Computation of Local ISS Lyapunov Functions for Discrete-time Systems Via Linear Programming☆

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Abstract

This paper presents a numerical algorithm for computing ISS Lyapunov functions for discrete-time systems which are input-to-state stable (ISS) on compact subsets of the state space. The algorithm relies on solving a linear optimization problem and delivers a continuous and piecewise affine ISS Lyapunov function on a suitable triangulation covering the given compact set excluding a small neighbourhood of the origin. The objective of the linear optimization problem is to minimize the ISS gain. It is shown that for every ISS system there exist a suitable triangulation such that the proposed algorithm terminates successfully.

Keywords: Discrete-time systems, Local input-to-state stability, Local ISS Lyapunov function, Lyapunov function, Linear programming

1. Introduction

The objective of this paper is to investigate how to compute Lyapunov functions which characterize input-to-state stability (ISS) for discrete-time systems with perturbations described by

$$x^+ = f(x, u) \tag{1}$$

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with vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, state $x \in \mathbb{R}^n$, and perturbation input $u \in \mathbb{R}^m$. We assume that f satisfies $f(0,0) = 0$ and one of the following two hypothesis.

(H1) The map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz continuous.

(H2) The vector field f is twice continuously differentiable.

The concept of input-to-state stability (ISS) for nonlinear continuous-time systems was originally introduced by Sontag [17] in the late 1980s. The concept defines a stability property of state trajectories with respect to initial states and inputs and implies that bounded inputs imply bounded outputs. Many results about ISS for continuous-time systems have been obtained, see [17, 18, 19, 21]. In [11], the ISS concept is extended to discrete-time systems. It is further proved in [11] that ISS is equivalent to the existence of ISS Lyapunov function for discrete-time system. ISS is particularly suitable for the stability analysis of large scale interconnected systems: if each subsystem of an interconnected system is ISS, then stability of the whole system may be analysed via ISS Lyapunov functions for the subsystems and small gain theorems [2, 3, 4, 10]. These results motivate us to consider the problem of computing ISS Lyapunov functions for locally ISS systems. The knowledge of an ISS Lyapunov function immediately leads to the knowledge of the ISS gain which can be used in stability analysis based on small gain theorems.

One way to compute ISS Lyapunov functions is by computing robust Lyapunov functions for suitable auxiliary systems, see, e.g., [20, 15, 6]. However, the introduction of auxiliary systems typically makes the gain functions very conservative. As a remedy, in the paper [14], an algorithm for the computation of ISS Lyapunov functions for continuous-time systems is proposed without resorting to auxiliary systems. The ISS Lyapunov function computed in this reference is a piecewise affine function on a triangular grid, whose values in the grid vertices are given by the solution of a linear optimization problem. The optimization objective includes minimizing the ISS gain of the Lyapunov function, hence on sufficiently fine triangulations one may expect to compute a near optimal gain parameter. The computed ISS Lyapunov functions are true functions which are not an approximation of a Lyapunov function, since the interpolation errors are incorporated in the algorithm. The approach to use linear programming for the computation of Lyapunov functions was first presented in [16]. For exponentially stable system, in [7] it was proved that the approach proposed in [16] always yields a solution.

Furthermore, this result was extended to asymptotically stable systems [8], to asymptotically stable, arbitrarily switched, non-autonomous systems [9], and to asymptotically stable differential inclusions [1]. The approaches proposed in these papers deliver true Lyapunov functions on compact subsets of the state space except possibly on arbitrarily small neighbourhood of the asymptotically stable equilibrium. Mainly inspired by results of [14], in this paper we will propose an analogous linear programming based algorithm for computing true ISS Lyapunov functions for locally ISS discrete-time systems. Discrete-time systems are of interest since they are widely used to study practical phenomena in many application fields such as engineering, chemistry and finance. Furthermore, because the solutions in discrete-time setting are sequence of points rather than continuous functions as in continuous-time setting, it is not possible to derive discrete-time results straightforwardly from their counterparts in continuous-time setting.

Since this paper can be seen as a discrete-time variant of [14], let us briefly describe the key differences between the main results of [14] and this paper, in order to demonstrate that the translation of [14] to discrete time is not entirely straightforward. The constraints on the linear optimization problem to be proposed in this paper will be stricter than that on the linear optimization problem (30) in [14], since the solution to the discrete-time system is not absolutely continuous. However, the conditions making sure the algorithm to be proposed always has a feasible solution are more relaxed than that of [14, Theorem 4.2]. The reason is that a difference quotient with fixed step size 1 of the Lyapunov function rather than its derivative is utilized for discrete-time system. In order to utilize directly the inequality delivered by the algorithm in the small gain theorem based stability analysis, we will moreover use the $\|\cdot\|_1$ norm for x and u instead of two different norms as in [14] for the definition of ISS Lyapunov function. Furthermore, we use a more elaborate expression for the decay rate of the ISS Lyapunov function than in [14]. Instead of $\|x\|_2$, we use the expression $\sigma\|x\|_1^2 + (1-\sigma)\|x\|_1$ (+"interpolation error"), which leads to a significantly less conservative ISS gain parameter. As an alternative, for computational domains with a small diameter we also discuss the use of expressions of the form $\|x\|_1^q$ (+"interpolation error") for $q \geq 3$.

The paper is organised as follows: in Section 2, the notation and preliminaries are introduced. In Section 3 the linear programming based algorithm for the computation of ISS Lyapunov functions for discrete-time systems is described. In Section 4 we discuss the main results of the paper: we prove that upon successful termination the algorithm yields an ISS Lyapunov function outside a small neighbourhood of the equilibrium, and that successful termination is guaranteed if the system admits a C^1 ISS Lyapunov function with bounded gradient and the simplicial grid is chosen appropriately. In Section 5, four numerical examples are presented to illustrate our algorithm and results. Some concluding remarks are discussed in Section 6.

2. Notations and Preliminaries

Let $\mathbb{Z}_+$, R_+ denote nonnegative integers, nonnegative real numbers respectively. Given a vector x in $\mathbb{R}^n$, let $x^\top$ denote its transpose. The standard inner product of $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y\rangle$. For a subset $\Omega \subset \mathbb{R}^n$, we denote the boundary, the closure and the complement of Ω by $\partial\Omega$, $\overline{\Omega}$ and Ω^C respectively. We use the standard *norms* $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$ and $\|x\|_\infty := \max_{i\in\{1,\ldots,n\}} |x_i|$ and let $B_p(z, r) := \{x \in \mathbb{R}^n \mid \|x - z\|_p < r\}$ denote the open ball of radius r around z in the norm $\|\cdot\|_p$. The *induced matrix norm* is defined by $\|A\|_p := \max_{\|x\|_p=1} \|Ax\|_p$. By $\|u\|_{\infty,p} := \sup_{k\in\mathbb{Z}_+} \|u(k)\|_p$ we denote the *supremum norm* of a function $u : \mathbb{Z}_+ \to \mathbb{R}^m$.

The admissible input values are given by $U_R := \overline{B_1(0, R)} \subset \mathbb{R}^m$ for a constant $R > 0$ and the admissible input functions by $u \in \mathcal{U}_R := \{u : \mathbb{Z}_+ \to U_R\}$. The k^{th} element of the solution sequence of (1) corresponding to an initial condition x_0 and an input $u \in \mathcal{U}_R$ is denoted by $x(k, x_0, u)$.

For the assumption (H1), we make a similar statement as discussed in [14]. Given a compact set $\mathcal{G} \subset \mathbb{R}^n$, we define the following notation: For each $u \in U_R$, $L_x(u)$ is the Lipschitz constant of the map $x \mapsto f(x, u)$, and for each $x \in \mathcal{G}$, $L_u(x)$ is the Lipschitz constant for the function $u \mapsto f(x, u)$. Then, by (H1) there exist constants $\overline{L}_x$ and $\overline{L}_u$ such that

$$\overline{L}_x \geq L_x(u) > 0,\ \overline{L}_u \geq L_u(x) > 0 \tag{2}$$

for all $x \in \mathcal{G}$, $u \in U_R$.

Let us recall comparison functions which are widely used in stability analysis. A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is called *positive definite* if it satisfies $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s > 0$. A positive definite function is of *class* $\mathcal{K}$ if it is strictly increasing and of *class* $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of *class* $\mathcal{L}$ if $\gamma(r)$ is strictly decreasing to 0 as $r \to \infty$ and we call a continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ of *class* $\mathcal{KL}$ if it is of class $\mathcal{K}_\infty$ in the first argument and of class $\mathcal{L}$ in the second argument. For more details about comparison functions, we recommend the reference [12].

The following definition describes the stability property concerned in this paper.

Definition 1. *System* (1) *is locally* input-to-state stable (ISS)*, if there exist* $\rho_x > 0$*,* $\rho_u > 0$*,* $\gamma \in \mathcal{K}_\infty$ *and* $\beta \in \mathcal{KL}$ *such that for all* $\|x_0\|_2 \le \rho_x$ *and* $\|u\|_\infty \le \rho_u$

$$\|x(k, x_0, u)\|_2 \le \beta(\|x_0\|_2, k) + \gamma(\|u\|_\infty), \quad \forall k \in \mathbb{Z}_+. \tag{3}$$

If $\rho_x = \rho_u = \infty$*, then system* (1) *is globally* input-to-state stable (ISS).

For $u \equiv 0$, it is obvious that ISS implies that the origin is an equilibrium of (1) which is asymptotically stable. The function $\gamma \in \mathcal{K}_\infty$ is referred to as *ISS gain*.

According to Theorem 1 in [11], the ISS property of (1) is equivalent to the existence of a smooth, i.e. C^∞, ISS-Lyapunov function for (1). In this following, we prove that this theorem makes sure our algorithm terminates successfully. However, the proposed algorithm delivers a continuous and piecewise affine and thus merely Lipschitz continuous Lyapunov functions. Hence, we do not require a priori differentiability in the following definition of an ISS Lyapunov function.

Definition 2. *Let* $\mathcal{G} \subseteq \mathbb{R}^n$ *with* $0 \in \operatorname{int} \mathcal{G}$*. A Lipschitz continuous function* $V : \mathcal{G} \to \mathbb{R}_+$ *is called an (local)* ISS Lyapunov function *for system* (1) *on* $\mathcal{G}$ *if there exist* $\mathcal{K}_\infty$ *functions* α_1*,* α_2*,* α *and* β *such that*

$$\alpha_1(\|x\|_1) \le V(x) \le \alpha_2(\|x\|_1) \tag{4}$$

$$V(f(x, u)) - V(x) \le -\alpha(\|x\|_1) + \beta(\|u\|_1) \tag{5}$$

hold for all $x \in \mathcal{G}$*,* $u \in U_R$*. If* $\mathcal{G} = \mathbb{R}^n$ *and* $R = \infty$ *then* V *is called a global* ISS Lyapunov function*. The function* $\beta \in \mathcal{K}_\infty$ *is called* Lyapunov ISS gain *or briefly* gain *in what follows. If* β *is linear, then the gain is said to be linear.*

Remark 3. *The importance of the chosen norm* $\|u\|_1$ *in* (5) *is that it will turn out to be useful in deriving easy estimates, see the last paragraph of the proof of Theorem 16. In contrast to [14, Definition 2.3], here* $\|x\|_1$ *is used, since it avoids conservative estimates in deriving an inequality for* $\|x\|_1$ *which is necessary for analysing stability of interconnected ISS systems by small gain theorems in [13, Section 1.6].*

For the purpose of making (5) algorithmically tractable, we have to restrict ourselves to particular classes of functions α and β. In this paper, we use the functions $\alpha(s) = \sigma s^2 + (1-\sigma)s$ and $\beta(s) = rs$, where $1 \geq \sigma \geq 0$ and $r \geq 0$ are fixed parameters which will be later become optimization variables. The following proposition proves that on compact subsets of the state space excluding a small ball around the origin this can be done without loss of generality. The reasons for excluding a small neighbourhood $B_2(0,\epsilon)$ of the origin will be explained in Remark 8.

Proposition 4. *If $W(x)$ is an ISS Lyapunov function for system (1) on a compact set $\mathcal{G} \subset \mathbb{R}^n$ with $0 \in \text{int}\,\mathcal{G}$, then for any $\epsilon > 0$ and $1 \geq \sigma \geq 0$ there exist positive constants $C, r > 0$ such that $V(x) := CW(x)$ satisfies*

$$V(x) \geq \|x\|_1 \quad \forall x \in \mathcal{G} \setminus B_2(0,\epsilon) \tag{6}$$

and

$$V(f(x,u)) - V(x) \leq -(\sigma\|x\|_1^2 + (1-\sigma)\|x\|_1) + r\|u\|_1 \tag{7}$$

for $x \in \mathcal{G} \setminus B_2(0,\epsilon)$, $u \in U_R$ with U_R from Definition 2.

Proof. Based on the assumption, there exist $\alpha_1, \alpha_2, \alpha, \beta \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x\|_1) \leq W(x) \leq \alpha_2(\|x\|_1) \tag{8}$$

$$W(f(x,u)) - W(x) \leq -\alpha(\|x\|_1) + \beta(\|u\|_1) \tag{9}$$

hold for $x \in \mathcal{G}$, $u \in U_R$.

Let

$$\begin{aligned} C = \min\{c \in \mathbb{R}_+ \,|\, c\alpha_1(\|x\|_1) \geq \|x\|_1 \text{ and} \\ c\alpha(\|x\|_1) \geq \sigma\|x\|_1^2 + (1-\sigma)\|x\|_1 + \epsilon, \forall x \in \mathcal{G} \setminus B_1(0,\epsilon)\}. \end{aligned}$$

It is obvious that such a C exists.

Define

$$r = \min\{r \in \mathbb{R}_+ \,|\, r\|u\|_1 \geq C\beta(\|u\|_1)\, \forall u \in U_{\mathbb{R}} \text{ with } C\beta(\|u\|_1) \geq \epsilon\}.$$

It is easy to check that such a r exists.

Using (8) and (9), we have

$$\|x\|_1 \leq C\alpha_1(\|x\|_1) \leq CW(x) \leq C\alpha_2(\|x\|_1) \tag{10}$$

$$\begin{aligned} CW(f(x,u)) - CW(x) &\leq -C\alpha(\|x\|_1) + C\beta(\|u\|_1) \\ &\leq -(\sigma\|x\|_1^2 + (1-\sigma)\|x\|_1) - \epsilon + \epsilon + r\|u\|_1 \\ &\leq -(\sigma\|x\|_1^2 + (1-\sigma)\|x\|_1) + r\|u\|_1. \end{aligned} \tag{11}$$

Therefore, $V(x) := CW(x)$ satisfies (6) and (7). This completes the proof. $\square$

3. The algorithm

In this section, we describe the algorithm to compute a local ISS Lyapunov function on a compact set $\mathcal{G} \subset \mathbb{R}^n$ with $0 \in \operatorname{int} \mathcal{G}$ for perturbation inputs from $U_R \subset \mathbb{R}^m$. The computed ISS Lyapunov function is a continuous and piecewise affine function defined on a simplicial grid. The algorithm consists in solving a linear optimization problem for the values of the Lyapunov functions on the vertices of the grid. Since the interpolation errors are incorporated in the algorithm, a true Lyapunov function rather than a numerical approximation is computed.

3.1. Definitions

Before introducing the algorithm, we state the following definitions described in [13, 14]: The *closed convex hull* of vectors $x_0, x_1, \ldots, x_m \in \mathbb{R}^n$ is defined by

$$\operatorname{co}\{x_0, \ldots, x_m\} := \left\{ \sum_{i=0}^{m} \lambda_i x_i : 0 \leq \lambda_i \leq 1, \sum_{i=0}^{m} \lambda_i = 1 \right\}.$$

A set of vectors $x_0, x_1, \ldots, x_m \in \mathbb{R}^n$ is said to be *affine independent* if $\sum_{i=1}^{m} \lambda_i (x_i - x_0) = 0$ implies $\lambda_i = 0$ for all $i = 1, \ldots, m$. This definition is independent of the numbering of the x_i, that is, of the choice of the reference point x_0.

A simplex in $\mathbb{R}^n$ is a set of the form $\Gamma = \operatorname{co}\{x_0, x_1, \ldots, x_j\}$, where $x_0, x_1, \ldots, x_j$ are affine independent. The faces of Γ are given by $\operatorname{co}\{x_{i_0}, \ldots, x_{i_k}\}$, where $\{x_{i_0}, \ldots, x_{i_k}\}$ ranges over the subsets of $\{x_0, x_1, \ldots, x_j\}$. An n-simplex is generated by a set of $n+1$ affine independent vertices.

Definition 5. *A collection $\mathcal{S} = \{\Gamma_\nu \mid \nu = 1, \ldots, N\}$ of n-simplices in $\mathbb{R}^n$ is called a suitable triangulation, if*

(i) for all $\Gamma_1, \Gamma_2 \in \mathcal{S}$ the intersection $\Gamma_1 \cap \Gamma_2$ is a face of both Γ_1 and Γ_2 or empty,
(ii) for $\mathcal{D}_\mathcal{S} = \cup_\nu \Gamma_\nu$, $\operatorname{int} \mathcal{D}_\mathcal{S}$ is a connected neighbourhood of the origin,
(iii) if $0 \in \Gamma \in \mathcal{S}$, then 0 is a vertex of $\Gamma \in \mathcal{S}$.

Given a simplex $\varGamma$, let $\mathrm{diam}(\varGamma) := \max_{x,y\in\varGamma} \|x-y\|_2$ denote the *diameter* of the simplex $\varGamma$.

In the following, we require that $\mathcal{G} \subset \mathbb{R}^n$ may be partitioned into finitely many n-simplices $\mathcal{T} = \{\varGamma_\nu \mid \nu = 1,\dots,N\}$, such that $\mathcal{T}$ defines a suitable triangulation. By assumption, we may also partition U_R into m-simplices $\mathcal{T}_u = \{\varGamma^u_\kappa \mid \kappa = 1,\dots,N_u\}$ defining a suitable triangulation. Let $h_{x,\nu} = \mathrm{diam}(\varGamma_\nu)$, $h_{u,\kappa} = \mathrm{diam}(\varGamma^u_\kappa)$ and $h_x = \max_{\nu=1,\dots,N} h_{x,\nu}$, $h_u = \max_{\kappa=1,\dots,N_u} h_{u,\kappa}$. For each $x \in \mathcal{G}$, we define the *active index set* $I_{\mathcal{T}}(x) := \{\nu \in \{1,\dots,N\} \mid x \in \varGamma_\nu\}$. For the simplices $\mathcal{T}_u$, we additionally assume that

(A1) for each simplex $\varGamma^u_\kappa \in \mathcal{T}_u$, the vertices of $\varGamma^u_\kappa$ are in the same closed orthant.

For our algorithm, we assume the existence of a simply connected and compact set $\mathcal{O} \subset \mathcal{G}$ satisfying the following property.

Property A

1. $\mathcal{O} \subset \mathcal{T}$, $0 \in \mathrm{int}\, \mathcal{O}$,
2. $x \in \mathcal{O}$ implies $f(x,u) \in \mathcal{G}$ for all $u \in \mathcal{T}_u$. Here f is from (1), and
3. there exists no $\varGamma_\nu$ with $x, y \in \varGamma_\nu$ satisfying $x \in \mathcal{O}$, $y \in \mathcal{G} \setminus \mathcal{O}$.

Remark 6. *Since the solution to equation* (1) *is not continuous, the constraints on the set $\mathcal{O}$ are necessary, in order to make sure that the point $f(x,u)$ is in the set $\mathcal{G}$ for $x \in \mathcal{O}$, $u \in \mathcal{T}_u$.*

Let $\mathrm{CPA}(\mathcal{T})$ denote the space of continuous functions $V : \mathcal{G} \to \mathbb{R}$ which are *linearly affine* on each simplex, i.e., there are $a_\nu \in \mathbb{R}$, $w_\nu \in \mathbb{R}^n$, $\nu = 1,\dots,N$, such that

$$V|_{\varGamma_\nu}(x) = \langle w_\nu, x\rangle + a_\nu \qquad \forall\, x \in \varGamma_\nu,\ \varGamma_\nu \in \mathcal{T} \tag{12}$$

$$\nabla V_\nu := \nabla V|_{\mathrm{int}\,\varGamma_\nu} = w_\nu \qquad \forall\, \varGamma_\nu \in \mathcal{T}. \tag{13}$$

Let $\nabla V_{\nu,k}$ ($k = 1,2,\dots,n$) denote the k-th component of the vector ∇V_ν for every $\varGamma_\nu \in \mathcal{T}$.

Remark 7. *Observe that a function $g \in \mathrm{CPA}(\mathcal{T})$ is uniquely determined by its values at vertices of simplices of $\mathcal{T}$ as follows: let $\varGamma_\nu = \mathrm{co}\{x_0,\dots,x_n\} \in \mathcal{T}$. Each $x \in \varGamma$ can be written as convex combination of its vertices, i.e., $x = \sum_{i=0}^{n} \lambda_i x_i$, $1 \geq \lambda_i \geq 0$ and $\sum_{i=0}^{n} \lambda_i = 1$. Then $g(x) = \sum_{i=0}^{n} \lambda_i g(x_i)$. It is clear that g is Lipschitz continuous.*

Similarly, we define $\mathrm{CPA}_u(\mathcal{T}_u)$. We note that (A1) implies that the map $u \mapsto \|u\|_1$ is contained in $\mathrm{CPA}_u(\mathcal{T}_u)$.

3.2. Interpolation errors

In order to compute a true Lyapunov function, we have to incorporate estimates for the interpolation errors on $\mathcal{T}$ — and on $\mathcal{T}_u$ — into the constraints of a linear optimization problem. Therefore, we analyse the error terms needed for this purpose in this section.

Let $x \in \Gamma_\nu = \text{co}\{x_0, x_1, \ldots, x_n\} \in \mathcal{T}$, $x = \sum_{i=0}^n \lambda_i x_i$, $1 \geq \lambda_i \geq 0$, $\sum_{i=0}^n \lambda_i = 1$ and $u \in \Gamma_\kappa^u = \text{co}\{u_0, u_1, \ldots, u_m\} \in \mathcal{T}_u$, $u = \sum_{j=0}^m \mu_j u_j$, $1 \geq \mu_j \geq 0$, $\sum_{j=0}^m \mu_j = 1$.

The basic idea of the algorithm is to impose conditions on $V \in \text{CPA}(\mathcal{T})$ in the vertices x_i of the simplices $\Gamma_\nu \in \mathcal{T}$ which make sure that the function V satisfies the inequalities (4) and (7) with σ calculated by the algorithm on the whole set $\mathcal{O} \setminus B_2(0, \epsilon)$.

From Remark 7, it is known that $V \in \text{CPA}(\mathcal{T})$ is completely determined by its values in the vertices of the simplices in $\mathcal{T}$.

In order to ensure the properness condition (4), we require

$$V(x_i) \geq \|x_i\|_1, \tag{14}$$

for every vertex $x_i \in \Gamma_\nu$, $V(0) = 0$ and $V \in \text{CPA}(\mathcal{T})$. According to (14), for $x \in \Gamma_\nu$ we have

$$V(x) = \sum_{i=0}^n \lambda_i V(x_i) \geq \sum_{i=0}^n \lambda_i \|x_i\|_1 \geq \|x\|_1 \,. \tag{15}$$

In order to make sure that $V(x)$ satisfies (7) for all $x \in \Gamma_\nu \subset \mathcal{O}$, $u \in \Gamma_\kappa^u \subset U_R$ via imposing inequalities in the node values $V(x_i)$, it is necessary to incorporate an estimate of the interpolation error into the inequalities. To this end, we demand

$$V(f(x_i, u_j)) - V(x_i) - r\|u_j\|_1 + \|\nabla V_\nu\|_1 A_{\nu,\kappa} + \sigma\|x_i\|_1^2 + (1-\sigma)\|x_i\|_1 + \Delta_\nu \leq 0 \tag{16}$$

for all $i = 0, 1, 2, \ldots, n$, $j = 0, 1, \ldots, m$. Here $A_{\nu,\kappa} \geq 0$ is a bound for the interpolation error of f which will be derived below, in the points (x, u) with $x \in \Gamma_\nu \subset \mathcal{O}$, $u \in \Gamma_\kappa^u \subset U_R$, $x \neq x_i$, $u \neq u_j$. The value $\Delta_\nu \geq 0$ is a bound for the interpolation error of the function $\sigma\|x\|_1^2 + (1-\sigma)\|x\|_1$ in the points x with $x \in \Gamma_\nu \subset \mathcal{O}$, $x \neq x_i$, which will be calculated below.

The next remark explains the two reasons for excluding a small neighbourhood of the origin in our numerical computation.

Remark 8. *Proposition 4 provides the theoretical feasibility of computing ISS Lyapunov functions satisfying* (4) *and* (7) *with* $1 \geq \sigma \geq 0$*,* $r > 0$ *for* $x \in \mathcal{O} \setminus B_2(0, \epsilon)$*,* $u \in U_R$ *and* $\epsilon > 0$*. However, Proposition 4 may fail to hold if* $\epsilon = 0$*. For instance, if* $\liminf_{s\to 0} \frac{\sigma s^2+(1-\sigma)s}{\alpha(s)} = +\infty$ *or* $\liminf_{s\to 0} \frac{s}{\alpha_1(s)} = +\infty$*, then* C *from Proposition 4 does not exists. Thus, we cannot ensure the existence of* V *with* (4) *and* (7) *from Proposition 4 for* $\epsilon = 0$*. This is the first reason for excluding a small neighbourhood of the origin.*

The second reason is the following: for $u_j = 0$*, if* x_i *is very close to the origin, then the value of* $V(f(x_i, 0)) - V(x_i)$ *in* (16) *is very close to* 0*. The sum of the positive terms* $\|\nabla V_\nu\|_1 A_{\nu,\kappa}$ *and* Δ_ν *in* (16) *may then become larger than* $|V(f(x_i, 0)) - V(x_i)|$*, such that* (16) *cannot hold. Thus, we have to exclude* x_i *near the origin in order to ensure feasibility of* (16)*.*

The inequalities (16) will be incorporated as an inequality constraint in the proposed linear optimization problem, thus it is necessary to derive estimates for $A_{\nu,\kappa}$ and Δ_ν before we formulate the algorithm. To this end, we recall [14, Proposition 3.4]. For a function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ which is twice continuously differentiable with respect to its first argument, the Hessian of $g(x, u)$ with respect to x at z is denoted by

$$H_g(z,u) = \begin{bmatrix} \left.\frac{\partial^2 g(x,u)}{\partial x_1^2}\right|_{x=z} & \cdots & \left.\frac{\partial^2 g(x,u)}{\partial x_1 \partial x_n}\right|_{x=z} \\ & \cdots & \\ \left.\frac{\partial^2 g(x,u)}{\partial x_n \partial x_1}\right|_{x=z} & \cdots & \left.\frac{\partial^2 g(x,u)}{\partial x_n^2}\right|_{x=z} \end{bmatrix}.$$

For the first argument $x \in \mathit{\Gamma}_\nu$, let

$$H_x(u) := \max_{z\in \mathit{\Gamma}_\nu} \|H_g(z, u)\|_2, \tag{17}$$

and let $K_x : U_R \to \mathbb{R}_+$, $\overline{K}_x$, respectively, denote a bounded function and a positive constant such that

$$\max_{\substack{z\in\mathit{\Gamma}_\nu \\ r,s=1,2,\ldots,n}} \left| \frac{\partial^2 g(z,u)}{\partial x_r \partial x_s} \right| \leq K_x(u) \leq \overline{K}_x \quad (u \in U_R). \tag{18}$$

Proposition 9 describes properties of a function $g : \mathcal{G} \times U_R \to \mathbb{R}^p$ with respect to its first argument. Analogous properties also hold with respect to the second argument.

Proposition 9 ([14], Proposition 3.4)**.** *Consider a convex combination* $x = \sum_{i=0}^{n} \lambda_i x_i \in \Gamma_\nu$, $\Gamma_\nu = \mathrm{co}\{x_0, x_1, \ldots, x_n\}$, $\sum_{i=0}^{n} \lambda_i = 1$, $1 \geq \lambda_i \geq 0$, $u \in U_R$ *and a function* $g : \mathcal{G} \times U_R \to \mathbb{R}^p$ *with components* $g(x,u) = (g_1(x,u), g_2(x,u), \ldots, g_p(x,u))$.

(a) If $g(x,u)$ *is Lipschitz continuous in* x *with the bounds* $L_x(u)$, $\overline{L}_x$ *from* (2)*, then*

$$\left\| g\left(\sum_{i=0}^{n} \lambda_i x_i, u\right) - \sum_{i=0}^{n} \lambda_i g(x_i, u) \right\|_\infty \leq L_x(u) h_{x,\nu} \leq \overline{L}_x h_{x,\nu} \tag{19}$$

holds for all $x \in \mathcal{G}$, $u \in U_R$.

(b) If $g_j(x,u)$ *is twice continuously differentiable with respect to* x *with the bound* $H_x(u)$ *from* (17) *on its second derivative for some* $j = 1, \ldots, p$, *then*

$$\left| g_j\left(\sum_{i=0}^{n} \lambda_i x_i, u\right) - \sum_{i=0}^{n} \lambda_i g_j(x_i, u) \right| \leq \tag{20}$$
$$\frac{1}{2} \sum_{i=0}^{n} \lambda_i H_x(u) \|x_i - x_0\|_2 \left(\max_{z \in \Gamma_\nu} \|z - x_0\|_2 + \|x_i - x_0\|_2 \right) \leq H_x(u) h_{x,\nu}^2.$$

Under the same differentiability assumption for all $j = 1, \ldots, p$, *the estimate*

$$\left\| g\left(\sum_{i=0}^{n} \lambda_i x_i, u\right) - \sum_{i=0}^{n} \lambda_i g(x_i, u) \right\|_\infty \leq n K_x(u) h_{x,\nu}^2 \leq n \overline{K}_x h_{x,\nu}^2 \tag{21}$$

holds for all $u \in U_R$ *by assuming the bounds from* (18).

Remark 10. *Using Proposition 9, we have*

$$A_{\nu,\kappa} \leq \overline{L}_x h_{x,\nu} + \overline{L}_u h_{u,\kappa} \qquad \textit{if } f \textit{ satisfies } (H1) \tag{22}$$
$$A_{\nu,\kappa} \leq n\overline{K}_x h_{x,\nu}^2 + m\overline{K}_u h_{u,\kappa}^2 \qquad \textit{if } f \textit{ satisfies } (H2) \tag{23}$$
$$\Delta_\nu \leq 2n\sigma h_{x,\nu}^2. \tag{24}$$

Now we explain why we utilize $\sigma\|x\|_1^2 + (1-\sigma)\|x\|_1$ in (7) and (16).

Remark 11. *In* (16)*, for* $\|x_i\| < 1$ *the inequality* $\|x_i\|_1^2 < \|x_i\|_1$ *holds while for* $\|x_i\|_1 > 1$ *the opposite inequality holds. As our goal is to compute small ISS gains* $r > 0$*, it is beneficial to make the terms* $\sigma\|x_i\|_1^2 + (1-\sigma)\|x_i\|_1$ *in* (16) *as small as possible. However, unless* $\mathcal{O} \subset B_1(0,1)$*, i.e.,* $\|x_i\|_1 < 1$ *for all vertices* x_i*, it is not clear which weight* $0 \le \sigma \le 1$ *will yield the smallest* r*. Hence, we use the weighted sum with* $0 \le \sigma \le 1$ *as optimization variable to determine decay rate which allows for the smallest possible ISS gain* r*.*

In case $\mathcal{O} \subset B_1(0,1)$*, the optimal* σ *will always be* $\sigma = 1$*, as this value minimizes* $\sigma\|x_i\|_1^2 + (1-\sigma)\|x_i\|_1$ *for each* x_i*, resulting in* $\sigma\|x_i\|_1^2 + (1-\sigma)\|x_i\|_1 = \|x_i\|_1^2$*. In this case, it may be beneficial to replace* $\|x_i\|_1^2$ *by higher powers* $\|x_i\|_1^q$*,* $q \ge 3$*, see Remark 20.*

3.3. The Algorithm

We are ready to formulate the linear programming algorithm for computing an ISS Lyapunov function V for system (1). In this algorithm, the values $V(x_i)$, σ are introduced as optimization variables. Since we want to reduce the influence of perturbation as much as possible, the objective of the linear optimization problem is to minimize the gain $\beta(s) = rs$ in (7), i.e., the number r.

As explained in Remark 8, we only consider x satisfying $x \in \mathcal{O} \setminus B_2(0,\epsilon)$ for a small $\epsilon > 0$. To this end we define the subsets

$$\mathcal{T}^\epsilon := \{\varGamma_\nu \mid \varGamma_\nu \cap B_2(0,\epsilon) = \emptyset\} \subset \mathcal{T} \quad \text{and} \quad \mathcal{O}^\epsilon := \bigcup_{\varGamma_\nu \in \mathcal{T}^\epsilon} (\varGamma_\nu \cap \mathcal{O}). \tag{25}$$

In the following algorithm, we will only impose the conditions (14) in the vertices $x_i \in \mathcal{G}$ and (16) in vertices $x_i \in \varGamma \in \mathcal{O}^\epsilon$. Furthermore, we utilize the estimates of the interpolation errors $A_{\nu,\kappa}$, Δ_ν obtained from Remark 10.

3.3.1. Algorithm

We solve the following linear optimization problem.

$$\text{Inputs:}\quad \begin{cases} \epsilon, \\ x_i \text{ for all vertices } x_i \text{ of each simplex } \mathit{\Gamma}_\nu \subset \mathcal{T}, \\ u_j \text{ for all vertices } u_j \text{ of each simplex } \mathit{\Gamma}^u_\kappa \in \mathcal{T}_u, \\ h_{x,\nu} \text{ of each simplex } \mathit{\Gamma}_\nu \subset \mathcal{T}, \\ h_{u,\kappa} \text{ of each simplex } \mathit{\Gamma}^u_\kappa \in \mathcal{T}_u, \\ \text{and one pair of the following two pairs of constants} \\ (1)\ \overline{L}_x, \overline{L}_u \text{ from (2) if } f \text{ satisfies (H1)}, \\ (2)\ \overline{K}_x, \overline{K}_u \text{ from (21) with respect to } x, u, \text{ respectively,} \\ \text{for } g(x,u) = f(x,u) \text{ from (1) if } f \text{ satisfies (H2).} \end{cases} \tag{26}$$

$$\text{Optimization variables:}\quad \begin{cases} V_{x_i} = V(x_i) \text{ for all vertices } x_i \text{ of} \\ \qquad \text{each simplex } \mathit{\Gamma}_\nu \subset \mathcal{T}, \\ C_{\nu,k} \text{ for } k = 1,2,\dots,n \text{ and every } \mathit{\Gamma}_\nu \subset \mathcal{T}, \\ r,\ \overline{C},\ \sigma \geq 0. \end{cases} \tag{27}$$

Optimization problem: (28)

minimize $\quad r$

subject to

$(C1):\ V_{x_i} \geq \|x_i\|_1$ for all vertices x_i of each simplex $\mathit{\Gamma}_\nu \subset \mathcal{G}$, and $V(0) = 0$,

$(C2):\ |\nabla V_{\nu,k}| \leq C_{\nu,k}$ for each simplex $\mathit{\Gamma}_\nu \subset \mathcal{G}$, $k = 1,2,\dots,n$,

$(C3):\ C_{\nu,k} \leq \overline{C}$ for each simplex $\mathit{\Gamma}_\nu \subset \mathcal{G}$, $k = 1,2,\dots,n$,

$(C4):\ f(x_i, u_j) \in \mathcal{T}$ for all vertices $x_i \in \mathcal{O}^\epsilon$, $u_j \in U_R^{\mathcal{T}}$,

$(C5):\ V_{x_i} < V_{x_j}$, for all vertices $x_i \in \partial(\mathcal{O} \setminus \mathcal{O}^\epsilon)$, $x_j \in \partial\mathcal{O}$,

$(C6):\ \sigma \leq 1$,

For all vertices x_i of each simplex $\mathit{\Gamma}_\nu \subset \mathcal{O}^\epsilon$, all vertices u_j of each simplex $\mathit{\Gamma}^u_\kappa \in \mathcal{T}_u$, one of the conditions $(C7)$, $(C8)$ is required:

$(C7):\ V(f(x_i,u_j)) - V(x_i) - r\|u_j\|_1 + n\overline{C}(\overline{L}_x h_{x,\nu} + \overline{L}_u h_{u,\kappa}) + \sigma(\|x_i\|_1^2 + 2nh_{x,\nu}^2) + (1-\sigma)\|x_i\|_1 \leq 0, \quad$ if f satisfies (H1),

$(C8):\ V(f(x_i,u_j)) - V(x_i) - r\|u_j\|_1 + n\overline{C}(n\overline{K}_x h_{x,\nu}^2 + m\overline{K}_u h_{u,\kappa}^2) + \sigma(\|x_i\|_1^2 + 2nh_{x,\nu}^2) + (1-\sigma)\|x_i\|_1 \leq 0, \quad$ if f satisfies (H2).

Remark 12. *The constraints here are stricter than of the linear optimization problem (30) for the continuous-time case in [14], since the solution of* (1) *is a sequence of points rather than an absolutely continuous function and may thus jump out of the domain* $\mathcal{G}$ *on which* V *is computed. The constraint* $(C6)$ *ensures the term* $(1-\sigma)\|x_i\|_1$ *is nonnegative. In the conditions* $(C7)$ *and* $(C8)$*, the terms* $\sigma(\|x_i\|_1^2+2nh_{x,\nu}^2)+(1-\sigma)\|x_i\|_1$ *instead of the linear term* $\|x_i\|_2$ *in [14] are used, reflecting the use of the decay rate* $\sigma\|x_i\|_1^2+(1-\sigma)\|x_i\|_1$ *in this paper.*

The following Remark 13- 15 are similar to Remark 3.4-3.6 of [14] respectively.

Remark 13. *(i) Using* (15) *we obtain that* $V(x) \geq \|x\|_1$ *for all* $x \in \mathcal{G}$*, and* $V(0) = 0$*.*

(ii) The condition $(C2)$ *defines linear constraints on the optimization variables* V_{x_i}*,* $C_{\nu,k}$*.*

(iii) Constraint $(C3)$ *is necessary since* $f(x,u)$ *and* x *may not be in the same simplex. The constant* $\overline{C}$ *plays an important role in the proof of Theorem 16.*

(iv) The condition $(C5)$ *ensures that the set* $B_2(0,\epsilon)$ *is a subset of the level set* $\{x \in \mathcal{O} | V(x) \leq \max_{x_i \in \partial(\mathcal{O} \setminus \mathcal{O}^\epsilon)} V(x_i)\}$*. If system* (1) *is locally ISS, the condition* $(C5)$ *is not necessary.*

Remark 14. *If the linear optimization problem* (28) *has a feasible solution, then the values* V_{x_i} *from this feasible solution at all vertices* x_i *of all simplices* $\Gamma_\nu \in \mathcal{T}$ *and the condition* $V \in \mathrm{CPA}(\mathcal{T})$ *uniquely define a continuous and piecewise affine function*

$$V : \mathcal{T} \to \mathbb{R}. \tag{29}$$

Remark 15. *According to Proposition 9, we may replace the term* $n\overline{K}_x h_{x,\nu}^2$ $+m\overline{K}_u h_{u,\kappa}^2$ *in (C8) with the sharper but more complicated estimate*

$$\begin{aligned}&\frac{n\overline{K}_x}{2}\left(\|x_i-x_0\|_2\left(\max_{k=1,2,\dots,n}\|x_k-x_0\|_2+\|x_i-x_0\|_2\right)\right)\\&+\frac{mK_u(x_i)}{2}\left(\|u_j-u_0\|_2\left(\max_{k=1,2,\dots,m}\|u_k-u_0\|_2+\|u_i-u_0\|_2\right)\right)\end{aligned}$$

with $K_u(x_i)$ *satisfying* (18) *with respect to* u*. The latter was used in our numerical experiments.*

4. Main results

In this section, we fist prove that any feasible solution of the proposed linear optimization problem defines an ISS Lyapunov function on $\mathcal{O}^\epsilon$. Furthermore, we discuss conditions under which the linear optimization problem has such a feasible solution.

Theorem 16. *If assumption (H1) or (H2) holds, $\mathcal{O}$ satisfies Property A and the linear optimization problem* (28) *has a feasible solution, then the function V from* (29) *is an ISS Lyapunov function for system* (1) *on $\mathcal{O}^\epsilon$, i.e., it satisfies* (4) *and* (7) *with $1 \geq \sigma \geq 0$ delivered by the algorithm for all $x \in \mathcal{O}^\epsilon$ and all $u \in U_R^{\mathcal{T}}$.*

Proof. Let $\Gamma_\nu = \mathrm{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T}$, $\Gamma_\nu \subset \mathcal{O}^\epsilon$, and $\Gamma_\kappa^u = \mathrm{co}\{u_0, u_1, \dots, u_m\} \in \mathcal{T}_u$. Consider the convex combinations $x = \sum_{i=0}^n \lambda_i x_i \in \Gamma_\nu$, $\sum_{i=0}^n \lambda_i = 1$, $1 \geq \lambda_i \geq 0$, and $u = \sum_{j=0}^m \mu_j u_j \in \Gamma_\kappa^u$, $\sum_{j=0}^m \mu_j = 1$, $1 \geq \mu_j \geq 0$.

Based on (15) we have $V(x) \geq \|x\|_1$ for all $x \in \mathcal{G}$. Thus in (4) we may choose α_1 to be the identity and the existence of α_2 follows by Lipschitz continuity.

In the following we prove that inequality (7) holds with fixed $1 \geq \sigma \geq 0$. We calculate

$$\begin{aligned}
V(f(x,u)) - V(x) &= V(f(x,u)) - \sum_{i=0}^n \lambda_i V(f(x_i,u)) + \sum_{i=0}^n \lambda_i V(f(x_i,u)) \\
&\quad - \sum_{i=0}^n \lambda_i V(x_i) \\
&\leq \sum_{i=0}^n \lambda_i n\overline{C} \|f(x,u) - f(x_i,u)\|_\infty \\
&\quad - \sum_{i=0}^n \lambda_i \sum_{j=0}^m \mu_j V(f(x_i,u_j)) + \sum_{i=0}^n \lambda_i V(f(x_i,u)) \\
&\quad + \sum_{i=0}^n \lambda_i \sum_{j=0}^m \mu_j V(f(x_i,u_j)) - \sum_{i=0}^n \lambda_i V(x_i) \\
&\leq \sum_{i=0}^n \lambda_i n\overline{C} \|f(x,u) - f(x_i,u)\|_\infty
\end{aligned}$$

$$+\sum_{i=0}^{n}\lambda_i\sum_{j=0}^{m}\mu_j n\overline{C}\|f(x_i,u)-f(x_i,u_j)\|_\infty$$
$$+\sum_{i=0}^{n}\lambda_i\sum_{j=0}^{m}\mu_j V(f(x_i,u_j))-\sum_{i=0}^{n}\lambda_i V(x_i).$$

According to Proposition 9, the constraints from $(C7)$ or $(C8)$ make sure that V satisfies

$$\begin{aligned}V(f(x,u))-V(x)\leq&-\sum_{i=0}^{n}\lambda_i[\sigma(\|x_i\|_1^2+2nh_{x,\nu}^2)+(1-\sigma)\|x_i\|_1]\\&+r\sum_{i=0}^{m}\mu_j\|u_j\|_1\\ \leq&-\sigma\|x\|_1^2-(1-\sigma)\|x\|_1+r\|u\|_1\,.\end{aligned}$$

In the last step we utilized the inequality $\|x_i\|_1^2+2nh_{x,\nu}^2\geq\|x\|_1^2$, and the equality $\sum_{i=0}^{m}\mu_j\|u_j\|_1=\|u\|_1$, which follows from (A1) and because 1-norm for u is used. Thus we have demonstrated (7) with $1\geq\sigma\geq 0$ holds for all $x\in\mathcal{O}^\epsilon$ and all $u\in U_R$. □

The next objective is to derive conditions under which the linear programming problem has a feasible solution. To this end, we require that the simplices in our grid satisfies a certain regularity property which is discussed in [13, 14]. In order to formalize these, we need the following notations.

For each $\Gamma_\nu=co\{x_0,x_1,\ldots,x_n\}\in\mathcal{T}$, let $y=x_0$, and define the $n\times n$ matrix $X_{\nu,y}$ by writing the components of the vectors x_1-y, x_2-y, …, x_n-y as row vectors consecutively, i.e.,

$$X_{\nu,y}=(x_1-y,x_2-y,\ldots,x_n-y)^\top. \tag{30}$$

Let $X_{\nu,y}^*:=\|X_{\nu,y}^{-1}\|_2$. It is proved that $X_{\nu,y}^*$ is independent of the order of $x_1,\ldots,x_n$ and $X_{\nu,y}^*=\lambda_{\min}^{-1}$ holds, where $\lambda_{\min}$ is the smallest singular value of $X_{\nu,y}$, see [1, Theorem 4.6]. Let

$$X_\nu^*:=\max_{y\text{ vertex of }\Gamma_\nu}\|X_{\nu,y}^{-1}\|_2,\qquad\text{and}\qquad\lambda^*:=\max_{\nu=1,2,\ldots,N}X_\nu^*. \tag{31}$$

The regularity property now demands that grids with arbitrarily flat simplices should be avoided, i.e., there exists a positive constant $R_1 > 0$ such that all simplices $\Gamma_\nu \in \mathcal{T}$ in the considered grids satisfy the inequality

$$X_\nu^* \cdot \operatorname{diam}(\Gamma_\nu) \leq \lambda^* h_x \leq R_1, \tag{32}$$

for X_ν^* and λ^* from (31), cf. [1, Remark 4.7], and h_x from Section 3.1.

Theorem 17. *Let S, $\mathcal{D}$, $\mathcal{G}$ be simply connected compact neighbourhoods of the origin such that $\overline{\operatorname{int} S} = S$, $\overline{\operatorname{int} \mathcal{D}} = \mathcal{D}$, $\overline{\operatorname{int} \mathcal{G}} = \mathcal{G}$, $\overline{\operatorname{int} S} \subset \operatorname{int} \mathcal{D}$, $\mathcal{D} \subset \operatorname{int} \mathcal{G}$ and $f(x,u) \in \mathcal{G}$ for $x \in \mathcal{D}$, $u \in U_R$. Consider system* (1) *which satisfies (H1) or (H2) and is ISS on $\mathcal{G}$. Let $\epsilon > 0$ and $R_1 > 0$. Then for every $R_1 > 0$ there exist $\delta_{R_1} > 0$, $\delta_u > 0$ and a compact set $\mathcal{O} \supset \operatorname{int} S$ satisfying Property A such that, for any suitable triangulations $\mathcal{T}$, $\mathcal{T}_u$ satisfying*

$$\max_{\Gamma_\nu \in \mathcal{T}} \operatorname{diam}(\Gamma_\nu) \leq \delta_{R_1}, \tag{33}$$

$$\max_{\Gamma_\kappa^u \in \mathcal{T}_u} \operatorname{diam}(\Gamma_\kappa^u) \leq \delta_u, \tag{34}$$

$$\lambda^* h_x \leq R_1, \quad \text{with } \lambda^* \text{ defined in } (31) \tag{35}$$

the linear optimization problem from our algorithm has a feasible solution and delivers an ISS Lyapunov function $V \in \operatorname{CPA}(\mathcal{T})$ on $\mathcal{O}^\epsilon$.

Proof. Since system (1) is ISS, there exists a C^1 ISS Lyapunov function $W : \mathbb{R}^n \to \mathbb{R}$ on $\mathcal{G}$ for which there exists a constant $C \geq 0$ such that $\|\nabla W(x)\|_2 \leq C$. According to Proposition 4 we may without loss of generality assume that W satisfies (6) and

$$W(f(x,u)) - W(x) \leq -2(\sigma\|x\|_1^2 + (1-\sigma)\|x\|_1) + r\|u\|_1 \tag{36}$$

for $x \in \mathcal{G} \setminus B_2(0,\epsilon)$, $u \in U_R$ and some $1 \geq \sigma \geq 0$, $r > 0$ (note that the factor 2 in (36) is easily obtained using $2C$ instead of C in Proposition 4).

Let $\epsilon_1 = \inf\{\|x-y\|_2 : x \in \operatorname{int} S, y \in \mathcal{D}^C\}$. Fix $R_1 > 0$ and $\epsilon_1 > \epsilon > 0$, and choose δ_u, δ_{R_1} so small that

$$\delta_u, \delta_{R_1} < \begin{cases} \min\{\frac{-D+\sqrt{D^2+8n\sigma Y}}{4n\sigma}, \epsilon_1\}, & \text{if } f \text{ satisfies (H1) and } \sigma \neq 0, \\ \min\{\frac{\epsilon}{D}, \epsilon_1\}, & \text{if } f \text{ satisfies (H1) and } \sigma = 0, \\ \min\{\frac{-C+\sqrt{C^2+4MY}}{2M}, \epsilon_1\}, & \text{if } f \text{ satisfies (H2)}, \end{cases} \tag{37}$$

where $\overline{C} = R_1 C$, $D = n\overline{C}(\overline{L}_x + \overline{L}_u) + C$, $Y = \sigma\epsilon^2 + (1-\sigma)\epsilon$, $M = 2n\sigma + n\overline{C}(n\overline{K}_x + m\overline{K}_u)$.

Let $\mathcal{T}$ be any suitable triangulation such that $\mathcal{G}_I = \mathrm{co}\{\mathcal{D} \cup f(\mathcal{D}, U_R)\} \subset \mathcal{T}$,

$$\max_{\mathit{\Gamma}_\nu \in \mathcal{T}} \mathrm{diam}(\mathit{\Gamma}_\nu) \leq \delta_{R_1}, \ \max_{\mathit{\Gamma}_\nu \in \mathcal{T}} \mathrm{diam}(\mathit{\Gamma}_\nu)\lambda^* \leq R_1.$$

Such a suitable triangulation exists and it can be constructed as $\mathcal{T}_{K,b}^{\mathcal{G}_I}$ from [5, Definition 13] with $K = 0$, $b = \delta_{R_1}/\sqrt{n}$. Similarly, let $\mathcal{T}_u$ be any simplicial complex such that $\mathcal{T}_u = U_R$,

$$\max_{\mathit{\Gamma}_\kappa^u \in \mathcal{T}_u} \mathrm{diam}(\mathit{\Gamma}_\nu) \leq \delta_u.$$

Let $\mathcal{O}$ be the union of the simplices in $\mathcal{T}$ that have a nonempty intersection with the interior of S , i.e.

$$\mathcal{O} := \bigcup_{\mathit{\Gamma}_\nu \cap \mathrm{int}\ S \neq \emptyset} \mathit{\Gamma}_\nu.$$

Based on (37), we have that $\mathcal{O} \subset \mathcal{D}$.

As proved in [5, Lemma 2], $\mathcal{O}$ is connected and $\overline{\mathrm{int}\ \mathcal{O}} = \mathcal{O}$. Therefore $\mathcal{O}$ satisfies Property A.

Consider an arbitrary but fixed $\mathit{\Gamma}_\nu = co\{x_0, x_1, \ldots, x_n\} \in \mathcal{T}^\epsilon$. Let $y = x_0$, and define

$$W_{\nu,y} := \begin{pmatrix} W(x_1) & - & W(y) \\ W(x_2) & - & W(y) \\ & \vdots & \\ W(x_n) & - & W(y) \end{pmatrix}.$$

Having these preliminary results at hand, we now assign values to the variables V_{x_i} and $C_{\nu,k}$ of the linear optimization problem from the algorithm and demonstrate that they fulfill the constraints.

For each vertex $x_i \in \mathit{\Gamma}_\nu \in \mathcal{T}$, let $V(x_i) = V_{x_i} := W(x_i)$. It is obvious that $(C5)$ is satisfied. Since W satisfies (6), we have $V(x_i) = V_{x_i} \geq \|x_i\|_1$ for $x \in \mathcal{T}$. In the following, we prove $(C2)$ and $(C3)$ hold.

To this end, considering one simplex $\mathit{\Gamma}_\nu = co\{x_0, x_1, \ldots, x_n\} \in \mathcal{T}$ and letting $y = x_0$, we have

$$\nabla V_\nu = X_{\nu,y}^{-1} W_{\nu,y}, \tag{38}$$

since V is linearly affine on the simplex $\mathit{\Gamma}_\nu$ and

$$V(x) = V(y) + \left\langle X_{\nu,y}^{-1} W_{\nu,y}, (x-y) \right\rangle = V(y) + W_{\nu,y}^\top (X_{\nu,y}^\top)^{-1}(x-y). \tag{39}$$

For the variables $C_{\nu,k}$, let

$$C_{\nu,k} := \|\nabla V_\nu\|_2 = \|X_{\nu,y}^{-1} W_{\nu,y}\|_2, \quad k = 1, \dots, n. \tag{40}$$

Therefore, $C_{\nu,k} \geq |\nabla V_{\nu,k}|$ for each $\Gamma_\nu \in \mathcal{T}$. Because of the boundedness of $\nabla W(x)$ on $\mathcal{G}$ and (32), there exists a positive constant $\overline{C}$ such that

$$\begin{aligned} C_{\nu,k} = \|X_{\nu,y}^{-1} W_{\nu,y}\|_2 &\leq \|X_{\nu,y}^{-1}\|_2 \max_{z \in \mathcal{G}_\epsilon} \|\nabla W(z)\|_2 h_x \\ &\leq R_1 \max_{z \in \mathcal{T}^\epsilon} \|\nabla W(z)\|_2 = \overline{C} \end{aligned} \tag{41}$$

holds for all ν and k.

In the following, we prove that $(C7)$ and $(C8)$ are satisfied for some r, σ.

Let x_i be an arbitrary vertex of an arbitrary simplex $\Gamma_\nu \subset \mathcal{O}^\epsilon$ and u_j of $\Gamma_\kappa^u \subset U_R^{\mathcal{T}}$. Since $f(x,u) \in \mathcal{T}$ for all $x \in \mathcal{O}^\epsilon$ and all $u \in U_R^{\mathcal{T}}$, there exists a simplex $\Gamma_\iota = \text{co}\{y_0, y_1, y_2, \dots, y_n\} \in \mathcal{T}$ such that $f(x_i, u_j) = \sum_{k=0}^{n} \lambda_k y_k \in \Gamma_\iota$ with $\sum_{k=0}^{n} \lambda_k = 1$. We have assigned $V(x) = W(x)$ for all vertices x of all simplices Γ_ν. Hence, for $x_i \in \mathcal{O}^\epsilon$ we have

$$\begin{aligned} V(f(x_i,u_j)) - V(x_i) &= \sum_{k=0}^{n} \lambda_k W(y_k) - W(x_i) \\ &= \sum_{k=0}^{n} \lambda_k W(y_k) - W(\sum_{k=0}^{n} \lambda_k y_k) + W(\sum_{k=0}^{n} \lambda_k y_k) - W(x_i) \\ &\leq C\delta_{R_1} - 2(\sigma\|x_i\|_1^2 + (1-\sigma)\|x_i\|_1) + r\|u_j\|_1. \end{aligned}$$

It is obvious that the chosen δ_{R_1}, δ_u ensure the following results

1. if f satisfies (H1), then

$$C\delta_{R_1} + n\overline{C}(\overline{L}_x\delta_{R_1} + \overline{L}_u\delta_u) + 2n\sigma\delta_{R_1}^2 \leq \sigma\|x_i\|_1^2 + (1-\sigma)\|x_i\|_1,$$

2. if f satisfies (H2), then

$$C\delta_{R_1} + n\overline{C}(n\overline{K}_x\delta_{R_1}^2 + m\overline{K}_u\delta_u^2) + 2n\sigma\delta_{R_1}^2 \leq \sigma\|x_i\|_1^2 + (1-\sigma)\|x_i\|_1.$$

hold for all $x_i \in \mathcal{O}^\epsilon$ and all $u_j \in U_R^{\mathcal{T}}$. Thus the theorem is proved. □

Remark 18. *In contrast to the proof of Theorem 4.2 in [11], here we only require that $W(x)$ is a C^1 function with bounded gradient instead of being a C^2 function, since here the inequality about the Hessian of $W(x)$ at x is not needed in the proof.*

5. Examples

In this section, we present four numerical examples to show how our proposed algorithm works. Our first example describes a two dimensional nonlinear dynamic system with one perturbation. Our second example illustrates that we can deal with the case of more than one perturbation. The third example describes a one dimensional system with one perturbation, in order to illustrate computation times for different dimensions of x and u. The fourth example demonstrates the influence of the value of σ in minimizing the gain parameter r.

The suitable triangulation can be obtained as described in [14]. Here we briefly explain how to get the suitable triangulation in two dimensions. We define an initial suitable triangulation with vertices at all integer coordinates in a rectangular region of $\mathbb{R}^2$ excluding a smaller rectangular region of $\mathbb{R}^2$, both with the origin in their interior. For instance, let N_i, M_i $(i = 1, 2, 3, 4)$ be positive integers and $M_i < N_i$. Then the vertices of the initial suitable triangulation are defined by all pairs of integers $(i, j) \in [-N_1, N_2] \times [-N_3, N_4] \setminus [-M_1, M_2] \times [-M_3, M_4]$. In order to obtain the suitable triangulation used in the following examples, the mapping

$$x \mapsto \omega x, \ \omega > 0, \tag{42}$$

is applied to all vertices of the initial suitable triangulation. Note that in [14], a nonlinear mapping instead of (42) is used. In this setting, $[-\omega M_1, \omega M_2] \times [-\omega M_3, \omega M_4]$ is the excluded small neighbourhood of the origin.

Now we describe the procedure for computation of ISS Lyapunov functions for system (1) by our proposed algorithm.

Computational Procedure:

1. Construct the suitable triangulations $\mathcal{T}$, $\mathcal{T}_u$ of $\mathcal{G}$ and U_R respectively.
2. Choose $\mathcal{O} \subset \mathcal{G}$ based on the considered domains of x and u and the function $f(x, u)$ such that $\mathcal{O}$ satisfies Property A. Introduce the variables and compile and store the constraints (C1)–(C6), (C7) or (C8).
3. Load the information about the constraints and variables into a matrix which can be read by the GNU Linear Programming Kit (GLPK)[1]. Set the objective of the linear optimization problem: $\min r$.

[1] `http://www.gnu.org/software/glpk/`

4. Solve the linear optimization problem by GLPK.
5. If the linear optimization problem has a feasible solution from Step 4, plot the figure of the Lyapunov function. Otherwise, adjust ω or M_i ($i = 1, \ldots, 4$), then redo Steps 1–4. Letting ω be smaller or $[-\omega M_1, \omega M_2] \times [-\omega M_3, \omega M_4]$ be larger increases the possibility that the algorithm terminates successfully.

5.1. Example 1

The system is described by the following difference equations

$$\begin{cases} x_1^+ = x_2, \\ x_2^+ = -0.2x_2 + 0.1\sin(x_1 + u) + sin(u), \end{cases} \tag{43}$$

where $x \in \mathcal{G} = [-0.225, 0.225]^2 \subset \mathbb{R}^2$, $U_R = [-0.12, 0.12] \subset \mathbb{R}$. We let $\mathcal{O} = [-0.195, 0.195]^2$.

A suitable triangulation of $\mathcal{G} = [-0.225, 0.225]^2$ is obtained as described above with $N_i = 15$, $M_i = 1$ ($i = 1, 2, 3, 4$) and the map (42) with $\omega = 0.015$. A suitable triangulation of $U_R = [-0.12, 0.12]$ is obtained with $N_i = 8$, $M_i = 1$ ($i = 1, 2$) and the map (42) with $\omega = 0.015$.

The algorithm for system (43) on $\mathcal{O} \setminus (-0.015, 0.015)^2$ yields the ISS Lyapunov function V_1 shown in Figure 1. As expected according to Remark 11, since $\mathcal{O} \subset B_1(0, 1)$ the algorithm delivers $\sigma = 1$. The gain parameter is $r = 0.55843$.

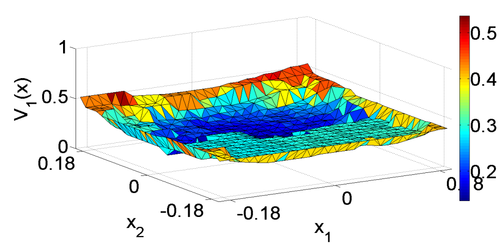

Figure 1: ISS Lyapunov function V_1 delivered by the algorithm for system (43) with $\sigma = 1$ and the gain parameter $r = 0.55843$.

5.2. Example 2

We consider the following system with two one dimensional perturbations

$$x^+ = 0.5x^2 + 0.1\sin u + 0.2w, \tag{44}$$

where $x \in \mathcal{G} = [-0.45, 0.45] \subset \mathbb{R}$, $u, w \in U_R = [-0.225, 0.225] \subset \mathbb{R}$. Let $\mathcal{O} = [-0.435, 0.435]$.

We partition the compact set $\mathcal{G}$ into a suitable triangulation as described above with $N_i = 30$, $M_i = 1$ $(i = 1, 2)$ and the map (42) with $\omega = 0.015$. Similarly, the compact sets U_R of perturbations sets are partitioned into suitable triangulations with $N_i = 15$, $M_i = 0$ $(i = 1, 2)$ and the map (42) with $\omega = 0.015$.

Here the objective of the linear optimization problem (28) is to minimize $r_1 + r_2$, where r_j is the parameter of the gain function $r_j|u_j|$ $(j = 1, 2)$. An ISS Lyapunov function V_2 is obtained by solving the linear optimization problem (28) for system (44) on $\mathcal{O} \setminus (-0.015, 0.015)$; it is shown in Figure 2. We obtain $\sigma = 1$ and the gain parameters $r_1 = 0.0155404$, $r_2 = 0.0311115$. Figure 2 indicates that the ISS Lyapunov function V_2 is not smooth in $\mathcal{O} \setminus (-0.015, 0.015)$.

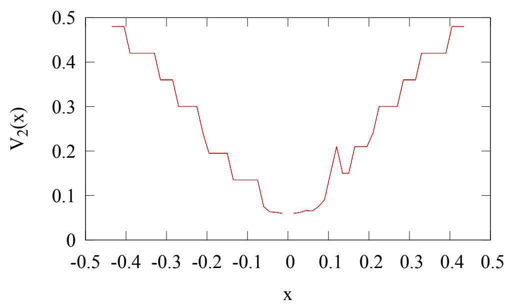

Figure 2: ISS Lyapunov function V_2 delivered by the algorithm for system (44) with $\sigma = 1$ and gain parameters $r_1 = 0.0155404$, $r_2 = 0.0311115$.

5.3. Example 3

We consider the following model adapted from [11] described by

$$x^+ = x^3 + u, \tag{45}$$

where $x \in \mathcal{G} = [-0.75, 0.75] \subset \mathbb{R}$, $u \in U_R = [-0.225, 0.225] \subset \mathbb{R}$. Let $\mathcal{O} = [-0.72, 0.72]$.

The set $\mathcal{G}$ is partitioned into a suitable triangulation by the similar way as above with $N_1 = 50$, $M_1 = 1$ and the map (42) with $\omega = 0.015$. We partition the set U_R into a suitable triangulation described as above with $N_1 = 15$, $M_1 = 1$ and the map (42) with $\omega = 0.015$.

The algorithm delivers an ISS Lyapunov function V_3 for system (45) on $\mathcal{O}\backslash[-0.015, 0.015]$ shown by Figure 3. We have $\sigma = 1$ and the gain parameter $r = 0.599036$. It can be seen from Figure 3 that V_3 is not smooth.

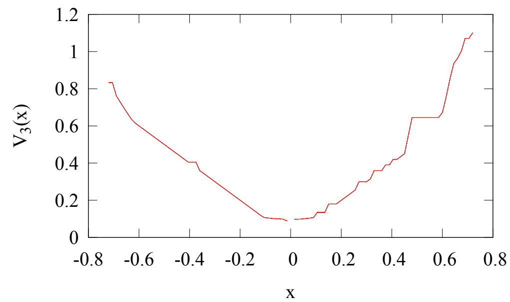

Figure 3: ISS Lyapunov function V_3 delivered by the algorithm for system (45) with $\sigma = 1$ and the gain parameter $r = 0.599036$.

5.4. Example 4

Consider the following system described by

$$x^+ = 0.1x + u \tag{46}$$

where $x \in \mathcal{G} = [-7.5, 7.5] \subset \mathbb{R}$, $u \in U_R = [-6, 6] \subset \mathbb{R}$. Let $\mathcal{O} = [-7.2, 7.2]$.

The suitable triangulation of $\mathcal{G}$ is obtained by the way described above with $N_1 = 50$, $M_1 = 1$ and the map (42) with $\omega = 0.15$. Similarly, we construct a suitable triangulation of U_R by letting $N_1 = 40$, $M_1 = 1$ and the map (42) with $\omega = 0.15$.

An ISS Lyapunov function V_4 is computed by solving the linear optimization problem (28) for system (46) on $\mathcal{O} \setminus [-0.15, 0.15]$. The ISS Lyapunov function V_4 is shown by Figure 4. $\sigma = 0.166964$ and the gain parameter $r = 1.80517$ are also delivered by the algorithm. Figure 4 clearly shows that the ISS Lyapunov function V_4 is not smooth.

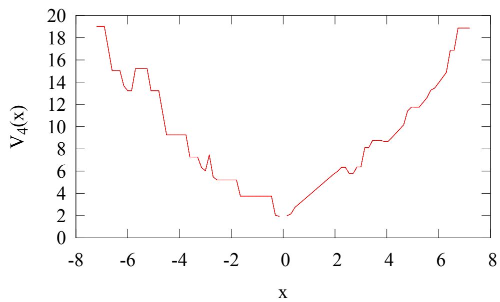

Figure 4: ISS Lyapunov function V_4 delivered by the algorithm for system (46) with $\sigma = 0.166964$ and the gain parameter $r = 1.80517$.

For the numerical computations we used the GNU Linear Programming Kit (GLPK). Table 1 shows the comparison of the computation times (denoted by CT) for the examples under different suitable triangulations.

System	x: N_i, M_i	u: N_i, M_i	$\mathcal{O}$	r	CT
(43)	15, 1	8, 1	$[-0.195, 0.195]^2$	0.55843	22228.8s
	15, 1	4, 1	$[-0.195, 0.195]^2$	0.422538	4482.21s
	8, 1	4, 1	$[-0.09, 0.09]^2$	0.436918	292.171s
(44)	30, 1	15, 0	$[-0.435, 0.435]$	0.0155404 0.0311115	1015.09s
(45)	50, 1	15, 1	$[-0.72, 0.72]$	0.599036	37.318s
	30, 1	15, 1	$[-0.42, 0.42]$	0.534445	17.98s
	25, 1	4, 1	$[-0.345, 0.345]$	0.196174	8.555s
(46)	50, 1	40, 1	$[-7.2, 7.2]$	1.80517	152.495s

Table 1: Computation times for examples under different suitable triangulations (PC: AMD Athlon II P360 Dual-Core 2.30 GHZ Processor with 2GB memory).

Remark 19. *From Table 1, it is obvious that the computation time becomes longer as the considered domains of x, u, and the dimensions increase. The size of the considered domains have influence on the value of r.*

Remark 20. *Table 2 shows that the value of σ plays an important role in minimizing the value of r. As Remark 11 states, if $\|x_i\|_1 < 1$ for all vertices of the triangulation, then $\sigma = 1$, i.e., the quadratic decay rate $\|x\|_1^2$ is the best choice. This is clearly reflected for the Systems* (43)*–*(45)*. For System* (46) *we have $\max_{x\in\mathcal{O}} \|x\|_1 > 1$, hence according to Remark 11 we do not*

System	x: N_i, M_i	u: N_i, M_i	$\mathcal{O}$	σ	r
(43)	15, 1	8, 1	$[-0.195, 0.195]^2$	1	0.55843
	...	...	...	0	3.19947
(44)	30, 1	15, 0	$[-0.435, 0.435]$	1	0.0155404 0.0311115
	...	...	...	0	0.201665 0.403595
(45)	50, 1	15, 1	$[-0.72, 0.72]$	1	0.599036
	...	...	...	0	2.08076
(46)	50, 1	40, 1	$[-7.2, 7.2]$	0.166964	1.80517
	...	...	...	0	1.9877
	...	...	...	1	5.87728

Table 2: For each example, comparison of the value of r under the same suitable triangulation and three different value of σ including the one delivered by the algorithm.

expect $\sigma = 1$ to be optimal. Indeed, the table shows that for this example $\sigma = 0.166964$ is the best choice.

According to Remark 11, if the considered domain satisfies $\mathcal{O} \subset B_1(0,1)$, then we expect that $\|x\|_1^q$ for $q \geq 3$ should yield an even smaller ISS gain r. Calculating the interpolation error similarly to Remark 10 as $\Delta_\nu \leq q(q-1)n \max_{x_i \in \mathcal{T}_\nu} \|x_i\|_i^{q-2} h_{x,\nu}^2$ we have recomputed the Lyapunov functions for Systems (43)–(45). Using the term $\|x_i\|_1^q + q(q-1)n \max_{x_i \in \mathcal{T}_\nu} \|x_i\|_i^{q-2} h_{x,\nu}^2$ instead of $(1-\sigma)\|x_i\|_1 + \sigma\|x_i\|_1^2 + \Delta_\nu$ in (C7) or (C8) and the same triangulations as before, the resulting values of r for selected values of q are shown in Table 3. One observes that r is indeed further reduced, though not significantly as for the different values of σ in Table 2. Unfortunately, we can not directly optimize the power q using our algorithm.

For $B_1(0,1) \subset \mathcal{O}$, one could try to use the more general function $a_1\|x\|_1 + a_2\|x\|_1^2 + \ldots + a_q\|x\|_1^q$ with $(\sum_{i=1}^q a_i = 1, a_i \geq 0, q \geq 3, q \in \mathbb{Z}_+)$ instead of $(1-\sigma)\|x\|_1 + \sigma\|x\|_1^2$. Unfortunately, this makes the linear optimization problem (28) difficult to solve. For instance, for $q = 3$, using the same triangulation as in the computation in Table 2 for (46), we did not get a feasible solution for (28). Trying to bring this more general approach into a solvable form will thus be subject of future research.

Based on the above analysis and results of these examples, we conclude that using $\sigma\|x\|_1^2 + (1-\sigma)\|x\|_1$ in (7) and (16) and thus in (C7) and (C8) is a very reasonable compromise.

System	x: N_i, M_i	u: N_i, M_i	$\mathcal{O}$	q	r
(43)	15, 1	8, 1	$[-0.195, 0.195]^2$	7	0.402855
(44)	30, 1	15, 0	$[-0.435, 0.435]$	8	0.0072805 0.0145773
(45)	50, 1	15, 1	$[-0.72, 0.72]$	6	0.558196

Table 3: Values of ISS gain r for decay rates $\|x\|_1^q$ with higher values of q. Compare with the values for $\sigma = 1$ in Table 2.

6. Conclusions

In this paper, the method for computation ISS Lyapunov functions for continuous-time systems from [14] is successfully extended to discrete-time systems. With suitable triangulations of state space and input value space, the algorithm delivers a true ISS Lyapunov function with a gain function for system on a compact set of state space excluding a small neighbourhood of the origin (Theorem 16). The optimization objective of minimizing the ISS gain parameter is expected to yield near minimal gains on sufficiently fine triangulations. However, the constraints of the linear optimization problem (28) are more restrictive than for the continuous time case since the solution of (1) is a sequence of points which is not absolutely continuous. If system (1) is ISS, then the algorithm will terminate successfully for sufficiently fine grids (Theorem 17). As Remark 18 describes, the conditions of Theorem 17 are a little more relaxed than of Theorem 4.2 in [14]. From the results presented in Section 5.2, we see that the linear programming based algorithm for the computation of an ISS Lyapunov function can be applied to systems with more than one type of input perturbations. The computed ISS Lyapunov functions and the obtained inequality can be directly used to analyse stability of interconnected ISS systems by small gain theorems (Remark 3). Beyond the extension to discrete time, the method from [14] is improved by using the term $\sigma\|x\|_1^2 + (1-\sigma)\|x\|_1$ instead of $\|x\|_1$ as decay rate in (7) and (16) and thus in (C7) and (C8). Herein, the parameter σ is directly optimized by the algorithm. This modification yields a significant improvement of the achievable ISS gain r, which for domains contained in $B_1(0,1)$ can be further improved using decay rates of the form $\|x_i\|_1^q$.

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