactic theory about deriving defeasible consequences from background information and default rules [52]. Although similar in some aspects, this syntactic theory is different than Makinson's general syntactic theory for non-monotonic consequence [69] and from Hansson's semantic model for preference [42].

The language $\mathcal{L}$ of default theory is a set of propositions inductively defined by a set of atomic propositions $p \in Atom$ (containing $\top$ as the true constant) and $\wedge, \vee, \Rightarrow$, and $\neg$ as the operations of conjunction, disjunction, material implication, and negation. The turnstile $\vdash$ indicates standard propositional logic consequence. So, unlike Chapter

3, here this consequence relation is entirely classical. Given $\varphi, \psi, \chi \cdots$ as propositions, and X as a set of propositions, then $X \vdash \varphi$ means that φ is derivable from X according to the rules of propositional logic. So $Cn(X) = \{\varphi \mid X \vdash \varphi\}$ is the logical closure of X.

A default theory Δ is a triple $\langle \mathcal{W}, \mathcal{I}, \mathcal{D}, < \rangle$ where $\mathcal{W}$ is a set of propositions, $\mathcal{I}$ is a set of agents, $\mathcal{D}$ is a set of default rules, and $<$ is a strict partial order on $\mathcal{D}$. A default (or default rule) δ_i for agent i is a pair (φ, ψ) for φ, ψ formulas of our propositional language. I often denote such a default as

$$\varphi \xrightarrow{\delta_i} \psi$$ [1]

I use the letters $\delta_i, \gamma_i, \cdots$ to denote defaults for agent i. Now I denote $\mathcal{D}_i$ as a subset of $\mathcal{D}$ such that $\mathcal{D}_i$ is a set of defaults δ_i for agent i. Given a default $\varphi \xrightarrow{\delta_i} \psi$, I say φ is the premise of δ_i, denoted as $Pre(\delta_i)$, and ψ is the conclusion of δ_i, denoted as $Con(\delta_i)$. Given a set $\mathcal{D}$ of defaults, the set of premises of $\mathcal{D}$ is defined as $Pre(\mathcal{D}) = \{Pre(\delta_i) \mid \delta_i \in \mathcal{D}\}$, and the set of conclusions of $\mathcal{D}$ is defined as $Con(\mathcal{D}) = \{Con(\delta_i) \mid \delta_i \in \mathcal{D}\}$. The order $<$ is used to represent the fact that certain defaults "override" others. So $\delta_i < \gamma_j$ indicates that the default γ_j for j has a higher priority than the default δ_i for i.

Horty adopts a reason-based reading of a given default rule $\varphi \xrightarrow{\delta_i} \psi$. In my agent-relative version of his theory this gives:

φ is a *prima facie* reason for i to ψ

Let me illustrate this by modeling part of the Chinese Tax Law Example. I come back to the German Driving Regulation later on. Suppose that s means that this is the tax season in China, d means a company submits the valid documentation to prove that it is a solar energy business, t means that the company files the CIT, and r that the company gets the preferential tax rate. In this case I have two defaults: $s \xrightarrow{\delta_c} t$ and $d \xrightarrow{\delta_a} r$. The default theory $\Delta_{CIT} = \langle \mathcal{W}, \mathcal{I}, \mathcal{D}, < \rangle$ is then defined as follows: $\mathcal{W} = \{s, \neg t \Leftrightarrow r\}$, $\mathcal{I} = \{a, c\}$, $\mathcal{D} = \{s \xrightarrow{\delta_c} t, d \xrightarrow{\delta_a} r\}$, with the priority $\delta_c < \delta_a$, reflecting that in event that both defaults are triggered then the tax reduction gets priority. See Figure 5.1.

A *scenario* based on a default theory $\Delta = \langle \mathcal{W}, \mathcal{I}, \mathcal{D}, < \rangle$ is a subset $\mathcal{S}$ of the set $\mathcal{D}$ of defaults in Δ. There are four scenarios in Δ_{CIT}: $\varnothing$, $\{\delta_c\}$, $\{\delta_a\}$, $\{\delta_c, \delta_a\}$. All things considered obligations for particular agents are derived from so-called *proper* scenarios, which are computed in the same way as in [52]. Proper scenarios should contain defaults that are be triggered, non-conflicted, and non-defeated at the same time.

5.1.1. DEFINITION. [Triggered] Given a scenario $\mathcal{S}$ based on $\Delta = \langle \mathcal{W}, \mathcal{I}, \mathcal{D}, < \rangle$, a default $(\varphi, \psi) \in \mathcal{D}$ is triggered in this scenario is defined as

$$(\varphi, \psi) \in Triggered_{\mathcal{W},\mathcal{D}}(\mathcal{S}) \Leftrightarrow \varphi \in Cn(\mathcal{W} \cup Con(\mathcal{S}))$$

[1] The notations in this chapter is slightly different than the previous ones. I use $\Rightarrow$ for the material conditional, and $\rightarrow$ for default rules.

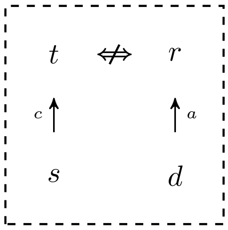


Figure 5.1: A default theory of a solar energy business having to file the regular CIT return to the local tax authority.

In Δ_{CIT} only δ_c triggered in all four scenarios. The default δ_a is never triggered, because its premise d is neither in the background information set $\mathcal{W}$ nor a conclusion of a default that could be included in a scenario.

5.1.2. DEFINITION. [Conflicted] Given a scenario $\mathcal{S}$ based on $\Delta = \langle \mathcal{W}, \mathcal{I}, \mathcal{D}, < \rangle$, a default $(\varphi, \psi) \in \mathcal{D}$ is conflicted in this scenario is defined as

$$(\varphi, \psi) \in Conflicted_{\mathcal{W},\mathcal{D}}(\mathcal{S}) \Leftrightarrow \neg\psi \in Cn(\mathcal{W} \cup Con(\mathcal{S}))$$

In my example the default δ_a is conflicted in all scenarios that contain δ_c. This default is itself conflicted in all scenarios containing δ_a.

5.1.3. DEFINITION. [Defeated] Given a scenario $\mathcal{S}$ based on $\Delta = \langle \mathcal{W}, \mathcal{I}, \mathcal{D}, < \rangle$, a default $(\varphi, \psi) \in \mathcal{D}$ is defeated in this scenario is defined as

$$\begin{aligned}(\varphi, \psi) \in Defeated_{\mathcal{W},\mathcal{D}}(\mathcal{S}) \Leftrightarrow\ & \exists(\varphi', \psi') \in Triggered_{\mathcal{W},\mathcal{D}}(\mathcal{S})\ s.t. \\ & (1)\ (\varphi, \psi) < (\varphi', \psi') \\ & (2)\ \neg\psi \in Cn(\mathcal{W} \cup \{\psi'\})\end{aligned}$$

In my example there is no defeated default, since the only potential defeater is δ_a but it is never triggered.

5.1.4. DEFINITION. [Proper Scenario] Given a scenario $\mathcal{S}$ based on the theory $\Delta = \langle \mathcal{W}, \mathcal{I}, \mathcal{D}, < \rangle$. A default $(\varphi, \psi) \in \mathcal{D}$ is proper in this scenario is defined as

$$\begin{aligned}(\varphi, \psi) \in Proper_{\mathcal{W},\mathcal{D}}(\mathcal{S}) \Leftrightarrow\ & (\varphi, \psi) \in Triggered_{\mathcal{W},\mathcal{D}}(\mathcal{S}), \text{ and} \\ & (\varphi, \psi) \notin Conflicted_{\mathcal{W},\mathcal{D}}(\mathcal{S}), \text{ and} \\ & (\varphi, \psi) \notin Defeated_{\mathcal{W},\mathcal{D}}(\mathcal{S})\end{aligned}$$

I say a scenario $\mathcal{S}$ is a proper scenario iff $\mathcal{S} = Proper_{\mathcal{W},\mathcal{D}}(\mathcal{S})$, which involves all those defaults satisfying all three conditions for propriety. As a summary of all three conditions, $\{\delta_c\}$ is the unique proper scenario in the previous running example.

5.1.1 Agent-Relative Defeasible Obligations

Prioritized default theories allow to distinguish two types of defeasible obligation, the so-called conflict and disjunctive accounts. Intuitively, given a default theory Δ, φ is obligatory in the disjunctive sense if φ follows from all proper scenarios of Δ. On the other hand, φ is obligatory in the conflict sense if it follows from some proper scenarios of Δ. The labels "disjunctive" and "conflict" come from their logical behavior in case a given theory has multiple, incompatible scenarios, viz. some in which φ follows, and some in which ψ follows, with $\psi \xrightarrow{\delta_i} \neg\varphi$. On the conflict account both φ will be obligatory, and also $\neg\varphi$, although not their conjunction. On the disjunctive account only $\varphi \vee \psi$ will be obligatory, but none of the disjuncts will.

5.1.5. DEFINITION. [Agent-Neutral Obligations - Conflict and Disjunctive Accounts] Let Δ be a default theory. Then:

- $\Delta \vDash O^c(\varphi)$, iff $\varphi \in Cn(\mathcal{W} \cup Con(\mathcal{S}))$ for some proper scenario S of Δ
- $\Delta \vDash O^d(\varphi)$, iff $\varphi \in Cn(\mathcal{W} \cup Con(\mathcal{S}))$ for all proper scenarios S of Δ

While both types of duties are agent-neutral, in the legal applications that I have in mind duties and permissions are agent-relative. In Chinese Tax Law, the duty to submit the tax report bears on company c. There is no such duty for the tax authority. Similarly, in the Right of Way example one Driver's permission overrides the other's. More generally, the Hohfeldian typology that I studied in the previous chapter also rests on the notion of correlative duties, which are agent-relative.

I capture agent-relative duties by a reducing them to a combination of agent-neutral duties and reason-supported actions. Agent i's ought to make it the case that φ will then be interpreted as that it ought to be the case that i makes it the case that φ. This is similar to the reduction of "ought to do" to "ought to be" that is proposed in [50], and further discussed in [19]. I define the latter notion using what I call *supported action sets*. Informally, making it the case that φ is a supported action for agent i given a proper scenario $\mathcal{S}$ whenever there is a reason for i in S to making it the case that φ, which comes down to saying that $\psi \xrightarrow{\delta_i} \varphi$ is in S, for some ψ. I will denote such supported actions with an additional piece of notation: formula $Do_i\varphi$ for each agent i, which should be read as "making it the case that φ is supported for agent i". The negation of this formula, $\neg Do_i\varphi$, should in turn be read as "making it the case that φ is conflicted for agent i".

5.1.6. DEFINITION. [Supported Action Sets - First order] Let $\mathcal{S}$ be a proper scenario for a default theory Δ, and $\mathcal{D}$ a set of defaults. The supported action set $\mathcal{A}_{\mathcal{S},\mathcal{D}}$ is defined as

$$\begin{aligned}\mathcal{A}_{\mathcal{S},\mathcal{D}} =&\{Do_{\{i_1,\cdots,i_n\}}\varphi \mid \exists(\psi_1,\varphi_1)\in\mathcal{D}_{i_1},\cdots,(\psi_n,\varphi_n)\in\mathcal{D}_{i_n}\text{s.t.}\\ &\qquad \varphi = \bigwedge_{1\le k\le n}\varphi_k \in Cn(\mathcal{W}\cup Con(\mathcal{S}))\}\\ &\cup\{\neg Do_{\{i_1,\cdots,i_n\}}\varphi \mid \exists(\psi_1,\varphi_1)\in\mathcal{D}_{i_1},\cdots,(\psi_n,\varphi_n)\in\mathcal{D}_{i_n}\text{s.t.}\\ &\qquad \neg\varphi = \neg\bigwedge_{1\le k\le n}\varphi_k \in Cn(\mathcal{W}\cup Con(\mathcal{S}))\}\end{aligned}$$

Observe that the agent-relative first-order supported actions cover all default rules for agents that are accepted in a given proper scenario. Namely, $Do_{\{i_1,\cdots,i_n\}}\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$ means that agents $i_1,\cdots,i_n$ together successfully support φ as an accepted conclusion in the proper scenario $\mathcal{S}$ of $\mathcal{D}$, while $\neg Do_{\{i_1,\cdots,i_n\}}\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$ indicates that agents $i_1,\cdots,i_n$'s joint support to φ is not accepted in the proper scenario $\mathcal{S}$ of $\mathcal{D}$. I simply write $Do_i\varphi$ as $Do_{\{i\}}\varphi$, and $\neg Do_i\varphi$ as $\neg Do_{\{i\}}\varphi$.

In multi-agent settings it could be that one agent's reason triggers obligations or permissions for another. This is the case in the informal presentation of the Chinese Tax Law example. Company c acting on its permission to submit the necessary documents triggers a reason for the Tax Authority to give c the preferential rate. In order to capture such cases I need to consider not only what I called first-order supported actions, but also second order ones:

5.1.7. DEFINITION. [Supported Action Sets - Second order] Let $\mathcal{S}$ be a proper scenario for a default theory Δ, and $\mathcal{D},\mathcal{D}'$ sets of defaults. The second-order supported action set $\mathcal{A}^{\mathcal{D}'}_{\mathcal{S},\mathcal{D}}$ is defined as

$$\begin{aligned}\mathcal{A}^{\mathcal{D}'}_{\mathcal{S},\mathcal{D}} =&\{\neg Do_{\{i_1,\cdots,i_m\}}\neg Do_{\{j_1,\cdots,j_n\}}\varphi \mid \exists(\varphi_1,\chi_1)\in\mathcal{D}'_{j_1},\cdots,(\psi_n,\chi_n)\in\mathcal{D}'_{j_n}\\ &\qquad \forall(\psi'_1,\chi'_1)\in\mathcal{D}_{i_1},\cdots,(\psi'_m,\chi'_m)\in\mathcal{D}_{i_m}\ \text{s.t.}\\ &\qquad \neg\varphi = \neg\bigwedge_{1\le k\le m}\chi'_k \in Cn(\mathcal{W}\cup\{\chi_1,\cdots,\chi_n\}) \Rightarrow (\psi'_1,\chi'_1),\cdots,(\psi'_m,\chi'_m)\notin\mathcal{S}\}\\ &\cup\{Do_{\{i_1,\cdots,i_m\}}\neg Do_{\{j_1,\cdots,j_n\}}\varphi \mid \exists(\psi_1,\chi_1)\in\mathcal{D}_{j_1},\cdots,(\psi_n,\chi_n)\in\mathcal{D}_{j_n}\\ &\qquad \exists(\psi'_1,\chi'_1)\in\mathcal{D}'_{i_1},\cdots,(\psi'_m,\chi'_m)\in\mathcal{D}'_{i_m}\ \text{s.t.}\\ &\qquad \neg\varphi = \neg\bigwedge_{1\le k\le m}\chi'_k \in Cn(\mathcal{W}\cup\{\chi_1,\cdots,\chi_n\}) \text{ and } (\psi'_1,\chi'_1),\cdots,(\psi'_m,\chi'_m)\in\mathcal{S}\}\end{aligned}$$

What I called second-order actions exhaust the relations of the non-defeated defaults and the defeated ones. More precisely, $\neg Do_{\{j_1,\cdots,j_n\}}\neg Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}^{\mathcal{D}'}_{\mathcal{S},\mathcal{D}}$ expresses that a proper scenario $\mathcal{S}$ asserts that agents $j_1,\cdots,j_n$ of $\mathcal{D}$ would not jointly defeat agents $i_1,\cdots,i_m$ of $\mathcal{D}'$'s potential joint support for performing φ. In other word, $i_1,\cdots,i_m$'s joint supporting to φ is not defeated by the other defaults in $\mathcal{S}$. Similarly, $Do_{\{j_1,\cdots,j_n\}}\neg Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}^{\mathcal{D}'}_{\mathcal{S},\mathcal{D}}$ addresses that $j_1,\cdots,j_n$ of $\mathcal{D}'$ together defeat

$i_1, \cdots, i_m$ of $\mathcal{D}$'s joint support to φ as an accepted conclusion in the proper scenario. As before, I simplify $\neg Do_{\{j\}}\neg Do_{\{i\}}\varphi$ as $\neg Do_j \neg Do_i \varphi$, and $Do_{\{j\}}\neg Do_{\{i\}}\varphi$ as $Do_j \neg Do_i \varphi$. If the $\mathcal{D}'$ and $\mathcal{D}$ in $\mathcal{A}^{\mathcal{D}'}_{\mathcal{S},\mathcal{D}}$ are the same, then I simply write $\mathcal{A}^{\mathcal{D}}_{\mathcal{S}}$.

To define agent-relative obligations I simply use supported action sets to re-define the disjunctive and the conflict accounts:

5.1.8. DEFINITION. [Agent-Relative Obligations - Conflict and Disjunctive Accounts] Let Δ be a default theory with the set $\mathcal{D}$ of defaults for conclusive reasons. Then:

- $\Delta \models O^c_{\{i_1,\cdots,i_m\}}([\neg]\varphi)$, iff $[\neg]Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$ for some proper scenario S of Δ
- $\Delta \models O^d_{\{i_1,\cdots,i_m\}}([\neg]\varphi)$, iff $[\neg]Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$ for all proper scenarios S of Δ

I write $O^c_i([\neg]\varphi)$ as $O^c_{\{i\}}([\neg]\varphi)$, and $O^d_i([\neg]\varphi)$ as $O^d_{\{i\}}([\neg]\varphi)$. When in a given theory Δ the conflict and the disjunctive account agree on all φ, I just write $\Delta \models O_{\{i_1,\cdots,i_m\}}\varphi$.

Let me illustrate this in my example again. $\mathcal{S} = \{\delta_c\}$ is the only proper scenario in Example 5.0.1. So we get that $Do_c t \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$. Now because δ_a is conflicted, I also get $\neg Do_a r \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$. These translate directly into obligations: $\Delta \models O_c(t)$ and $\Delta \models O_a(\neg r)$.

Agent-relative obligations, even under the disjunctive reading, do not behave like obligations in standard deontic logic. This is so because they are defined using the consequents of the defaults that constitute stable scenarios. Since there is no logical closure principle on defaults, I get that $\chi \xrightarrow{d_i} \varphi$ is an element of a proper scenario $\mathcal{S}$ does not entail that $\chi \xrightarrow{d_i} (\varphi \vee \psi)$ is also in $\mathcal{S}$.

5.1.9. THEOREM. *Let $\mathcal{S}$ be a proper scenario for the theory Δ. Then for O_i either in its conflict or disjunctive interpretation:*

- $\Delta \models O_i(\varphi)$ *does not imply* $\Delta \models O_i(\varphi \vee \psi)$.
- *There are default theories Δ for which $\Delta \models O_i(\top)$ fails.*

Proof:
The first statement is demonstrated by the case that a default $\chi \xrightarrow{d_i} \varphi$ in a proper scenario cannot entail $\chi \xrightarrow{d_i} (\varphi \vee \psi)$ is also in the same scenario. For the second statement, take the empty set of default rules as a proper scenario into account. In this case, $O_i(\top)$ fails because I cannot find any rules to conclude tautology as the conjunction of their conclusions. □

Apart from this, the disjunctive account of agent-relative obligations satisfies the so-called "agglomeration" principle [52, Chap. 4] for groups of agents. That is to say, having both $\psi \xrightarrow{d_i} \varphi$ and $\psi' \xrightarrow{d_j} \varphi'$ entails that $\varphi \wedge \varphi'$ is still in the conclusion of this proper scenario.

5.1.10. THEOREM. *Let $\mathcal{S}$ be a proper scenario for the theory Δ. Then for $O_{\{i_1,\cdots,i_m,j_1,\cdots,j_n\}}$ be in the disjunctive interpretation:*

1. *$Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$ and $Do_{\{j_1,\cdots,j_n\}}\varphi' \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$ iff $Do_{\{i_1,\cdots,i_m,j_1,\cdots,j_n\}}(\varphi \wedge \varphi') \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$.*
2. *$\Delta \vDash O_{\{i_1,\cdots,i_m\}}(\varphi)$ and $\Delta \vDash O_{\{j_1,\cdots,j_n\}}(\varphi')$ iff $\Delta \vDash O_{\{i_1,\cdots,i_m,j_1,\cdots,j_n\}}(\varphi \wedge \varphi')$.*

Proof:

1. LtR: Suppose $Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$ and $Do_{\{j_1,\cdots,j_n\}}\varphi' \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$. They entail that $\exists(\psi_1,\chi_1) \in \mathcal{D}_{i_1},\cdots,(\psi_m,\chi_m) \in \mathcal{D}_{i_n}$ s.t. $\varphi = \bigwedge_{1\leq k\leq m}\chi_k \in Cn(\mathcal{W}\cup Con(\mathcal{S}))$, and $\exists(\psi'_1,\chi'_1) \in \mathcal{D}_{i_1},\cdots,(\psi'_n,\chi'_n) \in \mathcal{D}_{i_n}$ s.t. $\varphi' = \bigwedge_{1\leq k\leq n}\chi'_k \in Cn(\mathcal{W} \cup Con(\mathcal{S}))$. Because of this, I can conclude that $\exists(\psi_1,\chi_1) \in \mathcal{D}_{i_1},\cdots,(\psi_m,\chi_m) \in \mathcal{D}_{i_n},(\psi'_1,\chi'_1) \in \mathcal{D}_{i_1},\cdots,(\psi'_n,\chi'_n) \in \mathcal{D}_{i_n}$ s.t. $\varphi \wedge \varphi' = \bigwedge_{1\leq k\leq m}\chi_k \wedge \bigwedge_{1\leq k\leq n}\chi'_k \in Cn(\mathcal{W} \cup Con(\mathcal{S}))$. This achieves the result $Do_{\{i_1,\cdots,i_m,j_1,\cdots,j_n\}}(\varphi \wedge \varphi') \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$. The proof for the other direction is similar.
2. The proof simply consists in unpacking the definition of agent-relative obligations.

□

As further logical properties, it should be clear that empty scenarios do not support any first-order action. Furthermore, second-order supported actions cohere with first-order action sets, and the latter with agent-neutral obligations:

5.1.11. THEOREM. *Let $\mathcal{S}$ be a proper scenario for a default theory Δ. Then:*

1. *$Do_{\{j_1,\cdots,j_n\}}\neg Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}^{\mathcal{D}'}_{\mathcal{S},\mathcal{D}}$ implies $\neg Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$.*
2. *$Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$ implies that $\Delta \vDash O_{\{i_1,\cdots,i_m\}}(\varphi)$.*

Proof:

1. Suppose $Do_{\{j_1,\cdots,j_n\}}\neg Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}^{\mathcal{D}'}_{\mathcal{S},\mathcal{D}}$. This indicates that $\exists(\psi_1,\chi_1) \in \mathcal{D}_{i_1},\cdots,(\psi_m,\chi_m) \in \mathcal{D}_{i_m}, \exists(\psi'_1,\chi'_1) \in \mathcal{D}'_{j_1},\cdots,(\psi'_n,\chi'_n) \in \mathcal{D}'_{j_n}$ s.t. $\neg\varphi = \neg\bigwedge_{1\leq k\leq n}\chi'_k \in Cn(\mathcal{W} \cup \{\chi_1,\cdots,\chi_m\})$ and $(\psi'_1,\chi'_1),\cdots,(\psi'_n,\chi'_n) \in \mathcal{S}$. As $(\psi'_1,\chi'_1),\cdots,(\psi'_n,\chi'_n) \in \mathcal{S}$, I then can conclude $\neg\bigwedge_{1\leq k\leq m}\chi_k \in Cn(\mathcal{W} \cup \{\chi'_1,\cdots,\chi'_n\}) \subseteq Cn(\mathcal{W} \cup Con(\mathcal{S}))$. This indicates that $\neg Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$.

2. This is proven according to the definition of agent-relative obligations.

□

This second property is reminiscent of, but different from, a well-known fact in default theories, namely that "what is, ought to be": if Δ is a default theory and $\varphi \in \mathcal{W}$ then $\Delta \models O\varphi$. This property holds for agent-neutral obligations here, but not for agent-relative ones, because unless a background assumption $\varphi \in \mathcal{W}$ is also the conclusion of a default δ_i in a proper scenario $\mathcal{S}$ it will not appear as $Do_i\varphi$ in the supported action set.

Observe that the interaction of the first-order and second-order supported actions reflects the core of stable semantics [28]. Recall the dominant principle in stable semantics: All accepted default rules are exactly the non-defeated ones. As first-order actions characterize *acceptability* and second-order actions capture the *non-defeatedness*, I get the following observation:

5.1.12. THEOREM. *Let $\mathcal{S}$ be a proper scenario for a default theory Δ with $\mathcal{W}$ for background. Then:*

1. *$Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$ implies that $\neg Do_{\{j_1,\cdots,j_n\}}\neg Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}^{\mathcal{D}}_{\mathcal{S},\mathcal{D}'}$ for any $j_1,\cdots,j_n$ of $\mathcal{D}'$.*

2. *Assume $\neg\varphi \notin Cn(\mathcal{W})$. If $\neg Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$ then $Do_{\{j_1,\cdots,j_n\}}\neg Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}^{\mathcal{D}'}_{\mathcal{S},\mathcal{D}}$ for some $j_1,\cdots,j_n$ of $\mathcal{D}'$.*

Proof:

1. Suppose $Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$. This means that $\exists(\psi_1,\chi_1) \in \mathcal{D}_{i_1},\cdots,(\psi_m,\chi_m) \in \mathcal{D}_{i_m}$ s.t. $\varphi = \bigwedge_{1\le k\le m}\chi_k \in Cn(\mathcal{W} \cup Con(\mathcal{S}))$. NTS: $\forall(\psi_1',\chi_1') \in \mathcal{D}_{j_1}',\cdots,(\psi_n',\chi_n') \in \mathcal{D}_{j_n}'$ s.t. if $\neg\varphi = \neg\bigwedge_{1\le k\le m}\chi_k' \in Cn(\mathcal{W} \cup \{\chi_1,\cdots,\chi_m\})$ then $(\psi_1',\chi_1'),\cdots,(\psi_n',\chi_n') \notin \mathcal{S}$. If not, then $\exists(\psi_1',\chi_1') \in \mathcal{D}_{j_1}',\cdots,(\psi_n',\chi_n') \in \mathcal{D}_{j_n}'$ s.t. $\neg\varphi = \neg\bigwedge_{1\le k\le n}\chi_k' \in Cn(\mathcal{W} \cup \{\chi_1,\cdots,\chi_m\})$ and $(\psi_1',\chi_1'),\cdots,(\psi_n',\chi_n') \in \mathcal{S}$. Taking $(\psi_1',\chi_1'),\cdots,(\psi_n',\chi_n') \in \mathcal{S}$ with the assumption, it then implies $\varphi,\neg\varphi \in Cn(\mathcal{W} \cup Con(\mathcal{S}))$. But this conflicts with $\mathcal{S}$ being a proper scenario. I can then conclude the result.

2. Assume $\neg\varphi \notin Cn(\mathcal{W})$ and $\neg Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{D}}$. This means that $\exists(\psi_1,\chi_1) \in \mathcal{D}_{i_1},\cdots,(\psi_m,\chi_m) \in \mathcal{D}_{i_m}$ s.t. $\neg\varphi = \neg\bigwedge_{1\le k\le m}\chi_k \in Cn(\mathcal{W} \cup Con(\mathcal{S}))$. As $\neg\varphi \notin Cn(\mathcal{W})$, it indicates that $\exists(\psi_1',\chi_1') \in \mathcal{D}_{j_1}',\cdots,(\psi_n',\chi_n') \in \mathcal{D}_{j_n}'$ s.t. $\neg\varphi = \neg\bigwedge_{1\le k\le n}\chi_k' \in Cn(\mathcal{W} \cup \{\chi_1,\cdots,\chi_m\})$ and $(\psi_1',\chi_1'),\cdots,(\psi_n',\chi_n') \in \mathcal{S}$.

□

The first result states that no accepted default is defeated by others, and the second that all non-accepted are defeated, if there are no conflicting defaults in the background.

I finally observe that there is a tight connection between defeated and non-defeated: If a negative default is defeated then its positive counterpart is not defeated.

5.1.13. THEOREM. *Let $\mathcal{S}$ be a proper scenario for a default theory Δ. Then:*

- $Do_{\{j_1,\cdots,j_n\}}\neg Do_{\{i_1,\cdots,i_m\}}\neg\varphi \in \mathcal{A}^{\mathcal{D}'}_{\mathcal{S},\mathcal{D}}$ *implies* $\neg Do_{\{j_1,\cdots,j_n\}}\neg Do_{\{i_1,\cdots,i_m\}}\varphi \in \mathcal{A}^{\mathcal{D}'}_{\mathcal{S},\mathcal{D}}$.

Proof:
Suppose $Do_{\{j_1,\cdots,j_n\}}\neg Do_{\{i_1,\cdots,i_m\}}\neg\varphi \in \mathcal{A}^{\mathcal{D}'}_{\mathcal{S},\mathcal{D}}$. This means that $\exists(\psi_1,\chi_1) \in \mathcal{D}_{i_1}$, $\cdots$, $(\psi_m,\chi_m) \in \mathcal{D}_{i_m}$, $\exists(\psi'_1,\chi'_1) \in \mathcal{D}'_{j_1}$, $\cdots$, $(\psi'_n,\chi'_n) \in \mathcal{D}'_{j_n}$ s.t. $\varphi = \bigwedge_{1\le k\le n}\chi'_k \in Cn(\mathcal{W}\cup\{\chi_1,\cdots,\chi_m\})$ and (ψ'_1,χ'_1), $\cdots$, $(\psi'_n,\chi'_n) \in \mathcal{S}$. This implies that $\varphi \in Cn(\mathcal{W}\cup Con(\mathcal{S}))$, and then $\bigwedge_{1\le k\le m}\chi_k \in Cn(\mathcal{W}\cup Con(\mathcal{S}))$. NTS: $\forall(\psi'_1,\chi'_1) \in \mathcal{D}'_{j_1}$, $\cdots$, $(\psi'_n,\chi'_n) \in \mathcal{D}'_{j_n}$ s.t. if $\neg\varphi = \neg\bigwedge_{1\le k\le m}\chi'_k \in Cn(\mathcal{W}\cup\{\chi_1,\cdots,\chi_m\})$ then (ψ'_1,χ'_1), $\cdots$, $(\psi'_n,\chi'_n) \notin \mathcal{S}$. If not, then $\exists(\psi''_1,\chi''_1) \in \mathcal{D}'_{j_1}$, $\cdots$, $(\psi''_n,\chi''_n) \in \mathcal{D}'_{j_n}$ s.t. $\neg\varphi = \neg\bigwedge_{1\le k\le n}\chi''_k \in Cn(\mathcal{W}\cup\{\chi_1,\cdots,\chi_m\})$ and (ψ''_1,χ''_1), $\cdots$, $(\psi''_n,\chi''_n) \in \mathcal{S}$. This indicates that $\neg\bigwedge_{1\le k\le m}\chi_k \in Cn(\mathcal{W}\cup Con(\mathcal{S}))$. But then it conflicts with the previous statement. □

5.2 Three Types of Permissions

In this section I consider three types of permissions that can be defined in default theories: weak, explicit and tacit permissions. This classification, and the intuitions underlying each of these, come from [71].

In doing so, I emphasize how important is the multi-agent aspect for the representation of the standard correlative relations between claim rights and duties. I furthermore compare the resulting permissions with weak permissions in SDL, strong permission in von Wright's sense, and free choice permission.

5.2.1 Weak Permissions

I start with weak permissions, the usual dual of obligations. Something is weakly permitted whenever it is not the case that it is forbidden, i.e. that its negation is mandatory. This type of permission is of course definable in Horty's prioritized default theory, and its logical behavior will depend on whether it is defined as the dual of the disjunctive or the conflict account.

5.2.1. DEFINITION. [Weak Permissions] Let Δ be a default theory with $\mathcal{D}$ for conclusive reasons.

- $\Delta \vDash P^c_{\{i_1,\cdots,i_m\}}(\varphi)$ iff $\neg Do_{\{i_1,\cdots,i_m\}}\varphi \notin \mathcal{A}_{\mathcal{S},\mathcal{D}}$ for all proper scenarios $\mathcal{S}$ based on Δ
- $\Delta \vDash P^d_{\{i_1,\cdots,i_m\}}(\varphi)$ iff $\neg Do_{\{i_1,\cdots,i_m\}}\varphi \notin \mathcal{A}_{\mathcal{S},\mathcal{D}}$ for some proper scenarios $\mathcal{S}$ based on Δ

I write $P_i^c(\varphi)$ as $P_{\{i\}}^c(\varphi)$, and $P_i^d(\varphi)$ as $P_{\{i\}}^d(\varphi)$. Similar to agent-relative obligations, I write $\Delta \vDash P_i(\varphi)$ when one unique proper scenario can be generated from Δ. Permission for negated formulas is defined similarly as for obligations. For the disjunctive account, for instance, this gives: $\Delta \vDash P_i^d(\neg\varphi)$ iff $Do_i\varphi \notin \mathcal{A}_{\mathcal{S},\mathcal{D}}$ for some proper scenarios $\mathcal{S}$ of Δ with $\mathcal{D}$.

Let me illustrate these notions of permissions in my running example. Recall that $\mathcal{S} = \{\delta_c\}$ is the only proper scenario above, and that I have $\Delta \vDash O_c(t)$ and $\Delta \vDash O_a(\neg r)$. From that I get that $\Delta \vDash P_c(t)$ and $\Delta \vDash P_c(\neg r)$, but also $\Delta \vDash P_c(d)$, as expected.

Looking a bit further I do find, however, counter-intuitive weak permissions, which speak in favor of looking at stronger notions. I do get, for instance, that $\Delta \vDash P_a(d)$, which might appear unwelcome since only the company but not the tax authority is permitted to submit the document. I rather see it as revealing how weak is the notion of weak permission, and showing the need for a stronger, explicit notion. Weak permissions do not distinguish between, on the one hand, an action not being forbidden for an agent because there are *some* reasons, possibly not conclusive, for doing it and, on the other hand, the same action not being forbidden simply because reasons are silent when it comes to that agent doing that action. We come back to this in the next section.

Now for the logical properties of weak permissions. As expected from the definition, $P^d_{\{i_1,\cdots,i_m\}}$ and $P^c_{\{i_1,\cdots,i_m\}}$ are the respective duals of $O^d_{\{i_1,\cdots,i_m\}}$ and $O^c_{\{i_1,\cdots,i_m\}}$. But just as for agent-relative obligations, few properties of the "diamond" P in SDL are retained here.

5.2.2. THEOREM. *Let Δ be a default theory.*

- $\Delta \vDash O_{\{i_1,\cdots,i_m\}}(\varphi)$ *iff* $\Delta \vDash \neg P_{\{i_1,\cdots,i_m\}}(\neg\varphi)$.
- $\Delta \vDash P_{\{i_1,\cdots,i_m\}}(\varphi)$ *or* $\Delta \vDash P_{\{j_1,\cdots,j_n\}}(\psi)$ *implies* $\Delta \vDash P_{\{i_1,\cdots,i_m,j_1,\cdots,j_n\}}(\varphi \vee \psi)$, *but not vice versa.*
- $\Delta \vDash O^d_{\{i_1,\cdots,i_m\}}(\varphi)$ *implies* $\Delta \vDash P^d_{\{i_1,\cdots,i_m\}}(\varphi)$.
- $\Delta \vDash O^c_{\{i_1,\cdots,i_m\}}(\varphi)$ *does not imply* $\Delta \vDash P^c_{\{i_1,\cdots,i_m\}}(\varphi)$.

Proof:
Here I only prove the interesting case that $\Delta \vDash P^c_{\{i_1,\cdots,i_m\}}(\varphi)$ or $\Delta \vDash P^c_{\{i_1,\cdots,i_m\}}(\psi)$ imply $\Delta \vDash P^c_{\{i_1,\cdots,i_m\}}(\varphi \vee \psi)$.

Suppose $\Delta \vDash P^c_{\{i_1,\cdots,i_m\}}(\varphi)$ or $\Delta \vDash P^c_{\{j_1,\cdots,j_n\}}(\psi)$. It indicates that $\neg Do_{\{i_1,\cdots,i_m\}}\varphi \notin \mathcal{A}_{\mathcal{S},\mathcal{D}}$ for all proper scenarios $\mathcal{S}$ based on Δ, or $\neg Do_{\{j_1,\cdots,j_n\}}\psi \notin \mathcal{A}_{\mathcal{S},\mathcal{D}}$ for all proper scenarios $\mathcal{S}$ based on Δ. This implies that $(\neg Do_{\{i_1,\cdots,i_m\}}\varphi \notin \mathcal{A}_{\mathcal{S},\mathcal{D}}$ or $\neg Do_{\{j_1,\cdots,j_n\}}\psi \notin \mathcal{A}_{\mathcal{S},\mathcal{D}})$, for all proper scenarios $\mathcal{S}$ based on Δ. According to the definition of first-order action sets, it means that $\forall(\chi_1,\varphi_1) \in \mathcal{D}_{i_1},\cdots,(\chi_m,\varphi_m) \in \mathcal{D}_{i_m},(\chi_1',\psi_1) \in \mathcal{D}_{j_1},\cdots,(\chi_n',\psi_n) \in \mathcal{D}_{j_n}$ s.t. $\neg\varphi = \neg\bigwedge_{1\le k\le m}\varphi_k \notin Cn(\mathcal{W} \cup Con(\mathcal{S}))$ or $\neg\psi = \neg\bigwedge_{1\le k\le n}\psi_k \notin Cn(\mathcal{W} \cup Con(\mathcal{S}))$. So then $\neg(\varphi\vee\psi) = \neg\bigwedge_{1\le k\le n}\varphi_k \wedge \neg\bigwedge_{1\le k\le n}\psi_k \notin$

$Cn(\mathcal{W} \cup Con(\mathcal{S}))$. From this I can conclude $\Delta \vDash P^c_{\{i_1,\cdots,i_m\}}(\varphi \vee \psi)$. □

Weak permission in the conflicted account further satisfies the so-called property of "agglomeration" in a way similar to the agent-relative obligation in the disjunctive account.

5.2.3. THEOREM. *Let* Δ *be a default theory.*

- $\Delta \vDash P^c_{\{i_1,\cdots,i_m,i_{m+1},\cdots,i_{m+n}\}}(\varphi \wedge \psi)$ *iff* $\Delta \vDash P^c_{\{i_1,\cdots,i_m\}}(\varphi) \wedge P^c_{\{i_{m+1},\cdots,i_{m+n}\}}(\psi)$.

Proof:
LtR: Suppose $\Delta \vDash P^c_{\{i_1,\cdots,i_m\}}(\varphi \wedge \psi)$. This means that $\neg Do_{\{i_1,\cdots,i_m\}}(\varphi \wedge \psi) \notin \mathcal{A}_{\mathcal{S},\mathcal{D}}$ for all proper scenarios $\mathcal{S}$ based on Δ. The former is equal to that $\forall(\chi_1, \chi_1^{'}) \in \mathcal{D}_{i_i}$, $\cdots$, $(\chi_m, \chi_m^{'}) \in \mathcal{D}_{i_m}$, $(\chi_{m+1}, \chi_{m+1}^{'}) \in \mathcal{D}_{i_{m+1}}$, $\cdots$, $(\chi_{m+n}, \chi_{m+n}^{'}) \in \mathcal{D}_{i_{m+n}}$, s.t. $\neg(\varphi \wedge \psi) = \neg \bigwedge_{1\leq k\leq m+n} \chi_k^{'} \notin Cn(\mathcal{W} \cup Con(\mathcal{S}))$. Assume that $\varphi = \bigwedge_{1\leq k\leq m} \chi_k^{'}$ and $\psi = \bigwedge_{m+1\leq k\leq m+n} \chi_k^{'}$. So I have

- $\forall(\chi_1, \chi_1^{'}) \in \mathcal{D}_{i_i}$, $\cdots$, $(\chi_m, \chi_m^{'}) \in \mathcal{D}_{i_m}$ s.t. $\neg\varphi = \neg \bigwedge_{1\leq k\leq m} \chi_k^{'} \notin Cn(\mathcal{W} \cup Con(\mathcal{S}))$, and
- $\forall(\chi_{m+1}, \chi_{m+1}^{'}) \in \mathcal{D}_{i_{m+1}}$, $\cdots$, $(\chi_{m+n}, \chi_{m+n}^{'}) \in \mathcal{D}_{i_{m+n}}$ s.t. $\neg\psi = \neg \bigwedge_{m+1\leq k\leq m+n} \chi_k^{'} \notin Cn(\mathcal{W} \cup Con(\mathcal{S}))$.

This concludes that $\Delta \vDash P^c_{\{i_1,\cdots,i_m\}}(\varphi) \wedge P^c_{\{i_{m+1},\cdots,i_{m+n}\}}(\psi)$.
The right-to-left direction is proven in a similar way. □

5.2.2 Explicit Permissions

Explicit permissions, as their name suggest, are permissions that are explicitly stated in a moral or legal code. These are the strong permissions in the sense of von Wright [118, p.31]. Here I follow [71] and stay at the intuitive level. Intuitively, the law-giver can forbid φ without further legal changes if φ is only weakly permitted, but not if it is explicitly permitted. This intuitive difference is also present in my running example. The permission for the foreign company c to submit the documents is explicit, while the same weak permission for the tax authority a is not. Such an example suggests the need to distinguish explicit from weak permissions.

In order to capture explicit permissions I extend Horty's account by splitting the set $\mathcal{D}$ into a set of permissive norms $\mathcal{P}$ and a set of mandatory norms $\mathcal{O}$, with $<$ the same partial order on $\mathcal{D}$ as usual.

I illustrate this idea through my running example. The explicit permission here is the following: the solar energy business *may* submit the documentation if it is the tax season. This permissive norm takes also the form of a default $s \xrightarrow{\delta_c^{'}} d$ for the company c. This default defeats the regular duty to pay full income tax. Formally, I get the default theory $\Delta'_{CIT} = \langle \mathcal{W}, \mathcal{I}, \mathcal{O} \cup \mathcal{P}, < \rangle$ defined as follows: $\mathcal{W} = \{s, \neg t \Leftrightarrow r\}$, $\mathcal{I} = \{a, c\}$, $\mathcal{O} = \{s \xrightarrow{\delta_c} t, d \xrightarrow{\delta_a} r\}$, $\mathcal{P} = \{s \xrightarrow{\delta'_c} d\}$, with the priority $\delta_c < \delta_a$, reflecting the priority of tax reduction. See Figure 5.2. With the addition of this permissive norm, the unique proper scenario becomes $\{\delta'_c, \delta_a\}$, as desired.

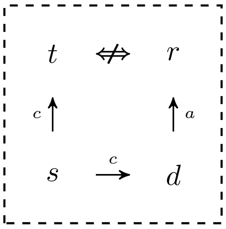


Figure 5.2: A default theory of a solar energy business c's claim to the right to a tax incentive against the local tax authority.

Now I turn to the formal definition.

5.2.4. DEFINITION. [Explicit Permissions] Given $\Delta = \langle \mathcal{W}, \mathcal{I}, \mathcal{O} \cup \mathcal{P}, < \rangle$, I say that $P_e(\varphi)$ is an explicit permission on this default theory, denoted as $\Delta \models P^e_{\{i_1, \cdots, i_m\}}(\varphi)$, iff

$$Do_{\{i_1, \cdots, i_m\}} \varphi \in \mathcal{A}_{\mathcal{S}, \mathcal{P}} \text{ for all proper scenarios } \mathcal{S} \text{ based on } \Delta$$

For example, with the permissive norm $s \xrightarrow{\delta'_c} d$ at hand, I get the explicit permission $P^e_c(d)$, indicating that the solar energy company may submit the valid documentation. Interestingly, the tax authority then has a correlative duty to grant the tax reduction. So this explicit permission $P^e_c(d)$ can be seen as a claim right, in Hohfeld's sense.

Explicit permissions give more intuitive results in this case than weak permissions. Recall that, as Figure 5.2 shows, since there is no conclusive reason for the company that forbids granting the tax reduction, I obtained the rather counterintuitive result: it is a weak permission for the company to reduce the tax rate. But there is no such explicit permission here, for the simple reason that this action may only be implemented by the tax authority. So weak permissions are too weak to model this important piece of Chinese Law.

The Chinese Law example shows that there can be a weak permission without an explicit one. Now I should see that the converse is also possible, by looking back at the Right of Way example. In this example all the defaults are permissive. Suppose that l is the driver who wants to turn left at an intersection on the priority street, r is the driver who signals to turn right at the same intersection of this street, t means that driver l's turns left, s means that driver r signals to turn right, and t' means that driver r turns right. I have three permissive norms in this case: $\top \xrightarrow{\delta_l} t$, $\top \xrightarrow{\delta_r} s$, and $s \xrightarrow{\delta'_r} t'$. This situation can then be captured in the following default theory: $\Delta = \langle \mathcal{W}, \mathcal{I}, \mathcal{O} \cup \mathcal{P}, < \rangle$,

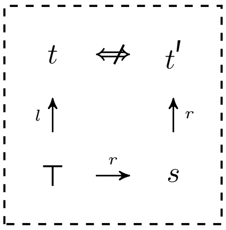


Figure 5.3: A default theory of the driver r's signal for turning right being a claim right overriding driver l's permission to turn left.

where $\mathcal{W} = \{\top, t' \Leftrightarrow \neg t\}$, $\mathcal{O} = \varnothing$, and $\mathcal{P} = \{\top \xrightarrow{\delta_l} t, \top \xrightarrow{\delta_r} s, s \xrightarrow{\delta_r'} t'\}$ with $\delta_l < \delta_r'$, representing that the driver turning right has priority. See Figure 5.3. In this default theory, the unique proper scenario is $\{\delta_r, \delta_r'\}$, only including permissive norms, which expresses that the driver r has the priority of turning right, as well as that the driver r has an explicit permission to turn right $P_r^e(t')$. However, he does not own a weak permission $P_r^c(t')$ to do so, for the simple reason that the set $\mathcal{O}$ for conclusive reasons is empty.

Similarly to weak permission, explicit permission also satisfies the "agglomeration" property. Though explicit permission and weak permission both have similar logical behavior, they are two independent categories of permission, as the previous arguments have shown, which do bring about different normative consequences in legal reasoning.

5.2.5. THEOREM. *Let Δ be a default theory.*

- $\Delta \vDash P^e_{\{i_1,\cdots,i_m\}}(\varphi) \wedge P^e_{\{i_1,\cdots,i_m\}}(\psi)$ *iff* $\Delta \vDash P^e_{\{i_1,\cdots,i_m\}}(\varphi \wedge \psi)$.

Proof:
As the logical structure similar to the agent-relative obligations in disjunctive interpretation, the proof here is also obvious. □

5.2.3 Tacit Permissions

What I call "tacit permission" follows the basic idea of a "silent" type of permission in [71, p.398]. The idea is that φ is tacitly permitted when φ cannot be coherently prohibited. Translated into the present framework, this notion of incoherence bears on the non-defeated permissive norms. Prohibiting φ hence leads to a contradiction when it defeats an otherwise undefeated permissive norm.

5.2.6. DEFINITION. [Tacit Permissions] Given $\Delta = \langle \mathcal{W}, \mathcal{I}, \mathcal{D} = \mathcal{O} \cup \mathcal{P}, < \rangle$, I say that $P^t_{\{i_1,\cdots,i_m\}}(\varphi)$ is a tacit permission based on Δ, denoted as $\Delta \vDash P^t_{\{i_1,\cdots,i_m\}}(\varphi)$, iff

$$Do_{\{i_1,\cdots,i_m\}}\neg\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{O}} \Rightarrow \neg Do_{\{j_1,\cdots,j_n\}}\neg Do_{\{i_1,\cdots,i_m\}}\varphi \notin \mathcal{A}^{\mathcal{P}}_{\mathcal{S},\mathcal{D}}$$

Let me illustrate this idea in my two running examples. In the tax reduction case, I get that $P^t_c(r)$, which means that the permission of granting the tax reduction by the authority is tacit. For the Right of Way example, driver r's permission to turn right is also tacit. This should show that tacit permission is neither weak nor explicit permission.

Tacit permission turns out to be free choice permission, but unlike weak and explicit permission it does not agglomerate.

5.2.7. THEOREM. *Let Δ be a default theory.*

- *If $\Delta \vDash P^t_{\{i_1,\cdots,i_m,j_1,\cdots,j_n\}}(\varphi \vee \psi)$ then $\Delta \vDash P^t_{\{i_1,\cdots,i_m\}}(\varphi) \wedge P^t_{\{j_1,\cdots,j_n\}}(\psi)$.*

Proof:
Suppose that $\Delta \vDash P^t_{\{i_1,\cdots,i_m,j_1,\cdots,j_n\}}(\varphi \vee \psi)$. This indicates that $Do_{\{i_1,\cdots,i_m,j_1,\cdots,j_n\}}\neg(\varphi \vee \psi) \in \mathcal{A}_{\mathcal{S},\mathcal{O}}$ can imply $\neg Do_{\{h_1,\cdots,h_l\}}\neg Do_{\{i_1,\cdots,i_m,j_1,\cdots,j_n\}}(\varphi \vee \psi) \notin \mathcal{A}^{\mathcal{P}}_{\mathcal{S},\mathcal{D}}$. Assume $Do_{\{i_1,\cdots,i_m\}}\neg\varphi \in \mathcal{A}_{\mathcal{S},\mathcal{O}}$ and $Do_{\{j_1,\cdots,j_n\}}\neg\psi \in \mathcal{A}_{\mathcal{S},\mathcal{O}}$. This means that $\exists(\chi_1,\varphi_1) \in \mathcal{O}_{i_1}$, $\cdots$, $(\chi_m,\varphi_m) \in \mathcal{O}_{i_m}$ s.t. $\neg\varphi = \bigwedge_{1\le k\le m}\varphi_k \in Cn(\mathcal{W} \cup Con(\mathcal{S}))$, and $\exists(\chi'_1,\psi_1) \in \mathcal{O}_{j_1}, \cdots, (\chi'_n,\psi_n) \in \mathcal{O}_{j_n}$ s.t. $\neg\psi = \bigwedge_{1\le k\le n}\psi_k \in Cn(\mathcal{W} \cup Con(\mathcal{S}))$. Taking them together, this indicates that $\exists(\chi_1,\varphi_1) \in \mathcal{O}_{i_1}$, $\cdots$, $(\chi_m,\varphi_m) \in \mathcal{O}_{i_m}$, $(\chi'_1,\psi_1) \in \mathcal{O}_{j_1}$, $\cdots$, $(\chi'_n,\psi_n) \in \mathcal{O}_{j_n}$ s.t. $\neg(\varphi \vee \psi) = \bigwedge_{1\le k\le m}\varphi_k \wedge \bigwedge_{1\le k\le n}\psi_k \in Cn(\mathcal{W} \cup Con(\mathcal{S}))$. So I have $Do_{\{i_1,\cdots,i_m,j_1,\cdots,j_n\}}\neg(\varphi \vee \psi) \in \mathcal{A}_{\mathcal{S},\mathcal{O}}$. According to the assumption, this implies that $\neg Do_{\{h_1,\cdots,h_l\}}\neg Do_{\{i_1,\cdots,i_m,j_1,\cdots,j_n\}}(\varphi \vee \psi) \notin \mathcal{A}^{\mathcal{P}}_{\mathcal{S},\mathcal{D}}$. I then have $\forall(\chi_1,\varphi_1) \in \mathcal{P}_{i_1}$, $\cdots$, $(\chi_m,\varphi_m) \in \mathcal{P}_{i_m}$, $(\chi'_1,\psi_1) \in \mathcal{P}_{j_1}$, $\cdots$, $(\chi'_n,\psi_n) \in \mathcal{P}_{j_n}$, $\exists(\chi''_1,\chi'''_1) \in \mathcal{D}_{h_1}$, $\cdots$, $(\chi''_l,\chi'''_l) \in \mathcal{D}_{h_l}$ s.t.

$$\neg\varphi \wedge \neg\psi = \neg(\varphi \vee \psi) = \neg \bigwedge_{1\le k\le l} \chi'''_k \in Cn(\mathcal{W} \cup \{\varphi_1,\cdots,\varphi_m,\psi_1,\cdots,\psi_n\})$$

and $(\chi''_1,\chi'''_1),\cdots,(\chi''_l,\chi'''_l) \in \mathcal{S}$. This then concludes that $\neg Do_{\{h_1,\cdots,h_l\}}\neg Do_{\{i_1,\cdots,i_m\}}\varphi \notin \mathcal{A}^{\mathcal{P}}_{\mathcal{S},\mathcal{D}}$ and $\neg Do_{\{h_1,\cdots,h_l\}}\neg Do_{\{j_1,\cdots,j_n\}}\psi \notin \mathcal{A}^{\mathcal{P}}_{\mathcal{S},\mathcal{D}}$ by unpacking the definitions. According to these, I can have the desired results. □

5.2.4 A Fourth Type: Protected Permissions

Having looked at weak, explicit and tacit permission, I now turn to notions of permissions arising from legal theory. This combines in a natural way the permissive norms introduced earlier and my multi-agent version of default theory. The crucial notion here is that of correlative permissions and duties. This can be illustrated in my Right of Way example. There, one's permissive right to turn at the crossroad correlates to other(s)' duty not to interfere. In short, this permissive right is protected. This reverses the standard relation between obligation and permission. As already mentioned in the introduction, in SDL obligation implies (weak) permission. This was the consistency requirement. With correlative duties this is the other way around: permission implies obligation. The so-called "protected permission" [121, 85] generates obligation.

5.2.8. DEFINITION. [Protections: Implicit Duties] Let a default theory Δ be a tuple of $\langle \mathcal{W}, \mathcal{I}, \mathcal{D} = \mathcal{O} \cup \mathcal{P}, < \rangle$. I say that $D_{\{i_1,\cdots,i_m\}}(\varphi)$ is a protection or an implicit duty, denoted as $\Delta \models D_{\{j_1,\cdots,j_n\}}(\varphi)$, iff

$$\neg Do_{\{j_1,\cdots,j_n\}} \neg Do_{\{i_1,\cdots,i_m\}} \varphi \in \mathcal{A}^{\mathcal{P}}_{\mathcal{S},\mathcal{D}} \text{ for some } i_1, \cdots, i_m \text{ of } \mathcal{P}$$

Protected permission can be seen as an implicit type of duty, because it aims at ensuring the other's explicit permissions so as to not be defeated by them. As a corollary of Theorem 5.1.12, I get that protected permission follows from explicit permission.

5.2.9. THEOREM. *Given a default theory* $\Delta = \langle \mathcal{W}, \mathcal{I}, \mathcal{D} = \mathcal{O} \cup \mathcal{P}, < \rangle$*, I have*

- $\Delta \models P^e_{\{i_1,\cdots,i_m\}}(\varphi) \Rightarrow \Delta \models D_{\{j_1,\cdots,j_n\}}(\varphi)$ *for all* $j_1, \cdots, j_n$ *of* $\mathcal{D}$*.*

This type of protected permission is particularly salient in the Right of Way example. I have seen that $P^e_r(t')$ is an explicit permission for the driver r to turn right. In terms of protected permission, I can then conclude that both drivers should not interfere with driver r's turning right.

5.3 Foundation in Argumentation Theory

After having developed four types of permission in my extended version of Horty's prioritized default theory, I now provide a theoretical foundation for it using *stable semantics* [28, 29, 14] and *preferred semantics* [16, 17, 42, 69, 40]. This section is organized as follows. I first introduce the basic notions of the fixed point characterization of the stable semantics. After that, I reconstruct the stable semantics in prioritized default theory, and then prove that it is a special case of proper scenarios. Finally, I present the preferred semantics as one alternative formulation of specificity, and show that, though it is computationally tractable, it cannot be a reasonable *guide* for real applications to concrete legal cases, as the stable semantics is.

5.3.1 A Primer on Stable Semantics

This section develops stable extensions for stable semantics in prioritized default theory, based on the framework developed by Dung [28, p.328]. A stable extension is a conflict free extension that defeats each argument which does not belong to this extension. The original account of stable semantics is one of the many extension-based models that originated in Dung's classic paper on abstract argumentation theory [28]. The crucial notions here are of the attack relation, modeling conflicts between arguments. This basic framework can be extended into the so-called structured argumentation theory [83], which strongly underlines the difference between attack and defeat, in the sense that the latter involves an explicit priority governing *arguments*. To keep the notations simple, I denote "arguments" by using default rules in this section.

5.3.1. DEFINITION. [Dung's Argumentation Framework and Attacks] Dung's argumentation framework (AF) [28] is a structure $\langle \mathcal{A}, \mathcal{C} \rangle$, where $\mathcal{C} \subseteq \mathcal{A} \times \mathcal{A}$ is a binary attack relation on the arguments $\mathcal{A}$. Then, for any $\delta, \gamma \in \mathcal{A}$,

- δ attacks γ iff $(\delta, \gamma) \in \mathcal{C}$
- $S \subseteq \mathcal{A}$ attacks γ iff $\exists \delta \in S$ s.t. $(\delta, \gamma) \in \mathcal{C}$
- $S \subseteq \mathcal{A}$ is *conflict free* iff $(\delta, \gamma) \notin \mathcal{C}$ for all $\delta, \gamma \in S$
- $\delta \in \mathcal{A}$ is *acceptable* w.r.t. some $S \subseteq \mathcal{A}$ iff $\forall \gamma \in \mathcal{A}$ s.t. if $(\gamma, \delta) \in \mathcal{C}$ then $\exists \gamma' \in S$ s.t. $(\gamma', \gamma) \in \mathcal{C}$

Argumentation theory captures specificity, either by an implicitly derived priority order in a given attack relation [29], or by an explicitly imposed priority order in it [83]. However, this section proposes a very simple variant of the explicit account [83] with an explicit strict partial priority order, which only involves the key notions of "argument," "attack," and priority in an argumentation framework [2]. In this simplified framework the defeat relation can be defined using the given attack relation and the priority: An argument δ is said to defeat an argument γ iff δ attacks γ and δ is "more specific" than γ [29, 83].

5.3.2. DEFINITION. [Structured Argumentation Framework and Defeats] A *structured argumentation framework* (SAF) is a tuple $\langle \mathcal{A}, \mathcal{C}, < \rangle$ where $\mathcal{A}$ is the set of all arguments, $\mathcal{C}$ is an attack relation, and $<$ is a strict partial priority order. Then

- δ defeats γ, denoted as $(\delta, \gamma) \in De$, iff $(\delta, \gamma) \in \mathcal{C}$ and $\gamma < \delta$
- $S \subseteq \mathcal{A}$ defeats γ iff $\exists \delta \in S$ s.t. $(\delta, \gamma) \in De$

De is the defeat relation that contains all the derived defeats. Like in default theory, a defeat relation can be derived by the strict partial priority order in a given SAF. But for simplification, I omit the standard distinction of undermining, rebutting and undercutting defeats in SAF [83].

Now I turn to the formal construction of extensions, which is based on the proposal by Dung [28, p.328], but follows the well-known formulation proposed by Prakken [83].

5.3.3. DEFINITION. [Extensions] Let $\langle \mathcal{A}, \mathcal{C}, < \rangle$ be a SAF, and $S \subseteq \mathcal{A}$ be *conflict free*. Then

[2]The derived priority order proposed in the implicit account [29] cannot characterize the twin example in [52]. No such derived strict partial priority order $<$ "*more specific*" exists. Otherwise, these two rules δ_1, δ_2 whose premises are tautologies and whose conclusions conflict with each other can derive these two relations: $\delta_1 < \delta_2$ and $\delta_2 < \delta_1$.

- S is an *admissible* extension iff $S \subseteq \{\delta \mid \delta$ is acceptable w.r.t. $S\}$
- S is a *preferred* extension iff S is a maximal (w.r.t. $\subseteq$) admissible extension
- S is a *stable* extension iff S is a preferred extension that defeats all arguments in $\mathcal{A}/S$

The stable extension defined here is similar to the one in [28], and is thus in the spirit of Dung's proposal, even if one adds an explicit defeat relation that provides a prioritized strength on arguments.

5.3.4. LEMMA. *Let S be conflict free. Then*

$$S \textit{ is a stable extension iff } S = \{\delta \mid \delta \textit{ is not defeated by } S\}.$$

Proof:
LtR: Assume that S is a stable extension

- Let $\delta \in S$. I need to show that δ is not defeated by S. Otherwise S defeats δ, namely $\exists\gamma \in S$ s.t. $\delta < \gamma$ and $(\gamma, \delta) \in \mathcal{C}$. Because $\delta \in S$, then it implies that S is not conflict free, which contradicts the assumption.
- Let δ is not defeated by S. I need to show that $\delta \in S$. Otherwise $\delta \in \mathcal{A}/S$. As S is a stable extension, δ is defeated by S, which contradicts the assumption again.

RtL: Assume that $S = \{\delta \mid \delta$ is not *defeated* by $S\}$. I want to show that S is a stable extension in the following three steps.

1. First, to show that S is admissible. Otherwise $\exists\delta \in S$ s.t. δ is not acceptable w.r.t. S. Namely $\exists\gamma \in \mathcal{A}$ s.t. $(\gamma, \delta) \in \mathcal{C}$ and $\forall\gamma' \in S$ s.t. $(\gamma', \gamma) \notin \mathcal{C}$. Since $(\gamma, \delta) \in \mathcal{C}$ and S is conflict free, I know that $\gamma \in \mathcal{A}/S$. Then γ is defeated by S, namely $\exists\gamma' \in S$ s.t. $\gamma' > \gamma$ and $(\gamma', \gamma) \in \mathcal{C}$. The "otherwise" does not hold, and thus S is admissible.
2. Next I want to show that S is maximal. Otherwise $\exists\delta \in \mathcal{A}/S$ s.t. δ is acceptable w.r.t. S. However, according to the assumption and $\delta \in \mathcal{A}/S$, δ is defeated by S, and so it cannot be acceptable. So no such δ exists.
3. In conclusion, it is obvious that $\forall\delta \in \mathcal{A}/S$ that δ is defeated by S, according to the assumption.

□

To conclude this section, I bridge prioritized default theory with stable semantics, by generating a prioritized default theory in SAF. This sheds new light on the construction of stable extensions in prioritized default theory, and on how this construction connects with the proper scenarios.

5.3.5. DEFINITION. A SAF-prioritized default theory is a tuple $\langle \mathcal{A}, \mathcal{C}, < \rangle$ where $\mathcal{A} = \langle \mathcal{W}, \mathcal{D} \rangle$, $<$ is a strict partial priority order over $\mathcal{D}$, and $\mathcal{C}$ is defined over $\mathcal{D}$ as follows:

$$(\delta, \gamma) \in \mathcal{C} \Leftrightarrow \neg Con(\gamma) \in Cn(\mathcal{W} \cup \{Con(\delta)\})$$

where $\delta, \gamma \in \mathcal{D}$.

In accordance with Definition 5.3.5, it is natural to represent acceptability by the combination of triggered and non-conflicted scenarios in prioritized default theory.

5.3.6. LEMMA. *Let $\mathcal{A}$ be a SAF- prioritized default theory where $\mathcal{A} = \langle \mathcal{W}, \mathcal{D}, \mathcal{C}, < \rangle$, and S be a non-empty set of all triggered and non-conflicted defaults w.r.t. $\mathcal{S}$. Namely $S = Triggered_{\mathcal{W},\mathcal{D}}(\mathcal{S}) \cap (\mathcal{D}/Conflicted_{\mathcal{W},\mathcal{D}}(\mathcal{S}))$. Then*

$$\delta \in S \textit{ iff } \forall \delta' \in \mathcal{D}.[(\delta', \delta) \in \mathcal{C} \Rightarrow \exists \delta'' \in S.(\delta'', \delta') \in \mathcal{C}].$$

Proof:
LtR: Assume that $\delta \in S$. Let $\delta' \in \mathcal{D}$ and $(\delta', \delta) \in \mathcal{C}$. So $\neg Con(\delta) \in Cn(\mathcal{W} \cup \{Con(\delta')\})$ according to Definition 5.3.5. As Cn satisfies contrapositivity, of course I have $\neg Con(\delta') \in Cn(\mathcal{W} \cup \{Con(\delta)\})$, which indicates $(\delta, \delta') \in \mathcal{C}$.

RtL: Let $\delta \notin S$. Assume that $\delta \in Triggered_{\mathcal{W},\mathcal{D}}(\mathcal{S})$. Then $\delta \in Conflicted_{\mathcal{W},\mathcal{D}}(\mathcal{S})$. Since S is non-empty, so $\exists \delta' \in S$ s.t. $\neg Con(\delta) \in Cn(\mathcal{W} \cup \{Con(\delta')\})$, and for any $\delta'' \in S$, of course, $(\delta'', \delta') \notin \mathcal{C}$, as both are contained in S. □

In other words, a default is triggered and non-conflicted iff it is acceptable w.r.t. a given scenario.

5.3.7. COROLLARY. *Let $\langle \mathcal{W}, \mathcal{D}, \mathcal{C}, < \rangle$ be a SAF-prioritized default theory, and scenario $\mathcal{S}$ is triggered and non-conflicted. Then*

$$\mathcal{S} \textit{ is a stable extension iff } \mathcal{S} = \{\delta \mid \delta \textit{ is not defeated by } \mathcal{S}\}.$$

Proof:
According to Lemma 5.3.6, this scenario $\mathcal{S}$ is an admission extension. And then it is conflict free. By using Lemma 5.3.4, this corollary is obvious. □

Together with Corollary 5.3.7, I can simply conclude that stable extensions are already one particular case of proper scenarios. The detailed connection is presented in the next section.

5.3.2 Safe Scenarios: A Representation of Stable Extensions, Providing Reasonable Explanations for Legal Permissibility

This section proposes a representation of stable extensions in prioritized default theory, what I called "safe scenarios," which undoubtedly gives rise to reasonable legal permissibilities in principle of "overall undefeated reasons." In doing so, two notions in stable extensions should be characterized, *acceptability* and *non-defeatedness*. As Definition 5.3.5 and Lemma 5.3.6 show, an "acceptable" default can be naturally represented as a combination of the *triggered* and *non-conflicted* default in prioritized default theory.

5.3.8. DEFINITION. [Acceptability] Given a scenario $\mathcal{S}$ based on $\Delta = \langle \mathcal{W}, \mathcal{D}, < \rangle$. A default δ is *acceptable* w.r.t. $\mathcal{S}$ iff δ is a triggered and non-conflicted default w.r.t. $\mathcal{S}$. A scenario is *acceptable* iff all defaults in it are acceptable w.r.t. itself, namely $\mathcal{S} \subseteq Triggered_{\mathcal{W},\mathcal{D}}(\mathcal{S}) \cap (\mathcal{D}/Conflicted_{\mathcal{W},\mathcal{D}}(\mathcal{S}))$.

In accordance to Lemma 5.3.6, an acceptable scenario is an admissible extension, and vice versa.

I then can define a scenario as *maximally acceptable* as it contains all the acceptable default w.r.t. itself.

5.3.9. DEFINITION. [Maximal Acceptability] Given a scenario $\mathcal{S}$ based on the theory $\Delta = \langle \mathcal{W}, \mathcal{D}, < \rangle$. This scenario $\mathcal{S}$ is a *maximally acceptable* scenario based on the theory Δ iff it contains all acceptable defaults w.r.t. $\mathcal{S}$, namely

$$\mathcal{S} = Triggered_{\mathcal{W},\mathcal{D}}(\mathcal{S}) \cap (\mathcal{D}/Conflicted_{\mathcal{W},\mathcal{D}}(\mathcal{S}))$$

I denote the set of all maximally acceptable scenarios based on Δ as $MA_{\mathcal{W},\mathcal{D}}$.

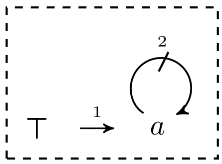
Each maximally acceptable scenario naturally presents the maximality of acceptability in the following sense.

5.3.10. LEMMA. *If* $\mathcal{S} \in MA_{\mathcal{W},\mathcal{D}}$ *then* $\mathcal{S} \cup \{(\varphi, \psi)\} \subseteq Conflicted_{\mathcal{W},\mathcal{D}}(\mathcal{S})$ *where* $(\varphi, \psi) \in Triggered_{\mathcal{W},\mathcal{D}}(\mathcal{S})$ *and* $(\varphi, \psi) \notin \mathcal{S}$.

So a maximally acceptable scenario is a preferred extension.

My construction of maximal acceptability can deal with the classical benchmark case for stable semantics: The vicious cycle example [52, Section 1.3.2]. In this example it returns an empty stable extension, which, as argued by Dung [28], is reasonable. I first apply the maximal acceptability in the following vicious cycles example.

5.3.11. EXAMPLE. [No Maximally Acceptable Scenarios: Vicious Cycles] Given a default theory $\Delta = \langle \mathcal{W}, \mathcal{D}, < \rangle$ where $\mathcal{W} = \{\top\}$, $\mathcal{D} = \{\top \xrightarrow{\delta_1} a, a \xrightarrow{\delta_2} \neg a\}$ with $\delta_1 < \delta_2$. See the following diagram:



In this diagram, $\{\delta_1\}$ is the unique element in $MA_{\mathcal{W},\mathcal{D}}$. For the other scenarios: neither $\varnothing$ nor $\{\delta_2\}$ is triggered, and $\{\delta_1, \delta_2\}$ is conflicted.

To return an empty extension in this vicious cycles case, I need to build a stable extension in prioritized default theory, which crucially uses the notion of non-defeated defaults.

5.3.12. DEFINITION. [Safe Scenarios] Given a default theory $\Delta = \langle \mathcal{W}, \mathcal{D}, < \rangle$, and a scenario $\mathcal{S} \in MA_{\mathcal{W},\mathcal{D}}$ as one of the maximally acceptable scenarios. I say $\mathcal{S} \in MA_{\mathcal{W},\mathcal{D}}$ is a safe scenario based on Δ, denoted as $\mathcal{S} \in Safe_{\mathcal{W},\mathcal{D}}$, iff

$$\mathcal{S} = \mathcal{D}/Defeated_{\mathcal{W},\mathcal{D}}(\mathcal{S})$$

The construction of safe scenarios is equal to that of stable extension in SAF, by taking Definition 5.3.9, Lemma 5.3.10, Lemma 5.3.6, and Corallary 5.3.7 together. I therefore conclude that safe scenarios are stable extensions in the sense of [28, 83].

Let me see how safe scenarios can be applied to the benchmark case of the vicious cycles example, which is a benchmark of plausibility.

5.3.13. EXAMPLE. In Example 5.3.11, even $\{\delta_1\}$ is the unique maximally consistent scenario based on Δ, there is no safe scenario based on this default theory. $\delta_2 \in Triggered_{\mathcal{W},\mathcal{D}}(\{\delta_1\})$ s.t. $\delta_2 > \delta_1$ and $\neg a \in Cn(\{\top\} \cup \{\neg a\})$. And so $\delta_1 \in Defeated_{\mathcal{W},\mathcal{D}}(\{\delta_1\})$. It implies that $\mathcal{D}/Defeated_{\mathcal{W},\mathcal{D}}(\{\delta_1\}) = \emptyset \neq \{\delta_1\}$. Hence, there is no safe scenario in this case.

This conclusion is intuitively plausible. In this case safe scenarios exclude all the non-reasonable ones.

Moreover, maximal acceptability is a necessary condition for safe scenarios, as the following lemma shows.

5.3.14. LEMMA. *If there is no scenario $\mathcal{S}$ in $MA_{\mathcal{W},\mathcal{D}}$, then there is no scenario $\mathcal{S}$ in $Safe_{\mathcal{W},\mathcal{D}}$.*

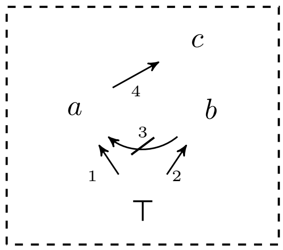
There is a close link between safe scenarios and proper scenarios: a safe scenario is a proper scenario.

5.3.15. THEOREM. *If $\mathcal{S} \in Safe_{\mathcal{W},\mathcal{D}}$ then $\mathcal{S} = Proper_{\mathcal{W},\mathcal{D}}(\mathcal{S})$.*

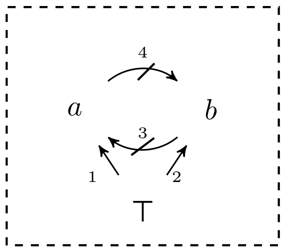
Proof:
Given a safe scenario $\mathcal{S}$ based on the default theory $\Delta = \langle \mathcal{W}, \mathcal{D}, < \rangle$. LtR: Let $(\varphi, \psi) \in \mathcal{S} \in Safe_{\mathcal{W},\mathcal{D}}$, so I have $(\varphi, \psi) \in Triggered_{\mathcal{W},\mathcal{D}}(\mathcal{S}) \cap (\mathcal{D}/Conflicted_{\mathcal{W},\mathcal{D}}(\mathcal{S}))$, and $(\varphi, \psi) \in \mathcal{D}/Defeated_{\mathcal{W},\mathcal{D}}(\mathcal{S})$. This results in $\mathcal{S} \subseteq Proper_{\mathcal{W},D}(\mathcal{S})$. RtL: Assume that $(\varphi, \psi) \notin \mathcal{S}$. As $\mathcal{S}$ is a safe scenario, by Definition 5.3.12, $(\varphi, \psi) \in Defeated_{\mathcal{W},\mathcal{D}}(\mathcal{S})$, and thus $(\varphi, \psi) \notin Proper_{\mathcal{W},D}(\mathcal{S})$. So I have $\mathcal{S} \supseteq Proper_{\mathcal{W},D}(\mathcal{S})$. I thus conclude that $\mathcal{S} = Proper_{\mathcal{W},D}(\mathcal{S})$. □

One significant feature of safe scenarios distinguished from proper scenarios is that: a maximally acceptable scenario including the highest priority is not necessarily safe.

5.3.16. EXAMPLE. Given a default theory $\Delta = \langle \mathcal{W}, \mathcal{D}, < \rangle$ where $\mathcal{W} = \{\top\}$, $\mathcal{D} = \{\top \xrightarrow{\delta_1} a, \top \xrightarrow{\delta_2} b, b \xrightarrow{\delta_3} \neg a, a \xrightarrow{\delta_4} c\}$ with $\delta_1 < \delta_2 < \delta_3 < \delta_4$. See the following diagram:



In this theory, $\{\delta_1, \delta_2, \delta_4\}$ and $\{\delta_2, \delta_3\}$ are the mere maximally acceptable scenarios. But $\{\delta_2, \delta_3\}$ is the unique safe scenario based on Δ, because no default rules in it are defeated. As $\delta_1 \in Defeated_{\mathcal{W},\mathcal{D}}(\{\delta_1, \delta_2, \delta_4\})$, I know that $\{\delta_1, \delta_2, \delta_4\}$ is not safe. However $\{\delta_1, \delta_2, \delta_4\}$ is a proper scenario. This result of a safe scenario reflects that, even if the maximally acceptable scenario contains the default in the highest priority, this condition cannot ensure this scenario is safe, because it may contain a default that is defeated. Compare with the following situation:



This diagram is a default theory $\Delta' = \langle \mathcal{W}', \mathcal{D}', <' \rangle$ where $\mathcal{W}' = \{\top\}$, $\mathcal{D}' = \{\top \xrightarrow{\delta_1} a, \top \xrightarrow{\delta_2} b, b \xrightarrow{\delta_3} \neg a, a \xrightarrow{\delta_4} \neg b\}$ with $\delta_1 <' \delta_2 <' \delta_3 <' \delta_4$. In this case, $\{\delta_1, \delta_4\}$ and $\{\delta_2, \delta_3\}$ are the two maximally acceptable scenarios based on Δ'. Moreover, $\{\delta_1, \delta_4\}$ is the unique safe scenario and the unique proper scenario based on Δ'.

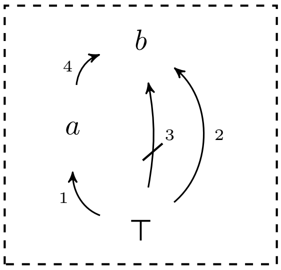
Example 5.3.16 in fact proves the following theorem, which asserts that a proper scenario may not be a safe scenario.

5.3.17. THEOREM. *Given $\Delta = \langle \mathcal{W}, \mathcal{D}, < \rangle$. It may have $\mathcal{S} = Proper_{\mathcal{W},\mathcal{D}}(\mathcal{S})$ but $\mathcal{S} \notin Safe_{\mathcal{W},\mathcal{D}}$.*

Putting together Definition 5.3.12, Theorem 5.3.15 and Theorem 5.3.17, I see that the proper scenario is a general characterization of stable extension.

Furthermore, I argue that a proper scenario is a better representation of *reasons*.

5.3.18. EXAMPLE. This counter example is similar to those in [52, section 8.3.3]. Given a default theory $\Delta = \langle \mathcal{W}, \mathcal{D}, < \rangle$ where $\mathcal{W} = \{\top\}$, $\mathcal{D} = \{\top \xrightarrow{\delta_1} a, \top \xrightarrow{\delta_2} b, \top \xrightarrow{\delta_3} \neg b, a \xrightarrow{\delta_4} b\}$ with $\delta_1 < \delta_2 < \delta_3 < \delta_4$. See the following diagram:



In this diagram, $\mathcal{S} = \{\delta_1, \delta_2, \delta_4\}$ is the unique maximally acceptable scenario based on Δ. But it is not a safe scenario. Because $Defeated_{\mathcal{W},\mathcal{D}}(\mathcal{S}) = \{\delta_2, \delta_3\}$, and then $\mathcal{S} \supset \mathcal{D}/Defeated_{\mathcal{W},\mathcal{D}}(\mathcal{S}) = \{\delta_1, \delta_4\}$. However, the unique proper scenario is $\mathcal{S}' = \{\delta_1, \delta_4\}$.

The proper scenario $\mathcal{S}'$ in the above has a reasonable representation of *reasons*. δ_2 is defeated in the above default theory by the triggered δ_3, and so it should not be contained as a reason.

In this section I reconstructed the standard stable extensions by safe scenarios in prioritized default theory, which is proved to be a special case of proper scenarios. The upside of safe scenario is that, as a standard model in stable semantics, it has a constructive fixed point. In other word, a safe scenario can be generated in an inductive process without "guessing." However, as the example shown in Example 5.3.18, it may be not able to exclude the unreasonable default in some concrete cases.

5.4 Explicit Permission vs. Exclusionary Permission

In this section I propose a formalization of Raz's exclusionary reason [86] in my extended version of Horty's prioritized default theory with exclusion. The upshot of this is that exclusionary permission is a particular instance of permission in my framework.

5.4.1 Default Theories with Exclusionary Reasons

To capture exclusionary reasons, I formulate exclusion as second-order reasons in prioritized default theory [52, p.121-130] [3], by developing "*proper expansions* on the background information" in the "fixed" prioritized default theory introduced particularly in Section 5.2. I choose the fixed prioritized default theory instead of the variable prioritized default theory as [52, p.121-130] does, because the former can explicitly express both permissive reasons and exclusionary reasons at the same time. I will build up my *exclusionary prioritized default theory* step by step.

Now I develop the exclusionary "fixed" prioritized default theory with the core concept *exclusions*. In doing so, the language of the theory has to contain sentences with the predicate Out, which is applied to the unique name d_δ for each given default

[3] See [52] for detailed discussion of "*exculsionary* reasons" in terms of undercutting defeats.

δ in Δ. The sentence $Out(d_\delta)$ indicates that δ is undercut, canceled or excluded from the consideration. I denote $Out(\Delta)$ as the set of all names of defaults which are undercut. Right now, given a prioritized default theory, the sentences in the background information and the set of default rules can both include exclusionary sentences. I call these type of theory *exclusionary prioritized default theories*. A sentence whose subsentence is under the scope of an Out-predicate is called an *exclusionary sentence*, and a sentence without is called a *pure sentence*. In this case, a default rule whose conclusion is scoped by the Out-predicate is called an *exclusionary reason*.

5.4.1. DEFINITION. [Exclusions] Let $\Delta = \langle \mathcal{W}, \mathcal{I}, \mathcal{D}, < \rangle$ be an exclusionary priority default theory. Then the set $Excluded_{\mathcal{S}}$ of default rules that are undercut in scenario $\mathcal{S}$ based on Δ is defined as follows:

$$\delta \in Excluded_{\mathcal{S}} \text{ iff } \mathcal{W} \cup Con(S) \vdash Out(d_\delta)$$

The exclusions $Excluded_{\mathcal{S}}$ in a given scenario $\mathcal{S}$ are those undercut default rules that are not applicable or triggered in this scenario any longer.

5.4.2. DEFINITION. [Triggered Defaults without Exclusions: Revised Definition] Given an exclusionary priority default theory $\Delta = \langle \mathcal{W}, \mathcal{I}, \mathcal{D}, < \rangle$ and a scenario $\mathcal{S}$ based on Δ. Then

$$(\varphi, \psi) = \delta \in Triggered_{\mathcal{W},\mathcal{D}}(\mathcal{S}) \Leftrightarrow \delta \notin Excluded_{\mathcal{S}} \ \&\ \varphi \in Cn(\mathcal{W} \cup Con(\mathcal{S}))$$

Here the proper scenarios based on Δ are defined as in Definition 5.1.4 but with this revised Definition 5.4.2 of triggered defaults.

I now present a revised version of the CIT example where I can apply my exclusionary default theory. In this revised example, the default theory contains the same background information, but I replace the permissive norm $d \xrightarrow{\delta_a} r$ with an exclusionary reason $d \xrightarrow{\delta_a} Out(d_{\delta_c})$, which indicates that submitting the document undercuts the regular tax rate. This exclusionary reason does not, however, deliver the correct result. In this modified example it is still permitted to reduce the tax rate.

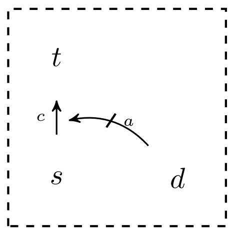


Figure 5.4: A revised default theory of CIT, in which the claim of the solar business is an exclusionary reason to undercut the regular tax rate.

5.4.3. EXAMPLE. [Revised CIT with Exclusionary Reasons] Given a theory Δ_{CIT_E} = $\langle \mathcal{W}, \mathcal{I}, \mathcal{D}, < \rangle$ as follows: $\mathcal{W} = \{s, d\}$, $\mathcal{I} = \{a, c\}$, $\mathcal{D} = \{s \xrightarrow{\delta_c} t, d \xrightarrow{\delta_a} Out(d_{\delta_c})\}$, in which the defaults contain the claim of the solar company as an exclusionary reason for tax reduction. See Figure 5.4. This default theory Δ has a unique proper scenario $\{\delta_a\}$, and thus, it can only imply the weak permission of filing regular tax, and of submitting the valid documentation. In other words, this exclusionary default theory cannot conclude the most important permission in this concrete case: it is permitted to reduce the regular tax rate.

In this section I reconstructed prioritized default theory, the so-called exclusionary prioritized default theory, by extending the object language with exclusions. By doing so, triggered default rules have to be re-defined accordingly. Comparing the results I just obtained with those in Section 5.2, it seems that in this particular legal case I obtain more reasonable conclusions with explicit permissive norms than with exclusionary permissions.

5.4.2 Alternative: e-Exclusionary Default Theories

In this reconstruction of exclusionary default theory, the basic idea is that the set of all exclusionary sentences only appears as a set of default rules, in order to make a sharp distinction of the respective effects of permissive norms and exclusionary reasons. To achieve this, I first collect all exclusionary sentences by the so-called exclusionary set $\mathcal{E}$ of default rules as follows. If an exclusionary sentence is a proposition $Out(d)$ consistent with the background information, then it is taken to have a corresponding default rule $\top \to Out(d)$ without priority. In this case, this proposition and its corresponding rule have the same effect on a given default theory. As all tautologies are derived from the background in the classical consequence relation, the corresponding rule is always triggered. Together with the fact that this rule should always be the least element in the proper scenario, it becomes included in $\mathcal{E}$. Unlike in the previous construction, all exclusionary sentences here can have their own given priority. All elements in $\mathcal{E}$ are called exclusionary defaults.

5.4.4. DEFINITION. [Alternative: e-Exclusionary Default Theories] Given a tuple Δ^e = $\langle \mathcal{W}, \mathcal{I}, \mathcal{D} \cup \mathcal{E}, < \rangle$. I say Δ^e is an e-exclusionary default theory iff $\mathcal{W}$ is a set of pure sentences, $\mathcal{I}$ is a set of agents, $\mathcal{D}$ is a set of default rules without exclusionary sentences, $\mathcal{E}$ is an exclusionary set of exclusionary defaults, and $<$ is a strict partial priority order over the union of $\mathcal{D}$ and $\mathcal{E}$.

The proper scenarios in this section are defined as in Section 5.4.1 with the revised Definition 5.4.2 of triggered defaults. I then have the following lemma which bridges the exclusionary default theory in Section 5.4.1 and the e-exclusionary default theory in this section. I first define a corresponding notion. I say $\Delta^e = \langle \mathcal{W}', \mathcal{I}, \mathcal{D}' \cup \mathcal{E}, < \rangle$ is a corresponding e-exclusionary default theory to $\Delta = \langle \mathcal{W}, \mathcal{I}, \mathcal{D}, < \rangle$, iff Δ^e is an e-exclusionary default theory s.t. $\mathcal{W}' \subseteq \mathcal{W}$ is the set of all pure sentences in $\mathcal{W}$, and $\mathcal{D}' \subseteq \mathcal{D}$ is the set of all pure sentences in $\mathcal{D}$, and $\mathcal{E}$ is the set of all corresponding exclusionary sentences in $\mathcal{W} \cup \mathcal{D}$. According to this construction, $\mathcal{W} \cup \mathcal{D} = \mathcal{W}' \cup \mathcal{D}' \cup \mathcal{E}$, and $<$ ranges over a subset of $\mathcal{D}' \cup \mathcal{E}$ related to $\mathcal{D}$.

5.4.5. LEMMA. *Let $\Delta = \langle \mathcal{W}, \mathcal{I}, \mathcal{D}, < \rangle$ be an exclusionary default theory in Section 5.4.1 and $\Delta^e = \langle \mathcal{W}', \mathcal{I}, \mathcal{D}' \cup \mathcal{E}, < \rangle$ be a corresponding e-exclusionary default theory to Δ in this section. Then $\mathcal{S}$ is a proper scenario based on Δ iff $\mathcal{S}$ is a proper scenario based on Δ^e. That is*

$$\mathcal{S} = Proper_{\mathcal{W},\mathcal{D}}(\mathcal{S}) \Leftrightarrow \mathcal{S} = Proper_{\mathcal{W}',\mathcal{D}' \cup \mathcal{E}}(\mathcal{S})$$

.

Proof:
According to $\mathcal{W} \cup \mathcal{D} = \mathcal{W}' \cup \mathcal{D}' \cup \mathcal{E}$ as their corresponding notions, and the same priority over rules, this result is obvious. □

Taking the last two observations together, they assert a relationship between permissive norms and exclusionary sentences. Comparing the e-exclusionary construction with the default theory with permissive norms, exclusionary sentences and permissive norms are two distinct types of default rule in a given default theory. However, I should keep in mind that, permissive norms may get the highest, while still being undercut by exclusionary sentences with lower priority. Therefore, a most prioritized explicit permission can be canceled by a less important exclusionary reason.

5.5 Conclusion

This chapter developed a formal theory for a rich typology of permissions by extending Horty's default theory. I showed that this theory can be applied to important examples in Chinese and in German Law.

Aside from this, my formal theory has highlighted the multi-agent aspect in legal reasoning by the so-called action sets. This turns to be an advantage for characterizing Hohfeldian correlative relations of claims and duties. I argued for this by displaying two case studies in explicit permission. However, a further argument for supporting this characterization should be given, in order to build up a firm philosophical foundation to link my normative concepts with Hohfeldian legal rights. I leave this to my future research.

This chapter also contributes to the formal studies of the theoretical foundation of my default theory, by connecting default theory with the other formal theories for *defeasibility* or *specificity*, and by justifying the idea of permissive norms as a response to [2, 86]. The investigation of these two issues provides me with a positive support for my formulation of default theory, by showing that my default theory can bring about intuitive normative consequences in the concrete cases for two items of Chinese and German Laws.

Chapter 6
Conclusion

This thesis has studied an abundant variety of notions of permissions and their logical structures in non-monotonic reasoning. Because actions are the subjects of permissions [117], the whole investigation can be divided into three broad action-theoretic ideas: conditionals, dynamics and defeasibility. Each of these showcase different properties of permissions, as well as a particular relationship with obligation.

6.1 Review of the Chapters

Chapter 2 argued that free choice permission is a plausible notion for normative reasoning about rational recommendations in games. In doing so, it presented a comparison between several existing non-normal modal logics, which share the idea of obligation as the weakest free choice permission. This comparison demonstrated the theoretical significance of this particular relationship between permission and obligation. They are, respectively, the sufficient and necessary conditions for rationality in games.

Chapter 3 examined the logical behaviors of free choice permission in natural language by using a family of substructural logics. In order to gain some deductive power, I augmented my substructural logic of action with the “rational monotony” assumption, and argued that this provides a plausible model of Free Choice Permission. The resulting logic is thus weaker than classical logic, but not too weak. The key conceptual contribution of this chapter was the interpretation of the conditional connective on action types in terms of "licensed instances".

Chapter 4 led me to a study of the dynamic behavior of permission in a legal context. In this chapter, the dynamic logic of permission is generalized from the well-established dynamic logics of lexicographic updates, and it is completely axiomatized. In contrast to the existing framework of power and immunity, this dynamic logic provides a complex reduction to static normative positions, while still viewing these rights as actions that change legal relations. This chapter also provided a computation of the dynamic normative positions, as a standard in the development of formal theories of Hohfeldian legal rights. Aside from this, the dynamic logic in this chapter can clearly distinguish the notions of "legal ability" and "legal permissibility," as illustrated through the analysis of a concrete example in German contract law.

Chapter 5 studied the defeasible character of actions and permissions in an extension of prioritized default theory, in order to capture two items of Chinese and German Law. The prioritized default theory developed here was based on two extensions: multi-agent and permissive norms. This can then define various independent types of permission. This agent-relative default theory with permissive norms can plausibly capture the case "Chinese Tax Incentives" and the "Rights of Way." Comparing the default theories with undercut defeaters, and other extension-based semantics, the prioritized default theory introduced in this chapter can helps us understand better these concrete legal situations.

6.2 Open Questions

The many facets of permissions that we encountered in this thesis open up a lot of questions for future research.

6.2.1 The Stability of Permission and Obligation

The first important open question in my view concerns the stability of norms. I understand "stability" here by analogy with the notion of entrenchment in belief revision [34, 35]. In this thesis, the permission in Chapter 2 can be viewed as the "all-or-nothing" permission. But what about those in the other chapters in non-monotonic frameworks? We know from dynamic doxastic logic [35] that belief and entrenchment are different. This distinction can arguably also be made for normative concepts. Here a promising research avenue would be to look at the notion of "trust" [66, 62] as sustainer of stability for deontic notions like permissions and obligations. If such an analogy would exist in deontic logic, what would be their definitions, and what are their logical behaviors and their relationship with the existing concepts in this thesis?

6.2.2 Aggregations on Permission and Obligation

From a multi-agent perspective, there is an interesting question about how to define permission and obligation for groups of agents, by analogy with distributed knowledge in epistemic logic [30]. Although this topic is not a no-man's land, most existing theories in multi-agent deontic logic address, for instance, the notions of group obligation based on collective obligation [91, 39] or joint action [50, 97]. In contrast, it would be interesting to look at the aggregation of permissions and obligations, such that each individual's permission and obligation contributed to an aggregate norm. In Chapter 4 and Chapter 5, a perspective of multi-agent interaction was addressed, but the permissions and obligations were still for individuals. These chapters have offered well-established frameworks for multi-agents; the next step is how to apply these formal frameworks to the topic of aggregating permissions and obligations.

Unlike the traditional viewpoint, this thesis has argued that non-standard permissions, e.g. free choice permission and explicit permission, are the key elements in some realistic cases, which cannot be captured by the standard notion of weak permission. I believe that all these non-standard views of permissions bear fruit beyond deontic logics, and they are the invaluable guidelines to develop normative systems, promising in the areas of ethics, economics, and policy making. And this is what has been argued in this thesis.

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Index

List of symbols

Zusammenfassung

Diese Arbeit erforschte verschiedene formale Theorien der taxonomischen Kategorie der Erlaubnis in nichtmonotonem Schließen, untersuchte die jeweiligen logischen Prinzipien und wandte diese verschiedenen Begriffe von Erlaubnis auf konkrete Fälle in natürlicher Sprache und Gesetzgebung an. Im Gegensatz zur singularen Form der Erlaubnis in SDL zeigte diese formale Untersuchung, wie wichtig plurale Sichtweisen auf Erlaubnis bei der Errichtung von normativen Systemen in der Praxis sind.

Kapitel 2 ist eine vergleichende Untersuchung von vier Logiken in zwei Darstellungen rationaler Verpflichtung als die schwächste Erlaubnis in Spielen; eine basiert auf Propositionen und die andere auf Handlungstypen. Sie lieferte einige Übersetzungsresultate bezüglich der deduktiven Stärke und zeigte die Ähnlichkeiten und Unterschiede zwischen ihnen auf.

Kapitel 3 ist eine neue Lösung des berüchtigten Rätsels des *free choice permission paradox*. Es untersuchte dabei eine Gruppe substruktureller Prinzipien, um *free choice permission* in natürlicher Sprache zu fassen, erweitert um ein nichtmonotones Prinzip aus handlungstheoretischer Sicht. Demgemäß entwickelte es eine Familie substruktureller Logiken mit hinreichender deduktiver Stärke. Diese Logiken werden vollständig axiomatisiert, dabei erfüllen einige den Gentzenschen Hauptsatz.

Kapitel 4 ist eine Pilotstudie zur Anwendung der etablierten Logik mit lexikographischen Updates auf Hohfeldsche dynamische Rechte. Anders als ihre Vorgänger stellt sie einen dritten Weg dar, um Macht und Immunität zu formulieren. Auf der einen Seite unterscheidet die dynamische Logik den dynamischen Charakter von Rechten von den statischen. Auf der anderen Seite verfügt sie als reduktive Darstellung über eine größere Menge von logischen Interaktionen zwischen statischen und dynamischen Rechten. Tatsächlich kann die hier entwickelte dynamische Logik in einem konkreten Fall aus dem deutschen Zivilrecht angewandt werden, um den Begriff des rechtlichen Könnens von dem des rechtlichen Dürfens zu unterscheiden.

Kapitel 5 ist eine formale Untersuchung von vier Arten von Erlaubnis auf der Grundlage von Hortys *prioritized default theory*, die verwendet werden kann um zwei bedeutende Teile deutschen und chinesischen Rechts zu analysieren. Dabei erweiterte sie Hortys Theorie um erlaubende Normen sowie *agency*. Darüber hinaus wurde bewiesen, dass diese formale Theorie im Vergleich zu Dungs Argumentationstheorie und der *exclusionary default theory* eine allgemeinere Formulierung von *defeasibility* darstellt.

Abstract

This thesis has studied various formal theories of the taxonomic category of permission in non-monotonic reasoning, examined their particular logical principles, and applied these various notions of permission to concrete cases from natural language and from legislation. In contrast to the singular form of permission in SDL, this formal study has pointed out how important plural views of permission are in building up a normative system in practice.

Chapter 2 is a comparative study to argue that free choice permission is a plausible notion about rational recommendations in games. In doing so, it presented a comparison between several existing non-normal modal logics, which share the idea of rational obligation as weakest permission. It provided several translation results regarding their deductive power, and pointed out the theoretical significance of this particular relationship between permission and obligation. They are, respectively, the sufficient and necessary conditions for rationality in games.

Chapter 3 is a new solution to the notorious puzzle of the "free choice permission paradox." In doing so, it studied a group of substructural principles for capturing free choice permission in natural language, augmented with a non-monotonic principle. Accordingly, it developed a family of substructural logics with sufficient deductive power. These logics are completely axiomatized, and some satisfy cut-elimination. The key conceptual contribution of this chapter was the interpretation of the conditional connective on action types in terms of "licensed instances."

Chapter 4 is a pilot study of applying the well-established logic with lexicographic updates on Hohfeldian dynamic rights. Different than its ancestors, it is a third way to formulate power and immunity. On the one hand, the dynamic logic distinguished the dynamic character of rights from the static ones. On the other hand, as a reductive account, it has a richer set of logical interactions between static and dynamic rights. Indeed, the dynamic logic developed here can be well applied to distinguish the notion of legal ability (*rechtliches Können*) from that of legal permissibility (*rechtliches Dürfen*), in a concrete case from the German civil code.

Chapter 5 is formal research that studies the defeasible character of actions and permissions in an extension of prioritized default theory, in order to capture two items in Chinese and German Law. The prioritized default theory developed here was based on two extensions: multi-agent and permissive norms. This can then define various independent types of permission. This agent-relative default theory with permissive norms can plausibly capture the case “Chinese Tax Incentives” and the “Rights of Way.” Comparing the exclusionary default theory, and other extension-based semantics, the prioritized default theory introduced in this chapter can help us understand better these concrete legal situations.

Curriculum Vitae

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Education

- **Ph.D candidate** Philosophy, Universität Bayreuth, Bayreuth, Germany, 2014-now.

- **Ph.D candidate** Philosophy, MCMP, Ludwig-Maximilians-Universität München, Munich, Germany, 2013-2014.

- **M.A.** Logic, Sun Yat-Sen University, Guangzhou, China, 2010-2013.

- **B.A.** Philosophy, Sun Yat-Sen University, Guangzhou, China, 2006-2010.

Research

Book Contribution

- Chapter 1, *Update Semantics and Dynamic Epistemic Logic*, 2011, Xiaowu Li(ed.).

Articles

- Huimin Dong and Olivier Roy. Three Deontic Logics for Rational Agency in Games. *Studies in Logic*, Vol. 8, No. 4 (2015): 7-31. 2015.

Proceedings

1. Huimin Dong and Olivier Roy. Dynamic logic of Power and Immunity. MIREL 2017. Forthcoming.

2. Xin Sun and Huimin Dong. Deontic Logic based on a Decidable PDL with Action Negation. *7th International Conference on Agents and Artificial Intelligence (ICAART) 2015*. ICAART 2015.

3. Albert J.J. Anglberger, Huimin Dong and Olivier Roy. Open Reading without Free Choice. *Deontic Logic and Normative Systems, Lecture Notes in Computer Science Volume 8554*. DEON 2014: 19-32.

4. Xin Sun and Huimin Dong. Stratified Action Negation, a Logic about Travel. *26th Benelux Conference on Artificial Intelligence (BNAIC) 2014*. BNAIC 2014: 81-87. ISSN 1568-7805.

5. Xin Sun and Huimin Dong. The Deontic Dilemma of Action Negation, and Its Solution. *11st Conferencen Logic and the Foundations of Game and Decision Theory*. LOFT 2014: 502-515.

6. Huimin Dong, Xiaowu Li. A Deontic Action Logic for Complex Actions. *Logic, Rationality, and Interaction, Lecture Notes in Computer Science Volume 8196*. LORI 2013: 311-315.

Manuscripts

- Xin Sun and Huimin Dong. Deontic logic based on a decidable PDL with action negation. 2015. Manuscript.

Presentations

1. The Variety of Permissions in Prioritized Default Theory, *The 3rd PIOTR Project Workshop*, Universität Bayreuth, Bayreuth, Germany, February 21st-22nd, 2017.

2. Open Reading for Free Choice Permission: A Perspective from Substructural Logics, *Non-classical Logics and Their Applications: The 8th International Workshop on Logic and Cognition (WOLC 2016)*, Sun Yat-Sen University, Guangzhou, China, December 5th-9th, 2016.

3. Open Reading for Free Choice Permission: A Perspective from Substructural Logics, *Colloquium Logicum 2016*, Universität Hamburg, Hamburg, Germany, September 12th, 2016.

4. Permissions and Obligations in Default Theory, *Tsinghua-Bayreuth Workshop*, Beijing, China, August 9th, 2016.

5. Open Reading for Free Choice Permission: A Perspective from Substructural Logics, *Tsinghua-Bayreuth Workshop*, Beijing, China, August 8th, 2016.

6. Open Reading for Free Choice Permission: A Perspective from Substructural Logics, *"Five Years MCMP: Quo Vadis, Mathematical Philosophy?" Workshop*, Munich, Germany, June 2nd, 2016.

7. Open Reading for Free Choice Permission: A Perspective from Substructural Logics, *21st Conference Applications of Logic*, Szklarska Poręba, Poland, May 9-13th, 2016.

8. Conditionals in Deontic Actions: Free Choice Permission with Substructural Logics, *Kick-off Seminar*, Lublin, Poland, February 17th, 2016.

9. Interpret Deontic Actions by Ternary Relation, *Logic in Progress*, Bayreuth, December 1st, 2015.

10. Ternary Interpretations For Deontic Actions, *Deontic Logic and Ethics 2015*, Venice Seminar 2015, Venice, November 25th, 2015.

11. Ternary Interpretations For Deontic Actions, *Jin Yuelin Conference on Dao, Logic and Epistemology*, Beijing, October 19th, 2015

12. A Substructural Logic for Free Choice Permission in Partial Interpretation, *Formal Ethics 2015*, Bayreuth, July 4th, 2015.

13. Permission on Deontic Action Logics in Open Reading. *The 1st Deontic Logic Event of the Polish-Bavaria Group*, Bayreuth, Germany, November 4th, 2014.

14. Stratified Action Negation, and Its Application. *The 11st Conferencen Logic and the Foundations of Game and Decision Theory*, LOFT 2014, Bergen, Norway, July 27-30th, 2014.

15. Open Reading without Free Choice. *The 12th International Conference on Deontic Logic and Normative Systems (DEON 2014)*, DEON 2014, Gent, Belgium, July 12-15th, 2014.

16. A λ-calculus for Deontic Actions: to Reject Free Choice. *The 1st Meeting of the Prague-Munich Group on Sub-Structural Epistemic Logic*, Prague, Czech Republic, May 13th, 2014.

17. Open Reading without Free Choice. *The 1st Meeting of the Bavaria Deontic Group*, Bayreuth, Germany, March 5th, 2014.
18. A Deontic Action Logic for Complex Actions. *The 4th International Workshop on Logic, Rationality and Interaction*, LORI 2013, Hangzhou, China, October 9-12th, 2013.

Projects

- *Member* of Research in *Permissions, Information and Institutional Dynamics, Obligations, and Rights* (supported by DFG-NCN program "Beethoven", No. RO 4548/4-1), 2016-2018.
- *Member* of Research in *Formal Studies of Communication in Cross-Culture* (supported by Humanities and Social Sciences Foundation of Ministry of Education of China, No.12JJD720005), 2012.
- *Member* of Research in *Update Semantics and Dynamic Epistemic Logic* (supported by National Social Science in China, No.08BZX050), 2008.

Teaching

- Teaching Assistant in *Formal Theories of Rights* with exercise session, Winter term 2015/2016.
- Teaching Assistant in *Diamonds are Philosophers' best Friends: An Introduction to Modal Logic* with exercise session, Summer term 2015.
- Teaching Assistant in *Logic for Norms and Obligation* with exercise session, Winter term 2014/2015.

Organizations

- Local Organizing Committee, *The 3rd PIOTR Project Workshop*, Bayreuth, February 21st-22nd, 2017.
- Local Organizing Committee, *The 13th International Conference on Deontic Logic and Normative Systems*, DEON 2016, Bayreuth, July 18-21st, 2016.
- Organizer, the internal research seminar *Philosophy Breakfast* in the Department of Philosophy, University of Bayreuth, 2014-now.
- Organizer, the *Deontic Logic Event*, The 2nd Munich-Bayreuth Deontic Logic Meeting, Bayreuth, November 4th, 2014.

- Assistant, The 1st Munich-Bayreuth Deontic Logic Meeting, Bayreuth, March 5th, 2014.

Academic Visit

- Research visit, Faculty of Philosophy, the John Paul II Catholic University of Lublin, Poland (from May 4th to June 1st, 2016), supported by project *Permissions, Information and Institutional Dynamics, Obligations, and Rights* (supported by DFG-NCN program "Beethoven").

Honors

- The CSC scholarship funded by the China Scholarship Council grant (CSC No. 201306380078), 2013-2017.
- The Full-Funded Scholarship of Sun Yat-Sen University, 2010-2012.
- The Excellent Graduates of Sun Yat-Sen University, 2010.
- The First-Class Scholarship of Sun Yat-Sen University, twice (top 5%, 2007, 2009).
- The Second-Class Scholarship of Sun Yat-Sen University, (top 10%, 2008).