UPPER BOUNDS FOR PARTIAL SPREADS

SASCHA KURZ*

ABSTRACT. A *partial t-spread* in $\mathbb{F}_q^n$ is a collection of t-dimensional subspaces with trivial intersection such that each non-zero vector is covered at most once. We present some improved upper bounds on the maximum sizes.

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1. INTRODUCTION

Let $q > 1$ be a prime power and n a positive integer. A *vector space partition* $\mathcal{P}$ of $\mathbb{F}_q^n$ is a collection of subspaces with the property that every non-zero vector is contained in a unique member of $\mathcal{P}$. If $\mathcal{P}$ contains m_d subspaces of dimension d, then $\mathcal{P}$ is of type $k^{m_k}\dots 1^{m_1}$. We may leave out some of the cases with $m_d = 0$. Subspaces of dimension 1 are called *holes*. If there is at least one non-hole, then $\mathcal{P}$ is called non-trivial.

A *partial t-spread* in $\mathbb{F}_q^n$ is a collection of t-dimensional subspaces such that the non-zero vectors are covered at most once, i.e., a vector space partition of type $t^{m_t}1^{m_1}$. By $A_q(n, 2t; t)$ we denote the maximum value of m_t [1]. Writing $n = kt + r$, with $k, r \in \mathbb{N}_0$ and $r \le t-1$, we can state that for $r \le 1$ or $n \le 2t$ the exact value of $A_q(n, 2t; t)$ was known for more than forty years [1]. Via a computer search the cases $A_2(3k+2, 6; 3)$ were settled in 2010 by El-Zanati et al. [5]. In 2015 the case $q = r = 2$ was resolved by continuing the original approach of Beutelspacher [13], i.e., by *considering* the set of holes in $(n-2)$-dimensional subspaces and some averaging arguments. Very recently, Năstase and Sissokho found a very clear generalized averaging method for the number of holes in $(n-j)$-dimensional subspaces, where $j \le t-2$, and general q, see [14]. Their Theorem 5 determines the exact values of $A_q(kt + r, 2t; t)$ in all cases where $t > \begin{bmatrix} r \\ 1 \end{bmatrix}_q := \frac{q^r-1}{q-1}$. Here, we streamline and generalize their approach leading to improved upper bounds on $A_q(n, 2t; t)$, c.f. [15].

2. SUBSPACES WITH THE MINIMUM NUMBER OF HOLES

Definition 2.1. A vector space partition $\mathcal{P}$ of $\mathbb{F}_q^n$ has *hole-type* (t, s, m_1), if it is of type $t^{m_t}\dots s^{m_s}1^{m_1}$, for some integers $n > t \ge s \ge 2$, $m_i \in \mathbb{N}_0$ for $i \in \{1, s, \dots, t\}$, and $\mathcal{P}$ is non-trivial.

Lemma 2.2. *(C.f.* [14, Proof of Lemma 9].*) Let $\mathcal{P}$ be a non-trivial vector space partition of $\mathbb{F}_q^n$ of hole-type (t, s, m_1) and $l, x \in \mathbb{N}_0$ with $\sum_{i=s}^{t} m_i = lq^s + x$. $\mathcal{P}_H = \{U \cap H : U \in \mathcal{P}\}$ is a vector space partition of type $t^{m'_t}\dots(s-1)^{m'_{s-1}}1^{m'_1}$, for a hyperplane H with $\widehat{m}_1$ holes (of $\mathcal{P}$). We have $\widehat{m}_1 \equiv \frac{m_1+x-1}{q} \pmod{q^{s-1}}$. If $s > 2$, then $\mathcal{P}_H$ is non-trivial and $m'_1 = \widehat{m}_1$.*

PROOF. If $U \in \mathcal{P}$, then $\dim(U) - \dim(U \cap H) \in \{0, 1\}$ for an arbitrary hyperplane H. Since $\mathcal{P}$ is non-trivial, we have $n \ge s$. For $s > 2$, counting the 1-dimensional subspaces of $\mathbb{F}_q^n$ and H, via $\mathcal{P}$ and $\mathcal{P}_H$, yields

$$(lq^s + x)\cdot \begin{bmatrix} s \\ 1 \end{bmatrix}_q + aq^s + m_1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q \quad \text{and} \quad (lq^s + x)\cdot \begin{bmatrix} s-1 \\ 1 \end{bmatrix}_q + a'q^{s-1} + \widehat{m}_1 = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$$

for some $a, a' \in \mathbb{N}_0$. Since $1 + q\cdot \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} n \\ 1 \end{bmatrix}_q = 0$ we conclude $1 + q\widehat{m}_1 - m_1 - x \equiv 0 \pmod{q^s}$. Thus, $\mathbb{Z} \ni \widehat{m}_1 \equiv \frac{m_1+x-1}{q} \pmod{q^{s-1}}$. For $s = 2$ we have

$$\left(lq^2 + x\right)\cdot (q+1) + aq^2 + m_1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q \quad \text{and} \quad \left(lq^2 + x\right)\cdot 1 + a'q + \widehat{m}_1 = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$$

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[1]The more general notation $A_q(n, 2t-2w; t)$ denotes the maximum cardinality of a collection of t-dimensional subspaces, whose pairwise intersections have a dimension of at most w. Those objects are called *constant dimension codes*, see e.g. [6]. For known bounds, we refer to `http://subspacecodes.uni-bayreuth.de` [10] containing also the generalization to *subspace codes* of mixed dimension.

leading to the same conclusion $\widehat{m}_1 \equiv \frac{m_1+x-1}{q} \pmod{q^{s-1}}$. □

Lemma 2.3. *(C.f.* [14, Proof of Lemma 9].*) Let $\mathcal{P}$ be a vector space partition of $\mathbb{F}_q^n$ of hole-type (t,s,m_1), $l,x\in\mathbb{N}_0$ with $\sum_{i=s}^{t} m_i = lq^s + x$, and $b,c\in\mathbb{Z}$ with $m_1 = bq^s + c \geq 1$. If $x\geq 1$, then there exists a hyperplane $\widehat{H}$ with $\widehat{m}_1 = \widehat{b}q^{s-1}+\widehat{c}$ holes, where $\widehat{c} := \frac{c+x-1}{q}\in\mathbb{Z}$ and $b>\widehat{b}\in\mathbb{Z}$.*

PROOF. Apply Lemma 2.2 and observe $m_1\equiv c \pmod{q^s}$. Let the number of holes in $\widehat{H}$ be minimal. Then,

$$\widehat{m}_1 \leq \text{average number of holes per hyperplane} = m_1\cdot \begin{bmatrix} n-1\\ 1\end{bmatrix}_q \Big/ \begin{bmatrix} n\\ 1\end{bmatrix}_q < \frac{m_1}{q}. \qquad (1)$$

Assuming $\widehat{b}\geq b$ yields $q\widehat{m}_1 \geq q\cdot(bq^{s-1}+\widehat{c}) = bq^s + c + x - 1 \geq m_1$, which contradicts Inequality (1). □

Corollary 2.4. *Using the notation from Lemma 2.3, let $\mathcal{P}$ be a non-trivial vector space partition with $x\geq 1$ and f be the largest integer such that q^f divides c. For each $0\leq j\leq s-\max\{1,f\}$ there exists an $(n-j)$-dimensional subspace U containing $\widehat{m}_1$ holes with $\widehat{m}_1\equiv\widehat{c} \pmod{q^{s-j}}$ and $\widehat{m}_1\leq (b-j)\cdot q^{s-j}+\widehat{c}$, where $\widehat{c} = \frac{c+\left[\begin{smallmatrix} j\\ 1\end{smallmatrix}\right]_q\cdot(x-1)}{q^j}$.*

Proof. Observe $\widehat{m}_1\equiv c\not\equiv 0 \pmod{q^{s-j}}$, i.e., $\widehat{m}_1\geq 1$, for all $j<s-f$. □

Lemma 2.5. *Let $\mathcal{P}$ be a non-trivial vector space partition of type $t^{m_t}1^{m_1}$ of $\mathbb{F}_q^n$ with $m_t = lq^t + x$, where $l = \frac{q^{n-t}-q^r}{q^t-1}$, $x\geq 2$, $t = \left[\begin{smallmatrix} r\\ 1\end{smallmatrix}\right]_q + 1 - z + u > r$, $q^f | x-1$, $q^{f+1}\nmid x-1$, and $f,u,z,r,x\in\mathbb{N}_0$. For $\max\{1,f\}\leq y\leq t$ there exists a $(n-t+y)$-dimensional subspace U with $L\leq (z+y-1)q^y + w$ holes, where $w = -(x-1)\left[\begin{smallmatrix} y\\ 1\end{smallmatrix}\right]_q$ and $L\equiv w \pmod{q^y}$.*

PROOF. Apply Corollary 2.4 with $s=t$, $j=t-y$, $b=\left[\begin{smallmatrix} r\\ 1\end{smallmatrix}\right]_q$, and $m_1 = \left[\begin{smallmatrix} r\\ 1\end{smallmatrix}\right]_q q^t - \left[\begin{smallmatrix} t\\ 1\end{smallmatrix}\right]_q (x-1)$. □

Lemma 2.6. *Let $\mathcal{P}$ be a vector space partition of $\mathbb{F}_q^n$ with $c\geq 1$ holes and a_i denote the number of hyperplanes containing i holes. Then, $\sum_{i=0}^{c} a_i = \left[\begin{smallmatrix} n\\ 1\end{smallmatrix}\right]_q$, $\sum_{i=0}^{c} ia_i = c\cdot\left[\begin{smallmatrix} n-1\\ 1\end{smallmatrix}\right]_q$ and $\sum_{i=0}^{c} i(i-1)a_i = c(c-1)\cdot\left[\begin{smallmatrix} n-2\\ 1\end{smallmatrix}\right]_q$.*

PROOF. Double-count the incidences of the tuples (H), (B_1,H), and (B_1,B_2,H), where H is a hyperplane and $B_1\neq B_2$ are points contained in H. □

Lemma 2.7. *Let $\Delta = q^{s-1}$, $m\in\mathbb{Z}$, and $\mathcal{P}$ be a vector space partition of $\mathbb{F}_q^n$ of hole-type (t,s,c). Then, $\tau_q(c,\Delta,m)\cdot\frac{q^{n-2}}{\Delta^2} - m(m-1)\geq 0$, where*

$$\tau_q(c,\Delta,m) = m(m-1)\Delta^2q^2 - c(2m-1)(q-1)\Delta q + c(q-1)\Big(c(q-1)+1\Big).$$

PROOF. Consider the three equations from Lemma 2.6. $(c-m\Delta)\Big(c-(m-1)\Delta\Big)$ times the first minus $\Big(2c-(2m-1)\Delta-1\Big)$ times the second plus the third equation, and then divided by $\Delta^2/(q-1)$, gives

$$(q-1)\cdot\sum_{h=0}^{\lfloor c/\Delta\rfloor}(m-h)(m-h-1)a_{c-h\Delta} = \tau_q(c,\Delta,m)\cdot\frac{q^{n-2}}{\Delta^2} - m(m-1)$$

due to Lemma 2.2. Finally, we observe $a_i\geq 0$ and $(m-h)(m-h-1)\geq 0$ for all $m,h\in\mathbb{Z}$. □

Lemma 2.8. *For integers $n>t\geq s\geq 2$ and $1\leq i\leq s-1$, there exists no vector space partition $\mathcal{P}$ of $\mathbb{F}_q^n$ of hole-type (t,s,c), where $c = i\cdot q^s - \left[\begin{smallmatrix} s\\ 1\end{smallmatrix}\right]_q + s - 1$.*[2]

PROOF. Assume the contrary and apply Lemma 2.7 with $m = i(q-1)$. Setting $y = s-1-i$ and $\Delta = q^{s-1}$ we compute

$$\tau_q(c,\Delta,m) = -q\Delta(y(q-1)+2) + (s-1)^2q^2 - q(s-1)(2s-5) + (s-2)(s-3).$$

Using $y\geq 0$ we obtain $\tau_2(c,\Delta,m)\leq s^2+s-2^{s+1}<0$. For $s=2$, we have $\tau_q(c,\Delta,m) = -q^2+q<0$ and for $q,s>2$ we have $\tau_q(c,\Delta,m)\leq -2q^s+(s-1)^2q^2<0$. Thus, Lemma 2.7 yields a contradiction. □

[2]For more general non-existence results of vector space partitions see e.g. [9, Theorem 1] and the related literature.

Theorem 2.9. *(C.f. [14, Lemma 10].) For integers $r \ge 1$, $k \ge 2$, $u \ge 0$, and $0 \le z \le \begin{bmatrix} r \\ 1 \end{bmatrix}_q/2$ with $t = \begin{bmatrix} r \\ 1 \end{bmatrix}_q + 1 - z + u > r$ we have $A_q(n, 2t; t) \le lq^t + 1 + z(q-1)$, where $l = \frac{q^{n-t}-q^r}{q^t-1}$ and $n = kt + r$.*

PROOF. Apply Lemma 2.5 with $x = 2 + z(q-1)$ and $y = z + 1$. If $z = 0$, then $L < 0$. For $z \ge 1$, apply Lemma 2.8. Thus, $A_q(n, 2t; t) \le lq^t + x - 1$. □

The known constructions for partial t-spreads give $A_q(kt + r, 2t; t) \ge lq^t + 1$, see e.g. [1] (or [13] for an interpretation using the more general multilevel construction for subspace codes). Thus, Theorem 2.9 is tight for $t \ge \begin{bmatrix} r \\ 1 \end{bmatrix}_q + 1$, c.f. [14, Theorem 5].

Theorem 2.10. *(C.f. [15, Theorem 6,7].) For integers $r \ge 1$, $k \ge 2$, $y \ge \max\{r, 2\}$, $z \ge 0$ with $\lambda = q^y$, $y \le t$, $t = \begin{bmatrix} r \\ 1 \end{bmatrix}_q + 1 - z > r$, $n = kt + r$, and $l = \frac{q^{n-t}-q^r}{q^t-1}$, we have*

$$A_q(n, 2t; t) \le lq^t + \left\lceil \lambda - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\lambda\left(\lambda - (z + y - 1)(q - 1) - 1\right)} \right\rceil.$$

PROOF. From Lemma 2.5 we conclude $L \le (z+y-1)q^y - (x-1)\begin{bmatrix} y \\ 1 \end{bmatrix}_q$ and $L \equiv -(x-1)\begin{bmatrix} y \\ 1 \end{bmatrix}_q \pmod{q^y}$ for the number of holes of a certain $(n-t+y)$-dimensional subspace U of $\mathbb{F}_q^n$. $\mathcal{P}_U := \{P \cap U \mid P \in \mathcal{P}\}$ is of hole-type (t, y, L) if $y \ge 2$. Next, we will show that $\tau_q(c, \Delta, m) \le 0$, where $\Delta = q^{y-1}$ and $c = iq^y - (x-1)\begin{bmatrix} y \\ 1 \end{bmatrix}_q$ with $1 \le i \le z+y-1$, for suitable integers x and m. Note that, in order to apply Lemma 2.5, we have to satisfy $x \ge 2$ and $y \ge f$ for all integers f with $q^f | x-1$. Applying Lemma 2.7 then gives the desired contradiction, so that $A_q(n, 2t; t) \le lq^t + x - 1$.

We choose[3] $m = i(q-1) - (x-1) + 1$, so that $\tau_q(c, \Delta, m) = x^2 - (2\lambda+1)x + \lambda(i(q-1)+2)$. Solving $\tau_q(c, \Delta, m) = 0$ for x gives $x_0 = \lambda + \frac{1}{2} \pm \frac{1}{2}\theta(i)$, where $\theta(i) = \sqrt{1 - 4i\lambda(q-1) + 4\lambda(\lambda-1)}$. We have $\tau_q(c, \Delta, m) \le 0$ for $|2x - 2\lambda - 1| \le \theta(i)$. We need to find an integer $x \ge 2$ such that this inequality is satisfied for all $1 \le i \le z + y - 1$. The strongest restriction is attained for $i = z + y - 1$. Since $z + y - 1 \le \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ and $u = q^y \ge q^r$, we have $\theta(i) \ge \theta(z + y - 1) \ge 1$, so that $\tau_q(c, \Delta, m) \le 0$ for $x = \left\lceil u + \frac{1}{2} - \frac{1}{2}\theta(z+y-1) \right\rceil$. (Observe $x \le \lambda + \frac{1}{2} + \frac{1}{2}\theta(z+y-1)$ due to $\theta(z+y-1) \ge 1$.) Since $x \le \lambda + 1$, we have $x - 1 \le \lambda = q^y$, so that $q^f | x-1$ implies $f \le y$ provided $x \ge 2$. The latter is true due to $\theta(z+y-1) \le \sqrt{1 - 4\lambda(q-1) + 4\lambda(\lambda-1)} \le \sqrt{1 + 4\lambda(\lambda-2)} < 2(\lambda - 1)$, which implies $x \ge \left\lceil \frac{3}{2} \right\rceil = 2$.

So far we have constructed a suitable $m \in \mathbb{Z}$ such that $\tau_q(c, \Delta, m) \le 0$ for $x = \left\lceil \lambda + \frac{1}{2} - \frac{1}{2}\theta(z+y-1) \right\rceil$. If $\tau_q(c, \Delta, m) < 0$, then Lemma 2.7 gives a contradiction, so that we assume $\tau_q(c, \Delta, m) = 0$ in the following. If $i < z + y - 1$ we have $\tau_q(c, \Delta, m) < 0$ due to $\theta(i) > \theta(z + y - 1)$, so that we assume $i = z + y - 1$. Thus, $\theta(z + y - 1) \in \mathbb{N}_0$. However, we can write $\theta(z + y - 1)^2 = 1 + 4\lambda\left(\lambda - (z + y - 1)(q - 1) - 1\right) = (2w-1)^2 = 1 + 4w(w-1)$ for some integer w. If $w \notin \{0, 1\}$, then $\gcd(w, w-1) = 1$, so that either $\lambda = q^y \mid w$ or $\lambda = q^y \mid w - 1$. Thus, in any case, $w \ge q^y$, which is impossible since $(z + y - 1)(q - 1) \ge 1$. Finally, $w \in \{0, 1\}$ implies $w(w-1) = 0$, so that $\lambda - (z + y - 1)(q - 1) - 1 = 0$. Thus, $z + y - 1 = \begin{bmatrix} y \\ 1 \end{bmatrix}_q \ge \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ since $y \ge r$. The assumptions $y \le t$ and $t = \begin{bmatrix} r \\ 1 \end{bmatrix}_q + 1 - z$ imply $z + y - 1 = \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ and $y = r$. This gives $t = r$, which is excluded. □

Setting $y = t$ in Theorem 2.10 yields [4, Corollary 8], which is based on [3, Theorem 1B]. And indeed, our analysis is very similar to the technique[4] used in [3]. Compared to [3, 4], the new ingredients essentially are lemmas 2.2 and 2.3, see also [14, Proof of Lemma 9]. [4, Corollary 8], e.g., gives $A_2(15, 12; 6) \le 516$, $A_2(17, 14; 7) \le 1028$, and $A_9(18, 16; 8) \le 3486784442$, while Theorem 2.10 gives $A_2(15, 12; 6) \le 515$, $A_2(17, 14; 7) \le 1026$, and $A_9(18, 16; 8) \le 3486784420$. Postponing the details and proofs to a more extensive and technical paper [12], we state:

- $2^4 l + 1 \le A_2(4k + 3, 8; 4) \le 2^4 l + 4$, where $l = \frac{2^{4k-1}-2^3}{2^4-1}$ and $k \ge 2$, e.g., $A_2(11, 8; 4) \le 132$;
- $2^6 l + 1 \le A_2(6k + 4, 12; 6) \le 2^6 l + 8$, where $l = \frac{2^{6k-2}-2^4}{2^6-1}$ and $k \ge 2$, e.g., $A_2(16, 12; 6) \le 1032$;
- $2^6 l + 1 \le A_2(6k + 5, 12; 6) \le 2^6 l + 18$, where $l = \frac{2^{6k-1}-2^5}{2^6-1}$ and $k \ge 2$, e.g., $A_2(17, 12; 6) \le 2066$;
- $3^4 l + 1 \le A_3(4k + 3, 8; 4) \le 3^4 l + 14$, where $l = \frac{3^{4k-1}-3^3}{3^4-1}$ and $k \ge 2$, e.g., $A_3(11, 8; 4) \le 2201$;

[3] Solving $\frac{\partial \tau_q(c,\Delta,m)}{\partial m} = 0$, i.e., minimizing $\tau_q(c, \Delta, m)$, yields $m = i(q-1) - (x-1) + \frac{1}{2} + \frac{x-1}{q^y}$. For $y \ge r$ we can assume $x - 1 < q^y$ due the known constructions for partial spreads, so that up-rounding yields the optimum integer choice. For $y < r$ the interval $\left[u + \frac{1}{2} - \frac{1}{2}\theta(i), u + \frac{1}{2} + \frac{1}{2}\theta(i)\right]$ may contain no integer.

[4] Actually, their analysis grounds on [16] and is strongly related to the classical second-order Bonferroni Inequality [2, 7, 8] in Probability Theory, see e.g. [11, Section 2.5] for another application for bounds on subspace codes.

- $3^5l+1 \le A_3(5k+3,10;5) \le 3^5l+13$, where $l=\frac{3^{5k-2}-3^5}{3^3-1}$ and $k\ge 2$, e.g., $A_3(13,10;5)\le 6574$;
- $3^5l+1 \le A_3(5k+4,10;5) \le 3^5l+44$, where $l=\frac{3^{5k-1}-3^4}{3^5-1}$ and $k\ge 2$, e.g., $A_3(14,10;5)\le 19727$;
- $3^6l+1 \le A_3(6k+4,12;6) \le 3^6l+41$, where $l=\frac{3^{6k-2}-3^4}{3^6-1}$ and $k\ge 2$, e.g., $A_3(16,12;6)\le 59090$;
- $3^6l+1 \le A_3(6k+5,12;6) \le 3^6l+133$, where $l=\frac{3^{6k-1}-3^5}{3^6-1}$ and $k\ge 2$, e.g., $A_3(17,12;6)\le 177280$;
- $3^7l+1 \le A_3(7k+4,14;7) \le 3^7l+40$, where $l=\frac{3^{7k-3}-3^4}{3^7-1}$ and $k\ge 2$, e.g., $A_3(18,14;7)\le 177187$;
- $4^5l+1 \le A_4(5k+3,10;5) \le 4^5l+32$, where $l=\frac{4^{5k-2}-4^3}{4^5-1}$ and $k\ge 2$, e.g., $A_4(13,10;5)\le 65568$;
- $4^6l+1 \le A_4(6k+3,12;6) \le 4^6l+30$, where $l=\frac{4^{6k-3}-4^3}{4^6-1}$ and $k\ge 2$, e.g., $A_4(15,12;6)\le 262174$;
- $4^6l+1 \le A_4(6k+5,12;6) \le 4^6l+548$, where $l=\frac{4^{6k-1}-4^5}{4^6-1}$ and $k\ge 2$, e.g., $A_4(17,12;6)\le 4194852$;
- $4^7l+1 \le A_4(7k+4,14;7) \le 4^7l+128$, where $l=\frac{4^{7k-3}-4^4}{4^7-1}$ and $k\ge 2$, e.g., $A_4(18,14;7)\le 4194432$;
- $5^5l+1 \le A_5(5k+2,10;5) \le 5^5l+7$, where $l=\frac{5^{5k-3}-5^2}{5^5-1}$ and $k\ge 2$, e.g., $A_5(12,10;5)\le 78132$;
- $5^5l+1 \le A_5(5k+4,10;5) \le 5^5l+329$, where $l=\frac{5^{5k-1}-5^4}{5^5-1}$ and $k\ge 2$, e.g., $A_5(14,10;5)\le 1953454$;
- $7^5l+1 \le A_7(5k+4,10;5) \le 7^5l+1246$, where $l=\frac{7^{5k-1}-7^2}{7^5-1}$ and $k\ge 2$, e.g., $A_7(14,10;5)\le 40354853$;
- $8^4l+1 \le A_8(4k+3,8;4) \le 8^4l+264$, where $l=\frac{8^{4k-1}-8^3}{8^4-1}$ and $k\ge 2$, e.g., $A_8(11,8;4)\le 2097416$;
- $8^5l+1 \le A_8(5k+2,10;5) \le 8^5l+25$, where $l=\frac{8^{5k-3}-8^2}{8^5-1}$ and $k\ge 2$, e.g., $A_8(12,10;5)\le 2097177$;
- $8^6l+1 \le A_8(6k+2,12;6) \le 8^6l+21$, where $l=\frac{8^{6k-4}-8^2}{8^6-1}$ and $k\ge 2$, e.g., $A_8(14,12;6)\le 16777237$;
- $9^3l+1 \le A_9(3k+2,6;3) \le 9^3l+41$, where $l=\frac{9^{3k-1}-9^2}{9^3-1}$ and $k\ge 2$, e.g., $A_9(8,6;3)\le 59090$;
- $9^5l+1 \le A_9(5k+3,10;5) \le 9^5l+365$, where $l=\frac{9^{5k-2}-9^3}{9^5-1}$ and $k\ge 2$, e.g., $A_9(13,10;5)\le 43047086$;

c.f. the web-page mentioned in footnote 1 for more numerical values and comparisons of the different upper bounds.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BAYREUTH, 95440 BAYREUTH, GERMANY
E-mail address: sascha.kurz@uni-bayreuth.de