Moduli Spaces of Varieties with Symmetries

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Abstract

In this thesis several topics on moduli problems of varieties with symmetries are treated.
We show the existence of the coarse moduli scheme $\mathfrak{M}_h[G]$ for Gorenstein canonical models with Hilbert polynomial h which admit an effective action by a given finite group G. We also introduce a canonical representation type decomposition $\mathcal{D}_h[G]$ of $\mathfrak{M}_h[G]$ which is useful in understanding the structure of $\mathfrak{M}_h[G]$.
We explain the method to determine the irreducible components of $\mathfrak{M}_g(G)$, the locus inside $\mathfrak{M}_g$ of smooth curves with an effective action by a finite group G. We do explicit computations for the cases where G is a cyclic or dihedral group.

Zusammenfassung

Ziel der vorliegenden Arbeit ist die Untersuchung von Modulraumproblemen für Varietäten mit einer effektiven Gruppenwirkung einer gegebenen endlichen Gruppe G. Die Arbeit besteht aus drei Teilen.

Im ersten Teil (Abschnitt 2 und 3) konstruieren wir das grobe Modulraum Schema für G-markierte Varietäten: Unter einer G-markierten Varietät verstehen wir ein Tripel (X, G, α), wobei X eine projektive Varietät, G eine endliche Gruppe und $\alpha : G \times X \to X$ eine treue Wirkung ist. Genauer definieren wir den Modulfunktor M_h^G, G-markierter kanonischer Modelle mit Hilbert Polynom h und beweisen, dass es ein quasiprojektives Schema $\mathfrak{M}_h[G]$ gibt, welches grober Modulraum für M_h^G ist. Des weiteren zeigen wir die Existenz eines eigentlichen, endlichen Morphismus von $\mathfrak{M}_h[G]$ auf den gewöhnlichen Modulraum $\mathfrak{M}_h$, so dass das Bild $\mathfrak{M}_h(G)$, welches Varietäten entspricht, die eine effektive Gruppenwirkung von G besitzen, abgeschlossen in $\mathfrak{M}_h$ ist. Für weitere Details, siehe Abschnitt 2 und 3.

Im zweiten Teil (Abschnitt 4 - Abschnitt 7) bestimmen wir durch explizite Berechnungen die irreduziblen Komponenten von $\mathfrak{M}_g(G)$ in $\mathfrak{M}_g$, dem Modulraum algebraischer Kurven vom Geschlecht $g \geq 2$, für bestimmte elementare Gruppen. Dies ist durch die Anwendung motiviert, die Struktur des singulären Orts von $\mathfrak{M}_g$ zu bestimmen. Es ist wohl bekannt, dass der singuläre Ort von $\mathfrak{M}_g$ aus Kurven mit nicht-trivialer

Automorphismengruppe besteht. In [Cor87] und [Cor08] gab Cornalba eine Beschreibung der irreduziblen Komponenten von Sing($\mathfrak{M}_g$), indem er die maximalen Orte glatter Kurven mit einer Wirkung einer zyklischen Gruppe von Primzahlordnung untersuchte. Später untersuchte Catanese in [Cat12] das analoge Problem für stabile Kurven und bestimmte die Komponenten des singulären Orts des Randes von $\overline{\mathfrak{M}}_g$, dem kompaktifizierten Modulraum.
In Abschnitt 4 wiederholen wir die allgemeine Theorie zur Bestimmung der irreduziblen Komponenten von $\mathfrak{M}_g(G)$, die wichtigsten Hilfsmitttel hier sind Hurwitz Vektoren, assoziiert zu Galois Überlagerungen von Kurven mit Galois Gruppe G.
In Abschnitt 5 bestimmen wir, unter Zuhilfenahme der Strukturtheorie zyklischer Überlagerungen, die nicht-vollen zyklischen Untergruppen der Abbildungsklassengruppe Map_g.
In Abschnitt 6 und 7 bestimmen wir die irreduziblen Komponenten von $\mathfrak{M}_g(D_n)$, wobei D_n die Diedergruppe ist, basierend auf den Resultaten für Äquivalenzklassen von D_n-Hurwitz Vektoren in [CLP11].

Im dritten Teil (Abschnitt 8) führen wir die kanonische Zerlegung nach dem Darstellungstyp $\mathcal{D}_h[G]$ von $\mathfrak{M}_h[G]$ ein. Wir verwenden Hilbert Auflösungen des kanonischen Rings G-markierter Varietäten um $\mathcal{D}_h[G]$ zu untersuchen. Des Weiteren, im Falle algebraischer Kurven, wenden wir die Chevalley-Weil Formel auf G-markierte Kurven an und geben viele interessante Beispiele.

Key words: Coarse moduli space, G-marked variety, Gorenstein canonical model, Moduli of curves.

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1. **Introduction**

The aim of this thesis is to investigate moduli problems of varieties which admit an effective action by a given finite group G. The thesis consists of three parts: in the first part (Section 2 and Section 3) we construct the coarse moduli scheme $\mathfrak{M}_h[G]$ for G-marked canonical models with Hilbert polynomial h; in the second part (Section 4 - Section 7) we do explicit computations to determine the irreducible components of $\mathfrak{M}_g(G)$, the locus inside $\mathfrak{M}_g$ of the curves of genus g admitting an effective action by certain elementary groups G; in the third part (Section 8) we introduce the canonical representation type decomposition $\mathcal{D}_h[G]$ of $\mathfrak{M}_h[G]$.

1.1. **G-marked moduli spaces.** The moduli theory of algebraic varieties was motivated by the attempt to fully understand Riemann's assertion in [Rie57] that the isomorphism classes of Riemann surfaces of genus $g > 1$ depend on $(3g-3)$ parameters (called "moduli"). The modern approach to moduli problems via functors was developed by Grothendieck and Mumford (cf. [MF82]), and later by Gieseker, Kollár, Viehweg, et al (cf. [Gie77], [Kol13], [Vie95]). The idea is to define a moduli functor for the given moduli problem and study the representability of the moduli functor via an algebraic variety or some other geometric object. For instance, in the case of smooth projective curves of genus $g \geq 2$, we consider the (contravariant) functor $\mathcal{M}_g$ from the category of schemes to the category of sets, such that

(1) For any scheme T, $\mathcal{M}_g(T)$ consists of the T-isomorphism classes of flat projective families of curves of genus g over the base T.
(2) Given a morphism $f : S \to T$, $\mathcal{M}_g(f) : \mathcal{M}_g(T) \to \mathcal{M}_g(S)$ is the map associated to the pull back.

It has been shown by Mumford that there exists a quasi-projective *coarse moduli scheme* $\mathfrak{M}_g$ for the functor $\mathcal{M}_g$ (cf. [Mum62]), in the following sense:
there exists a natural transformation $\eta : \mathcal{M}_g \to \mathrm{Hom}(-, \mathfrak{M}_g)$, such that

$$\eta_{\mathbf{Spec}(\mathbb{C})} : \mathcal{M}_g(\mathbf{Spec}(\mathbb{C})) \to \mathrm{Hom}(\mathbf{Spec}(\mathbb{C}), \mathfrak{M}_g)$$

is bijective and η is universal among such natural transformations. This means that the closed points of $\mathfrak{M}_g$ are in one-to-one correspondence

with the isomorphism classes of curves of genus g and given a family $\mathfrak{X} \to T$ of curves of genus g, we have a morphism (induced by η) from T to $\mathfrak{M}_g$ such that any (closed) point $t \in T$ is mapped to $[\mathfrak{X}_t]$ in $\mathfrak{M}_g$. The definitions are the same in higher dimensions, if one replaces curves of genus $g \geq 2$ by Gorenstein varieties with ample canonical classes. The existence of a coarse moduli space is then more difficult to prove, we refer to [Vie95] and [Kol13] for further discussions.
For several purposes, it is important to generalize the method to moduli problems of varieties which admit an effective action by a given finite group G. Here we consider the concept of a *G-marked variety*, which is a triple (X,G,α) such that X is a projective variety and $\alpha : G \times X \to X$ is a faithful action. The isomorphisms between G-marked varieties are G-equivariant isomorphisms (for more details, see Definition 2.1). In similarity to the case of $\mathcal{M}_g$, we study in this thesis the moduli functor M_h^G of G-marked Gorenstein canonical models with Hilbert polynomial h such that, for any scheme T, $\mathsf{M}_h^G(T)$ is the set of T-isomorphism classes of G-marked flat families of Gorenstein canonical models with Hilbert polynomial h over the base T, and given a morphism $f : S \to T$, $\mathsf{M}_h^G(f)$ is the map associated to the set of pull-backs (cf. Definition 2.4). We refer to the recently published survey article [Cat15, Section 10], for some applications in the case of algebraic curves and surfaces; there the author discusses several topics on the theory of G-marked curves and sketches the construction of the moduli space of G-marked canonical models of surfaces.
The main result of the first part is the following (also see Theorem 3.1):

Theorem 1.1. *Given a finite group G and a Hilbert polynomial $h \in \mathbb{Q}[t]$, there exists a quasi-projective coarse moduli scheme $\mathfrak{M}_h[G]$ for M_h^G, the moduli functor of G-marked Gorenstein canonical models with Hilbert polynomial h.*

This part is arranged as follows:
In Section 2 we introduce the definition of "G-marked varieties" and the associated moduli problem by defining the moduli functor M_h^G for a given group G and Hilbert polynomial h.
In Section 3 we first study two basic properties (boundedness and local closedness) of the moduli functor M_h^G.
Recall that a moduli functor of varieties M is called *bounded* if the objects in $\mathsf{M}(\mathbf{Spec}(\mathbb{C}))$ are parameterized by a finite number of families

(cf. Definition 3.2). In Corollary 3.23 we show that M_h^G is bounded by a family $U^G_{N,h'} \to H^G_{N,h'}$ over an appropriate subscheme of a Hilbert scheme.
However the family $U^G_{N,h'} \to H^G_{N,h'}$ that we get in Corollary 3.23 may not belong to $\mathsf{M}_h^G(H^G_{N,h'})$, i.e., not every fibre of the family is a G-marked canonical model. Here comes the problem of local closedness: roughly speaking, a moduli functor M of varieties is called *locally closed* if for any flat projective family $\mathfrak{X} \to T$, the subset $\{t \in T | [\mathfrak{X}_t] \in \mathsf{M}(\mathbf{Spec}(\mathbb{C}))\}$ is locally closed in T (see Definition 3.25 for more details). We solve this problem in Proposition 3.26 by taking a locally closed subscheme $\bar{H}^G_{N,h'}$ of $H^G_{N,h'}$ and considering the restriction of $U^G_{N,h'} \to H^G_{N,h'}$ to $\bar{H}^G_{N,h'}$.
Then we apply Geometric Invariant Theory, obtaining the quotient $\mathfrak{M}_h[G]$ of $\bar{H}^G_{N,h'}$ by some reductive groups and prove that $\mathfrak{M}_h[G]$ is the coarse moduli scheme for our moduli functor M_h^G.

1.2. **Irreducible components of $\mathfrak{M}_g(G)$.** In the second part we introduce the method to determine the irreducible components of $\mathfrak{M}_g(G)$, the locus inside $\mathfrak{M}_g$ of the curves admitting an effective action by a given finite group G. Moreover we do explicit computations for cyclic and dihedral groups.
The computation is motivated by the application in determining the structure of the singular locus of $\mathfrak{M}_g$. It is well-known that the singular locus of $\mathfrak{M}_g$ consists of curves with non-trivial automorphism groups. In [Cor87] and [Cor08] Cornalba gave a description of the irreducible components of $Sing(\mathfrak{M}_g)$ by studying the maximal loci of smooth curves admitting actions by cyclic groups of prime order. Later in [Cat12] Catanese studied the analogous problem for stable curves and determined the components of the singular locus of the boundary of $\overline{\mathfrak{M}_g}$, the compactified moduli space.
Given two G-marked projective curves (C_1, G) and (C_2, G) of genus g, they are said to have the same *unmarked topological type* if the underlying (ramified) topological covers of $C_1 \to C_1/G$ and $C_2 \to C_2/G$ are isomorphic (see Section 4). Given a topological type $[\rho]$, the locus $\mathfrak{M}_{g,\rho}(G)$ inside $\mathfrak{M}_g(G)$ of curves admitting an effective action by G with the topological type $[\rho]$ is irreducible and closed (cf. [CLP15],

Theorem 2.3), hence we see that $\mathfrak{M}_g(G)$ is a union of irreducible closed subsets:

$$\mathfrak{M}_g(G) = \bigcup_{[\rho]} \mathfrak{M}_{g,\rho}(G).$$

Then the problem of determining the irreducible components of $\mathfrak{M}_g(G)$ is then equivalent to determining when one locus of the form $\mathfrak{M}_{g,\rho}(G)$ contains another. Using Teichmüller theory, this problem can be interpreted as classifying the pair of subgroups H, H' of Map_g, the mapping class group, satisfying the following condition (cf. Section 4):

$(**)$ H is isomorphic to G, $H \neq G(H)$ and $G(H)$ has a subgroup H',

which is isomorphic to G and different from H,

where $G(H) := \bigcap_{C \in Fix(H)} Aut(C)$ and H acts on the Teichmüller space $\mathcal{T}_g$ as a subgroup of Map_g.
The main tool we use in this part is the *Hurwitz vector*, roughly speaking, a Hurwitz vector associated to a covering $C \to C/G$ is a vector with entries in G which records the ramification behavior of the covering (cf. Definition 4.2). The set of topological types is then in one-to-one correspondence with the set of orbits of Hurwitz vectors by the action of certain groups (for more details, see Definition 4.4).
The main result of this part is the following (cf. Theorem 7.14):

Theorem 1.2. *Let $G = D_n$ be the dihedral group of order $2n$ and let H, H' be subgroups of Map_g, satisfying the condition $(**)$ and assume $\delta_H := \dim Fix(H) \geq 1$.*

Then $G(H) \simeq D_n \times \mathbb{Z}/2$ and H corresponds to $D_n \times \{0\}$. The group H' and the topological action of the group $G(H)$ (i.e. its Hurwitz vector) are as listed in the tables in 7.4.

In Section 4 we recall the general theory concerning the determination of the irreducible components of $\mathfrak{M}_g(G)$.
In Section 5, using the structure theory of cyclic covers, we determine the non-full cyclic subgroups of the mapping class group Map_g.
In Sections 6 and 7 we determine the irreducible components of $\mathfrak{M}_g(D_n)$, where D_n is the dihedral group, based on the classification results of the equivalence classes of D_n-Hurwitz vectors given in [CLP11].

1.3. **The canonical representation type decomposition of $\mathfrak{M}_h^G$.** In this part we study the moduli scheme $\mathfrak{M}_h[G]$ (cf. Theorem 1.1) that we have obtained in the first part via its canonical representation type decomposition.

For a G-marked canonical model (X, G, ρ) (with Hilbert polynomial h) and a natural number k, we have an induced representation $\rho_k : G \to Aut(H^0(X, \omega_X^k))$. The *canonical representation type* of (X, G, ρ) is the set of representations $\{\rho_k\}$ for all k large enough (more precisely, the k's satisfying Matsusaka's big theorem, cf. [Mat86]).
One observes that the set of varieties inside $\mathfrak{M}_h[G]$ with a fixed representation type is a union of connected components of $\mathfrak{M}_h[G]$ (cf. [Cat13, Prop37]), hence we obtain a *canonical representation type decomposition* $\mathcal{D}_h[G]$ of $\mathfrak{M}_h[G]$ (cf. Definition 8.1) which decomposes $\mathfrak{M}_h[G]$ into subsets consisting of varieties of the same canonical representation type.

The first result of this part is that the decomposition $\mathcal{D}_h[G]$ depends only on finitely many ρ_k's. To be more precise, in Proposition 8.8 we show that there exists a natural number $N = N(h, G)$ (in fact, we give an effective bound), depending on the Hilbert polynomial h and the group G, such that for any $k \geq N$, the representation ρ_k is determined by the ρ_i's with $i \leq N$. The main idea here is to consider the Hilbert resolutions of the canonical rings of G-marked varieties with Hilbert polynomial h (cf. Lemma 8.4).

Then we study the case of G-marked curves, where we have better estimates and several interesting examples. The first recipe we use here is the Chevalley-Weil formula (cf. Theorem 8.10): in Corollary 8.12 we show that ρ_k determines $\rho_{k+|G|}$, while in Example 8.13 we give an example showing that ρ_i and ρ_j may determine different decompositions if $|i-j| < |G|$.
We are also interested in how far is the decomposition $\mathcal{D}_g[G]$ from the decomposition of $\mathfrak{M}_g[G]$ into connected components. In the case where G is a nonabelian metacyclic group, we give an estimate of the number of connected components inside $(\mathfrak{M}_g[G])_{r,r}$, the component of the regular representation (cf. Definition 8.15), showing that the component of $\mathcal{D}_g[G]$ may not be connected (cf. Proposition 8.17).

2. **G-marked varieties**

In this paper we work over the complex field $\mathbb{C}$. By a "scheme" we mean a separated scheme of finite type over $\mathbb{C}$, a point in a scheme is assumed to be a closed point. Moreover, G shall always denote a finite group.

Definition 2.1 ([Cat15], Definition 181)**.** (1) A *G-marked (projective) variety (resp. scheme)* is a triple (X, G, ρ) where X is a projective variety (resp. scheme) and $\rho : G \to \mathrm{Aut}(X)$ is an injective homomorphism. Or equivalently, it is a triple (X, G, α) where $\alpha : X \times G \to X$ is a faithful action of G on X.
(2) A morphism f between two G-marked varieties (X, G, ρ) and (X', G, ρ') is a G-equivariant morphism $f : X \to X'$, i.e., $\forall g \in G, f \circ \rho(g) = \rho'(g) \circ f$.
(3) A family of G-marked varieties (resp. schemes) is a triple $((p : \mathfrak{X} \to T), G, \rho)$, where G acts faithfully on $\mathfrak{X}$ via an injective homomorphism $\rho : G \to \mathrm{Aut}(\mathfrak{X})$ and trivially on T; p is flat, projective, G-equivariant; and $\forall t \in T$, the induced triple $(\mathfrak{X}_t, G, \rho_t)$ is a G-marked variety (resp. scheme).
(4) A morphism between two G-marked families $((p : \mathfrak{X} \to T), G, \rho)$ and $((p' : \mathfrak{X}' \to T'), G, \rho')$ is a commutative diagram:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\tilde{f}} & \mathfrak{X}' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ T & \xrightarrow{f} & T' \end{array}$$

where $\tilde{f} : \mathfrak{X} \to \mathfrak{X}'$ is a G-equivariant morphism.
(5) Let $((p : \mathfrak{X} \to T), G, \rho)$ be a G-marked family and let $f : S \to T$ be a morphism. Denoting by $\mathfrak{X}_S$ (or $f^*\mathfrak{X}$) the fibre product of f and p, the action ρ induces a G-action ρ_S (or $f^*\rho$) on $\mathfrak{X}_S$ such that $((p_S : \mathfrak{X}_S \to S), G, \rho_S) =: f^*((p : \mathfrak{X} \to T), G, \rho)$ is again a G-marked family.

Remark 2.2. Observe that, given a flat family of varieties $\mathfrak{X} \to T$ with a group G acting on each fibre, we do not yet have a G-marked family, i.e., we may not find an action of G on $\mathfrak{X}$. For any point $t \in T$, we can find a suitable analytic neighbourhood D_t such that the action of G on $\mathfrak{X}_t$ can be extended to an action on $\mathfrak{X}|_{D_t} \to D_t$. However if one wants to extend the action to the whole family, there comes the problem of monodromy: for another point $t' \in T$, the extensions along two different paths connecting t and t' may not result in the same action on $\mathfrak{X}_{t'}$.

Definition 2.3. A normal projective variety X is called a *canonical model* if X has canonical singularities (cf. [Rei87]) and K_X is ample.

Definition 2.4. Denote by $\mathfrak{Sch}$ the category of schemes (over $\mathbb{C}$). The *moduli functor of G-marked Gorenstein canonical models* with Hilbert polynomial $h \in \mathbb{Q}[t]$ is a contravariant functor:

$$\mathsf{M}_h^G : \mathfrak{Sch} \to \mathfrak{Sets}, \ \textit{such that}$$

(1) For any scheme T,

$$\mathsf{M}_h^G(T) := \{((p : \mathfrak{X} \to T), G, \rho) |\ p \text{ is flat and projective, all fibres of p are canonical models, } \omega_{\mathfrak{X}/T} \text{ is invertible, } \forall t \in T, \forall k \in \mathbb{N}, \chi(\mathfrak{X}_t, \omega_{\mathfrak{X}_t}^k) = h(k)\}/\simeq$$

where "$\simeq$" is the equivalence relation given by the isomorphisms of G-marked families over T (i.e., in the commutative diagram of 2.1 (4), take $T' = T$ and $f = id_T$).
(2) Given $f \in \mathrm{Hom}(S, T)$, $\mathsf{M}_h^G(f) : \mathsf{M}_h^G(T) \to \mathsf{M}_h^G(S)$ is the map associated to the pull back, i.e.,

$$[((p : \mathfrak{X} \to T), G, \rho)] \mapsto [((p_S : \mathfrak{X}_S \to S), G, \rho_S)].$$

Remark 2.5. In this article, whenever we write $((\mathfrak{X} \to T), G, \rho) \in \mathsf{M}_h^G(T)$, we mean choosing a representative $((\mathfrak{X} \to T), G, \rho)$ from the isomorphism class $[((\mathfrak{X} \to T), G, \rho)] \in \mathsf{M}_h^G(T)$.

In the case where G is trivial, we denote by M_h the corresponding functor.

3. The coarse moduli space $\mathfrak{M}_h[G]$

We have defined the moduli functor M_h^G of G-marked Gorenstein canonical models with Hilbert polynomial h in the previous section (cf. 2.4), in this section we show the existence of a coarse moduli scheme $\mathfrak{M}_h^G$ for M_h^G. The main theorem of this section is the following:

Theorem 3.1. *Given a finite group G and a Hilbert polynomial $h \in \mathbb{Q}[t]$, there exists a quasi-projective coarse moduli scheme $\mathfrak{M}_h[G]$ for M_h^G, the moduli functor of G-marked Gorenstein canonical models with Hilbert polynomial h.*

3.1. Basic properties of M_h^G.

In this section we study two important properties of the moduli functor M_h^G: boundedness and local closedness. The main results are (3.22), (3.23) for boundedness and (3.26) for local closedness.

Definition 3.2. A moduli functor M of varieties is called *bounded* if there exists a flat and projective family $\mathfrak{U} \to S$ over a scheme S such that $\forall [X] \in \mathsf{M}(\mathbf{Spec}(\mathbb{C}))$, X is isomorphic to a fibre $\mathfrak{U}_s$ for some $s \in S$. (For a stronger definition, see [Kov09], Definition 5.1)

In the case where G is trivial boundedness is already known (cf. [Kar00], [Mat86]). However we can not apply it directly to the general case since we have an action by G. Here we introduce the notion of "bundle of G-frames" to solve this problem.
Let Y be a scheme and $\mathcal{E}$ a locally free sheaf of rank n on Y. Set

$$\mathbb{V}(\mathcal{E}) := \mathbf{Spec}_Y Sym(\mathcal{E}^\vee),$$

the geometric vector bundle associated to $\mathcal{E}$ over Y (cf. [Har77][1], Exercise II.5.18).

Definition 3.3 (Frame Bundle)**.** Let $\mathcal{E}$ be a locally free sheaf of rank n on a scheme Y. In this paper we call (what is in bundle theory called) the principal bundle associated to $\mathbb{V}(\mathcal{E})$ the *frame bundle* $\mathcal{F}(\mathcal{E})$ of $\mathcal{E}$ over Y. For any $y \in Y$, the fibre $\mathcal{F}(\mathcal{E})_y$ over y is called the set of frames (i.e., bases) for the vector space $\mathcal{E} \otimes \mathbb{C}(y)$.
Hence $\mathcal{F}(\mathcal{E})$ is the open subscheme of $\mathbb{V}(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y^n, \mathcal{E}))$ such that $\forall y \in Y$, the fibre $\mathcal{F}(\mathcal{E})_y$ corresponds to the invertible homomorphisms.

[1]The bundle $\mathbb{V}(\mathcal{E})$ defined here is in fact dual to that of [Har77].

We denote a point in $\mathcal{F}(\mathcal{E})$ as a pair (y, ψ), where y is a point in Y and $\psi : \mathbb{C}^n \to \mathcal{E} \otimes \mathbb{C}(y)$ is an isomorphism of $\mathbb{C}$-vector spaces.

Remark 3.4. Let $\mathcal{E}$ be a locally free sheaf of rank n on Y: $\mathbb{V}(\mathcal{E})$ admits a local trivialization $(\{U_\alpha\}, g_{\alpha\beta})$, where $\{U_\alpha\}$ is an open covering of Y such that $\mathbb{V}(\mathcal{E})|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^n$, and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{C})$ are cocycles for $\mathbb{V}(\mathcal{E})$. We have that $\mathcal{F}(\mathcal{E})|_{U_\alpha} \simeq U_\alpha \times GL(n, \mathbb{C})$ and the maps

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \times GL(n, \mathbb{C}) \to U_\alpha \cap U_\beta \times GL(n, \mathbb{C}),\ (y, M) \mapsto (y, g_{\alpha\beta}(y)M)$$

are the gluing morphisms of $\mathcal{F}(\mathcal{E})$.

Proposition 3.5. *Let $\mathcal{E}$ be a locally free sheaf of rank n on a scheme Y and $p : \mathcal{F}(\mathcal{E}) \to Y$ the natural projection. There exists a tautological isomorphism $\phi_{\mathcal{E}} : \mathcal{O}^n_{\mathcal{F}(\mathcal{E})} \to p^*\mathcal{E}$ of sheaves on $\mathcal{F}(\mathcal{E})$ such that for any point $z := (y, \psi) \in \mathcal{F}(\mathcal{E})$, $\phi_{\mathcal{E}}|_{\{z\}} = \psi$ via the isomorphism $Hom(\mathbb{C}^n, p^*\mathcal{E} \otimes \mathbb{C}(z)) \simeq Hom(\mathbb{C}^n, \mathcal{E} \otimes \mathbb{C}(y))$.*

Proof. This proposition is well known (the idea is similar to that of [Gro58]). Observe that $p^*\mathcal{E}$ has n global sections $s_1(\mathcal{E}), ..., s_n(\mathcal{E})$ such that for any $z = (y, \psi) \in \mathcal{F}(\mathcal{E})$, $s_i(\mathcal{E}) \otimes \mathbb{C}(z) = \psi(e_i)$, where $\{e_i\}_{i=1}^n$ is the canonical basis of $\mathbb{C}^n$ and we identify $p^*\mathcal{E} \otimes \mathbb{C}(z)$ with $\mathcal{E} \otimes \mathbb{C}(y)$. Then the universal basis morphism $\phi_{\mathcal{E}} := (s_1(\mathcal{E}), ..., s_n(\mathcal{E})) : \mathcal{O}^n_{\mathcal{F}(\mathcal{E})} \to p^*\mathcal{E}$ is an isomorphism of locally free sheaves. □

Remark 3.6. The set of sections $\{s_i(\mathcal{E})\}_{i=1}^n$ (or equivalently, the isomorphism $\phi_{\mathcal{E}}$) satisfies the following compatibility conditions:
(1) Let $f : X \to Y$ be a morphism and let $f_{\mathcal{F}} : \mathcal{F}(f^*\mathcal{E}) \to \mathcal{F}(\mathcal{E})$ be the induced morphism: we have that $f^*_{\mathcal{F}}(s_i(\mathcal{E})) = s_i(f^*\mathcal{E})$.
(2) Given an isomorphism $l : \mathcal{E}_1 \to \mathcal{E}_2$ of locally free sheaves on Y, the induced isomorphism $l_{\mathcal{F}} : \mathcal{F}(\mathcal{E}_1) \to \mathcal{F}(\mathcal{E}_2)$ commutes with the projections $p_j : \mathcal{F}(\mathcal{E}_j) \to Y$, $j = 1, 2$. We have that $l^*_{\mathcal{F}}(\phi_{\mathcal{E}_2}) = p_1^*(l) \circ \phi_{\mathcal{E}_1}$.

Definition 3.7. Let $\mathcal{E}$ be a locally free sheaf of rank n on a scheme Y: we say that a group G acts *faithfully and linearly* on $\mathcal{E}$ if
(1) the action is given by an injective homomorphism $\rho : G \hookrightarrow \mathrm{Aut}_{\mathcal{O}_Y}(\mathcal{E})$;
(2) $\forall y \in Y$, the induced action ρ_y is a faithful G-representation on $\mathbb{C}^n$.
In this case we call the pair $(\mathcal{E}, \rho)$ a *locally free G-sheaf.*

Definition 3.8. (1) Given $\phi \in \mathrm{Aut}(Y)$, let $\Gamma_\phi : Y \to Y \times Y$ be the graph map of ϕ. The *fixpoints scheme* of ϕ (denoted by $\mathrm{Fix}(\phi)$) is the

(scheme-theoretic) inverse image of Δ by Γ_ϕ, where Δ is the diagonal subscheme of $Y \times Y$.
(2) Given a G-action on Y, the *fixpoints scheme* of G on Y is:

$$Y^G := \cap_{g \in G} Fix(\phi_g),$$

where $\phi_g : Y \to Y, y \mapsto gy$.

Remark 3.9. (1) Let $f : X \to Y$ be a G-equivariant morphism between two schemes on which G acts: we have a natural restriction morphism $f|_{X^G} : X^G \to Y^G$.
(2) Given a G-action on Y and a subgroup H of G, there is an induced $C(H)$-action on Y^H, where $C(H)$ is the centralizer group of H in G.

Definition 3.10. Let $(\mathcal{E}, \rho)$ be a locally free G-sheaf of rank n on Y. Given a faithful linear representation $\beta : G \to GL(n, \mathbb{C})$, we define an action (β, ρ) of G on $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y^n, \mathcal{E})$: $\forall g \in G$, open subset $U \subset Y, \phi \in \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y^n, \mathcal{E})(U)$ *and* $s \in \mathcal{O}_Y^n(U)$; $(g\phi)(s) := \rho(g)\phi(\beta(g^{-1})s)$. The action (β, ρ) restricts naturally to $\mathcal{F}(\mathcal{E})$, we denote by $\mathcal{F}(\mathcal{E}, G, \rho; \beta)$ the corresponding fixpoints scheme: it is called the *bundle of G-frames of $\mathcal{E}$ associated to the action ρ with respect to β.*

Remark 3.11. (1) Denoting by $C(G, \beta)$ the centralizer group of $\beta(G)$ in $GL(n, \mathbb{C})$, an easy observation is that $\forall y \in Y$, the fibre $\mathcal{F}(\mathcal{E}, G, \rho; \beta)_y$ corresponds to the set of G-equivariant isomorphisms between the G-linear representations β and ρ_y. Therefore we have that either

$$\mathcal{F}(\mathcal{E}, G, \rho; \beta)_y = \emptyset, \text{ } or \text{ } \mathcal{F}(\mathcal{E}, G, \rho; \beta)_y \simeq C(G, \beta).$$

(2) If $\beta, \beta' : G \to GL(n, \mathbb{C})$ are equivalent representations (i.e., there exists $g \in GL(n, \mathbb{C})$ such that $\beta' = g\beta g^{-1}$), then we have that

$$\mathcal{F}(\mathcal{E}, G, \rho; \beta) \simeq \mathcal{F}(\mathcal{E}, G, \rho; \beta').$$

Observe that if Y is connected and there exists $y \in Y$ such that $\mathcal{F}(\mathcal{E}, G, \rho; \beta)_y \simeq C(G, \beta)$, then $\mathcal{F}(\mathcal{E}, G, \rho; \beta)_{y'} \simeq C(G, \beta)$ for all $y' \in Y$ (See [Cat13], Prop 37), hence we have the following definition:

Definition 3.12. Let Y be a connected scheme and $(\mathcal{E}, \rho)$ a locally free G-sheaf of rank n on Y. We say that $(\mathcal{E}, \rho)$ (or $\mathcal{E}$ if the action is clear from the context) has *decomposition type* β, where $\beta : G \to GL(n, \mathbb{C})$ is a faithful representation, if there exists $y \in Y$, such that $\rho_y \simeq \beta$.

Definition 3.13 (Bundle of G-frames)**.** Let $(\mathcal{E}, \rho)$ be a locally free G-sheaf of rank n on a scheme Y. We define the *bundle of G-frames of $\mathcal{E}$ associated to ρ*, denoted by $\mathcal{F}(\mathcal{E}, G, \rho)$ (or $\mathcal{F}(\mathcal{E}, G)$ when ρ is clear from the context), as follows:
If Y is connected and $\mathcal{E}$ has decomposition type β, then $\mathcal{F}(\mathcal{E}, G, \rho) := \mathcal{F}(\mathcal{E}, G, \rho; \beta)$.
In general we decompose Y into the union of connected components $Y = \sqcup Y_i$ and $\mathcal{F}(\mathcal{E}, G, \rho)$ is the (disjoint) union of all the $\mathcal{F}(\mathcal{E}|_{Y_i}, G, \rho|_{Y_i})$.

Remark 3.14. Since we can vary β in its equivalence class, we see from (3.11-2) that $\mathcal{F}(\mathcal{E}, G)$ is unique up to isomorphisms.

Definition 3.15. Let $(\mathcal{E}, \rho)$ be a *free* G-sheaf of rank n on a scheme Y. The action is said to be *defined over* $\mathbb{C}$ if there exists a G-equivariant isomorphism $\phi : (\mathcal{O}_Y^n, \beta) \to (\mathcal{E}, \rho)$, where $\beta : G \to GL(n, \mathbb{C})$ is a faithful representation.

Proposition 3.16. *Let $(\mathcal{E}, \rho)$ be a locally free G-sheaf of rank n on a connected scheme Y with decomposition type β. The projection $p : \mathcal{F}(\mathcal{E}, G) \to Y$ induces an action $p^*\rho$ on $p^*\mathcal{E}$. Then $(p^*\mathcal{E}, p^*\rho)$ is defined over $\mathbb{C}$: the morphism $\phi_{\mathcal{E},G} := \phi_{\mathcal{E}}|_{\mathcal{F}(\mathcal{E},G)} : (\mathcal{O}^n_{\mathcal{F}(\mathcal{E},G)}, \beta) \to (p^*\mathcal{E}, p^*\rho)$ is a G-equivariant isomorphism, where $\phi_{\mathcal{E}}$ is the universal basis morphism defined in* (3.5).

Proof. It is clear that $\phi_{\mathcal{E},G}$ is an isomorphism of sheaves, what remains to show is that $\phi_{\mathcal{E},G}$ is G-equivariant. Since $\phi_{\mathcal{E},G}$ is an isomorphism of locally free sheaves, it suffices to show that $\forall (y, \psi) \in \mathcal{F}(\mathcal{E}, G)$, $\phi_{\mathcal{E},G}|_{\{(y,\psi)\}}$ is G-equivariant. By our construction in (3.5), we have that $p^{-1}(y) \subset \mathbb{V}(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y^n, \mathcal{E}))_y^G \simeq \mathrm{Hom}(\mathbb{C}^n, \mathcal{E} \otimes \mathbb{C}(y))^G$, where the G-action is (β, ρ_y). Under this isomorphism the point (y, ψ) corresponds exactly to $\phi_{\mathcal{E},G}|_{\{(y,\psi)\}}$, hence $\phi_{\mathcal{E},G}|_{\{(y,\psi)\}}$ is G-equivariant. □

Remark 3.17. Given a locally free G-sheaf $(\mathcal{E}, \rho)$ of rank n on Y, setting $s_i(\mathcal{E}, G) := s_i(\mathcal{E})|_{\mathcal{F}(\mathcal{E},G)}$, then $\{s_i(\mathcal{E}, G)\}$ and $\phi_{\mathcal{E},G}$ have similar properties as $\{s_i(\mathcal{E})\}$ and $\phi_{\mathcal{E}}$ have in (3.6).

Proposition 3.18. *Assume that Y is connected and $(\mathcal{E}, \rho)$ is a locally free G-sheaf of rank n on Y with decomposition type β. Then there is a natural $C(G, \beta)$-action on $\mathcal{F}(\mathcal{E}, G)$ and Y is a categorical quotient of $\mathcal{F}(\mathcal{E}, G)$ by $C(G, \beta)$.*

Proof. To see the $C(G,\beta)$-action, it suffices to notice that the actions β and ρ on $\mathcal{F}(\mathcal{E})$ commute, i.e., $\forall g \in G$, $\beta(g)\rho(g) = \rho(g)\beta(g)$ as elements in $\mathrm{Aut}(\mathcal{F}(\mathcal{E}))$.
From the definition of $\mathcal{F}(\mathcal{E},G)$, one observes that the projection $p : \mathcal{F}(\mathcal{E},G) \to Y$ is affine and $C(G,\beta)$-equivariant, therefore we may assume that $Y, \mathcal{F}(\mathcal{E},G)$ are affine schemes and A (resp. B) is the coordinate ring of Y (resp. $\mathcal{F}(\mathcal{E},G)$). Since p is surjective and $C(G,\beta)$-equivariant, we have that $A \subset B^{C(G,\beta)} \subset B$. Noting that B is a finitely generated $\mathbb{C}$-algebra and $C(G,\beta)$ is a reductive group (cf. 3.20), we conclude that $B^{C(G,\beta)}$ is a finitely generated $\mathbb{C}$-algebra and $\mathbf{Spec} B^{C(G,\beta)}$ is the universal categorical quotient of $\mathcal{F}(\mathcal{E},G)$ by $C(G,\beta)$ (see [MF82], p.27). Now since every fibre of p is a closed $C(G,\beta)$-orbit (in fact isomorphic to $C(G,\beta)$), which must be mapped to a point in $\mathbf{Spec} B^{C(G,\beta)}$, for dimensional reasons we conclude that $B^{C(G,\beta)}$ is a finite A-module. For any maximal ideal m of A, by the proposition of a universal categorical quotient (cf. [MF82], p.4) we see that $\mathbf{Spec}(B^{C(G,\beta)} \otimes_A \mathbb{C}(m))$ is the categorical quotient of $p^{-1}(\mathbf{Spec}(\mathbb{C}(m))) \simeq C(G,\beta)$ by $C(G,\beta)$, hence $B^{C(G,\beta)} \otimes_A \mathbb{C}(m) = \mathbb{C}$, which implies that $(B^{C(G,\beta)}/A) \otimes_A \mathbb{C}(m) = 0$. By Nakayama's lemma, we have that $(B^{C(G,\beta)}/A)_m = 0$, which implies that $A = B^{C(G,\beta)}$. □

Before stating the Boundedness theorem, let us first recall the action of general linear groups on Hilbert schemes (cf. [Vie95], Section 7.1). Denote by $H_{n,h}$ the Hilbert scheme of closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial h and by $U_{n,h} \subset H_{n,h} \times \mathbb{P}^n$ the universal family. Let $\Phi : GL(n+1,\mathbb{C}) \times \mathbb{P}^n \to \mathbb{P}^n$ be the natural action, so that there is an action $\Psi : GL(n+1) \times H_{n,h} \to H_{n,h}$ such that $\forall g \in GL(n+1,\mathbb{C})$, $U_{n,h}$ is invariant under the morphism $\Psi_g \times \Phi_g$.

Given a (finite) group G, a faithful representation of G on $V := \mathbb{C}^{n+1}$ is given by an injective homomorphism $\beta : G \to GL(n+1,\mathbb{C})$, or equivalently, by a decomposition $V = \bigoplus_{\rho\in Irr(G)} W_\rho^{n(\rho)}$. Two representations are equivalent (i.e. the images of G are conjugate as subgroups of $GL(n+1,\mathbb{C})$) if and only if they have the same decomposition type (cf. [Ser77], Chap.2), hence the set of equivalence classes $\mathcal{B}_n$ of G-representations on V is finite.

Definition 3.19. Given $\beta : G \to GL(n+1,\mathbb{C})$ a faithful representation, it induces an action $\Psi|_{\beta(G)}$ of G on $H_{n,h}$. Define $H_{n,h}^{G,\beta}$ as the fixpoints scheme of the $\beta(G)$-action on $H_{n,h}$ and denote by $U_{n,h}^{G,\beta}$ the restriction of $U_{n,h}$ from $H_{n,h}$ to $H_{n,h}^{G,\beta}$.

Remark 3.20. (1) We have already seen that $C(G,\beta)$, the centralizer group of $\beta(G)$ in $GL(n+1,\mathbb{C})$, has a natural action on $H_{n,h}^{G,\beta}$ (cf. 3.9). By Schur's Lemma one obtains that $C(G,\beta) \simeq \Pi_{\rho\in Irr(G)} GL(n(\rho),\mathbb{C})$, hence $C(G,\beta)$ is reductive.
(2) Let β, β' be two equivalent representations, such that $\beta' = g\beta g^{-1}$ for some $g \in GL(n+1,\mathbb{C})$, then $H_{n,h}^{G,\beta}$ is isomorphic to $H_{n,h}^{G,\beta'}$ via Ψ_g as subschemes of $H_{n,h}$.
(3) Since $U_{n,h}^{G,\beta}$ (as a subscheme of $H_{n,h}^{G,\beta} \times \mathbb{P}^n$) is invariant under the action $id \times (\Phi|_{\beta(G)})$, we obtain a G-marked family $((p_\beta : U_{n,h}^{G,\beta} \to H_{n,h}^{G,\beta}), G, \beta)$.

Definition 3.21. Let V be a $\mathbb{C}$-vector space of dimension $n+1$. Denoting by $\mathcal{B}_n$ the set of equivalence classes of G-linear representations on V, we pick one representative in each equivalence class of $\mathcal{B}_n$ and define:

$$((p : U_{n,h}^G \to H_{n,h}^G), G, \mathcal{B}_n) := \bigsqcup_{[\beta]\in\mathcal{B}_n} ((p_\beta : U_{n,h}^{G,\beta} \to H_{n,h}^{G,\beta}), G, \beta),$$

where "$\bigsqcup$" means a disjoint union.
Note that two different choices of representatives result in isomorphic families.

By Matsusaka's big theorem ([Mat86], Theorem 2.4), there exists an integer k_0 such that $\forall [X] \in \mathsf{M}_h(\mathbf{Spec}\mathbb{C})$, $\omega_X^{k_0}$ is very ample and has vanishing higher cohomology groups, we fix one such k_0 for the rest of this section (we refer to [Siu93], [Dem96] and [Siu02] for effective bounds on k_0). Given a family $(p : \mathfrak{X} \to T) \in \mathsf{M}_h(T)$, by "Cohomology and Base change" (cf. [Mum70], II.5), $p_*(\omega_{\mathfrak{X}/T}^{k_0})$ is a locally free sheaf of rank $h(k_0)$. Moreover we have a surjection $p^*p_*(\omega_{\mathfrak{X}/T}^{k_0}) \twoheadrightarrow \omega_{\mathfrak{X}/T}^{k_0}$, which induces a T-embedding $i : \mathfrak{X} \hookrightarrow \mathbb{P}(p_*(\omega_{\mathfrak{X}/T}^{k_0}))$ such that $\omega_{\mathfrak{X}/T}^{k_0} \simeq i^*(\mathcal{O}_{\mathbb{P}(p_*(\omega_{\mathfrak{X}/T}^{k_0}))}(1))$ (cf. [Har77], II.7.12). Assuming in addition that $p_*(\omega_{\mathfrak{X}/T}^{k_0})$ is trivial, the T-embedding becomes $i : \mathfrak{X} \hookrightarrow T \times \mathbb{P}^N$ ($N := h(k_0) - 1$). Setting $h'(k) := h(k_0 k)$, there exists a morphism $f : T \to H_{N,h'}$ such that $\mathfrak{X} \simeq f^*U_{N,h'}$. Now taking the group action into account, we have the following:

Proposition 3.22 (Boundedness)**.** *Given* $((p:\mathfrak{X}\to T),G,\rho)\in \mathsf{M}_h^G(T)$, *denote by* $\bar\rho$ *the induced action of* G *on* $p_*(\omega_{\mathfrak{X}/T}^{k_0})$. *Assume that* $p_*(\omega_{\mathfrak{X}/T}^{k_0})$ *is trivial and* $\bar\rho$ *is defined over* $\mathbb{C}$, *then there exists* $f:T\to H_{N,h'}^G$, *such that* $((\mathfrak{X}\to T),G,\rho)\simeq f^*((U_{N,h'}^G\to H_{N,h'}^G),G,\mathcal{B}_N)$, *and* $\omega_{\mathfrak{X}/T}^{k_0}\simeq \mathcal{O}_{T\times\mathbb{P}^N}(1)|_{\mathfrak{X}}$.

Proof. It suffices to prove the statement on each connected component of T, hence we may assume that T is connected and $p_*(\omega_{\mathfrak{X}/T}^{k_0})$ has decomposition type β.
The action $\bar\rho$ induces an action of G on $\mathbf{Proj}_T(p_*(\omega_{\mathfrak{X}/T}^{k_0}))=T\times\mathbb{P}^N$ such that the embedding $i:\mathfrak{X}\to T\times\mathbb{P}^N$ is G-equivariant. Since by assumption $\bar\rho$ is defined over $\mathbb{C}$, we may require that the action on $T\times\mathbb{P}^N$ is given by $\pi_2^*(\beta)$, where $\pi_2:T\times\mathbb{P}^N\to\mathbb{P}^N$ is the projection onto the second factor. Now by the universal property of the Hilbert scheme, there exists $f:T\to H_{N,h'}$, such that $i(\mathfrak{X})=(f\times Id_{\mathbb{P}^N})^*U_{N,h'}$. To complete the proof, it remains to show that f factors through $H_{N,h'}^{G,\beta}$, which is equivalent to the property that $\forall g\in G, \Psi_{\beta(g)}\circ f=f$; again by the universal property of the Hilbert scheme, this is equivalent to showing that $\forall g\in G$, $((\Psi_{\beta(g)}\circ f)\times id_{\mathbb{P}^N})^*U_{N,h'}=i(\mathfrak{X})$. However we have that

$$((\Psi_{\beta(g)}\circ f)\times id_{\mathbb{P}^N})^*U_{N,h'}=(f\times id_{\mathbb{P}^N})^*(\Psi_{\beta(g)}\times id_{\mathbb{P}^N})^*U_{N,h'}$$

$$=(f\times id_{\mathbb{P}^N})^*(id_{U_{N,h'}}\times\Phi_{\beta(g)^{-1}})^*U_{N,h'}=(id_T\times\Phi_{\beta(g)^{-1}})^*(f\times id_{\mathbb{P}^N})^*U_{N,h'}$$

$$=(id_T\times\Phi_{\beta(g)^{-1}})^*(i(\mathfrak{X})),$$

which is simply $i(\mathfrak{X})$ as the embedding $i:\mathfrak{X}\to T\times\mathbb{P}^N$ is G-equivariant. □

Combining (3.16) with (3.22), we have the following corollary:

Corollary 3.23. *For any scheme* T *and* $((p:\mathfrak{X}\to T),G,\rho)\in\mathsf{M}_h^G(T)$, *let* $q:\mathcal{F}(p_*(\omega_{\mathfrak{X}/T}^{k_0}),G)\to T$ *be the bundle of* G*-frames of* $p_*(\omega_{\mathfrak{X}/T}^{k_0})$ *over* T. *Then the isomorphism* $\phi_{p_*(\omega_{\mathfrak{X}/T}^{k_0}),G}$ *induces a morphism*

$$f_{\mathfrak{X}/T,k_0,G}:\mathcal{F}(p_*(\omega_{\mathfrak{X}/T}^{k_0}),G)\to H_{N,h'}^G$$

such that

$$\mathsf{M}_h^G(q)((\mathfrak{X}\to T),G,\rho)\simeq f_{\mathfrak{X}/T,k_0,G}^*((U_{N,h'}^G\to H_{N,h'}^G),G,\mathcal{B}_N),$$

where $N:=h(k_0)-1$, $h'(k):=h(k_0k)$.

Remark 3.24. Given an isomorphism $((p : \mathfrak{X}_1 \to T), G, \rho_1) \simeq ((p : \mathfrak{X}_2 \to T), G, \rho_2)$, we have an induced isomorphism $l : p_*(\omega^{k_0}_{\mathfrak{X}_1/T}) \to p_*(\omega^{k_0}_{\mathfrak{X}_2/T})$ of G-sheaves on T. Both $p_*(\omega^{k_0}_{\mathfrak{X}_1/T})$ and $p_*(\omega^{k_0}_{\mathfrak{X}_2/T})$ have decomposition type β. Then l induces a $C(G, \beta)$-equivariant isomorphism: $l_{\mathcal{F}} : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}_1/T}), G) \to \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}_2/T}), G)$. From (3.6), (3.17) and the proof of (3.22) we have that $f_{\mathfrak{X}_1/T,k_0,G} = f_{\mathfrak{X}_2/T,k_0,G} \circ l_{\mathcal{F}}$.

We have already shown that M^G_h is bounded (in the sense of 3.23). However in general $H^G_{N,h'}$ may not be a parameterizing space for M^G_h, i.e., some fibre of $((U^G_{N,h'} \to H^G_{N,h'}), G, \mathcal{B}_N)$ may not be a canonical model. We will see that the set of points in $H^G_{N,h'}$ over which the fibre is a Gorenstein canonical model forms a locally closed subscheme $\bar{H}^G_{N,h'}$. In general such problems correspond to studying the local closedness of the moduli functor.

Definition 3.25 ([Kov09], 5.C)**.** A moduli functor of polarized varieties M is called *locally closed* if the following condition holds: For every flat family of polarized varieties $(\mathfrak{X} \to T, \mathfrak{L})$, there exists a locally closed subscheme $i : T' \hookrightarrow T$ such that if $f : S \to T$ is any morphism then $f^*(\mathfrak{X} \to T, \mathfrak{L}) \in \mathsf{M}(S)$ iff f factors through T'.

Here we do not state a general "G-version" of local closedness, but only consider the case of Hilbert schemes. For a general discussion, see [Kol08], Corollary 24.

Proposition 3.26. *Using the same notations as in (3.22), there exists a locally closed subscheme $\bar{H}^G_{N,h'}$ of $H^G_{N,h'}$, satisfying the following conditions:*
(1) $((\bar{U}^G_{N,h'} \to \bar{H}^G_{N,h'}), G, \mathcal{B}_N) := ((U^G_{N,h'} \to H^G_{N,h'}), G, \mathcal{B}_N)|_{\bar{H}^G_{N,h'}} \in \mathsf{M}^G_h(\bar{H}^G_{N,h'})$.
(2) The morphism f that we obtained in (3.22) factors through $\bar{H}^G_{N,h'}$.

Proof. In the case where G is trivial the existence of $\bar{H}_{N,h'}$ follows from the facts that the subset

$$\{x \in H_{N,h'} |\ (\omega^{k_0}_{U_{N,h'}/H_{N,h'}})_x \simeq (\mathcal{O}_{H_{N,h'} \times \mathbb{P}^N}(1)|_{U_{N,h'}})_x\}$$

is closed in $H_{N,h'}$ (cf.[Mum70], II.5, Corollary 6) and being canonical and Gorenstein is an open property (cf.[Elk81]).
In general we set $\bar{H}^{G,\beta}_{N,h'} := \bar{H}_{N,h'} \bigcap H^{G,\beta}_{N,h'}$ and $\bar{H}^G_{N,h'} := \coprod \bar{H}^{G,\beta}_{N,h'}$. For condition (1), the fact that $(\bar{U}_{N,h'} \to \bar{H}_{N,h'}) \in \mathsf{M}_h(\bar{H}_{N,h'})$ implies that $(\bar{U}^G_{N,h'} \to \bar{H}^G_{N,h'}) \in \mathsf{M}_h(\bar{H}^G_{N,h'})$, now taking the action of G into account,

we have that $((\bar{U}^G_{N,h'} \to \bar{H}^G_{N,h'}), G, \mathcal{B}_N) \in \mathsf{M}^G_h(\bar{H}^G_{N,h'})$. Condition (2) is satisfied for similar reasons. □

Remark 3.27. (1) Given $(X_1, G, \rho_1), (X_2, G, \rho_2) \in \mathsf{M}^G_h(\mathbf{Spec}\mathbb{C})$ such that $H^0(\omega^{k_0}_{X_1})$ and $H^0(\omega^{k_0}_{X_2})$ have the same decomposition type β, by (3.26) there exist $f_i : \mathbf{Spec}(\mathbb{C}) \to \bar{H}^{G,\beta}_{N,h'}$ such that

$$(X_i, G, \rho_i) \simeq \mathsf{M}^G_h(f_i)((\bar{U}^{G,\beta}_{N,h'} \to \bar{H}^{G,\beta}_{N,h'}), G, \beta) \text{ for } i = 1, 2.$$

From the proof of (3.22) we see that X_1 and X_2 are isomorphic as G-marked varieties $\iff$ $\exists g \in C(G, \beta)$ such that $f_1(\mathbf{Spec}(\mathbb{C})) = \Psi_g f_2(\mathbf{Spec}(\mathbb{C}))$.

(2) Notations as in (3.23). Assume that T is connected and $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ has decomposition type β and denote by Ψ' the action of $C(G, \beta)$ on $\mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G)$. From the proof of (3.16) we see that $\forall g \in C(G, \beta)$, $\Psi'_g \times \Phi_g$ leaves $q^*\mathfrak{X} \simeq f^*_{\mathfrak{X}/T,k_0,G}(\bar{U}^{G,\beta}_{N,h'})$ invariant as a subscheme of $\mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \times \mathbb{P}^N$, i.e., $(\Psi'_g \times \Phi_g) f^*_{\mathfrak{X}/T,k_0,G}(\bar{U}^{G,\beta}_{N,h'}) = f^*_{\mathfrak{X}/T,k_0,G}(\bar{U}^{G,\beta}_{N,h'})$. This implies that

$$(\Psi'_g \times id) f^*_{\mathfrak{X}/T,k_0,G}(\bar{U}^{G,\beta}_{N,h'}) = f^*_{\mathfrak{X}/T,k_0,G}((id \times \Phi_{g^{-1}})(\bar{U}^{G,\beta}_{N,h'})) = f^*_{\mathfrak{X}/T,k_0,G}((\Psi_g \times id)(\bar{U}^{G,\beta}_{N,h'})).$$

Therefore we conclude that the morphism obtained in (3.23),

$$f_{\mathfrak{X}/T,k_0,G} : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \to \bar{H}^{G,\beta}_{N,h'},$$

is $C(G, \beta)$-equivariant.

3.2. **The Construction of $\mathfrak{M}_h[G]$.** We have obtained a parameterizing space $\bar{H}^G_{N,h'}$ for the moduli functor M^G_h, now we construct $\mathfrak{M}_h[G]$ as a quotient space of $\bar{H}^G_{N,h'}$ and show that it is the coarse moduli scheme for M^G_h.

In (3.20) we have seen that the group $C(G, \beta)$ acts on $H^{G,\beta}_{N,h'}$: it is clear that the subscheme $\bar{H}^{G,\beta}_{N,h'}$ is invariant under this action. The first goal of this section is to show that the quotient $\bar{H}^{G,\beta}_{N,h'}/C(G, \beta)$ exists (as a scheme).

Setting $SC(G, \beta) := SL(N+1, \mathbb{C}) \bigcap C(G, \beta)$ and denoting by $PGC(G, \beta)$ the image of $C(G, \beta)$ under the natural homomorphism $GL(N+1, \mathbb{C}) \to PGL(N+1, \mathbb{C})$, we have a central extension:

$$1 \to \mathbb{C}^* \to C(G, \beta) \to PGC(G, \beta) \to 1.$$

Since $\mathbb{C}^*$ acts trivially on $\bar{H}^{G,\beta}_{N,h'}$, we have

$$\bar{H}^{G,\beta}_{N,h'}/C(G,\beta) \simeq \bar{H}^{G,\beta}_{N,h'}/PGC(G,\beta).$$

On the other hand $SC(G,\beta)$ maps surjectively onto $PGC(G,\beta)$, hence we have that $\bar{H}^{G,\beta}_{N,h'}/PGC(G,\beta) \simeq \bar{H}^{G,\beta}_{N,h'}/SC(G,\beta)$. Therefore from now on we consider $\bar{H}^{G,\beta}_{N,h'}/SC(G,\beta)$ instead. (It is not difficult to show that $SC(G,\beta)$ is reductive.)

Lemma 3.28. *$SC(G,\beta)$ acts properly on $\bar{H}^{G,\beta}_{N,h'}$ and $\forall x \in \bar{H}^{G,\beta}_{N,h'}$, the stabilizer subgroup $Stab(x)$ is finite.*

Proof. In the case where G is trivial the lemma is known by studying the separatedness of the corresponding functor (cf. [Vie95], 7.6, 8.21; [Kov09], 5.D). Now since $SC(G,\beta)$ is a closed subgroup of $SL(N+1,\mathbb{C})$ and $\bar{H}^{G,\beta}_{N,h'}$ is a closed subscheme of $\bar{H}_{N,h'}$ which stays invariant under the action of $SC(G,\beta)$, the lemma follows immediately. □

In order to apply Geometric Invariant theory, we have to find an $SC(G,\beta)$-linearized invertible sheaf on $\bar{H}^{G,\beta}_{N,h'}$ and verify certain stability conditions (cf. [MF82], Chap.1).
Let us first look at the case where G is trivial: let $p : \bar{U}_{N,h'} \to \bar{H}_{N,h'}$ be the universal family and define

$$\lambda_{k_0} := \det(p_*(\omega^{k_0}_{\bar{U}_{N,h'}/\bar{H}_{N,h'}})).$$

A result of Viehweg (see [Vie95], 7.17) states that λ_{k_0} admits an $SL(N+1,\mathbb{C})$-linearization and

$$\bar{H}_{N,h'} = (\bar{H}_{N,h'})^s(\lambda_{k_0}),$$

where $(\bar{H}_{N,h'})^s(\lambda_{k_0})$ denotes the set of $SL(N+1,\mathbb{C})$-stable points with respect to λ_{k_0}. Then it is easy to obtain the following proposition:

Proposition 3.29. *There exists a geometric quotient $(\mathfrak{M}^{G,\beta}_{k_0,h},\pi_\beta)$ of $\bar{H}^{G,\beta}_{N,h'}$ by $SC(G,\beta)$, moreover:*
(1) The quotient map $\pi_\beta : \bar{H}^{G,\beta}_{N,h'} \to \mathfrak{M}^{G,\beta}_{k_0,h}$ is an affine morphism.
(2) There exists an ample invertible sheaf $\mathcal{L}$ on $\mathfrak{M}^{G,\beta}_{k_0,h}$ such that $\pi^*_\beta\mathcal{L} \simeq (\lambda^{G,\beta}_{k_0})^n$ for some $n > 0$, where setting $p_\beta := p|_{\bar{U}^{G,\beta}_{N,h'}} : \bar{U}^{G,\beta}_{N,h'} \to \bar{H}^{G,\beta}_{N,h'}$, $\lambda^{G,\beta}_{k_0} := \det((p_\beta)_*(\omega^{k_0}_{\bar{U}^{G,\beta}_{N,h'}/\bar{H}^{G,\beta}_{N,h'}}))$.

Proof. Noting that $\omega^{k_0}_{\bar{U}_{N,h'}/\bar{H}_{N,h'}}|_{\bar{U}^{G,\beta}_{N,h'}} \simeq \omega^{k_0}_{\bar{U}^{G,\beta}_{N,h'}/\bar{H}^{G,\beta}_{N,h'}}$ (cf. [HK04], Lemma 2.6) and applying "cohomology and base change", we have that

$$\lambda^{G,\beta}_{k_0} \simeq \lambda_{k_0}|_{\bar{H}^{G,\beta}_{N,h'}}.$$

Since $\bar{H}^{G,\beta}_{N,h'}$ (as a subscheme of $\bar{H}_{N,h'}$) is invariant under the $SC(G,\beta)$-action, the $SL(N+1,\mathbb{C})$-linearization of λ_{k_0} induces a natural $SC(G,\beta)$-linearization of $\lambda^{G,\beta}_{k_0}$. By Lemma (3.28), we have that $SL(N+1,\mathbb{C})$ acts properly on $\bar{H}_{N,h'}$ and $SC(G,\beta)$ acts properly on $\bar{H}^{G,\beta}_{N,h'}$. Noting that a one-parameter subgroup $\mu : \mathbb{C}^* \to SC(G,\beta)$ is also a subgroup of $SL(N+1,\mathbb{C})$ and that $\bar{H}^{G,\beta}_{N,h'}$ is closed in $\bar{H}_{N,h'}$, we see that for any $x \in \bar{H}^{G,\beta}_{N,h'}$, $\lim_{t\to 0}(\mu(t)x)$ exists in $\bar{H}^{G,\beta}_{N,h'}$ if and only if it exists in $\bar{H}_{N,h'}$. Now by applying the Hilbert-Mumford criterion (cf. [MF82], Theorem 2.1), we see that

$$(\bar{H}_{N,h'})^s(\lambda_{k_0}) = \bar{H}_{N,h'} \Rightarrow (\bar{H}^{G,\beta}_{N,h'})^s(\lambda^{G,\beta}_{k_0}) = \bar{H}^{G,\beta}_{N,h'}.$$

Then the proposition follows from standard GIT methods (cf. [MF82], Theorem 1.10). □

We are ready to prove the main theorem (3.1):

Proof of (3.1). We set

$$\mathfrak{M}_h[G] := \bigsqcup_{[\beta]\in\mathcal{B}_N} \mathfrak{M}^{G,\beta}_{k_0,h} \tag{1}$$

(note that if $\mathsf{M}^G_h(\mathbf{Spec}\mathbb{C}) = \emptyset$ then $\mathfrak{M}_h[G] = \emptyset$).
Let us make the following convention: for any natural transformation $\theta : \mathsf{M}^G_h \to \mathrm{Hom}(-,Q)$, scheme T and $[((p : \mathfrak{X} \to T), G, \rho)] \in \mathsf{M}^G_h(T)$, we write $\theta_T(\mathfrak{X})$ or simply $\theta(\mathfrak{X})$ as an abbreviation for $\theta_T([((p : \mathfrak{X} \to T), G, \rho)])$.
Step 1. Construction of a natural transformation $\eta : \mathsf{M}^G_h \to \mathrm{Hom}(-, \mathfrak{M}_h[G])$: Given T a scheme and $((p : \mathfrak{X} \to T), G, \rho) \in \mathsf{M}^G_h(T)$, it suffices to define η on each connected component of T, hence we assume furthermore that T is connected. We have the bundle of G-frames of $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ over T, $q : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \to T$. By (3.23) and (3.26) there exists a morphism $f_{\mathfrak{X}/T,k_0,G} : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \to \bar{H}^{G,\beta}_{N,h'}$ such that

$$\mathsf{M}^G_h(q)((\mathfrak{X} \to T), G, \rho) \simeq \mathsf{M}^G_h(f_{\mathfrak{X}/T,k_0,G})((\bar{U}^{G,\beta}_{N,h'} \to \bar{H}^{G,\beta}_{N,h'}), G, \beta)$$

for some $[\beta] \in \mathcal{B}_N$. Setting

$$\bar{f}_{\mathfrak{X}/T,k_0,G} := \pi_\beta \circ f_{\mathfrak{X}/T,k_0,G} : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \to \mathfrak{M}^{G,\beta}_{k_0,h},$$

by (3.27-2) we see that $\bar{f}_{\mathfrak{X}/T,k_0,G}$ is $C(G,\beta)$-equivariant (where we take the trivial action on $\mathfrak{M}^{G,\beta}_{k_0,h}$). Since T is the quotient of $\mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}),G)$ by $C(G,\beta)$ (cf. 3.18), there exists a (unique) morphism $\eta_T(\mathfrak{X}):T\to\mathfrak{M}_h[G]$ such that $\bar{f}_{\mathfrak{X}/T,k_0,G}=\eta_T(\mathfrak{X})\circ q$. Note that by (3.24) $\eta_T(\mathfrak{X})$ is independent of the representative family $((p:\mathfrak{X}\to T),G,\rho)$ that we choose, hence $\eta_T(\mathfrak{X})$ is well defined.

In order to show that η is a natural transformation, let $l\in\mathrm{Hom}(S,T)$ and let $((p:\mathfrak{X}\to T),G,\rho)\in\mathsf{M}^G_h(T)$: it suffices to show that

$$\eta_S(\mathfrak{X}_S)=\eta_T(\mathfrak{X})\circ l.$$

Without loss of generality we assume that S and T are connected and $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ has decomposition type β, now considering the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}((p_S)_*(\omega^{k_0}_{\mathfrak{X}_S/S}),G) & \xrightarrow{\tilde{l}} & \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}),G) \\ \big\downarrow{\scriptstyle q_S} & & \big\downarrow{\scriptstyle q} \\ S & \xrightarrow{l} & T \end{array},$$

from (3.6-1) and (3.17) we see that $\bar{f}_{\mathfrak{X}_S/S,k_0,G}=\bar{f}_{\mathfrak{X}/T,k_0,G}\circ\tilde{l}$. Since $\bar{f}_{\mathfrak{X}_S/S,k_0,G}$, $\bar{f}_{\mathfrak{X}/T,k_0,G}$ and $\tilde{l}$ are all $C(G,\beta)$-equivariant, hence we have $\eta_S(\mathfrak{X}_S)=\eta_T(\mathfrak{X})\circ l$ by (3.18).

Step 2. $\mathfrak{M}_h[G]$ is the coarse moduli scheme for M^G_h:

(1) $\eta_{\mathbf{Spec}\mathbb{C}}$ induces a one-to-one correspondence between $\mathsf{M}^G_h(\mathbf{Spec}\mathbb{C})$ and the set of (closed) points of $\mathfrak{M}_h[G]$.

Surjectivity follows from (3.26), and injectivity follows from (3.27-1).

(2) The universal property of η.

Let $\theta:\mathsf{M}^G_h\to\mathrm{Hom}(-,Q)$ be another natural transformation: we show that there exists a unique morphism $\gamma:\mathfrak{M}_h[G]\to Q$ such that $\theta=\mathrm{Hom}(\gamma)\circ\eta$.

For any $[\beta]\in\mathcal{B}_N$, the universal family $((\bar{U}^{G,\beta}_{N,h'}\to\bar{H}^{G,\beta}_{N,h'}),G,\beta)\in\mathsf{M}^G_h(\bar{H}^{G,\beta}_{N,h'})$ induces a morphism $\theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'}):\bar{H}^{G,\beta}_{N,h'}\to Q$. For any $g\in C(G,\beta)$, we have that

$$(\Psi_g\times id_{\mathbb{P}^N})((\bar{U}^{G,\beta}_{N,h'}\to\bar{H}^{G,\beta}_{N,h'}),G,\beta)=(id_{\bar{H}^{G,\beta}_{N,h'}}\times\Phi_{g^{-1}})((\bar{U}^{G,\beta}_{N,h'}\to\bar{H}^{G,\beta}_{N,h'}),G,\beta)$$

as subschemes of $\bar{H}^{G,\beta}_{N,h'}\times\mathbb{P}^N$, noting that the right hand side is isomorphic to $((\bar{U}^{G,\beta}_{N,h'}\to\bar{H}^{G,\beta}_{N,h'}),G,\beta)$ as G-marked families, we see that

$$\theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'})=\theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'})\circ\Psi_g.$$

This implies that $\theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'})$ is $C(G,\beta)$-equivariant, hence it induces a (unique) morphism $\gamma_\beta : \mathfrak{M}^{G,\beta}_{k_0,h} \to Q$ such that

$$\theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'}) = \gamma_\beta \circ \eta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'}).$$

Now we can define $\gamma : \mathfrak{M}_h[G] \to Q$ such that the restriction of γ to each $\mathfrak{M}^{G,\beta}_{k_0,h}$ is γ_β.
From the construction of γ we already saw that γ must be unique, it remains to show that $\theta = \mathrm{Hom}(\gamma) \circ \eta$. Given $((p : \mathfrak{X} \to T), G, \rho) \in \mathsf{M}^G_h(T)$, let $q : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \to T$ be the bundle of G-frames of $p_*(\omega^{k_0}_{\mathfrak{X}/T})$, we assume again that T is connected and $p_*(\omega^{k_0}_{\mathfrak{X}/T})$ has decomposition type β. By (3.23) and (3.26) there exists

$$f_{\mathfrak{X}/T,k_0,G} : \mathcal{F}(p_*(\omega^{k_0}_{\mathfrak{X}/T}), G) \to \bar{H}^{G,\beta}_{N,h'}$$

such that

$$\mathsf{M}^G_h(q)((\mathfrak{X} \to T), G, \rho) \simeq \mathsf{M}^G_h(f_{\mathfrak{X}/T,k_0,G})((\bar{U}^{G,\beta}_{N,h'} \to \bar{H}^{G,\beta}_{N,h'}), G, \beta),$$

hence we have that

$$\theta(q^*\mathfrak{X}) = \theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'}) \circ f_{\mathfrak{X}/T,k_0,G} = \gamma_\beta \circ \eta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'}) \circ f_{\mathfrak{X}/T,k_0,G} = \gamma_\beta \circ \eta(q^*\mathfrak{X}),$$

where the first and third equalities hold since θ and η are natural transformations, the second equality holds by the construction of γ_β. Finally the fact that $f_{\mathfrak{X}/T,k_0,G}$ and $\theta_{\bar{H}^{G,\beta}_{N,h'}}(\bar{U}^{G,\beta}_{N,h'})$ are $C(G,\beta)$-equivariant $\Rightarrow \theta(q^*\mathfrak{X})$ is also $C(G,\beta)$-equivariant. By (3.18) $\exists!\, l' \in \mathrm{Hom}(T, Q)$ such that $\theta(q^*\mathfrak{X}) = l' \circ q$, which implies that $\theta_T(\mathfrak{X}) = l' = \gamma_\beta \circ \eta_T(\mathfrak{X})$. □

As an application of our results, we show that the locus $\mathfrak{M}_h(G)$ inside $\mathfrak{M}_h$ of varieties which admit an effective action by a group G is closed. This has been proven in [Cat83], Theorem 1.8 for the case of surfaces, the idea there generalizes naturally to higher dimensional cases.
Given a faithful representation $\beta : G \to GL(N+1, \mathbb{C})$, we have a natural inclusion $i_\beta : \bar{H}^{G,\beta}_{N,h'} \subset \bar{H}_{N,h'}$. Noting that the restriction of the quotient map $\pi : \bar{H}_{N,h'} \to \mathfrak{M}_h$ to $\bar{H}^{G,\beta}_{N,h'}$ is $SC(G,\beta)$-equivariant, we obtain an induced morphism $u^{G,\beta}_{k_0,h} : \mathfrak{M}^{G,\beta}_{k_0,h} \to \mathfrak{M}_h$. We define a morphism $u^G_h : \mathfrak{M}_h[G] \to \mathfrak{M}_h$ such that $u^G_h|_{\mathfrak{M}^{G,\beta}_{k_0,h}} = u^{G,\beta}_{k_0,h}$. We denote by $\mathfrak{M}_h(G)$ the (scheme-theoretic) image of u^G_h in $\mathfrak{M}_h$. Then we can interpret the problem into showing that u^G_h maps $\mathfrak{M}_h[G]$ surjectively onto $\mathfrak{M}_h(G)$.

Corollary 3.30. *The morphism* $u_h^G : \mathfrak{M}_h[G] \to \mathfrak{M}_h$ *is finite and maps* $\mathfrak{M}_h[G]$ *surjectively onto* $\mathfrak{M}_h(G)$*;* $\mathfrak{M}_h(G)$ *is a closed subscheme of* $\mathfrak{M}_h$*.*

Proof. It is easy to see that u_h^G is quasi-finite: given a point $[X] \in \mathfrak{M}_h$, since $\mathrm{Aut}(X)$ is finite, then the set of injective homomorphisms $\rho : G \to \mathrm{Aut}(X)$ is finite, hence $(u_h^G)^{-1}([X])$, which corresponds to the set of isomorphism classes of G-markings on X, is also finite.
For the remaining statements, it suffices to show that u_h^G is proper, which is equivalent to showing that $u_{k_0,h}^{G,\beta} : \mathfrak{M}_{k_0,h}^{G,\beta} \to \mathfrak{M}_h$ is proper for each $[\beta] \in \mathcal{B}_N$. Applying the valuative criterion of properness, we have to prove that for every pointed curve (C, O) (not necessarily complete) and for any commutative diagram

$$\begin{array}{ccc} C^\star & \xrightarrow{f'} & \mathfrak{M}_{k_0,h}^{G,\beta} \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle u_{k_0,h}^{G,\beta}} \\ C & \xrightarrow{f} & \mathfrak{M}_h \end{array}$$

where $C^\star := C - \{O\}$, there exists a unique $l : C \to \mathfrak{M}_{k_0,h}^{G,\beta}$ making the whole diagram commute.
By GIT we know that $\mathfrak{M}_{k_0,h}^{G,\beta}$ is quasi-projective and hence separated, therefore the uniqueness of l is clear. For the existence of l, since $\pi_\beta : \bar{H}_{N,h'}^{G,\beta} \to \mathfrak{M}_{k_0,h}^{G,\beta}$ is a quotient map of quasi-projective schemes, it suffices to show that there exists a finite morphism $v : (B, O') \to (C, O)$ and a morphism $l' : B \to \bar{H}_{N,h'}^{G,\beta}$ such that

$$(*)\ u_{k_0,h}^{G,\beta} \circ \pi_\beta \circ l' = f \circ v \ \textit{and}\ \pi_\beta \circ (l'|_{B^\star}) = f' \circ (v|_{B^\star}),$$

where $B^\star := B - \{O'\}$.
Considering the quotient map $\pi : \bar{H}_{N,h'} \to \mathfrak{M}_h$, we can assume without loss of generality that we have a morphism $m : C \to \bar{H}_{N,h'}$ such that $f = \pi \circ m$. Then we obtain a family $(\mathfrak{X} \to C) := m^*(\bar{U}_{N,h'}) \in \mathsf{M}_h(C)$ such that $\mathfrak{X} \subset C \times \mathbb{P}^N$. The idea of constructing the morphism $v : (B, O') \to (C, O)$ is similar to that of (3.16). We consider first the subspace

$$Z := \{(t, A(t) | A(t)\mathfrak{X}_t \textit{ corresponds to a point in } \bar{H}_{N,h'}^{G,\beta}\} \subset C \times GL(N+1, \mathbb{C}).$$

By assumption we see that $p_1 : Z - p_1^{-1}(O) \to C^\star$ is surjective, where $p_1 : C \times GL(n+1, \mathbb{C}) \to C$ is the projection onto the first factor, hence we can find a curve B' inside Z, such that $p_1|_{B'} : B' \to C^\star$ is surjective. For similar reasons as in (3.16), we get a G-marked family

$((p_1|_{B'})^*\mathfrak{X}^\star \to B'), G, \beta)$, where $\mathfrak{X}^\star := \mathfrak{X} - \mathfrak{X}_O$. After possibly taking the normalization of B', we can extend the morphism $p_1|_{B'}$ to a morphism $v : (B, O') \to (C, O)$ and we see that $(((v|_{B^\star})^*\mathfrak{X}^\star \to B^\star), G, \beta)$ is a G-marked family, where $B^\star := B - \{O'\}$.

We claim that the G-action on $(v|_{B^\star})^*\mathfrak{X}^\star \to B^\star$ can be extended to an action on $(\mathfrak{X}' \to B) := (v^*\mathfrak{X} \to B)$. Since $\omega^{k_0}_{\mathfrak{X}'/B}$ induces an embedding $i : \mathfrak{X}' \to B \times \mathbb{P}^N$, we see that the claim is equivalent to that $i(\mathfrak{X}')$ is invariant under the action $\pi_2^*(\beta)$, where $\pi_2 : B \times \mathbb{P}^N \to \mathbb{P}^N$ is the projection on to the second factor. After possibly shrinking B, we can assume that B is connected and hence $i(\mathfrak{X}')$ is irreducible. The fact that $((v|_{B^\star})^*\mathfrak{X}^\star \to B^\star), G, \beta)$ is a G-marked family implies that $i((v|_{B^\star})^*\mathfrak{X}^\star)$ is invariant under $\pi_2^*(\beta)$, now from the irreducibility of $i(\mathfrak{X}')$ we see that $i(\mathfrak{X}')$ is also invariant under the action $\pi_2^*(\beta)$.

Now we have a G-marked family $((\mathfrak{X}' \to B), G, \beta)$, by (3.22) we obtain a morphism $l' : B \to \bar{H}^{G,\beta}_{N,h'}$, it is easy to check that l' satisfies $(*)$. $\square$

4. **Irreducible components of $\mathfrak{M}_g(G)$**

In the second part of this article (section 4 - section 7) we shall study the locus $\mathfrak{M}_g(G)$ inside $\mathfrak{M}_g$, the coarse moduli space of algebraic curves of genus $g \geq 2$, for certain elementary groups G. In this section we recall some prerequisite results and give rough ideas about what we are going to do in later sections.
A good approach to understanding the irreducible components of $\mathfrak{M}_g(G)$ is to view $\mathfrak{M}_g$ as the quotient of the Teichmüller space $\mathcal{T}_g$ by the natural action of the mapping class group Map_g:

$$\pi : \mathcal{T}_g \to \mathcal{T}_g/Map_g = \mathfrak{M}_g.$$

Observe that

$$\mathfrak{M}_g(G) = \bigcup_{[\rho]} \mathfrak{M}_{g,\ \rho}(G),$$

where $\rho : G \hookrightarrow Map_g$ is an injective homomorphism, $\mathfrak{M}_{g,\ \rho}(G)$ is the image of the fixed locus of $\rho(G)$ under the natural projection π and $\rho \sim \rho'$ iff they are equivalent by the equivalence relation generated by the automorphisms of G and the conjugations by Map_g. We call this equivalence class an *unmarked topological type* (cf. [CLP15], section 2). Since each $\mathfrak{M}_{g,\ \rho}(G)$ is an irreducible (Zariski) closed subset of $\mathfrak{M}_g$ (cf. [CLP15], Theorem 2.3), in order to determine the irreducible components of $\mathfrak{M}_g(G)$, it suffices to determine the maximal loci of the form $\mathfrak{M}_{g,\ \rho}(G)$, i.e., to figure out when one locus contains another.
To be precise, our problem is: for which ρ and ρ', does $\mathfrak{M}_{g,\ \rho'}(G)$ contain $\mathfrak{M}_{g,\ \rho}(G)$? Hence we determine the loci $\mathfrak{M}_{g,\ \rho}(G)$ which are not maximal whence the irreducible decomposition of $\mathfrak{M}_g(G)$. The above problem is equivalent to the classification of subgroups H, H' of Map_g ($g \geq 2$), where H and H' satisfy the following condition:

$$(*)\ H, H' \simeq G, H \neq H' \text{ and } Fix(H) \subset Fix(H').$$

Definition 4.1. For any finite subgroup $H \subset Map_g$, set $\delta_H :=\dim Fix(H)$ and let $G(H) := \bigcap_{C \in Fix(H)} Aut(C)$ ($Fix(H)$ corresponds to the complex structures for which the action of H is holomorphic, whereas $G(H)$ is the common automorphism group of all the curves in $Fix(H)$).
If $H = G(H)$ we call H *full*.

It is easy to see that condition $(*)$ is equivalent to the condition:

$(**)$ H is isomorphic to G and not full, $G(H)$ has a subgroup H'

which is isomorphic to G and different from H.

For explicit computations, we need some general theory of Galois covers of Riemann surfaces (cf. [Cat15], Section 5).

Definition 4.2 ([CLP11], Definition 1)**.** A *G-Hurwitz vector* is an ordered sequence

$$v = (a_1, b_1, ..., a_{g'}, b_{g'}; c_1, ..., c_r) \in G^{2g'+r}$$

such that the following conditions are satisfied:

(i) $ord(c_i) = m_i > 1$ for all i;
(ii) G is generated by the components of v, usually we write $G = \langle v \rangle$;
(iii) $\prod_{i=1}^{g'}[a_i, b_i] \prod_{j=1}^{r} c_j = 1$.

We call the sequence $(m_i)_{i=1}^{r}$ the type of v.

Definition 4.3. (1) Let G be a finite group acting effectively on a curve C of genus $g \geq 2$: we obtain a Galois cover $p : C \to C/G =: C'$ branched in r points on C' with branching indices $m_1, ..., m_r$. Denoting by g' the genus of C', the *orbifold fundamental group* of the cover is a group with the following presentation:

$$T(g'; m_1, ..., m_r) := \langle \alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'}; \gamma_1, ..., \gamma_r \mid \Pi[\alpha_j, \beta_j]{\cdot}\Pi\gamma_i = 1, \gamma_i^{m_i} = 1\rangle.$$

In the case of $g' = 0$, we set $T(m_1, ..., m_r) := T(0; m_1, ..., m_r)$.
(2) The cover $C \to C/G$ is (topologically) determined by a surjective homomorphism

$$\mu : T(g'; m_1, ..., m_r) \twoheadrightarrow G$$

such that $f(\gamma_j)$ has order m_j inside G (cf. [Cat15], Section 6). One sees immediately that

$$v := [\mu(\alpha_1), \mu(\beta_1), ..., \mu(\alpha_{g'}), \mu(\beta_{g'}); \mu(\gamma_1), ..., \mu(\gamma_r)]$$

is a G-Hurwitz vector, and we call it the Hurwitz vector associated to μ.

In the rest of this section we only consider the case of $g' = 0$.
Given a surjective homomorphism $\mu : T(m_1, ..., m_r) \twoheadrightarrow G$, the Hurwitz vector associated to μ is not uniquely determined, since we can choose different presentations for $T(m_1, ..., m_r)$. For instance consider the group $T(m_1, ..., m_r)$ with the presentation $\langle \gamma_1, ...\gamma_r | \Pi\gamma_i = 1, \gamma_i^{m_i} = 1\rangle$. For any $1 \leq k < r$, we have a set of generators $\{\delta_{i,k}\}$: $\delta_{i,k} := \alpha_i$ if

$i \neq k, k+1$; $\delta_{k,k} := \alpha_k \alpha_{k+1} \alpha_k^{-1}$ and $\delta_{k+1,k} := \alpha_k$. This induces an isomorphism between $T(m_1, ..., m_r)$ and $T(l_1, ..., l_r)$, where $l_i = m_i$ if $i \neq k, k+1$; $l_k = m_{k+1}$ and $l_{k+1} = m_k$. Different choices of generators correspond to the following braid group action on the set of Hurwitz vectors.
Recall that Artin's *braid group on r strands* has the presentation

$$\mathfrak{B}_r := \langle \sigma_1, ..., \sigma_{r-1} | \forall 1 \leq i \leq r-2,\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}; \forall |j-i| \geq 2, \sigma_i \sigma_j = \sigma_j \sigma_i \rangle.$$

The group $\mathfrak{B}_r$ acts on the set of Hurwitz vectors of length r as follows:

$$(v_1, ..., v_i, v_{i+1}, ..., v_r) \overset{\sigma_i}{\mapsto} (v_1, ..., v_i v_{i+1} v_i^{-1}, v_i, ..., v_r).$$

On the other hand, for any $h \in Aut(G)$, we can compose μ with h, which induces an $Aut(G)$-action on the set of G-Hurwitz vectors: given $v = (v_1, ..., v_r)$ a G-Hurwitz vector, define $h(v) := (h(v_1), ..., h(v_r))$.
Since these actions (by $\mathfrak{B}_r$ and by $Aut(G)$) commute, they induce an action of the group $\mathfrak{B}_r \times Aut(G)$ on the set of G-Hurwitz vectors of length r.

Definition 4.4. Given two G-Hurwitz vectors v, v' of length r, we say that v and v' are B.A.-equivalent if they are in the same $\mathfrak{B}_r \times Aut(G)$-orbit.

Remark 4.5. Two Hurwitz vectors v and v' determine the same unmarked topological type iff they are B.A.-equivalent (cf. [CLP15], section 2).

A natural question raises: when are two Hurwitz vectors B.A.-equivalent? In the case where G is abelian, it is easy to answer this question since the braid action corresponds to the permutation of entries and the automorphism group of G is also clear.
If the group G is not abelian, the question becomes much more difficult: even for the dihedral group D_n, it took effort to solve the problem (cf. [CLP11], [CLP15]). Later in section 6 we shall come back to this topic.

Now let us look again at the condition (∗∗): in order to do computation via Hurwitz vectors, we still have two questions.
(1) Denoting by v_H(resp. $v_{G(H)}$) the Hurwitz vector for the cover $C \to C/H$(resp. $C \to C/G(H)$), which types of Hurwitz vectors $v_H, v_{G(H)}$ may occur?

(2) Given a finite subgroup $H \subset Map_g$ which is not full, what can we say about $G(H)$?
Fortunately, the first question was answered by Magaard, Shaska, Shpectorov, and Völklein in [MSSV02], we have the following theorem:

Theorem 4.6 ([MSSV02], 4.1)**.** *Let $H \subsetneqq K$ be two (finite) subgroups of Map_g such that $\delta_H = \delta_K =: \delta$. Given a general curve $C \in Fix(H)$, one of the following holds:*
I) $\delta_H = 3$, $[K{:}H] = 2$, and $C \to C/K$ is a covering of $\mathbb{P}^1$ branched on 6 points, $P_1, \ldots, P_6$, with branching indices all equal to 2. Moreover the subgroup H corresponds to the unique genus two double cover of $\mathbb{P}^1$ branched on the 6 points.
II) $\delta_H = 2$, $[K{:}H] = 2$, and $C \to C/K$ is a covering of $\mathbb{P}^1$ branched on five points, $P_1, \ldots, P_5$, with branching indices $2, 2, 2, 2, c_5$. Moreover the subgroup H corresponds to a double cover of $\mathbb{P}^1$ branched on the 4 points $P_1, \ldots, P_4$ with branching index 2.
III) $\delta_H = 1$, there are 3 possibilities:
$III-a$) H has index 2 in K, and $C \to C/K$ is a covering of $\mathbb{P}^1$ branched on 4 points, $P_1, \ldots, P_4$, with branching indices $2, 2, 2, 2d_4$, where $d_4 > 1$. Moreover the subgroup H corresponds to the unique genus one double cover of $\mathbb{P}^1$ branched on the 4 points $P_1, \ldots, P_4$.
$III-b$) H has index 2 in K, and $C \to C/K$ is a covering of $\mathbb{P}^1$ branched on 4 points, $P_1, \ldots, P_4$, with branching indices $2, 2, c_3, c_4$, where $c_3 \leq c_4$ and $c_4 > 2$. Moreover the subgroup H corresponds to a genus zero double cover of $\mathbb{P}^1$ branched on two points with branching index 2.
$III-c$) H is normal in K, $K/H \cong (\mathbb{Z}/2)^2$, and $C \to C/K$ is a covering of $\mathbb{P}^1$ branched on 4 points, $P_1, \ldots, P_4$, with branching indices $2, 2, 2, c_4$, where $c_4 > 2$. Moreover the subgroup H corresponds to the unique genus zero cover of $\mathbb{P}^1$ with Galois group $(\mathbb{Z}/2)^2$ branched on the 3 points P_1, P_2, P_3 with branching index 2.

It is easy to see that in all cases of (4.6), we have that $K = G(H)$, hence we call the cases in (4.6) the *cover types* (of H and $G(H)$).

The second question is purely group-theoretic, from (4.6) we see that there are 2 possibilities:
(i) for cases I), II), III-a), III-b) we have $[G(H) : H] = 2$;
(ii) for case III-c) we have that $H \lhd G(H)$ and $G(H)/H \cong (\mathbb{Z}/2)^2$.
Given a finite group H, we call the possibilities of $G(H)$ the *group types*

(of H and $G(H)$).
Unfortunately, there is no general result on group types for an arbitrary finite group, we will see the group types for cyclic groups in section 5 and for dihedral groups in section 7.

Now we have all the ingredients: given a finite group H, we compute the possible equivalence classes of Hurwitz vectors for each cover type and group type. For more details, see section 7 for the case of dihedral groups.

5. **Cyclic covers of curves**

In this section the we consider the case of cyclic groups: $G = \mathbb{Z}/d$, and we fix a generator γ of $\mathbb{Z}/d$. We start by recalling the structure theorem for cyclic covers of Riemann surfaces, then we compute the group types for cyclic groups and determine the corresponding Hurwitz vectors.

Theorem 5.1 ([Cat12], Theorem 1.1). (1) *Let C be a smooth irreducible projective curve, the datum of a pair (B, γ), where B is also a smooth irreducible projective curve and γ is an order d automorphism of B such that $B/\langle\gamma\rangle \cong C$, is equivalent to the datum of reduced effective divisors $D_1, ..., D_{d-1}$ on C without common components, and of a divisor class L such that we have the following linear equivalence*

$$dL \equiv \sum_i iD_i$$

and moreover, setting $m := \gcd\{d, \{i|D_i \neq 0\}\}$, either $m = 1$ or, setting $d = mn$, the divisor class

$$L' := \frac{d}{m}L - \sum_i \frac{i}{m}D_i$$

has order precisely m in $Pic(C)$.
(2) *Let $\mathbb{L}$ be the geometric line bundle whose sheaf of regular sections is $\mathcal{O}_C(L)$. Then B is the normalization of the singular covering*

$$B' \subset \mathbb{L}, B' := \{(y, z)|z^d = \prod_{i=1}^{d-1} \delta_i^i(y)\}$$

where $y \in C$, $z \in \mathbb{L}_y$ and δ_i is a section of $\mathcal{O}_C(D_i)$ such that the zero set of δ_i is D_i. Moreover the action of γ on B' is given by $z \mapsto e^{\frac{2\pi\sqrt{-1}}{d}}z$.

Remark 5.2. Given a point $p \in D_i$, we can choose t, a suitable local coordinate near p, such that in an analytic neighbourhood of p we have $B' = \{(t, z)|z^d = t^i\}$. Setting $m_i := \gcd(d, i)$, $d = d_i m_i$ and $i = i' m_i$, we see that B' has m_i branches at the point $(0, 0)$, say one branch is $B'_1 := \{(t, z)|z^{d_i} = t^{i'}\}$. The morphism $\mathbb{C} \supset U \to B'_1 : x \mapsto (x^{d_i}, x^{i'})$ gives a normalization of B'_1. Hence for the covering map $B \to C$, the inverse image of p has m_i points and locally the map is given by $x \mapsto x^{d_i}$. We have the following observations.
(1) Let $q \in B$ be a point lying over p: the stabilizer group of q is generated by γ^{m_i}, hence the branching index at the point p is equal to

$order(\gamma^{m_i}) = d_i$.
(2) Locally around q, the action of γ^{m_i} is given by $x \mapsto e^{\frac{2\pi\sqrt{-1}}{d_i}} x$.

Definition 5.3 ([Cat12], Definition 2.2)**.** Let C be a smooth irreducible projective curve of genus g on which $\mathbb{Z}/d$ acts faithfully, and set $C' = C/(\mathbb{Z}/d)$, $h := genus(C')$.
Denote by $k_i = deg(D_i)$ for $i = 1, ..., d-1$, and by $(k_1, ..., k_{d-1})$ the *branching sequence* of γ. A change of generator of $\mathbb{Z}/d$ corresponds to a $(\mathbb{Z}/d)^*$-action on the sequence, we denote the resulting equivalence class by $[(k_1, ..., k_{d-1})]$.

Definition 5.4 ([Cat12], Definition 2.3)**.** Given a branching datum corresponding to a sequence $[(k_1, ..., k_{d-1})]$, set

$$h := 1 + \frac{2(g-1)}{2d} - \frac{1}{2}\sum_i k_i(1 - \frac{\gcd(i,d)}{d}).$$

The branching datum is said to be *admissible* for d and g if the following two conditions are satisfied:
(1) $\sum_i k_i i \equiv 0 \pmod d$,
(2) h is a positive integer; or $h = 0$, $\gcd\{d, \{i | k_i \neq 0\}\} = 1$.

Remark 5.5. In section 4 we have introduced the notion of the $B.A.-$ equivalence relation of Hurwitz vectors. Now assuming the base curve C' has genus 0, it is easy to see that in the cyclic cover case $B.A.-$equivalence classes of Hurwitz vectors are in one to one correspondence with equivalence classes of admissible branching sequences.
Starting with an admissible branching sequence $(k_1, ..., k_{d-1})$, we construct a vector

$$v := (\underbrace{\gamma, ..., \gamma}_{k_1 \ times}, ..., \underbrace{\gamma^{d-1}, ..., \gamma^{d-1}}_{k_{d-1} \ times}).$$

Condition (1) of (5.4) is equivalent to that the product of all the entries in v is 1, condition (2) is equivalent to that entries of v generate the group $\mathbb{Z}/d$; hence v is indeed a Hurwitz vector. Since $\mathbb{Z}/d$ is abelian, the braid action on Hurwitz vectors corresponds to the permutation of entries. Then one checks easily that the above construction induces a $1-1$ correspondence between equivalence classes of admissible branching sequences and $B.A.-$equivalence classes of Hurwitz vectors.

Given an admissible branching datum $[(k_1, ..., k_{d-1})]$, there is a connected complex manifold $\mathcal{T}_{g;d,[(k_1,...k_{d-1})]}$ parameterizing the pair $(C, \mathbb{Z}/d)$

with branching datum $[(k_1, ..., k_{d-1})]$ (cf.(2.4) in [Cat12]). The rough idea there is to consider $\mathcal{T}_{h,k}$, the Teichmüller space of curves of genus h with $k := \sum_i k_i$ marked points, there is a universal family of curves over $\mathcal{T}_{h,k}$ with k sections. Then by applying a relative version of (5.1), we obtain a family of curves with the given branching datum.

We denote by $\mathfrak{M}_{g;d,[(k_1,...,k_{d-1})]}$ the image of $\mathcal{T}_{g;d,[(k_1,...k_{d-1})]}$ in $\mathfrak{M}_g$. Since $\mathcal{T}_{g;d,[(k_1,...k_{d-1})]}$ is a connected manifold, we see that $\mathfrak{M}_{g;d,[(k_1,...,k_{d-1})]}$ is an irreducible subvariety of $\mathfrak{M}_g$. Again we have that

$$\mathfrak{M}_g(\mathbb{Z}/d) = \bigcup_{[(k_1,...,k_{d-1})] \ admissible} \mathfrak{M}_{g;d,[(k_1,...,k_{d-1})]}.$$

Therefore to determine the irreducible components of $\mathfrak{M}_g$ is equivalent to determining the maximal loci of the form $\mathfrak{M}_{g;d,[(k_1,...,k_{d-1})]}$.

Assume that $\mathfrak{M}_{g;d,[(k_1,...,k_{d-1})]} \subset \mathfrak{M}_{g;d,[(k'_1,...,k'_{d-1})]}$ and $[(k_1, ..., k_{d-1})] \neq [(k'_1, ..., k'_{d-1})]$. We pick a general curve $C \in \mathfrak{M}_{g;d,[(k_1,...,k_{d-1})]}$, then there exist two distinct subgroups H and H' of $Aut(C)$ such that both H and H' are isomorphic to $\mathbb{Z}/d$ and the image of $Fix(H)$ (resp. $Fix(H')$) in $\mathfrak{M}_g$ is $\mathfrak{M}_{g;d,[(k_1,...,k_{d-1})]}$ (resp. $\mathfrak{M}_{g;d,[(k'_1,...,k'_{d-1})]}$). This implies that H is a proper subgroup of $Aut(C)$ and $Aut(C) = G(H)$. Therefore we can apply (4.6).

First we determine the group type of $(H, G(H))$. We have two cases: $G(H)/H \simeq \mathbb{Z}/2$ or $G(H)/H \simeq (\mathbb{Z}/2)^2$.

If $G(H)/H \simeq \mathbb{Z}/2$, let γ be a generator of H and choose $\delta \in G(H) - H$, we have that $\delta^2 = \gamma^k$ for some $0 \leq k \leq d-1$. Since H is a normal subgroup, we have that $\delta\gamma\delta^{-1} = \gamma^l$, where l is an integer such that $\gcd(l, d) = 1$. Noting that $\gamma = \delta^2\gamma\delta^{-2} = \gamma^{l^2}$ and $\gamma^k = \delta\gamma^k\delta^{-1} = \gamma^{kl}$, we obtain the condition that $l^2 \equiv 1 \mod d$ and $d|(kl - k)$. It is easy to check that these data determine the pair $(H, G(H))$:

Lemma 5.6. *Let $G(H)$ be a group containing an index 2 cyclic subgroup H of order d. Then $G(H)$ has the presentation:*

$$\{\alpha, \beta | \alpha^d = 1, \alpha^k = \beta^2, \beta\alpha = \alpha^l\beta\}$$

such that $0 \leq k, l < d$, $\gcd(l, d) = 1$, $d|((l-1)k)$ and $l^2 \equiv 1 \mod d$. Moreover, $\gamma := \bar{\alpha}$ is a generator of H.

For the case of $G(H)/H \simeq (\mathbb{Z}/2)^2$, using similar arguments and with additional assumptions it is easy to get the following:

Lemma 5.7. *Let $G(H)$ be a group containing a normal cyclic subgroup H of order d such that $G(H)/H \simeq (\mathbb{Z}/2)^2$. Assume in addition that there exist three elements $b_1, b_2, b_3 \in G(H) - H$ such that b_i has order 2 and the product $b_1b_2b_3$ is contained in H. Then $G(H)$ has the presentation:*

$$\{\alpha, \beta_1, \beta_2 | \alpha^d = 1, \beta_1^2 = \beta_2^2 = 1, \beta_1\alpha = \alpha^{l_1}\beta_1, \beta_2\alpha = \alpha^{l_2}\beta_2, \beta_1\beta_2 = \beta_2\beta_1\alpha^{e_{1,2}}\}$$

such that $0 \leq l_1, l_2, e_{1,2} < d$, $\gcd(l_i, d) = 1$, $l_i^2 \equiv 1 \mod d$, $d|(l_i+1)e_{1,2}$, for $i = 1, 2$ and $\gcd(d, l_1l_2 + 1)|e_{1,2}$.
Moreover, $\gamma := \bar{\alpha}$ is a generator of H; $b_i = \bar{\beta}_i$, $b_i\gamma b_i = \gamma^{l_i}$ for $i = 1, 2$ and $b_2b_1b_2 = b_1\gamma^{e_{1,2}}$; $b_3 = b_1b_2\gamma^f$, where f is an integer such that $0 \leq f < d$ and $d|((l_1l_2 + 1)f + e_{1,2})$.

Now we are ready to determine the non-maximal loci.

Proposition 5.8. *Assume that $g \geq 2$, $d \geq 3$ and $dim_{\mathbb{C}}\mathfrak{M}_{g;d,[(k_1,...,k_{d-1})]} \geq 1$, then $\mathfrak{M}_{g;d,[(k_1,...,k_{d-1})]}$ is maximal for all admissible sequences $[(k_1, ..., k_{d-1})]$ except for the following (possible) cases:*
(1) Case $III - b)$ (of 4.6).
(i) $d = 2d'$, $G(H) \simeq H \times \mathbb{Z}/2$. Denoting by γ a generator of H, the Hurwitz vectors for $G(H)$ are:

$$((e, 1), (e, 1), (\gamma^{-1}, 0), (\gamma, 0))$$
$$\text{and } ((e, 1), (\gamma^{d'}, 1), (\gamma^{d'-1}, 0), (\gamma, 0)).$$

(ii) Following the notations in (5.6), $d = 2d' = 8\tilde{d}$, $k = 2$ and $l = d'+1$, the Hurwitz vectors for $G(H)$ are:

$$(\delta\gamma^{2\tilde{d}-1}, \delta\gamma^{2\tilde{d}-1}, \gamma^{-1}, \gamma)$$
$$\text{and } (\delta\gamma^{2\tilde{d}-1}, \delta\gamma^{6\tilde{d}-1}, \gamma^{4\tilde{d}-1}, \gamma).$$

(2) Case $III - c)$ (of 4.6).
$d = 2d'$, following the notations in (5.7), the Hurwitz vector for $G(H)$ is

$$(b_1, b_2, b_3, b_3b_2b_1).$$

There are three possibilities:
(i) $(l_1, l_2, e_{1,2}) = (1, d - 1, 0)$;
(ii) $2|d'$, $(l_1, l_2, e_{1,2}) = (1, d' - 1, d')$;
(iii) $2|d'$, $(l_1, l_2, e_{1,2}) = (1 + d', d - 1, d')$ or $(1 + d', d' - 1, 0)$.

Moreover, in the above three cases f must satisfy the condition that $\gcd(f,d)=1$.

Proof. Let H be an order d cyclic subgroup of Map_g such that $Fix(H)$ maps surjectively onto $\mathfrak{M}_{g;d,[(k_1,\ldots,k_{d-1})]}$. If $\mathfrak{M}_{g;d,[(k_1,\ldots,k_{d-1})]}$ is not maximal, then there exists a subgroup $H' \subset G(H)$ such that $H' \simeq H$ and $H' \neq H$. As we have seen in (4.6), there are two possibilities:
(a) $[G(H):H]=2$ (which corresponds to cases $I, II, III-a, III-b$ in 4.6), we see that $[H' : H\cap H']=2$, hence d must be an even number, say $d=2d'$. In this case $H\cap H' =<\gamma^2>$. Applying (5.6) and choosing an appropriate generator δ of H', we have that $\delta^2=\gamma^2$ and $\delta\gamma\delta^{-1}=\gamma^l$ such that $\gcd(l,d)=1$, $l^2\equiv 1 \mod d$ and $d|2(l-1)$. The condition $d|2(l-1)$ is equivalent to that $d'|(l-1)$, hence we have either $l=1$ or $l=d'+1$, in the latter case we must have $2|d'$ since $\gcd(l,d)=1$.
We determine all the elements of oder 2 in $G(H)$. It is clear that $\gamma^{d'}$ and $\delta^{d'}$ have order two, and they are equal iff $2|d'$. The other candidates are of the form $\delta\gamma^m$ $(0<m<d)$, and the equality $1=(\delta\gamma^m)^2=\gamma^{(l+1)m+2}$ holds if and only if $\gcd(d,l+1)|2$. Since d is even and $\gcd(d,l)=1$, we have that $2|\gcd(d,l+1)$, hence the only possibility is $\gcd(d,l+1)=2$.
If $l=1$, this is automatically true, and we get $m=2d'-1$ or $d'-1$.
If d' is even and $l=d'+1$, we see that $\gcd(d,l+1)=2 \Leftrightarrow \gcd(d',\frac{d'}{2}+1)=1 \Leftrightarrow 4|d'$. In this case we get $m=\frac{d'}{2}-1$ or $\frac{3d'}{2}-1$.

For case I of (4.6), if there exists a homomorphism $\mu : T(2,2,2,2,2,2) \to G(H)$ which determines the cover $C \to C/G(H)$ (for some general curve $C \in Fix(G(H))$, denoting by $(v_1,\ldots,v_6)$ the Hurwitz vector associated to μ, then v_i has order 2 for $i=1,2,\ldots,6$. However from the proceeding argument we see that the elements of order 2 in $G(H)$ generate a subgroup of order 2 or 4, hence f can not be surjective, this case is excluded. Similarly one can also exclude cases II, III-a of (4.6).

For case III-b of (4.6), picking a general curve $C \in Fix(H)$, the cover $C \to C/G(H)$ has an associated Hurwitz vector $v=(v_1,v_2,v_3,v_4)$, such that $G(H) =< v_i >$, $v_3,v_4 \in H$, $v_1,v_2 \notin H$, $\Pi v_i = 1$ and $c_1=c_2=2$, where $c_i := order(v_i)$. If $G(H)$ has only one element of order 2, which must be $\gamma^{d'}$, we immediately get a contradiction, hence we have the following two cases:
(α) $\delta\gamma\delta^{-1}=\gamma$, i.e., $G(H)$ is commutative. We have that $v_1,v_2 \in$

$\{\delta\gamma^{-1}, \delta\gamma^{d'-1}\}$. If $v_1 = v_2 = \delta\gamma^{-1}$, then we have $v_3^{-1} = v_4 = \gamma^k$ for some integer k, the condition that $G(H) =< v_i >$ implies that $\gcd(k, d) = 1$. Now by applying the automorphism $\gamma^k \mapsto \gamma, \delta\gamma^{-1} \to \delta\gamma^{-1}$, we see that

$$v \sim_{B.A.} (\delta\gamma^{-1}, \delta\gamma^{-1}, \gamma^{-1}, \gamma).$$

With similar argument, we get two more Hurwitz vectors

$$(\delta\gamma^{-1}, \delta\gamma^{d'-1}, \gamma^{d'-1}, \gamma)$$

and

$$(\delta\gamma^{d'-1}, \delta\gamma^{d'-1}, \gamma^{-1}, \gamma),$$

the latter being B.A.-equivalent to $(\delta\gamma^{-1}, \delta\gamma^{-1}, \gamma^{-1}, \gamma)$ via the automorphism $\gamma \mapsto \gamma, \delta\gamma^{-1} \mapsto \delta\gamma^{d'-1}$.
(β) $d' = 4\tilde{d}$ and $\delta\gamma\delta^{-1} = \gamma^{d'+1}$. Note that for any integer k which is coprime to d, the map $\delta \mapsto \delta^k, \gamma \mapsto \gamma^k$ defines an automorphism of $G(H)$ fixing H. As in (α), we can show that v is B.A.-equivalent to either

$$(\delta\gamma^{2\tilde{d}-1}, \delta\gamma^{2\tilde{d}-1}, \gamma^{-1}, \gamma)$$

or

$$(\delta\gamma^{2\tilde{d}-1}, \delta\gamma^{6\tilde{d}-1}, \gamma^{4\tilde{d}-1}, \gamma).$$

(b) If $G(H)/H \simeq (\mathbb{Z}/2)^2$ (which corresponds to the case III-c in 4.6), let γ' be a generator of H': we have that $H \cap H' =< \gamma'^2 >$ or $< \gamma'^4 >$, the latter being impossible since $\bar{\gamma'}$ can not have order 4 in $G(H)/H$. We also see that d is an even number, say $d = 2d'$. Following the notations in (5.7), any element which is not contained in H has the form $b_i\gamma^{m_i}$, $i = 1, 2, 3$. Since $b_i\gamma^{m_i} \notin H$ and $(b_i\gamma^{m_i})^2 = \gamma^{(l_i+1)m_i}$, the order of $b_i\gamma^{m_i}$ is $\frac{2d}{\gcd(d,(l_i+1)m_i)}$, where $l_3 :\equiv l_1 l_2 \mod d$.
We assume first that H' has a generator of the form $b_1\gamma^{m_1}$ for some integer m_1, then we have $d = order(b_1\gamma^{m_1}) = \frac{2d}{\gcd(d,(l_i+1)m_1)}$, which implies that $2 = \gcd(d, (l_1+1)m_1)$, or equivalently $1 = \gcd(d', \frac{l_1+1}{2}m_1)$, hence $\gcd(d', \frac{l_1+1}{2}) = 1$. Combining with the condition that $d|(l_1^2 - 1)$, we get $d'|(l_1 - 1)$, therefore we have that $l_1 = 1$ or $d' + 1$, for the latter case d' must be even. If H' has a generator of the form $b_2\gamma^{m_2}$, we reduce it to the known case: setting $b'_1 = b_2$, $b'_2 = b_2 b_1 b_2$ and $b'_3 = b_3$, we see that $(b'_1, b'_2, b'_3, b'_3 b'_2 b'_1)$ is $B.A.$-equivalent to $(b_1, b_2, b_3, b_3 b_2 b_1)$ and b'_1, b'_2, b'_3 also satisfy the condition of (5.7). If H' has a generator of the form $b_3\gamma^{m_3}$, similarly we can reduce it to the known case.
We need to determine the subgroup $K := \langle b_1, b_2, b_3\rangle$. By (5.7) we have $b_2 b_1 b_3 = \gamma^f$, where f satisfies the condition that $d|((l_1 l_2+1)f + e_{1,2})$, let

$e_{i,3}$ be an integer such that $b_3 b_i b_3 = b_i \gamma^{e_{i,3}}$ for $i = 1, 2$. One computes easily that $d|((l_2+1)f - e_{1,3})$ and $d|((l_1+1)f - e_{2,3})$. Then we conclude that any element in K must be of the form $\gamma^{jf}, b_1\gamma^{j_1 f}, b_2\gamma^{j_2 f}$ or $b_3\gamma^{j_3 f}$ for some integers j, j_1, j_2, j_3. Therefore we see that the condition $K = G(H)$ is equivalent to that $\gcd(d, f) = 1$. We have several cases:
(i) $l_1 = 1, e_{1,2} = 0$. Then we have that d divides $(l_1 l_2 + 1)f + e_{1,2} \equiv (l_2+1)f \mod d$. Since $\gcd(d, f) = 1$, we get $d|(l_2+1)$, hence $l_2 = d-1$. In this case we have that $G(H) \simeq D_d \times \mathbb{Z}/2$, where D_d corresponds to the dihedral group generated by γ and b_2.
Using similar arguments as in $(b) - (i)$, we obtain the following:
(ii) $l_1 = 1, e_{1,2} = d'$. Then $l_2 = d' - 1$ providing that $2|d'$.
(iii) $2|d'$ and $l_1 = 1 + d'$. We have that $(e_{1,2}, l_2) = (0, d' - 1)$ or $(d', 2d' - 1)$.

□

6. **Results on dihedral covers**

In this section we have a quick review of the results of F. Catanese, M. Lönne and F. Perroni on the $B.A.$-equivalence classes of D_n-Hurwitz vectors. All contents of this section come from [CLP11] and [CLP15]. Recall that D_n, the dihedral group of order $2n$, has a standard presentation:

$$D_n = \langle x, y|\ x^n = y^2 = (xy)^2 = 1\rangle.$$

We call the elements x^i *rotations* and the elements yx^i *reflections.* Sometimes x^iy will also be denoted by i for $0 \leq i \leq n-1$.

Remark 6.1. We have the following easy observations.
(1) The conjugacy class of x^i is $\{x^i,\ x^{-i}\}$.
(2) If n is odd, any two reflections are conjugate; if n is even, two reflections yx^i and yx^j are conjugate $\iff$ $2|(i-j)$.

Definition 6.2. Given a finite group G, denote by $\xi_1, ..., \xi_l$ the non-trivial conjugacy classes of G, where l is the number of non-trivial conjugacy classes of G. The *Nielsen function* $\nu(v)$ of a G-Hurwitz vector (cf. 4.2) $v = (a_1, b_1, ..., a_{g'}, b_{g'}; c_1, ..., c_r)$ is a vector

$$\nu(v) := (k_{\xi_1}, ..., k_{\xi_l}),$$

where k_{ξ_j} is the number of the c_i's in the conjugacy class ξ_j.
In the case of $G = D_n$, we make it more explicit.
If $n = 2n'+1$ is odd, $\nu(v) = (k, k_1, ..., k_{n'})$, where k (resp. k_i) is the number of the c_i's in the conjugacy class of y (resp. x^i).
If $n = 2n'$ is even, $\nu(v) = (k_y, k_{xy}, k_1, ..., k_{n'})$, where k_y (resp. k_{xy}, k_i) is the number of the c_i's in the conjugacy class of y, (resp. xy, x^i).
The $\mathrm{Aut}(G)$-action on the set of Hurwitz vectors induces an action of $\mathrm{Aut}(G)$ on the set $\mathcal{N} := \{\nu(v)|v\ is\ a\ Hurwitz\ vector\}$, the equivalence class of $\nu(v)$ in $\mathcal{N}/Aut(G)$ will be denoted by $[\nu(v)]$.

Let $\pi : X \to Y$ be a Galois cover of compact connected Riemann surfaces with Galois group D_n. Denote by g the genus of X and by g' the genus of Y. Let $\{y_1, ..., y_r\} \subset Y$ be the branch locus of π with branching indices $m_1, ..., m_r$. As explained in (4.3), π is (topologically) determined by a surjective homomorphism:

$$\mu : T(g'; m_1, ..., m_r) \twoheadrightarrow D_n.$$

In the case that $n = 2n'$ is even, let $\epsilon : D_{2n'} \to (\mathbb{Z}/2)^2$ be the canonical surjection onto the Abelianization, and let $p : Z \to Y$ be the degree 4

covering associated to the composition $\mu' := \epsilon \circ \mu$.
Observe that p is unramified iff none of the elements c_i is a reflection or a rotation x^i with odd exponent i.
By the Hurwitz formula applied to μ and μ', the geometrical property that p is unramified is just a property of μ.

Definition 6.3 ([CLP11], Definition 2)**.** The *numerical type* of a D_n-Hurwitz vector v is defined as follows.
If $n = 2n' + 1$ is odd, it is the pair $(g', [\nu(v)])$.
If $n = 2n'$ is even, there are two cases:
(I) If p is ramified, then the numerical type is again the pair $(g', [\nu(v)])$.
(II) if p is unramified, then consider the two dimensional subspace U of $H^1(Y, \mathbb{Z}/2)$ dual to the surjection $H_1(Y, \mathbb{Z}/2) \to (\mathbb{Z}/2)^2$ through which μ' factors.
Define $\iota \in \{0, 1\}$ to be $= 0$ if U is isotropic, and $= 1$ otherwise.
Then the numerical type is defined as the triple $(g', [\nu(v)], \iota)$.
Note that for $g' = 0$, only case (I) occurs.

For our later use, we are mainly interested in the case of $g' = 0$, we have the following theorem, which is the main result of [CLP11].

Theorem 6.4 ([CLP11], Theorem 2)**.** *The group $\mathfrak{B}_r \times \mathrm{Aut}(D_n)$ acts transitively on the set of Hurwitz vectors of a fixed numerical type, hence dihedral covers of $\mathbb{P}^1$ of a fixed numerical type form an irreducible closed subvariety of the moduli space.*
More precisely, given v with $\nu(v) = (k, k_1, ..., k_{n'})$ (resp. $\nu(v) = (k_y, k_{xy}, k_1, ..., k_{n'})$), set $R := \sum_i k_i$, and assume (w.l.o.g.) $\{h, k\} = \{k_y, k_{xy}\}, h \leq k$ (observe that k, resp. $k + h$ is even).
We have then, assuming throughout $0 < l_i \leq l_{i+1} \leq n'$, $\underline{l} = (l_1, ..., l_R)$ and setting $|\underline{l}| :\equiv \sum_i l_i \mod n$:

i) $v \sim_{B.A.} (\underbrace{0, ..., 0, 1 + |\underline{l}|}_{k}, x^{l_1}, ..., x^{l_R})$, *if* $n = 2n' + 1$.

ii) $v \sim_{B.A.} (\underbrace{0, ..., 0}_{h}, \underbrace{1, ..., 1, \lambda}_{k}, x^{l_1}, ..., x^{l_R})$, *if* $n = 2n'$ *and* $h \neq 0$.
Here $\lambda = |\underline{l}| + \epsilon$, *where* $\epsilon \in \{0, 1\}$, $\epsilon + k \equiv 1 \mod 2$.

iii) $v \sim_{B.A.} (\underbrace{1, ..., 1, 3, \lambda}_{k}, x^{l_1}, ..., x^{l_R})$, *if* $n = 2n'$ *and* $h = 0$.
Here $\lambda = |\underline{l}| + 3$.

The following lemma is important for our later computations.

Lemma 6.5. *Every D_n-Hurwitz vector of length r of the form*

$$v = (v_1, ..., yx^a, yx^b, yx^c, ..., v_r)$$

is B.A.-equivalent to

$$v' = (v_1, ..., yx^{a'}, yx^{a'}, yx^{c'}, ..., v_r)$$

or

$$v'' = (v_1, ..., yx^{a''}, yx^{b''}, yx^{b''}, ..., v_r)$$

via braid moves that only affect the triple (yx^a, yx^b, yx^c).

Proof. See [CLP11], Lemma 2.1. □

Remark 6.6. In regard to the case of $g' \geq 1$, in [CLP15] the authors have defined a new homological invariant which allows them to tell when two G-Hurwitz vectors are not B.A.-equivalent; for the case of $H = D_n$, the dihedral group, they also found one representative vector for each unmarked topological type.

7. **Irreducible components of $\mathfrak{M}_g(D_n)$**

In this section we determine the irreducible components of $\mathfrak{M}_g(D_n)$, the main result is theorem (7.14), which is a joint work with Sascha Weigl (cf. [LW16]).

As explained in section 4, the decomposition problem is equivalent to the classification of subgroups H, H' of Map_g $(g \geq 2)$, where H and H' satisfy the following condition:

$$(**) \text{ H is isomorphic to } D_n \text{ and not full, } G(H) \text{ has a subgroup } H',$$

$$\text{which is isomorphic to } D_n \text{ and different from } H.$$

The content of this section is arranged as follows.
First we apply (4.6) and the Riemann-Hurwitz formula to obtain pairs of dimensions $(\delta_H, \delta_{H'})$, which can occur under condition $(**)$.
Then we compute the group types of H and $G(H)$. This is done by classifying the index 2 subgroups of $G(H)$, where $G(H)$ is a finite group containing two distinct index 2 subgroups which are isomorphic to D_n.
In the end of this section we classify the B.A.-equivalence classes of Hurwitz vectors of the map $C \to C/G(H) \simeq \mathbb{P}^1$ for each cover type and group type, by giving one representative vector for each equivalence class. The results are presented through tables.

7.1. **A rough classification.** Given a general curve $C \in Fix(H)$ such that $C \to C/H$ is a Galois cover branched on r points, we have that $\delta_H = 3g_{C/H} - 3 + r$ (cf. [CLP15], Theorem 2.3).
The case of $\delta_H = \delta_{H'}$ was studied in Corollary 7.2 of [CLP2]. We only consider the case of $\delta_H < \delta_{H'}$.

Since we have the condition $(**)$, which implies that $\delta_{G(H)} = \delta_H$, we can apply Theorem (4.6). Moreover we apply the Riemann-Hurwitz formula to each cover type to get the possible pairs $(\delta_H, \delta_{H'})$.

Corollary 7.1. *Assume $(**)$ and moreover $\delta_H < \delta_{H'}$. Then the following pairs of dimensions $(\delta_H, \delta_{H'})$ can occur:*
I) $(3, 4)$, $(3, 5)$.
II) $(2, 3)$, $(2, 4)$.
$III - a$) $(1, 2)$.

$III-b)$ $(1,2),(1,3)$.
$III-c)$ *None.*

Proof. $I)$ $\delta_H = 3$.
By the Riemann-Hurwitz formula, we have

$$2g(C) - 2 = |G(H)|(-2 + 6 \cdot \frac{1}{2}) = |H'|(2(g_{C/H'} - 1) + k/2),$$

where k is the number of branching points of the cover $C \to C/H'$.
It is easy to see that $(g_{C/H'}, k) = (2,0), (1,4)$ or $(0,8)$, corresponding to $\delta_{H'} = 3, 4, 5$. Since we require $\delta_H < \delta_{H'}$, the possible pairs are (3,4) and (3,5).
$II)$ $\delta_H = 2$.
In this case $C/H' \to \mathbb{P}^1$ is a double cover branched in at most 5 points. Using the Riemann-Hurwitz formula, there are two cases:
(i) $g_{C/H'} = 0$ and $C/H' \to \mathbb{P}^1$ is branched on 2 of the 5 points with branching indices 2,2.
If $c_5 = 2$ or P_5 is not a branching point, we have $\delta_{H'} = 3$;
otherwise c_5 is even and bigger than 2 and P_5 is a branching point, we get $\delta_{H'} = 4$.
(ii) $g_{C/H'} = 1$ and $C/H' \to \mathbb{P}^1$ is branched in 4 of the 5 points with branching indices 2,2,2,2.
The only possible case in which $\delta_{H'} > 2$ is that c_5 is even and bigger than 2 and P_5 is one of the branching points. In this case $\delta_{H'} = 3$.
$III)$ $\delta_H = 1$.
$III-a)$ Similar to case $II)$, one gets that $g_{C/H'} = 0$, $C/H' \to \mathbb{P}^1$ is a double cover with one of the branching points P_4 and $\delta_{H'} = 2$.
$III-b)$ $i)$ If $c_3 = 2$, the only possibility is that c_4 being even, $g_{C/H'} = 0$ and $C/H' \to \mathbb{P}^1$ is a double cover with one of the branching points P_4, here $\delta_{H'} = 2$.
$ii)$ $c_3 > 2$, there are three possibilities:
$\alpha)$ c_3 or c_4 is even, one and only one point of P_3 and P_4 is a branching point. This case is similar to $III-b)-i)$, $\delta_{H'} = 2$.
$\beta)$ Both c_3 and c_4 are even, $g_{C/H'} = 0$, and $C/H' \to \mathbb{P}^1$ is a double cover branched on P_3 and P_4. We have $\delta_{H'} = 3$.
$\gamma)$ Both c_3 and c_4 are even, $g_{C/H'} = 1$, and $C/H' \to \mathbb{P}^1$ is a double cover branched on 4 points $P_1, \ldots, P_4$. We have $\delta_{H'} = 2$.
$III-c)$ We will give the proof in Lemma (7.7).

□

7.2. **The group type.** From Theorem (4.6) we know that $[G(H){:}H] = 2$ except for the case $III - c)$. Such a pair induces an exact sequence

$$1 \to H \to G(H) \to \mathbb{Z}/2 \to 1.$$

This type of extension, where $H = D_n$ and $G(H)$ has another subgroup H' isomorphic to D_n, has been classified in [CLP15], Proposition 7.4. There are 3 group types:

Group type 1) $G(H) \cong D_n \times \mathbb{Z}/2$, and H corresponds to the subgroup $D_n \times \{0\}$.
Group type 2) $n = 2d$, $G(H) \cong D_{2n} = \langle z, y | z^{2n} = y^2 = 1, yzy = z^{-1}\rangle$ and $H = \langle x := z^2, y\rangle$.
Group type 3) $n = 4h$, where h is odd, and $G(H)$ is the semidirect product of $H \cong D_n$ with $\langle \beta_2 \rangle \cong \mathbb{Z}/2$, such that the conjugation by β_2 acts as follows:

$$y \mapsto yx^2, x \mapsto x^{2h-1}.$$

For each group type, we determine the index 2 subgroups of $G(H)$ and find out which of them are isomorphic to D_n.

Group type 1) Recall the standard presentation $D_n = \langle x, y | x^n = y^2 = 1, yxy^{-1} = x^{-1}\rangle$ and set $C_n := \mathbb{Z}/n$.
Note that $H \cap H'$ is an index 2 subgroup of H and that $H \cap H' \triangleleft G(H)$. In order to determine all the possible subgroups H', we have to understand the index 2 subgroups K of H such that $K \triangleleft G(H)$.
$a)$ $K = C_n \times \{0\}$ (this is the only case when n is odd).
Since $G(H)/K \cong (\mathbb{Z}/2)^2$, there are two more index 2 subgroups of $G(H)$: $H_{1,1} := \langle K, (e, 1)\rangle$ and $H_{1,2} := \langle K, (y, 1)\rangle \cong D_n$.
$b)$ If $n = 2d$, there are two more cases: $K = \langle (x^2, 0), (y, 0)\rangle$ or $K = \langle (x^2, 0), (yx, 0)\rangle$ (both are isomorphic to D_d).
Here we have 4 more index 2 subgroups of $G(H)$: $H_{1,3} := \langle (x^2, 0), (y, 0), (e, 1)\rangle$, $H_{1,4} := \langle (x^2, 0), (y, 0), (x, 1)\rangle$, $H_{1,5} := \langle (x^2, 0), (yx, 0), (e, 1)\rangle$, and $H_{1,6} := \langle (x^2, 0), (yx, 0), (x, 1)\rangle$. One checks easily that $H_{1,4}$ and $H_{1,6}$ are isomorphic to D_n and that $H_{1,3}$ and $H_{1,5}$ are isomorphic to D_n if and only if d is odd.
Group type 2) Using similar arguments as for group type 1), we obtain 2 more index 2 subgroups of $G(H)$: $H_{2,1} := \langle x\rangle \cong C_{2n}$ and $H_{2,2} := \langle z^2, yz\rangle \cong D_n$.
Group type 3) There are 6 more index 2 subgroups of $G(H)$: $H_{3,1} :=$

$\langle C_n, (e, \beta_2)\rangle$, $H_{3,2} := \langle C_n, (y, \beta_2)\rangle$, $H_{3,3} := \langle (x^2, 0), (y, 0), (e, \beta_2)\rangle$, $H_{3,4} := \langle (x^2, 0), (y, 0), (x, \beta_2)\rangle$, $H_{3,5} := \langle (x^2, 0), (yx, 0), (e, \beta_2)\rangle$ and $H_{3,6} := \langle (x^2, 0), (yx, 0), (x, \beta_2)\rangle$. Only $H_{3,3}$ is isomorphic to D_n (since $H_{3,3} = \langle (y, \beta_2), (e, \beta_2)\rangle$).

7.3. **Hurwitz vectors for $C \to C/G(H)$.** We are ready to determine the pairs $(H, G(H))$ satisfying condition $(**)$. As explained in section 4, we find out all the possible Hurwitz vectors for the cover $C \to C/G(H)$. Note that according to (4.6), we have that $C/G(H) \simeq \mathbb{P}^1$.

Definition 7.2. Let $C \to C/G(H) \cong \mathbb{P}^1$ be a Galois cover of a given group type and cover type. We call a homomorphism

$$\mu : T(m_1, \dots, m_r) \to G(H)$$

admissible if it satisfies the following two conditions:
(1) μ is surjective, $T(m_1, ..., m_r)$ is isomorphic to the orbifold fundamental group of $C \to C/G(H)$ and $\mu(\gamma_i)$ has order m_i in $G(H)$.
(2) $\mu_H := \pi_H \circ f : T(m_1, \dots, m_r) \to G(H)/H$ corresponds to the cover $C/H \to \mathbb{P}^1$, where $\pi_H : G(H) \to G(H)/H$ is the quotient homomorphism.

Definition 7.3. Let $\mu : T(m_1, ...m_r) \to G$ and $\mu' : T(l_1, ...l_r) \to G$ be admissible for a given cover type and group type. We say that μ is equivalent to μ' if their corresponding Hurwitz vectors are in the same $\mathfrak{B}_r \times \mathrm{Aut}(G(H))_H$-orbit, where $\mathrm{Aut}(G(H))_H$ denotes the subgroup of $\mathrm{Aut}(G(H))$ which leaves H invariant.

Remark 7.4. An admissible μ determines both the covers $C \to C/G(H)$ and $C \to C/H$, hence we require the equivalence relation to be generated by $\mathfrak{B}_r$ and $Aut(G(H))_H$. It can happen that two admissible homomorphisms have $B.A.$-equivalent Hurwitz vectors, but are not equivalent (cf. Remark 7.13).

Example 7.5. Cover type $III - b$) and group type 1) (cf. Corollary 7.1)

i) $c_3 = 2$, assume that n is even and $c_4 = n$.
Consider the homomorphism $\mu : T(2, 2, 2, c_4) \to D_n \times \mathbb{Z}/2$, $\gamma_1 \mapsto (yx, 1)$, $\gamma_2 \mapsto (e, 1)$, $\gamma_3 \mapsto (y, 0)$, $\gamma_4 \mapsto (x, 0)$.
One computes easily that $\delta_{H_{1,2}} = \delta_{H_{1,6}} = 1$ and $\delta_{H_{1,4}} = 2$.
ii) $c_3 > 2$, assume we have an admissible μ, then it is easy to see

that $\mu(\gamma_3) = (x^{i_3}, 0)$ and $\mu(\gamma_4) = (x^{i_4}, 0)$. On the other hand, we have $\mu(\gamma_1), \mu(\gamma_2) \in \{(yx^k, 1), k \in \mathbb{Z}; (x^{n/2}, 1)(if\ n\ is\ even)\}$. Since $\Pi\mu(\gamma_i) = 1$, there are two possibilities:
(a) $\mu(\gamma_1), \mu(\gamma_2) = (x^{n/2}, 1)$, which implies $\mathrm{Im}(\mu) \subset \langle(x, 0), (0, 1)\rangle$, a contradiction.
(b) $\mu(\gamma_1) = (yx^{i_1}, 1)$ and $\mu(\gamma_2) = (yx^{i_2}, 1)$, which implies $\mathrm{Im}(\mu) \subset \langle(x, 0), (y, 1)\rangle$, again a contradiction.

Now we classify all admissible μ's for the Galois cover $C \to C/G(H)$, in the following way: for each cover type and group type, we construct all possible Hurwitz vectors according to their branching behavior, as given in Theorem (4.6).

Lemma 7.6. *Group type* 2) *has no admissible* μ *for any cover type.*

Proof. Cover type I)
Assume that we have an admissible $\mu : T(2, 2, 2, 2, 2, 2) \to D_{2n}$, then $\mu_H(\gamma_i) = 1$ for $i = 1, \ldots, 6$, which implies that $\mu(\gamma_i) \in \{yz^{2k+1}, z^{2l+1}, k, l \in \mathbb{Z}\}$. Moreover $\mu(\gamma_i)$ has order two, thus $\mu(\gamma_i) \in \{yz^{2k+1}, k \in \mathbb{Z}\}$. We find that $\mathrm{Im}(\mu) \subset H_{2,2}$, a contradiction.
Cover type II)
If there exists an admissible $\mu : T(2, 2, 2, 2, c_5) \to D_{2n}$, then we see that $\mu(\gamma_i) \in \{yz^{2k+1}, k \in \mathbb{Z}\}$ for $i = 1, 2, 3, 4$ and $\mu(\gamma_5) \in \{z^{2l}, l \in \mathbb{Z}\}$ (since $\Pi\mu(\gamma_i) = 1$), which implies that $\mathrm{Im}(\mu) \subset H_{2,2}$, a contradiction.
Cover type III-a)
Given an admissible $\mu : T(2, 2, 2, 2d_4) \to D_{2n}$, we have that $\mu(\gamma_i) \in \{yz^{2k+1}, k \in \mathbb{Z}\}\}$ for $i = 1, 2, 3$ and $\mu(\gamma_4) \in \{z^{2l+1}, l \in \mathbb{Z}\}$. However the product $\Pi\mu(\gamma_i) \neq 1$, a contradiction.
Cover type III-b)
i) $c_3 = 2$. We have $\mu(\gamma_i) = yz^{2k_i+1}$ for $i = 1, 2$, $\mu(\gamma_3) = yz^{2k_3}$ or z^n and $\mu(\gamma_4) = z^{2k_4}$. If $\mu(\gamma_3) = yz^{2k_3}$ we find $\Pi\mu(\gamma_i) \neq 1$; otherwise we have $\mu(\gamma_3) = z^n$, which implies $\mathrm{Im}(\mu) \subset \langle yz, z^2\rangle$. In both cases we have no admissible μ.
ii) $c_3 > 2$. We have $(\mu(\gamma_1), \mu(\gamma_2), \mu(\gamma_3), \mu(\gamma_4)) = (yz^{2k_1+1}, yz^{2k_2+1}, z^{2k_3}, z^{2k_4})$. We see that $\mathrm{Im}(\mu) \subset \langle yz, z^2\rangle$, a contradiction. □

Lemma 7.7. *Group type* 3) *has no admissible* μ *for any cover type.*

Proof. First of all we determine the order 2 elements of the form (a, β_2) in $G(H)$. One computes easily that $(x^j, \beta_2)^2 = (x^{2jh}, 0)$ and $(yx^k, \beta_2)^2 = (x^{2kh-2k+2}, 0) \neq (e, 0)$. Therefore we conclude that (a, β_2) has order two $\Leftrightarrow a = x^j$ and j is even.

Cover type I)

Now assume that we have an admissible μ, which implies that $\mu(\gamma_i) = (x^{2j_i}, \beta_2)$. On the other hand these elements are contained in the proper subgroup $\langle (x^2, 0), (e, \beta_2) \rangle$, we see that μ can not be surjective, a contradiction.

Cover type II)

If there exists an admissible μ, then we must have $\mu(\gamma_i) = (x^{2j_i}, \beta_2)$ for $i = 1, 2, 3, 4$, and since $\Pi\mu(\gamma_i) = 1$ it follows that $\mathrm{Im}(\mu) \subset \langle (x^2, 0), (e, \beta_2) \rangle$, a contradiction.

Cover type III-a)

Assuming we have an admissible μ, we see that $\mu(\gamma_i) = (x^{2j_i}, \beta_2)$ for $i = 1, 2, 3$. Since $\Pi\mu(\gamma_i) = 1$ it follows that $\mathrm{Im}(\mu) \subset \langle (x^2, 0), (e, \beta_2) \rangle$, again a contradiction.

Cover type III-b)

i) $c_3 = 2$. We must have $\mu(\gamma_1) = (x^{2j_1}, \beta_2)$, $\mu(\gamma_2) = (x^{2j_2}, \beta_2)$, $\mu(\gamma_3) = (x^{2h}, 0)$ or $(yx^k, 0)$ and $\mu(\gamma_4) = (x^l, 0)$ for some integer $l \neq 2h$. If $\mu(\gamma_3) = (x^{2h}, 0)$, then $\mathrm{Im}(\mu) \subset \langle (x, 0), (0, \beta_2) \rangle$; if $\mu(\gamma_3) = (yx^k, 0)$ we see that $\Pi\mu(\gamma_i) \neq 1$. In both cases we can not get an admissible μ.
ii) $c_3 > 2$. Given an admissible μ, we have $\mu(\gamma_1) = (x^{2j_1}, \beta_2)$, $\mu(\gamma_2) = (x^{2j_2}, \beta_2)$, $\mu(\gamma_3) = (x^{k_3}, 0)$ and $\mu(\gamma_4) = (x^{k_4}, 0)$ $(k_3, k_4 \neq 2h)$. One sees immediately that $\mathrm{Im}(\mu) \subset \langle (x, 0), (0, \beta_2) \rangle$, a contradiction. □

Lemma 7.8. *Cover type $III - c)$ has no admissible μ.*

Proof. Assume that we have an admissible $\mu : T(2, 2, 2, c_4) \to G(H)$. Setting $(b_1, b_2, b_3, b_4) := (\mu(\gamma_1), \mu(\gamma_2), \mu(\gamma_3), \mu(\gamma_4))$, we have the following observations.
(1) $b_1^2 = b_2^2 = b_3^2 = 1$. Since $b_4 \in H$ and $order(b_4) = c_4 > 2$, we see that b_4 must lie in the (order n) cyclic subgroup of H, say $b_4 = x^k$; and we also find $n > 2$.
(2) The fact that H is normal in $G(H)$ implies that there exist integers k_i such that $b_i x b_i = x^{k_i}$ and $\gcd(k_i, n) = 1$ for $i = 1, 2, 3$. Then one computes easily that $x^k b_i = b_i x^{kk_i}$.
(3) The condition $b_1 b_2 b_3 b_4 = 1 \Rightarrow b_1 b_2 = x^{-k} b_3$, moreover $(b_1 b_2)^2 = x^{-k} b_3 x^{-k} b_3 = x^{-k-kk_3}$.

Any element in $\mathrm{Im}(\mu)$ has the form $\Pi\beta_i$, where $\beta_i \in \{x^k, x^{-k}, b_1, b_2, b_3\}$. Since $b_1b_2b_3b_4 = 1$, without loss of generality we can assume $\beta_i \in \{x^k, x^{-k}, b_1, b_2\}$, which means that every element in $\mathrm{Im}(\mu)$ is a word in these four elements.
Using (2), we can "move" all the $x^{\pm k}$ terms to the right end. Taking (1) into account, we see that the elements are of the forms $(b_1b_2)^s x^t$, $b_2(b_1b_2)^s x^t$ or $(b_1b_2)^s b_1 x^t$ for some integers s and t, now using (3), one sees immediately that elements in $\mathrm{Im}(\mu)$ have the form x^j, b_1x^j, b_2x^j or b_3x^j. It turns out that $y \notin \mathrm{Im}(\mu)$, a contradiction. □

From the preceding arguments, we know that the only group type to consider is Group type I), which means $G(H) = D_n \times \mathbb{Z}/2$ and $H = D_n \times \{0\}$. We denote by $(e, 0)$ the neutral element of $D_n \times \mathbb{Z}/2$, where $\mathbb{Z}/2$ is additively generated by 1. The results are presented via the Hurwitz vectors associated to the admissible $\mu' s$.

Lemma 7.9. *Classification of cover type I)*

In this case the only admissible μ for n ***odd*** *has the associated Hurwitz vector*

$$v = ((y, 1), (y, 1), (yx, 1), (yx, 1), (e, 1), (e, 1)).$$

For n ***even*** *(n=2m) there are the following possibilities:*

$$v = ((y, 1), (y, 1), (yx, 1), (yx, 1), (e, 1), (e, 1)),$$

$$v = ((y, 1), (yx^m, 1), (yx, 1), (yx, 1), (x^m, 1), (e, 1)),$$

$$v = ((y, 1), (yx^m, 1), (yx^2, 1), (yx^2, 1), (x^m, 1), (e, 1)), m \text{ odd}.$$

For $n = 2$ there are the following:

$$v = ((y, 1), (y, 1), (x, 1), (x, 1), (e, 1), (e, 1)),$$

$$v = ((y, 1), (yx, 1), (x, 1), (x, 1), (x, 1), (e, 1)).$$

Proof. Since the cover $C/H \to \mathbb{P}^1$ is branched in 6 points (cf. 4.6) we need a Hurwitz vector with all entries having second component equal to 1. So we have

$$v = ((y^{k_1}x^{l_1}, 1), (y^{k_2}x^{l_2}, 1), (y^{k_3}x^{l_3}, 1), (y^{k_4}x^{l_4}, 1), (y^{k_5}x^{l_5}, 1), (y^{k_6}x^{l_6}, 1))$$

The first observation is that the condition $\langle v\rangle = G(H)$ implies that there must exist j such that $k_j = 1$. Therefore up to automorphism we can assume

$$v = ((y,1),(y^{k_2}x^{l_2},1),(y^{k_3}x^{l_3},1),(y^{k_4}x^{l_4},1),(y^{k_5}x^{l_5},1),(y^{k_6}x^{l_6},1))$$

We consider the two cases n odd and n even separately.

i) n odd: Not all k_j can be equal to 1, otherwise we cannot generate the element $(y,0)$. Now the only element of order two of the form $(x^l,1)$ in G is $(e,1)$. So because of the product one condition v either looks like

$$v = ((y,1),(yx^{l_2},1),(yx^{l_3},1),(yx^{l_4},1),(e,1),(e,1))$$

or

$$v = ((y,1),(y,1),(e,1),(e,1),(e,1),(e,1)),$$

the latter being excluded, since $G(H) \neq \langle v\rangle$.

The product one condition gives $l_2 + l_4 \equiv l_3 \mod n$. The condition $\langle v\rangle = G(H)$ implies $\gcd(l_2,l_3,l_4,n) = \gcd(l_2,l_4,n) = 1$. Since the second factor $\mathbb{Z}/2$ of $G(H)$ is abelian, we can apply Lemma (6.5) to achieve that $l_3 = l_4$. Now v looks like

$$v = ((y,1),(yx^{l_2},1),(yx^{l_4},1),(yx^{l_4},1),(e,1),(e,1))$$

and again by the product one condition we obtain $l_2 \equiv 0 \mod n$ and therefore $1 = \gcd(l_2,l_4,n) = \gcd(l_4,n)$.

So we can apply the automorphism $(x^{l_4},0) \mapsto (x,0),(y,0) \mapsto (y,0)$ to v and we can take

$$v = ((y,1),(y,1),(yx,1),(yx,1),(e,1),(e,1))$$

as a Hurwitz vector for the covering $C \to \mathbb{P}^1$.

ii) n=2m is even: Recall the general form:

$$v = ((y,1),(y^{k_2}x^{l_2},1),(y^{k_3}x^{l_3},1),(y^{k_4}x^{l_4},1),(y^{k_5}x^{l_5},1),(y^{k_6}x^{l_6},1))$$

Again, first we distinguish the possible Hurwitz vectors by the (even and positive) number of k_j that are equal to 1. We call the element y^kx^l a reflection if $k \equiv 1 \mod 2$.
In the current case there exists $m = n/2$, which gives an extra order

2 element $(x^m, 1) \in G(H)$. As in the odd case, 6 reflections cannot occur. For the case of 2 reflections, assume, up to ordering,

$$v = ((y,1),(yx^{l_2},1),(x^{l_3},1),(x^{l_4},1),(x^{l_5},1),(x^{l_6},1)).$$

As before, $(l_3,l_4,l_5,l_6) = (0,0,0,0)$ is impossible. In the cases $(l_3,l_4,l_5,l_6) = (m,m,0,0)$ and $(l_3,l_4,l_5,l_6) = (m,m,m,m)$ we get $l_2 = 0$. In the first case we can only have $\langle v \rangle = G(H)$ if $n = 2$. Also in the second case we must have $n = 2$ but the elements $(y,1)$ and $(x,1)$ cannot generate $G(H)$ since the element $(e,1)$ is missing. In the cases $(l_3,l_4,l_5,l_6) = (m,m,m,0)$ and $(l_3,l_4,l_5,l_6) = (m,0,0,0)$ we get $l_2 = m$ which also implies that $n = 2$. So if $n > 2$ these cases do not occur. The corresponding Hurwitz vectors are:

$$v = ((y,1),(y,1),(x,1),(x,1),(e,1),(e,1)),$$

$$v = ((y,1),(yx,1),(x,1),(x,1),(x,1),(e,1))$$

and

$$v = ((y,1),(yx,1),(x,1),(e,1),(e,1),(e,1)),$$

the third one being equivalent to the second one by an automorphism of $G(H)$ that fixes H.

Assume, for the case of 4 reflections, up to ordering

$$v = ((y,1),(yx^{l_2},1),(yx^{l_3},1),(yx^{l_4},1),(x^{l_5},1),(x^{l_6},1)).$$

Here we have the 3 cases: $l_5 = l_6 = m$, $l_5 = l_6 = 0$ and $l_5 = m$, $l_6 = 0$.

In the first two cases from the product one condition we get $l_2 + l_4 \equiv l_3 \mod n$. To generate $G(H)$ we must have $\gcd(l_2,l_3,l_4,n) = \gcd(l_2,l_4,n) = 1$.

Using Lemma (6.5) again, we arrive at

$$v = ((y,1),(yx^{l_2},1),(yx^{l_4},1),(yx^{l_4},1),(x^m,1),(x^m,1))$$

resp.

$$v = ((y,1),(yx^{l_2},1),(yx^{l_4},1),(yx^{l_4},1),(e,1),(e,1))$$

and so we get $l_2 \equiv 0 \mod n$. Now we have $\gcd(l_2,l_4,n) = \gcd(l_4,n) = 1$ and we can apply the automorphism $(x^{l_4},0) \mapsto (x,0), (y,0) \mapsto (y,0)$ to v to arrive at

$$v = ((y,1),(y,1),(yx,1),(yx,1),(x^m,1),(x^m,1))$$

resp.

$$v = ((y,1),(y,1),(yx,1),(yx,1),(e,1),(e,1)).$$

Using the automorphism $(e,1) \mapsto (x^m,1),(y,0) \mapsto (yx^{-m},0)$ we see that these two are equivalent.

It remains to consider the case that $l_5 = m$ and $l_6 = 0$, i.e.

$$v = ((y,1),(yx^{l_2},1),(yx^{l_3},1),(yx^{l_4},1),(x^m,1),(e,1)).$$

We apply (6.5) again and it follows $l_2 = m$. So we get

$$v = ((y,1),(yx^m,1),(yx^l,1),(yx^l,1),(x^m,1),(e,1)),$$

where $\gcd(l,m) = 1$.

We have two sub-cases, i.e. $\gcd(l,n) = 1$ and $\gcd(l,n) = 2$. In the first case we can use the automorphism $(x^l,0) \mapsto (x,0),(y,0) \mapsto (y,0)$ to obtain

$$v = ((y,1),(yx^m,1),(yx,1),(yx,1),(x^m,1),(e,1)).$$

In the second case (where m must be odd) we can achieve

$$v = ((y,1),(yx^m,1),(yx^2,1),(yx^2,1),(x^m,1),(e,1)).$$

□

Lemma 7.10. *Classification of cover type II)*
Up to equivalence, the admissible μ's are given by the Hurwitz vectors:
(1) $c_5 = 2$,

$$v = ((y,1),(yx,1),(yx,1),(e,1),(y,0)).$$

(2) $c_5 > 2$,

$$v = ((y,1),(yx^{-1},1),(e,1),(e,1),(x,0)), c_5 = n;$$

$$v = ((y,1),(yx^{m-1},1),(x^m,1),(e,1),(x,0)), n = 2m, c_5 = n;$$

$$v = ((y,1),(yx^{m-2},1),(x^m,1),(e,1),(x^2,0)), n = 2m, m \textit{ is odd}, c_5 = m.$$

Proof. Assume that we have an admissible $\mu : T(2,2,2,2,c_5) \to D_n \times \mathbb{Z}/2$ of cover type II). Then we must have:

$$v := (\mu(\gamma_1),\mu(\gamma_2),\mu(\gamma_3),\mu(\gamma_4),\mu(\gamma_5)) = ((a_1,1),(a_2,1),(a_3,1),(a_4,1),(a_5,0))$$

There are two possibilities:
(1) $\underline{c_5 = 2}$.
As argued in the previous lemma, we do the classification in terms of

the number of reflections in $\{a_i\}$, which can be either 2 or 4.

(i) There are 2 reflections.
(a) a_5 is a reflection, W.L.O.G. we can assume that a_1 is another reflection and $a_1 = yx^l, a_5 = y$. It is clear that $a_2, a_3, a_4 \in \{e, x^{n/2}$(if n is even)$\}$.
There are 4 cases (up to ordering): $\alpha)$ $(a_2, a_3, a_4) = (e, e, e)$, $\beta)$ $(a_2, a_3, a_4) = (x^{n/2}, e, e)$, $\gamma)$ $(a_2, a_3, a_4) = (x^{n/2}, x^{n/2}, e)$, $\delta)$ $(a_2, a_3, a_4) = (x^{n/2}, x^{n/2}, x^{n/2})$.
In cases $\alpha)$ and $\delta)$ we get no admissible μ since μ can not be surjective.
For cases $\beta) and \gamma)$ (where n is even) we get that μ is admissible $\iff$ $n = 2$.
(b) a_5 is not a reflection, first we conclude that n must be even and $a_5 = x^{n/2}$. Using similar arguments as in (a), one finds that

$$v = ((y, 1), (yx^l, 1), (a_3, 1), (a_4, 1), (x^{n/2}, 0)),\ a_3, a_4 \in \{e, x^{n/2}\}.$$

There are three cases, and one checks easily that in each case μ is admissible if and only if $n = 2$.
(ii) There are 4 reflection.
$a)$ a_5 is a reflection. W.L.O.G. we assume

$$v = ((yx^{l_1}, 1), (yx^{l_2}, 1), (yx^{l_3}, 1), (a_4, 1), (y, 0)),\ a_4 \in \{e, x^{n/2} \textit{(if n is even)}\}.$$

Again we apply Lemma (6.5) so that we can assume $l_2 = l_3$. Since μ is admissible, using similar arguments as in the previous lemma, we have:
Case $\alpha)$ If $a_4 = e$, then $l_1 \equiv 0 \mod n$ and $\gcd(l_2, n) = 1$. Under the automorphism $(x^{l_2}, 0) \mapsto (x, 0)$, $(y, 0) \mapsto (y, 0)$, we get

$$v \sim ((y, 1), (yx, 1), (yx, 1), (e, 1), (y, 0)).$$

Case $\beta)$ $n = 2m$ is even and $a_4 = x^m$. One gets $l_1 \equiv m \mod (2m)$ and $\gcd(l_2 - m, 2m) = 1$. Using the automorphism $(x^{l_2 - m}, 0) \mapsto (x, 0)$, $(y, 0) \mapsto (y, 0)$, we can achieve

$$v \sim ((yx^m, 1), (yx^{m+1}, 1), (yx^{m+1}, 1), (x^m, 1), (y, 0)).$$

Using the automorphism (of $G(H)$): $(x, 0) \mapsto (x, 0)$, $(y, 0) \mapsto (y, 0)$, $(e, 1) \mapsto (x^m, 1)$, one finds out that Case $\beta)$ is equivalent to Case $\alpha)$.
$b)$ a_5 is not a reflection.
In this case we see that n must be even and

$$v = ((y, 1), (yx^{l_2}, 1), (yx^{l_3}, 1), (yx^{l_4}, 1), (x^{n/2}, 0)).$$

It is easy to check that μ can not be surjective since $(y,0)$ is not contained in $Im(\mu)$.
Up to now we have got all the admissible μ's for the case $n=2$ (since $n=2$ implies that $c_5=2$). One checks easily that they are equivalent to each other, since in this case $G(H)$ is abelian.

(2) $\underline{c_5>2}$.
The element a_5 must lie in the order n cyclic subgroup of H, say $a_5 = x^k$ $(k \neq \frac{n}{2}$ if n is even $)$.
(i) There are 2 reflections, W.L.O.G. we may assume

$v = ((y,1),(yx^l,1),(a_3,1),(a_4,1),(x^k,0)),\ a_3,a_4 \in \{e, x^{n/2}(if\ n\ is\ even)\}.$

There are 3 cases:
Case α) $(a_3,a_4)=(e,e)$.
We get that $l+k\equiv 0 \mod n$ and $\gcd(k,n)=1$. Applying the automorphism $(x^k,0)\mapsto(x,0)$, $(y,0)\mapsto(y,0)$ we get

$$v \sim ((y,1),(yx^{-1},1),(e,1),(e,1),(x,0)).$$

Moreover we see that $c_5=n$.
Case β) $n=2m$ and $(a_3,a_4)=(x^m,e)$.
We get that $l+k\equiv m \mod 2m$ and $\gcd(k,m)=1$.
If $\gcd(k,n)=1$ (which is the unique case if $2|m$), then

$$v \sim ((y,1),(yx^{m-1},1),(x^m,1),(e,1),(x,0)).$$

Here we find $c_5=n$.
Otherwise $\gcd(k,n)=2$ (which may happen only if $2\nmid m$), we have that

$$v \sim ((y,1),(yx^{m-2},1),(x^m,1),(e,1),(x^2,0))$$

and $c_5=m$.
Case γ) $n=2m$ and $(a_3,a_4)=(x^m,x^m)$.
We get that $l+k\equiv 0 \mod n$, $\gcd(k,n)=1$ and

$$v \sim ((y,1),(yx^{-1},1),(x^m,1),(x^m,1),(x,0)).$$

Moreover, we see that $c_5=n$.
Using the automorphism $(x,0)\mapsto(x,0)$, $(y,0)\mapsto(yx^{-m},0)$, $(e,1)\mapsto(x^m,1)$, we see that case α) is equivalent to Case γ).
(ii) There are 4 reflections.
One checks easily that μ can not be surjective since $(y,0)\notin \mathrm{Im}(\mu)$. $\square$

Lemma 7.11. *Classification of cover type III-a)*
We have that $n = 2m$ and $d_4 = m$. Up to equivalence there is a unique admissible μ with associated Hurwitz vector

$$v = ((y, 1), (yx^{-1}, 1), (e, 1), (x, 1)).$$

Proof. Assume that $\mu : T(2, 2, 2, 2d_4) \to D_n \times \mathbb{Z}/2$ is admissible and of cover type III-a). Then we must have

$$v := (\mu(\gamma_1), \mu(\gamma_2), \mu(\gamma_3), \mu(\gamma_4)) = ((a_1, 1), (a_2, 1), (a_3, 1), (a_4, 1)).$$

The condition $d_4 > 1$ implies that $a_5 = x^k$ for some integer k ($k \neq n/2$ *if n is even*).
There can only be 2 reflections among a_1, a_2, a_3. W.L.O.G. we can assume

$$v = ((y, 1), (yx^l, 1), (a_3, 1), (x^k, 1)), a_3 \in \{e, x^{n/2}(if\ n\ is\ even)\}.$$

Case a) $a_3 = e$.
We get that $l + k \equiv 0 \mod n$, $\gcd(k, n) = 1$ and

$$v \sim ((y, 1), (yx^{-1}, 1), (e, 1), (x, 1)).$$

In this case $2d_4 = n$, it turns out that n must be even.
Case b) $n = 2m$ and $a_3 = x^m$.
We get that $l + k \equiv m \mod n$, $\gcd(l, n) = 1$ and

$$v \sim ((y, 1), (yx, 1), (x^m, 1), (x^{m-1}, 1)).$$

Using the automorphism $(x, 0) \mapsto (x^{-1}, 0)$, $(y, 0) \mapsto (yx^{-m}, 0)$, $(e, 1) \mapsto (x^m, 1)$, we find that case a) is equivalent to case b). □

Lemma 7.12. *Classification of cover type III-b)*
We have that $c_3 = 2$ and $c_4 = n$. Up to equivalence there is a unique admissible μ with associated Hurwitz vector

$$v = ((yx, 1), (e, 1), (y, 0), (x, 0)).$$

Proof. In Example (7.5) we saw that if a type $III - b$) cover has group type 1), c_3 must be 2, combining with the proof of Corollary (7.1) we have that the case $(\delta_H, \delta_{H'}) = (1, 3)$ does not occur.
Let $\mu : T(2, 2, 2, c_4) \to D_n \times \mathbb{Z}/2$ be admissible with cover type III-b). We have that

$$v := (\mu(\gamma_1), \mu(\gamma_2), \mu(\gamma_3), \mu(\gamma_4)) = ((a_1, 1), (a_2, 1), (a_3, 0), (a_4, 0)).$$

Since $c_4 > 2$ we get that $a_4 = x^k$ for some integer k. It is obvious that there are two (and only two) reflections among a_1, a_2, a_3.
(1) a_3 is not a reflection. In this case n must be even, say $n = 2m$, and we have $a_3 = x^m$. W.L.O.G. we may assume

$$v = ((y, 1), (yx^l, 1), (x^m, 0), (x^k, 0)).$$

It is easy to see that $(y, 0) \notin \mathrm{Im}(\mu)$, therefore in this case there is no admissible μ.
(2) a_3 is a reflection. W.L.O.G we may assume

$$v = ((yx^l, 1), (a_2, 1), (y, 0), (x^k, 0)), a_2 \in \{e, x^{n/2}(if\ n\ is\ even)\}.$$

There are two possibilities:
(i) $a_2 = e$, we get that $k \equiv l \mod n$, $\gcd(k, n) = 1$ and

$$v \sim ((yx, 1), (e, 1), (y, 0), (x, 0)), c_4 = n.$$

(ii) $n = 2m$ is even and $a_2 = x^m$, we get that $k \equiv l + m \mod n$, $\gcd(k, n) = 1$ and

$$v \sim ((yx^{m+1}, 1), (x^m, 1), (y, 0), (x, 0)), c_4 = n.$$

Using the automorphism $(x, 0) \mapsto (x, 0)$, $(y, 0) \mapsto (y, 0)$, $(e, 1) \mapsto (x^m, 1)$, we see that case (i) is equivalent to case (ii). □

Remark 7.13. If we drop the restriction on μ_H, it is easy to check that the two Hurwitz vectors in $III - a)$ and $III - b)$ are $B.A.$-equivalent (Consider the automorphism of $G(H)$: $(x, 0) \mapsto (x, 1)$, $(y, 0) \mapsto (yx, 0)$, $(e, 1) \mapsto (e, 1)$).

7.4. **Results.** We present our results through tables. We have obtained the representative vectors for the cover $C \to C/G(H)$ in (7.9), (7.10), (7.11) and (7.12), there will be one table for each representative vector . For the reader's convenience we present a short list of notation:

v	Hurwitz vector for the cover $C \to C/G(H)$
$v_{G/H'}$	Hurwitz vector for the double cover $C/H' \to C/G(H) = \mathbb{P}^1$
$g_{C/H'}$	Genus of C/H'
$\delta_{H'}$	Dimension of $Fix(H')$
$v_{H'}$	Hurwitz vector for the cover $C \to C/H'$

We will use the following subgroups of $D_n \times \mathbb{Z}/2$, where $D_n = \langle x, y \mid x^n = y^2 = 1, yxy^{-1} = x^{-1} \rangle$ and e denotes the neutral element of D_n.

Subgroup	Generators
K	$(x,0)$
$H_{1,1}$	$K,(e,1)$
$H_{1,2}$	$K,(y,1)$
$H_{1,3}$	$(x^2,0),(y,0),(e,1)$
$H_{1,4}$	$(x^2,0),(y,0),(x,1)$
$H_{1,5}$	$(x^2,0),(yx,0),(e,1)$
$H_{1,6}$	$(x^2,0),(yx,0),(x,1)$

For compactness, we make the following conventions:
Whenever the groups $H_{1,4}, H_{1,6}, H_{1,3}, H_{1,5}$ occur, we assume that $n = 2m$, in the last 2 cases we additionally assume m to be odd. If $H_{1,1}$ appears we are in the case $n = 2$. We identify the groups $H_{1,3}$ and $H_{1,5}$ with D_n by sending their respective generators in the given order to x^{m+1}, y, x^m.
The cover types are those which appear in Theorem (4.6).

Theorem 7.14. *Let H, H' be subgroups of Map_g, satisfying condition $(**)$ and $\delta_H \geq 1$. Then $G(H) \simeq D_n \times \mathbb{Z}/2$, H corresponds to $D_n \times \{0\}$. The group H' and the topological action of the group $G(H)$ (i.e. its Hurwitz vector) are as listed in the following tables.*

Remark 7.15. Given a cover $C \to C/H$, the data consisting of $g_{C/H}$ and the branching indices are called the signature of the cover. In [BCGG03], Section 3 the authors computed the signatures for the possible non-maximal loci of the form $\mathfrak{M}_{g,\ \rho}(D_n)$, which is a direct corollary of our result.

Cover type I)
($\delta_H = 3$, $g_{C/H} = 2$, $C \to C/H$ is unramified)

$v = ((y,1),(y,1),(yx,1),(yx,1),(e,1),(e,1))$.

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	(0,0,0,0,1,1)	0	5	(y,y,y,y,yx,yx,yx,yx)
$H_{1,3}$	(0,0,1,1,0,0)	0	5	$(yx^m, yx^{m-2}, yx^m, yx^{m+2}, x^m, x^m, x^m, x^m)$
$H_{1,4}$	(1,1,0,0,1,1)	1	4	(e,y;yx,yx,yx,yx)
$H_{1,5}$	(1,1,0,0,0,0)	0	5	$(yx^m, yx^m, yx^m, yx^m, x^m, x^m, x^m, x^m)$
$H_{1,6}$	(0,0,1,1,1,1)	1	4	(e,yx;y,y,y,y)

$v = ((y,1),(yx^m,1),(yx,1),(yx,1),(x^m,1),(e,1))$, $n = 2m$.

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,0,0,1,1)$	0	5	$(y, y, yx^m, x^m y, yx, yx, yx, yx)$
$H_{1,3}$	$(0,1,1,1,1,0)$	1	4	$(x^{m+1}, x^{m-1}; x^m, x^m, yx^m, yx^m)$
$H_{1,4}$(m odd)	$(1,0,0,0,0,1)$	0	5	$(yx^m, x^m y, yx, yx^3, yx, xy, x^m, x^m)$
$H_{1,4}$(m even)	$(1,1,0,0,1,1)$	1	4	$(x^m, x^m y; yx, yx, yx, yx)$
$H_{1,5}$	$(1,0,0,0,1,0)$	0	5	$(yx^{\frac{m^2-1}{2}}, yx^{\frac{m^2-1}{2}}, yx^m, yx^m, yx^m, yx^m, x^m, x^m)$
$H_{1,6}$(m odd)	(0,1,1,1,0,1)	1	4	$(x^{m+1}, x^{m-1}; x^m, x^m, y, y)$
$H_{1,6}$(m even)	(0,0,1,1,1,1)	1	4	$(e, x^{m-1}y; y, y, x^m y, yx^m)$

$v = ((y,1),(yx^m,1),(yx^2,1),(yx^2,1),(x^m,1),(e,1))$, $n = 2m$, m odd.

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,0,0,1,1)$	0	5	$(y, y, yx^m, yx^m, yx^2, yx^2, yx^2, yx^2)$
$H_{1,3}$	$(0,1,0,0,1,0)$	0	5	$(x^m, x^m, yx^m, yx^m, yx^{-1}, yx^{-3}, yx^{-1}, yx)$
$H_{1,4}$	$(0,0,0,0,1,1)$	0	5	$(y, y, yx^m, yx^m, yx^2, yx^2, yx^2, yx^2)$
$H_{1,5}$	$(1,0,1,1,1,0)$	1	4	$(x^2, x^{-2}; x^m, x^m, x^m y, x^m y)$
$H_{1,6}$	(0,1,0,0,0,1)	0	5	$(y, y, x^2 y, x^6 y, x^2 y, yx^2, x^m, x^m)$

For $n = 2$ we have two extra cases:

$$v = ((y,1),(y,1),(x,1),(x,1),(e,1),(e,1)).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,1}$	$(1,1,0,0,0,0)$	0	5	(x,x,x,x,y,y,y,y)
$H_{1,2}$	$(0,0,1,1,1,1)$	1	4	$(e,x;y,y,y,y)$
$H_{1,3}$	$(0,0,1,1,0,0)$	0	5	(yx,yx,yx,yx,x,x,x,x)
$H_{1,4}$	$(1,1,0,0,1,1)$	1	4	$(e,y;x,x,x,x)$
$H_{1,5}$	$(1,1,1,1,0,0)$	1	4	$(e,yx;x,x,x,x)$
$H_{1,6}$	(0,0,0,0,1,1)	0	5	(y,y,y,y,x,x,x,x)

$$v = ((y,1),(yx,1),(x,1),(x,1),(x,1),(e,1)).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,1}$	$(1,1,0,0,0,0)$	0	5	(yx,yx,yx,yx,yx,yx,y,y)
$H_{1,2}$	$(0,0,1,1,1,1)$	1	4	$(e,e;y,y,yx,yx)$
$H_{1,3}$	$(0,1,1,1,1,0)$	1	4	$(y,y;x,x,yx,yx)$
$H_{1,4}$	$(1,0,0,0,0,1)$	0	5	(yx,yx,x,x,x,x,x,x,x,x)
$H_{1,5}$	$(1,0,1,1,1,0)$	1	4	$(e,y;x,x,x,x)$
$H_{1,6}$	(0,1,0,0,0,1)	0	5	(y,y,x,x,x,x,x,x,x,x)

Cover type II)
($\delta_H = 2,\ g_{C/H} = 1$)

(1) $c_5 = 2$.

$$v = ((y,1),(yx,1),(yx,1),(e,1),(y,0)),\ \ v_H = (x,x^{-1};y,y).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,0,1,1)$	0	3	(y,y,yx,yx,yx,yx)
$H_{1,3}$	$(0,1,1,0,0)$	0	3	$(yx^m,yx^{-1},x^m,x^m,y,yx^{m+1})$
$H_{1,4}$	$(1,0,0,0,1)$	0	3	(yx,yx,yx,yx,y,y)
$H_{1,5}$	$(1,0,0,0,1)$	0	3	$(yx^m,yx,yx^m,yx^{-1},x^m,x^m)$
$H_{1,6}$	$(0,1,1,1,1)$	1	2	$(e,yx;y,y)$

(2) $c_5 > 2$.

$v = ((y,1),(yx^{-1},1),(e,1),(e,1),(x,0)),\ c_5 = n,\ v_H = (x^{-1},y;x,x)$.

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,1,1,0)$	0	3	$(y,y,yx^{-1},yx^{-3},x,x)$
$H_{1,3}$	$(0,1,0,0,1)$	0	4	$(yx^m,yx^{-1},x^m,x^m,x^m,x^m,x^{m+1})$
$H_{1,4}$	$(1,0,1,1,1)$	1	3	$(y,y;x^2,yx^3,yx)$
$H_{1,5}$	$(1,0,0,0,1)$	0	4	$(yx^{-1},yx^{m-2},x^m,x^m,x^m,x^m,x^{m+1})$
$H_{1,6}$	$(0,1,1,1,1)$	1	3	$(yx^{-1},yx^{-1};x^2,yx^2,y)$

$v = ((y,1),(yx^{m-1},1),(x^m,1),(e,1),(x,0)),\ n = 2m,\ v_H = (x^{m-1},yx^m;x,x)$.

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,1,1,0)$	0	3	$(y,y,yx^{m-1},yx^{m-3},x,x)$
$H_{1,3}$	$(0,0,1,0,1)$	0	4	$(yx^m,yx^{m+2},yx,yx,x^m,x^m,x^{-2})$
$H_{1,4}$ (m odd)	$(1,1,0,1,1)$	1	3	$(x^{m-1},y;x^2,x^m,x^m)$
$H_{1,4}$ (m even)	$(1,0,1,1,1)$	1	3	$(yx^m,y;x^2,yx^{m+3},yx^{m+1})$
$H_{1,5}$	$(1,1,1,0,1)$	1	3	$(x^{m+1},y;x^{-2},x^m,x^m)$
$H_{1,6}$ (m odd)	$(0,0,0,1,1)$	0	4	$(y,yx^{-2},yx^{m-1},yx^{m-1},x^m,x^m,x^2)$
$H_{1,6}$ (m even)	$(0,1,1,1,1)$	1	3	$(yx^{-1},yx^{m-1};x^2,yx^2,y)$

$v = ((y,1),(yx^{m-2},1),(x^m,1),(e,1),(x^2,0)),\ n = 2m,\ m \text{ odd},\ c_5 = m,$

$$v_H = (x^{m-2},yx^m;x^2,x^2).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,1,1,0)$	0	3	$(y,y,yx^{m-2},yx^{m-6},x^2,x^2)$
$H_{1,3}$	$(0,1,1,0,0)$	0	3	$(yx^m,yx^{m-4},x^m,x^m,x^2,x^2)$
$H_{1,4}$	$(1,0,0,1,0)$	0	3	$(yx^{m-2},yx^{m-6},x^m,x^m,x^2,x^2)$
$H_{1,5}$	$(1,0,1,0,0)$	0	3	$(y,yx^{-4},x^m,x^m,x^2,x^2)$
$H_{1,6}$	$(0,1,0,1,0)$	0	3	$(y,yx^{-4},x^m,x^m,x^2,x^2)$

For $n = 2$ we have one extra case.

$$v = ((yx, 1), (x, 1), (e, 1), (e, 1), (y, 0)),\ v_H = (y, yx; y, y).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,1}$	$(1, 0, 0, 0, 1)$	0	3	(x, x, y, y, y, y)
$H_{1,2}$	$(0, 1, 1, 1, 1)$	1	2	$(x, x; yx, yx)$
$H_{1,3}$	$(1, 1, 0, 0, 0)$	0	3	(x, x, x, x, y, y)
$H_{1,4}$	$(0, 0, 1, 1, 0)$	0	3	(yx, yx, x, x, y, y)
$H_{1,5}$	$(0, 1, 0, 0, 1)$	0	3	(yx, yx, x, x, x, x)
$H_{1,6}$	$(1, 0, 1, 1, 1)$	1	2	$(yx, yx; x, x)$

Cover type III-a)
$(\delta_H = 1,\ g_{C/H} = 1)$

$$v = ((y, 1), (yx^{-1}, 1), (e, 1), (x, 1)),\ 2d_4 = n = 2m,\ v_H = (x^{-1}, y; x^2).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0, 0, 1, 1)$	0	2	$(y, yx^2, yx^{-1}, yx^{-1}, x^2)$
$H_{1,3}$	$(0, 1, 0, 1)$	0	2	$(yx^m, yx^{-1}, x^m, x^m, x^{m+1})$
$H_{1,4}$	$(1, 0, 1, 0)$	0	1	(yx^{-1}, yx^{-3}, x, x)
$H_{1,5}$	$(1, 0, 0, 1)$	0	2	$(yx^{-1}, yx^{m-2}, x^m, x^m, x^{m+1})$
$H_{1,6}$	$(0, 1, 1, 0)$	0	1	(x, x, y, yx^{-2})

Cover type III-b)
$(\delta_H = 1,\ g_{C/H} = 0)$

$$v = ((yx, 1), (e, 1), (y, 0), (x, 0)),\ c_4 = n = 2m,\ v_H = (y, yx^{-2}, x, x).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0, 1, 1, 0)$	0	1	(yx, yx^{-1}, x, x)
$H_{1,3}$	$(1, 0, 0, 1)$	0	2	$(x^m, x^m, y, yx^{m-1}, x^{m+1})$
$H_{1,4}$	$(0, 1, 0, 1)$	0	2	(yx, yx^{-1}, y, y, x^2)
$H_{1,5}$	$(0, 0, 1, 1)$	0	2	$(yx^m, yx^{-1}, x^m, x^m, x^{m+1})$
$H_{1,6}$	$(1, 1, 1, 1)$	1	1	$(yx, x; x^2)$

8. **Decompositions of $\mathfrak{M}_h[G]$**

In the proof of theorem 3.1 (cf. 1) we saw that $\mathfrak{M}_h[G]$ has a decomposition which depends upon the choice of a sufficiently large natural number k:

$$\mathcal{D}^G_{k,h} : \ \mathfrak{M}_h[G] = \bigsqcup_{[\beta]\in\mathcal{B}_N} \mathfrak{M}^{G,\beta}_{k,h},$$

Since given a G-marked family $((\mathfrak{X} \to T), G, \rho)$ over a connected base T, the induced G-representations on $H^0(\omega^k_{\mathfrak{X}_t})$ are all isomorphic to each other for any $t \in T$ (cf. [Cat13], Prop 37), each component of the decomposition is a union of connected components of $\mathfrak{M}_h[G]$.

Definition 8.1. (1) Given a space X with two decompositions $\mathcal{D}_1$: $X = \bigsqcup_{i\in I} Y_i$ and $\mathcal{D}_2 : X = \bigsqcup_{j\in J} W_j$, where each Y_i, W_j is a union of connected components of X, their minimal refinement is defined as:

$$\mathcal{D}_1 \cap \mathcal{D}_2 : X = \bigsqcup_{(i,j)\in I\times J} Y_i \cap W_j.$$

(2) The *canonical representation type decomposition* of $\mathfrak{M}_h[G]$ is the minimal refinement of all the above decompositions:

$$\mathcal{D}_h[G] := \cap_{k\in\mathcal{K}} \mathcal{D}^G_{k,h},$$

where $\mathcal{K}$ denotes the set of natural numbers satisfying Matsusaka's big theorem (cf. [Mat86], Theorem 2.4).

Remark 8.2. Since $\mathfrak{M}_h[G]$ is a quasi-projective scheme, we see immediately that there exists a minimal natural number $N(h,G)$ and integers $k_1, ..., k_{N(h,G)}$ such that

$$\mathcal{D}_h[G] = \cap_{i=1}^{N(h,G)} \mathcal{D}^G_{k_i,h},$$

Several natural questions arise:
Question 1. What is an explicit bound for $N(h,G)$?
Question 2. Are the components of $\mathcal{D}_h[G]$ connected? or how many connected components do they have?

To answer question 1, we provide first a method which works in general, the main idea is to consider suitable Hilbert resolutions of the canonical rings of varieties with a fixed Hilbert polynomial h (cf. [Cat92], Section 2). Then in the case of algebraic curves we use the Chevalley-Weil formula to obtain a more precise bound for $N(h,G)$.

Since the functor M_h is bounded, there exists a minimal natural number $m = m(h)$ such that $\forall X \in \mathsf{M}_h(\mathbf{Spec}(\mathbb{C}))$, $H^i(X, \omega_X^m) = 0$ for any $i > 0$ and the m-th pluricanonical map of X, $\phi_m : X \to \mathbb{P}^n$, is an embedding, where $n := h(m) - 1$. Recall that the canonical ring of X is:

$$\mathcal{R} = \mathcal{R}(X, \omega_X) := \bigoplus_{k \geq 0} H^0(X, \omega_X^k)$$

Since ω_X is ample, $\mathcal{R}$ is a finite graded module over the graded ring $\mathcal{A} := Sym(H^0(X, \omega_X^m))$. The degree k direct summand of $\mathcal{R}$ (resp. $\mathcal{A}$) is denoted by $\mathcal{R}_k$ (resp. $\mathcal{A}_k$).

Remark 8.3. Assuming a group G acts on X, we have naturally induced actions on $\mathcal{R}$ and $\mathcal{A}$. It is easy to see that these actions are compatible in the following sense:
(1) $\forall k \in \mathbb{N}$, $\mathcal{A}_k$ (resp. $\mathcal{R}_k$) is a G-invariant subspace of $\mathcal{A}$ (resp. $\mathcal{R}$).
(2) $\forall g \in G, a_{k_1} \in \mathcal{A}_{k_1}$ and $a_{k_2} \in \mathcal{A}_{k_2}$, $g(a_{k_1}a_{k_2}) = (ga_{k_1})(ga_{k_2})$ (the same holds for $\mathcal{R}$).
(3) $\forall g \in G, a_{k_1} \in \mathcal{A}_{k_1}$ and $u_{k_2} \in \mathcal{R}_{k_2}$, $g(a_{k_1}u_{k_2}) = (ga_{k_1})(gu_{k_2})$.

Denoting by δ the depth of $\mathcal{R}$ as an $\mathcal{A}$-module, by Hilbert's syzygy theorem we have a minimal free resolution of $\mathcal{R}$ of length $n+1-\delta$ (cf. [Gre89], Theorem 1.2):

$$0 \to L_{n+1-\delta} \to L_{n-\delta} \to \cdots \to L_1 \to L_0 \to \mathcal{R} \to 0.$$

Now taking the action of G into account, we have the following:

Lemma 8.4. *Let $\mathcal{A} = \mathbb{C}[x_0, ..., x_n]$ and let $\mathcal{M}$ be a finite graded $\mathcal{A}$-module. Assuming that we have actions of G on $\mathcal{A}$ and $\mathcal{M}$ such that 8.3* (1)*,* (2) *and* (3) *are satisfied, then there exists a minimal G-equivariant free resolution of $\mathcal{M}$:*

$$0 \to L_{n+1-\delta} \to L_{n-\delta} \to \cdots \to L_1 \to L_0 \to \mathcal{M} \to 0,$$

where δ is the depth of $\mathcal{M}$ as an $\mathcal{A}$-module. Moreover, L_i is a direct sum:

$$L_i = \bigoplus_{\chi \in Irrchar(G)} \bigoplus_{j=1}^{s_\chi} \mathcal{A}(-n_{\chi,i,j}) \otimes V_\chi,$$

where $Irrchar(G)$ denotes the set of irreducible characters of G and V_χ is the irreducible representation associated to χ.

Proof. Since $\mathcal{M}$ is a finitely generated $\mathcal{A}$-module, there exists a minimal integer k_1 such that $\mathcal{M}_{k_1} \neq 0$. We have a natural G-equivariant $\mathcal{A}$-module morphism:

$$\psi_1 : \mathcal{A}(-k_1) \otimes \mathcal{M}_{k_1} \to \mathcal{M},$$

where the action on the left hand side is: $g(a \otimes m) = (ga) \otimes (gm)$.
Now $\mathcal{M}/Im(\psi_1)$ is again a finitely generated graded $\mathcal{A}$-module and G-module, hence we have $\bar{\eta}_2 : \mathcal{A}(-k_2) \otimes (\mathcal{M}/Im(\psi_1))_{k_2} \to \mathcal{M}/Im(\psi_1)$, which can be lifted to a G-equivalent homomorphism $\eta_2 : \mathcal{A}(-k_2) \otimes \mathcal{M}'_{k_2} \to \mathcal{M}$, where $k_2 > k_1$ is the minimal integer such that $(\mathcal{M}/Im(\psi_1))_{k_2} \neq 0$, and $\mathcal{M}'_{k_2}$ is a G-invariant subspace of $\mathcal{M}_{k_2}$ which maps isomorphically onto $(\mathcal{M}/Im(\psi_1))_{k_2}$. We repeat the process and (since $\mathcal{M}$ is finitely generated) after a finite number of steps we obtain $L_0 = \oplus_{\nu=1}^{l_0} \mathcal{A}(-k_\nu) \otimes \mathcal{M}'_{k_\nu}$ (we set $\mathcal{M}'_{k_1} = \mathcal{M}_{k_1}$) and a surjective G-equivariant morphism $d_0 : L_0 \twoheadrightarrow \mathcal{M}$. By decomposing $\mathcal{M}'_{k_j}$ into irreducible G-subspaces we get the promised form of L_0. From our construction, we see that L_0 is a finitely generated graded-$\mathcal{A}$-module and G-module satisfying 8.3 (3).
We define L_i and d_i for $i \geq 1$ inductively: assuming that we already have $d_{i-1} : L_{i-1} \to L_{i-2}$ and $\ker(d_{i-1})$ is a finitely generated graded $\mathcal{A}$-module and G-module satisfying 8.3 (3), we repeat the construction process of d_0 and get $d_i : L_i \twoheadrightarrow \ker(d_{i-1}) \subset L_{i-1}$. By Hilbert's syzygy theorem we have $L_{n-\delta+2} = 0$. From our construction it is clear that the resulting resolution is minimal. □

Setting $N'(h, G) := m + \max\{n_{\chi,i,j}\}$, from (8.4) we have the following:

Proposition 8.5. *For any $k > N'(h, G)$, the G-representation on $\mathcal{R}_k$ is determined by the representations on $\mathcal{R}_1, ..., \mathcal{R}_{N'(h,G)}$, hence $N(h, G) \leq N'(h, G)$.*

In order to find an explicit bound on $N(h, G)$, we estimate the integers m and $\max\{n_{\chi,i,j}\}$ separately.

The problem of finding an effective bound on m is the so called "effective Matsusaka problem". Kollár has shown in [Kol93] that $m \leq 2(d+3)(d+2)!(2+d)$, where $d := deg(h) = dimX$. If we only consider canonically polarized manifolds, we have better results (cf. [Dem96], [Siu02]): we would like to mention the result by Angehrn and Siu, they have shown that $m \leq (d+1)(d^2+d+4)/2+2$ (cf.[AS95]).

To determine $\max\{n_{\chi,i,j}\}$, we recall first the notion of the Castelnuovo-Mumford regularity (cf. [Mum66], Lecture 14).

Definition 8.6. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$: $\mathcal{F}$ is said to be *s-regular* if $H^i(\mathbb{P}^n, \mathcal{F}(s-i)) = 0$ for all $i > 0$, the *regularity* of $\mathcal{F}$ is the minimal natural number with this property.
The regularity of a graded $\mathcal{A}$-module $\mathcal{M}$ is the regularity of its associated sheaf $\widetilde{\mathcal{M}}$.

Let s be the regularity of $\mathcal{R}$ as an $\mathcal{A}$-module: we have the following inequalities.

Lemma 8.7. *Notations as in (8.4). For any i, j and χ,*

$$i \leq n_{\chi,i,j} \leq i+s.$$

Proof. See [Cat92], Section 2. □

An immediate consequence is that

Proposition 8.8. $\max\{n_{\chi,i,j}\} \leq s+n+1-\delta \leq s+n+1.$

We refer to [Mum66], Lecture 14 for the fact that given a Hilbert polynomial h, $\forall X \in \mathsf{M}_h(\mathbf{Spec}(\mathbb{C}))$, the regularity of $\mathcal{R}(X, \omega_X)$ (as an $\mathcal{A}$-module) is bounded by a polynomial in the coefficients of $h(mx)$.

Observe that, the ring $\mathcal{R}$ is a direct sum of graded $\mathcal{A}$-submodules:

$$\mathcal{R} = \bigoplus_{j=0}^{m-1} \mathcal{R}(j),$$

where $\mathcal{R}(j) := \bigoplus_{i\geq 0} \mathcal{R}_{j+mi}$. Hence we have the following proposition:

Proposition 8.9. *For large k, the G-representation on $\mathcal{R}_k$ is determined by the representation on $\mathcal{R}_m$ and the representations on the lower degree summands $\mathcal{R}_l$ whose degree l lies in the same modulo m congruence class of k.*

In the rest of this section we answer question 1 and question 2 for algebraic curves using topological methods. In the case of curves we use genera instead of Hilbert polynomials, for instance, for curves of genus $g \geq 2$ the corresponding moduli space is denoted by $\mathfrak{M}_g[G]$.

From now on C shall denote a smooth projective curve of genus $g \geq 2$. Moreover we assume that a finite group G acts effectively on C, we denote by C' the quotient curve C/G and by g' the genus of C'. The Galois cover $p : C \to C'$ is branched in r points ($r = 0$ if p is unramified) on C' with branching indices $m_1, ..., m_r$. The cover p has an associated Hurwitz vector $v = (a_1, b_1, ...a_{g'}, b_{g'}; c_1, ...c_r)$ of type (m_i) (cf. 4.3).

The main ingredient in comparing $\mathcal{D}^G_{k,g}$ for different k's is the following version of the Chevalley-Weil formula:

Theorem 8.10 (Chevalley-Weil, cf.[CW34])**.** *Let (C, G, ϕ) be a G-marked curve and let $v = (a_1, b_1, ...a_{g'}, b_{g'}; c_1, ...c_r)$ be a Hurwitz vector associated to the cover $C \to C/\phi(G)$.*
Denote by χ_{ϕ_k} the character of the representation $\phi_k : G \to H^0(\omega_C^k)$ which is induced naturally by ϕ, and let χ_ρ be the character of an irreducible representation $\rho : G \to GL(W_\rho)$. We have the following formulae:

$$(1)\ \langle \chi_{\phi_1}, \chi_\rho \rangle = \chi_\rho(1_G)(g' - 1) + \sum_{i=1}^{r} \sum_{\alpha=1}^{m_i-1} \frac{\alpha N_{i,\alpha}}{m_i} + \sigma,$$

where setting $\xi_{m_i} := \exp(2\pi i/m_i)$, $N_{i,\alpha}$ is the multiplicity of $\xi^\alpha_{m_i}$ as eigenvalue of $\rho(c_i)$, and $\sigma = 1$ if ρ is trivial, otherwise $\sigma = 0$.

$$(2)\ \langle \chi_{\phi_k}, \chi_\rho \rangle = \frac{2k}{|G|}\chi_\rho(1_G)(g-1) - \chi_\rho(1_G)(g'-1) - \sum_{i=1}^{r} \sum_{\alpha=0}^{m_i-1} N_{i,\alpha} \frac{[-\alpha - k]_{m_i}}{m_i}.$$

Here $k \geq 2$, $[n]_{m_i} \in \{0, ..., m_i - 1\}$ is the congruence class of the integer n modulo m_i.

Remark 8.11. The Chevalley-Weil formula given in 8.10 is not in the original form of [CW34], but in the form of [FG15], theorem 1.11.[2]

Using (8.10), we see that

$$\langle \chi_{\phi_{k+|G|}}, \chi_\rho \rangle - \langle \chi_{\phi_k}, \chi_\rho \rangle = 2\chi_\rho(1_G)(g - 1) \text{ for } k \geq 2$$

and

$$\langle \chi_{\phi_{1+|G|}}, \chi_\rho \rangle - \langle \chi_{\phi_1}, \chi_\rho \rangle = 2\chi_\rho(1_G)(g - 1) - \sigma,$$

[2]The authors proved the formula for $k = 1$, but with their method one easily obtains the formula for any k. I am thankful to C. Gleißner for bringing these formulae to my attention.

which is independent of the action ϕ. Hence we have the following:

Corollary 8.12. *For curves of genus $g \geq 2$ and a finite group G, $\mathcal{D}^G_{k,g}$ and $\mathcal{D}^G_{k+|G|,g}$ give the same decomposition of $\mathfrak{M}_g[G]$ for any $k \geq 1$. Therefore we have $\mathcal{D}_g[G] = \cap_{k=1}^{|G|}\mathcal{D}^G_{k,g}$ and $N(g,G) \leq |G|$.*

The next example shows that $\mathcal{D}^G_{k_1,g}$ and $\mathcal{D}^G_{k_2,g}$ could be different if $|k_1 - k_2| < |G|$, hence $\mathcal{D}_g[G]$ might be a proper refinement of each $\mathcal{D}^G_{k,g}$.

Example 8.13. In this example $G = \mathbb{Z}/3\mathbb{Z} = \{0, \bar{1}, \bar{2}\}$ and $g = 6$. Let $\chi_i : G \to \mathbb{C}^*, \bar{1} \mapsto \xi_3^i$, $i = 1, 2$ be the nontrivial irreducible characters of G. Consider two G-marked curves (C_1, G, ϕ) and (C_2, G, ϕ') of genus g with associated Hurwitz vectors

$$v = (\bar{1}, 0, 0, \bar{2}; \bar{2}, \bar{1}) \text{ and}$$

$$v' = (\bar{1}, \bar{1}, \bar{2}, \bar{2}, \bar{1}, \bar{1}, \bar{2}, \bar{2}).$$

Using (8.10), one computes easily that $\chi_{\phi_1} = 2\chi_{triv} + 2\chi_1 + 2\chi_2$, $\chi_{\phi_2} = 5\chi_{triv} + 5\chi_1 + 5\chi_2$, $\chi_{\phi_3} = 9\chi_{triv} + 8\chi_1 + 8\chi_2$; $\chi_{\phi'_1} = 3\chi_1 + 3\chi_2$, $\chi_{\phi'_2} = 5\chi_{triv} + 5\chi_1 + 5\chi_2$ and $\chi_{\phi'_3} = 11\chi_{triv} + 7\chi_1 + 7\chi_2$. Hence we see that $\mathcal{D}^G_{2,6}$ is different from $\mathcal{D}^G_{1,6}$ and $\mathcal{D}^G_{3,6}$.

We answer now Question 2. The idea is to consider the topological types of G-actions on curves. The following observations are important:

Remark 8.14.

1) Using (8.10), we see that if two G-marked curves have the same marked[3] topological type (i.e., the equivalence class of the topological G-actions on a compact Riemann surface with a given genus, cf. [Cat15], 11.2), then they must have the same representation type for all $k \geq 1$.
2) Given a G-marked family of curves over a connected base, then the marked topological types are all the same for any G-marked curve in the family (cf. [Cat15], chapter 11).
3) Given two G-marked curves of genus g such that G acts freely on both curves, from (8.10) we see that the respectively induced G-representations on $H^0(\omega^k)$ are the same for all k. Moreover both representations on $H^0(\omega^k)$ are direct sum of regular G-representations for $k \geq 2$.

[3] The word "marked" means that we do not allow automorphisms of G.

Definition 8.15. (1) Given a finite group G, we denote by $\chi_{r.r}$ the character of the *regular representation* of G on the group algebra $\mathbb{C}[G]$ induced by left translation of G.
(2) The *component of the regular representation* $(\mathfrak{M}_g[G])_{r.r}$ of $\mathfrak{M}_g[G]$ (with respect to the decomposition $\mathcal{D}_g[G]$) is the subscheme of $\mathfrak{M}_g[G]$ consisting of the G-marked curves $[(C, G, \phi)]$ of genus g such that there exists a sequence of natural numbers $\{n_k\}$, such that $\chi_{\phi_k} = n_k\chi_{r.r}$ for all $k \geq 2$.

Recall that a split metacyclic group G is a split extension of two cyclic groups, or equivalently G has the following presentation:

$$G = \langle x, y | x^m = y^n = 1, yxy^{-1} = x^r \rangle$$

where m, n and r are positive integers such that $r^n \equiv 1 \mod m$.

Using the above observations and assuming G is a nonabelian split metacyclic group, we give a lower bound for the number of connected components of $(\mathfrak{M}_g[G])_{r.r}$.

Denote by $\mathfrak{MTF}(G, g)$ the set of marked topological types of *free* G-actions on a compact Riemann surface of genus g. By 8.14. 2) and 3) we see that $(\mathfrak{M}_g[G])_{r.r}$ has at least $|\mathfrak{MTF}(G, g)|$ connected components.

In the case that G is a nonabelian split metacyclic group, we have the following result of Edmonds.

Theorem 8.16 ([Edm83], Theorem 1.7)**.** *Given G a nonabelian split metacyclic group, then there is a bijection* $\mathbf{B} : \mathfrak{MTF}(G, g) \to H_2(G, \mathbb{Z})$.

With our preceding discussion, we immediately have the following:

Proposition 8.17. *Let G be a nonabelian split metacyclic group: $\forall g \geq 2$, $(\mathfrak{M}_g[G])_{r.r}$ has at least $|H_2(G, \mathbb{Z})|$ connected components.*

In the end we provide a formula to compute $H_2(G, \mathbb{Z})$ for a split metacyclic group G.

Lemma 8.18. *$H_2(G) = \mathbb{Z}/d\mathbb{Z}$, where* $d = \frac{\gcd(m,\ r-1)\gcd(m,\ \Sigma_{i=0}^{n-1} r^i)}{m}$.

Proof. See [Edm83], Lemma 1.2. □

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