CHARACTERIZATION OF THRESHOLD FUNCTIONS: STATE OF THE ART, SOME NEW CONTRIBUTIONS AND OPEN PROBLEMS

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Abstract. The study of the characterization of threshold functions within the class of switching functions is an important problem that goes back at least to the mid–20th century. Due to different motivations switching and threshold functions have been investigated in a variety of different mathematical contexts: Boolean or switching functions, neural networks, hypergraphs, coherent structures, Sperner families, clutters, secret sharing and simple games or binary voting systems.

The paper revises the state of the art about this significant problem and proposes some new contributions concerning asummability and invariant asummability, a refinement of asummability. It also includes several questions and conjectures for future research whose solution would mean a new breakthrough.

Keywords: switching functions, Boolean functions; linear separability problem; threshold functions; asummability conditions; simple games
MSC: 91A12; 06E30; 94C10; 68T27; 92B20

1. Introduction

The study of switching functions goes back at least to Dedekind's 1897 work, in which he determined the exact number of simple games with four or fewer players. Since that time these structures have been investigated in a variety of different contexts either theoretically [28, 31, 30, 32, 6] in the context of Boolean functions or because of their numerous applications: neural networks [1], simple games [47, 36, 37, 50], threshold logic [43], hypergraphs [53], coherent structures [52], learning theory [40], complexity theory [3], and secret sharing [56, 58, 2]. Several books on neural networks have studied these structures: [48, 54, 57, 51]. To the best of our knowledge the first work linking threshold logic and simple games is due to Dubey and Shapley [11] and a compact study encompassing knowledge in both fields is due to Taylor and Zwicker [61].

One of the most fundamental questions in all these areas is to characterize which monotonic switching functions are weighted threshold. This problem is known in threshold logic as the linear separability problem. This question has also been posed in other research fields by using different, but essentially being equivalent, terminologies. Three different treatments to solve this problem have been considered.

The first consists in studying the consistency of a system of inequalities. Each inequality is formed by the inner product of two vectors: a non-negative integer vector of weights which represents the unknown variables and the vector formed by the subtraction of a true vector minus a false vector. The system is formed by considering all possible subtractions of true and false vectors. If the switching function is a threshold function then each inequality must be positive and the system of inequalities is consistent. This quite natural procedure is hardly a characterization, it is only a translation. A theorem on the existence of solutions for systems of linear inequalities was given in [8]. Linear programming is also a useful tool as shown in [5, 39, 19].

The second treatment, very close to the previous one, is a geometric approach based on the existence of a separating hyperplane that separates true vectors from false vectors. This procedure is elegant but not very efficient in practice. An use of the geometrical approach can be found in [12] and [34] for a variant of it.

The third method is an algebraic approach based on some combinatorial features. The idea behind this approach lies in the consideration of exchanges among vectors and the possibility to convert some true vectors into false vectors, no matter the number of vectors involved in these exchanges. The early works of [13] and [9] reexamined for simple games in [60] are the central point of this work.

Logic gates, switching functions or Boolean functions can be thought as simple games, with weighted games playing the role of threshold functions. The class of threshold functions admits a structural characterization, the asummability property, that is both natural and elegant. It states that a switching function is a threshold function if and only if it satisfies the property of asummability. Some of the deepest results on this subject were done in the area of threshold logic during the late 1950s and early 1960s by people such as Chow, Elgot, Gabelman, and Winder, as reported by Hu [35] and Muroga [43]. Some of this work was anticipated by Isbell [36, 37] in the field of simple games. In the book by Taylor and Zwicker [61] the authors propose the property of trade-robustness, which is equivalent to the property of asummability but is more transparent in the voting-theoretic context since it gives rise to some intuitions—most particularly, the idea of trading players among winning coalitions. Moreover, this trading notion reveals a number of important properties of simple games. Freixas and Molinero [21] propose a relaxation of trade-robustness for regular switching functions and called it invariant-trade robustness, which is less costly because only considers trades with a type of minimal winning coalitions.

For the sake of simplicity, clarity, and for being coherent with the historical performed studies we start to write the paper in the language of Boolean algebra (very similar to that of neural networks or threshold logic) and will continue exposing the main results in the language of simple games. As suggested in [61] some notions, as the concept of trade robustness, naturally arise in the political or economical context. Tables 1 and 2 contain the main equivalences which are also recalled in one language—that of switching functions— in the rest of this section.

Table 1. Variables and vectors versus players and coalitions.

variable or node	player or voter
irrelevant variable	null player
essential variable	vetoer
vector	coalition
true vector	winning coalition
false vector	losing coalition
minimal true vector	minimal winning coalition
maximal false vector	maximal losing coalition
shift-minimal true vector	shift-minimal winning coalition

Table 2. Types of functions versus types of simple games.

switching function	non-monotonic simple game
monotonic switching function	monotonic simple game
threshold function	weighted game
k-out-of-n switching function	symmetric simple game
2-monotonic function	complete game
k-summable	not k-trade robust
k-asummable	k-trade robust
k-invariant summable	not k-invariant trade robust
k-invariant asummable	k-invariant trade robust

Quite recently some variants of the linear separability problem have been studied in depth in the context of neural networks [4, 15, 14, 20].

Non-monotonic switching functions are more natural in areas such as threshold logic than they are in simple games. For instance, the XOR function is a switching function which is not a threshold function. However, the non-monotonicity of the XOR function prevents it of being interpreted in the context of simple games, since the condition of monotonicity is inherent in simple games or cooperative games more generally. Throughout this paper we exclusively deal with monotonic switching functions, or equivalently, simple games.

The paper is organized as follows. The basic terminology is recalled in the rest of this section concluding with the statement of a result dating from the sixties which is the starting point of our research. Section 2 begins with the exposition of the computational difficulties of testing the condition of asummability. We continue with the consideration of regular switching functions and with the exposition of a known characterization for them, that of swap asummability. We conclude the section by providing a significant refinement of asummability, that of invariant asummability, which constitutes a complementary test for ascertaining if a given switching function is a threshold function. Equivalent formulations for asummability in the context of simple games are given at the end of this section.

In section 3 we will turn our attention to symmetries, situations for which at least two components or players play an equivalent role in the function, i.e., they are substitutes. Such symmetries allow a compact description that facilitates the computations of the two previous tests. A parametrization result for classifying all regular functions, or complete simple games, up to isomorphisms, which will be intensively used in the next sections, is presented at the end of this section.

The central question in section 4 is whether the condition of 2-invariant asummability, the minimum requirement for invariant asummability, is sufficient to ensure that a regular switching function is weighted threshold. The parameters associated to a regular switching function with such a property are identified.

Section 5 and 6 respectively consider other parameters for which further conditions for k-invariant asummability and k-asummability (with $k > 2$) are sufficient to ensure that a regular switching function is weighted threshold. Some findings are shown and several questions and conjectures are proposed for future research. In section 7 we conclude the paper.

1.1. **Terminology and a fundamental result.** Let $N = \{1, 2, \dots, n\}$ denote the finite set of indices for the variables. A function $f : \{0,1\}^n \to \{0,1\}$ is called a *switching function* (or Boolean function). Let $x = (x_1, \dots, x_n) \in \{0,1\}^n$ be a *vector*. Given two vectors x, y, we have $y \leq x$ when $y_i \leq x_i$ for every variable $i = 1, \dots, n$. A switching function is *monotonic* (or positive) if $f(y) \leq f(x)$ for every pair of vectors x, y such that $y \leq x$.

A monotonic switching function is completely determined by its sets of minimal true vectors: a *true vector* $x \in f^{-1}(\{1\})$ is *minimal* if $f(y) = 0$ for all $y \in \{0,1\}^n$ such that $y \leq x$, $y \neq x$. Analogously, $x \in \{0,1\}^n$ is a *false vector* if $x \in f^{-1}(\{0\})$ and is *maximal* if moreover $f(y) = 1$ for all $y \in \{0,1\}^n$ such that $y \geq x$, $y \neq x$. We denote the set of true vectors of a function as $W = f^{-1}(\{1\})$ and the set of false vectors of a function as $L = f^{-1}(\{0\})$. Analogously, we denote the set of minimal true vectors of a function as $W^m \subseteq W$, and the set of maximal false vectors of a function as $L^M \subseteq L$. From now on we only deal with monotonic switching functions so that the term monotonic is omitted in what follows.

Given a vector x, let $X = \{i \in N : x_i = 1\}$ be a subset of the variables set. There is a one-to-one correspondence between any vector x and any *coalition* X.[1] Some types of coalitions are now highlighted: $X \in \mathcal{W}$ if and only if $x \in W$; $X \in \mathcal{L}$ if and only if $x \in L$; $X \in \mathcal{W}^m$ if and only if $x \in W^m$; $X \in \mathcal{L}^M$ if and only if $x \in L^M$. In terms of simple games the sets $\mathcal{W}$, $\mathcal{L}$, $\mathcal{W}^m$, and $\mathcal{L}^M$ are respectively the sets of: *winning* coalitions, *losing* coalitions, *minimal* winning coalitions, and *maximal* losing coalitions.

Some special type of variables deserve to be highlighted. Variable i is *irrelevant* in f if $f(x) = f(y)$ for all x with $x_i = 1$ and y with $y_k = \begin{cases} x_k, & \text{if } k \neq i \\ 0, & \text{if } k = i \end{cases}$. Variable i is *essential* in f if $f(x) = 1$ implies $x_i = 1$.

Variable i dominates j, denoted by $i \succsim j$, in a switching function f whenever $f(x) = 1$ with $x_i = 0$, $x_j = 1$ implies $f(y) = 1$ where $y_k = \begin{cases} x_k, & \text{if } k \neq i, j \\ 1, & \text{if } k = i \\ 0, & \text{if } k = j \end{cases}$. The *dominance*

[1] We adopt this term taken from the game theory context where players form coalitions with the purpose of passing proposals or amendments submitted to a vote.

relation[2] $\succsim$ is reflexive and transitive, but not necessarily complete. Thus, it is a preordering. Variables i and j are *symmetric* in f, denoted $i \sim j$, if $i \succsim j$ and $j \succsim i$. Symmetric variables form an equivalence class and N can be partitioned into equivalence classes of variables, namely $N_1, \ldots, N_t$. If $i \succsim j$ but $i \sim j$ is not true, then i *strictly dominates* j, denoted $i \succ j$, and they respectively belong to different equivalence classes, N_p and N_q in which $k \succ l$ for all $k \in N_p$ and $l \in N_q$, so that we use the notation $N_p > N_q$.

A particular case of switching functions are *regular* (or 2-monotonic) ones. A switching function $f : \{0,1\}^n \to \{0,1\}$ is said to be regular if either $i \succsim j$ or $j \succsim i$ for all $i, j \in N$. That is in a non-regular function there always exists a pair of variables i, j such that neither $i \succsim j$ nor $j \succsim i$. Note that if f is regular the equivalence classes of variables are linearly ordered, i.e., $N_1 > \cdots > N_t$. If, moreover, f has either essential variables or irrelevant variables, then N_1 is the set of essential variables, while N_t (with $t > 1$) is the set of irrelevant variables.

A particular case of regular switching functions are *weighted threshold* ones. A monotonic switching function $f : \{0,1\}^n \to \{0,1\}$ is said to be weighted threshold if there exist n positive integers, called *weights*, $w_1, \ldots, w_n \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^n w_i(x_i - y_i) > 0$ for all vectors $x \in W$ and $y \in L$. For a weighted threshold function a positive number $t \in \mathbb{Z}_{>0}$ can be inserted between the weights of true and false vectors, i.e., $\sum_{i=1}^n w_i x_i \geq t > \sum_{i=1}^n w_i y_i$ for all $x \in W$ and $y \in L$. The weights of true vectors are the numbers $w \cdot x$ for all $x \in W$ where "$\cdot$" denotes the inner product of vectors $w = (w_1, \ldots, w_n)$ and x. Analogously, the weights of false vectors are the numbers $w \cdot y$ for all $y \in L$.

This definition itself suggests two ways to test whether a switching function is weighted threshold. The first is based on solving the consistency of a linear system of inequalities with the weights as unknowns. The second is based on a geometrical approach by trying to find a hyperplane, $\sum_{i=1}^n w_i z_i = t$, with $z \in \{0,1\}^n$ and the weights and the threshold as unknowns such that: $\sum_{i=1}^n w_i x_i \geq t$ for every true vector x and $\sum_{i=1}^n w_i y_i < t$ for every false vector y. Theorems on the existence of solutions for systems of linear inequalities go back to the early 1900s. An important related result, mentioned in von Neumann and Morgenstern [47] p. 138 and extensively treated in [16] p. 31, is the Theorem of the Alternative.

For these quite direct characterizations of weighted threshold functions there exist some techniques, see e.g., Muroga [43] for the former one. The third (algebraic) characterization for which we are concerned in this paper uses a combinatorial treatment, which is very natural in the game theory context, there called trade robustness.[3]

Let f be a switching function, if there exist $2k$ vectors (not necessarily different) $x^1, \ldots, x^k$ being true vectors and $y^1, \ldots, y^k$ being false vectors such that $x^1 + \ldots + x^k = y^1 + \ldots + y^k$, where $+$ denotes the componentwise addition of vectors, then we say that f is *k-summable.* If f is not j-summable, for $j = 1, \ldots, k$, we say that f is *k-asummable.* If f is k-asummable for every $k \geq 1$ we say that f is *asummable.*

Independently, Elgot [13] and Chow [9] proved the following characterization of weighted threshold functions.

Theorem 1.1. ([13, 9]) *Let f be a switching function. Then, f is weighted threshold $\Leftrightarrow$ f is asummable.*

The proof of the sufficiency of asummability for weightedness follows from an old result in the theory on linear inequalities. One can find it in Muroga ([43], p. 192, lemma 7.2.1). Theorem 1.1 actually provides a fairly simple and uniform procedure for showing that certain functions are not weighted threshold: one produces a sequence of true vectors and indicates suitable trades of variables among these true vectors to convert all of them into false vectors.

This result is the starting point of our study. In the next section we introduce some known and quite recent refinements of Theorem 1.1.

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[2]In game theory it is known as the *desirability* relation which was introduced for simple games by Isbell (see [36] as well as [37]) and later on generalized to cooperative games by Maschler and Peleg [41]).

[3]Indeed, trades among players are very natural in some human activities. Assume a soccer team who has two outstanding goalkeepers but a weakness in the center-forward position, oppositely another team has the lack of a good goalkeeper but has two outstanding center-forwards who do not play well playing together. In this situation it is convenient for both teams to exchange goalkeepers and center forwards, so that both teams have both positions well covered.

2. Refinements of the theorem by Elgot and Chow

2.1. **Some known upper bounds for testing asummability.** A naïve checking of weightedness in Theorem 1.1 is an infinite process in principle. The next theorem shows that the finiteness of N allows to transform it into a finite, albeit lengthy, process.

Theorem 2.1. *([4,6]) Let f be a switching function. The following are equivalent,*

(*i*) *f is weighted threshold.*
(*ii*) *f is asummable.*
(*iii*) *f is $2^{2^{|N|}}$-asummable.*

Notice that a naive checking of (*iii*) is a finite (albeit lengthy) process, 2^{2^n} in (*iii*) can be lowered to $2^{\binom{n}{n/2}}$ where $|N| = n$ since, in a switching function of n variables, $\binom{n}{n/2}$ is either the maximal size for W^m or an upper bound for it. The potential length 2^{2^n} from (*iii*) was lowered to $(n+1)2^{\frac{1}{2}n\log_2 n}$ in [29]. The determination of the true worst-case order of magnitude in (*iii*) is a challenging open problem. So far it is not known whether the concept of asummability allows a polynomial sized certificate for non-weightedness. We remark that such a certificate is possible based on the dual of a linear program implementing the conditions from the notion of weighted threshold function.

Depending on the number of variables it is well-known that each switching function up to 8 players is either weighted or not 2-asummable. For 9 players one can find a switching function which is 3-assumable but 4-summable.

Gabelman [27] provides a sequence of switching functions with m^2 variables, with as many variables as equivalence classes in N, being $(m-1)$-asummable but m-summable for each positive integer m. So, m-asummability has to be considered at least for $m \in \Omega(\sqrt{|N|})$. This lower bound was increased to $\left\lfloor \frac{|N|-1}{2} \right\rfloor$ for $|N| \geq 9$ in Gvodzeva and Slinko [29]. Again, every equivalence class is a singleton in the used construction of the respective switching function.

2.2. **Regular functions.** A particular case of 2-asummability consists of a single swap of two variables in the two true vectors chosen. We say that f is *swap asummable* if for all true vectors x^1, x^2 with $x_i^1 = 1$, $x_j^1 = 0$ and $x_i^2 = 0$, $x_j^2 = 1$ for some arbitrary variables i, j it occurs that: either y^1 is a true vector or y^2 is a true vector, where

$$y_k^1 = \begin{cases} x_k^1, & \text{if } k \neq i,j \\ 0, & \text{if } k = i \\ 1, & \text{if } k = j \end{cases} \quad \text{and} \quad y_k^2 = \begin{cases} x_k^2, & \text{if } k \neq i,j \\ 1, & \text{if } k = i \\ 0, & \text{if } k = j \end{cases}.$$

Note that if vectors x^1, x^2 as defined do not exist, the function f is regular. The interest of this especial case is due to:

Theorem 2.2. *([59]) Let f be a switching function. Then, f is regular $\Leftrightarrow$ f is swap asummable.*

Clearly a weighted threshold function is regular, since $w_i \geq w_j$ implies $i \succsim j$. Thus, every non-regular function is not weighted threshold. Hence, only the restriction to regular switching functions makes sense to ascertain whether a given switching function is weighted threshold. But, Theorem 2.2 still says more: non-regular functions are all swap summable, i.e., there always exist variables i, j such that after swapping them in two given true vectors x^1, x^2 they can be converted into false vectors y^1 and y^2.

Example 2.3. The switching function defined as $f(x_1, x_2, x_3, x_4) = x_1 \cdot x_2 \vee x_3 \cdot x_4$ is not regular. Indeed, $(1,1,0,0)$ and $(0,0,1,1)$ are true vectors whereas $(1,0,1,0)$ and $(0,1,0,1)$ are false vectors. These vectors provide a certificate of swap summability.

For the class of regular switching functions we can find functions being k-asummable but $(k+1)$-summable for $k > 1$. Due to the previous result this is not possible for the class of non-regular switching functions.

A true vector is *shift-minimal*, denoted $x \in W^s$, in a regular function if there exists an ordering, say $(j_1, j_2, \ldots, j_n)$ of the variables such that $f(x) = 1$ with $x_{j_i} = 1$, $x_{j_{i+1}} = 0$

implies $f(y) = 0$ where $y_k = \begin{cases} x_k, & \text{if } k \neq j_i, j_{i+1} \\ 0, & \text{if } k = j_i \\ 1, & \text{if } k = j_{i+1} \end{cases}$. Note that the given ordering should respect the dominance relation of variables, i.e., $j_i \succsim j_{i+1}$ for all $i = 1, \ldots, n-1$.

In a regular function the set of shift-minimal winning coalitions is denoted by $\mathcal{W}^s$ and it suffices to generate the set of all winning coalitions. Indeed, if X is a shift-minimal winning coalition the coalitions obtained from X by replacing some $j \in X$ by some $i \notin X$ with $i \succsim j$ are minimal winning coalitions. If additionally some player is added in X and possibly transformed according to the previous step, then the resulting coalition is winning but not minimal.

Having all this in mind, the Chow and Elgot result can still be relaxed within the class of regular functions. In fact, Freixas and Molinero [21] prove that only shift-minimal true vectors suffice to check whether a switching function is weighted threshold. Indeed, if for a regular function f, there exist $2k$ vectors (not necessarily different) $x^1, \ldots, x^k$ being shift-minimal true vectors and $y^1, \ldots, y^k$ being false vectors such that $x^1 + \ldots + x^k = y^1 + \ldots + y^k$, then we say that the regular function f is *k-shift summable*. If f is not j-shift summable, for $j = 1, \ldots, k$, we say that f is *k-shift asummable*. If f is k-shift asummable for every $k \geq 1$ we say that f is *shift asummable*.

Theorem 2.4. *(Freixas and Molinero's Theorem 4.3 in* [21]*) Let f be a regular switching function. Then, f is m-asummable $\Leftrightarrow$ f is k-shift asummable for some $k \geq m$.*

As non-regular functions are swap summable, it makes sense to study asummability only within the class of regular switching functions and inside this class Theorem 2.4 refines Theorem 1.1.

Corollary 2.5. *Let f be a regular switching function. Then, f is weighted threshold $\Leftrightarrow$ f is shift asummable.*

Note the following. First, if a given regular function is m-summable (with m being the minimum value for summability) but also m-shift summable for some m then some unnecessary checking is avoided in using shift summability instead of summability, therefore Corollary 2.5 constitutes a significant simplification with respect to Theorem 1.1. Second, if f is m-summable (with m being the minimum value for summability) and k-shift summable with $k > m$ and being the minimum value for shift summability, then it is not clear which of the two tests involves less operations; consequently the two Theorems 1.1 and 2.4 are complementary. Third, if a given regular function is weighted threshold both processes are lengthy, but again complementary.

Two respective relaxations of Theorems 1.1 and 2.4 can be obtained if the number of equivalence classes is lower than the number of voters, i.e., $t < |N|$. The idea of such reformulations consists in grouping several "equivalent" vectors (or coalitions) into a single coalitional type of vector representing all of them. Before proceeding, we introduce two real–world examples of simple games (see Taylor and Pacelli [59] for a thorough presentation of these examples). From now on we will use the standard terminology of voting for simple games.

Example 2.6. The United Nations Security Council. The voters in this system are the fifteen countries that make up the Security Council, five of which are called permanent members whereas the other ten are called non-permanent members. Passage requires a total of at least nine of the fifteen possible votes, subject to a veto due to a nay vote from any one of the five permanent members. This example as presented here ignores abstention. For a treatment of this example considering the possibility of abstention we refer the reader to [26].

There is a fundamental difference between the permanent and the non-permanent members in the United Nations Security Council, while members within each of these groups play equivalent roles. As the equivalence relation $\sim$ partitions the player set N in $1 \leq t \leq |N|$ equivalence classes $N_1, \ldots, N_t$. We speak of t types of equivalent voters, e.g., we have two types of voters in Example 2.6: N_1 is formed by the five permanent members (vetoers) and N_2 is formed by the ten non-permanent members.

Example 2.7. The System to amend the Canadian Constitution. Since 1982, an amendment to the Canadian Constitution can become law only if it is approved by at least seven of the ten Canadian provinces, subject to the proviso that the approving provinces have, among them, at least half of Canada's population. It was first studied in Kilgour [38]. An old census (in percentages) for the Canadian provinces was: Prince Edward Island (1%), Newfoundland (3%), New Brunswick (3%), Nova Scotia (4%), Manitoba (5%), Saskatchewan (5%), Alberta (7%), British Columbia (9%), Quebec (29%) and Ontario (34%).

For example observe that coalitions (from now on we use abridgments to denote the provinces)

$$X_1 = \{PEI, New, Man, Sas, Alb, BC, Que\}$$

and

$$X_2 = \{NB, NS, Man, Sas, Alb, BC, Ont\}$$

are minimal winning coalitions because they both have exactly 7 provinces and their total population surpasses the 50%. Instead, coalitions $Y_1 = \{Man,\ Sas,\ Alb,\ BC,\ Que,\ Ont\}$ and $Y_2 = \{PEI,\ New,\ NB,\ NS,\ Man,\ Sas,\ Alb,\ BC\}$ are both losing because T_1 does not have 7 or more members and T_2 does not reach the 50% of the total Canada's population.

In example 2.6 we have again two equivalence classes: N_1 which is formed by the two big provinces, Ontario and Quebec, and N_2 which is formed by the other eight provinces.

2.3. **Trade robustness and invariant trade robustness.** The notion of asummability has a natural translation when coalitions are used instead of vectors. This new version is more natural in game theory and in economic applications because it involves the idea of trades among coalitions, see [60, 61]. Suppose $(N, \mathcal{W})$ is a simple game. Then a *trading transform* is a coalition sequence $\langle X_1, \ldots, X_k, Y_1, \ldots, Y_k \rangle$ of even length satisfying the following condition:

$$|\{i : a \in X_i\}| = |\{i : a \in Y_i\}| \quad \text{for all } a \in N.$$

The Xs are called the pre-trade coalitions and the Ys are called the post-trade coalitions. A *k-trade* for a simple game $(N, \mathcal{W})$ is a trading transform $\langle X_1, \ldots, X_j, Y_1, \ldots, Y_j \rangle$ with $j \leq k$. The simple game $(N, \mathcal{W})$ is *k-trade robust* if there is no trading transform for which all the Xs are winning in $(N, \mathcal{W})$ and all the Ys are losing in $(N, \mathcal{W})$. If $(N, \mathcal{W})$ is k-trade robust for all k, then $(N, \mathcal{W})$ is said to be *trade robust.*

Loosely speaking, $(N, \mathcal{W})$ is k-trade robust if a sequence of k or fewer (not necessarily distinct) winning coalitions can never be rendered losing by a trade.

Trivially, Theorem 1.1 can be reformulated in an equivalent way.

Theorem 2.8. *([60]) Let $(N, \mathcal{W})$ be a simple game. Then, $(N, \mathcal{W})$ is weighted $\Leftrightarrow$ $(N, \mathcal{W})$ is trade robust.*

Analogously, if we are restricted to complete simple games, equivalent to regular switching functions, and only allow pre-trades of shift-minimal winning coalitions, then we may refer to the property of *invariant-trade robustness* instead of trade robustness and again Theorem 2.4 can be reformulated in an equivalent way.[4]

Theorem 2.9. *([21]) Let $(N, \mathcal{W})$ be a complete simple game. Then, $(N, \mathcal{W})$ is weighted $\Leftrightarrow$ $(N, \mathcal{W})$ is invariant-trade robust.*

As seen in Example 2.7 the trading transform $\langle X_1, X_2 | Y_1, Y_2 \rangle$ certifies a failure of 2-invariant-trade robustness and therefore this complete simple game is not weighted. It is also trivial to see that the simple game described in Example 2.6 is invariant-trade robust and therefore weighted.

We adopt from now these denominations. In the next sections we will express them in a more compact form which avoid many computations if the number of the equivalence classes for the game is lower than the number of players, i.e., $t < |N|$.

[4]The original version of this result was given in the context of simple games first.

3. Symmetries and a parametrization of complete simple games

The simple game from Example 2.6 has $\binom{10}{4} = 210$ minimal winning coalitions, each consisting of all five permanent members and four arbitrary non-permanent members. The simple game from Example 2.7 has 112 minimal winning coalitions, 56 of them formed by one of the two big provinces and six other provinces, and 56 additional ones formed by the two big provinces and five other provinces. For $t < |N|$ types of voters we can represent coalitions in a more compact way.

Let $(N, \mathcal{W})$ be a simple game and $N_1, \ldots, N_t$ be a partition of the player set into t equivalence classes of voters. A *coalition type* (or *coalition vector*) is a vector $\overline{s} = (s_1, \ldots, s_t) \in (\mathbb{N} \cup \{0\})^t$ with $0 \le s_i \le |N_i|$ for all $1 \le i \le t$. We say that a coalition $S \subseteq N$ has type $\overline{s}$ if $s_i = |S \cap N_i|$ for all $1 \le i \le t$. A coalition type $\overline{s}$ is called winning if the coalitions of that type are winning. Analogously, the notions of minimal winning, shift-minimal winning, losing, maximal losing and shift-maximal losing are translated similarly for coalitional types.

So, the simple game from Example 2.6 can be described by the unique minimal winning coalition type $(5, 4)$ which represents all coalitions with 5 permanent members and 4 non-permanent members. The notion of a trading transform can be transferred to coalitional types.

3.1. **Coalitional types.** Let $(N, \mathcal{W})$ be a simple game and $N_1, \ldots, N_t$ be a partition into t equivalence classes of players. A *vectorial trading transform* for G is a sequence $\langle \overline{x}_1, \ldots, \overline{x}_j; \overline{y}_1, \ldots, \overline{y}_j \rangle$ of coalition types of even length such that

$$\sum_{i=1}^{j} x_{i,k} = \sum_{i=1}^{j} y_{i,k} \quad \text{for all } 1 \le k \le t. \tag{1}$$

The definition of a vectorial trading transform means that for each component $1 \le k \le t$, the sum of the k^{th} $\overline{x}$s components coincides with the sum of the k^{th} $\overline{y}$s components.

A *vectorial m-trade* is a vectorial trading transform with $j \le m$ such that the $\overline{x}_i$s are winning and after trades, as described in 1, convert into $\overline{y}_i$.

A given m-trade can easily be converted into a vectorial m-trade. The following lemma shows that the converse is also true.

Lemma 3.1. *For each pair of vectors $\overline{a} = (a_1, \ldots, a_r) \in \mathbb{N}_{>0}^r$, $\overline{b} = (b_1, \ldots, b_s) \in \mathbb{N}_{>0}^s$ with $\sum_{i=1}^r a_i = \sum_{i=1}^s b_i$ and $m = \max(\max_i a_i, \max_i b_i)$ there exist two sequences of sets $A_1, \ldots, A_r \subseteq \{1, \ldots, m\}$ and $B_1, \ldots, B_s \subseteq \{1, \ldots, m\}$ with $|A_i| = a_i$, $|B_i| = b_i$ and*

$$|\{i : j \in A_i\}| = |\{i : j \in B_i\}|$$

for all $j \in \{1, \ldots, m\}$.

Proof. W.l.o.g. we assume $a_1 \ge \cdots \ge a_r$ and $b_1 \ge \cdots \ge b_s$. We prove the statement by induction on $\sigma = \sum_{i=1}^r a_i$. For $\sigma = 1$ we have $r = s = a_1 = b_1 = m = 1$ and can choose $A_1 = B_1 = \{1\}$. We remark that the statement is also true for $\sigma = 0$, i.e., where $r = s = 0$.

If there exist indices i, j with $a_i = b_j$, then we can choose $A_i = \{1, \ldots, a_i\}$, $B_j = \{1, \ldots, b_j = a_i\}$ and apply the induction hypothesis on $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r)$ and $(b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_s)$.

In the remaining cases we assume w.l.o.g. $a_1 = m$ and $b_1 < m$. Now let l be the maximal index with $a_l = m$. Since $\sum_{i=1}^r a_i = \sum_{i=1}^s b_i$ we have $s \ge l$. So, we can consider the reduction to $(a_1 - 1, \ldots, a_l - 1, a_{l+1}, \ldots, a_r)$ and $(b_1 - 1, \ldots, b_l - 1, b_{l+1}, \ldots, b_s)$, where we possibly have to remove some zero entries and the maximum entry decreases to $m - 1$. Let $A'_1, \ldots, A'_r, B'_1, \ldots, B'_s \subseteq \{1, \ldots, m-1\}$ be suitable coalitions (allowing $A'_i = \emptyset$ or $B'_i = \emptyset$ for the ease of notation). Adding player m to the first l coalitions in both cases yields the desired sequences of coalitions. □

The construction in Lemma 3.1 for each equivalence class of voters separately converts a vectorial m-trade into an m-trade. Also for vectorial m-trades we may assume that the winning coalition types are minimal winning or that the losing coalition types are maximal losing. Since the number of coalition types is at most as large as the number of coalitions we can computationally benefit from considering vectorial m-trades if the number of types of voters is less than the number of voters.

3.2. **A parametrization of complete simple games.** In a complete simple game $(N, \mathcal{W})$ we have a strict ordering between voters from different equivalence classes. So we denote by $N_1 > \cdots > N_t$ the equivalence classes which form the unique partition of N where $a \succ b$ for all $a \in N_i$ and $b \in N_j$ with $i < j$. Let $\overline{n} = (n_1, \ldots, n_t)$ where $n_i = |N_i|$ for all $i = 1, \ldots, t$. Consider

$$\Lambda(\overline{n}) = \{\overline{s} \in (\mathbb{N} \cup \{0\})^t : \overline{n} \geq \overline{s}\},$$

where $\geq$ stands for the ordinary componentwise ordering, that is, $\overline{a} \geq \overline{b}$ if and only if $a_k \geq b_k$ for every $k = 1, ..., t$. and also consider the weaker ordering $\succeq$ given by comparison of partial sums, that is,

$$\overline{a} \succeq \overline{b} \text{ if and only if } \sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i \text{ for } k = 1, .., t.$$

If $\overline{a} \succeq \overline{b}$ we say that $\overline{a}$ *dominates* $\overline{b}$.

The couple $(\Lambda(\overline{n}), \succeq)$ is a distributive lattice and possesses a maximum (respectively, minimum) element, namely $\overline{n} = (n_1, ..., n_t)$ (resp. $\overline{0} = (0, ..., 0)$). As abbreviations we use $\overline{a} \succ \overline{b}$ for the cases where $\overline{a} \succeq \overline{b}$ but $\overline{a} \neq \overline{b}$ and $\overline{a} \bowtie \overline{b}$ for the cases where neither $\overline{a} \succeq \overline{b}$ nor $\overline{b} \succeq \overline{a}$.

The interpretation of $\overline{a} \succeq \overline{b}$ is as follows. If $\overline{b}$ is a winning coalitional vector and $\overline{a} \succeq \overline{b}$, then also $\overline{a}$ is winning. Similarly, if $\overline{a}$ is losing then $\overline{b}$ is losing too for all $\overline{a} \succeq \overline{b}$.

A winning coalitional vector $\overline{a}$ such that $\overline{b}$ is losing for all $\overline{a} \succ \overline{b}$ is called shift-minimal winning. Similarly, a losing coalition type $\overline{b}$ such that $\overline{a}$ is winning for all $\overline{a} \succ \overline{b}$ are called shift-maximal losing. Each complete simple game can uniquely described by either its set of shift-minimal winning coalition types or its set of shift-maximal losing coalition types.

Based on this insight, Carreras and Freixas ([7] pp. 148-150) provided a classification theorem for complete simple games that allow to enumerate all these games up to isomorphism by listing the possible values of certain invariants. Indeed, to each complete simple game $(N, \mathcal{W})$ one can associate the vector $\overline{n} \in \mathbb{N}^t$ as defined above and the list of shift-minimal winning coalitional vectors: $\overline{m}_p = (m_{p,1}, m_{p,2}, \ldots, m_{p,t})$ for $1 \leq p \leq r$.

Recall that two simple games $(N, \mathcal{W})$ and $(N', \mathcal{W}')$ are said to be *isomorphic* if there exists a bijective map $f : N \to N'$ such that $S \in \mathcal{W}$ if only if $f(S) \in \mathcal{W}'$.

Theorem 3.2. *([7]) (a) Given a vector $\overline{n} \in \mathbb{N}^t$ and a matrix $\mathcal{M}$ whose rows $\overline{m}_p = (m_{p,1}, m_{p,2}, \ldots, m_{p,t})$ for $1 \leq p \leq r$ satisfy the following properties:*
(i) $0 \leq \overline{m}_p \leq \overline{n}$ for $1 \leq p \leq r$;
(ii) $\overline{m}_p$ and $\overline{m}_q$ are not $\succeq$–comparable if $p \neq q$; i.e., $\overline{m}_p \bowtie \overline{m}_q$
(iii) if $t = 1$, then $m_{1,1} > 0$; if $t > 1$, then for every $k < t$ there exists some p such that

$$m_{p,k} > 0, \; m_{p,k+1} < n_{k+1};$$

and
(iv) $\mathcal{M}$ is lexicographically ordered by partial sums, if $p < q$ either $m_{p,1} > m_{q,1}$ or there exists some $k \geq 1$ such that $m_{p,k} > m_{q,k}$ and $m_{p,i} = m_{q,i}$ for $h < k$.

Then, there exists a complete simple game $(N, \mathcal{W})$ associated to $(\overline{n}, \mathcal{M})$.
(b) Two complete games $(N, \mathcal{W})$ and $(N', \mathcal{W}')$ are isomorphic if and only if $\overline{n} = \overline{n'}$ and $\mathcal{M} = \mathcal{M}'$.

The pair $(\overline{n}, \mathcal{M})$ is referred as the *characteristic invariants* of game $(N, \mathcal{W})$. The authors prove that these parameters determine the game in the sense that one is able to define a unique up to isomorphism complete simple game which possesses these invariants. The characteristic invariants allow us to count and generate all these games for small values of n. Other applications of the characteristic invariants are to considerably reduce the calculus of some solutions, as values or power indices, of the game (see e.g., [24] for the nucleolus [55]) or to study whether a game admits a representation as a weighted game by studying the consistency of a system of inequalities as we will see below.

If matrix $\mathcal{M}$ has only one row, i.e. a unique shift-minimal coalitional vector, then the characteristic invariants reduce to the couple $(\overline{n}, \overline{m})$ with

$$\begin{array}{rcccl} 1 & \leq & m_1 & \leq & n_1 \\ 1 & \leq & m_k & \leq & n_k - 1 \quad \text{if } 2 \leq k \leq t-1, \\ 0 & \leq & m_t & \leq & n_t - 1, \end{array}$$

and where the first subindex in matrix $\mathcal{M}$ is omitted. It is said, see [24], that $(\overline{n}, \overline{m})$ is a *complete simple game with minimum.*

We sketch here how to obtain the characteristic invariants $(\overline{n}, \mathcal{M})$ for the complete game from winning coalitions and reciprocally.

Given a simple game $(N, \mathcal{W})$, for each coalition S we consider *the vector or coalitional type*

$$\overline{s} = (|S \cap N_1|, ..., |S \cap N_t|),$$

in $\Lambda(\overline{n})$ where N_i are the equivalence classes with $N_1 > ... > N_t$. The vector $\overline{n}$ is $(|N_1|, ..., |N_t|)$. The rows of matrix $\mathcal{M}$ are those $\overline{s}$ such that any S is a shift-minimal winning coalition in the lattice $(\Lambda(\overline{n}), \succeq)$. Observe that each vector of indices that $\succeq$–dominates a row of $\mathcal{M}$ corresponds to winning coalitions.

Conversely, given $(\overline{n}, \mathcal{M})$ the game $(N, \mathcal{W})$ can be reconstructed, up to isomorphism, as follows. The cardinality of N is $n = \sum_{i=1}^{t} n_i$, the elements of N are denoted by $\{1, 2, \ldots, n\}$. The equivalent classes of $(N, \mathcal{W})$ are $N_1 = \{1, \ldots, n_1\}$, $N_2 = \{n_1+1, \ldots, n_1+n_2\}$, and so on.

Each $S \subseteq N$ with vector $\overline{s} = (|S \cap N_1|, ..., |S \cap N_t|)$ is a winning coalition if $\overline{s} \succeq \overline{m}$ for some $\overline{m}$ being a row of $\mathcal{M}$. Hence, the set of winning coalitions is

$$\mathcal{W} = \{S \subseteq N \,:\, \overline{s} \succeq \overline{m}_p, \text{ where } \overline{m}_p \text{ is a row of } \mathcal{M}\}.$$

Notice that a model $\overline{r}$ is winning if and only if each coalition representative R is winning. In particular, the shift-minimal winning coalitions are those with a vector being a row of $\mathcal{M}$. Precisely,

$$\mathcal{W}^s = \{S \subseteq N : \overline{s} = \overline{m}_p \text{ for some } p = 1, \ldots r\}.$$

Analogously, one can define the models of shift–maximal losing coalitions which can be written as rows in a matrix $\mathcal{Y}$ lexicographically ordered, as requested also for $\mathcal{M}$, to preserve uniqueness. These models are the maximal vectors which are not $\succeq$-comparable among them and do not dominate by $\succeq$ any row of $\mathcal{M}$.

Some particular forms of the pair $(\overline{n}, \mathcal{M})$ reveal the presence of players being either vetoers or nulls. For instance, if $m_{p,t} = 0$ for all $p = 1, \ldots, r$ the game has n_t null players. If $m_{p,1} = n_1$ for all $p = 1, \ldots, r$ the game has n_1 vetoers.

Using the well known fact that any weighted game admits *normalized* representations, where $i \sim j$ if and only if $w_i = w_j$, we will consider from now on, $w = (w_1, ..., w_t)$, the vector of weights to be assigned to the members of each equivalence class. Using normalized representations a weighted game may be expressed as $[q; w_1(n_1), \ldots, w_t(n_t)]$ in which repetition of weights is indicated within parentheses and q stands for the quota or threshold. However, these parentheses will be omitted provided that $\overline{n} = (n_1, \ldots, n_t)$ is a known vector. A complete simple game, $(N, \mathcal{W})$, is weighted *if and only if* there is a vector $w = (w_1, ..., w_t)$, such that $w_1 > ... > w_t \geq 0$, which satisfies the system of inequalities

$$(\overline{m}_p - \overline{\alpha}_q) \cdot w > 0 \quad \text{for all} \quad p = 1, 2, ..., r, \quad q = 1, \ldots, s$$

where r is the number of rows of $\mathcal{M}$, s the number of rows of $\mathcal{Y}$, and $\overline{\alpha}_q$ are the rows of $\mathcal{Y}$.

Only for $n \geq 6$ there are complete simple games which are not weighted. The following example is the smallest possible illustration of a complete simple game with minimum, i.e., with one shift-minimal winning vector, that is not a weighted game. It helps us to understand better this kind of games, which are extensively used in the next section.

Example 3.3. (1) (Example 2.6 revisited) The characteristic invariants for this example are: $\overline{n} = (5, 10)$ and $\mathcal{M} = (5\ 4)$. Thus,

$$\mathcal{W} = \{(5, x) \in \Lambda(5, 10) \,:\, x \geq 4\}$$

$$\mathcal{W}^m = \mathcal{W}^s = \{(5, 4)\}$$

Note also that $\mathcal{Y} = \begin{pmatrix} 5 & 3 \\ 4 & 10 \end{pmatrix}$ whose rows are the shift-maximal coalitional types. Trivially this game is weighted. In the next section we will show that to prove this it suffices to verify k-invariant trade robustness, where k is 2.

(2) (Example 2.7 revisited) The characteristic invariants for this example are: $\overline{n} = (2, 8)$ and $\mathcal{M} = (1\ 6)$. Thus,

$$\mathcal{W} = \{(x, y) \in \Lambda(2, 8) \,:\, x \geq 1 \text{ and } x + y \geq 7\}$$
$$\mathcal{W}^m = \{(2, 5),\ (1, 6)\}$$
$$\mathcal{W}^s = \{(1, 6)\}$$

Note also that $\mathcal{Y} = \begin{pmatrix} 2 & 4 \\ 0 & 8 \end{pmatrix}$ whose rows are the shift-maximal coalitional types.

Note that Example 2.7 is not 2-invariant trade robust since the coalitional type trading transform $< (1, 6), (1, 6) | (2, 4), (0, 8) >$ is a certificate for it. Hence, the game is not weighted.

3.3. **Particular parameters of a complete simple game.** Two parameters for a complete simple game are significant for our studies: r the number of rows of $\mathcal{M}$ or number of shift-minimal coalitional vectors and t the number of equivalence classes of players in the game. The conditions that $\mathcal{M}$ must fulfill are described in Theorem [7]. The question we pose here is the following: Are there some values for r and t for which 2-invariant trade robustness is conclusive? The purpose of the section 4 is to prove that the posed question has an affirmative answer for either $t = 1$ or $r = 2$, while in section 5 we investigate the remaining cases.

Let us remark that the number of complete and weighted games as a function of $|N|$ up to isomorphisms has been determined for these two parameters. We use below the notations $cg(n, r)$, $cg(n, t)$, $wg(n, r)$, and $wg(n, t)$ depending on whether we consider complete or weighted games or parameter r or parameter t. The first exact counting can be traced back at least to May [42] which establishes the number of symmetric or anonymous simple games. Each of such games admits $[q; \underbrace{1, 1, \ldots, 1}_{n}]$ as a weighted representation where $q \in \{1, \ldots, n\}$. As $t = 1$ implies $r = 1$ we have $cg(n, t = 1) = wg(n, t = 1) = n$.

For $r = 1$, we have $cg(n, r = 1) = 2^n - 1$ (see [25]) complete simple games with minimum with n players up to isomorphism and the number of weighted games with minimum, $wg(n, r = 1)$, is given by

$$wg(n, r = 1) = \begin{cases} 2^n - 1, & \text{if } n \leq 5 \\ \dfrac{n^4 - 6n^3 + 23n^2 - 18n + 12}{12}, & \text{if } n \geq 6 \end{cases}$$

cf. [18].

For $t = 2$ we have the nice formula $cg(n, t = 2) = F(n + 6) - (n^2 + 4n + 8)$ (cf. [23]) where $F(n)$ are the Fibonacci numbers which constitute a well–known sequence of integer numbers defined by the following recurrence relation: $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n - 1) + F(n - 2)$ for all $n > 1$. Quite curiously the addition of trivial voters, as null voters or vetoers, in complete games with two equivalence classes formed by non-trivial voters give new larger Fibonacci sequences (cf. [17]). Up to now there is not a known formula for $wg(n, t = 2)$ although it has been proved in [19] that $wg(n, t = 2) \leq \frac{n^5}{15} + 4n^4$.

Concerning general enumeration for simple, complete and weighted games it should be said that in the successive works by Muroga et al. [45, 44, 46] the number of such games was determined up to eight voters. Only the numbers of complete and weighted games for $n = 9$ voters have been determined since then, cf. [22] for the number of complete games for $n = 9$ and cf. [39]for the number of weighted games for $n = 9$. An asymptotic upper bound for weighted games is given in [10] and an asymptotic lower bound for complete games in [49].

4. Cases for which the test of **2**-invariant trade robustness is conclusive

Note first that each simple game with a unique equivalence class of voters, $t = 1$, is anonymous (symmetric), and thus weighted. Non-complete games are not swap robust and therefore they are obviously not weighted. Hence, we can limit our study to complete simple games.

Prior to study them let us consider the null effect on invariant trade robustness of removing either null or veto players in a given complete simple game.

Since adding and removing null players does not change a coalition from winning to losing or the other way round, we can state:

Lemma 4.1. *Let G be a complete simple game and G' be the game arising from G by removing its null players. With this we have that G is m-invariant trade robust if and only if G' is m-invariant trade robust.*

And a similar result, not as immediate, concerns veto players.

Lemma 4.2. *Let G be a complete simple game and G' be the game arising from G by removing its veto players. If G' is a simple game, then G is m-invariant trade robust if and only if G' is m-invariant trade robust.*

Proof. If veto players are present, then each winning coalition of a simple game must contain all veto players. So, in any m-trade every involved losing coalition must also contain all veto players. Given a simple game $G = (N, \mathcal{W})$, where $\emptyset \neq V \subseteq N$ is the set of veto players. If $V = N$ the game G is the unanimity game and therefore weighted. Otherwise we can consider $G' = (N', \mathcal{W}')$, where $N' = N \backslash V$ and $N' \supseteq S \in \mathcal{W}'$ if and only if $S \cup V \in \mathcal{W}$. If $\emptyset \in \mathcal{W}'$, then the players in $N \backslash V$ are nulls in G' and the game is indeed weighted. Otherwise G' is a simple game too. If G is complete, then G' is complete too, see e.g. [17]. Given an m-trade for G', we can obtain an m-trade for G by adding V to all coalitions. For the other direction removing all veto players turns an m-trade for G into an m-trade for G'. □

4.1. 2-invariant characterization for $\mathbf{r = 1}$.

Theorem 4.3. *Each complete simple game G with $r = 1$ shift-minimal winning coalition type is either weighted or not 2-invariant trade robust.*

Proof. Due to Lemma 4.1 and Lemma 4.2 we can assume that G contains neither nulls nor vetoers, since also the number of shift-minimal winning coalition types is preserved by the transformations used in the respective proofs.

For $t \geq 3$ types of players let the invariants of G be given by $\overline{n} = (n_1, \ldots, n_t)$ and $\mathcal{M} = \begin{pmatrix} m_1 & \ldots & m_t \end{pmatrix}$, where we abbreviate the unique shift-minimal winning coalitional vector by $\overline{m}$. From the conditions of the general parametrization theorem in [7] we conclude $1 \leq m_1 \leq n_1$, $0 \leq m_t \leq n_t - 1$, and $1 \leq m_i \leq n_i - 1$ for all $1 < i < t$. If $m_1 = n_1$ then G contains veto players and if $m_t = 0$ then G contains null players (cf. [17]). So, we have $1 \leq m_i \leq n_i - 1$ for all $1 \leq i \leq t$ in our situation. We can easily check that $\overline{a} = (m_1 - 1, m_2 + 1, m_3 + 1, m_4, \ldots, m_t)$ and $\overline{b} = (m_1 + 1, m_2 - 1, m_3 - 1, m_4, \ldots, m_t)$ are losing. Thus, $< \overline{m}, \overline{m}; \overline{a}, \overline{b} >$ is a 2-trade and G is not 2-invariant trade robust.

For $t = 2$ types of players let the invariants of G be given by $\overline{n} = (n_1, n_2)$ and $\mathcal{M} = \begin{pmatrix} m_1 & m_2 \end{pmatrix}$, where again we abbreviate the unique shift-minimal winning coalitional vector by $\overline{m}$. From the conditions of the general parametrization theorem in [7] we conclude $1 \leq m_1 \leq n_1$ and $0 \leq m_2 \leq n_2 - 1$. If $m_1 = n_1$ then G contains veto players and if $m_t = 0$ then G contains null players. So, we have $1 \leq m_i \leq n_i - 1$ for all $1 \leq i \leq 2$ in our situation.

If $2 \leq m_2 \leq n_2 - 2$, then $\overline{a} = (m_1 - 1, m_2 + 2)$ and $\overline{b} = (m_1 + 1, m_2 - 2)$ are losing. Thus, $< \overline{m}, \overline{m}; \overline{a}, \overline{b} >$ is a 2-trade and G is not 2-invariant trade robust.

If $m_2 = 1$ or $m_2 = n_2 - 1$, then both games are weighted.

Indeed, if $m_2 = 1$, then $\mathcal{Y} = \begin{pmatrix} m_1 & 0 \\ m_1 - 1 & n_2 \end{pmatrix}$, and the weights $(w_1, w_2) = (n_2, 1)$ may be assigned to players in each class respectively, so that a quota of $m_1 \cdot w_1 + w_2 = m_1 \cdot n_2 + 1$ separates weights of winning and losing coalition types.

If $m_2 = n_2 - 1$, then $\mathcal{Y} = \begin{pmatrix} c_1 & c_2 \\ m_1 - 1 & n_2 \end{pmatrix}$ where $c_1 = \min(n_1, m_1 + n_2 - 2)$ and $c_2 = \max(m_1 + n_2 - 2 - n_1, 0)$.

Now, we have two subcases to consider:

If $c_1 = n_1$ a solution is $(w_1, w_2) = (n_1 - m_1 + 2, n_1 - m_1 + 1)$ with quota $q = m_1 \cdot w_1 + (n_2 - 1) \cdot w_2 = m_1 \cdot (n_1 - m_1 + 2) + (n_2 - 1) \cdot (n_1 - m_1 + 1)$.

If $c_1 = m_1 + n_2 - 2$ then $c_2 = 0$ and a solution is $(w_1, w_2) = (n_2, n_2 - 1)$ with quota $q = m_1 \cdot w_1 + (n_2 - 1) \cdot w_2 = m_1 \cdot n_2 + (n_2 - 1)^2$. □

So, complete simple games with $r = 1$ have the property that they are either weighted or not 2-invariant trade robust. Now we are going to see that this characterization is also true for $t = 2$.

4.2. **2-invariant characterization for t = 2.** Freixas and Molinero [21] prove that there is a sequence of complete simple games G_m with *three* types of equivalent voters, i.e., $t = 3$, and *three* types of shift-minimal winning types, i.e., $r = 3$, such that G_m is m-invariant trade but not $(m+1)$-invariant trade robust for each positive integer m. Moreover, they state in Conjecture 6.1 of their paper that any complete game with $t = 2$ types of equally desirable voters is either weighted or not 2-invariant trade robust. In this subsection we prove this conjecture. Prior to stating the result let us introduce some characterizations for weightedness that will be used in the sequel. The definition of a weighted game can be rewritten to a quota-free variant:

Lemma 4.4. *Let $(N, \mathcal{W})$ be a simple game. Then,*
$(N, \mathcal{W})$ weighted $\iff$ there are n nonnegative integers $w_1, \ldots, w_n$ such that

$$\sum_{i \in S} w_i > \sum_{i \in T} w_i \tag{2}$$

for all $S \in \mathcal{W}$ and all $T \in \mathcal{L}$.

Moreover, we can use a single weight for equivalent players, i.e., a common weight w_i for each voter $p \in N_i$ where N_i is an equivalence class of players according to the desirability relation. If the game is complete we have a total order among the equivalence classes, $N_1 > \cdots > N_t$. Assume from now on $t = 2$ so that $N_1 \neq \emptyset$ and $N_2 \neq \emptyset$ is a partition of N. By $\mathcal{W}^v$ we denote the set of winning coalition types and by $\mathcal{L}^v$ the set of losing coalition types. For instance, $(x, y) \in \mathcal{W}^v$ means that all coalition $S \subseteq N$ such that $|S \cap N_1| = x$ and $|S \cap N_2| = y$ is winning. With this, Lemma 4.4 can be rewritten to:

Lemma 4.5. *Let $(N, \mathcal{W})$ be a complete simple game with two types of voters. Then,*
$(N, \mathcal{W})$ weighted $\iff$ there are two integers $w_1, w_2 \geq 0$ such that

$$[(x, y) - (x', y')] \cdot (w_1, w_2) > 0 \tag{3}$$

for all $(x, y) \in \mathcal{W}^v$ and all $(x', y') \in \mathcal{L}^v$ and "$\cdot$" stands here for the inner product.

For the proof of the theorem for $t = 2$ two special parameters of a complete simple game will play a key role so that we give even another reformulation of Lemma 4.4:

Lemma 4.6. *Let $(N, \mathcal{W})$ be a complete simple game with two types of voters. Then,*
$(N, \mathcal{W})$ weighted $\iff$ there are two integers $w_1, w_2 \geq 0$ such that

$$w_2 > M w_1 \quad \text{and} \quad w_1 > P w_2, \tag{4}$$

where

$$M = \max_{(x,y) \in \mathcal{W}^v,\, (x',y') \in \mathcal{L}^v \,:\, x' \geq x} \frac{x' - x}{y - y'}$$

and

$$P = \max_{(x,y) \in \mathcal{W}^v,\, (x',y') \in \mathcal{L}^v \,:\, x' < x} \frac{y' - y}{x - x'}$$

fulfill $0 \leq M < 1$ and $P \geq 1$.

Proof. Let $(x, y) \in \mathcal{W}^v$ and $(x', y') \in \mathcal{L}^v$. If $x' \geq x$, then $x + y > x' + y'$, so that $y - y' > x' - x \geq 0$. Thus, M is well defined and we have $0 \leq M < 1$. Also P is well defined, since we assume $x' < x$ in its definition. For $r = 2$ in matrix $\mathcal{M}$ in Theorem 3.2 we conclude the existence of a shift-minimal winning type (a, b) with $a > 0$ and $b < |N_2|$, i.e., $(a - 1, b + 1)$ is losing. Thus, we have $P \geq \frac{(b+1)-b}{a-(a-1)} = 1$.

It remains to remark that all inequalities of the definition or weighted game are implied by the ones in (4). □

Corollary 4.7. *Let $(N, \mathcal{W})$ be a complete simple game with two types of voters. Using the notation from Lemma 4.6, we have*

$$(N, \mathcal{W}) \text{ weighted} \quad \Longleftrightarrow \quad M\,P < 1. \tag{5}$$

We still need an additional technical lemma.

Lemma 4.8. *Let $s, u \in \mathbb{R}_{\ge 0}$ and $t, v \in \mathbb{R}_{>0}$. If $t > v$ and $\frac{s}{t} \ge \frac{u}{v}$, then we have $\frac{s-u}{t-v} \ge \frac{s}{t}$.*

Proof. From $\frac{s}{t} \ge \frac{u}{v}$ we sequentially conclude $vs \ge ut$ and $st - ut \ge st - vs$. Dividing both sides by $(t-v)t$ yields the stated inequality. □

Let us finally prove the result of this subsection, which was previously stated as Conjecture 6.1 in [21, page 1507].

Theorem 4.9. *Let $G = (N, \mathcal{W})$ be a complete simple game with two types of voters. Then, G is weighted if and only if G is 2-invariant trade robust.*

Proof. The direct part is immediate since G being weighted implies G satisfies m-invariant trade robustness for all $m > 1$. For the other part we start by proving that if G is a complete simple game with $t = 2$ types of voters and G is 2-invariant trade robust, then it is 2-trade robust.

Let $\langle (a_1, b_1), (a_2, b_2); (u_1, v_1), (u_2, v_2) \rangle$ be a 2-trade of G such that (a_1, b_1) and (a_2, b_2) are minimal winning. If both coalition types are shift-minimal, we have finished. In the remaining cases we construct a 2-trade with one shift-minimal winning coalition type more than before. W.l.o.g. we assume that (a_1, b_1) is not shift-minimal, so that we consider the shift to $(a_1 - 1, b_1 + 1)$. If $u_1 \ge 1$ and $v_1 \le n_2 - 1$ then we can replace (u_1, v_1) by the losing coalitional vector $(u_1 - 1, v_1 + 1)$. By symmetry the same is true for (u_2, v_2). Thus, for the cases, where we can not shift one of the losing vectors, we have

$$(u_1 = 0 \ \vee \ v_1 = n_2) \wedge (u_2 = 0 \ \vee \ v_2 = n_2).$$

(1) $u_1 = 0$, $u_2 = 0$:
Since $u_1 + u_2 = a_1 + a_2$ we have $a_1 = a_2 = 0$. Since $(0, b_1)$, $(0, b_2)$ are winning and $(0, v_1)$, $(0, v_2)$ are losing, we have $\min(b_1, b_2) > \max(v_1, v_2)$, which contradicts $b_1 + b_2 = v_1 + v_2$.

(2) $v_1 = n_2$, $v_2 = n_2$:
Since $b_1 + b_2 = v_1 + v_2$ we have $b_1 = b_2 = n_2$. Since (a_1, n_2), (a_2, n_2) are winning and (u_1, n_2), (u_2, n_2) are losing, we have $\min(a_1, a_2) > \max(u_1, u_2)$, which contradicts $a_1 + a_2 = u_1 + u_2$.

(3) $u_1 = 0$, $v_2 = n$:
Since $u_1 + u_2 = a_1 + a_2$ we have $a_2 \le u_2$. Comparing the winning coalitional vector (a_2, b_2) with the losing vector (u_2, n_2), yields $b_2 > n_2$, which is not possible.

(4) $u_2 = 0$, $v_1 = n$:
Similar to case (3).

Thus, a shift of one of the losing vectors is always possible, if not both winning vectors are shift-minimal.

According to Theorem 2.8 it remains to prove that for $t = 2$ it is not possible for G to be 2-trade robust but not weighted.

Let (a, b) and (a', b') be two winning vectors, (c, d) and (c', d') be two losing vectors such that

$$M = \frac{c-a}{b-d} \qquad \text{and} \qquad P = \frac{d'-b'}{a'-c'}, \tag{6}$$

where we assume that the vectors are chosen in such a way that both $c - a$ and $d' - b'$ are minimized. We remark $a' - c' > 0$, $d' - b' > 0$, $b - d > 0$ and $c - a \ge 0$. The latter inequality can be strengthened to $c - a > 0$, since $c - a$ implies $M = 0$ and $MP < 1$, which is a contradiction to the non-weightedness of G.

Corollary 4.7 implies $MP \ge 1$, so that

$$\frac{c-a}{b-d} \ge \frac{a'-c'}{d'-b'}. \tag{7}$$

With this, we have only the following three cases:

(a) $c - a \geq a' - c'$ and $b - d \leq d' - b'$.
(b) $c - a > a' - c'$ and $b - d > d' - b'$.
(c) $c - a < a' - c'$.

If $c - a = a' - c'$ then we have $b - d \leq d' - b'$ according to Inequality (7), i.e., we are in case (a). If $c - a > a' - c'$ then either case (a) or case (b) applies. The remaining cases are summarized in (c).

(a) Since $c + c' \geq a + a'$ and $d + d' \geq b + b'$, we can delete convenient units of some coordinates of (c, d) and (c', d') to obtain two well-defined losing vectors satisfying $(c'', d'') \leq (c, d)$ and $(c''', d''') \leq (c', d')$ with $c'' + c''' = a + a'$ and $d'' + d''' = b + b'$. Thus,
$$\langle (a, b), (a', b'); (c'', d''), (c''', d''') \rangle$$
certificates a failure of 2-trade robustness.

(b) Consider $(c'', d'') = (a + a' - c', b + b' - d')$. Since $a' - c' > 0$ and $c - a > a' - c'$ we have $a < c'' < c$. Since $b - d > d' - b'$ and $d' - b' > 0$ we have $d < d'' < b$. Thus, (c'', d'') is a well-defined coalition type. Assuming that (c'', d'') is winning, we obtain
$$\frac{c - c''}{d'' - d} = \frac{\overbrace{c - a}^{>0} - (\overbrace{a' - c'}^{>0})}{\underbrace{b - d}_{>0} - (\underbrace{d' - b'}_{>0})} \overset{\text{Lemma 4.8}}{\geq} \frac{c - a}{b - d} = M,$$
using $b - d > d' - b'$ and Inequality (7). Since $c - c'' < c - a$ we have either a contradiction to the maximality of M or the minimality of $c - a$. Thus, (c'', d'') has to be losing and
$$\langle (a, b), (a', b'); (c', d'), (c'', d'') \rangle$$
certificates a failure of 2-trade robustness.

(c) With $c - a < a' - c'$ Inequality (7) implies $d' - b' > b - d$. Consider $(a'', b'') = (c + c' - a, d + d' - b)$. Since $c - a > 0$ and $c - a < a' - c'$ we have $c' < a'' < a'$. Since $d' - b' > b - d$ and $b - d > 0$ we have $b' < b'' < d'$. Thus, (a'', b'') is a well-defined coalition type. Assuming that (a'', b'') is losing, we obtain
$$\frac{b'' - b'}{a' - a''} = \frac{\overbrace{d' - b'}^{>0} - (\overbrace{b - d}^{>0})}{\underbrace{a' - c'}_{>0} - (\underbrace{c - a}_{>0})} \overset{\text{Lemma 4.8}}{\geq} \frac{d' - b'}{a' - c'} = P$$
using $c - a < a' - c'$ and Inequality (7). Since $b'' - b' < d' - b'$ we have either a contradiction to the maximality of P or the minimality of $d' - b'$. Thus, (a'', b'') has to be winning and
$$\langle (a, b), (a'', b''); (c, d), (c', d') \rangle$$
certificates a failure of 2-trade robustness. □

Let us have a look at Example 2.7 again. We have already observed that this game is not weighted. Nevertheless it can be represented as the intersection $[7; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] \cap [12; 6, 6, 1, 1, 1, 1, 1, 1, 1, 1]$, i.e., there are only two types of provinces – the large ones, Ontario and Quebec, and the small ones, see [23]. Indeed the game is complete and the minimal winning vectors are given by $(2, 5)$ and $(1, 6)$. The maximal losing vectors are given by $(2, 4)$, $(1, 5)$, and $(0, 8)$, so that we have $M = \frac{1}{2}$ and $P = 3$. These values are uniquely attained by the coalition types $(1, 6)$, $(2, 5)$ and $(2, 4)$, $(0, 8)$. Thus we are in case (c) of the proof of Theorem 4.9 and determine the winning coalitional vector $(a'', b'') = (1, 6)$. Indeed
$$\langle (1, 6), (1, 6); (2, 4), (0, 8) \rangle$$
certificates a failure of 2-trade robustness. We remark that our previous argument for non-weightedness was exactly of this form and that the coalition type $(1, 6)$ is shift-minimal. Let's finally conclude this subsection by recalling that for Theorem 4.9 establishes that a complete game is weighted if and only it is 2-invariant trade robust, property that requires

fewer computations than 2-trade robustness, which was proved in [33] to be sufficient for testing weightedness.

5. Further invariant trade characterizations

We have seen in the previous section that complete simple games with either $t = 2$ or $r = 1$ have the property that they are either weighted or not 2-invariant trade robust.

For other combinations of r and t it is interesting to ascertain which is the maximum integer m such that m-invariant trade robustness for the given game with parameters r and t guarantees that it is weighted. Note first that $t = 1$ implies $r = 1$ so that the pairs $(r, t) = (r, 1)$ for $r > 1$ are not feasible. The results in the previous section allow us to conclude that for $(r, t) = (1, t)$ with t arbitrary or for $(r, t) = (r, 2)$ with r arbitrary such an m is given by 2.

The existence of a sequence of games being m-invariant trade robust but not $(m + 1)$-invariant trade robust is proven for $m \geq 4$ by using complete games with parameters $(r, t) = (3, 3)$ in [21]. We wonder what is happening for the remaining cases.

Consider first the smallest case: $(r, t) = (2, 3)$.

Lemma 5.1. *For $m \geq 3$ the sequence of complete simple games uniquely characterized by $\overline{n} = (2, m, m)$ and $\mathcal{M} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & m-1 \end{pmatrix}$ is $(m-1)$-invariant trade robust but not m-invariant trade robust.*

Proof. For brevity we set $w_1 = (2, 0, 1)$ and $w_2 = (1, 1, m - 1)$. The maximal losing coalition types are given by $l_1 = (2, 0, 0)$, $l_2 = (1, 0, m)$, $l_3 = (1, 1, m - 2)$, and $l_4 = (0, m, m)$. Since $m \cdot w_2 = 1 \cdot l_1 + (m - 2) \cdot l_2 + 1 \cdot l_4$, the game is not m-invariant trade robust.

Now assume that there are non-negative integers a, b, c, d, e, f with $a+b = d+e+f > 0$ and

$$a \cdot w_1 + b \cdot w_2 \leq c \cdot l_1 + d \cdot l_2 + e \cdot l_3 + f \cdot l_4.$$

We conclude

$$2a + b \leq 2c + d + e, \tag{8}$$

$$b \leq e + m \cdot f, \text{ and} \tag{9}$$

$$a + (m - 1) \cdot b \leq (m - 1) \cdot (d + e + f). \tag{10}$$

Assuming $f = 0$, we conclude $b \leq e$ from Inequality (9), so that we have $a \geq c + d$ due to $a + b = d + e + f$. Inequality (8) then yields $a = c$, $b = e$, and $d = 0$. By inserting this into Inequality (10), we conclude $c = e = 0$, which contradicts $d + e + f > 0$. Thus, we have $f \geq 1$.

Inequality (8) yields $c \geq a + f \geq 1$. Assuming $b \leq d + e + f$ we conclude $a \geq c$ from $a + b = d + e + f$, which is a contradiction to $c \geq a + f$ and $f \geq 1$. Thus, we have $b \geq d + e + f + 1$.

Inequality (10) yields

$$\frac{d + f - e}{m - 1} - a \geq 1, \tag{11}$$

so that $d + f \geq m - 1$. Since $c \geq 1$, we have $c + d + e + f \geq m$, i.e., the game is $m - 1$-invariant trade robust. □

We remark that the smallest complete simple game with $t = 3$, $r = 2$ being 3-invariant trade robust, but not 4-invariant trade robust, is given by $\overline{n} = (2, 2, 3)$ and $\mathcal{M} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$, as already observed in [21]. A certificate for a failure of 4-invariant trade robustness is given by $\langle w_1, w_1, w_2, w_2; l_1, l_1, l_1, l_2 \rangle$, where $w_1 = (2, 1, 0)$, $w_2 = (1, 0, 3)$, $l_1 = (2, 0, 1)$, and $l_2 = (0, 2, 3)$. The smallest complete simple game with $t = 3$, $r = 2$ being 4-invariant trade robust, but not 5-invariant trade robust, is attained by Lemma 5.1 for $m = 5$.

Lemma 5.2. *For $m \geq 3$ the sequence of complete simple games uniquely characterized by $\overline{n} = (2, m, m)$ and $\mathcal{M} = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 2 \\ 1 & 0 & m \\ 0 & m & m-1 \end{pmatrix}$ is m-invariant trade robust but not $(m+1)$-invariant trade robust.*

Proof. For brevity we set $w_1 = (2, 1, 0)$, $w_2 = (2, 0, 2)$, $w_3 = (1, 0, m)$, and $w_4 = (0, m, m-1)$. The maximal losing coalition types are given by $l_1 = (2, 0, 1)$, $l_2 = (1, 0, m-1)$, $l_3 = (1, 1, m-3)$, $l_4 = (0, m, m-2)$, and $l_5 = (0, m-1, m)$. Since $(m-1)\cdot w_1 + 2\cdot w_3 = m\cdot l_1 + 1\cdot l_5$, the game is not $(m+1)$-invariant trade robust.

Now assume that there are non-negative integers $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$, and b_5 with $\sum_{i=1}^{4} a_i = \sum_{i=1}^{5} b_i > 0$ and

$$k = \sum_{i=1}^{4} a_i \cdot w_i \leq \sum_{i=1}^{5} b_i \cdot l_i. \tag{12}$$

It suffices to consider the cases where $k \leq m$. We conclude

$$2a_1 + 2a_2 + a_3 \leq 2b_1 + b_2 + b_3, \tag{13}$$
$$a_1 + ma_4 \leq b_3 + m(b_4 + b_5) - b_5, \text{ and} \tag{14}$$
$$2a_2 + ma_3 + (m-1)a_4 \leq b_1 + (m-1)(\sum_{i=2}^{5} b_i) - 2b_3 - b_4 + b_5. \tag{15}$$

Let us first assume $a_4 = 0$. Using Inequality (12) and Inequality (13) we obtain

$$a_3 \geq b_2 + b_3 + 2b_4 + 2b_5. \tag{16}$$

Inserting this into Inequality (15) yields after rearranging

$$2a_2 + b_2 + 3b_3 + (m+2)b_4 + mb_5 \leq b_1. \tag{17}$$

Since $k \leq m$, we have $b_4 = b_5 = 0$. (For $k = m+1$ we have the solution $b_1 = m$, $b_2 = b_3 = b_4 = 0$, $b_5 = 1$, $a_1 = m-1$, $a_3 = 2$, and $a_2 = a_4 = 0$.) With this, Inequality (14) simplifies to $b_3 \geq a_1$ and Inequality (15) simplifies to $b_1 + (m-1)b_2 + (m-3)b_3 \geq 2a_2 + ma_3$. Twice the first plus the second inequality gives

$$b_1 + (m-1)b_2 + (m-1)b_3 \geq 2a_1 + 2a_2 + ma_3. \tag{18}$$

Inserting Inequality (12) yields

$$-b_1 + (m-3)(b_2 + b_3) \geq (m-3)a_3 + a_3. \tag{19}$$

Using Inequality (16) we conclude $a_3 = b_1 = 0$. Using Inequality (16) again, we conclude $b_2 = b_3 = 0$, which is a contradiction to $k = b_1 + b_2 + b_3 + b_4 + b_5 > 0$. Thus, we have $a_4 \geq 1$ in all cases.

$2m-2$ times Inequality (13) plus twice Inequality (14) plus Inequality (15) minus $3m-2$ times Inequality (12) yields

$$ma_1 + ma_2 + b_2 + b_3 + a_4 \leq (m-1)b_1.$$

Since $a_4 \geq 1$ and $b_1 \in \mathbb{Z}_{\geq 0}$, we have $b_1 \geq 1$.

$3m-4$ times Inequality (13) plus 4 times Inequality (14) plus twice Inequality (15) minus $6m-4$ times Inequality (12) yields

$$ma_3 \geq 2a_4 + 2b_1 + (m-2)b_2 + (m-2)b_3.$$

Since $a_4, b_1 \geq 1$ and $a_3 \in \mathbb{Z}_{\geq 0}$, we have $a_3 \geq 1$.

$m - \frac{3}{2}$ times Inequality (13) plus Inequality (14) plus Inequality (15) minus $2m-2$ times Inequality (12) yields

$$a_2 + \frac{a_3}{2} + a_4 \leq -\frac{b_2}{2} - \frac{3b_3}{2} + b_5. \tag{20}$$

Since $a_3, b_4 \geq 1$ and $b_5 \in \mathbb{Z}_{\geq 0}$, we have $b_5 \geq 2$.

$k = a_1 + a_2 + a_3 + a_4 \leq m$ plus m times Inequality (13) plus Inequality (15) minus $2m+1$ times Inequality (12) yields

$$2a_2 - (m+1)a_4 \leq -2b_2 - 4b_3 - (m+3)b_4 - (m+1)b_5 + m. \tag{21}$$

Since $b_5 \geq 2$ and $a_4 \in \mathbb{Z}_{\geq 0}$, we have $a_4 \geq 2$.

From Inequality (20) and $a_2, b_2, b_3 \geq 0$ we conclude $b_5 \geq \frac{a_3}{2} + a_4$, so that $b_5 \geq a_4 + 1$ due to $a_3 \geq 1$ and $b_5 \in \mathbb{Z}_{\geq 0}$. From Inequality (21) and $a_2, b_2, b_3, b_4 \geq 0$ we conclude $(m+1)a_4 \geq (m+1)b_5 - m$. Inserting $b_5 \geq a_4 + 1$ finally yields the contradiction $(m+1)a_4 \geq (m+1)a_4 + 1$. □

We remark that the proof of Lemma 5.2 looks rather technical and complicated at first sight. However, the underlying idea is very simple. We have to show that the parametric ILP given by inequalities (12)-(15) and 9 non-negative integer variables $a_1, \ldots, a_4, b_1, \ldots, b_5$ has a minimum value of $m + 1$ for the target function $a_1 + a_2 + a_3 + a_4$. By relaxing the integrality conditions we obtain a corresponding linear program. Minimizing a suitable variable yields a fractional lower bound that can be rounded up. The corresponding dual multipliers are used to conclude the respective lower bounds directly.

Conjecture 5.3. *For each $r \geq 5$, i.e. at least 5 coalitional types of shift-minimal winning coalitions, there exists a sequence $(G_m^r)_{m\geq 3}$ of complete simple games, such that G_m^r is m-invariant trade robust but not $(m+1)$-invariant trade robust.*

Lemma 5.4. *Let $G = (\overline{n}, \mathcal{M})$ be a complete simple game with t types of voters and r shift-minimal winning coalition types, being m-invariant trade robust, but not $(m+1)$-invariant trade robust for some $m > 1$. Then, there exists a complete simple game G' with $t+1$ types of voters and r shift-minimal winning coalition types, which is m-invariant trade robust, but not $(m+1)$-invariant trade robust.*

Proof. Let $\widetilde{m}^1, \ldots, \widetilde{m}^r$ denote the rows of $\mathcal{M}$. If G contains nulls, i.e., if $\widetilde{m}_t^i = 0$ for all $1 \leq i \leq r$, we set $\widehat{m}_j^i = \widetilde{m}_j^i$, $\widehat{m}_t^i = 1$, $\widehat{m}_{t+1}^i = 0$, $\widehat{n}_j = \overline{n}_j$, $\widehat{n}_t = 2$, and $\widehat{n}_{t+1} = \overline{n}_t$ for all $1 \leq j \leq t-1$, $1 \leq i \leq r$. Otherwise we set $\widehat{m}_j^i = \widetilde{m}_j^i$, $\widehat{m}_t^i = 1$, $\widehat{n}_j = \overline{n}_j$, and $\widehat{n}_{t+1} = 2$ for all $1 \leq j \leq t$, $1 \leq i \leq r$.

With this, we choose $G' = (\widehat{n}, \mathcal{M}')$, where $\mathcal{M}'$ is composed of the r rows $\widehat{m}^1, \ldots, \widehat{m}^r$. We can easily check that G' is indeed weighted. Let $l = (l_1, \ldots, l_{t+1})$ be a losing coalitional vector in G'. If G contains no nulls, then $(l_1, \ldots, l_t)$ is a losing vector in G. Otherwise, $(l_1, \ldots, l_{t-1}, l_{t+1})$ is a losing coalitional vector in G. Thus, a possible certificate for the failure of m-invariant trade robustness for G' could be converted into a certificate for the failure of m-invariant trade robustness for G by deleting the $(t-1)$th or tth column of the corresponding vectors – a contradiction. Similarly, we can convert a certificate for the failure of $(m+1)$-invariant trade robustness for G into a certificate for the failure of $(m+1)$-invariant trade robustness for G' by inserting ones into the $(t-1)$th or tth column of the corresponding vectors. □

The same proof is literally valid in the case of trade robustness:

Lemma 5.5. *Let $G = (\overline{n}, \mathcal{M})$ be a complete simple game with t types of voters and r shift-minimal winning coalition types, being m-trade robust, but not $(m+1)$-trade robust for some $m > 1$. Then, there exists a complete simple game G' with $t+1$ types of voters and r shift-minimal winning coalition types, which is m-trade robust, but not $(m+1)$-trade robust.*

With these results at hand we may prove that larger classes of games according to parameters r and t never reduce the largest failures of invariant-trade robustness. Table 3 summarizes the invariant trade robust test to be used for a game to determine whether this is weighted. Looking at this table we conclude that 2-invariant trade robustness is conclusive exactly for the cases determined in Section 4 (conjectured values are printed in bold face), while for others there is no combination of r and t for which some $m > 2$ be enough to ensure that the game is weighted.

In words, if one wishes to study the class of complete games with a given pair (r, t) then 2-invariant trade robustness is a very powerful tool to check weightedness for $t \leq 2$ and $r = 1$, but for the rest of combinations (r, t) we need to look at trade-robustness, which is the purpose of the next section.

TABLE 3. W: weighted; −: not possible; 2-I-T-R: either weighted or not 2-invariant trade robust; ∞-I-T-R: there are games being not m-invariant trade robust for all m; NW: not a weighted game; conjectured values in **bold face**.

$r\downarrow \mid t\rightarrow$	1	2	3	4	...
1	W	2-I-T-R	2-I-T-R	2-I-T-R	NW
2	-	2-I-T-R	∞-I-T-R	∞-I-T-R	∞-I-T-R
3	-	2-I-T-R	∞-I-T-R	∞-I-T-R	∞-I-T-R
4	-	2-I-T-R	∞-I-T-R	∞-I-T-R	∞-I-T-R
...	-	2-I-T-R	∞-**I-T-R**	∞-**I-T-R**	∞-**I-T-R**

6. FURTHER TRADE CHARACTERIZATIONS

It is well known that all simple games with up to 3 voters are weighted while there are non-weighted simple games for $n \geq 4$ voters. Restricting the class of simple games to swap robust simple games, i.e. complete simple games, one can state that up to 5 voters each such game is weighted while for $n \geq 6$ voters there are non-weighted complete simple games. Going over to 2-invariant trade robustness does not help too much. As shown in [21], precisely 3 of the 60 non-weighted complete simple games with $n = 6$ voters are 2-invariant trade robust but not 3-invariant trade robust. For the classical trade robustness the same authors have shown that all 2-trade robust complete simple games with up to seven voters are weighted. By an exhaustive enumeration we have shown that the same statement is true for $n = 8$ voters, i.e., there are exactly 2 730 164 weighted games and the remaining 13 445 024 complete simple games are not 2-trade robust. As shown in [29], there are complete simple games with $n = 9$ voters, which are 3-trade robust but not 4-trade robust. The corresponding example, belonging to a parametric family, consists of nine different types of players, i.e., no two players are equivalent.

If the number t of types of players is restricted we can obtain tighter weighted characterizations. For $t = 1$ the games are always weighted and for $t = 2$ weightedness is equivalent with 2-trade robustness (or 2-invariant trade robustness for complete simple games). Based on this characterization one can computationally determine the number of complete simple games with two types of voters which are either weighted, i.e. 2-invariant trade robust, or not weighted, i.e. not 2-invariant trade robust. In [23] this calculation was executed for $n \leq 40$ voters. It turns out that the fraction of non 2-invariant trade robust complete simple games quickly tends to 1. An exact, easy-to-evaluate, and exponentially growing formula for the number of complete simple games with two types of voters is proven in [23, 39]. From the upper bound $n^5/15 + 4n^4$, see [19], for the number of weighted games with two types of voters, we can conclude that this is generally true. We remark that it is not too hard to compute the number of 2-invariant trade robust complete simple games with $t = 2$ for $n \leq 200$, see [17], so that we abstain from giving a larger table.

For $t = 3$ types of voters we have checked by an exhaustive enumeration that up to $n = 10$ voters each complete simple game is either weighted or not 2-trade robust. For $n = 11$ voters we have the four examples given by $\overline{n} = (3, 3, 5)$, $\mathcal{M}_1 = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 5 \end{pmatrix}$, $\mathcal{M}_2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix}$, $\mathcal{M}_3 = \begin{pmatrix} 3 & 3 & 0 \\ 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 3 & 5 \end{pmatrix}$, and $\mathcal{M}_4 = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 0 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{pmatrix}$, which are all 2-trade robust but not 3-trade robust. This resolves an open problem from [21]. We have computationally checked all complete simple games with three types of voters, i.e. $t = 3$, and up to 15 voters, i.e. $|N| \leq 15$, see Table 6. It seems that the number of games which are 2-trade robust but not 3-trade robust grows rather slowly. Indeed, up to 15 players every 3-trade robust such game is weighted.

For $t = 4$ types and $n = 9$ voters there are several complete simple games which are 2-trade robust but not 3-trade robust, e.g. the one given by $\overline{n} = (1, 2, 3, 3)$ and $\mathcal{M} =$

Table 4. Classification of complete simple games with three types of up to 15 voters. Parameters: size (n), number of complete simple games ($\#CG$), number of weighted simple games ($\#WG$), number of non 2-trade robust complete simple games ($\#N\text{-}2T$), number of non 3-trade, but 2-trade, robust complete simple games ($\#N\text{-}3T$).

n	$\#CG$	$\#WG$	$\#N\text{-}2T$	$\#N\text{-}3T$
3	0	0	0	0
4	6	6	0	0
5	50	50	0	0
6	262	256	6	0
7	1114	976	138	0
8	4278	3112	1166	0
9	15769	8710	7059	0
10	58147	22084	36063	0
11	221089	51665	169420	4
12	886411	113211	773186	14
13	3806475	234649	3571788	38
14	17681979	463872	17218019	88
15	89337562	879989	88457385	188
16	492188528	1610011	490578137	380
17	2959459154	2852050	2956606348	756

$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & 2 \end{pmatrix}$. For $t = 4$ and $n = 10$ there are already 120 complete simple games which are 2-trade robust but not 3-trade robust.

The next cases to look at, are $t = 3$ and $r = 2$. For both cases we have already presented examples which are 2-trade robust but not 3-trade robust. Still we found no example which is 3-trade robust but not 4-trade robust.

Question 6.1. *Is every 3-trade robust complete simple game with $t = 3$ types of voters weighted?*

Question 6.2. *Is every 3-trade robust complete simple game with $r = 2$ shift-minimal winning coalition types weighted?*

As a first step into the direction of these two questions, we have looked at the intersection of both classes, i.e., complete simple games with $t = 3$ and $r = 2$. The game corresponding to the previously presented matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ for $n = 11$ voters are of this type and can be generalized:

Lemma 6.3. *For each $k_1, k_2, k_3, l \in \mathbb{N}$ the games uniquely characterized by $\overline{n}_1 = (n_1, n_2, n_3)$, $\mathcal{M}_1 = \begin{pmatrix} n_1 - (l+1) & n_2 - 1 & n_3 - (l+2) \\ n_1 - 2(l+1) & n_2 - 1 & n_3 \end{pmatrix}$, where $n_1 = 3 + k_1 + 2l$, $n_2 = 3 + k_2$, $n_3 = 5 + k_3 + 2l$, and $\overline{n}_2 = (n_1, n_2, 5 + 2l)$, $\mathcal{M}_2 = \begin{pmatrix} l+1 & 1 & l+2 \\ 0 & 1 & 2(l+2) \end{pmatrix}$ are 2-trade robust but not 3-trade robust.*

We skip the easy but somewhat technical and lengthy proof. Having the nice parametrization at hand, we can easily state the corresponding generating function

$$x^{11}\left(\frac{1}{(1-x)^3(1-x^4)} + \frac{1}{(1-x)^2(1-x^4)}\right) = \frac{x^{11}(x-2)}{(1-x)^3(1-x^4)}$$

counting the number of such examples, i.e., asymptotically there are $\frac{n^3}{24} + O(n^2)$ such games.

Conjecture 6.4. *All 3-trade robust complete simple games with $t = 3$ and $r = 2$ are weighted. Additionally, the 2-trade robust but not 3-trade robust games are exactly those from Lemma 6.3.*

By an exhaustive enumeration we have checked Conjecture 6.4 up to $n = 20$ voters.

From the previous results it is not clear whether a small number of types or shift-minimal winning coalition types allows to restrict the check of m-trade robustness to a finite m.

Question 6.5. *For which values of t does a sequence G_k of complete simple games with t types of voters exist such that G_k is k-trade robust but not $(k+1)$-trade robust for all $k \geq 2$?*

Question 6.6. *For which values of r does a sequence G_k of complete simple games with r shift-minimal winning coalition types exist such that G_k is k-trade robust but not $(k+1)$-trade robust for all $k \geq 2$?*

Any progress concerning answers for either the conjecture or the questions posed would be of interest.

In Table 5 we combine the results from Section 4 with the questions of this section. For $r = 2$ or $t = 3$ we have not found any example being 3-trade robust but not weighted, but this should be checked formally and become conjectures for future work (in Table 5 it appears in black).

TABLE 5. W: weighted; $-$: not possible; 2-I-T-R: either weighted or not 2-invariant trade robust; **3-T-R** for small values of n all games are either weighted or not 3-trade robust – still a conjecture; NW: not a weighted game; ?: it is not known if some $m > 2$ suffices to assert that m-trade robustness implies weighted.

$r \downarrow \mid t \rightarrow$	1	2	3	4	...
1	W	2-I-T-R	2-I-T-R	2-I-T-R	NW
2	-	2-I-T-R	**3-T-R**	**3-T-R**	**3-T-R**
3	-	2-I-T-R	**3-T-R**	?	?
4	-	2-I-T-R	**3-T-R**	?	?
...	-	2-I-T-R	**3-T-R**	?	?

7. Conclusion

This paper looks at the characterization of threshold functions within the class of switching functions. We have tried to gather results and efforts that have taken place in different areas of study. The new results presented in this paper have been exposed in the simple game terminology since some significant advances have been held in this area in the last two decades. To study the main problem we have restricted ourselves to the class of regular functions since non-regular functions are swap summable and therefore not threshold functions.

For regular functions the test of asummability can be computationally relaxed to invariant asummability. The strongest condition for invariant asummability, 2-invariant asummability, is conclusive for deciding if a given regular function is a threshold function if the regular function has either a unique coalitional type of shift-minimal winning true vectors or two types of components—two equivalence classes—. Larger values for the number of shift-minimal winning true vectors or for the number of equivalence classes show that the tests of asummability and invariant asummability are complementary. We have found some conspicuous examples of non-threshold functions being k-asummable (or k'-invariant asummable for some $k' \geq k$) but $k+1$-summable (or $k'+1$-invariant summable). We have incorporated a number of open questions in hopes of others taking up the challenges that we have left where over.

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