THE ORDER OF THE AUTOMORPHISM GROUP OF A BINARY q-ANALOG OF THE FANO PLANE IS AT MOST TWO

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ABSTRACT. It is shown that the automorphism group of a binary q-analog of the Fano plane is either trivial or of order 2.
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1. INTRODUCTION

Motivated by the connection to network coding, q-analogs of designs have received an increased interest lately. Arguably the most important open problem in this field is the question for the existence of a q-analog of the Fano plane [6]. Its existence is open over any finite base field $\mathrm{GF}(q)$. The most important single case is the binary case $q = 2$, as it is the smallest one. Nonetheless, so far the binary q-analog of the Fano plane has withstood all computational or theoretical attempts for its construction or refutation.

Following the approach for other notorious putative combinatorial objects as, e.g., a projective plane of order 10 or a self-dual binary $[72, 36, 16]$ code, the possible automorphisms of a binary q-analog of the Fano plane have been investigated in [4]. As a result [4, Theorem 1], its automorphism group is at most of order 4, and up to conjugacy in $\mathrm{GL}(7, 2)$ it is represented by a group in the following list:

(a) The trivial group.
(b) The group of order 2

$$G_2 = \left\langle \begin{pmatrix} 0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle.$$

(c) One of the following two groups of order 3:

$$G_{3,1} = \left\langle \begin{pmatrix} 0&1&0&0&0&0&0\\ 1&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \quad \text{and} \quad G_{3,2} = \left\langle \begin{pmatrix} 0&1&0&0&0&0&0\\ 1&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle.$$

(d) The cyclic group of order 4

$$G_4 = \left\langle \begin{pmatrix} 1&1&0&0&0&0&0\\ 0&1&1&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&1&0&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle.$$

For the groups of order 2, the above result was achieved as a special case of a more general result on restrictions of the automorphisms of order 2 of a binary

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q-analog of Steiner triple systems [4, Theorem 2]. All the remaining groups have been excluded computationally applying the method of Kramer and Mesner.

In this article, we will extend these results as follows. In Section 3 automorphisms of order 3 of general binary q-analogs of Steiner triple systems $\mathrm{STS}_2(v)$ will be investigated. The main result is Theorem 2, which excludes about half of the conjugacy types of elements of order 3 in $\mathrm{GL}(v, 2)$ as the automorphism of an $\mathrm{STS}_2(v)$. In the special case of ambient dimension 7, the group $\mathrm{GL}(7, 2)$ has 3 conjugacy types $G_{3,1}$, $G_{3,2}$ and $G_{3,3}$ of subgroups of order 3. Theorem 2 shows that the group $G_{3,2}$ is not the automorphism group of a binary q-analog of the Fano plane. Furthermore, Theorem 2 provides a purely theoretical argument for the impossibility of $G_{3,3}$, which previously has been shown computationally in [4].

In Section 4, the groups G_4 and $G_{3,1}$ will be excluded computationally by showing that the Kramer-Mesner equation system does not have a solution. Both cases are fairly large in terms of computational complexity. To bring the problems to a feasible level, the solution process is parallelized and executed on the high performance Linux cluster of the University of Bayreuth. For the latter and harder case $G_{3,1}$, we additionally make use of the inherent symmetry of the search space given by the normalizer of the prescribed group, see also [8].

Finally, the combination of the results of Sections 3 and 4 yields

Theorem 1. *The automorphism group of a binary q-analog of the Fano plane is either trivial or of order* 2. *In the latter case, up to conjugacy in* $\mathrm{GL}(7, 2)$ *the automorphism group is represented by*

$$\left\langle \begin{pmatrix} 0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle.$$

2. Preliminaries

Throughout the article, V is a vector space over $\mathrm{GF}(2)$ of finite dimension v.

2.1. **The subspace lattice.** For simplicity, a subspace of V of dimension k will be called a *k-subspace*. The set of all k-subspaces of V is called the *Grassmannian* and is denoted by $\begin{bmatrix} V \\ k \end{bmatrix}_q$. As in projective geometry, the 1-subspaces of V are called *points*, the 2-subspaces *lines* and the 3-subspaces *planes*. Our focus lies on the case $q = 2$, where the 1-subspaces $\langle \mathbf{x} \rangle_{\mathrm{GF}(2)} \in \begin{bmatrix} V \\ 1 \end{bmatrix}_2$ are in one-to-one correspondence with the nonzero vectors $\mathbf{x} \in V \setminus \{\mathbf{0}\}$. The number of all r-subspaces of V is given by the Gaussian binomial coefficient

$$\#\begin{bmatrix} V \\ k \end{bmatrix}_q = \begin{bmatrix} v \\ k \end{bmatrix}_q = \begin{cases} \frac{(q^v-1)\cdots(q^{v-r+1}-1)}{(q^r-1)\cdots(q-1)} & \text{if } k \in \{0, \ldots, v\}; \\ 0 & \text{otherwise.} \end{cases}$$

The set $\mathcal{L}(V)$ of all subspaces of V forms the subspace lattice of V. There are good reasons to consider the subset lattice as a subspace lattice over the unary "field" GF(1) [5].

By the fundamental theorem of projective geometry, for $v \geq 3$ the automorphism group of $\mathcal{L}(V)$ is given by the natural action of $\mathrm{P\Gamma L}(V)$ on $\mathcal{L}(V)$. In the case that q is prime, the group $\mathrm{P\Gamma L}(V)$ reduces to $\mathrm{PGL}(V)$, and for the case of our interest $q = 2$, it reduces further to $\mathrm{GL}(V)$. After a choice of a basis of V, its elements are represented by the invertible $v \times v$ matrices A, and the action on $\mathcal{L}(V)$ is given by the vector-matrix-multiplication $\mathbf{v} \mapsto \mathbf{v}A$.

2.2. Designs.

Definition 1. *Let t, v, k be integers with $0 \leq t \leq k \leq v$ and λ another positive integer. A set $D \subseteq \begin{bmatrix} V \\ k \end{bmatrix}_q$ is called a t-$(v, k, \lambda)_q$* subspace design *if each t-subspace of V is contained in exactly λ elements (called* blocks*) of D. When $\lambda = 1$, D is called a q-*Steiner system. *If additionally $t = 2$ and $k = 3$, D is called a q-*Steiner triple system *and denoted by* $\mathrm{STS}_q(v)$.

Classical combinatorial designs can be seen as the limit case $q = 1$ of a design over a finite field. Indeed, quite a few statements about combinatorial designs have a generalization to designs over finite fields, such that the case $q = 1$ reproduces the original statement [3, 9, 10, 15].

One example of such a statement is the following [18, Lemma 4.1(1)]: If D is a t-$(v, k, \lambda)_q$ design, then D is also an s-$(v, k, \lambda_s)_q$ for all $s \in \{0, \dots, t\}$, where

$$\lambda_s := \lambda \frac{\begin{bmatrix} v-s \\ t-s \end{bmatrix}_q}{\begin{bmatrix} k-s \\ t-s \end{bmatrix}_q}.$$

In particular, the number of blocks in D equals

$$\#D = \lambda_0 = \lambda \frac{\begin{bmatrix} v \\ t \end{bmatrix}_q}{\begin{bmatrix} k \\ t \end{bmatrix}_q}.$$

So, for a design with parameters t-$(v, k, \lambda)_q$, the numbers $\lambda \begin{bmatrix} v-s \\ t-s \end{bmatrix}_q / \begin{bmatrix} k-s \\ t-s \end{bmatrix}_q$ necessarily are integers for all $s \in \{0, \dots, t\}$ (*integrality conditions*). In this case, the parameter set t-$(v, k, \lambda)_q$ is called *admissible*. It is further called *realizable* if a t-$(v, k, \lambda)_q$ design actually exists.

For designs over finite fields, the action of $\mathrm{Aut}(\mathcal{L}(V)) \cong \mathrm{P\Gamma L}(V)$ on $\mathcal{L}(V)$ provides a notion of isomorphism. Two designs in the same ambient space V are called *isomorphic* if they are contained in the same orbit of this action (extended to the power set of $\mathcal{L}(V)$). The *automorphism group* $\mathrm{Aut}(D)$ of a design D is its stabilizer with respect to this group action. If $\mathrm{Aut}(D)$ is trivial, we will call D *rigid*. Furthermore, for $G \leq \mathrm{P\Gamma L}(V)$, D will be called G-invariant if it is fixed by all elements of or equivalently, if $G \leq \mathrm{Aut}(D)$. Note that if D is G-invariant, then D is also H-invariant for all subgroups $H \leq G$.

2.3. Steiner triple systems.

For an $\mathrm{STS}_q(v)$ we have

$$\lambda_1 = \frac{\begin{bmatrix} v-1 \\ 2-1 \end{bmatrix}_q}{\begin{bmatrix} 3-1 \\ 2-1 \end{bmatrix}_q} = \frac{q^{v-1} - 1}{q^2 - 1} \quad \text{and}$$

$$\lambda_0 = \frac{\begin{bmatrix} v \\ 2 \end{bmatrix}_q}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q} = \frac{(q^v - 1)(q^{v-1} - 1)}{(q^3 - 1)(q^2 - 1)}.$$

As a consequence, the parameter set of an ordinary or a q-analog Steiner triple system $\mathrm{STS}_q(v)$ is admissible if and only if $v \equiv 1, 3 \bmod 6$ and $v \geq 3$. For $q = 1$, the existence question is completely answered by the result that a Steiner triple system is realizable if and only if it is admissible [11]. However in the q-analog case, our current knowledge is quite sparse. Apart from the trivial $\mathrm{STS}_q(3)$ given by $\{V\}$, the only decided case is $\mathrm{STS}_2(13)$, which has been constructed in [1].

The smallest admissible case of a non-trivial q-Steiner triple system is $\mathrm{STS}_q(7)$, whose existence is open for any prime power value of q. It is known as a *q-analog of the Fano plane*, since the unique Steiner triple system $\mathrm{STS}_1(7)$ is the Fano plane. It is worth noting that there are cases of Steiner systems without a q-analog, as the famous large Witt design with parameters 5-$(24, 8, 1)$ does not have a q-analog for any prime power q [9].

2.4. **Group actions.** Let G be a group acting on a set X via $x \mapsto x^g$. The stabilizer of x in G is given by $G_x = \{g \in G \mid x^g = x\}$, and the G-orbit of x is given by $x^G = \{x^g \mid g \in G\}$. By the action of G, the set X is partitoned into orbits. For all $x \in X$, there is the correspondence $x^g \mapsto G_x g$ between the orbit x^G and the set $G_x \backslash G$ of the right cosets of the stabilizer G_x in G. For finite orbit lengths, this implies the orbit-stabilizer theorem stating that $\#x^G = [G : G_x]$. In particular, the orbit lengths $\#x^G$ are divisors of the group order $\#G$.

For all $g \in G$ we have

$$G_{x^g} = g^{-1} G_x g. \tag{1}$$

This leads to the following observations:

(a) The stabilizers of elements in the same orbit are conjugate in G, and any conjugate subgroup of G_x is the G-stabilizer of some element in the G-orbit of x.
(b) Equation (1) shows that $G_{x^g} = G_x$ for all $g \in N_G(G_x)$, where N_G denotes the normalizer in G. Consequentely, for any subgroup $H \leq G$ the normalizer $N_G(H)$ acts on the elements of $x \in X$ with $N_x = H$.

The above observations greatly benefit our original problem, which is the investigation of all the subgroups H of $G = \mathrm{GL}(7, 2)$ for the existence of a binary q-analog D of the Fano plane whose stabilizer G_D equals H: By observation 2.4, we may restrict the search to representatives of subgroups of G up to conjugacy. Furthermore, having fixed some subgroup H, by observation 2.4 the normalizer $N = N_G(H)$ is acting on the solution space. Consequently, we can notably speed up the search process by applying isomorph rejection with resprect to the action of N.

2.5. **The method of Kramer and Mesner.** The method of Kramer and Mesner [13] is a powerful tool for the computational construction of combinatorial designs. It has been successfully adopted and used for the construction of designs over a finite field [2, 14]. For example, the hitherto only known q-analog of a Steiner triple system in [1] has been constructed by this method. Here we give a short outline, for more details we refer the reader to [2]. The *Kramer-Mesner matrix* $M_{t,k}^G$ is defined to be the matrix whose rows and columns are indexed by the G-orbits on the set $\begin{bmatrix} V \\ t \end{bmatrix}_q$ of t-subspaces and on the set $\begin{bmatrix} V \\ k \end{bmatrix}_q$ of k-subspaces of V, respectively. The entry of $M_{t,k}^G$ with row index T^G and column index K^G is defined as $\#\{K' \in K^G \mid T \leq K'\}$. Now there exists a G-invariant t-$(v, k, \lambda)_q$ design if and only if there is a zero-one solution vector $\mathbf{x}$ of the linear equation system

$$M_{t,k}^G \mathbf{x} = \lambda \mathbf{1}, \tag{2}$$

where $\mathbf{1}$ denotes the all-one column vector. More precisely, if $\mathbf{x}$ is a zero-one solution vector of the system (2), a t-$(v, k, \lambda)_q$ design is given by the union of all orbits K^G where the corresponding entry in $\mathbf{x}$ equals one. If $\mathbf{x}$ runs over all zero-one solutions, we get all G-invariant t-$(v, k, \lambda)_q$ designs in this way.

3. Automorphisms of order 3

In this section, automorphisms of order 3 of binary q-analogs of Steiner triple systems are investigated. While the techniques are not restricted to $q = 2$ or order 3, we decided to stay focused on our main case of interest. In parts, we follow [4, Section 3] where automorphisms of order 2 have been analyzed.

We will assume that $V = \mathrm{GF}(2)^v$, allowing us to identify $\mathrm{GL}(V)$ with the matrix group $\mathrm{GL}(v, 2)$.

Lemma 1. *In* $\mathrm{GL}(v,2)$*, there are exactly* $\lfloor v/2 \rfloor$ *conjugacy classes of elements of order* 3*. Representatives are given by the block-diagonal matrices* $A_{v,f}$ *with* $f \in \{0,\ldots,v-1\}$ *and* $v-f$ *even, consisting of* $\frac{v-f}{2}$ *consecutive* 2×2 *blocks* $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$*, followed by a* $f \times f$ *unit matrix.*

Proof. Let $A \in \mathrm{GL}(v,2)$ and $m_A \in \mathrm{GF}(2)[X]$ be its minimal polynomial. The matrix is of order 3 if and only if m_A divides $X^3-1=(X+1)(X^2+X+1)$ but $m_A \neq X+1$. Now the enumeration of the possible rational normal forms of A yields the stated classification. □

For a matrix A of order 3, the unique conjugate $A_{v,f}$ given by Lemma 1 will be called the *type* of A. The action of $\langle A_{v,f}\rangle$ partitions the point set $\begin{bmatrix}\mathrm{GF}(2)^v\\1\end{bmatrix}_2$ into orbits of size 1 or 3. An orbit of length 3 may either consist of three collinear points (*orbit line*) or of a triangle (*orbit triangle*).

Lemma 2. *The action of* $\langle A_{v,f}\rangle$ *partitions* $\begin{bmatrix}\mathrm{GF}(2)^v\\1\end{bmatrix}_2$ *into*

(i) 2^f-1 *fixed points;*
(ii) $\frac{2^{v-f}-1}{3}$ *orbit lines;*
(iii) $\frac{(2^{v-f}-1)(2^f-1)}{3}$ *orbit triangles.*

Proof. Let $G=\langle A_{v,f}\rangle$. The eigenspace of $A_{v,f}$ corresponding to the eigenvalue 1 is of dimension f and equals $F=\langle \mathbf{e}_{v-f+1},\mathbf{e}_{v-f+2},\ldots,\mathbf{e}_v\rangle$. The fixed points are exactly the 2^f-1 elements of $\begin{bmatrix}F\\1\end{bmatrix}_2$. Furthermore, for a non-zero vector $\mathbf{x}\in\mathrm{GF}(2)^v$ the orbit $\langle\mathbf{x}\rangle^G_{\mathrm{GF}(2)}$ is an orbit line if and only if $A^2_{v,f}\mathbf{x}+A_{v,f}\mathbf{x}+\mathbf{x}=\mathbf{0}$ or equivalently,

$$\mathbf{x}\in K := \ker(A^2_{v,f}+A_{v,f}+I_v)=\langle \mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_{v-f}\rangle.$$

Thus, the number of orbit lines is $\begin{bmatrix}\dim(K)\\1\end{bmatrix}_2/3=(2^{v-f}-1)/3$. The remaining $\begin{bmatrix}v\\1\end{bmatrix}_2 - \begin{bmatrix}f\\1\end{bmatrix}_2 - \begin{bmatrix}v-f\\1\end{bmatrix}_2 = (2^{v-f}-1)(2^f-1)$ points are partitioned into orbit triangles. □

Example 1. *We look at the classical Fano plane as the points and lines in* $\mathrm{PG}(2,2)=\mathrm{PG}(\mathrm{GF}(2)^3)$*. Its automorphism group is* $\mathrm{GL}(3,2)$*. By Lemma 1, there is a single conjugacy class of automorphisms of order* 3*, represented by*

$$A_{3,1}=\begin{pmatrix}0&1&0\\1&1&0\\0&0&1\end{pmatrix}.$$

By Lemma 2, the action of $\langle A_{3,1}\rangle$ *partitions the point set* $\begin{bmatrix}\mathrm{GF}(2)^3\\1\end{bmatrix}_2$ *into the fixed point*

$$\langle(0,0,1)\rangle_{\mathrm{GF}(2)},$$

the orbit line

$$\{\langle(1,0,0)\rangle_{\mathrm{GF}(2)},\quad \langle(0,1,0)\rangle_{\mathrm{GF}(2)},\quad \langle(1,1,0)\rangle_{\mathrm{GF}(2)}\},$$

and the orbit triangle

$$\{\langle(1,0,1)\rangle_{\mathrm{GF}(2)},\quad \langle(0,1,1)\rangle_{\mathrm{GF}(2)},\quad \langle(1,1,1)\rangle_{\mathrm{GF}(2)}\}.$$

Now we look at planes E fixed under the action of $\langle A_{v,f}\rangle$. Here, the restriction of the automorphism $\mathbf{x}\mapsto A_{v,f}\mathbf{x}$ to E yields an automorphism of $E\equiv\mathrm{GF}(2)^3$ whose order divides 3. If its order is 1, then E consists of 7 fixed points and we call E of *type 7*. Otherwise, the order is 3. So, by Example 1 it is of type $A_{3,1}$, and E consists of 1 fixed point, 1 orbit line and 1 orbit triangle. Here, we call E of *type 1*.

Lemma 3. *Under the action of* $\langle A_{v,f}\rangle$,

$$\#\text{fixed planes of type } 7 = \begin{bmatrix} f \\ 3 \end{bmatrix}_2 = \frac{(2^f-1)(2^{f-1}-1)(2^{f-2}-1)}{21};$$

$$\#\text{fixed planes of type } 1 = \#\text{orbit triangles} = \frac{(2^f-1)(2^{v-f}-1)}{3}.$$

Proof. The fixed planes of type 7 are precisely the planes in the space of all fixed points of dimension f. Each fixed plane of type 3 is uniquely spanned by an orbit triangle. □

Example 2. *By Lemma 1, the conjugacy classes of elements of order* 3 *in* $\mathrm{GL}(7,2)$ *are represented by*

$$A_{7,1} = \begin{pmatrix} 0&1&0&0&0&0&0\\ 1&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \quad A_{7,3} = \begin{pmatrix} 0&1&0&0&0&0&0\\ 1&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \quad A_{7,5} = \begin{pmatrix} 0&1&0&0&0&0&0\\ 1&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}.$$

By Lemma 2 and Lemma 3, we get the following numbers:

	$A_{7,1}$	$A_{7,3}$	$A_{7,5}$
#fixed points	1	7	31
#orbit lines	21	5	1
#orbit triangles	21	35	31
#fixed planes of type 7	0	1	155
#fixed planes of type 1	21	35	31

In the following, D denotes an $\mathrm{STS}_2(v)$ with an automorphism $A_{v,f}$ of order 3. From the admissibility we get $v \equiv 1,3 \mod 6$ and hence f odd. The fixed points are given by the 1-subspaces of the eigenspace of $A_{v,f}$ corresponding to the eigenvalue 1, which will be denoted by F. The set of fixed planes in D of type 7 and 1 will be denoted by F_7 and F_1, respectively.

Lemma 4. *Let* $L \in \begin{bmatrix} V \\ 2 \end{bmatrix}_2$ *be a fixed line. Then the block passing through* L *is a fixed block.*

Proof. From the design property, there is a unique block $B \in D$ passing through L. For all $A \in \langle A_{v,f}\rangle$, we have $B \cdot A \in D$ and $B \cdot A > L \cdot A = L$, so $B \cdot A = B$ by the uniqueness of B. Hence B is a fixed block. □

Lemma 5. *The blocks in* F_7 *form an* $\mathrm{STS}_2(f)$ *on* F.

Proof. Obviously, each fixed block of type 7 is contained in F. Let $L \in \begin{bmatrix} F \\ 2 \end{bmatrix}_2$. By Lemma 4, there is a unique fixed block $B \in D$ passing through L. Since L consists of 3 fixed points, B must be of type 7. Hence $B \leq F$. □

The admissibility of $\mathrm{STS}_2(f)$ yields $f \equiv 1,3 \equiv 6$, so:

Corollary 1. *An* $\mathrm{STS}_2(v)$ *does not have an automorphism of order* 3 *of type* $A_{v,f}$ *with* $f \equiv 2 \bmod 3$.

In particular, a binary q-analog of the Fano plane does not have an automorphism of order 3 and type $A_{7,5}$. This gives a theoretical confirmation of the computational result of [4], where the group $\langle A_{7,5}\rangle$ has been excluded computationally.

Lemma 6.

$$\#F_7 = \frac{(2^f-1)(2^{f-1}-1)}{21}; \tag{3}$$

$$\#F_1 = \#\text{orbit lines} = \frac{2^{v-f}-1}{3}. \tag{4}$$

Proof. By Lemma 5, the number $\#F_7$ equals the λ_0-value of an $\mathrm{STS}_2(f)$.

For $\#F_1$, we double count the set X of all pairs (L, B) where L is an orbit line, $B \in F_1$ and $L < B$. By Lemma 2, the number of choices for L is $\frac{2^{v-f}-1}{3}$. Lemma 4 yields a unique fixed block B passing through L. Since B contains the orbit line L, B has to be of type 1. So $\#X = \frac{2^{v-f}-1}{3}$. On the other hand, there are $\#F_1$ possibilities for B and each such B contains a single orbit line. So $\#X = \#F_1$, verifying Equation (4). □

Lemma 7. *An* $\mathrm{STS}_2(v)$ *with* $v \geq 7$ *does not have an automorphism of order* 3 *of type* $A_{v,f}$ *with* $f > (v-3)/2$ *and* $f \not\equiv v \mod 3$.

Proof. Assume that $v \geq 7$ and $f \not\equiv v \mod 3$. Let $P \in \begin{bmatrix} F \\ 1 \end{bmatrix}_2$ and X be the set of all blocks passing through P which are not of type 7. The number of blocks passing through P is $\lambda_1 = \frac{2^{v-1}-1}{3}$. By Lemma 5, F_7 is an $\mathrm{STS}_2(f)$ on F. So the number of blocks of type 7 passing through P is given by the λ_1-value of an $\mathrm{STS}_2(f)$, which equals $\frac{2^{f-1}-1}{3}$. Hence $\#X = \frac{2^{v-1}-2^{f-1}}{3}$. Since P is a fixed point, the action of $\langle A_{v,f} \rangle$ partitions X into orbits of size 1 and 3. Depending on v and f, the remainder of $\#X$ modulo 3 is shown below:

	$f \equiv 1 \bmod 6$	$f \equiv 3 \bmod 6$	$f \equiv 5 \bmod 6$
$v \equiv 1 \bmod 6$	0	1	2
$v \equiv 3 \bmod 6$	2	0	1

In our case $f \not\equiv v \bmod 3$, we see that $\#X$ is not a multiple of 3, implying the existence of at least one fixed block in X, which must be of type 1. Thus, it contains only 1 fixed point, showing that the type 1 blocks coming from different points $P \in \begin{bmatrix} F \\ 1 \end{bmatrix}_2$ are pairwise distinct. In this way, we see that

$$2^f - 1 = \#\text{fixed points} \leq \#F_1 = \frac{2^{v-f}-1}{3},$$

where the last equality comes from Lemma 6. Using the preconditions $v \geq 7$ and v, f odd, we get that this inequality is violated for all $f > (v-3)/2$. □

Remark 1. *[(a)]*

(1) *The condition* $v \geq 7$ *cannot be dropped since the automorphism group of the trivial* $\mathrm{STS}_2(3)$ *is the full linear group* $\mathrm{GL}(3,2)$ *containing an automorphism of type* $A_{3,1}$.
(2) *In the case that the remainder of* $\#X$ *modulo* 3 *equals* 2, *we could use the stronger inequality* $2(2^f - 1) \leq \#F_1$. *However, the final condition on* f *is the same.*

Lemma 7 allows us to exclude one of the groups left open in [4, Theorem 1]:

Corollary 2. *There is no binary q-analog of the Fano plane invariant under* $G_{3,2} := \langle A_{7,3} \rangle$.

As a combination of Lemma 1, Corollary 1 and Lemma 7, we get:

Theorem 2. *Let D be an* $\mathrm{STS}_2(v)$ *with an automorphism A of order* 3. *Then A has the type* $A_{v,f}$ *with* $f \not\equiv 2 \mod 3$. *If* $f \equiv v \mod 3$, *then either* $v = 3$ *or* $f \leq (v-3)/2$.

Example 3. *Theorem 2 excludes about half of the conjugacy types of elements of order 3. Below, we list the remaining ones for small admissible values of v:*

	$A_{7,1}$	$A_{9,1}$	$A_{9,3}$	$A_{13,1}$	$A_{13,3}$	$A_{13,7}$
#fixed points	1	1	7	1	7	127
#orbit lines	21	85	21	1365	341	21
#orbit triangles	21	85	147	1365	2387	2667
#fixed planes of type 7	0	0	1	0	1	11811
#fixed planes of type 1	21	85	147	1365	2387	2709
$\#F_7$	0	0	1	0	1	381
$\#F_1$	21	85	21	1365	341	21

We conclude this section with an investigation of the case $A_{v,1}$, which has not been excluded for any value of v. The computational treatment of the open case $A_{7,1}$ in Section 4 will make use of the structure result of the following lemma.

Lemma 8. *Let D be a $\mathrm{STS}_2(v)$ with an automorphism of type $A_{v,1}$. Then D contains $\frac{2^{v-1}-1}{3}$ fixed blocks of type 1. The remaining blocks of D are partitioned into orbits of length 3. Furthermore, V can be represented as $V = W + X$ with $\mathrm{GF}(2)$ vector spaces W and X of dimension $v-1$ and 1, respectively, such that the fixed blocks of type 1 are given by the set $\{L + X : L \in \mathcal{L}\}$, where $\mathcal{L}$ is a Desarguesian line spread of $\mathrm{PG}(W)$.*

Proof. Let $W = \mathrm{GF}(2^{v-1})$, which will be considered as a $\mathrm{GF}(2)$ vector space if not stated otherwise. Let $\zeta \in W$ be a primitive third root of unity. We consider the automorphism $\varphi : \mathbf{x} \mapsto \zeta\mathbf{x}$ of W of order 3. Since φ does not have fixed points in $\begin{bmatrix} W \\ 1 \end{bmatrix}_2$, φ is of type $A_{v-1,0}$. The set $\mathcal{L} = \begin{bmatrix} W \\ 1 \end{bmatrix}_4$ is a Desarguesian line spread of $\mathrm{PG}(W)$. It consists of all lines of $\mathrm{PG}(W)$ with $\varphi(L) = L$. Since $\mathrm{PG}(W)$ does not contain any fixed points under the action of φ, $\mathcal{L}$ is the set of the $(2^{f-1}-1)/3$ orbit lines.

Now let X be a $\mathrm{GF}(2)$ vector space of dimension 1. The map $\hat{\varphi} = \varphi \times \mathrm{id}_X$ is an automorphism of $V = W \times X$ of order 3 and type $A_{v,1}$. Let $\hat{\mathcal{L}} = \{L + X \mid L \in \mathcal{L}\}$. Under the action of $\hat{\varphi}$, the elements of $\hat{\mathcal{L}}$ are fixed planes of type 1. By Lemma 3, the total number of fixed planes of type 1 equals $\#\hat{\mathcal{L}} = \#\mathcal{L}$, so $\hat{\mathcal{L}}$ is the full set of fixed planes of type 1. Moreover, Lemma 6 gives $\#F_1 = (2^{f-1}-1)/3 = \#\hat{\mathcal{L}}$, on the one hand, so all these planes have to be blocks of D, and $\#F_7 = 0$ on the other hand, so the remaining blocks are partitioned into orbits of length 3. □

4. Computational results

The automorphism groups $G_{3,1}$ and G_4 of a tentative $\mathrm{STS}_2(7)$ are excluded computationally by the method of Kramer and Mesner from Section 2.5. The matrix $M_{t,k}^{G_4}$ consists of 693 rows and 2439 columns, the matrix $M_{t,k}^{G_{3,1}}$ has 903 rows and 3741 columns. In both cases, columns containing entries larger than 1 had been ignored since from equation (2) it is immediate that the corresponding 3-orbits cannot be part of a Steiner system.

One of the fastest method for exhaustively searching all 0/1 solutions of such a system of linear equations where all coefficients are in $\{0, 1\}$ is the backtrack algorithm *dancing links* [12]. We implemented a parallel version of the algorithm which is well suited to the job scheduling system *Torque* of the Linux cluster of the University of Bayreuth. The parallelization approach is straightforward: In a first step all paths of the dancing links algorithm down to a certain level are stored. In the second step every such path is started as a separate job on the computer cluster, where initially the algorithm is forced to start with the given path.

For the group G_4 the search was divided into 192 jobs. All of these determined that there is no $\mathrm{STS}_2(7)$ with automorphism group G_4. Together, the exhaustive search of all these 192 sub-problems took approximately 5500 CPU-days.

The group $G_{3,1}$ was even harder to tackle. The estimated run time (see [12]) for this problem is 27 600 000 CPU-days.

In order to break the symmetry of this search problem and avoid unnecessary computations, the normalizer $N(G_{3,1})$ of $G_{3,1}$ in $\mathrm{GL}(7,2)$ proved to be useful. According to GAP [7], the normalizer is generated by

$$N(G_{3,1}) = \left\langle \begin{pmatrix} 0&1&0&0&0&0&0\\ 1&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&1&0&1&0\\ 1&0&1&0&1&0&0\\ 1&1&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&1&0&1&0\\ 1&1&1&1&1&1&0\\ 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle$$

and has order 362 880.

As discussed in Section 2.4, if for a prescribed group G, s_1, s_2 are two solutions of the Kramer-Mesner equations (2), then s_1 and s_2 correspond to two designs D_1 and D_2 both having G as full automorphism group. A permutation σ_n which maps the 1-entries of s_1 to the 1-entries of s_2 can be represented by an element $n \in \mathrm{GL}(7,2)$. In other words, $D_1^n = D_2$. Since G is the full automorphism group of D_1 and D_2 it follows for all $g \in G$:

$$D_1^{ng} = D_2^g = D_2 = D_1^n.$$

This shows that $n \in N(G)$.

This can be used as follows in the search algorithm. We force one orbit K_i^G to be in the design. If dancing links shows that there is no solution which contains this orbit, all k-orbits in $(K_1^G)^N$ can be excluded from being part of a solution, i.e. the corresponding columns of $M_{t,k}^G$ can be removed.

In the case $G_{3,1}$, the set of k-orbits is partitioned into four orbits under the normalizer $N(G_{3,1})$. Two of this four orbits, let's call them K_1^G and K_2^G, can be excluded with dancing links in a few seconds. The third orbit K_3^G needs more work, see below. After excluding the third orbit, also the fourth orbit is excluded in a few seconds.

For the third orbit K_3^G we iterate this approach and fix two k-orbits simultaneously, one of them being K_3^G. That is, we consider all cases of fixed pairs (K_3^G, K_i^G), where $K_i^G \notin (K_1^G)^N \cup (K_2^G)^N$. If there is no design which contains this pair of k-orbits, all k-orbits of the orbit $(K_i^G)^S$ can be excluded too, where $S = G_{K_3^G}$ is the stabilizer of the orbit K_3^G under the action of $N(G)$.

This process could be repeated for triples, but run time estimates show that fixing pairs of k-orbits minimizes the computing time.[1] Under the stabilizer of K_3^G, the set of pairs (K_3^G, K_i^G) of k-orbits is partitioned into 14 orbits. Seven of these 14 pairs representing the orbits lead to problems which could be solved in a few seconds. The remaining seven sub-problems were split into 49 050 separate jobs with the above approach for parallelization. These jobs could be completed by dancing links in approximately 23 600 CPU-days on the computer cluster, determining that there is no $\mathrm{STS}_2(7)$ with automorphism group $G_{3,1}$.

For the group G_2 the estimated run time is 3 020 000 000 000 000 CPU-days which seems out of reach with the methods of this paper.

[1] If iterated till the end, this type of search algorithm is known as *orderly generation*, see e.g. [16, 17].

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References

[1] Braun, M., Etzion, T., Östergård, P., Vardy, A., Wassermann, A.: Existence of q-analogs of steiner systems (2013)
[2] Braun, M., Kerber, A., Laue, R.: Systematic construction of q-analogs of t-(v, k, λ)-designs. Des. Codes Cryptogr. **34**(1), 55–70 (2005). DOI 10.1007/s10623-003-4194-z
[3] Braun, M., Kiermaier, M., Kohnert, A., Laue, R.: Large sets of subspace designs (2014)
[4] Braun, M., Kiermaier, M., Nakić, A.: On the automorphism group of a binary q-analog of the Fano plane. European J. Combin. **51**, 443–457 (2016). DOI 10.1016/j.ejc.2015.07.014
[5] Cohn, H.: Projective geometry over $\mho_1$ and the Gaussian binomial coefficients. Amer. Math. Monthly **111**(6), 487–495 (2004). DOI 10.2307/4145067
[6] Etzion, T., Storme, L.: Galois geometries and coding theory. Des. Codes Cryptogr. **78**(1), 311–350 (2016). DOI 10.1007/s10623-015-0156-5
[7] The GAP Group: GAP – Groups, Algorithms, and Programming, Version 4.8.3 (2016). URL http://www.gap-system.org
[8] Kaski, P.: Isomorph-free exhaustive generation of designs with prescribed groups of automorphisms. SIAM J. Discrete Math. **19**(3), 664–690 (2005).
[9] Kiermaier, M., Laue, R.: Derived and residual subspace designs. Adv. Math. Commun. **9**(1), 105–115 (2015). DOI 10.3934/amc.2015.9.105
[10] Kiermaier, M., Pavčević, M.O.: Intersection numbers for subspace designs. J. Combin. Des. **23**(11), 463–480 (2015). DOI 10.1002/jcd.21403
[11] Kirkman, T.P.: On a problem in combinatorics. Cambridge and Dublin Math. J. **2**, 191–204 (1847)
[12] Knuth, D.E.: Dancing links. In: J. Davies, B. Roscoe, J. Woodcock (eds.) Millennial Perspectives in Computer Science: Proceedings of the 1999 Oxford-Microsoft Symposium in Honour of Sir Tony Hoare. Palgrave (2000)
[13] Kramer, E.S., Mesner, D.M.: t-designs on hypergraphs. Discrete Math. **15**(3), 263–296 (1976). DOI 10.1016/0012-365X(76)90030-3
[14] Miyakawa, M., Munemasa, A., Yoshiara, S.: On a class of small 2-designs over GF(q). J. Combin. Des. **3**(1), 61–77 (1995). DOI 10.1002/jcd.3180030108
[15] Nakić, A., Pavčević, M.O.: Tactical decompositions of designs over finite fields. Des. Codes Cryptogr. **77**(1), 49–60 (2015). DOI 10.1007/s10623-014-9988-7
[16] Read, R.C.: Every one a winner or how to avoid isomorphism search when cataloguing combinatorial configurations. Ann. Discrete Math. **2**, 107–120 (1978)
[17] Royle, G.F.: An orderly algorithm and some applications in finite geometry. Discrete Math. **185**(1–3), 105–115 (1998). DOI http://dx.doi.org/10.1016/S0012-365X(97)00167-2
[18] Suzuki, H.: On the inequalities of t-designs over a finite field. European J. Combin. **11**(6), 601–607 (1990). DOI 10.1016/S0195-6698(13)80045-5

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