Stability Analysis of Unconstrained Receding Horizon Control Schemes

Von der Universität Bayreuth
zur Erlangung des akademischen Grades eines

Doktor der Naturwissenschaften (Dr. rer. nat.)

genehmigte Abhandlung

vorgelegt von

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Tag der Einreichung:	15. Dezember 2011
Tag des Kolloquiums:	27. April 2012

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Deutsche Zusammenfassung

Das Thema dieser Dissertation ist die modellprädiktive Regelung (MPC) — im Englischen auch “receding horizon control” genannt. Typischerweise wird diese Methodik eingesetzt, um ein auf einem unendlichen Zeithorizont gestelltes Optimalsteuerungsproblem approximativ zu lösen, beispielsweise um eine gegebene Regelstrecke an einem Arbeitspunkt zu stabilisieren. Allerdings sind Optimalsteuerungsprobleme mit einem unendlichen Optimierungshorizont im Allgemeinen kaum oder nur mit sehr hohem Rechenaufwand lösbar. Deshalb wird der Zeithorizont abgeschnitten und folglich das Ausgangsproblem durch eines auf einem endlichen Horizont ersetzt. In der modellprädiktiven Regelung werden die folgenden drei Schritte durchgeführt:

- Das Verhalten der Regelstrecke wird, ausgehend von einem Modell und der zuletzt vorgenommenen Messung, prädiziert, um das Optimalsteuerungsproblem zu lösen und damit einhergehend eine Folge von Steuerwerten zu berechnen.
- Das erste Element dieser Folge wird an der Strecke implementiert.
- Der Startzustand des betrachteten Optimalsteuerungsproblems aus dem ersten Schritt wird aktualisiert. Zudem wird der Optimierungshorizont vorwärts in der Zeit verschoben, was den englischen Namen des Verfahrens erklärt.

Dieses Vorgehen wird ad infinitum wiederholt. So wird eine Steuerfolge auf dem unendlichen Zeithorizont erzeugt. Die modellprädiktive Regelung generiert also eine Folge von Optimalsteuerungsproblemen mit endlichen Optimierungshorizont, um die gesuchte Lösung zu approximieren.

Insbesondere die Möglichkeit Steuer- und Zustandsbeschränkungen explizit zu berücksichtigen hat in den letzten Jahrzehnten zu verstärktem Interesse an dieser Methodik geführt. Des Weiteren wächst die Anzahl der Industrieanwendungen stetig, siehe [33,100]. Ein weiterer Vorteil dieser Lösungsstrategie ist die inhärente Robustheit eines geschlossenen Regelkreises — zum Beispiel gegenüber externen Störeinflüssen oder Modellierungsfehlern, siehe [102].

Trotz der weiten Verbreitung modellprädiktiver Regelungsverfahren in der Anwendung ist die zugehörige Stabilitätsanalyse nicht einfach. Die ersten Ansätze basierten auf (künstlichen) Endbedingungen und -kosten, siehe [17, 66]. Diese durch die theoretische Analyse motivierten Problemmodifikationen schaffen zusätzliche Einflussmöglichkeiten, um Stabilitätseigenschaften des geschlossenen Regelkreises zu verbessern. Weil die industrielle Praxis jedoch weitestgehend auf den Einsatz dieser Hilfsmittel verzichtet, beschäftigen wir uns mit der so genannten unrestringierten modellprädiktiven Regelung, die weder Endbedingungen noch Endkosten in die Problemformulierung aufnimmt. Diesbezüglich kann der in [39] vorgestellte Ansatz als unser Ausgangspunkt betrachtet werden. In diesem wird ein Optimierungsproblem konzipiert, um asymptotische Stabilität des bzw. Güteabschätzungen an den mittels modellprädiktiver Regelung geschlossenen

Regelkreis herzuleiten. Positivität des zugehörigen Suboptimalitätsgrades ist eine notwendige und hinreichende Stabilitätsbedingung für die Systemklasse, welche eine vorausgesetzte Kontrollierbarkeitsbedingung erfüllt.

Gliederung und eigener Beitrag

Diese Arbeit ist in fünf Kapitel gegliedert. Die ersten zwei führen in grundlegende Konzepte sowie die Problemstellung ein. Anschließend wird in Abschnitt 3.1 die in [39] entwickelte Methodik kurz zusammengefasst, welche als Ausgangspunkt für das weitere Vorgehen angesehen werden kann. Danach werden eigene Resultate dargestellt.

Diese Gliederung soll sowohl eine Inhaltsübersicht bieten als auch den Beitrag der in dieser Arbeit entwickelten Resultate zu der Analyse unrestringierter modellprädiktiver Regelungsverfahren erläutern.

☞ Im ersten Abschnitt von Kapitel 1 wird das grundlegende Konzept eines Kontrollsystems eingeführt. Dabei wird unter anderem die Zulässigkeit von Kontrollfolgen behandelt. Zusätzlich wird die optimale Wertefunktion definiert. In Abschnitt 1.2 wird die eingeführte Terminologie verwendet, um die wesentlichen Unterschiede eines geschlossenen Regelkreises im Vergleich zur offenen Regelkette herauszuarbeiten. So erlaubt der geschlossene Regelkreis beispielsweise auf äußere Störungen oder Meßfehler zu reagieren. In diesem Zusammenhang wird der Begriff der asymptotischen Stabilität benötigt, um die allgemeine Problemstellung zu definieren. In den letzten beiden Abschnitten von Kapitel 1 beschäftigen wir uns sowohl mit Abtast- als auch mit Netzwerksystemen — zwei wichtige Systemklassen, an denen die Ergebnisse der nächsten Abschnitte demonstriert werden. Dabei wird insbesondere gezeigt, wie von Differentialgleichungen induzierte Systeme als zeitdiskrete Systeme behandelt werden können. Zum Abschluss des Kapitels wird der für diese Arbeit wichtige Begriff der Rückkopplung bzgl. mehrerer Abtastintervalle definiert.

☞ In Kapitel 2 wird die modellprädiktive Regelung — eine Methodik um Optimalsteuerungsprobleme auf unendlichem Zeithorizont approximativ zu lösen — in ihren verschiedenen Facetten betrachtet. Beginnend mit der modellprädiktiven Regelung in ihrer einfachsten Form: unrestringiertes MPC. Anschließend wird die gleiche Kontrollstrategie um zusätzliche Endkosten oder -bedingungen erweitert. Die Berücksichtigung dieser künstlich zu den in jedem Iterationsschritt zu lösenden Optimalsteuerungsproblemen hinzugefügten Komponenten führt zu verbesserten Stabilitätseigenschaften des MPC Algorithmus. Der dafür zu zahlende Preis ist die schwierige Aufgabe, passende Endkosten zu entwerfen. Genau dieser Nachteil ist der Grund dafür, dass in der Industrie hauptsächlich unrestringiertes MPC zum Einsatz kommt. Ein weiterer wichtiger Aspekt ist die Zulässigkeit modellprädiktiver Regelungsverfahren. Dazu werden die wesentlichen Ideen aus [99] in groben Zügen skizziert.

☞ Am Beginn des folgenden dritten Kapitels wird die in [39] entwickelte Methodik kurz vorgestellt. Diese erlaubt es, basierend auf einer Kontrollierbarkeitsannahme, eine relaxierte Lyapunov-Ungleichung sicherzustellen — ein wesentliches Hilfsmittel, um Stabilität des geschlossenen Regelkreises nachzuweisen. Darüber hinaus liefert der umrissene Ansatz einen Suboptimalitätsindex, der angibt, wie gut die mit MPC erzielte Regelgüte im Vergleich zur bestmöglichen ist. Im folgenden Abschnitt 3.2

wird der entsprechende Stabilitätsbeweis auf zeitvariante Kontrollhorizonte verallgemeinert, eine kleine Modifikation, die insbesondere im Netzwerkkontext genutzt werden kann, um nicht vernachlässigbare Verzögerungen sowie Paketausfälle auszugleichen, siehe [47, 48]. Zudem wird sich diese Erweiterung für die Herleitung weiterer Ergebnisse als hilfreich erweisen.

Um die eingeführte Methodik anzuwenden, wird die Lösung eines linearen Programms benötigt, dessen Größe dem Optimierungshorizont in der modellprädiktiven Regelung entspricht. In Abschnitt 3.3 wird eine Lösungsformel für dieses Optimierungsproblem hergeleitet, welche einer der Eckpfeiler für die folgende Stabilitätsanalyse unrestringierter MPC-Schemata ist. Um die wesentlichen Beweisschritte besser darstellen zu können, wurden einige technische Details in einen Hilfsunterabschnitt ausgegliedert. Anschließend wird die bereits erwähnte Lösungsformel genutzt, um zu zeigen, dass MPC das Regelungsproblem auf unendlichem Zeithorizont beliebig gut approximiert — vorausgesetzt der Optimierungshorizont ist hinreichend groß, ein Resultat im Einklang mit [32, 120]. Im folgenden Abschnitt werden die bisherigen Ergebnisse anhand der linearen Wellengleichung veranschaulicht. Insbesondere wird instantane Kontrollierbarkeit rigoros gezeigt. Instantan bedeutet hier, dass der MPC-Algorithmus mit kleinstmöglichem Optimierungshorizont ausgeführt wird. Dieser Abschnitt basiert auf einer Zusammenarbeit mit Nils Altmüller, siehe [4, 5].

Die wichtigsten Beiträge von Kapitel 3 sind

➥ eine analytische Lösungsformel für das lineare Programm,

➥ ein Beweis für instantane Kontrollierbarkeit der linearen Wellengleichung und

➥ die Verallgemeinerung des Stabilitätsbeweises aus [39] auf den Fall zeitvarianter Kontrollhorizonte.

Einige Resultate dieses Kapitels wurden bereits in [45, 46] in einer Vorabversion veröffentlicht. Jedoch wurden insbesondere die Beweise gründlich überarbeitet, um deren Nachvollziehbarkeit zu erleichtern.

☞ In Kapitel 4 wird eine Sensitivitätsanalyse bzgl. der wichtigsten Parameter durchgeführt: Optimierungs- und Kontrollhorizont. Insbesondere die Bedeutung des Letzteren sollte man nicht unterschätzen. Wir beginnen mit dem Optimierungshorizont. Die in Kapitel 3 hergeleitete Formel wird dazu verwendet parameterabhängige Stabilitätsgebiete zu berechnen. Dies erlaubt Rückschlüsse auf den unterschiedlichen Einfluss des Überschwing- und Abklingverhaltens und folglich auf den Entwurf geeigneter Stufenkosten für MPC, siehe [6, 39]. Des Weiteren wird der minimale stabilisierende Horizont, also der kleinste Optimierungshorizont, der asymptotische Stabilität garantiert, genauer untersucht. In diesem Zusammenhang wird — für passend gewählte Kontrollhorizonte — lineares Wachstum bzgl. der akkumulierten Wachstumsschranken aus der vorausgesetzten Kontrollierbarkeitsbedingung gezeigt, was einer qualitativen Verbesserung im Vergleich zu den Abschätzungen aus [120] entspricht. Im darauffolgenden Abschnitt betrachten wir Kontrollhorizonte. Hier werden insbesondere nützliche Symmetrie- und Monotonieeigenschaften gezeigt, welche für die Algorithmenentwicklung in Abschnitt 4.4 eine wichtige Rolle spielen. Abschnitt 4.2 besteht aus zwei Teilen. Im ersten Teil werden die Ergebnisse zusammengefasst während im zweiten, der die Unterabschnitte 4.2.2

und 4.2.3 umfasst, die entsprechenden Beweise dargestellt werden. Für diese wird eine ausgefeilte Beweistechnik benötigt.

Abschnitt 4.3 ist in drei eigenständige Teile gegliedert. Zuerst beschäftigen wir uns mit der vorausgesetzten Kontrollierbarkeitsbedingung. Danach wird ein Beispiel eines linearen Pendels auf einem Wagen betrachtet. Die durchgeführten numerischen Tests bestätigen unsere theoretischen Resultate bzgl. des Kontrollhorizonts. Als drittes Thema werden Endgewichte und ihre Auswirkungen auf den Suboptimalitätsgrad behandelt. In Abschnitt 4.4 werden Algorithmen auf Basis der durchgeführten Sensitivitätsanalyse entwickelt. Weil der Rechenaufwand bei wachsendem Optimierungshorizont schnell steigt, wird dieser Parameter typischerweise als Schlüsselgröße in MPC aufgefasst. Die vorgestellten Algorithmen nutzen das Konzept des Kontrollhorizonts, um Abschätzungen für die garantierte Regelgüte zu verbessern — ohne den Optimierungshorizont zu verlängern. Zudem wird der entwickelte Grundalgorithmus weiter ausgefeilt, um ein verbessertes Robustheitsverhalten zu erzielen. Um die Vorteile der in diesem Abschnitt entwickelten Algorithmen besser herauszustreichen, wird das Beispiel des synchronen Generators eingehend studiert, siehe [28, 34, 94].

Die Hauptresultate dieses Kapitels sind

- Sensitivitätsanalyse bezüglich des Optimierungshorizonts $\rightsquigarrow$ asymptotische Abschätzungen für den minimalen stabilisierenden Horizont,
- Sensitivitätsanalyse bezüglich des Kontrollhorizonts $\rightsquigarrow$ Symmetrie- und Monotonieeigenschaften unserer Suboptimalitätsabschätzungen und
- Design zweier Algorithmen basierend auf den theoretischen Resultaten, um den benötigten Optimierungshorizont und folglich den Rechenaufwand zu reduzieren.

☞ Das letzte Kapitel dieser Dissertationsschrift wird mit einer Fallstudie einer Reaktions-Diffusions-Gleichung begonnen, um das weitere Vorgehen zu motivieren. In diesem Zusammenhang wird eine zeitkontinuierliche Version unserer Kontrollierbarkeitsbedingung eingeführt. Weil aus abgetasteten Differentialgleichungen abgeleitete zeitdiskrete Regelstrecken ein Kernanwendungsgebiet von MPC sind, werden Effekte untersucht, die mit der Verwendung feinerer Diskretisierungen verbunden sind. Hierbei werden neben positiven Auswirkungen auch mögliche Fallstricke sehr kurzer Abtastraten beleuchtet — sehr schnelle Abtastung kann erforderlich sein, um wesentliche Eigenschaften des Ausgangssystem auf sein abgetastetes Pendant zu übertragen. Insbesondere wird gezeigt, dass der Ansatz aus [39] für klassisches MPC in Kombination mit beliebig feiner Diskretisierung nicht anwendbar ist. Beliebig feine Diskretisierung entspricht hier einer gegen Null strebenden Abtastzeit. Des Weiteren wird der Grenzwert dieses Diskretisierungsprozesses berechnet. Dieser Grenzwert stimmt mit seinem zeitkontinuierlichen Pendant aus [103, 104] überein, was klärt, wie die Ansätze [39] und [104] zusammenhängen.

Um die beobachteten Probleme für sehr schnelle Abtastung zu beheben, wird eine Wachstumsbedingung eingeführt. Mit Hilfe dieser Bedingung können zum Beispiel Stetigkeitseigenschaften, wie sie typischerweise für Abtastsysteme gelten, in unserer Stabilitätsanalyse berücksichtigt werden. Dazu wird die Methodik aus [39] um diese Annahme erweitert. Anschließend wird gezeigt, dass dieses Vorgehen das beobachtete Problem löst. Zudem werden einfach nachprüfbare Bedingungen hergeleitet, um diese zusätzliche Voraussetzung zu verifizieren.

In Abschnitt 5.4 werden so genannte akkumulierte Schranken als alternative Kontrollierbarkeitsannahme eingeführt und in unsere Technik zur Bestimmung von Güteabschätzungen eingebaut. Diese akkumulierten Schranken stammen aus [120]. Um deren Auswirkungen zu untersuchen, wird das Beispiel der Reaktions-Diffusions-Gleichung wieder aufgegriffen. Insgesamt führt dieses Vorgehen auf verbesserte Güteabschätzungen für den mittels MPC geschlossenen Regelkreis. Im abschließenden Abschnitt wird die in dieser Dissertationsschrift entwickelte Methodik mit alternativen Ansätzen aus [90] sowie [120] verglichen. Dabei werden insbesondere Unterscheidungsmerkmale herausgestellt. Die in [90] eingeführte Methodik liefert, falls anwendbar, die besten Abschätzungen. Allerdings ist ihr Anwendungsgebiet auf lineare endlich-dimensionale Systeme beschränkt und erfordert zusätzliches Wissen über die optimale Wertefunktion — eine restriktive Zusatzbedingung. Die anderen beiden Ansätze lassen die Behandlung allgemeiner nichtlinearer sowie unendlich-dimensionaler Systeme inklusive Kontroll- und Zustandsbeschränkungen zu. Obwohl vergleichbare Annahmen benötigt werden, sind die Güteabschätzungen aus [120] häufig deutlich konservativer im Vergleich zu unserem Ansatz, der folglich überlegen erscheint.

Die Hauptbeiträge aus Kapitel 5 sind:

- Untersuchung der aus der Verwendung feinerer Diskretisierungen resultierenden Auswirkungen auf unsere Güteabschätzungen sowie die Berechnung des Grenzwertes eines entsprechenden Verfeinerungsprozesses.
- Aufstellen einer Wachstumsbedingung, die dazu führt, dass der vorgestellte Ansatz trotz sehr schneller Abtastung gute Ergebnisse liefert.
- Verwendung akkumulierter Schranken, um unsere Güteabschätzungen weiter zu verbessern.
- Vergleich mit anderen Ansätzen.

Statt eines separaten Beispielkapitels werden die hergeleiteten Resultate direkt in ihren jeweiligen Abschnitten mit Beispielen verbunden, um ihre Aussagen zu veranschaulichen und so die theoretischen Ergebnisse besser nachvollziehbar zu machen. Einige Resultate dieser Dissertationsschrift wurden bereits in Vorabversionen veröffentlicht, siehe [6,45–47], [41,50], [4,5] und [97].

Danksagung

Besonderer Dank gilt meinem Doktorvater Prof. Dr. Lars Grüne für seine hervorragende Betreuung sowie seine wertvollen Anregungen, ohne die diese Arbeit in ihrer jetzigen Form nicht möglich gewesen wäre. Des Weiteren möchte ich mich bei Prof. Dr. Frank Lempio und Prof. Dr. Hans Josef Pesch sowie Nils Altmüller, Marcus von Lossow, Dr. Jürgen Pannek, Marcus Reble, Michael Schamel und Dr. Martin Seehafer bedanken.

Ebenfalls besonderer Dank gilt meinen Eltern Ingrid und Dr. Wilhelm Worthmann, meinen Geschwistern Elke und Dr. Hans Worthmann sowie Johannes Hertel, Ekue-sse Situ Tomety und meiner Partnerin Anja Kleinhenz für ihre Unterstützung in jedweder Hinsicht. Zudem möchte ich mich bei meinen Freunden bedanken!

Summary

In this thesis we are concerned with receding horizon control (RHC), also known as model predictive control. Typically, this methodology is employed in order to approximately solve optimal control problems on an infinite time horizon whose goal is to stabilize a given system at a set point. Since optimal control problems on an infinite time horizon are, in general, computationally intractable, the original problem is replaced by a problem on a truncated and, thus, finite time horizon. Receding horizon control proceeds in the following three steps:

- ➢ Based on a model and the most recent known measurement the system behavior is predicted in order to solve an optimal control problem on a finite time horizon and, thus, to compute an open loop sequence of control values (or an input function in a continuous time setting).
- ➢ The first element of this sequence (or the first portion of the computed control function) is implemented at the plant.
- ➢ The current state, which corresponds to the initial state of the optimal control problem considered in the first step, is updated. In addition, the optimization horizon is shifted forward in time which explains the terminology receding or moving horizon control.

Repeating the described procedure ad infinitum yields a sequence of control values on the infinite time horizon. Hence, RHC iteratively generates a sequence of optimal control problems on a finite time horizon in order to approximate the desired solution.

Due to its ability to explicitly incorporate control and state constraints, this control technique has attracted considerable attention during the last decades. Furthermore, its beneficial use in many industrial applications is reported, cf. [33, 100]. Besides being a solution strategy for the introduced class of problems, another advantage, which leads to an increased interest in RHC, results from generating a closed loop solution which ensures an inherent robustness with respect to, e.g. external disturbances or modelling errors, cf. [102].

Despite the widespread use of RHC in applications, the stability analysis is far from trivial. The first cornerstones in order to deal with this issue employ (artificial) terminal constraints or costs, cf. [17, 66]. These theoretically motivated extensions allow to exert additional influence in order to enforce stability properties of the RHC closed loop. However, since the industrial practice hardly takes these stabilizing constraints into account, we concentrate on the stability behavior of so called unconstrained RHC schemes which neither incorporate terminal constraints nor costs. To this end, the approach proposed in [39] is considered as a starting point. Here, an optimization problem is set up in order to deduce asymptotic stability of and performance bounds for the receding horizon closed loop. Positivity of the resulting suboptimality degree is a necessary and sufficient stability condition on the class of systems satisfying an assumed controllability condition.

Outline and Contribution

This thesis is subdivided into five chapters. The two initial chapters introduce basic concepts and the problem setting. Then, the ensuing Section 3.1 begins with a short summary of the methodology proposed in [39] which may be regarded as our starting point. In the remaining part new results are presented.

The goal of the following outline is twofold: on the one hand a concise overview of the content is provided. On the other hand the contribution of the results developed in this thesis to the analysis of unconstrained RHC schemes is explained.

- ☞ In the first section of Chapter 1 the basic concept of control systems is introduced. Inter alia, attention is paid to admissibility of input sequences. In addition, the optimal value function is defined. In Section 1.2, using this terminology the main differences between open and closed loop control are considered. For instance, closed loop control allows to react to external disturbances or measurement errors. In this context, the general problem setting is defined for which the notion of asymptotic stability is required. In the final two sections of Chapter 1 sampled data as well as networked control systems are dealt with which represent important classes of control systems and constitute application areas for the results presented in the ensuing chapters. Here, we explain how to interpret systems governed by differential equations in our discrete time setting. The chapter is concluded by giving a precise definition of (multistep) feedback laws which play a decisive role for this thesis.

- ☞ In chapter 2 we are concerned with RHC — a methodology in order to deal with optimal control problems on an infinite time horizon — in its various shapes. We begin with RHC in its simplest version: unconstrained RHC. Subsequently, the same control strategy extended by additional terminal costs or constraints is considered. Incorporating these artificial ingredients in the underlying optimal control problems to be solved in each iteration step equips the receding horizon algorithm with improved stability properties. However, one has to face the challenging task of designing appropriate terminal costs which gives reason to the observation that unconstrained RHC is predominantly used in industries. In order to conclude this chapter, the main ideas from [99] in order to ensure feasibility of unconstrained RHC schemes are briefly sketched in Section 2.4.

- ☞ In the ensuing Chapter 3 we begin with a concise survey on the methodology from [39] which enables us, based on a controllability assumption, to ensure a relaxed Lyapunov inequality — our main tool in order to conclude stability of the receding horizon closed loop. Furthermore, this approach yields a suboptimality index which allows to compare the receding horizon performance with the costs attributed to the optimal control problem on the infinite time horizon. In the following Section 3.2 the corresponding stability proof is extended to time varying control horizons — a slight modification which is of particular interest in the networked control setting in order to compensate for non negligible delays and packet dropouts, cf. [47, 48], but which also turns out to be very beneficial in order to derive further results.

  Applying the proposed technique requires to solve a linear program whose dimension equals the optimization horizon of the receding horizon scheme. In Section 3.3 we derive a solution formula for this optimization problem which forms a cornerstone for the ensuing results. In order to structure the involved proof more clearly, some technical details are postponed to an auxiliary subsection which enables us to concentrate on the key steps. Then, this formula is used in order to show that RHC approximates the optimal control on an infinite time horizon for a sufficiently large horizon arbitrarily well — a result in consonance with [32, 120]. In the ensuing section the presented results are illustrated by means of the linear wave equation. In particular, instantaneous controllability is shown rigorously, i.e. RHC stabilizes the system based on the shortest possible optimization horizon. This section is joint work with Nils Altmüller, cf. [4, 5].

The main contributions of Chapter 3 are

- ➥ extension of the stability proof from [39] to time varying control horizons,
- ➥ analytical solution formula for the linear program, and
- ➥ proof of instantaneous controllability for the linear wave equation.

Preliminary versions of some of the results in this chapter were previously published in [45, 46]. However, the proofs are carefully revised and rearranged in this thesis in order to facilitate their accessibility.

☞ In Chapter 4, a complete sensitivity analysis is carried out with respect to the most important parameters in our RHC strategy: the optimization and the control horizon. In particular, the latter turns out to be much more meaningful than it might appear at first glance. Beginning with the optimization horizon, the formula deduced in Chapter 3 is exploited in order to compute parameter depending stability regions which enables us to draw conclusions on the different impact of the overshoot and the decay rate and, thus, on the design of suitable stage costs for RHC, cf. [6, 39]. Furthermore, the minimal stabilizing horizon, i.e. the smallest optimization horizon guaranteeing asymptotic stability, is subject to investigation. In this context, we establish linear growth in terms of the accumulated bound from the proposed controllability condition with suitably chosen control horizons which improves the estimates from [120] qualitatively. In the subsequent section, we focus on the control horizon and point out interesting symmetry and monotonicity properties which pave the way in order to develop algorithms in Section 4.4. This section is composed of two parts. The first part provides a summary of the results while the second consisting of Subsections 4.2.2 and 4.2.3 contains the corresponding proofs which are based on a sophisticated technique.

The ensuing Section 4.3 is subdivided into three independent parts. Firstly, we comment on the supposed controllability condition. Secondly, the linear pendulum on a cart example is considered. Here, numerical experiments confirm our theoretically derived results vis-à-vis the control horizon. Thirdly, attention is paid to the impact of terminal weights in the considered setting. In Section 4.4, algorithms based on the results of the carried out sensitivity analysis are set up. Since the computational expenditure grows rapidly for increasing optimization horizon, this parameter is typically regarded as the key quantity in RHC. The proposed algorithm exploits the concept of control horizons in order to improve the guaranteed performance without prolonging the optimization horizon. In addition, a more elaborate version of this algorithm is introduced in order to enhance robustness. In order to indicate benefits of the developed algorithms, the example of a synchronous generator is considered in detail, cf. [28, 34, 94].

The key results of Chapter 4 are

- sensitivity analysis with respect to the optimization horizon which yields, e.g. asymptotic estimates on minimal stabilizing horizons,
- sensitivity analysis with respect to the control horizon showing symmetry and monotonicity properties of the proposed suboptimality estimates, and
- development of two algorithms which exploit the theoretically deduced results in order to reduce the optimization horizon and, thus, the computational costs.

☞ In the final chapter of this thesis a case study of a reaction diffusion equation is carried out first in order to motivate the ensuing investigations. In this context, a continuous time version of our controllability condition is introduced. Since RHC for discrete time systems induced by a sampled differential equation is a driving force behind the proposed analysis, effects linked to employing more accurate discretizations are analyzed. In particular, we do not only observe positive effects of very fast sampling — which may be necessary in order to preserve essential features in a sampled data setting, cf. [91] — but also point out possible pitfalls. More precisely, we rigorously prove that, for classical RHC, the approach from [39] fails for arbitrarily fine discretizations, i.e. for letting the sampling time tend to zero. Furthermore, the continuous time limit of a discretization procedure is deduced which coincides with results derived in [103, 104] for a continuous time setting. As a consequence, the approach originating from [39] is unified with its counterpart based on a continuous time setting from [104].

In order to overcome the observed drawbacks for very fast sampling, a growth condition is introduced which reflects, e.g. continuity properties typically present in a sampled data system. Then, we generalize the technique from [39] to this setting and show that the growth condition is a suitable tool in order to resolve the observed problem. Furthermore, easily checkable sufficient conditions for guaranteeing this additional prerequisite are presented.

In Section 5.4, accumulated bounds, which represent an alternative controllability assumption from [120], are introduced and incorporated in our setting. In order to investigate their ramifications, the examples of the reaction diffusion equation and the synchronous generator are considered again. In conclusion, the corresponding suboptimality estimates are improved. In the final section the methodology developed in this thesis is compared with alternative approaches from [90] and [120]. In particular, distinguishing factors are pointed out. The technique proposed in [90] yields, if applicable, the best results. However, its application is limited to linear finite dimensional systems and necessitates additional knowledge on the optimal value function — a restrictive extra condition. The other two methodologies allow to deal with nonlinear and infinite dimensional systems including state and control constraints. But although similar assumptions are used, the performance bounds resulting from [120] are often more conservative in comparison to our approach which, thus, seems to be superior.

The main contributions of Chapter 5 are

- investigation of the impact of using more accurate discretizations,
- derivation of a formula for the limit of an iterative refinement process,
- introduction of a growth condition which resolves problems occurring for very fast sampling,

- definition of accumulated bounds in order to generate tighter performance estimates, and
- comparison with other approaches

In order to facilitate understanding of the theoretical results, several illustrating examples are incorporated throughout the text, i.e. we do not present a separate example chapter but rather interconnect the derived assertions with examples in order to directly demonstrate their impact. Some results of this thesis were already published in preliminary versions, cf. [6, 45–47], [41, 50], [4, 5], and [97].

Acknowledgement

I want to thank Prof. Dr. Andrew Richard Teel for his valuable and helpful comments after reviewing this thesis.

Furthermore, I am grateful that I was supported by the DFG priority research program 1305 "Control Theory of Digitally Networked Dynamical Systems", grant. no. Gr1569/12-1, and the German National Academic Foundation (Studienstiftung des deutschen Volkes).

Chapter 1

Control Systems, Stability, and Feedback

In this chapter the problem formulation of this thesis is presented. To this end, control systems, admissible sequences of control values, and an optimal value function are defined in the first Section 1.1. In the ensuing section the concept of stability, which characterizes the long-term behavior of systems evolving in time, is introduced. The theory of Lyapunov which allows to rigorously deduce asymptotic stability is of particular interest in this context. Furthermore, the basic ideas of closed loop control are presented. Then, in Sections 1.3 and 1.4 sampled-data and networked control systems are dealt with in order to motivate our discrete time setting as well as the proposed multistep feedback.

The set of real numbers is denoted by $\mathbb{R}$ and the set of integers by $\mathbb{Z}$. Furthermore, $\mathbb{N}$ stands for the natural numbers, i.e. $\mathbb{Z}_{>0}$, as well as $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$, i.e. the set of non-negative integers. We require the following definition, cf. [106].

Definition 1.1 (Metric space)
A metric space is a set X with a metric or distance function $d : X \times X \to \mathbb{R}$ such that the following properties are satisfied for all $x, y, z \in X$:

- *definiteness, i.e. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,*
- *symmetry $d(x, y) = d(y, x)$, and*
- *triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$.*

1.1 Control Systems and Problem Formulation

In this thesis we are concerned with control systems. The state of a control system evolves depending on its current state and a control input. This input parameter can be chosen in order to exert influence on the system. A classical example is the inverted pendulum on a cart, cf. Figure 1.1 and Section 1.3. Here, the state consists of the angle Φ of the pendulum, the position of the cart and the corresponding velocities. The movement is determined by the current state and an external force u acting on the cart.

The concept of a control system is formalized in the following definition.

Definition 1.2 (Control system)
Let X and U be metric spaces. A control system is a quadruplet $\Sigma = (\mathbb{T}, X, U, f)$ consisting of a time domain $\mathbb{T} = \{Tk \,|\, k \in \mathbb{N}_0\}$, $T > 0$, a state space X, a set of control values U,

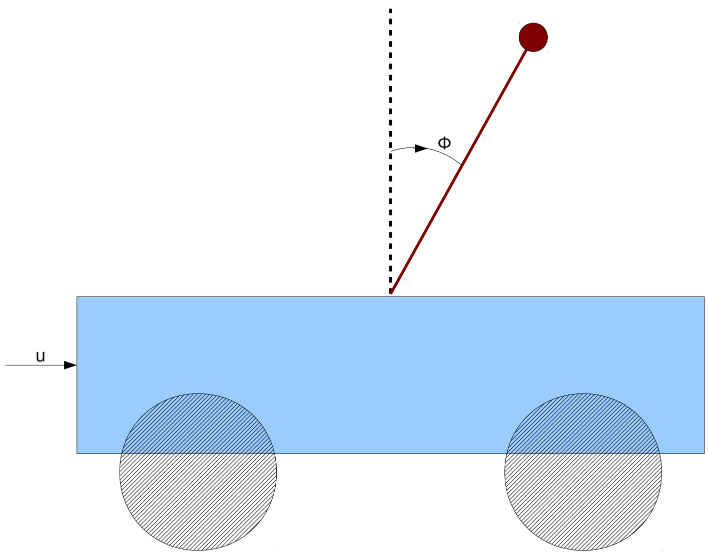


Figure 1.1: Schematic illustration of the inverted pendulum on a cart, cf. [37].

and a transition map $f : \mathcal{D}_f \to X$*. The transition map* $f(\cdot,\cdot)$ *is defined on a subset* $\mathcal{D}_f$ *of* $X \times U$*.*

The state space X need not satisfy the definition of a linear space, which can be found, e.g., in [72]. Since control systems are defined forward in time, the time domain $\mathbb{T}$ is a subset of the positive real axis.

In order to investigate the class of control systems, we typically consider models which capture the dynamical behavior of an underlying process, cf. Section A.2 for a mathematical model of the inverted pendulum on a cart. These models are employed in order to deduce a suitable transition map. Since the concept of control systems is used in order to describe dynamics of practically motivated systems, the states and control values are often restricted. For instance, the set of control values may have to be bounded. The following definition allows for incorporating constraints in our setting.

Definition 1.3 (State and control constraints)
Let nonempty sets $\mathbb{X} \subseteq X$ *and* $\mathbb{U} \subseteq U$ *denote the set of feasible states and controls, respectively. A sequence* $u(\cdot) = (u(n))_{n \in \{0,1,\ldots,N-1\}} \in U^N$*,* $N \in \mathbb{N}$*, is called admissible for* $x_0 \in \mathbb{X}$ *if*

$$u(n) \in \mathbb{U} \qquad \text{and} \qquad f(x_u(n;x_0), u(n)) \in \mathbb{X} \qquad \text{holds for all } n \in \{0,1,\ldots,N-1\}.$$

Here, $x_u(n;x_0)$ *is defined recursively by the system dynamics*

$$x_u(n+1;x_0) := f(x_u(n;x_0), u(n)) \qquad \text{for } n \in \mathbb{N}_0 \text{ with } x_u(0;x_0) := x_0. \tag{1.1}$$

$\mathcal{U}^N(x_0)$ *denotes the set* $\{u(\cdot) \in U^N : u(\cdot) \text{ is admissible for } x_0\}$ *and a sequence* $u(\cdot) = (u(n))_{n \in \mathbb{N}_0} \in U^{\mathbb{N}}$ *is called admissible for* $x_0 \in \mathbb{X}$*, i.e.* $u(\cdot) \in \mathcal{U}^\infty(x_0)$*, if* $(u(n))_{n \in \{0,1,\ldots,N-1\}} \in \mathcal{U}^N(x_0)$ *holds for each* $N \in \mathbb{N}$*.*

The abbreviations $x(n) = x_u(n) = x_u(n;x_0)$ are used when the parameters x_0 and $u(\cdot)$ clearly follow from the context. Furthermore, the states $x(n)$, $n \in \mathbb{N}_0$, are enumerated without stating the scaling factor T resulting from the time domain $\mathbb{T}$ explicitly. The set $\mathbb{X}$ characterizes all feasible states, e.g. we may choose $X = \mathbb{R}^n$ and $\mathbb{X} = \{x \in X \mid h(x) \leq 0\}$ for $h : \mathbb{R}^n \to \mathbb{R}$ in order to model state constraints. We make the following assumption which ensures that, for each feasible state $x_0 \in \mathbb{X}$, an admissible sequence of control values $u(\cdot) \in \mathcal{U}^\infty(x_0)$ exists on the infinite time horizon, cf. [120, Assumption A3].

Assumption 1.4 (Controlled forward invariance)
For each state $x \in \mathbb{X}$, let a control value $u \in \mathbb{U}$ exist such that $f(x,u) \in \mathbb{X}$ holds.

Assumption 1.4 is also termed weak forward invariance or viability, cf. [35]. Suppose that Assumption 1.4 does not hold. Then, the state constraints are violated for a feasible state $x_0 \in \mathbb{X}$ for all $u \in \mathbb{U}$. Hence, the task of steering the control system with initial value x_0 is not well-posed.

The sequence of control values $u(\cdot) : \mathbb{N}_0 \to U$ is interpreted as an input, i.e. $u(\cdot)$ is constructed in order to suitably manipulate the behavior of the system. In this thesis, our goal is to stabilize a given plant at a desired position which is, in general, specified in advance. This type of problem is called set point stabilization and fits well to the example of the inverted pendulum on a cart, in which the upright position is the desired state. Typically, these particular positions are so called equilibria $x^\star \in \mathbb{X} \subseteq X$ satisfying

$$f(x^\star, u^\star) = x^\star \tag{1.2}$$

for at least one control value $u^\star \in \mathbb{U}$, cf. [108, Section 5.4]. Trajectories emanating from an equilibrium $x^\star \in \mathbb{X}$ may be balanced at this position by a suitably chosen control input.

We aim at steering the system to its equilibrium $x^\star$, at least asymptotically. If more than one trajectory converges asymptotically to the desired equilibrium, the transient behavior of the system may be taken into account in order to assess the quality of the induced behavior of the system to be controlled, cf. [58, Section 5.5]. To this end, we define a cost functional which is based on so called stage costs, cf. [7, Subsection 1.6.1].

Definition 1.5 (Cost functional and stage costs)
Let a control system $(\mathbb{T}, X, U, f)$ as well as feasible sets $\mathbb{X} \subseteq X$ and $\mathbb{U} \subseteq U$ be given. Then, the cost functional $J_\infty : X \times U^{\mathbb{N}} \to \mathbb{R}_0^+ \cup \{\infty\}$ is defined by

$$J_\infty(x_0, u(\cdot)) = \sum_{n=0}^{\infty} \ell(x_u(n; x_0), u(n)) \tag{1.3}$$

with stage (running) costs $\ell : X \times U \to \mathbb{R}_0^+ \cup \{\infty\}$ which are continuous on $\mathbb{X} \times \mathbb{U}$. Here, the system dynamics are given by (1.1).

Hence, our goal is to minimize the cost functional (1.3) and to stabilize the considered control system asymptotically at a given set point $x^\star$. In order to state this task mathematically, these two objectives are coupled by the stage costs. To this end, the following definition of a comparison function is required, cf. [115, Exercise 7.3.11], [35,39], and [58, Definition 3.2.1].

Definition 1.6 ($\mathcal{K}_\infty$-function)
A continuous function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be of class $\mathcal{K}$ if $\alpha(\cdot)$ is strictly increasing and $\alpha(0) = 0$. If, additionally, the property $\lim_{r\to\infty} \alpha(r) = \infty$ holds, $\alpha(\cdot)$ is said to be of class $\mathcal{K}_\infty$.

We point out that each function $\alpha(\cdot) \in \mathcal{K}_\infty$ is invertible, cf. [70]. The following assumption consists of two parts. The first ensures that staying at the desired equilibrium $x^\star$ forever at zero cost is possible. The second, which uses Definition 1.6, incorporates the stabilization task in the cost functional (1.3) because not tending to $x^\star$ causes infinite costs.

Assumption 1.7
Let an equilibrium $x^\star$ exist which satisfies:

(i) $u \in \mathbb{U}$ with $f(x^\star, u) = x^\star$ and $\ell(x^\star, u) = 0$ exists.

(ii) $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ exist such that the inequalities

$$\alpha_1(\|x\|_{x^\star}) \leq \ell^\star(x) := \inf_{u \in \mathbb{U}: f(x,u) \in \mathbb{X}} \ell(x,u) = \inf_{u \in \mathcal{U}^1(x_0)} \ell(x,u) \leq \alpha_2(\|x\|_{x^\star}) \tag{1.4}$$

hold for each $x \in \mathbb{X}$ where $\|x\|_{x^\star} := d_X(x, x^\star)$.

We remark that condition (ii) can be relaxed in various ways, e.g. it could be replaced by a detectability condition similar to the one used in [32]. However, in order to keep the presentation technically simple, we work with Assumption 1.7(ii). Moreover, the equilibrium $x^\star$ may be replaced by a closed set A at which the system has to be stabilized, cf. [39].

Typical stage costs are, e.g. $\ell(x,u) := d_X(x,x^\star)^2 + \lambda\, d_U(u,u^\star)^2$. Here, $\lambda \in \mathbb{R}_{\geq 0}$ denotes a regularization parameter and d_X, d_U metrics on X, U, respectively. If the metric space X exhibits the structure of a linear space [72], the desired equilibrium $x^\star$ is supposed to be located at the origin 0_X of this space, cf. [38, Remark 2.4].[1] The contribution of the regularization parameter λ is twofold: firstly, it allows for penalizing the control effort which is used in order to steer the system in the desired direction. Secondly, in particular for systems governed by partial differential equations, it implies some regularity for the corresponding solutions, cf. [119].

Our goal is to find, for a given initial value $x_0 \in \mathbb{X}$, an admissible sequence of control values $u(\cdot) \in \mathcal{U}^\infty(x_0)$ which minimizes a cost functional of type (1.3). In order to tackle this task, the optimal value function is defined.

Definition 1.8 (Optimal value function)
Let a control system $(\mathbb{T}, X, U, f)$, a set of feasible states $\mathbb{X} \subseteq X$, and a set of feasible control values $\mathbb{U} \subseteq U$ be given. Then, for a given state $x_0 \in \mathbb{X}$, the optimal value function $V_\infty(\cdot) : \mathbb{X} \to \mathbb{R}_0^+ \cup \{\infty\}$ is defined by

$$V_\infty(x_0) := \inf_{u(\cdot) \in \mathcal{U}^\infty(x_0)} J_\infty(x_0, u(\cdot)) \tag{1.5}$$

with the set of admissible input sequences $\mathcal{U}^\infty(x_0)$ from Definition 1.3.

Substituting the objective of stabilizing the plant at a set point by tracking a reference signal is possible. To this end, the stage costs as well as the cost functional have to explicitly depend on the time, cf. [107, Section 3.2]. The results of this thesis are generalizable to this setting, cf. [44].

Let us suppose that the optimal value function is finite for each feasible state, i.e. $V_\infty(x_0) < \infty$ holds for all $x_0 \in \mathbb{X}$. Otherwise, the considered minimization problem is

[1]Often we omit the subscript X and write 0 for the origin of the respective (linear) metric space.

either not feasible or the computed control causes infinite costs and is, thus, not distinguishable from an infeasible one. In both cases the optimization problem is not well-posed. Since $V_\infty(x_0) < \infty$ on $\mathbb{X}$ implies the existence of an admissible sequence of control values $u(\cdot) \in \mathcal{U}^\infty(x_0)$ for each $x_0 \in \mathbb{X}$, Assumption 1.4 is ensured.

Summarizing, we want to find an admissible sequence of control values $u(\cdot)$ which stabilizes the considered control system with minimal costs. The qualitative goals of steering the system feasibly and stabilizing it at the desired equilibrium are coupled with the quantitative objective of minimizing a performance criterion via the optimal value function $V_\infty(\cdot)$. Since the coupling is done by the stage costs, modelling these appropriately is an important task.

1.2 Closed Loop Control and Asymptotic Stability

In the previous section the basic problem formulation was given. In order to sketch the upcoming approach, the following assumption is made in order to avoid technical difficulties. Assumption 1.9 is used only for illustrative purposes in the first chapter of this thesis.

Assumption 1.9
For each $x_0 \in \mathbb{X} \subseteq X$, let the infimum in Definition 1.8 be a minimum, i.e. a sequence of control values $u^\star_{x_0}(\cdot) \in \mathcal{U}^\infty(x_0)$ satisfies

$$J_\infty(x_0, u^\star_{x_0}(\cdot)) = V_\infty(x_0). \tag{1.6}$$

Let $u^\star_{x_0}(\cdot) = (u^\star_{x_0}(n))_{n\in\mathbb{N}_0} \in \mathcal{U}^\infty(x_0)$ denote an admissible sequence of control values depending on the initial value $x_0 \in \mathbb{X}$ which satisfies (1.6). The corresponding solution $x_{u^\star_{x_0}}(\cdot; x_0)$ emanating from x_0 is called open loop trajectory. Since model uncertainties or disturbances are typically present while applying the sequence of control values $u^\star_{x_0}(\cdot)$, the generated trajectory $x_{u^\star_{x_0}}(\cdot; x_0)$ might not be stable - even for arbitrary small perturbations, cf. [36, Example 5.2]. Hence, in order to obtain a solution which compensates at least for small perturbations, so called closed loop solutions are considered, cf. Figure 1.2.

Applying the first element $u^\star_{x_0}(0)$ of the computed open loop control, yields the equality

$$J_\infty(x_0, u^\star_{x_0}(\cdot)) = \sum_{n=0}^{\infty} \ell(x_{u^\star_{x_0}}(n; x_0), u^\star_{x_0}(n)) = \ell(x_0, u^\star_{x_0}(0)) + \sum_{n=1}^{\infty} \ell(x_{u^\star_{x_0}}(n; x_0), u^\star_{x_0}(n)).$$

Furthermore, the next state $x_1 := x_{u^\star_{x_0}}(1; x_0) = f(x_0, u^\star_{x_0}(0))$ is determined. Then, the following optimization problem can be considered:

$$\text{Minimize} \quad J_\infty(x_1, u(\cdot)) = \sum_{n=0}^{\infty} \ell(x_u(n; x_1), u(n)) \quad \text{w.r.t.} \quad u(\cdot) \in \mathcal{U}^\infty(x_1).$$

Let the corresponding solution be denoted by $u^\star_{x_1}(\cdot)$. Concatenating $u^\star_{x_0}(0)$ and $u^\star_{x_1}(\cdot)$ yields a control sequence $\tilde{u}(\cdot) \in \mathcal{U}^\infty(x_0)$ with $\tilde{u}(0) = u^\star_{x_0}(0)$ and $\tilde{u}(n) = u^\star_{x_1}(n-1)$ for $n \in \mathbb{N}$. Since $u^\star_{x_0}(\cdot)$ satisfies (1.6), $J_\infty(x_0, u^\star_{x_0}(\cdot)) \leq J_\infty(x_0, \tilde{u}(\cdot))$ is known. Now, suppose that the strict inequality $J_\infty(x_0, u^\star_{x_0}(\cdot)) < J_\infty(x_0, \tilde{u}(\cdot))$ holds. Then,

$$J_\infty(x_0, \tilde{u}(\cdot)) \;=\; \sum_{n=0}^{\infty} \ell(x_{\tilde{u}}(n; x_0), \tilde{u}(n))$$

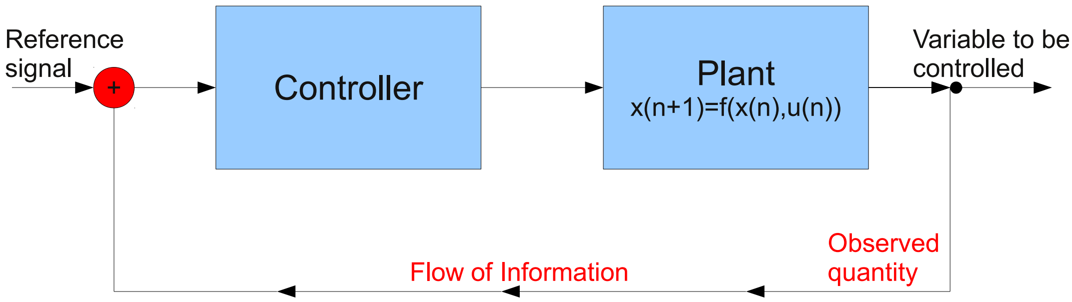


Figure 1.2: Scheme of open and closed loop control. The distinctive features are drawn in red: in the closed loop an observed quantity and, thus, information about the current state is compared with a reference signal, e.g. the distance from the desired equilibrium, and transmitted to the controller — the control loop is closed. Based on this information the control signal may be updated. Without integrating this flow of information in the control loop a reaction to disturbances or modelling errors is not possible.

$$\begin{aligned} &= \ell(x_0, u^\star_{x_0}(0)) + \sum_{n=0}^{\infty} \ell(x_{u^\star_{x_1}}(n; x_{u^\star_{x_0}}(1; x_0)), u^\star_{x_1}(n)) \\ &> \ell(x_0, u^\star_{x_0}(0)) + \sum_{n=0}^{\infty} \ell(x_{u^\star_{x_0}}(1+n; x_0), u^\star_{x_0}(1+n)) = J_\infty(x_0, u^\star_{x_0}(\cdot)) \end{aligned}$$

is obtained which contradicts the definition of $u^\star_{x_1}(\cdot)$. As a consequence, the optimal value function $V_\infty(\cdot)$ satisfies

$$\begin{aligned} V_\infty(x_0) &= J_\infty(x_0, u^\star_{x_0}(\cdot)) = J_\infty(x_0, \tilde{u}(\cdot)) \\ &= \ell(x_0, u^\star_{x_0}(0)) + J_\infty(x_1, u^\star_{x_1}(\cdot)) \\ &= \ell(x_0, u^\star_{x_0}(0)) + V_\infty(x_1) = \ell(x_0, u^\star_{x_0}(0)) + V_\infty(f(x_0, u^\star_{x_0}(0))). \end{aligned}$$

The fact that $u^\star_{x_0}(\cdot)$ depends only on the current state x_0 enables us to define a static state feedback $F_\infty : \mathbb{X} \to \mathbb{U}$ by $F_\infty(x_0) := u^\star_{x_0}(0)$. Plugging this definition into the last chain of equalities yields

$$V_\infty(x_0) = \ell(x_0, F_\infty(x_0)) + V_\infty(f(x_0, F_\infty(x_0))). \tag{1.7}$$

Indeed, (1.7) characterizes an optimal feedback value for the optimization problem for a given state $x_0 \in \mathbb{X}$ on the infinite time horizon and allows for an iterative computation of an optimal sequence of control values. This technique is called dynamic programming, cf. [113] and [81] for its use as a computational tool. It is based on Bellman's principle of optimality which states that tails of optimal trajectories are again optimal, cf. [9]. Reformulating (1.7) provides the Lyapunov equation

$$V_\infty(f(x_0, F_\infty(x_0))) = V_\infty(x_0) - \ell(x_0, F_\infty(x_0)). \tag{1.8}$$

In order to illustrate the presented ideas, a simple discrete time control system is considered, which was introduced in [112] and further investigated in [39, 90]. Note that this example does not exhibit any control or state constraints which makes the analysis much easier.

Example 1.10
Let $\mathbb{U} = U = \mathbb{R}$, $\mathbb{X} = X := \mathbb{R}^2$, and $\ell(x,u) = x^T Qx + u^T Ru$ be given. Then, $\mathcal{U}^\infty(x_0) = U^{\mathbb{N}}$ holds for each $x_0 \in \mathbb{X} = X$. The following optimal control problem is considered:

$$\min_{u(\cdot)\in\, U^{\mathbb{N}}} \sum_{n=0}^{\infty} x(n)^T Qx(n) + u(n)^T Ru(n) = \min_{u(\cdot)\in\, U^{\mathbb{N}}} \sum_{n=0}^{\infty} x(n)^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(n) + u(n)^T u(n)$$

subject to the linear dynamics

$$x(n+1) = Ax(n) + Bu(n) = \begin{pmatrix} 1 & 1.1 \\ -1.1 & 1 \end{pmatrix} x(n) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(n).$$

For this example, the optimal value function is computable via $V_\infty(x_0) = x_0^T P x_0$ where P is the unique positive definite solution of the algebraic Riccati equation (ARE)

$$P = A^T PA - A^T PB(B^T PB + R)^{-1} B^T PA + Q.$$

Moreover, $F_\infty(x_0) = u^\star_{x_0}(0)$ is given by

$$F_\infty(x_0) = -(B^T PB + R)^{-1} B^T PAx_0,$$

cf. [74, 90] and [8]. Here, this leads approximately to

$$P \approx \begin{pmatrix} 5.09839937 & 3.210349330 \\ 3.21034933 & 7.406837723 \end{pmatrix} \qquad \text{and} \qquad F_\infty(x_0) \approx \begin{pmatrix} 0.58728054 \\ -1.301110161 \end{pmatrix}^T x_0.$$

Using this feedback, we obtain the closed loop system

$$x(n+1) = Ax(n) + BF_\infty(x(n)) = (A + BF_\infty)x(n). \tag{1.9}$$

Hence, for $x_0 = (1\ 1)^T \in \mathbb{X}$, (1.8) corresponds to

$$16.416 \approx V_\infty((A + BF_\infty)x_0) = V_\infty(x_0) - \ell(x_0, F_\infty(x_0)) \approx 18.926 - 2.510.$$

Supposing that a static state feedback map $F : X \to U$ satisfying

$$F(x) \in \mathbb{U} \quad \text{and} \quad f(x, F(x)) \in \mathbb{X} \qquad \text{for all } x \in \mathbb{X} \tag{1.10}$$

is given, the resulting closed loop trajectory $x_F(\cdot) = (x_F(n))_{n\in\mathbb{N}_0}$ is generated by $x_F(n+1; x_0) = f(x_F(n; x_0), F(x_F(n; x_0)))$, $n \in \mathbb{N}_0$, with $x_F(0; x_0) = x_0$. The conditions given in (1.10) ensure that the corresponding sequence of control values $F(x_F(\cdot; x_0)) = (F(x_F(n; x_0)))_{n\in\mathbb{N}_0}$ is contained in $\mathcal{U}^\infty(x_0)$ for $x_0 \in \mathbb{X}$ and, thus, admissible. Hence, assuming that (1.10) holds, system dynamics $\tilde{f} : X \to X$ depending solely on the state can be defined by $\tilde{f}(x) := f(x, F(x))$. This map $\tilde{f}$ defines a dynamical system, cf. [53, 58, 117].

Definition 1.11 (Dynamical system)
A dynamical system on X is a triple $(X, \mathbb{T}, x)$ which consists of the time domain $\mathbb{T} := \mathbb{N}_0$, the state space X, and a map $x : \mathbb{T} \times X \to X$ such that

- *$x(0, x_0) = x_0$ for all $x_0 \in X$ (consistency),*
- *$x(\tau, x(t, x_0)) = x(\tau + t, x_0)$ for all $x_0 \in X$ and $t, \tau \in \mathbb{T}$ (group property).*

The restriction to the time domain $\mathbb{N}_0$ is not necessary but fits well for our purposes. Since the time domain is contained in $\mathbb{R}_0^+$, $(X, \mathbb{T}, x)$ is said to be a semi dynamical system in some references, cf. [35]. Next, we want to introduce the concept of asymptotic stability for a dynamical system. To this end, comparison functions $\beta \in \mathcal{KL}_0$ are required, cf. [40].

Definition 1.12 ($\mathcal{KL}$- and $\mathcal{KL}_0$-functions)
A function $\beta : \mathbb{R}_0^+ \times \mathbb{N}_0 \to \mathbb{R}_0^+$ is said to be of class $\mathcal{KL}$ if

- *for each $t \in \mathbb{R}_0^+$, $\beta(\cdot, t) : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathcal{K}_\infty$ and*
- *for each $r \geq 0$, $\beta(r, \cdot) : \mathbb{N}_0 \to \mathbb{R}_0^+$ is decreasing with $\lim_{t\to\infty} \beta(r,t) = 0$.*

Furthermore, a function $\beta : \mathbb{R}_0^+ \times \mathbb{N}_0 \to \mathbb{R}_0^+$ is said to be of class $\mathcal{KL}_0$ if

- *for each $t \in \mathbb{R}_0^+$, $\beta(\cdot, t) : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathcal{K}_\infty$ or $\beta(\cdot, t) \equiv 0$ and*
- *for each $r > 0$, $\lim_{t\to\infty} \beta(r,t) = 0$.*

Since discrete time systems are dealt with, $\beta(\cdot, \cdot)$ from Definition 1.12 is, in contrast to [58, Definition 3.2.1], defined on $\mathbb{R}_0^+ \times \mathbb{N}_0$ instead of $\mathbb{R}_0^+ \times \mathbb{R}_0^+$. Each $\beta(\cdot, \cdot) \in \mathcal{KL}_0$ may be extended to a continuous function on $\mathbb{R}_0^+ \times \mathbb{R}_0^+$, e.g. by linear interpolation. Vice versa, taking a continuous $\mathcal{KL}_0$-function defined on $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ as a starting point allows to define a corresponding restriction canonically. This mapping is tacitly used in order to avoid technical details for discrete time systems originating from continuous time ones.

Since each continuous $\mathcal{KL}_0$-function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ can be overbounded by a function $\tilde{\beta}(\cdot, \cdot) \in \mathcal{KL}$, e.g. by setting $\tilde{\beta}(r,t) = \sup_{\tau \geq t} \beta(r, \tau) + e^{-t} r$, this can also be done for functions defined according to Definition 1.12. Two important representatives of class $\mathcal{KL}_0$-functions $\beta(\cdot, \cdot)$ are given in the following example.

Example 1.13
The first example is in fact contained in $\mathcal{KL} \subset \mathcal{KL}_0$.

- *Let an overshoot bound $C \geq 1$ and a decay rate $\sigma \in (0,1)$ be given. Then, exponentially decaying functions are defined by*

$$\beta(r, n) = C\sigma^n r. \tag{1.11}$$

While the second requires the more general class $\mathcal{KL}_0$.

- *A function $\beta(\cdot, \cdot) : \mathbb{R}_0^+ \times \mathbb{N}_0 \to \mathbb{R}_0^+$ is linear in its first argument and equal to zero for sufficiently large second arguments if a finite number $n_0 \in \mathbb{N}_0$ and a sequence $(c_n)_{n \in \mathbb{N}_0} \subset \mathbb{R}_0^+$ satisfying $c_n = 0$ for all $n \geq n_0$ exist such that*

$$\beta(r, n) = r \cdot c_n \qquad \text{for all } n \in \mathbb{N}_0 \tag{1.12}$$

  holds. Such a function can be defined by choosing only finitely many elements c_n, $n \in \{0, 1, \ldots, n_0 - 1\}$.

Note that each function of the second class of Example 1.13 may be overbounded by an exponentially decaying one. However, using the larger class $\mathcal{KL}_0$ allows for employing tighter bounds in order to estimate the actual behavior of the system, cf. [39].

The following submultiplicativity property will be required in this thesis in order to characterize the stability behavior of a considered class of systems better

$$\beta(r, n+m) \leq \beta(\beta(r,n), m) \qquad \forall\, n, m \in \mathbb{N}_0 \quad \text{and} \quad r \geq 0. \tag{1.13}$$

For $\beta(r, n+m) = C\sigma^{n+m}r \leq C^2\sigma^n\sigma^m r = C \cdot \sigma^m(C\sigma^n r) = \beta(\beta(r,n),m)$ with $C \geq 1$, Property (1.13) is always satisfied. While it is satisfied for the second class if and only if $c_{n+m} \leq c_n c_m$ holds. If needed, this property can be assumed without loss of generality by applying Sontag's $\mathcal{KL}$-Lemma, cf. [115]. Further comments on $\mathcal{KL}_0$-functions can be found in [39, Section 3].

Using class $\mathcal{KL}$-functions $\beta(\cdot,\cdot)$ allows to define asymptotic stability, cf. [44].

Definition 1.14 (Asymptotic stability)
Let a dynamical system $(X, \mathbb{N}_0, x)$, a set $\mathbb{X} \subseteq X$, and an equilibrium $x^\star$ be given, i.e. $x(n, x^\star) = x^\star$ for $n \in \mathbb{N}_0$. The equilibrium is said to be asymptotically stable on $\mathbb{X} \subseteq X$ if a $\mathcal{KL}$-function β exists such that, for each $x \in \mathbb{X}$, the state trajectory $x(n; x_0)$, $n \in \mathbb{N}_0$, is contained in $\mathbb{X}$ and, in addition, satisfies the inequality

$$\|x(n;x_0)\|_{x^\star} = d_X(x(n;x_0), x^\star) \leq \beta(d_X(x_0;x^\star), n) = \beta(\|x_0\|_{x^\star}, n), \quad n \in \mathbb{N}_0. \tag{1.14}$$

Definition 1.14 implies two important properties:

- stability (in the sense of Lyapunov), i.e. for any $\varepsilon > 0$, $\delta = \delta(\varepsilon) > 0$ exists such that $x(n; x_0) \in \mathbb{X}$ and $d_X(x(n;x_0), x^\star) < \varepsilon$, $n \in \mathbb{N}_0$, hold for all $x_0 \in \mathbb{X}$ satisfying $d_X(x_0, x^\star) < \delta$, i.e. trajectories stay arbitrarily close to the equilibrium $x^\star$ if their initial state is feasible and located in a sufficiently small neighborhood of $x^\star$.

- attraction, i.e. the state trajectory converges to $x^\star$ since $d_X(x(n;x_0), x^\star)$ tends to zero for n approaching infinity for all $x_0 \in \mathbb{X}$.

Next, the concept of Lyapunov functions, which will be employed in order to conclude stability of a control system operated in closed loop, is introduced, cf. [44, Definition 2.18]. A Lyapunov function may be seen as an energy norm, i.e. it measures the energy present in the system. Hence, a Lyapunov inequality ensures a "loss of energy" and, thus, characterizes the desired equilibrium as a state of the system at which energy is vanished, cf. [115, p.348].

Definition 1.15 (Lyapunov function)
Let $x^\star = 0$ be an equilibrium point for a dynamical system $(X, \mathbb{N}_0, x)$ and $\mathbb{X} \subseteq X$ be a subset of the state space. Then, a function $V : \mathbb{X} \to \mathbb{R}_0^+$ is said to be a Lyapunov function on $\mathbb{X}$ if

- *$\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ exist such that the following condition holds*

$$\alpha_1(\|x_0\|_{x^\star}) \leq V(x_0) \leq \alpha_2(\|x_0\|_{x^\star}) \qquad \forall\ x_0 \in \mathbb{X} \tag{1.15}$$

- *and, in addition, a $\mathcal{K}$-function $W : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ exists such that*

$$V(x(1;x_0)) \leq V(x_0) - W(V(x_0))$$

  holds for all $x_0 \in \mathbb{X}$ satisfying $x(1;x_0) \in \mathbb{X}$.

Furthermore, if $\mathbb{X} = X$, then $V(\cdot)$ is called global Lyapunov function.

For instance, the first inequality in Condition (1.15) can be verified for a closed loop system if the inequalities $\alpha_1(\|x\|_{x^\star}) \leq \ell^\star(x) \leq \ell(x, F(x)) \leq V(x) < \infty$ hold for all $x \in \mathbb{X} \subseteq X$. Here, in contrast to [95,115], continuity of the Lyapunov function $V(\cdot)$ is not assumed which allows to deal, e.g. with state constraints. Often, even further regularity assumptions are imposed on the Lyapunov function $V(\cdot)$, cf. [21, 70, 87]. However, in view of the fact that the considered dynamical systems are generated by a control system and, thus, the closed loop is based on a static state feedback $F : X \to U$, which may be discontinuous, these cannot be expected, cf. [20, 114].

The main purpose of the following Theorem 1.16 is to illustrate the connection between a Lyapunov equation (or inequality) and asymptotic stability of the respective closed loop system, cf. [11, Section V.2] and [78]. For a proof, we refer to [44, Theorem 2.19].

Theorem 1.16
Let $x^\star$ be an equilibrium point and $\mathbb{X} \subseteq X$, $x^\star \in \mathbb{X}$, be a forward invariant set for a dynamical system $(X, \mathbb{N}_0, x)$. If a function $V(\cdot)$ satisfying the conditions of Definition 1.15 on $\mathbb{X}$ exists, the equilibrium $x^\star \in \mathbb{X}$ is asymptotically stable.

We like to point out that existence of a Lyapunov function can also be concluded assuming asymptotic stability in the sense of Defintion 1.14, see, e.g., the construction carried out in [92, Lemma 4]. The question which assumptions are, in general, needed in order to ensure existence of a Lyapunov function is adressed in the so called converse Lyapunov theory, for details we refer to [67, 68].

Example 1.10 is considered once more in order to illustrate the results of this section.

Example 1.17
The control system from Example 1.10 with static state feedback $F_\infty(\cdot)$ is considered. Plugging $F_\infty(\cdot)$ into the transition map $f(\cdot,\cdot)$ yields the system dynamics (1.9) for the closed loop. We verify the conditions of Definition 1.15. Then, Theorem 1.16 is applied in order to conclude asymptotic stability.

Since the matrix P is positive definite, [36, Lemma 3.9] yields the existence of constants c_1, $c_2 \in \mathbb{R}_{>0}$ such that

$$c_1\|x\|^2 \leq x^T P x \leq c_2 \|x\|^2 \qquad \forall\, x \in \mathbb{R}^2.$$

Since the spectrum $\sigma(P)$ of the linear map $P : \mathbb{R}^2 \to \mathbb{R}^2$ consists of the two eigenvalues $\underline{\lambda} \approx 2.841$ and $\bar{\lambda} \approx 9.664$, the constants $c_1 := 5/2 \leq \underline{\lambda}$ and $c_2 := 10 \geq \bar{\lambda}$ may be chosen. As a consequence, defining $\alpha_1(r) = c_1 r$, $\alpha_2(r) = c_2 r$ guarantees the validity of

$$\alpha_1(\|x\|) = c_1 x^T x \leq x^T P x = V(x) \leq c_2 x^T x = \alpha_2(\|x\|), \tag{1.16}$$

i.e. the first chain of inequalities needed in Definition 1.15. Setting $W(r) = \alpha_2^{-1}(r) = c_2^{-1} r$ ensures the second inequality:

$$\begin{aligned} V(x) - W(V(x)) &= x^T P x - \alpha_2^{-1}(V(x)) \\ &\geq x^T P x - \alpha_2^{-1}(\alpha_2(\|x\|)) \\ &= x^T P x - x^T Q x \\ &\geq x^T P x - \ell(x, F_\infty(x)) \overset{(1.8)}{=} V(f(x, F_\infty)), \end{aligned}$$

which is exactly the desired inequality for the dynamical system defined by the corresponding closed loop. Hence, asymptotic stability of the feedback loop with $F_\infty(\cdot)$ is ensured by Theorem 1.16.

In order to further illustrate the introduced concepts, Inequality (1.14) is shown. Using the monotonicity of $\alpha_1(\cdot)$ and (1.16), we obtain

$$\|x(n; x_0)\| \leq \alpha_1^{-1}(V(x(n; x_0))) \;\leq\; \alpha_1^{-1}(V(x(n-1; x_0)) - W(V(x(n-1; x_0))))$$

$$
\begin{aligned}
&= c_1^{-1}(1-c_2^{-1})V(x(n-1;x_0)) \\
&\leq c_1^{-1}(1-c_2^{-1})^n V(x_0) \\
&\leq c_1^{-1}(1-c_2^{-1})^n \alpha_2(\|x_0\|) = c_1^{-1}c_2(1-c_2^{-1})^n\|x_0\|,
\end{aligned}
$$

i.e. (1.14) *with* $\beta(r,n) := C\sigma^n r$, $C := c_1^{-1}c_2$, $\sigma := 1 - c_2^{-1} \in (0,1)$. $\beta(\cdot,\cdot)$ *is of class* $\mathcal{KL}$ *and ensures exponential stability which implies asymptotic stability, cf. Example 1.13.*

The $\mathcal{KL}_0$*-function* $\beta(\cdot,\cdot)$ *represents upper bounds on the distance of the trajectory* $(x(n;x_0))_{n\in\mathbb{N}_0}$ *from the set point* $x^\star$. *The deduced estimate ensures asymptotic stability but provides conservative bounds. Since we are going to use the derived* $\mathcal{KL}$*-function* $\beta(\cdot,\cdot)$ *in the ensuing chapters, a second, tighter estimate is deduced. To this end, note that* $\|x(n;x_0)\| = \|(A+BF_\infty)^n x_0\|$ *holds. The eigenvalues of the matrix* $(A+BF_\infty)$ *are* $a \pm ib$ *with* $a \approx 0.34944$ *and* $b \approx 0.37519$. *Hence, a matrix* Q *exists which represents a change of coordinates transforming the matrix* $(A+BF_\infty)$ *to its Jordan canonical form, i.e.*

$$
Q^{-1}(A+BF_\infty)Q = J = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}
$$

with transformation matrix

$$
Q^{-1} \approx \begin{pmatrix} 1 & 0 \\ 1.7339 & 2.9318 \end{pmatrix} \qquad \text{and} \qquad Q \approx \begin{pmatrix} 1 & 0 \\ -0.59141 & 0.34108 \end{pmatrix}.
$$

Since we are interested in a representation consisting of entries which are real numbers and, thus, $Q \in \mathbb{R}^{2\times 2}$, *a similarity transformation is performed. The following property of the matrix* J *can be observed:*

$$
\|Jx\| = \left\| \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = \sqrt{a^2+b^2}\|x\|.
$$

In addition, $\|Q\| \approx 1.1755$ *and* $\|Q^{-1}\| \approx 3.4464$ *hold. As a consequence, the inequality*

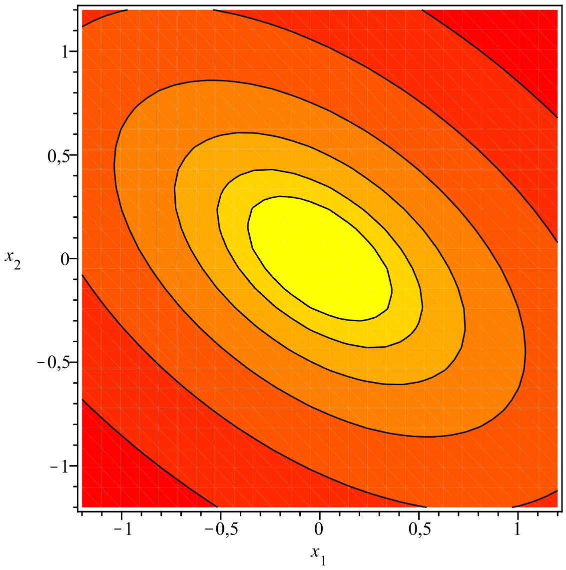
$$
\begin{aligned}
\|x(n;x_0)\| &= \|QQ^{-1}(A+BF_\infty)^n x_0\| \\
&= \|QQ^{-1}(A+BF_\infty)QQ^{-1}(A+BF_\infty)^{n-1}QQ^{-1}x_0\| \\
&\leq \|Q\|\|J^n Q^{-1}x_0\| \leq \|Q\|(\sqrt{a^2+b^2})^n\|Q^{-1}\|\|x_0\|
\end{aligned}
$$

holds. Hence, $\beta(r,n) = C\sigma^n r$ *with* $C = \|Q\|\|Q^{-1}\| \approx 4.0512432$ *and* $\sigma = \sqrt{a^2+b^2} \approx 0.51271945$ *can be employed in order to characterize the stability behavior of the considered closed loop, cf. Figure 1.3.*

Since measuring the state $x(\cdot)$ is, in general, not possible, an output $y(\cdot)$ is commonly incorporated in mathematical system theory, e.g. $y(n) = Cx(n) + Du(n)$ with $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$ for linear control systems with $X \subseteq \mathbb{R}^n$. However, in this thesis observability of the whole state is assumed, e.g. $C = Id$, $D = 0$ for the inverted pendulum on a cart example.

1.3 Sampled-Data Systems

In this section sampled-data systems are introduced. These discrete time systems represent continuous time control systems governed by differential equations in our discrete time framework. Here, the inverted pendulum on a cart serves as an example.



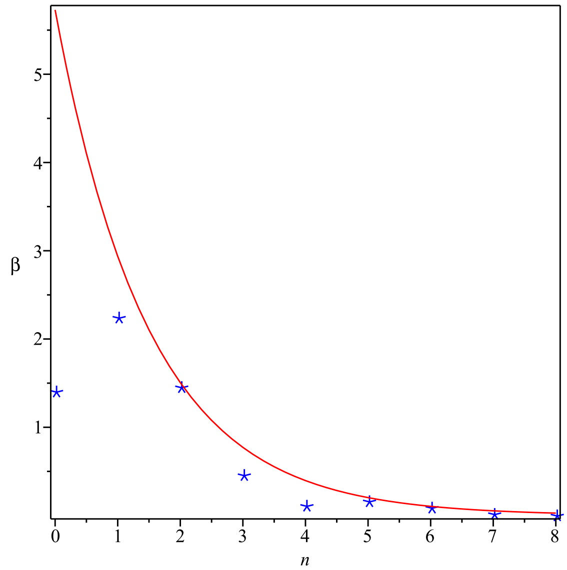


Figure 1.3: Graphical illustration of the optimal value function $x^T Px$ with level set $\mathcal{N}_i = \{x \in \mathbb{X} : V(x) = x^T Px \leq 2^i\}$, $i \in \{-1, 0, 1, 2, 3, 4\}$ on the left. On the right, $\|x(n; x_0)\|$ and $\beta(\|x_0\|, n)$ are depicted for initial value $x_0 = (1\ 1)^T$.

Let a Banach space X and a metric space W be given. Then, control systems generated by controlled — finite or infinite dimensional — differential equations of type

$$\dot{x}(t) = \frac{d}{dt}x(t) = g(x(t), \tilde{u}(t)) \tag{1.17}$$

are considered. State and control constraints are modeled by $\mathbb{X} \subseteq X$ and $\mathbb{W} \subseteq W$, respectively. The function $\tilde{u}(\cdot) : \mathbb{R}_0^+ \to W$ plays the role of an input.

Our goal is to rewrite this continuous time control system as a discrete time one according to Definition 1.2. To this end, let a sampling period $T > 0$ be fixed. Then, the time domain $\mathbb{T}$ is set to $T\mathbb{N}_0$. Furthermore, note that the solution $\Phi(T; x_0, \tilde{u}(\cdot))$ at time T of the differential equation (1.17) is determined by its initial state x_0 and a control function $\tilde{u}(\cdot) \in \mathcal{L}^1([0, t), W)$ — provided that a unique solution exists on the interval $[0, T]$ which is assumed in this thesis.[2] Suitable conditions to ensure existence and uniqueness depend on the system dynamics (1.17), e.g. (local) Lipschitz continuity with respect to the state for ordinary differential equations, cf. [115, Appendix C.3]. Defining $U := \mathcal{L}^1([0, T), W)$ and, consequently, $\mathbb{U} := \mathcal{L}^1([0, T), \mathbb{W})$, a discrete time transition map $f : X \times U \to X$ can be defined by

$$f(x_0, u) := \Phi(T; x_0, \tilde{u}(\cdot)) \qquad \text{with } \tilde{u}(\cdot)|_{[0,T)} := u. \tag{1.18}$$

Then, the successor state $x_1 = f(x_0, u)$ equals the one of the continuous time system at time T. A discrete time control value $u \in U$ represents a control function on the interval $[0, T)$. Since the Lebesgue integral is used, cf. [93], the value of the control function $\tilde{u}(\cdot)$ at time T does not play a role for the solution $\Phi(\cdot; x_0, u) = \Phi(\cdot; x_0, \tilde{u}(\cdot))$ at time T. The obtained discrete time system is called sampled-data system and yields snapshots of the corresponding continuous time system at the sampling instants $0, T, 2T, \ldots$. We emphasize that allowing an arbitrary metric space for the set of control values U is essential for this

[2]For a definition of $\mathcal{L}^p$ spaces, $1 \leq p \leq \infty$, we refer to [61, chapter I.4].

construction. In addition, since the state space X may be a Banach space, this setting is applicable to ordinary and partial differential equations. Summarizing, a suitably chosen sampled-data discrete time control system can be considered instead of its continuous time counterpart.

We did not give a precise definition of a continuous time control system.[3] Nevertheless, we stress the fact that existence of a solution $\Phi(\cdot; x_0, \tilde{u}(\cdot))$ at time T has to be verified for systems governed by differential equations of type (1.17) and, thus, their discrete time counterparts.

Remark 1.18
Assumption 1.4 guarantees, for each $x \in \mathbb{X}$, existence of a feasible input value such that the successor state is contained in the feasible set $\mathbb{X}$. This implies, in particular, that the induced solution emanating from $x_0 \in \mathbb{X}$ exists at the next time instant. Iterating this argument ensures existence of the state trajectory for all times $n \in \mathbb{N}_0$. For systems governed by differential equations and the respective continuous time control systems a so called non-triviality condition is typically assumed, cf. [115]. This condition merely ensures the existence of a sequence of times $(t_i)_{i\in\mathbb{N}_0} \subset \mathbb{R}_{>0}$ and a corresponding sequence of input functions $(\tilde{u}_i(\cdot))_{i\in\mathbb{N}_0}$, $\tilde{u}_i(\cdot) \in \mathcal{L}_1([t_i, t_{i+1}), W)$ such that $\Phi(\sum_{i=0}^{n} t_i; x_0, \tilde{u}(\cdot))$ exists for each $n \in \mathbb{N}_0$. Here, $\tilde{u}(\cdot)$ stands for the concatenation of the control functions $\tilde{u}_0[0, t_0), \tilde{u}_1[t_0, t_1), \ldots, \tilde{u}_n[t_{n-1}, t_n)$. However, e.g. finite escape times are not excluded, cf. the supplementary Section A.1. Hence, existence of a sequence of time instants satisfying $t_i \to \infty$ for i tending to infinity cannot be concluded and, thus, existence of solutions is an issue. One remedy for this problem is indicated in Sections 4.4 and A.1.

In order to illustrate how to deal with a continuous time control system in our discrete time setting, the inverted pendulum, which was introduced in Section 1.1, is considered. In Section A.2 the system dynamics of this example are deduced, which shows that the inverted pendulum example is governed by an ordinary differential equation of type (1.17).

Example 1.19 (Continuous time system)
The inverted pendulum on a cart is considered, cf. Section A.2. Let a discretization parameter T be given and the vector field $g : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ be defined by

$$g(x, u) := \begin{pmatrix} x_2 \\ \frac{1}{M(x_3)} \left[(J + ml^2) \left(\beta u - c x_2 - m l x_4^2 \sin x_3 \right) - ml \cos x_3 \left(c_P x_4 - mgl \sin x_3 \right) \right] \\ x_4 \\ \frac{1}{M(x_3)} \left[ml \cos x_3 \left(\beta u - c x_2 - m l x_4^2 \sin x_3 \right) - (M + m) \left(c_P x_4 - mgl \sin x_3 \right) \right] \end{pmatrix} \tag{1.19}$$

with $M(x_3) = (M + m)J + Mml^2 + m^2 l^2 \sin^2 x_3^2$, cf. (A.7). *We refer to [76] for a definition of vector fields. Existence and well-posedness of a solution of the corresponding differential equation follow from [115, chapter C]. Defining the state space $X := \mathbb{R}^4$ and $W := \mathbb{R}$, the system dynamics* (1.17) *are determined by $g : X \times W \to X$. Then, setting $U := \mathcal{L}^1([0, T), W)$ the discrete time dynamics are, for control input $u \in U$, given by* (1.18).

The inverted pendulum on a cart has two equilibria, the downward and the upright position. Here, our goal is to stabilize the pendulum at its unstable position, i.e. at the

[3]A definition can be found in [115].

upright position, which is located at the origin in our model. Possible choices for the stage costs $\ell : X \times U \to \mathbb{R}_0^+$ *are*

$$\ell(x,u) = T\|x\|^2 + \lambda \int_0^T u(t)^2 \, dt \qquad \text{and} \qquad \ell(x,u) = \int_0^T \|\Phi(t;x,u)\|^2 \, dt + \lambda \int_0^T u(t)^2 \, dt.$$

Both stage costs $\ell(\cdot,\cdot)$ presented in Example 1.19 penalize the control effort. However, the first choice only takes the states at the time instants $0, T, 2T, \ldots$ into account, whereas the second also accommodates the intersampling behavior of the system, cf. [56]. On the other hand, evaluating the norm of the trajectory on the interval $[0,T)$ causes, in general, additional numerical effort in contrast to the first variant. For sufficiently small sampling periods T, picking one or the other of these optimization criteria is mainly a matter of taste in view of continuity properties of the respective solutions. For details, the reader is referred to [115, Chapter C].

Suppose that, for given initial state $x_0 \in \mathbb{X} \subseteq X$, a sequence of control values $(u^\star_{x_0}(n))_{n\in\mathbb{N}_0} \subset \mathcal{L}^1([0,T),W)$ satisfying (1.6) is computed analogously to Section 1.2. Then, a control function $\tilde{u}(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_0^+, W)$ can be constructed by concatenating this sequence $u^\star_{x_0}(\cdot)$ which may be interpreted as the continuous time counterpart of our discrete time sequence. In order to generate a closed loop solution, the first portion $\tilde{u}(\cdot)|_{[0,T)} = u^\star_{x_0}(0)(\cdot)$ of this control function which, in combination with the initial value x_0, uniquely determines the next state

$$x_1 = x_{u^\star_{x_0}}(1;x_0) = f(x_0, u^\star_{x_0}(0)) = \Phi(T; x_0, u^\star_{x_0}(0)(\cdot)) = \Phi(T; x_0, \tilde{u}(\cdot)|_{[0,T)})$$

has to be implemented at the considered plant. However, since this has to be done by a digital computer, a control input $u \in \mathcal{L}^1([0,T),W)$ cannot be used in general — even if u is admissible. Hence, the class of feasible input signals has to be adjusted in order to allow for an implementation at the respective plant. In order to keep the presentation technically simple, we restrict ourselves to piecewise constant control functions. Hence, the control input can change its value only at the sampling instants, cf. [115, Subsections 1.3 and 2.10].

Definition 1.20 (Sampled-data system with zero order hold)
Let X denote a Banach space and U a metric space. Let a controlled differential equation of type (1.17) *and a sampling period $T \in \mathbb{R}_{>0}$ be given. Then, the sampled-data system $\Sigma_{[T]}$ with zero order hold is the discrete time system $\Sigma_{[T]} = (T\mathbb{N}_0, X, U, f)$ with transition map $f : X \times U \to X$ defined by*

$$f(x,u) = \Phi(T; x, \tilde{u}(\cdot)) \qquad \text{with } \tilde{u}(t) = u \in U \text{ for all } t \in [0,T).$$

Furthermore, let $\mathbb{X} \subseteq X$ and $\mathbb{U} \subseteq U$ be given. Then, the sequence of control values $(u(n))_{n\in\{0,1,\ldots,N-1\}} \in U^N$ is admissible for $x_0 \in \mathbb{X}$ if $(u(n))_{n\in\{0,1,\ldots,N-1\}} \in \mathcal{U}^N(x_0)$ holds and the state trajectory $x(n)$ exists for $n \in \{1,2,\ldots,N\}$.

Note that admissibility of $(u(n))_{n\in\mathbb{N}_0}$ implies existence of the state trajectory for all positive times. The discrete time n corresponds to the continuous time $t = nT$, cf. Remark 1.18. Detailed introductions in sampled-data systems can be found in [1,18,27]. Zero order hold implementations of sampled-data systems are widespread in applications. In order to compute the solution $\Phi(T;x,u)$ for a nonlinear ordinary differential equation, numerical methods, e.g. Runge-Kutta methods, may be employed, cf. [13]. More sophisticated methods like higher order explicit or implicit Runge-Kutta schemes including step-size control may also be used, cf. [54, 55]. We emphasize that restricting the applied control functions on a sampling interval to piecewise constant ones may shrink the feasible region — in particular for large sampling periods. However, since sampled-data systems typically inherit some continuity properties, sufficiently fast sampling often resolves this problem.

Modelling errors may cause deviations of the considered control system from the actual behavior of the plant. But even in the unrealistic scenario that no computation or modelling errors occur, the necessity to approximate computed input signals, e.g. by zero order hold with sufficiently small sampling periods, leads to further perturbations, which are not negligible from a practitioner's point of view. In order to cope with such perturbations, closed loop control is preferable in comparison to its open loop counterpart.

In the following example the impact of implementing a sampled-data system in a zero order hold fashion is investigated. In order to separate errors induced by using a zero order hold control from those resulting from numerical computations, Example 1.19 is linearized at the origin, i.e. the desired equilibrium. This enables us to solve the resulting differential equation exactly and, thus, to exclude numerical effects. In addition, the analysis is simplified by considering the linearized version. Note that the equilibrium located at the downward position is removed by this linearization. However, we are only interested in the upright equilibrium.

We start by stating the linearized version of the inverted pendulum on a cart example.

Example 1.21 (Linearized inverted pendulum)
The inverted pendulum on a cart from Example 1.19 is linearized at the origin $0_{\mathbb{R}^4}$ with respect to the angle $\varphi = -x_3$. This leads to the linear differential equation

$$\dot{x}(t) = \frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = Ax(t) + bu(t) \qquad \text{with} \qquad b := \begin{pmatrix} 0 \\ (J+ml^2)\beta \\ 0 \\ ml\beta \end{pmatrix} \in \mathbb{R}^4, \quad (1.20)$$

matrix $A \in \mathbb{R}^{4\times 4}$,

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -M_0^{-1}(J+ml^2)c & M_0^{-1}m^2l^2g & -M_0^{-1}mlc_P \\ 0 & 0 & 0 & 1 \\ 0 & -M_0^{-1}mlc & M_0^{-1}(M+m)mgl & -M_0^{-1}(M+m)c_P \end{pmatrix},$$

and constant $M_0 := (M+m)J + ml^2M$, cf. [23, Sections 2 and 3].[4] The solution of this ordinary differential equation is given by

$$\Phi(t; x_0, u(\cdot)) = e^{At}x_0 + \int_0^t e^{A(t-s)}bu(s)\, ds, \quad (1.21)$$

cf. [22, chapter 2]. We aim at representing this ordinary differential equation as a sampled-data system. To this end, a discretization parameter $T \in \mathbb{R}_{>0}$ is chosen. Then, proceeding analogously to Example 1.19 yields a transition map $f : X \times U \to X$ on $\mathbb{T} = T\mathbb{N}_0$ with $X = \mathbb{R}^4$, $U = \mathcal{L}^1([0,T), \mathbb{R})$.

The linearized inverted pendulum describes its nonlinear counterpart very accurately at the equilibrium. Although the model is restricted to a plane, it is an appropriate

[4] For sufficiently small values φ the approximations $\sin(x_3) \approx x_3$, $\cos(x_3) \approx 1$, $\sin^2(x_3) \approx 0$, and $\dot{x_3}^2 \sin(x_3) \approx 0$ are used in order to linearize the system at the origin.

model for small angular deviations because the dynamics can be treated separately for each coordinate direction, cf. [23, pp. 9 – 10]. Hence, we focus on small angular deviations, e.g. $|\varphi|$ is not permitted to exceed one degree of arc, cf. [23, p.17].

Example 1.22 (Constraints for Example 1.21)
In order to take into account that the angle is restricted to small values, the state constraint $\|x_3(t)\| < c$ for a sufficiently small constant $c \in \mathbb{R}_{>0}$ may be imposed on $x(t)$. Hence, $\mathbb{X} := \{x \in X : \|x_3\| < c\}$ is chosen as the set of feasible states. Assuming an unbounded set of feasible controls, e.g. setting $\mathbb{W} := W = \mathbb{R}$, allows to arbitrarily influence the angular velocity $x_4(\cdot)$. Hence, for each $x_0 \in X$, a control value $u_{x_0} \in \mathbb{U} := \mathcal{L}^1([0,T), \mathbb{W})$ exists such that $f(x_0, u_{x_0}) = \Phi(T; x, u_{x_0}) \in \mathbb{X}$ holds which ensures that the imposed state constraints can be satisfied. Of course, this cannot be done in the considered practical application, i.e. the set of admissible control values $\mathbb{W}$ will be confined to some real interval, $[a,b]$, $a < 0 < b$. However, since the initial value of the angular velocity is located in a small neighborhood of the origin, the system can be steered such that the original state constraint $\|x_3(\cdot)\| < c$ is satisfied and, in addition, the angular velocity $x_4(\cdot)$ remains sufficiently small. This allows for ensuring feasibility of the system by choosing the control input appropriately.

The example of the inverted pendulum on a cart is considered once more in order to investigate the impact of using a zero order hold feedback.

Example 1.23 (Example 1.21 continued)
Let a sampling period $T > 0$ as well as parameters $c_P = 0$, $m = 1$, $l = 1$, $g = 9.81$, $M = 0$, $J = 2$, $c = 1/10$, and $\beta = 0.5$ be given. The sampled-data system with zero order hold of the linearized inverted pendulum on a cart model is considered, cf. Example 1.21 for a linearized version of the nonlinear pendulum model of Example 1.19. Since a zero order hold feedback is assumed, the control value may change only at the time instants $0, T, 2T, \ldots$. Hence, the constant control function $\tilde{u}(\cdot) \equiv u$ is identified with the corresponding control value u. Then, the following linear system dynamics are obtained

$$x(n+1) = \Phi(T; x(n), u(n)) \overset{(1.21)}{=} e^{AT} x(n) + u(n) \int_0^T e^{As}\, ds. \tag{1.22}$$

A feedback control $u(n) = Fx(n)$ is used in order to obtain a closed loop system. Note that F is a linear map represented by a matrix. Hence, $Fx(n)$ is written instead of $F(x(n))$. Plugging this into (1.22) yields the closed loop

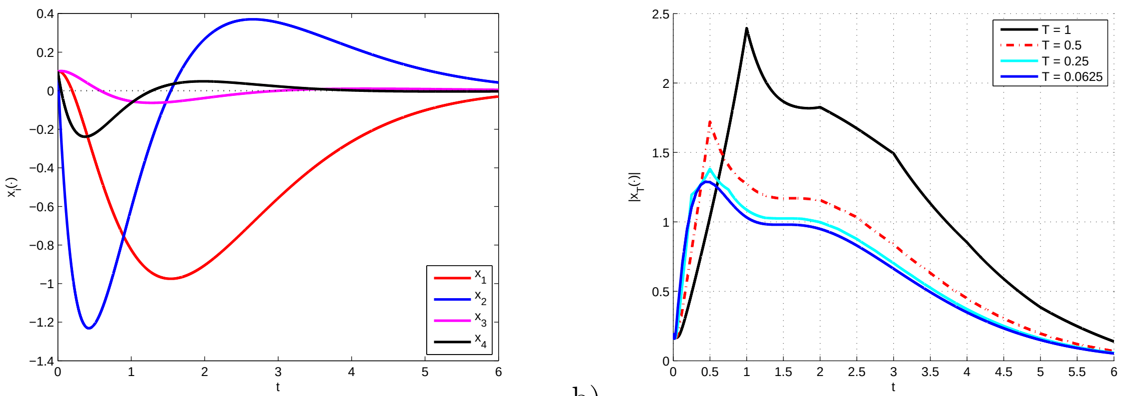
$$x(n+1) = e^{AT} x(n) + \int_0^T e^{As}\, ds \cdot Fx(n) = \left(e^{AT} + \int_0^T e^{As}\, dsF \right) x(n).$$

Furthermore, let the following stage costs $\ell : X \times U \to \mathbb{R}_0^+$ be given, cf. Example 1.19:

$$\ell(x,u) := T(x^T Q x + u^T R u) = T(x^T x + u^T u) = T\|x\|^2 + T\|u\|^2.$$

The cost functional is given by $J_\infty(x_0, u(\cdot)) = \sum_{n=0}^{\infty} \ell(x_u(n; x_0), u(n))$. Incorporating the sampling period T in the stage costs, allows for a comparison of the resulting closed loops in dependence on the sampling period T because $J_\infty(x_0, u(\cdot))$ approximates the integral $\int_0^\infty x(t; x_0, u(t))^T Q x(t; x_0, u(t)) + u(t)^T R u(t)\, dt$. Since using a smaller sampling period implies the possibility of changing the control value more often, a decrease of the cost functional is expected for smaller sampling periods.

We point out that constraints are not considered in this example which allows to employ the matlab *routine* dlqr *in order to solve the corresponding minimization problem. The abbreviation* dlqr *stands for discrete linear quadratic regulator. This* matlab *routine provides, in addition to a matrix P satisfying $V_\infty(x_0) = x_0^T P x_0$, also a feedback matrix F. Note that P as well as the feedback law represented by the matrix F depend on the sampling period T. For our numerical computations, the initial value $x_0 = \frac{1}{10}(1\ 1\ 1\ 1)^T$ is picked. Trajectories for the very small sampling period $T = 2^{-16}$ are drawn in Figure 1.4. For $T = 2^{-i}$, $i \in \{1, 2, \ldots, 16\}$, convergence to the desired equilibrium, i.e. $x_T(n; x_0) \to x^\star = 0$ is observed. In order to illustrate this, the norm of the solution on the interval $[0, 6]$ is computed for different sampling periods T, cf. Figure 1.4 b).*



a) b)

Figure 1.4: In a) a graphical illustration of the dynamical behavior of the linearized inverted pendulum on a cart is given ($T = 2^{-16}$). In b), the norm of sampled-data systems with zero order hold is illustrated for different sampling periods T.

Using smaller T and, thus, evaluating the feedback law more often leads to an improved behavior. This observation is substantiated by the forth and fifth column of Table 1.1 in which the Euclidean and the infinity norm are computed at $t = 6$. For sampling periods $T \leq 2^{-6}$, the impact of zero order hold seems to be negligible, cf. Table 1.1. In the second column of Table 1.1, the optimal value function $V_\infty^T(x_0)$ is approximated. The optimal value function grows for increasing sampling period T. Choosing T too large leads to a deteriorate dynamical behavior of the resulting closed loop.

As seen in Example 1.17, the eigenvalues of the closed loop transition map have to be determined in order to find suitable parameters $C \geq 1$ (overshoot) and $\sigma \in (0, 1)$ (decay rate) of a $\mathcal{KL}$-function $\beta(\cdot, \cdot)$ which enables us to show asymptotic stability of the closed loop. However, in order to assess the solutions based on the different sampling periods T, only comparing the eigenvalues which are attributed to the respective closed loop is insufficient. Instead, the eigenvalue is taken to the (T^{-1})-th power, e.g λ^4 for $T = 0.25$, cf. the third column of Table 1.1. This scaling of the eigenvalues and, thus, the corresponding decay rates leads to a measure for the decrease after one time unit, i.e. T^{-1} times the sampling period T. Constants for the overshoot bound may be computed analogously to Example 1.10.

Using the very small value $T = 2^{-16}$ allows to generate results which can be interpreted as a reference solution which is not affected by the zero order hold implementation. For this sampling period, Figure 1.5 shows level sets of the optimal value function $V_\infty(x) = x^T P x$ at $x_3 = x_4 = 0$ on the left and $x_1 = x_2 = 0$ on the right. Taking the range of values

Sampling period T	$x_0^T P x_0$	$\Vert\lambda(A+BF)\Vert^{1/T}$	$\Vert x_T(\cdot)\vert_{t=6}\Vert_2$	$\Vert x_T(\cdot)\vert_{t=6}\Vert_\infty$
1.0000000000000000	19.67513368	0.44677011	0.13877609	0.10871640
0.5000000000000000	10.21853967	0.43498765	0.06912694	0.05567655
0.2500000000000000	8.34728912	0.43168006	0.05627710	0.04564705
0.1250000000000000	7.90776081	0.43082527	0.05331032	0.04331718
0.0625000000000000	7.79907101	0.43060972	0.05258338	0.04274543
0.0312500000000000	7.77169915	0.43055572	0.05240256	0.04260316
0.0156250000000000	7.76470700	0.43054221	0.05235741	0.04256763
0.0078125000000000	7.76288128	0.43053883	0.05234612	0.04255875
0.0039062500000000	7.76238581	0.43053799	0.05234330	0.04255653
0.0019531250000000	7.76224241	0.43053778	0.05234260	0.04255598
0.0009765625000000	7.76219680	0.43053773	0.05234242	0.04255584
0.0004882812500000	7.76218051	0.43053771	0.05234238	0.04255581
0.0002441406250000	7.76217400	0.43053771	0.05234237	0.04255580
0.0001220703125000	7.76217115	0.43053771	0.05234236	0.04255580
0.0000610351562500	7.76216983	0.43053771	0.05234236	0.04255579
0.0000305175781250	7.76216919	0.43053771	0.05234236	0.04255579
0.0000152587890625	7.76216888	0.43053771	0.05234236	0.04255579

Table 1.1: Numerical results for the linearized inverted pendulum on a cart in dependence on the sampling period T for initial value $x_0 = \frac{1}{10}(1\ 1\ 1\ 1)^T$.

into account shows that the optimal value function is much more sensitive with respect to changes in the angle and its velocity (x_3 and x_4) than in the position of the cart and its velocity (x_1 and x_2). Both plots indicate that the chosen initial value with solely positive values makes the stabilization problem more difficult.

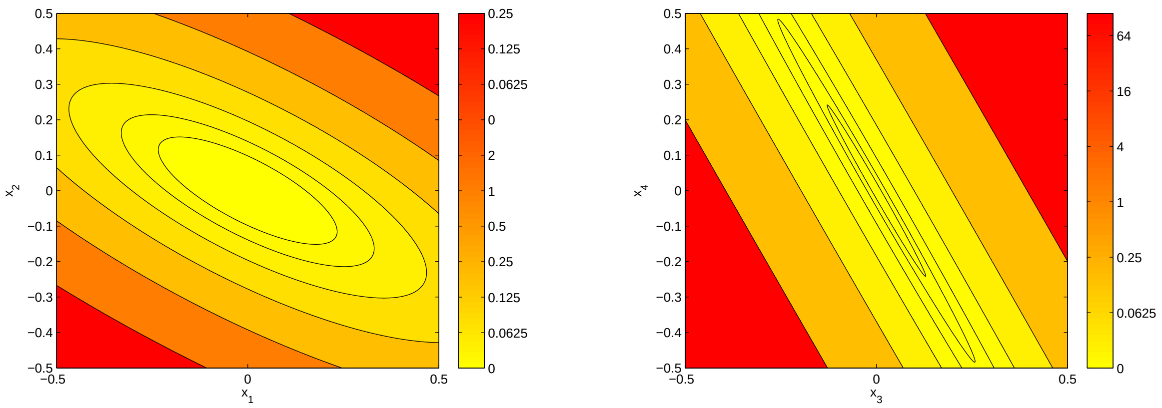


Figure 1.5: Level sets of the optimal value function $V_\infty(\cdot)$ for the linearized inverted pendulum on a cart example for sampling period $T = 2^{-16}$: on the left the third and forth component of x_0 are set to zero, i.e. $V_\infty^T(x_0)$ for $x_{0,3} = x_{0,4} = 0$ is depicted. On the right, the same is repeated for $x_{0,1} = x_{0,2} = 0$.

In order to ensure that the dynamical behavior of a sampled-data system with zero order hold converges to the one of the continuous time system, sufficiently fast sampling and, thus, small sampling periods T are required, cf. [91]. Furthermore, it is possible to allow for more than one control value per sampling interval, i.e. multirate sampling, cf. [63,80]. That means the current state is measured, then a sequence of, let's say m, control values is computed and applied on the sampling interval, i.e. the first is implemented on the interval $[0, T/m)$, the second on $[T/m, 2T/m)$ and so forth. Hence, m (possibly different) control values are implemented during one sampling period.

1.4 Networked Systems and Multistep Feedback

In this section a networked control setting is introduced in order to motivate the definition of a multistep feedback. This type of a static state feedback will turn out to be helpful also for other applications, cf. Chapter 4.

Due to lower implementation costs and greater interoperability networked control systems (NCS) are increasingly used, particularly in the automotive and aeronautical industries, cf. [47]. The situation of a networked control system shown in Figure 1.6 is considered. The controller uses a network channel at every time instant $n \in \mathbb{N}$ in order to transmit the feedback control value $u(n) = \mu(x(n))$ to the actuator. Since, in contrast to, e.g. [118,122], no particular protocol like round–robin (RR) or try–once–discard (TOD) is assumed, a packet either arrives unperturbed and with negligible delay over the channel or is treated as a dropout. A dropout means that the control value sent by the controller does not arrive at the actuator.

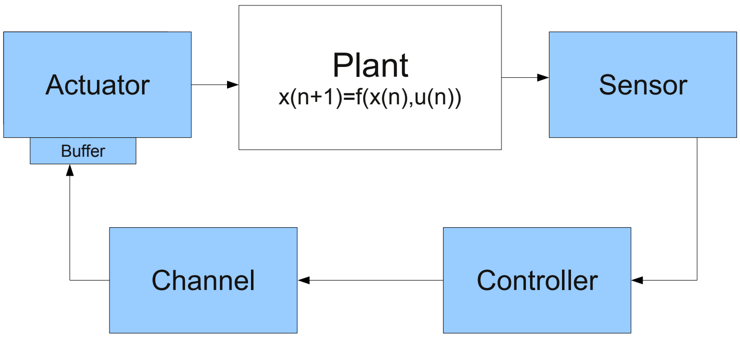


Figure 1.6: Scheme of the considered networked control system. The communication between the controller and the actuator is carried out via a channel. Integrating this additional element in the control loop may lead to packet dropouts as well as delays.

In order to compensate for dropouts, we add a buffer device in the actuator and adapt the controller design: at each time instant n, instead of a single control value $u(n) = \mu(x(n)) \in U$, a sequence $\mu(x(n), 0), \mu(x(n), 1), \ldots, \mu(x(n), m^\star - 1)$ of control values is sent. In the actuator, the elements of this sequence are buffered and used until the next sequence arrives.

In the ideal case when no packet dropouts occur, the actuator applies the control sequence

$$\mu(x(n), 0), \mu(x(n+1), 0), \mu(x(n+2), 0), \mu(x(n+3), 0), \ldots.$$

If, however, transmission is successful at, e.g. time n and $n+3$ but fails at time $n+1$ and $n+2$, the actuator applies

$$\mu(x(n), 0), \mu(x(n), 1), \mu(x(n), 2), \mu(x(n+3), 0), \ldots.$$

In order to formalize this idea, we define a sequence $(m_i)_{i\in\mathbb{N}_0}$ of *control horizons*, which counts the time instants between the i-th and the $(i+1)$-st successful transmission. For these sequences the following definitions are introduced.

Definition 1.24
Let a set $M \subseteq \{1,2,\ldots,m^\star\}$, $m^\star \in \mathbb{N}$, be given. A sequence of control horizons $(m_i)_{i\in\mathbb{N}_0}$ is said to be admissible *if $m_i \in M$ holds for all $i \in \mathbb{N}_0$. For $k,n \in \mathbb{N}_0$, the following expressions are defined:*

$$\begin{aligned}\sigma(k) &:= \sum_{j=0}^{k-1} m_i \quad \textit{(using the convention } \textstyle\sum_{j=0}^{-1} = 0\textit{)},\\ \varphi(n) &:= \max\{\sigma(k)\,|\,k\in\mathbb{N}_0, \sigma(k)\leq n\}.\end{aligned}$$

Here $\sigma(k)$ denotes the k-th successful transmission time while $\varphi(n)$ denotes the largest successful transmission time before or at time instant n. Note that by convention, time $n=0$ coincides with the first successful transmission.

Using this notation, the control sequence applied by the actuator can be expressed as

$$\mu(x(\sigma(k)),0),\ldots,\mu(x(\sigma(k)),m_k-1),\mu(x(\sigma(k+1)),0),\ldots$$

in which m_k is unknown at time $\sigma(k)$. Note that this notation is a posteriori and only used in order to analyze the resulting scheme afterward. Although the control loop is not closed at each sampling instant, measurements are used in order to update the sequence of control values which allows to react to disturbances or modelling errors. Nevertheless, in the networked control setting, we aim at closing the loop as often as possible in order to robustify the closed loop behavior of the considered system. Hence, more than the first element of the open loop sequence of control values is only implemented if a packet dropout occurs. Using the precomputed sequence of control values should be favorable in comparison to using a default control value.

In order to be consistent with the scheme introduced above, the term feedback control is used in the following general sense.

Definition 1.25
Let $m^\star \in \mathbb{N}$ and $M \subseteq \{1,2,\ldots,m^\star\}$ be given. A multistep feedback law is a map $\mu : \mathbb{X}\times\{0,\ldots,m^\star-1\}\to\mathbb{U}$ which, for an admissible control horizon sequence $(m_i)_{i\in\mathbb{N}_0}\subset M$, is applied according to the rule $x_\mu(0)=x_0$,

$$x_\mu(n+1) = f(x_\mu(n),\mu(x_\mu(\varphi(n)),n-\varphi(n))). \tag{1.23}$$

For details about this setting we refer to [48]. We point out that the concept of multistep feedbacks will turn out to be beneficial also in a setting without delays and packet dropouts in order to enhance stability properties of closed loop controlled systems, cf. Section 4.4.

Chapter 2

Receding Horizon Control

In this chapter we present the main idea of receding horizon control (RHC) which is also called model predictive control (MPC).[1] Then, we discuss the stability analysis of receding horizon control schemes with terminal constraints and, if necessary, terminal costs. Furthermore, a feasibility proof from [90] for unconstrained RHC schemes is sketched. In this context, unconstrained means that neither terminal constraints nor terminal costs are added to the basic receding horizon setting.

2.1 Introduction

In the last chapter we dealt with the optimization problem

$$\min_{u(\cdot)\in\mathcal{U}^\infty(x_0)} \quad J_\infty(x_0;u(\cdot)) = \sum_{n=0}^{\infty} \ell(x_u(n;x_0),u(n)) \tag{2.1}$$

$$\text{subject to} \quad x_u(n+1;x_0) = f(x_u(n;x_0),u(n)) \text{ with } x_u(0;x_0) = x_0 \in \mathbb{X}, \tag{2.2}$$

$$\mathcal{U}^\infty(x_0) := \left\{ (u(n))_{n\in\mathbb{N}_0} \,\middle|\, \begin{array}{l} u(n) \in \mathbb{U} \\ x_u(n+1;x_0) \in \mathbb{X} \end{array} \text{ for all } n \in \mathbb{N}_0 \right\} \tag{2.3}$$

with the convention $V_\infty(x_0) := \inf_{u(\cdot)\in\mathcal{U}^\infty(x_0)} J_\infty(x_0;u(\cdot)) = \infty$ when either the optimal trajectory causes costs summing up to infinity or the set $\mathcal{U} = \mathcal{U}^\infty = \mathcal{U}^\infty(x_0)$ of admissible sequences of control values is empty, i.e. there does not exist a sequence $u(\cdot) = (u(n))_{n\in\mathbb{N}_0}$ of control values satisfying the control constraints $u(n) \in \mathbb{U}$, $n \in \mathbb{N}_0$, such that the state constraints $x_u(n+1;x_0) \in \mathbb{X}$ are maintained for all $n \in \mathbb{N}_0$. Since $V_\infty(x_0) = \infty$ characterizes the optimal control problem as not well-defined, the following assumption is made in order to exclude these scenarios from our analysis.

Assumption 2.1
Let $V_\infty(x_0) < \infty$ hold for each $x_0 \in \mathbb{X}$.

Assumption 2.1 implies, among others, the existence of a sequence of control values $u_{x_0}(\cdot) \in \mathcal{U}^\infty(x_0)$ such that $V_\infty(x_0) \leq J_\infty(x_0;u_{x_0}(\cdot)) < \infty$ holds. Since $u_{x_0}(\cdot) \in \mathcal{U}^\infty(x_0)$, the state constraints $x_{u_{x_0}}(n;x_0) \in \mathbb{X}$, $n \in \mathbb{N}_0$, are satisfied.

[1]The terms moving or rolling horizon can also be found in literature.

Summarizing, our goal is to solve the minimization problem (2.1) - (2.3), i.e. to minimize the cost functional subject to the system dynamics and the control and state constraints. However, solving problem (2.1) - (2.3) is, in general, intractable because its solution involves solving a Hamilton-Jacobi difference equation. In particular, this holds for systems whose dynamics are either nonlinear or defined on a space of infinite dimension. For example, control systems whose dynamics are governed by partial differential equations belong to the latter category. Hence, we aim at approximating the desired solution or, at least, solving the closely related stabilization problem, i.e. looking for a sequence of control values $u(\cdot) \in \mathcal{U}^\infty(x_0)$ which stabilizes the system at the equilibrium $x^\star$. To this end, the desired state has to be characterized appropriately by the stage costs $\ell(\cdot,\cdot)$, i.e. $\ell^\star(x) = \min_{u \in \mathcal{U}^1(x)} \ell(x,u) = 0$ if and only if $x = x^\star$, cf. (1.4). If the aforementioned task may be fulfilled by more than one sequence of control values, we pick some $u(\cdot) \in \mathcal{U}^\infty(x_0)$ which minimizes the cost functional $J(x_0;\cdot)$ or, at least, yields a performance which does not deviate too much from the optimal one. To be more precise, our objective is that the computed control $u(\cdot)$ induces costs $J_\infty(x_0;u(\cdot))$ which are bounded by the optimal costs $V_\infty(x_0)$ multiplied by a certain factor $1/\alpha$, i.e.

$$J_\infty(x_0;u(\cdot)) \leq \frac{1}{\alpha} \cdot V_\infty(x_0).$$

For example, $\alpha = 1/2$ means that the costs associated with the chosen control $u(\cdot)$ are at most twice as much as the optimal ones. The optimal value $V_\infty(x_0)$ coincides with the minimal costs. Nevertheless more than one control may exist which induces exactly this amount of costs. Here, we tacitly agree in picking one of these whenever we use the term optimal control. Furthermore, note that such a sequence of control values, for which the infimum in the problem formulation is attained, may not exist at all.

Before we tackle the raised questions, the basic ideas of receding horizon control, which represents a remedy in order to deal with the described problem setting, are presented. To this end, we consider the auxiliary problem with *optimization horizon* $N \in \mathbb{N}$:

$$\min_{u(\cdot)\in\mathcal{U}^N(x_0)} \quad J_N(x_0;u(\cdot)) = \sum_{n=0}^{N-1} \ell(x_u(n;x_0),u(n)) \tag{2.4}$$

$$\text{subject to} \quad x_u(n+1;x_0) = f(x_u(n;x_0),u(n)) \text{ with } x_u(0;x_0) = x_0 \in \mathbb{X}, \tag{2.5}$$

$$\mathcal{U}^N(x_0) := \left\{ (u(n))_{n\in\mathbb{N}_0} \;\middle|\; \begin{array}{l} u(n) \in \mathbb{U} \\ x_u(n+1;x_0) \in \mathbb{X} \end{array} \text{ for } 0 \leq n \leq N-1 \right\}. \tag{2.6}$$

The corresponding optimal value function is given by

$$V_N(x_0) = \inf_{u(\cdot)\in\mathcal{U}^N(x_0)} J_N(x_0;u(\cdot)). \tag{2.7}$$

Note that this problem differs from problem (2.1) - (2.3): the time horizon is truncated, i.e. the cost functional evaluates the stage costs only at the first N time instants. Moreover, the set $\mathcal{U}^N(x_0)$, which contains the control and state constraints, is adapted as well, e.g. the trajectory only has to be feasible until time N. In the next sections additional terminal constraints and costs are incorporated in this setting in order to ensure feasibility of the resulting receding horizon closed loop. In this subsection, however, we consider the conceptually simplest receding horizon approach imposing neither terminal costs nor terminal constraints which makes it easier to carve out some of the basic principles of moving horizon schemes. Furthermore, this scheme is predominant in industrial applications, cf. [100], which motivates its analysis in the following chapters. For questions concerning feasibility we refer to Section 2.4.

Already in the last chapter, we indicated how to obtain a closed loop system assuming being able to compute a sequence of control values which solves the original problem, i.e. satisfies $J_\infty(x_0; u(\cdot)) = V_\infty(x_0)$ in view of Assumption 1.9. Here, we proceed analogously with the problem posed on the truncated time horizon, i.e. we solve Problem (2.4) - (2.6) in order to obtain a sequence of N control values. This sequence may be extended by concatenation with the sequence which is identically zero. Note that the values $u(N), u(N+1), \ldots$ do not play a role for the problem on the truncated horizon. Problem (2.4) - (2.6) is, e.g. for dynamics governed by a nonlinear ordinary differential equation, a nonlinear optimal control problem. Using the introduced concept of sampled-data systems with zero order hold and, thus, discretizing the control function $u(\cdot)$ transforms this optimal control problem to an optimization problem which is comparatively easy to solve, cf. [29, 84, 119]. We point out that this approach involves a prediction of the future states $x_u(n; x_0)$, $n = 1, 2, \ldots, N$, which motivates the term "predictive" in model predictive control. In addition, the method is based on a model which is employed in order to predict the trajectory on the interval $[0, NT)$ in dependence on the control $u(\cdot)$. Next, we implement the first $m \in \{1, \ldots, N-1\}$ elements of the computed sequence. In order to streamline the presentation of the main idea, let us set $m = 1$, i.e. implementing only the first element of the sequence of control values which may be called "classical" MPC.

This situation is illustrated in Figure 2.1: x_0 denotes the current state of the state evolution which is induced by a sampled-data system, cf. Section 1.3. Hence, we use a (multistep) feedback control according to Definition 1.25, e.g. $\mu_N(0; x_0) = u(0)$ for $m = 1$, and implement this at the plant which yields the new initial state $x_0 := x_{\mu_N}(m; x_0)$ for the optimization problem (2.4) - (2.6). Note that $x_{\mu_N}(m; x_0)$ may differ from the predicted state $x_u(m; x_0)$, e.g. due to modelling errors. Then, the procedure is repeated, i.e. an optimization with respect to the optimization horizon N is carried out which, again, yields our receding horizon feedback, cf. Figure 2.2. This shifting of the optimization horizon forward in time explains the term "receding horizon".

Summarizing, we define a multistep feedback law $\mu_{N,m^\star}$ by picking the first m elements of the optimal control sequence based on the finite horizon optimal value function $V_N(x_0)$. This course of action is repeated after shifting the horizon. In order to formalize this concept, the following definition is given.

Definition 2.2
For $m \geq 1$ and $N \geq m+1$ a multistep MPC feedback law is defined by $\mu_{N,m}(x_0, n) = u^\star(n)$, where $u^\star(\cdot)$ is a minimizing control for problem (2.4) - (2.6) with initial value x_0. Although the open loop optimal control $u^\star(\cdot) = u^\star_N(\cdot; x_0)$ depends on the initial state x_0 and the optimization horizon N, the subscript N and the corresponding initial state x_0 are often not listed.

Using this feedback leads to a receding horizon closed loop. Note that the following is supposed.

Remark 2.3
We assume that there is no model plant mismatch and neglect disturbances. Hence, the actual state $x_{\mu_N}(m; x_0)$ coincides with the predicted state $x_u(m; x_0)$, cf. [26, p.12]. Supposing this, the resulting closed loop is investigated with respect to so called nominal stability. In

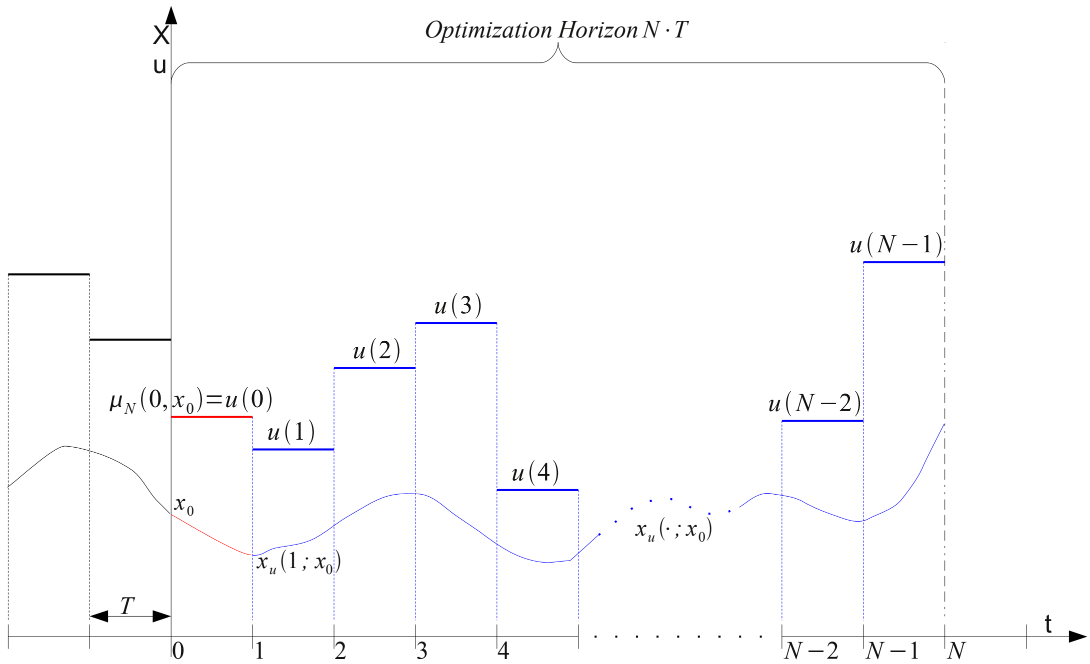


Figure 2.1: Graphical illustration of the main idea (I/II): the computed control values and the corresponding predicted trajectory are drawn in blue. In classical MPC the first control value is implemented as a feedback with respect to the initial state x_0. Hence, $x_{\mu_N}(1; x_0) = f(x_0, \mu_N(0; x_0)) = x_u(1; x_0)$ is obtained by applying $\mu_N(0; x_0) = u(0)$, which is indicated in red.

order to emphasize this, the term "nominal closed loop" is sometimes used. For robustness issues we refer to [10], [14, Chapter 8], [85, chapter 8], [109, chapters 9-11], and [12]. In particular, we emphasize that robustness may get lost by incorporating additional terminal constraints, cf. [31].

Assumption 1.9, which was used for illustrative purposes, is replaced by the following, weaker assumption.

Assumption 2.4
Let the infimum of Problem (2.4) - (2.6) be attained, i.e., for each $x_0 \in \mathbb{X} \subseteq X$, let a sequence of control values $u^\star(\cdot)$ exist such that $J_N(x_0; u^\star(\cdot)) = V_N(x_0)$ holds.

Assumption 2.4 ensures that the infimum of the optimal value function (2.7) is a minimum. In the following, let us suppose that an optimization algorithm is at our disposal which finds the global minimum. Since the optimizer computes, in general, only a local minimum, this is, in particular for non-convex optimization problems, no matter of course.[2] The motivation for this assumption is mainly to avoid technical details which distract the reader from the main ideas of the presented methodologies.

[2]The problem of not being able to provide a global minimum — independently of whether the reason goes back to being stuck in a local minimum or aborting the computation prematurely in order to reduce

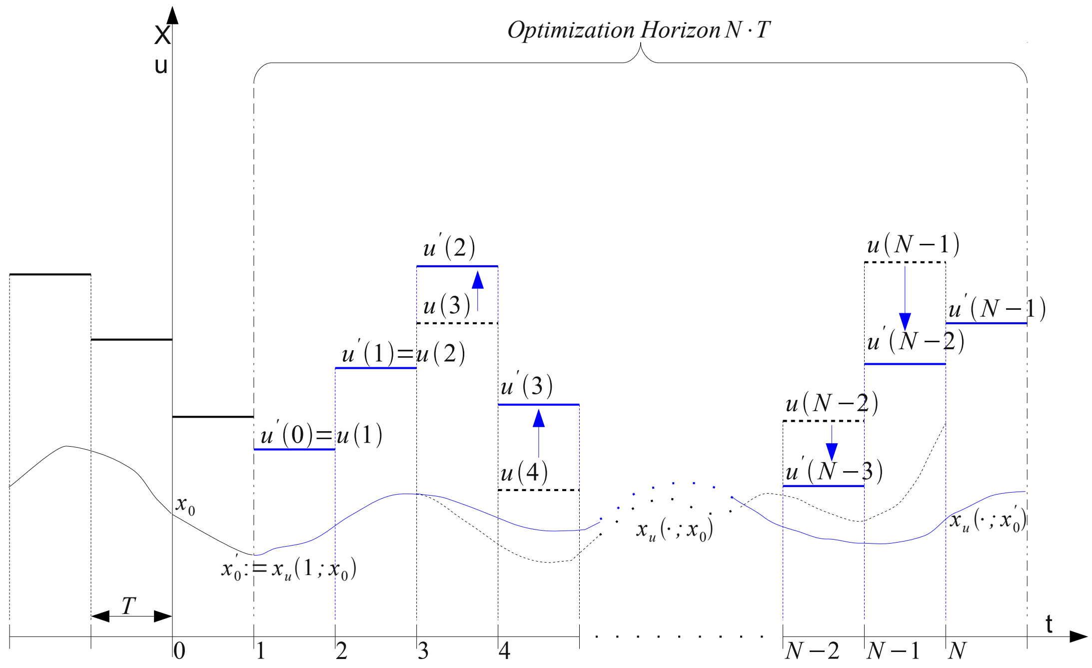


Figure 2.2: Graphical illustration of the main idea (II/II): the first element is already implemented, cf. Figure 2.1. The current state is defined as our new initial state and the optimization problem (2.4) - (2.6) is solved with respect to the new initial state. Note that the value $u'(N-1)$ has to be computed from scratch whereas the former computed control values may be a sensible initial guess for the optimization. The resulting control values may coincide with the ones from the preceding step, cp. $u'(0)$ and $u'(1)$, or differ since the optimization horizon takes additional states into account, cp. $u'(n)$, $n = 2, \dots, N-2$. Hence, the predicted trajectory changes as well.

The following remarkable consequence holds for optimal trajectories.

Remark 2.5
As mentioned in Section 1.2, tails of optimal trajectories are again optimal for the respective optimal control problem. For the problem on a finite time horizon, this reads as follows: let $u^\star(\cdot)$ denote a sequence of control values satisfying $J_N(x_0; u^\star(\cdot)) = V_N(x_0)$, then

$$J_{N-1}(x_{u^\star}(1;x_0), u^\star(1+\cdot)) = V_{N-1}(x_{u^\star}(1;x_0))$$

holds — the tail $u^\star(1+\cdot)$ of the optimal sequence of control values $u^\star(\cdot)$ is an optimal control for the problem on the shortened horizon $N-1$ with initial value $x_{u^\star}(1;x_0)$, i.e. the state at the next time instant of the trajectory emanating from x_0 generated by $u^\star(0)$, cf. [44, Corollary 3.16] for a proof.

Furthermore, the following is pointed out for networked control systems.

the computational effort and, thus, the time spend for solving the corresponding optimization problem — is tackled in [43].

Remark 2.6
Let us consider the networked control system setting given in Section 1.4. Since in each MPC optimization step an optimal control sequence is computed, RHC is ideally suited to implement the compensation strategy based on the proposed multistep feedback, cf. [47,48].

2.2 Terminal Equality Constraints

In this and the following section two techniques are presented which can be used in order to ensure stability of the resulting receding horizon closed loop. In order to keep the presentation technically simple, we focus on the case $m = 1$. However, the results are easily generalizable to time varying control horizons, i.e. sequences $(m_i)_{i\in\mathbb{N}_0}$ with $m_i \in M \subseteq \{1, \ldots, m^\star\}$ for each $i \in \mathbb{N}_0$.

In order to take care of feasibility and stability, one has to keep in mind that problem (2.4) - (2.6) does not guarantee stability and feasibility for the resulting states of the receding horizon closed loop — although the optimization problems are feasible at each time instant. In this chapter we consider two concepts which ensure, if applicable, stability and feasibility. The first one, which stems back to [66, 75], adds an additional terminal equality constraint to the optimization Problem (2.4) - (2.6), i.e. the set of admissible controls is modified to

$$\mathcal{U} = \mathcal{U}^N_{x^\star}(x_0) := \left\{ (u(n))_{n\in\mathbb{N}_0} \;\middle|\; \begin{array}{l} u(n) \in \mathbb{U} \\ x_u(n+1; x_0) \in \mathbb{X} \;\; \text{for } 0 \leq n \leq N-1 \\ x_u(N; x_0) = x^\star \end{array} \right\}. \tag{2.8}$$

The predicted trajectory still has to satisfy the state constraints $x_u(n + 1; x_0) \in \mathbb{X}$ for $n \in \{0, 1, \ldots, N-1\}$ but, additionally, also the terminal constraint $x_u(N; x_0) = x^\star$, i.e. the predicted final state $x_u(N; x_0)$ has to be the desired equilibrium. Remember that the corresponding optimal solution is denoted by $u^\star(\cdot) = u^\star_N(\cdot; x_0)$, cf. Definition 2.2. Using the RHC feedback $\mu_{N,1}(x_0, 0)$ leads to the state $x_{\mu_{N,1}}(1; x_0) \in \mathbb{X}$, cf. Remark 2.3. Since $f(x^\star, u^\star) = x^\star$ holds according to (1.2), defining the sequence of control values $(\tilde{u}_N(n))_{n\in\{0,1,\ldots,N-1\}}$ by $\tilde{u}_N(n) = u^\star_N(n+1; x_0)$, $n \in \{0, 1, 2, \ldots, N-2\}$, and $\tilde{u}_N(N-1) = u^\star$ yields an admissible control for the optimization problem consisting of (2.4), (2.5), and (2.8) with initial value $x_0 := x_{\mu_{N,1}}(1; x_0)$. Hence, we obtain

$$\begin{aligned} V_N(x_0) &= \ell(x_0, u^\star_N(0; x_0)) + J_{N-1}(x_{u^\star_N}(1; x_0), u^\star_N(1 + \cdot)) \\ &= \ell(x_0, u^\star_N(0; x_0)) + J_{N-1}(x_{u^\star_N}(1; x_0), \tilde{u}_N(\cdot)) \\ &= \ell(x_0, u^\star_N(0; x_0)) + J_N(x_{u^\star_N}(1; x_0), \tilde{u}_N(\cdot)) \\ &\geq \ell(x_0, u^\star_N(0; x_0)) + V_N(x_{u^\star_N}(1; x_0)). \end{aligned}$$

The third equality holds because $x_{u^\star_N}(N; x_0) = x_{\tilde{u}_N}(N-1; x_{u^\star_N}(1; x_0)) = x^\star$, $\tilde{u}(N-1) = u^\star$, and $\ell(x^\star, u^\star) = 0$. Furthermore, $\tilde{u}_N(\cdot)$ is admissible for Problem (2.4), (2.5), and (2.8) with initial value $x_{u^\star_N}(1; x_0)$ which implies the final inequality. Comparing this inequality with Equality (1.8) one observes that the structure of the optimization problem incorporating the terminal equality constraint, in combination with (1.15) for $V_N(\cdot)$ instead of $V(\cdot)$, implies the validity of a Lyapunov inequality which ensures a decrease in the amount of a certain minimum quantity $\alpha_1(\|x_0\|) \leq \ell^\star(x_0) \leq \ell(x_0, u)$ for all $u \in \mathbb{U}$. Note that validating $V_N(x_0) \leq \alpha_2(\|x_0\|)$ is significantly easier in the setting based on a finite time horizon which exhibits a finite sum in the cost functional $J_N(x_0, \cdot)$.

In order to illustrate this, Example 1.10 is revisited.

Example 2.7
The discrete time system from Examples 1.10 and 1.17 is considered. This control system is governed by the linear dynamics

$$x(n+1) = Ax(n) + Bu(n) = \begin{pmatrix} 1 & 1.1 \\ -1.1 & 1 \end{pmatrix} x(n) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(n).$$

In order to apply the receding horizon scheme specified in (2.4), (2.5), *and* (2.8), *one has to verify that the added terminal equality constraint can be satisfied. Note that this corresponds to finite time controllability, i.e. asymptotically stabilizable systems with a* $\mathcal{KL}_0$*-function* $\beta(r,n) = rc_n$ *in the sense of Remark 1.13. To this end, the first two iterates* $x_u(1;x_0)$ *and* $x_u(2;x_0)$ *are calculated in dependence on the initial condition* $x_0 := (x_{01}, x_{02})^T$*:*

$$\begin{aligned} x(1) &= Ax(0) + Bu(0) = \begin{pmatrix} x_{01} + 1.1x_{02} \\ -1.1x_{01} + x_{02} + u(0) \end{pmatrix}, \\ x(2) &= Ax(1) + Bu(1) = \begin{pmatrix} -0.21x_{01} + 2.2x_{02} + 1.1u(0) \\ -2.2x_{01} - 0.21x_{02} + u(0) + u(1) \end{pmatrix}. \end{aligned}$$

The choice $u(0) = 21x_{01}/110 - 2x_{02}$ *implies that the first component of* $x(2)$ *equals zero. Then, using the control* $u(1) = 2.2x_{01} + 0.21x_{02} - u(0) = 221x_{01}/110 + 221x_{02}/100$ *sets the state vector* $x(2)$ *equal to the desired equilibrium, i.e. the origin. Since this line of arguments holds for arbitrary initial values* x_0*, an optimization horizon* $N \geq 2$ *allows for incorporating the terminal equality constraint. Note that this is the shortest possible horizon in our setting.*

Hence, the considered RHC scheme is used in order to tackle the optimization problem which was already solved in an optimal fashion, cf. Examples 1.10 and 1.17. Here, we observe the different behavior of the closed loop in dependence on the optimization horizon, cf. Figure 2.3.[3]

Using the comparatively short horizon $N = 2$ *limits the set of feasible controls and, as a consequence, forces the optimization algorithm to choose a significantly worse control in view of the overall performance of the resulting RHC closed loop, cf. Table 2.1.*

Note that existence of a feasible initial solution of the considered optimization problem is tacitly assumed. Then, feasibility is obtained from this so called initial feasibility which is, since (2.6) is substituted by (2.8), a nontrivial assumption as shown in the following example.

Example 2.8 (Linear wave equation)
In the following we will change the notation to be consistent with the usual PDE notation: $x \in \Omega$ *is the independent space variable while the unknown function* $y(\cdot, t) : \Omega \to \mathbb{R}$ *represents the state. We consider the one-dimensional linear wave equation with homogeneous Dirichlet boundary condition on the left and Neumann boundary control on the right boundary*

$$y_{tt}(x,t) - c^2 y_{xx}(x,t) = 0 \qquad \text{on } (0,L) \times (0,\infty) \tag{2.9}$$

$$y(0,t) = 0 \qquad \text{on } (0,\infty) \tag{2.10}$$

[3] We point out that the receding horizon closed loop for optimization horizon $N = 2$ does, in general, not render the system to the equilibrium in two steps. Here, this is attributed to the particular structure of the considered example.

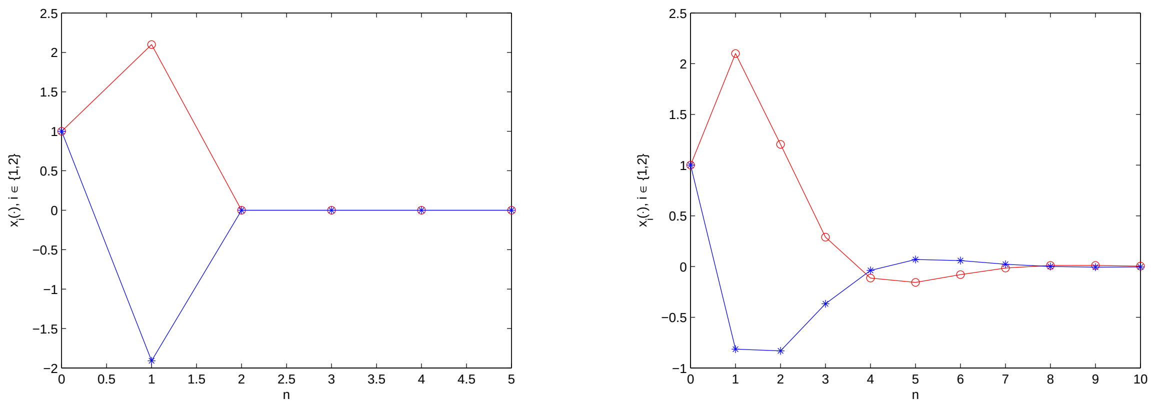


Figure 2.3: Illustration of the impact of different optimization horizons for Example 2.7 with initial value $x_0 = (1\ 1)^T$. On the left, RHC with terminal equality constraints and horizon $N = 2$ is used, whereas on the right, the optimization horizon is increased to $N = 9$. The resulting costs decrease significantly by not steering the plant to the desired equilibrium as fast as possible, cf. Table 2.1. The first component ($\circ$) of the depicted trajectories is drawn in red.

N	$J_\infty(x_0; \mu_{N,1}(\cdot,\cdot))$	$u(0)$	$u(1)$
2	31.128166	-1.809091	4.219091
3	19.223223	-0.880706	2.566194
4	18.931820	-0.710534	2.309525
5	18.925980	-0.715834	2.295242
6	18.926044	-0.717288	2.296095
7	18.925941	-0.714462	2.293226
8	18.925938	-0.713791	2.292244
9	18.925936	-0.713848	2.292351

Table 2.1: The resulting costs for the RHC scheme with terminal equality constraint in dependence on the optimization horizon N for Example 2.7. For $N = 2$ the additional constraint is very restrictive. Moreover, one observes a significant change in the applied control values.

$$y_x(L,t) = u(t) \qquad \text{on } (0,\infty) \tag{2.11}$$

Here $c \neq 0$ denotes the propagation speed of the wave. The initial data are given by $y(x,0) = y_0(x)$ and $y_t(x,0) = y_1(x)$ with $(y_0, y_1) \in \mathcal{C}([0,L]) \times \mathcal{L}^2([0,L])$. The solution space is given by

$$X = \{y : y \in \mathcal{L}^2(0,t^*; \mathcal{H}^1([0,L])) \text{ with } y_t \in \mathcal{L}^2(0,t^*; \mathcal{L}^2([0,L]))\ \forall t^* > 0\}$$

and $u \in \mathcal{L}^\infty([0,\infty))$. $\mathcal{H}^1(\Omega)$ consists of all measurable functions which are square integrable and whose weak derivative is also measurable and an element of $\mathcal{L}^2(\Omega)$. Note that the boundary values of this class of functions are well defined by means of the trace operator, cf. [25, 119].

We aim at steering the system given in Example 2.8 to the origin $y \equiv 0$ which is an equilibrium for (2.9) - (2.11). It is well known that this evolution equation is exactly controllable for optimization horizons $\bar{T} \geq 2L/c$, cf. [125], i.e. for each initial data (y_0, y_1) and each desired state $y^\star = (y_0^\star, y_1^\star)$ satisfying certain regularity assumptions a control function $u(\cdot)$ exists such that $y(\cdot, 2L/c) = y^\star$ holds. In particular, the system is finite time controllable to the origin, i.e. there exists a $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$ according to Definition 1.12 such that

$$\|y(\cdot,t)\|_{\mathcal{H}_1(\Omega)} = \|y(\cdot,t)\|_{\mathcal{L}^2(\Omega)} + \|y_t(\cdot,t)\|_{\mathcal{L}^2(\Omega)} \leq \beta(\|y_0\|_{\mathcal{L}^2(\Omega)} + \|y_1\|_{\mathcal{L}^2(\Omega)}, t)$$

with the property $\beta(r,t) = 0$ for $t \geq 2L/c$. Our cost functional is given by

$$\sum_{n=0}^{N-1} \frac{1}{4} \int_0^L \varrho(y_x(x,nT), y_t(x,nT))\, dx + \lambda \int_0^{NT} u(t)\, dt \tag{2.12}$$

with

$$\varrho(y_x(\cdot,t), y_t(\cdot,t)) = \omega_1(\cdot)(y_x(\cdot,t) + y_t(\cdot,t)/c)^2 + \omega_2(\cdot)(y_x(\cdot,t) - y_t(\cdot,t)/c)^2.$$

Here $\omega_i : [0,L] \to \mathbb{R}$, $i = 1,2$, denote appropriate weight functions.

Remark 2.9
Note that our cost functional consists of two parts. The first is related to the energy of the system and evaluates the motion to the left and the right boundary separately, e.g. $y_x(\cdot,t) + y_t(\cdot,t)/c$ represents the movement to the left boundary. For the special case $\omega_1 = \omega_2 \equiv 1$ we obtain exactly the energy of the system. The second term in (2.12) *penalizes the control effort with weight $\lambda > 0$.*

Since our results are formulated in a discrete time setting, the continuous time system given in Example 2.8 is rewriten as

$$y(n+1) = f(y(n), u(n))$$

with state $y(n) := y(\cdot, nT)$ and control $u(n) \in U := \mathcal{L}^\infty([0,T), \mathbb{R})$. This enables us to treat this partial differential equation as a discrete time system. Note that allowing arbitrary metric spaces is essential for this choice of U. Here, the discrete time n corresponds to the continuous time nT. Hence, the running costs are given by

$$\ell(y(n), u(n)) := \frac{1}{4} \int_0^L \varrho(y_x(x,nT), y_t(x,nT))\, dx + \lambda \int_0^T u(n)(t)^2\, dt \tag{2.13}$$

with the weight functions $\omega_1(\cdot)$ and $\omega_2(\cdot)$, which still have to be specified. In order to obtain finite time controllability, we choose the particular control

$$u(n)(t) := \frac{1}{2}\left(y_x(L-ct, nT) - \frac{y_t(L-ct, nT)}{c} \right) \qquad \forall\, t \in [0,T) \tag{2.14}$$

which ensures that no reflections occur on the right boundary at which the control is located. By using this control the solution of (2.9) - (2.11) coincides with the uncontrolled solution of the wave equation on a semi-infinite interval $[0,\infty)$, cf. [4]. The corresponding solution can be calculated by D'Alembert's method, cf. [116],

$$y(x,t) \quad = \quad \frac{1}{2}\Big[y_0(x+ct) + y_0(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} y_1(s)\, ds \qquad \text{for} \quad x > ct,$$

$$y(x,t) \;=\; \frac{1}{2}\Big[y_0(ct+x) - y_0(ct-x)\Big] + \frac{1}{2c}\int_{ct-x}^{ct+x} y_1(s)\,ds \qquad \text{for} \quad x < ct. \quad (2.15)$$

Furthermore, let a sampling period $T \ll 2L/c$ be given. Supposing that the sampling period is comparatively small seems to be a realistic assumption because it ensures that the cost functional takes the transient behavior adequately into account. In addition, the implementation has to be done in a sampled-data fashion. Hence, the chosen control function $u(\cdot) \in \mathcal{L}^\infty([0,\infty])$ has to be represented appropriately on a sampling interval, e.g. being constant for zero order hold which is the most frequently used implementation strategy. Hence, the sampling period has to be set to a sufficiently small value. As a consequence, a very large optimization horizon N is required in order to satisfy the terminal equality constraint which is contrary to one of the main advantages of RHC, i.e. reducing the computational effort significantly. The decisive property of this example, which is exploited in order to illustrate this drawback of RHC schemes based on terminal equality constraints, is the finite propagation speed of the waves which leads to the effect that the entire state can not be influenced arbitrarily fast and, thus, makes this RHC scheme extremely restrictive.

Remark 2.10
The phenomenon of not being able to steer the system arbitrarily fast to the desired equilibrium also occurs for many other systems whose control input is restricted by control constraints and constraints coupling the state and the control. Hence, long optimization horizons are often required in order to satisfy the additional terminal constraint (2.8)*.*

Summarizing, the RHC scheme incorporating the terminal equality constraint is applicable for the linear wave equation (2.9) - (2.11) but requires a very large optimization horizon. In the following chapter, we aim at designing a RHC scheme which provably solves this example with a significantly shorter horizon N.

Remark 2.11
Note that the feasibility of the terminal state constraint does not depend on the chosen stage costs and, consequently, the cost functional at all. Moreover, the stated control indeed steers the system to the desired equilibrium state as fast as possible. Nevertheless, a horizon of a length of at least $2L/c$ is needed in order to reach this state for initial data chosen from an arbitrarily small neighborhood measured, e.g. in the $\mathcal{L}^2$-norm.

In particular for nonlinear examples, exact controllability is a restrictive assumption. For linear infinite dimensional systems, e.g. Example 2.8, this condition may be satisfied, cf. [71] for further examples. However, the applicability of the respective RHC scheme remains questionable and often exhibits a poor performance due to unnecessarily large optimization horizons as will be shown for the linear wave equation below.

2.3 Terminal Inequality Constraints and Costs

Next, we focus on toning down the drawbacks observed in the previous section by using a more elaborate RHC scheme. This subsection, which is mainly based on [102], weakens the terminal equality constraint introduced in Subsection 2.2 by using a terminal region in combination with an additional terminal cost. This is typically chosen as a Lyapunov function whose purpose is to provide an estimate for the remaining cost to go.

To this end, a bounded set $\mathbb{X}_f \subseteq \mathbb{X}$ containing the desired target state $x^\star$ is defined as a terminal region. Additionally, a local (control) Lyapunov function is required, i.e. a continuous function $V_f : \mathbb{X}_f \to \mathbb{R}_0^+$ satisfying

$$\min_{u \in \mathbb{U}} \{V_f(f(x,u)) + \ell(x,u) : f(x,u) \in \mathbb{X}_f\} \leq V_f(x) \qquad \forall\, x \in \mathbb{X}_f.^4 \tag{2.16}$$

Typically, the shape of the terminal region $\mathbb{X}_f$ resembles a level set $\{x \in \mathbb{X} : V_f(x) \leq c\}$, $c \in \mathbb{R}_{>0}$ of the employed (control) Lyapunov function $V_f(\cdot)$. Hence, the choices of $\mathbb{X}_f$ and $V_f(\cdot)$ are coupled.

Note that assuming (2.16) implicitly implies controlled forward invariance of the terminal region $\mathbb{X}_f$. The following finite horizon optimization problem is based on the combination of a terminal region and a (control) Lyapunov function:

$$\min_{u(\cdot) \in \mathcal{U}^N_{\mathbb{X}_f}(x_0)} \quad J^f_N(x_0; u(\cdot)) = \sum_{n=0}^{N-1} \ell(x_u(n; x_0), u(n)) + V_f(x_u(N; x_0)) \tag{2.17}$$

$$\text{subject to} \quad x_u(n+1; x_0) = f(x_u(n; x_0), u(n)) \text{ with } x_u(0; x_0) = x_0 \in \mathbb{X}, \tag{2.18}$$

$$\mathcal{U} = \mathcal{U}^N_{\mathbb{X}_f}(x_0) := \left\{ (u(n))_{n \in \mathbb{N}_0} \,\middle|\, \begin{array}{l} u(n) \in \mathbb{U} \\ x_u(n+1; x_0) \in \mathbb{X} \ \text{ for } 0 \leq n \leq N-1 \\ x_u(N; x_0) \in \mathbb{X}_f \end{array} \right\}. \tag{2.19}$$

The cost functional was modified by adding the respective (control) Lyapunov function. Since the domain of $V_f(\cdot)$ consists only of a subset $\mathbb{X}_f$ of the feasible set $\mathbb{X}$, one has to ensure that the final state $x_u(N; x_0)$ of the predicted trajectory $x_u(\cdot; x_0)$ is contained in $\mathbb{X}_f$. In order to take this requirement into account, the set of admissible sequences of control values is adjusted, cf. (2.19). Again, as in the previous section, we did not change the condition $x_u(n+1; x_0) \in \mathbb{X}$, $n \in \{0, 1, \ldots, N-1\}$ although $x_u(N; x_0) \in \mathbb{X}$ is automatically satisfied because of the added constraint $x_u(N; x_0) \in \mathbb{X}_f \subseteq \mathbb{X}$. The system dynamics remain the same as in (2.5). Furthermore, we like to point out that the corresponding optimal value function for this problem, denoted by $V^f_N(x_0)$, is composed of the sum of the first N stage costs as well as the (control) Lyapunov function $V_f(\cdot)$. In order to emphasize that the cost functional also takes the (control) Lyapunov function $V_f(\cdot)$ into account, $J^f_N(\cdot, \cdot)$ is written instead of $J_N(\cdot, \cdot)$.

Let, for given $x_0 \in \mathbb{X}$, $u^\star_N(\cdot) = u^\star_N(\cdot; x_0)$ be an optimal sequence of control values for Problem (2.17) - (2.19). Then, the following line of arguments establishes a Lyapunov inequality, which enables us to deduce — in combination with the usual inequality conditions for $V^f_N(\cdot)$, cf. (1.15) — asymptotic stability of the resulting closed loop. $\hat{u} \in \mathbb{U}$ is chosen such that

$$V_f(f(x_{u^\star_N}(N; x_0), \hat{u})) + \ell(x_{u^\star_N}(N; x_0), \hat{u}) \leq V_f(x_{u^\star_N}(N; x_0))$$

holds. Since validity of (2.16) is assumed, such a control value exists. Let $\tilde{u}_N(\cdot)$ be defined by $(u^\star_N(1), u^\star_N(2), \ldots, u^\star_N(N-1), \hat{u})$. Now, the argumentation is similar to that of the previous subsection:

$$V^f_N(x_0) \quad = \quad \sum_{n=0}^{N-1} \ell(x_{u^\star_N}(n; x_0), u^\star_N(n)) + V_f(x_{u^\star_N}(N; x_0))$$

[4] Again, we use a minimum in order to keep the presentation technically simple.

$$
\begin{aligned}
&\geq \sum_{n=0}^{N-1} \ell(x_{u_N^\star}(n;x_0), u_N^\star(n)) + \ell(x_{u_N^\star}(N;x_0), \hat{u}) + V_f(f(x_{u_N^\star}(N;x_0), \hat{u})) \\
&= \ell(x_0, u_N^\star(0;x_0)) + \sum_{n=0}^{N-1} \ell(x_{\tilde{u}_N}(n; x_{u_N^\star}(1;x_0)), \tilde{u}_N(n)) + V_f(f(x_{u_N^\star}(N;x_0), \hat{u})) \\
&= \ell(x_0, u_N^\star(0;x_0)) + J_N^f(x_{u_N^\star}(1;x_0), \tilde{u}_N(\cdot)) \\
&\geq \ell(x_0, u_N^\star(0;x_0)) + V_N^f(x_{u_N^\star}(1;x_0)).
\end{aligned}
$$

Inequality (2.16) guarantees that $f(x_{u_N^\star}(N;x_0), \hat{u}) \in \mathbb{X}_f$ and, thus, the admissibility of $\tilde{u}(\cdot)$ which ensures the last inequality. Hence, recursive feasibility of the RHC problem is a consequence of the assumed initial feasibility. Stability can be deduced by the standard Lyapunov arguments. As a consequence, initial feasibility in combination with (2.16) guarantees stability of the receding horizon closed-loop.

The main advantage of this RHC scheme in comparison to the one with terminal equality constraints is the relaxation of the terminal constraint. Note that the schemes coincide for $\mathbb{X}_f = \{x^\star\}$, $V_f(\cdot) \equiv 0$. The scheme based on a terminal region and a (control) Lyapunov function does not require exact controllability to the desired equilibrium.

Remark 2.12
In particular for nonlinear systems, finding a suitable terminal region $\mathbb{X}_f$ which is control forward invariant and satisfies (2.16) *is challenging. For systems governed by time invariant ordinary differential equation, a linearization at the set point, i.e. the desired equilibrium, often allows to compute a locally stabilizing feedback as well as a local (control) Lyapunov function. Note that* (2.16) *has to be satisfied for this feedback $K : \mathbb{X}_f \to \mathbb{U}$, i.e.*

$$
V_f(f(x, K(x))) + \ell(x, K(x)) \leq V_f(x) \quad \text{and} \quad f(x, K(x)) \in \mathbb{X}_f \qquad \forall\, x \in \mathbb{X}_f.
$$

Hence, one looks for a control sequence steering the nonlinear system "sufficiently close" to $x^\star$. Once, the trajectory has entered the terminal region $\mathbb{X}_f$, the control input may be switched to the predefined feedback which ensures the validity of the desired (control) Lyapunov inequality — this strategy is also termed dual mode, cf. [73, p.8]. Feasibility of the resulting closed loop is, as already mentioned, ensured by supposing initial feasibility.

In order to illustrate these MPC schemes, the example of the nonlinear inverted pendulum on a cart is considered as a sampled-data system with zero order hold.

Example 2.13
Our goal is to stabilize the nonlinear inverted pendulum on a cart at the origin, i.e. our desired equilibrium. In order to apply RHC based on additional terminal costs, a local (control) Lyapunov function has to be specified. To this end, the Lyapunov function $V_f(x) = x^T P x$, which was calculated for the linearized model in Example 1.23, is employed. The stage costs and parameters are also taken from this example in order to ensure consistency with $V_f(\cdot)$. The terminal region $\mathbb{X}_f$ is implicitly defined by $\{x \in \mathbb{R}^4 : x^T P x \leq c\}$. For sufficiently small parameter $c \in \mathbb{R}_{>0}$, these choices heuristically ensure the desired Lyapunov Inequality (2.16)*. This claim is substantiated by our numerical results, below.*

Let the initial value $x_0 = (0.1\ 0.1\ 0.1\ 0.1)^T$, the sampling parameter $T = 0.0625$, and the terminal region $\mathbb{X}_f = \{x \in \mathbb{R}^4 : V_f(x) \leq 0.1\}$ be given. The predicted trajectories are computed by means of the MATLAB routine ode15 *which is an implicit Runge-Kutta method with step size control. Since a constrained nonlinear minimization problem is dealt*

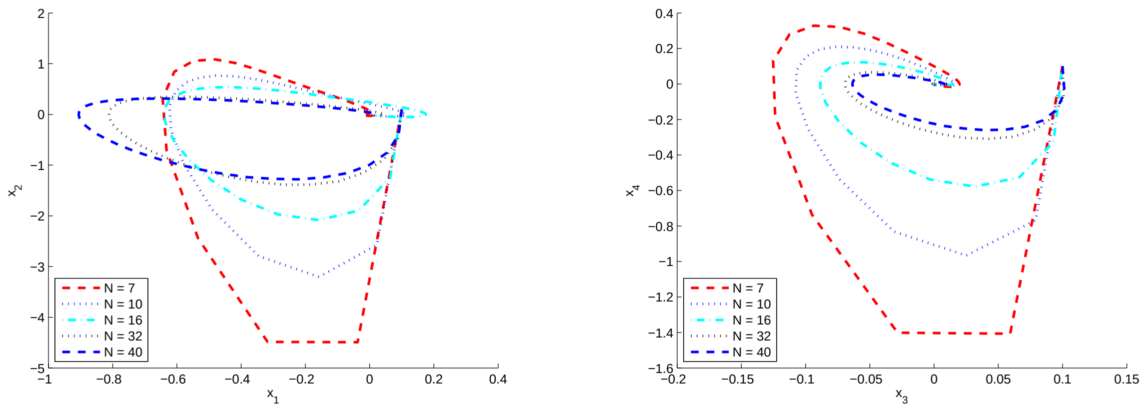


Figure 2.4: Trajectories generated by receding horizon control for various optimization horizons N with initial value $x_0 = 0.1 \cdot (1,1,1,1)^T$.

with, the routine fmincon *is used in order to solve the involved optimization problems. The resulting trajectories are depicted in Figure 2.4.*

Our numerical computations show that $N = 7$ *is the smallest optimization horizon which allows for an initially feasible trajectory, i.e. computing a sequence of* N *control values such that* $x_u(N;x_0) \in \{x \in \mathbb{R}^n : x^T P x \leq c\}$ *holds. However, this leads to* $J_7(x_0) = 1628.51369$*. We point out that the contribution of the additional terminal cost is limited to* $c = 0.1$ *and, thus, negligible. Rather, the large value of the cost functional has to be ascribed to the terminal constraint* $x_u(N;x_0) \in \mathbb{X}_f$ *whose satisfaction demands a comparatively large control effort. The actual costs of the corresponding receding horizon closed loop sum up to* 870.6461.[5] *Increasing the optimization horizon, which implicitly enlarges the feasible set of the optimization, significantly reduces the costs associated with the first* 128 *steps, cf. Figure 2.5.*

For the chosen initial condition, the static state feedback computed for the linearized version may also be used in order to stabilize the system, however, without taking the terminal constraint into account. In doing so, costs amounting to 60.7659 *are produced. RHC outperforms this feedback only for a sufficiently large optimization horizon, e.g.* $N = 20$*. Hence, using a terminal region has a stabilizing effect but may shrink the set* $\mathcal{U} = \mathcal{U}^N(x_0)$ *of admissible controls* $u(\cdot)$ *and, as a consequence, may cause higher costs. At the extreme,* $\mathcal{U}$ *equals the empty set and the optimization problem* (2.17) - (2.19) *becomes infeasible, e.g.* $N \leq 6$*. For larger horizons, the impact of incorporating a terminal constraint in* $\mathcal{U}$ *is reduced, which results in an enlarged set* $\mathcal{U}$ *and lower costs on the infinite horizon. Note that RHC with smaller optimization horizons steers the closed loop trajectory, in general, faster into the terminal region* $\mathbb{X}_f$*, cf. Table 2.2. The optimal value function* $V_N^f(\cdot)$ *decreases strictly along the receding horizon closed loop solution in our numerical computations, cf. Figure 2.5.*[6] *The desired Lyapunov inequality is, however, only satisfied for the first steps of the RHC solution due to our heuristic choice of the terminal cost* $V_f(\cdot)$*.*

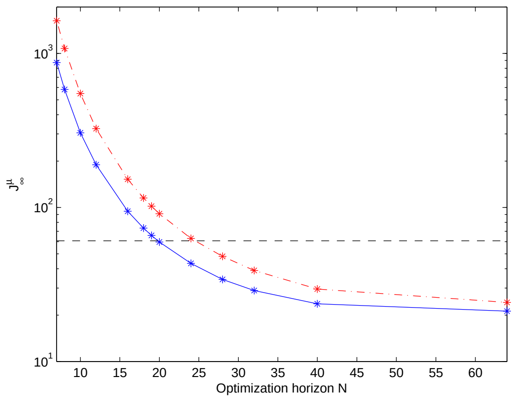
The purpose of the incorporated local (control) Lyapunov function is to appropriately

[5]The closed loop costs are only measured on the interval $[0,8]$ instead of $[0,\infty)$. However, at $t = 8$ the state is already very close to the desired set point such that this truncation of the time horizon does not distort the numerical results.

[6]This claim does not hold for the trajectory generated by the static state feedback for $N \leq 32$.

N	7	10	12	16	24	30	32	40	64
time t	1.125	1.5625	1.9375	2.8750	2.6875	3.0000	3.0625	3.8125	5.4375

Table 2.2: Time elapsed until the terminal constraint, i.e. $x_{\mu_N}(t) \in \mathbb{X}_f$, is, depending on the optimization horizon N, satisfied.



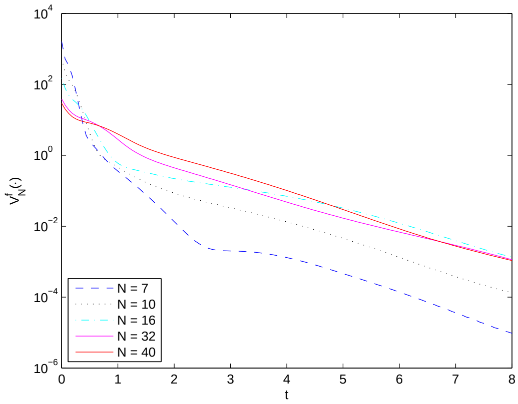


Figure 2.5: On the left, the overall costs J^{μ}_{∞} and $V^f_N(x_0)$ depending on the optimization horizon N are drawn in blue and red, respectively. The costs associated with the precomputed feedback are indicated by the dashed line. On the right, the difference $V^f_N(x_{\mu_N}(n+1)) - V^f_N(x_{\mu_N}(n))$ is illustrated for various optimization horizons N.

estimate the cost to go, i.e. to give an upper bound for the remaining costs which are needed in order to render the system asymptotically stable. Often, such a Lyapunov function, which has to satisfy Inequality (2.16), is constructed by a linearization at the desired set point, cf. Example 2.13. As a consequence, the terminal region $\mathbb{X}_f$ has to be chosen sufficiently small which makes the terminal constraint $x_u(N; x_0) \in \mathbb{X}_f$ more restrictive. Finding a (control) Lyapunov function such that Inequality (2.16) is satisfied globally, i.e. for all $x \in \mathbb{X}$, allows to neglect the terminal constraint entirely. However, this is, in particular for systems governed by nonlinear ordinary or partial differential equations, a challenging task and, in general, not possible.

Summarizing, a large domain of attraction requires, in general, a large optimization horizon N in RHC schemes with terminal constraints. Furthermore, for each initial condition $x_0 \in \mathbb{X}$, an initially feasible solution of (2.17) - (2.19), i.e. a trajectory emanating from x_0 and reaching the terminal region after at most N time steps, has to be found. Hence, the presumably most difficult problem has to be tackled at the beginning. On the other hand, feasibility and stability of the receding horizon closed loop are guaranteed. In conclusion, finding a terminal region equipped with an appropriate local (control) Lyapunov function and ensuring initial feasibility is demanding and often too restrictive from a practical point of view, cf. [100] — although these prerequisites are already easier to verify compared to the terminal equality constraints from the previous section. In addition, MPC with terminal constraints may give asymptotic stability without any robustness, as shown in [31]. Hence, we shift our focus to unconstrained RHC schemes.

2.4 Feasibility

In Sections 2.2 and 2.3 constraints were introduced whose satisfaction guarantees feasibility and stability of the respective RHC schemes. However, finding an initially feasible trajectory and, if terminal costs are used, designing a suitable (control) Lyapunov function, which is used in order to estimate the cost to go, is challenging. Furthermore, these approaches may render initial conditions infeasible for horizons N for which RHC schemes without terminal constraints and costs stabilize the system. The linear wave equation is such an example in which the finite propagation speed prevents the system from reaching a neighborhood of the origin fast whereas so called unconstrained RHC fulfills the proposed task of stabilizing the system even for extremely short optimization horizons N, cf. [62].

In this thesis we are concerned mainly with the stability analysis for unconstrained RHC. However, since these schemes do not guarantee feasibility of the resulting RHC closed loop right from the beginning, the system may become infeasible although a Lyapunov inequality was satisfied for the truncated optimal value function $V_N(\cdot)$ in each of the preceding steps. The phenomenon of not being able to detect feasibility problems on time, is often termed short-sightedness of the receding horizon closed loop, cf. [2, p.178] and [44, Example 8.1]. In order to ensure feasibility, Assumption 1.4 is supposed which fits in well with our standard assumption that the optimal value function is finite for each state x_0 of the feasible set $\mathbb{X}$.

Here, a sketch of a feasibility proof from [99] is presented which outlines a way to encounter the feasibility problem without Assumption 1.4. We point out that the main idea of rendering a level set of the value function $V_N(\cdot)$ invariant with respect to the employed receding horizon strategy is also used in order to ensure feasibility for an example considered in Section 4.4. Since the examples which are investigated for infinite dimensional systems do not exhibit tight state constraints, we restrict ourselves mainly to finite dimensional systems. Nevertheless, we emphasize that the concepts presented in this section can not be transferred to infinite dimensional systems because some conclusions can not be drawn analogously. For example the unit sphere is bounded and closed but not compact in $\mathcal{L}^2(\mathbb{R}, \mathbb{R}^n)$, cf. [119] and [110, Corollary 4.5]. Furthermore, we like to point out that [99] only deals with systems governed by linear dynamics. The ideas, however, may be generalized to the nonlinear setting. More elaborate techniques in order to ensure feasibility of unconstrained RHC schemes are discussed, e.g. in [44].

A necessary condition for feasibility of unconstrained RHC with optimization horizon N is feasibility on the infinite horizon which is characterized by a finite value of the respective optimal value function $V_\infty(x_0)$. Hence, the first step towards a feasibility analysis is taking a closer look at this set. The linear setting is considered, i.e. system dynamics $x(n+1) = Ax(n) + Bu(n)$ with a controllable pair $[A, B]$ and constraints given by $Ex + Fu \leq \psi$. Neglecting the constraints, assuming that $[A, B]$ is a controllable pair implies that every $x_0 \in \mathbb{R}^n$ is exactly controllable to the origin in a finite number of steps which is less or equal the dimension $n \in \mathbb{N}$ of the state space, cf. [58] for a precise definition. However, due to the constraint which may model simple box constraints for the control input and, thus, excluding unsaturated controls, this does not hold for the whole space. Hence, we define the set $I_0 = \{0\}$ and the sets

$$I_{k+1} := \{x \in \mathbb{R}^n : \exists\, u \text{ such that } Ax + Bu \in I_k \text{ and } Ex + Fu \leq \psi\}.$$

Thus, I_1 contains all points which may be steered to the origin in one step without violating the imposed constraints. Moreover, $I_k \subseteq I_{k+1}$ due to the construction. Defining the set $I_\infty := \bigcup_{k=0}^{\infty} I_k$ we obtain the following result.

Theorem 2.14
Let the pair $[A, B]$ be controllable and $(0, 0)$ be an interior point of the constraint set $\{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : Ex + Bu \leq \psi\}$. Furthermore, let the stage costs satisfy $\ell(x, u) \geq \alpha(\|x\|)$ for a $\mathcal{K}_\infty$-function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, e.g. $x^T Qx + u^T Ru$ with positive definite matrix Q and positive semi-definite matrix R. Then the following equivalence holds:

$$x_0 \in I_\infty \Longleftrightarrow V_\infty(x_0) < \infty.$$

The main ideas of the proof are sketched. Then, possibilities in order to generalize Theorem 2.14 to the nonlinear setting are indicated and briefly discussed in Remark 2.15.

Supposing $x_0 \in I_\infty$ ensures the existence of an index K such that $x_0 \in I_K$. Consequently, the definition of the set I_K allows us to construct a sequence of control values $(u(k))_{k\in\{0,1,\ldots,K-1\}}$ which feasibly steers the system from $x_0 \in I_K$ to I_0. Hence, $V_\infty(x_0)$ is bounded by $\sum_{n=0}^{K-1} \ell(x_u(n; x_0), u(n)) < \infty$.

Let $x_0 \notin I_\infty$ be given. Since $[A, B]$ is controllable and $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ is an interior point of the constraint set, deadbeat control may be carried out. To be more precise, every state contained in a sufficiently small neighborhood of the origin may be steered to the origin in at most n steps. Let $\mathcal{B}_\delta(0) \subset \mathbb{R}^n$, $\delta \in \mathbb{R}_{>0}$, denote a ball with radius δ completely contained in this neighborhood. Hence, since $x_0 \notin I_\infty$, there does not exist a sequence of control values steering x_0 into $\mathcal{B}_\delta(0)$. As a consequence, for each feasible $(u(n))_{n\in\mathbb{N}_0}$ the estimate $\ell(x_u(n; x_0), u(n)) \geq \alpha(\|x_u(n; x_0)\|) > \alpha(\delta/2) > 0$ holds for all $n \in \mathbb{N}_0$. Hence, $J_\infty(x_0, u(\cdot)) = \sum_{n=0}^{\infty} \ell(x_u(n; x_0), u(n)) \geq \sum_{n=0}^{\infty} \alpha(\delta/2) = \infty$ for every feasible $(u(n))_{n\in\mathbb{N}_0}$.

Remark 2.15
Deadbeat control is a restrictive assumption in the nonlinear setting, cp. RHC with terminal equality constraints in Section 2.2. Hence, this prerequisite should be weakened, e.g. assuming the existence of a neighborhood of the desired equilibrium such that each point contained in this set is stabilizable inducing finite costs. This seems to be a reasonable option in order to generalize the proposed characterization of the feasible set for the problem on an infinite time horizon to a nonlinear setting. Furthermore, we emphasize that RHC with additional terminal constraints and costs requires a similar, even stronger assumption anyway, cp. Inequality (2.16).

We continue with the main result concerning feasibility from [99].

Theorem 2.16
Let the assumptions of Theorem 2.14 be satisfied and a parameter $\mu \in \mathbb{R}_{>0}$ be given. The μ sub-level set S_μ of $V_\infty(\cdot)$ is defined by $\{x \in \mathbb{R}^n : V_\infty(x) \leq \mu\}$. Then, an optimization horizon $N' \in \mathbb{N}_{\geq 2}$ exists such that S_μ is an invariant set under any RHC feedback resulting from the optimization problem (2.4) – (2.6) *with horizon $N \geq N'$.*

The proof, which can be found in [99, Appendix], consists of two parts and relies essentially on the monotonicity of the value function $V_N(\cdot)$ with respect to the optimization horizon length N. To be more precise, $V_N(\cdot)$ has to be monotonically increasing in N — a characteristic which is automatically fulfilled for unconstrained RHC, cf. Section 2.1.

We start by two auxiliary claims in order to prepare the ground for the actual proof.

- Let $\beta \in (0, \mu)$ be chosen such that $\{x \in \mathbb{R}^n : \alpha(\|x\|) \leq \beta\} \subseteq S_\mu$. Then, the following calculation shows that $x_{\mu_N}(1; x_0) \in S_\mu$ holds for all $x_0 \in S_\beta$ and $N \in \mathbb{N}_{\geq 2}$:

$$\beta \geq V_\infty(x_0) \geq V_N(x_0) \geq V_{N-1}(x_{\mu_N}(1; x_0)) \geq \ell^\star(x_{\mu_N}(1; x_0)) \geq \alpha(\|x_{\mu_N}(1; x_0)\|).$$

- Let the set W be defined by $\{x \in \mathbb{R}^n : \alpha(\|x\|) \leq \mu\}$. Then, $x_{\mu_N}(1; x_0) \in W \cap I_\infty$ holds for all $x_0 \in S_\mu$ and sufficiently large horizons N. Note that $x_{\mu_N}(1; x_0) \in I_\infty$ guarantees $V_\infty(x_{\mu_N}(1; x_0)) < \infty$.

  Repeating the computation used in order to establish the previous assertion with β substituted by μ yields $x_{\mu_N}(1; x_0) \in W$. In order to show $x_{\mu_N}(1; x_0) \in I_\infty$, a line of arguments similar to the proof of Theorem 2.14 is employed: initially, for $x \notin I_\infty$, a lower bound for the stage costs is established in order to derive a contradiction for sufficiently large N, cf. [99, Lemma 12] for details.

Taking the first assertion into account allows us to focus on states $x \in S_\mu \backslash S_\beta$ in order to prove Theorem 2.16. Suppose that a horizon length N' satisfying the claim of Theorem 2.16 does not exist. Then, for each $j \in \mathbb{N}$, a horizon length $N_j \geq j$ and a state $x_0^j \in S_\mu \backslash S_\beta$ exist such that $x_{\mu_{N_j}}(1; x_0^j) \in W \backslash S_\mu \subset W$. Since W is compact, $(x_{\mu_{N_j}}(1; x_0^j))_{j \in \mathbb{N}}$ has a convergent subsequence $(\tilde{x}_k)_{k \in \mathbb{N}} := (x_{\mu_{N_{j_k}}}(1; x_0^{j_k}))_{k \in \mathbb{N}}$, $(j_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ with $j_{k+1} > j_k$ for all $k \in \mathbb{N}$, with $\tilde{x}_k \to \tilde{x}_\infty$ for k tending to infinity.

If $\tilde{x}_\infty$ is not contained in I_∞, $V_\infty(\tilde{x}_\infty) = \infty$ holds. Otherwise, $V_\infty(\tilde{x}_\infty) = \lim_{k \to \infty} V_\infty(\tilde{x}_k) \geq \mu$ is ensured by the second assertion in view of Theorem 2.14. Combining these assertions, yields $V_\infty(\tilde{x}_\infty) \geq \mu$. Hence, for every $\varepsilon > 0$, a horizon length N exists such that

$$V_N(\tilde{x}_\infty) > \mu - \varepsilon/4. \tag{2.20}$$

Next, we prove $\infty > V_N(\tilde{x}_\infty)$ by contradiction. To this end, suppose that $\tilde{x}_\infty$ is not feasible for the optimization problem (2.4) - (2.6) with horizon length N. The constraints specify a bounded set for any N', cf. [99, Lemma 10], which is shrinking for larger N. Hence, $V_N(\tilde{x}_\infty) = \infty$ implies the existence of an open neighborhood of $\tilde{x}_\infty$ which is not feasible for all $N' \geq N$ — a contradiction to the convergence $\tilde{x}_k \to \tilde{x}_\infty$ for $k \to \infty$.

Choose $\varepsilon = \inf_{x_0 \in S_\mu \backslash S_\beta} \alpha(\|x_0\|) \leq \inf_{x_0 \in S_\mu \backslash S_\beta} \ell^\star(x_0)$. Then, $N \in \mathbb{N}$ exists such that (2.20) with $\infty > V_N(\tilde{x}_\infty)$ holds. Since $V_N(\cdot)$ is continuous, picking $N_{j_k} > N$ large enough ensures

$$V_{N_{j_k}}(\tilde{x}_k) = V_{N_{j_k}}(x_{\mu_{N_{j_k}}}(1; x_0^{j_k})) \geq V_N(x_{\mu_{N_{j_k}}}(1; x_0^{j_k})) \geq \mu - \varepsilon/2. \tag{2.21}$$

Hence, we obtain the following inequality which leads to a contradiction and completes the proof of Theorem 2.16[7]

$$\begin{aligned} \mu &\geq V_{N_{j_k}}(x_0^{j_k}) = \ell(x_0^{j_k}, u_{N_{j_k}}(0)) + V_{N_{j_k}-1}(x_{\mu_{N_{j_k}}}(1; x_0^{j_k})) \\ &\geq \ell^\star(x_0^{j_k}) + V_N(x_{\mu_{N_{j_k}}}(1; x_0^{j_k})) \geq \varepsilon + \mu - \varepsilon/2 = \mu + \varepsilon/2. \end{aligned}$$

The main ideas of this proof are generalizable to the nonlinear setting. However, generalizing this feasibility result to the infinite dimensional setting may cause additional (technical) problems, e.g. compactness of the set W can not be expected. Instead one has to use the concept of weak sequential compactness, cf. [79, Section 10.2], and, as a consequence, only obtains a weakly convergent subsequence. Note that the different compactness terms are equivalent for normed spaces, cf. [106, Eberlein-Šmulian Theorem], and that, e.g. the unit sphere is weakly sequentially compact in every reflexive space, cf. [106, Theorem 2.8.2]. Again, we refer to [44] for more elaborate results with respect to feasibility, in particular for a generalization to the nonlinear case.

[7] The first equality is flawed in [99]. Since using the optimization horizon $N_k + 1$ leads, in general, not to $x_{\mu_{N_k}}(1; x_0^k)$ as the next state.

We like to point out that a standard assumption for stability results in RHC is the relation of the function $\alpha_1(\cdot) \in \mathcal{K}_\infty$ and the stage costs via (1.4). Hence, the prerequisites of Theorems 2.14 and 2.16 are not too restrictive.

Summarizing, RHC with either terminal constraints or costs ensures feasibility a priori but at the expense of assuming an initially feasible solution — independently of whether feasibility issues play a role or not. Using unconstrained RHC may lead to feasibility problems, in particular for short optimization horizons N due to its "short sightedness", cf. [2, p.178]. On the other hand, neglecting terminal constraints enlarges the set of admissible controls significantly and, thus, may improve the closed loop performance.

In this thesis, however, we do not focus on feasibility issues. This motivates Assumption 1.4, which may be weakened. Assumption 1.4 ensures, for each initial value $x_0 \in \mathbb{X}$, the existence of a sequence of control values which satisfies the constraints. Nevertheless, RHC may cause infinite costs in the long run.

Chapter 3

Stability and Suboptimality of RHC Schemes

In this thesis we are concerned with receding horizon schemes which incorporate neither terminal constraints nor additional terminal costs. These schemes exhibit a decisive advantage in contrast to their counterparts which take terminal costs or constraints into account: the optimal value function $V_N(\cdot)$ increases, for each feasible initial value $x_0 \in \mathbb{X}$, monotonically in the optimization horizon N — an inherent monotonicity property which allows us to exploit Lyapunov type inequalities in order to estimate, in addition to concluding stability, the performance of the resulting RHC closed loop.

Assumption 2.1 ensures boundedness of $V_\infty(\cdot)$ on $\mathbb{X}$ which is a necessary condition for well-posedness of the optimal control problem on the infinite time horizon because otherwise either the constraints are inevitably violated or the stage costs are not summable. The latter indicates that the cost functional does not provide a suitable criterion for stabilizing the system at the desired equilibrium and is, thus, inadequately designed. Hence, the monotonically increasing sequence $(V_N(x_0))_{N\in\mathbb{N}_{\geq 2}}$ is bounded from above by $V_\infty(x_0)$.

In Section 3.1 a relaxed Lyapunov inequality is introduced which forms the core of our stability and suboptimality results. Based on a controllability condition and Bellman's principle of optimality, a nonlinear program is deduced which gives us a sufficient condition in order to validate this Lyapunov inequality. In the ensuing section our main stability theorem is presented. In Section 3.3 the proposed optimization problem is solved for an important subclass, which provides an easily checkable stability and performance criterion. Then, the introduced methodology is demonstrated. To this end, our key assumption, i.e. Assumption 3.2, is verified for the linear wave equation which allows to ensure instantaneous controllability for this hyperbolic partial differential equation rigorously.

3.1 Relaxed Lyapunov Inequality

In Section 1.4 networked control systems were introduced and the notation of a multistep feedback law $\mu : \mathbb{X} \times \{0, 1, \ldots, m^\star - 1\} \to \mathbb{U}$ with $m^\star \in \mathbb{N}$ was specified. Using a receding horizon controller based on optimization problem (2.4) – (2.6) yields a sequence of N input values for a given initial value x_0. Since we intend to employ these values in order to construct the feedback law $\mu_N(\cdot, \cdot)$, the condition $m^\star \leq N - 1$ has to be satisfied. The parameter $m^\star$ determines the maximal number of control values which may be applied before the optimization problem has to be solved again in order to update — based on a measurement of the current state — the sequence of control values. Hence, $m^\star$ limits the maximal time the system may stay in open loop. Whereas the set $M \subseteq \{1, 2, \ldots, m^\star\}$ from Definition 1.25 mainly places some flexibility at our disposal, which might be convenient for the networked control setting, e.g. the network topology may force us only to use odd numbers of elements of the computed sequence of control values due to transmission specifications. Nevertheless, one may think of $M = \{1, 2, \ldots, m^\star - 1\}$ in the sequel.

Let an admissible control horizon sequence $(m_i)_{i\in\mathbb{N}_0}$ be given. Then, using the notation from Definition 1.24, the corresponding costs on the infinite time interval are given by

$$V_\infty^{\mu,(m_i)}(x_0) := \sum_{n=0}^{\infty} \ell(x_\mu(n), \mu(x_\mu(\varphi(n)), n - \varphi(n))).$$

Our approach relies on the following result from relaxed dynamic programming [83,101], which is a generalization of [39, Proposition 2.4].

Proposition 3.1
Let a multistep feedback law $\tilde{\mu} : \mathbb{X} \times \{0, 1, \ldots, m^\star - 1\} \to \mathbb{U}$, a set $M \subseteq \{1, 2, \ldots, m^\star\}$, and a function $\widetilde{V} : \mathbb{X} \to \mathbb{R}_0^+$ be given. Suppose that, for each $x_0 \in \mathbb{X}$, the solution $x_{\tilde{\mu}}(\cdot) = x_{\tilde{\mu}}(\cdot; x_0)$ with $x_{\tilde{\mu}}(0) = x_0$ satisfies $x_{\tilde{\mu}}(n; x_0) \in \mathbb{X}$, $n \in \{0, 1, \ldots, N-1\}$, and

$$\widetilde{V}(x_0) \geq \widetilde{V}(x_{\tilde{\mu}}(m)) + \alpha \sum_{k=0}^{m-1} \ell(x_{\tilde{\mu}}(k), \tilde{\mu}(x_0, k)) \qquad \forall\, m \in M \tag{3.1}$$

for some $\alpha \in (0, 1]$. Then, for all $x_0 \in \mathbb{X}$ and all admissible sequences $(m_i)_{i\in\mathbb{N}_0}$ of control horizons, the estimate

$$\alpha V_\infty(x_0) \leq \alpha V_\infty^{\tilde{\mu},(m_i)}(x_0) \leq \widetilde{V}(x_0) \tag{3.2}$$

holds.

Proof: Consider $x_0 \in \mathbb{X}$ and the trajectory $x_{\tilde{\mu}}(\cdot) = x_{\tilde{\mu},(m_i)}(\cdot; x_0)$ generated by the closed loop system using the multistep feedback $\tilde{\mu}(\cdot, \cdot)$ associated with $(m_i)_{i\in\mathbb{N}_0}$. Since $x_{\tilde{\mu}}(n; x_{\tilde{\mu}}(\sigma(k); x_0)) \in \mathbb{X}$, $n \in \{0, 1, \ldots, N-1\}$, implies $x_{\tilde{\mu}}(\sigma(k+1); x_0)$, (3.1) yields

$$\alpha \sum_{n=\sigma(k)}^{\sigma(k+1)-1} \ell(x_{\tilde{\mu}}(n), \tilde{\mu}(x_{\tilde{\mu}}(\varphi(n)), n - \varphi(n))) \leq \widetilde{V}(x_{\tilde{\mu}}(\sigma(k))) - \widetilde{V}(x_{\tilde{\mu}}(\sigma(k+1)))$$

for all $k \in \mathbb{N}_0$. Summing over the transmission times $\sigma(k)$, $k = 0, 1, \ldots, k^\star$, yields

$$\begin{aligned} \alpha \sum_{n=0}^{\sigma(k^\star)-1} \ell(x_{\tilde{\mu}}(n), \tilde{\mu}(x_{\tilde{\mu}}(\varphi(n)), n - \varphi(n))) &= \alpha \sum_{k=0}^{k^\star-1} \sum_{n=\sigma(k)}^{\sigma(k+1)-1} \ell(x_{\tilde{\mu}}(n), \tilde{\mu}(x_{\tilde{\mu}}(\varphi(n)), n - \varphi(n))) \\ &\leq \widetilde{V}(x(0)) - \widetilde{V}(x(\sigma(k^\star)) \leq \widetilde{V}(x(0)). \end{aligned}$$

For $k^\star \to \infty$ this shows that $\widetilde{V}(x_0)$ is an upper bound for $\alpha V_\infty^{\tilde{\mu},(m_i)}(x_0)$. Since the definition of the optimal value function $V_\infty(\cdot)$ ensures the first inequality in (3.2) directly, this completes the proof.

□

Our goal consists of establishing (3.1) for $\widetilde{V}(\cdot) = V_N(\cdot)$ and the corresponding RHC controller $\tilde{\mu}(\cdot,\cdot) = \mu_N(\cdot,\cdot)$. Then, using the monotonicity of $V_N(\cdot)$ in N yields

$$\alpha V_\infty^{\mu_N,(m_i)}(x_0) \leq V_N(x_0) \leq V_\infty(x_0),$$

which guarantees that the RHC closed loop produces at most $1/\alpha$ as much costs as the optimal feedback on the infinite time horizon, i.e. a suboptimality estimate.

Our key ingredient in order to deduce (3.1) is the following controllability assumption from [39]. The relation between Assumption 3.2 and other controllability conditions, e.g. the one used in [32], is discussed in Section 5.5, below.

Assumption 3.2
Let a function $\beta(\cdot,\cdot) \in \mathcal{KL}_0$ be given. Suppose that, for each $x_0 \in \mathbb{X}$, an admissible control function $u_{x_0}(\cdot) \in \mathcal{U} = \mathcal{U}^\infty(x_0) \subseteq U^{\mathbb{N}_0}$ exists, which satisfies

$$\ell(x_{u_{x_0}}(n), u_{x_0}(n)) \leq \beta(\ell^\star(x_0), n) \qquad \text{for all } n \in \mathbb{N}_0. \tag{3.3}$$

Important representatives of class $\mathcal{KL}_0$-functions lead to exponential or finite time controllability, cf. Remark 1.13. In addition to Assumption 3.2, the useful property (1.13) is assumed which ensures that any sequence of the form $\lambda_n = \beta(r,n)$, $r > 0$, fulfills $\lambda_{n+m} \leq \beta(\lambda_n, m)$, cf. Section 1.2.

Assumption 3.2 is verified for the discrete time system from Example 1.10 in order to illustrate the meaning of Condition (3.3). In particular, the example shows that the involved sequence of control values $u_{x_0} \in \mathcal{U} = \mathcal{U}^\infty(x_0)$ does not need to be optimal.

Example 3.3
Example 1.10 is considered. The stage costs are given by

$$\ell(x,u) = x^T Q x + u^T R u = x^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + u^T u = \|x\|^2 + \|u\|^2.$$

In Example 1.17 we derived the estimate $\|x(n;x_0)\| \leq C\sigma^n \|x_0\|$ for the static state feedback BF_∞. Hence, using the feedback BF_∞ applied to the current state yields

$$\begin{aligned} \ell(x(n),u(n)) &= \|x(n)\|^2 + \|BF_\infty x(n)\|^2 \leq (1 + \|BF_\infty\|^2)\|x(n)\|^2 \\ &\leq (1 + \|BF_\infty\|^2) C^2 \sigma^{2n} \|x_0\|^2 = \tilde{C}\tilde{\sigma}^n \ell^\star(x_0) \end{aligned}$$

with $\tilde{C} := (1 + \|BF_\infty\|^2)C^2$, $\tilde{\sigma} := \sigma^2$, i.e. exponential controllability with respect to the stage costs or, equivalently, Assumption 3.2 with $\beta(r,n) = \tilde{C}\tilde{\sigma}^n \cdot r$. Note that the $\mathcal{KL}$-function β is linear in its first argument.

Alternatively, one may show that this example is finite time controllable, cf. Example 2.7. In view of these results, we obtain

$$\begin{aligned} \ell(x(0),u(0)) &= \|x_0\|^2 + \|\begin{pmatrix} 21/110 & -2 \end{pmatrix} x_0\|^2 \leq 60941/12100 \cdot \ell^\star(x_0) < 5.04 \cdot \ell^\star(x_0), \\ \ell(x(1),u(1)) &= \left\| \begin{pmatrix} 1 & 1.1 \\ -10/11 & -1 \end{pmatrix} \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} \right\|^2 + \left\| \begin{pmatrix} 221/110 & 2.21 \end{pmatrix} \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} \right\|^2 \\ &\leq 15677961/1210000 \cdot \ell^\star(x_0) < 12.96 \cdot \ell^\star(x_0). \end{aligned}$$

Hence, $\beta(r,0) = c_0 \cdot r$, $\beta(r,1) = c_1 \cdot r$, and $\beta(r,n) = 0$ for $n \in \mathbb{N}_{\geq 2}$ with $c_0 = 5.04$ and $c_1 = 12.96$.

Remark 3.4
Analogously to the previous example, Assumption 3.2 can be shown for Example 1.23. Further examples are given below. Note that asymptotic stability of the closed loop usually implies Assumption 3.2. Since designing $\ell(\cdot,\cdot)$ appropriately may be beneficial in order to verify Assumption 3.2, incorporating the stage costs weakens our controllability condition, cf. Subsection 4.3.1.

In order to ease notation, we define

$$B_N(r) := \sum_{n=0}^{N-1} \beta(r,n) \tag{3.4}$$

for any $r \geq 0$ and any $N \in \mathbb{N}_{\geq 1}$. A consequence of Assumption 3.2 and Bellman's principle of optimality, i.e. $V_N(x) = \min_{u\in\mathbb{U}}\{\ell(x,u) + V_{N-1}(f(x,u))\}$, is the following lemma from [39].

Lemma 3.5
Suppose that Assumption 3.2 holds. Let $x_0 \in \mathbb{X}$ and an optimal control $u^\star(\cdot)$ for the finite horizon optimal control problem (2.4) – (2.6) *with optimization horizon $N \geq 2$ be given. Then, for each $m = 1, \ldots, N-1$, the following inequalities hold:*

$$\begin{aligned}
V_N(x_0) \quad &\leq \quad B_N(\ell^\star(x_0)), && (3.5)\\
J_{N-j}(x_{u^\star}(j), u^\star(j+\cdot)) \quad &\leq \quad B_{N-j}(\ell^\star(x_{u^\star}(j))), \quad j = 1,\ldots,N-2 && (3.6)\\
V_N(x_{u^\star}(m)) \quad &\leq \quad J_j(x_{u^\star}(m), u^\star(m+\cdot)) + B_{N-j}(l^\star(x_{u^\star}(m+j))),\\
&\qquad\qquad j = 0,\ldots,N-m-1. && (3.7)
\end{aligned}$$

The inequalities stated in Lemma 3.5 are based on the fact that tails of optimal trajectories are again optimal. For instance in (3.7), Estimate (3.3) is used after following the respective optimal trajectory emanating from $x_{u^\star}(m)$ for j steps. Summarizing, Lemma 3.5 links the cost attributed to some time instant with quantities deduced from optimality and Assumption 3.2.

Next, we provide a constructive approach in order to compute α in (3.1) for systems satisfying Assumption 3.2. Note that (3.1) depends only on m_0 and not on the remainder of the control horizon sequence. This enables us to perform the computation separately for each control horizon m and, consequently, allows for determining the desired α for time varying control horizons by minimizing with respect to the obtained values for all admissible m. To this end, we consider arbitrary values $\lambda_0, \ldots, \lambda_{N-1}, \nu > 0$ and start by deriving necessary conditions under which these values coincide with an optimal sequence $\ell(x_{u^\star}(n), u^\star(n))$ and an optimal value $V_N(x_{u^\star}(m))$, respectively.

Proposition 3.6
Suppose that Assumption 3.2 holds and consider $N \geq 2$, $m \in \{1,\ldots,N-1\}$, a sequence $\lambda_n > 0$, $n = 0,\ldots,N-1$, and a value $\nu > 0$. Consider $x_0 \in \mathbb{X}$ and assume that a minimizing control $u^\star(\cdot) \in \mathcal{U}$ for (2.4) – (2.6) *exists such that λ_n equals $\ell(x_{u^\star}(n), u^\star(n))$ for all $n \in \{0,\ldots,N-1\}$. Then*

$$\sum_{n=k}^{N-1} \lambda_n \leq B_{N-k}(\lambda_k), \quad k = 0,\ldots,N-2 \tag{3.8}$$

holds true. If, in addition, $\nu = V_N(x_{u^\star}(m))$ holds, we have

$$\nu \leq \sum_{n=0}^{j-1} \lambda_{n+m} + B_{N-j}(\lambda_{j+m}), \quad j = 0,\ldots,N-m-1. \tag{3.9}$$

Proof: If the stated conditions hold, then λ_n and ν meet the inequalities in Lemma 3.5, which is exactly (3.8) and (3.9).

$\square$

Using this proposition a sufficient condition for suboptimality of the RHC feedback law $\mu_{N,m}$ is given in the following Theorem from [39].

Theorem 3.7
Consider $\beta \in \mathcal{KL}_0$, $N \geq 2$, $m \in \{1, \dots, N-1\}$, and assume that all sequences $\lambda_n > 0$, $n = 0, \dots, N-1$ and values $\nu > 0$ fulfilling (3.8), (3.9) *satisfy the inequality*

$$\sum_{n=0}^{N-1} \lambda_n - \nu \geq \alpha \sum_{n=0}^{m-1} \lambda_n \tag{3.10}$$

for some $\alpha \in (0, 1]$. Then, for each optimal control problem (2.4) - (2.6) *satisfying Assumption 3.2, the assumptions of Proposition 3.1 are satisfied for the multistep MPC feedback law $\mu_{N,m}(\cdot, \cdot)$. In particular, the inequality*

$$\alpha V_\infty(x) \leq \alpha V_\infty^{\mu_{N,m}}(x) \leq V_N(x)$$

holds for all $x \in \mathbb{X}$.

In view of Theorem 3.7, the value α can be interpreted as a performance bound which indicates how good the receding horizon strategy approximates the infinite horizon cost. In the remainder of this section we present an optimization based approach for computing α. To this end, consider the following optimization problem.

Problem 3.8
Let $\beta(\cdot, \cdot) \in \mathcal{KL}_0$, $N \geq 2$, and $m \in \{1, \dots, N-1\}$ be given. Compute

$$\alpha_{N,m} = \alpha^1_{N,m} := \inf_{\lambda_0, \dots, \lambda_{N-1}, \nu} \frac{\sum_{n=0}^{N-1} \lambda_n - \nu}{\sum_{n=0}^{m-1} \lambda_n}$$

subject to the constraints (3.8), (3.9), *and $\lambda_0, \dots, \lambda_{N-1}, \nu > 0$.*

The following is a corollary from Theorem 3.7.

Corollary 3.9
Consider $\beta(\cdot, \cdot) \in \mathcal{KL}_0$, $N \geq 2$, and $m \in \{1, \dots, N-1\}$ and assume that Problem 3.8 has an optimal value $\alpha \in (0, 1]$. Then, for each optimal control problem given by (3.8), (3.9) *satisfying Assumption 3.2 the assumptions of Theorem 3.7 are satisfied and, consequently, the corresponding assertions hold.*

The considered setting can be extended to the setting including an additional weight $\omega \geq 1$ on the final term, i.e. altering our finite time cost functional by adding $(\omega - 1)\ell(x_u(N-1), u(N-1))$. Note that the original form of the functional $J_N(\cdot, \cdot)$ is obtained by setting $\omega = 1$. All results in this section remain valid if the statements are suitably adapted. In particular, (2.4) and (3.4) become

$$J_N(x_0, u) \;:=\; \sum_{n=0}^{N-2} \ell(x_u(n), u(n)) + \omega \ell(x_u(N-1), u(N-1))$$

$$B_N(r) \;:=\; \sum_{n=0}^{N-2} \beta(r,n) + \omega\beta(r,N-1). \tag{3.11}$$

Problem 3.8 is changed to the following optimization problem.

Problem 3.10
Let $\beta(\cdot,\cdot) \in \mathcal{KL}_0$, $N \in \mathbb{N}_{\geq 2}$, and $m \in \{1,2,\ldots,N-1\}$ be given. Compute

$$\alpha^{\omega}_{N,m} := \inf_{\lambda_0,\ldots,\lambda_{N-1},\nu} \frac{\sum_{n=0}^{N-2}\lambda_n + \omega\lambda_{N-1} - \nu}{\sum_{n=0}^{m-1}\lambda_n}$$

$$\textit{subject to} \quad \sum_{n=k}^{N-2}\lambda_n + \omega\lambda_{N-1} \leq B_{N-k}(\lambda_k), \quad k=0,\ldots,N-2 \tag{3.12}$$

$$\nu \leq \sum_{n=0}^{j-1}\lambda_{n+m} + B_{N-j}(\lambda_{j+m}), \quad j=0,\ldots,N-m-1 \tag{3.13}$$

and $\lambda_0,\ldots,\lambda_{N-1},\nu > 0$.

Remark 3.11
Note that incorporating an additional weight on the final term may cancel out the mentioned monotonicity of $V_N(\cdot)$ with respect to the optimization horizon N. As a consequence, $V_N(x_0) > V_\infty(x_0)$ is not excluded. Hence, the interpretation of the computed index $\alpha^{\omega}_{N,m}$ as a performance index becomes more difficult. For example, although Corollary 3.9 implies $V_\infty^{\mu_N,(m_i)}(x_0) \leq 1/\alpha^{\omega}_{N,m} V_N(x_0)$, the conclusion $V_\infty^{\mu_N,(m_i)}(x_0) \leq 1/\alpha^{\omega}_{N,m} V_\infty(x_0)$ may be wrong.

3.2 Asymptotic Stability

In this section, which extends [39, Section 5] to time varying control horizons, it is shown how the performance bound $\alpha = \alpha^{\omega}_{N,m}$ can be used in order to conclude asymptotic stability of the receding horizon closed loop. Assumption 1.7 ensures global asymptotic stability of $x^\star$ under the optimal feedback for the infinite horizon problem, provided $\beta(r,\cdot)$ is summable. The results of this section are generalizable to stage costs whose level set $\mathcal{L} := \{x \in \mathbb{X} : \exists\, u \in \mathbb{U} \text{ with } f(x,u) \in \mathbb{X} \text{ satisfying } \ell(x,u) = 0\} = \{x \in \mathbb{X} : \ell^\star(x) = 0\}$ consists not only of a desired set point $x^\star$. Furthermore, the condition (ii) of Assumption 1.7 can be relaxed in various ways, cf. [39].

Our main stability result is formulated in the following theorem.

Theorem 3.12
Consider $\beta(\cdot,\cdot) \in \mathcal{KL}_0$, $m^\star \geq 1$, $N \geq m^\star + 1$, and a set $M \subseteq \{1,\ldots,m^\star\}$ and suppose that Assumptions 3.2 and 1.7 are satisfied. Furthermore, assume that $\alpha^\star := \min_{m\in M}\{\alpha^{\omega}_{N,m}\} > 0$ where $\alpha^{\omega}_{N,m}$ denotes the optimal value of Problem 3.10. Then, the multistep RHC feedback law $\mu_{N,m^\star}(\cdot,\cdot)$, which is based on the optimal control problem consisting of (2.4) – (2.6)*, asymptotically stabilizes the equilibrium $x^\star$ for all admissible control horizon sequences $(m_i)_{i\in\mathbb{N}_0}$. In addition, the function $V_N(\cdot)$ is a Lyapunov function at the transmission times $\sigma(k)$ in the sense that*

$$V_N(x_{\mu_{N,m^\star}}(\sigma(k+1))) \leq V_N(x_{\mu_{N,m^\star}}(\sigma(k))) - \alpha^\star V_{m_k}(x_{\mu_{N,m^\star}}(\sigma(k))) \tag{3.14}$$

holds for all $k \in \mathbb{N}_0$ and $x_0 \in \mathbb{X}$.

Proof: From (1.4) and Lemma 3.5 we immediately obtain the inequality

$$\alpha_1(\|x\|_{x^\star}) \leq V_N(x) \leq B_N(\alpha_2(\|x\|_{x^\star})). \tag{3.15}$$

Note that $B_N \circ \alpha_2$ is again a $\mathcal{K}_\infty$–function. The stated Lyapunov Inequality (3.14) follows immediately from the definition of $\alpha^\star$ and (3.8) which holds according to Corollary 3.9 for all $m \in M$. Again, using (1.4) we obtain $V_m(x) \geq \alpha_1(\|x\|_{x^\star})$ and thus a standard construction, see e.g. [91], yields a $\mathcal{KL}$–function $\rho(\cdot, \cdot)$ for which the inequality $V_N(x_{\mu_{N,m^\star}}(\sigma(k))) \leq \rho(V_N(x), k) \leq \rho(V_N(x), \lfloor \sigma(k)/m^\star \rfloor)$ holds. In addition, using the definition of $\mu_{N,m^\star}$, for $p = 1, \ldots, m_k - 1$, $k \in \mathbb{N}_0$, and abbreviating $x(n) = x_{\mu_{N,m^\star}}(n)$ we obtain

$$\begin{aligned}
V_N(x(\sigma(k)+p)) &\leq \sum_{n=\sigma(k)+p}^{\sigma(k+1)-1} \ell(x(n), \mu_{N,m^\star}(x(\varphi(n)), n - \varphi(n))) + V_{N-m_k+p}(x(\sigma(k+1))) \\
&\leq \sum_{n=\sigma(k)}^{\sigma(k+1)-1} \ell(x(n), \mu_{N,m^\star}(x(\varphi(n)), n - \varphi(n))) + V_{N-m_k+p}(x(\sigma(k+1))) \\
&\leq V_N(x(\sigma(k))) + \omega V_N(x(\sigma(k+1))) \ \leq \ (1+\omega) V_N(x(\sigma(k)))
\end{aligned}$$

where (3.14) was used in the last inequality. Hence, the estimate

$$V_N(x_{\mu_{N,m^\star}}(n)) \leq (1+\omega)\rho(V_N(x), \lfloor \varphi(n)/m^\star \rfloor)$$

is obtained which implies

$$\begin{aligned}
\|x_{\mu_{N,m^\star}}(n)\|_{x^\star} &\leq \alpha_1^{-1}(V_N(x_{\mu_{N,m^\star}}(n))) \\
&\leq \alpha_1^{-1}((1+\omega)\rho(V_N(x), \lfloor \varphi(n)/m^\star \rfloor)) \\
&\leq \alpha_1^{-1}((1+\omega)\rho(B_N(\alpha_2(\|x\|_A)), \lfloor (n - m^\star)/m^\star \rfloor))
\end{aligned}$$

and thus asymptotic stability with $\mathcal{KL}$-function $\tilde{\beta}$ given by

$$\tilde{\beta}(r, n) = \alpha_1^{-1}((1+\omega)\rho(B_N(\alpha_2(r)), \lfloor (n - m^\star)/m^\star \rfloor)) + re^{-n}.$$

□

Remark 3.13 *(i) For the "classical" RHC case $m^\star = 1$ and $\beta(\cdot, \cdot)$ satisfying* (1.13) *it is shown in [39, Theorem 5.3] that the criterion from Theorem 3.12 is tight in the sense that if $\alpha^\star < 0$ holds then a control system, which satisfies Assumption 3.2 but which is not stabilized by the RHC scheme, exists. We conjecture that this assertion remains valid for $m^\star \geq 2$.*

(ii) Note that, in Theorem 3.12, we use a criterion for arbitrary but fixed $m \in M$ in order to conclude asymptotic stability for time varying $(m_i)_{i \in \mathbb{N}_0} \subseteq M$. This is possible since our proof yields V_N as a common Lyapunov function, cf. also [82, Section 2.1.2].

3.3 Linear Program

The goal of this section is to solve Problem 3.8 or its extended version, i.e. Problem 3.10, which allows for an additional final weight. While this is an optimization problem of much lower complexity than the original RHC optimization problem (2.4) - (2.6), still, it is in general nonlinear. However, it becomes a linear program if $\beta(r,n)$ and, thus, $B_N(r)$ from (3.4) and (3.11) is linear in r, cf. Example 3.3.

Lemma 3.14
Let $\beta(r,t)$ be linear in its first argument. Then Problem 3.10 yields the same optimal value $\alpha^{\omega}_{N,m}$ as

$$\min_{\lambda_0,\lambda_1,\ldots,\lambda_{N-1},\nu} \sum_{n=0}^{N-2} \lambda_n + \omega\lambda_{N-1} - \nu$$

subject to the (now linear) constraints (3.12), (3.13) *with $B_N(k)$ from* (3.11) *and*

$$\lambda_0,\ldots,\lambda_{N-1},\nu \geq 0, \quad \sum_{n=0}^{m-1} \lambda_n = 1. \tag{3.16}$$

For a proof we refer to [39, Remark 4.3 and Lemma 4.6], observing that this proof is easily extendable to $\omega \geq 1$. The following remark comments on the assumed linearity of the $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$.

Remark 3.15
A $\mathcal{KL}_0$-function $\beta(\cdot,\cdot) : \mathbb{R}_0^+ \times \mathbb{N}_0$, which is linear in its first argument, may be written as $\beta(r,n) = rc_n$ for a suitably chosen sequence $(c_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$. Hence, we obtain the following for (3.11)*:*

$$B_N(r) = \sum_{n=0}^{N-2} \beta(r,n) + \omega\beta(r,N-1) = r \cdot \left(\sum_{n=0}^{N-2} c_n + \omega c_{N-1} \right).$$

In order to exploit this representation, we define

$$\gamma_N := B_N(r)/r = \sum_{n=0}^{N-2} c_n + \omega c_{N-1}. \tag{3.17}$$

We point out that Assumption 3.2 implies $c_0 \geq 1$ for $\mathcal{KL}_0$-functions which are linear in their first argument. Moreover, we assume without loss of generality that $c_n = 0$ for an arbitrary index n implies $c_{n+i} = 0$ for all $i \in \mathbb{N}$ since $c_n = 0$ in combination with (3.3) *ensures that the respective trajectory has already reached the desired equilibrium exactly at time n.*

The following lemma is based on the observation that the optimum of the optimization problem posed in Lemma 3.14 satisfies Inequality (3.13), $j = N - m - 1$, with equality. Using this fact and the condition $\sum_{n=0}^{m-1} \lambda = 1$ allows to rewrite the optimization problem to be solved.

Proposition 3.16
Let $\beta(\cdot,\cdot)$ be linear in its first argument and let γ_k be defined according to (3.17)*. Then the optimal value of Problem 3.10 equals the optimal value of the optimization problem*

$$\min_{\lambda} \quad 1 - (\gamma_{m+1} - \omega)\,\lambda_{N-1}$$

subject to $\lambda = (\lambda_1, \ldots, \lambda_{N-1})^T \geq 0$ *componentwise and the linear constraints*

$$\begin{aligned}
\gamma_N \sum_{n=1}^{m-1} \lambda_n + \sum_{n=m}^{N-2} \lambda_n + \omega \quad \lambda_{N-1} \quad &\leq \quad \gamma_N - 1 && && (3.18)\\
\sum_{n=j}^{N-2} \lambda_n - \gamma_{N-j} \quad \lambda_j + \omega \quad \lambda_{N-1} \quad &\leq \quad 0 && (j = \quad 1, \ldots, N-2) && (3.19)\\
\sum_{n=j}^{N-2} \lambda_n - \gamma_{N-j+m}\, \lambda_j + \gamma_{m+1}\lambda_{N-1} \quad &\leq \quad 0 && (j = \quad m, \ldots, N-2). && (3.20)
\end{aligned}$$

Proof: We proceed from the linear optimization problem stated in Lemma 3.14 and show that Inequality (3.13), $j = N-m-1$, is active in the optimum, i.e. (3.13) is an equality in the optimum for $j = N-m-1$. To this end, we assume the opposite and deduce a contradiction.

Suppose $\lambda_{N-1} > 0$. Since (3.13) is not active and due to the continuity of $B_{m+1}(\lambda_{N-1}) = \gamma_{m+1}\lambda_{N-1}$ with respect to λ_{N-1}, this allows for reducing the value of λ_{N-1} without violating (3.13), $j = N-m-1$. As a consequence, the objective function decreases strictly whereas all other constraints remain valid — a contradiction to the assumed optimality. Hence, $\lambda_{N-1} = 0$. Then, since $\lambda_{N-2} \leq B_{m+2}(\lambda_{N-2}) = \gamma_{m+2}\lambda_{N-2}$ (3.12), $k = N-2$, holds trivially and the validity of (3.13), $j = N-m-1$ ensures (3.13), $j = N-m-2$. This allows us to derive $\lambda_{N-2} = 0$ analogously to $\lambda_{N-1} = 0$. Iterative application of this line of arguments provides $\lambda_m = \ldots = \lambda_{N-2} = \lambda_{N-1} = 0$. But then the right hand side of (3.13), $j = N-m-1$, sums up to zero which – in combination with $\nu \geq 0$ – leads to the claimed contradiction.

Hence, we treat (3.13), $j = N-m-1$, as an equality constraint. In conjunction with the non-negativity conditions imposed on $\lambda_m, \ldots, \lambda_{N-1}$ this ensures $\nu \geq 0$. We point out that the special case $\gamma_N - 1 = 0$ leads to $\lambda_2 = \lambda_3 = \ldots = \lambda_{N-1}$ via (3.12), $k = 0$, and $\nu = 0$ via (3.13), $j = N-m-1$, and, thus, to $\alpha^{\omega}_{N,m} = 1$. This is also reflected by the optimization problem formulated in Proposition 3.16. Thus, we assume w.l.o.g. $\gamma_N - 1 > 0$, cf. Remark 3.15. As a consequence, (3.12), $k = 0$, in combination with the linearity of $B_N(\cdot)$ guarantees $\lambda_0 \geq 0$ for all feasible points.

Next, we utilize (3.16) and (3.13), $j = N-m-1$, in order to eliminate ν and λ_0 from the considered optimization problem. Using these equalities and the definition of γ_{m+1}, the objective function from Lemma 3.14 is converted into the desired form. Furthermore, (3.16) provides the equivalence of (3.12), $k = 0$, and (3.18). Taking (3.13), $j = N-m-1$, into account yields

$$\sum_{n=m+j}^{N-2} \lambda_n + \gamma_{m+1}\lambda_{N-1} - \gamma_{N-j}\lambda_{m+j} \leq 0$$

for (3.13), $j = 0, \ldots, N-m-2$. Shifting the control variable j shows the equivalence to (3.20), $j = m, \ldots, N-2$. Paraphrasing (3.12) provides (3.19) for $k = 1, \ldots, N-2$.

□

Before we proceed, we formulate Problem 3.17 by dropping Inequalities (3.19), $j = m, \ldots, N-2$. The solution of this relaxed (optimization) problem paves the way for dealing with Problem 3.10: suppose that the $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$ from Assumption 3.2 fulfills Property (1.13). Then, the optimum of Problem 3.17 is also feasible for Problem

3.10. Otherwise, its optimal value can still be used as a lower bound for the suboptimality degree of the receding horizon closed loop.

Problem 3.17
Minimize $1 - (\gamma_{m+1} - \omega)\,\lambda_{N-1}$ *subject to* $\lambda = (\lambda_1, \ldots, \lambda_{N-1})^T \geq 0$ *componentwise and* $A\lambda \leq \bar{b}$, *where*

$$A := \begin{pmatrix} a_1 & a_2 & \ldots & a_{N-2} & \omega \\ d_1 & 1 & \ldots & 1 & b_1 \\ 0 & d_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & b_{N-3} \\ 0 & \ldots & 0 & d_{N-2} & b_{N-2} \end{pmatrix} \quad \text{and} \quad \bar{b} := \begin{pmatrix} \gamma_N - 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

with

$$a_j = \begin{cases} \gamma_N & \text{for } j < m \\ 1 & \text{otherwise} \end{cases} \qquad b_j = \begin{cases} \omega & \text{for } j < m \\ \gamma_{m+1} & \text{otherwise} \end{cases} \qquad d_j = \begin{cases} 1 - \gamma_{N-j} & \text{for } j < m \\ 1 - \gamma_{N-j+m} & \text{otherwise} \end{cases}$$

Theorem 3.18
Let $\beta(\cdot,\cdot) \in \mathcal{KL}_0$ *from Assumption 3.2 be linear in its first argument and satisfy* (1.13). *Then the optimal value* $\alpha = \alpha^{\omega}_{N,m}$ *of Problem 3.10 for given optimization horizon* N, *control horizon* m, *and weight* ω *on the final term satisfies* $\alpha^{\omega}_{N,m} = 1$ *if and only if* $\omega \geq \gamma_{m+1}$. *Otherwise, we get*

$$\alpha^{\omega}_{N,m} = 1 - \frac{(\gamma_{m+1} - \omega) \prod\limits_{i=m+2}^{N} (\gamma_i - 1) \prod\limits_{i=N-m+1}^{N} (\gamma_i - 1)}{\left(\prod\limits_{i=m+1}^{N} \gamma_i - (\gamma_{m+1} - \omega) \prod\limits_{i=m+2}^{N} (\gamma_i - 1) \right) \left(\prod\limits_{i=N-m+1}^{N} \gamma_i - \prod\limits_{i=N-m+1}^{N} (\gamma_i - 1) \right)}. \tag{3.21}$$

Proof: We showed that the linear optimization problem stated in Proposition 3.16 yields the same optimal value as Problem 3.10 for $\mathcal{KL}_0$-functions which are linear in their first argument. Technically, this is posed as a minimization problem. Taking the restriction $\lambda_{N-1} \geq 0$ into account, leads to the question, whether the coefficient of λ_{N-1} in the objective function is positive or not. As a consequence, the aim is either minimizing or maximizing λ_{N-1}. In the first case, i.e. $\gamma_{m+1} - \omega \leq 0$, choosing $\lambda_1 = \ldots = \lambda_{N-1} = 0$ solves the considered task and provides $\alpha^{\omega}_{N,m} = 1$.

Hence, we suppose $\lambda_{m+1} - \omega > 0$. In order to prove the assertion, i.e. the stated formula, we solve the relaxed Problem 3.17 and show that its optimum is also feasible for the original problem, i.e. Problem 3.10.

The linear system of equations $A\lambda = \bar{b}$ with A and $\bar{b}$ from Problem 3.17 is satisfied at the optimum — a crucial property which is shown by Lemma 3.22. This allows us to deduce expressions for $\lambda_{N-2}, \lambda_{N-3}, \ldots, \lambda_1$ depending (only) on λ_{N-1}. Inserting the obtained terms into $A_1\lambda = \bar{b}_1$ allows for solving this equation with respect to variable λ_{N-1}. Plugging this expression for λ_{N-1} into the objective function of the optimization problem in consideration, yields Formula (3.18).

Suppose $N - m \geq 2$. Then λ_{N-j}, $j = 2, 3, \ldots, N-m$ is given by

$$\lambda_{N-j} = \frac{\prod_{i=m+1}^{m+j-1} \gamma_i}{\prod_{i=m+2}^{m+j} (\gamma_i - 1)} \cdot \lambda_{N-1}. \tag{3.22}$$

We show this claim by induction over $j = 2, 3, \ldots, N-m$. For $j = 2$, the assertion follows directly from $A_{N-1}\lambda = \bar{b}_{N-1} = 0$. Thus, we continue with the induction step using Lemma 3.21 with $m+2$, $m+j-1$ instead of m, M. Using $-d_{N-j} = -(1-\gamma_{N-(N-j)+m}) = \gamma_{m+j}-1$ and $b_{m-j} = \gamma_{m+1}$ yields

$$\begin{aligned}
\lambda_{N-j} &= \frac{\gamma_{m+1}\lambda_{N-1} + \sum_{i=2}^{j-1}\lambda_{N-i}}{\gamma_{m+j}-1} \\
&\overset{I.A.}{=} \frac{\gamma_{m+1}\left[\prod_{i=m+2}^{m+j-1}(\gamma_i - 1) + \sum_{i=2}^{j-1}\left(\prod_{k=m+2}^{m+i-1}\gamma_k \prod_{k=m+i+1}^{m+j-1}(\gamma_k - 1)\right)\right]}{\prod_{i=m+2}^{m+j}(\gamma_i - 1)} \cdot \lambda_{N-1} \\
&= \frac{\gamma_{m+1}\left[\prod_{i=m+2}^{m+j-1}(\gamma_i - 1) + \sum_{i=m+2}^{m+j-1}\left(\prod_{k=m+2}^{i-1}\gamma_k \prod_{k=i+1}^{m+j-1}(\gamma_k - 1)\right)\right]}{\prod_{i=m+2}^{m+j}(\gamma_i - 1)} \cdot \lambda_{N-1} \\
&\overset{(3.32)}{=} \frac{\prod_{i=m+1}^{m+j-1}\gamma_i}{\prod_{i=m+2}^{m+j}(\gamma_i - 1)} \cdot \lambda_{N-1}.
\end{aligned}$$

Suppose $m \geq 2$. Then λ_{m-j}, $j = 1, 2, \ldots, m-1$ is given by

$$\lambda_{m-j} = \frac{\prod_{i=N-m+1}^{N-m+j-1}\gamma_i}{\prod_{i=N-m+1}^{N-m+j}(\gamma_i - 1)}\left(\omega\lambda_{N-1} + \sum_{i=2}^{N-m}\lambda_{N-i}\right). \tag{3.23}$$

We show (3.23) by induction over $j = 1, 2, \ldots, m-1$. For an index j chosen from the specified range, $-d_{m-j} = \gamma_{N-m+j} - 1$ and $b_{m-j} = \omega$ hold. Hence, considering $A_m\lambda = \bar{b}_m = 0$ provides the assertion for $j = 1$. Using Lemma 3.21 with $N-m+1$, $N-m+j$ instead of m, M, we perform the induction step in order to show the assertion:

$$\begin{aligned}
\lambda_{m-j} &= \frac{\omega\lambda_{N-1} + \sum_{i=2}^{N-m}\lambda_{N-i} + \sum_{i=1}^{j-1}\lambda_{m-i}}{(\gamma_{N-m+j} - 1)} \\
&\overset{I.A.}{=} \frac{\prod\limits_{i=N-m+1}^{N-m+j-1}(\gamma_i - 1) + \sum\limits_{i=1}^{j-1}\left(\prod\limits_{k=N-m+1}^{N-m+i-1}\gamma_k \prod\limits_{k=N-m+i+1}^{N-m+j-1}(\gamma_k - 1)\right)}{\prod_{i=N-m+1}^{N-m+j}(\gamma_i - 1)}\left(\omega\lambda_{N-1} + \sum_{i=2}^{N-m}\lambda_{N-i}\right) \\
&= \frac{\prod\limits_{i=N-m+1}^{N-m+j-1}(\gamma_i - 1) + \sum\limits_{i=N-m+1}^{N-m+j-1}\left(\prod\limits_{k=N-m+1}^{i-1}\gamma_k \prod\limits_{k=i+1}^{N-m+j-1}(\gamma_k - 1)\right)}{\prod_{i=N-m+1}^{N-m+j}(\gamma_i - 1)}\left(\omega\lambda_{N-1} + \sum_{i=2}^{N-m}\lambda_{N-i}\right) \\
&\overset{(3.32)}{=} \frac{\prod_{i=N-m+1}^{N-m+j-1}\gamma_i}{\prod_{i=N-m+1}^{N-m+j}(\gamma_i - 1)}\left(\omega\lambda_{N-1} + \sum_{i=2}^{N-m}\lambda_{N-i}\right).
\end{aligned}$$

Before we proceed, we further investigate the second factor of (3.23). Again, Lemma 3.21 with $m+2$, N instead of m, M is of helpful:

$$\begin{aligned}
\sum_{j=2}^{N-m}\lambda_{N-j} + \omega\lambda_{N-1} &\overset{(3.22)}{=} \sum_{j=2}^{N-m}\frac{\prod_{i=m+1}^{m+j-1}\gamma_i}{\prod_{i=m+2}^{m+j}(\gamma_i - 1)}\lambda_{N-1} + \omega\lambda_{N-1} \\
&= \left[\omega + \frac{\sum_{j=2}^{N-m}\left(\prod_{i=m+1}^{m+j-1}\gamma_i \prod_{m+j+1}^{N}(\gamma_i - 1)\right)}{\prod_{i=m+2}^{N}(\gamma_i - 1)}\right]\lambda_{N-1}
\end{aligned}$$

$$\begin{aligned}
&= \left[\omega + \frac{\gamma_{m+1}\sum_{j=m+2}^{N}\left(\prod_{i=m+2}^{j-1}\gamma_i\prod_{j+1}^{N}(\gamma_i-1)\right)}{\prod_{i=m+2}^{N}(\gamma_i-1)}\right]\lambda_{N-1}\\
&\overset{(3.32)}{=} \frac{\omega\prod\limits_{i=m+2}^{N}(\gamma_i-1)+\gamma_{m+1}\left(\prod\limits_{i=m+2}^{N}\gamma_i-\prod\limits_{i=m+2}^{N}(\gamma_i-1)\right)}{\prod_{i=m+2}^{N}(\gamma_i-1)}\cdot\lambda_{N-1}\\
&= \left(\frac{\prod_{i=m+1}^{N}\gamma_i-(\gamma_{m+1}-\omega)\prod_{i=m+1}^{N}(\gamma_i-1)}{\prod_{i=m+2}^{N}(\gamma_i-1)}\right)\lambda_{N-1}. \qquad (3.24)
\end{aligned}$$

Now, we prepared the ground in order to extract an explicit expression for λ_{N-1} from $A_1\lambda = \bar{b}_1 = \gamma_N - 1$ by applying (3.23). To this end, we consider the left hand side of this equation, i.e.

$$\begin{aligned}
A_1\lambda &= \gamma_N\sum_{i=1}^{m-1}\lambda_i+\sum_{i=m}^{N-2}\lambda_i+\omega\lambda_{N-1} = \gamma_N\sum_{j=1}^{m-1}\lambda_{m-j}+\sum_{j=2}^{N-m}\lambda_{N-j}+\omega\lambda_{N-1}\\
&\overset{(3.23)}{=} \left[\gamma_N\sum_{j=1}^{m-1}\frac{\prod_{i=N-m+1}^{N-m+j-1}\gamma_i}{\prod_{i=N-m+1}^{N-m+j}(\gamma_i-1)}+1\right]\left(\omega\lambda_{N-1}+\prod_{j=2}^{N-m}\lambda_{N-j}\right). \qquad (3.25)
\end{aligned}$$

The first factor of the left hand side is rewritten by means of Lemma 3.21 applied with $N-m+1$, $N-1$ instead of m, M:

$$\begin{aligned}
\gamma_N\sum_{j=1}^{m-1}\frac{\prod_{i=N-m+1}^{N-m+j-1}\gamma_i}{\prod_{i=N-m+1}^{N-m+j}(\gamma_i-1)}+1 &= \frac{\gamma_N\sum_{j=1}^{m-1}\left(\prod_{i=N-m+1}^{N-m+j-1}\gamma_i\prod_{i=N-m+j+1}^{N-1}(\gamma_i-1)\right)}{\prod_{i=N-m+1}^{N-1}(\gamma_i-1)}+1\\
&= \frac{\gamma_N\sum_{j=N-m+1}^{N-1}\left(\prod_{i=N-m+1}^{j-1}\gamma_i\prod_{i=j+1}^{N-1}(\gamma_i-1)\right)}{\prod_{i=N-m+1}^{N-1}(\gamma_i-1)}+1\\
&\overset{(3.32)}{=} \frac{\gamma_N\left(\prod_{i=N-m+1}^{N-1}\gamma_i-\prod_{i=N-m+1}^{N-1}(\gamma_i-1)\right)}{\prod_{i=N-m+1}^{N-1}(\gamma_i-1)}+1\\
&= \frac{\prod_{i=N-m+1}^{N}\gamma_i-\prod_{i=N-m+1}^{N}(\gamma_i-1)}{\prod_{i=N-m+1}^{N-1}(\gamma_i-1)}. \qquad (3.26)
\end{aligned}$$

Hence, inserting (3.24) and (3.26) into (3.25) and solving $A_1\lambda = \gamma_N - 1$ with respect to λ_{N-1} yields

$$\lambda_{N-1} = \frac{(\gamma_N-1)\prod_{i=N-m+1}^{N-1}(\gamma_i-1)}{\prod_{i=N-m+1}^{N}\gamma_i-\prod_{i=N-m+1}^{N}(\gamma_i-1)}\,\frac{\prod_{i=m+2}^{N}(\gamma_i-1)}{\prod_{i=m+1}^{N}\gamma_i-(\gamma_{m+1}-\omega)\prod_{i=m+1}^{N}(\gamma_i-1)}.$$

Taking this expression for λ_{N-1} into account shows that the optimal value of Problem 3.17 is given by (3.21).

However, the assertion claims this to be the optimal value of Problem 3.10 as well. In order to prove this, it is sufficient to show that the optimum of Problem 3.17 satisfies (3.19), $j = m, \ldots, N-2$. As a consequence, it solves the optimization problem stated in Proposition 3.16 which is equivalent to Problem 3.10. As a byproduct, this covers the necessity of the previously considered condition $\gamma_{m+1} - \omega \leq 0$ in order to obtain $\alpha_{N,m}^{\omega} = 1$.

To this end, we perform a pairwise comparison of (3.20) and (3.19) for $j \in \{m, \ldots, N-2\}$ in order to show that the constraints given by (3.19), $j = m, \ldots, N-2$, are dispensable. Since

$$(\gamma_{m+1} - \omega)\lambda_{N-1} \geq (\gamma_{N-j+m} - \gamma_{N-j})\lambda_j \qquad j = m, \ldots, N-2 \tag{3.27}$$

ensures

$$\sum_{n=j}^{N-2} \lambda_N - \gamma_{N-j}\lambda_j + \omega\lambda_{N-1} \leq \sum_{n=j}^{N-2} \lambda_N - \gamma_{N-j+m}\lambda_j + \gamma_{m+1}\lambda_{N-1},$$

it suffices to establish (3.27) for the obtained optimum in order to show the assertion. (3.22) characterizes the components λ_j, $j = m, \ldots, N-2$, in the optimum of Problem 3.17 by means of the equation

$$\left(\prod_{i=m+2}^{N-j+m} (\gamma_i - 1)\right) \lambda_j = \gamma_{m+1} \left(\prod_{i=m+2}^{N-j+m-1} \gamma_i\right) \lambda_{N-1}, \qquad j = m, \ldots, N-2.$$

Using this representation of λ_j which (only) depends on λ_{N-1} (3.27) is equivalent to

$$(\gamma_{m+1} - \omega) \prod_{i=m+2}^{N-j+m} (\gamma_i - 1) \geq (\gamma_{N-j+m} - \gamma_{N-j}) \prod_{i=m+1}^{N-j+m-1} \gamma_i, \qquad j = m, \ldots, N-2.$$

Since the left hand side of this expression is equal to

$$(\gamma_{m+1}-\omega) \prod_{i=m+2}^{N-j+m-1} (\gamma_i-1)(c_0-1)+(\gamma_{m+1}-\omega) \prod_{i=m+2}^{N-j+m-1} (\gamma_i-1) \left[\sum_{n=1}^{N-j+m-2} c_n + \omega c_{N-j+m-1}\right],$$

$(c_0 - 1) \geq 0$, and $(\gamma_{N-j+m} - \gamma_{N-j}) = \sum_{n=N-j-1}^{N-j+m-2} c_n + \omega c_{N-j+m-1} - \omega c_{N-j-1}$, applying Lemma 3.23 with $k = 1$ completes the proof.

□

Remark 3.19
Even if Property (1.13) *is not satisfied, the proof of Theorem 3.18 shows that Formula* (3.21) *provides the optimal value of the relaxed Problem 3.17 and, thus, a lower bound for Problem 3.10. Suppose that Assumption 3.2 is satisfied with a $\mathcal{KL}_0$-function which is linear in its first argument. Then, the $\alpha^{\omega}_{N,m}$-value of Theorem 3.18 may still be used as a lower bound for the suboptimality degree of the receding horizon closed loop.*

Theorem 3.18 allows us to easily compute performance bounds which are needed in Theorem 3.12 in order to prove stability provided $\beta(\cdot, \cdot)$ is known. However, even if $\beta(\cdot, \cdot)$ is not known exactly, we can deduce valuable information. The following corollary is obtained by a careful analysis of the fraction in (3.21).

Corollary 3.20
Let m and $\omega \geq 1$ be given. Then, for each summable $\mathcal{KL}_0$-function $\beta(\cdot, \cdot)$ which is linear in its first argument, i.e. $B_N(r) = r \cdot \gamma_N$ and $\lim_{N\to\infty} \gamma_N < \infty$, the convergence $\lim_{N\to\infty} \alpha^{\omega}_{N,m} \to 1$ holds.

Proof: Without loss of generality we assume $\gamma_{m+1} - \omega > 0$. Otherwise Theorem 3.18 yields the assertion for all $N \geq m+1$. Hence, we have to show that the subtrahend of the difference in formula (3.21) converges to zero as the optimization horizon N tends to infinity. To this end, the considered term is divided into the factors

$$\frac{(\gamma_{m+1} - \omega) \prod_{i=m+2}^{N} (\gamma_i - 1)}{\left(\prod_{i=m+1}^{N} \gamma_i - (\gamma_{m+1} - \omega) \prod_{i=m+2}^{N} (\gamma_i - 1) \right)} \quad \text{and} \quad \frac{\prod_{i=N-m+1}^{N} (\gamma_i - 1)}{\left(\prod_{i=N-m+1}^{N} \gamma_i - \prod_{i=N-m+1}^{N} (\gamma_i - 1) \right)}. \tag{3.28}$$

Since $\beta(r, n)$ is linear in its first and summable with respect to its second argument, $(\gamma_N)_{N \in \mathbb{N}_{\geq 2}}$ is a Cauchy sequence. Hence, an index $\bar{N} = \bar{N}(\varepsilon)$ exists such that $\omega \sum_{n=\bar{N}}^{\infty} c_n \leq \varepsilon < 1$ and, thus,

$$\gamma_{\bar{N}} - (\omega - 1)c_{\bar{N}-1} \leq \gamma_i \leq \gamma_{\bar{N}} - (\omega - 1)c_{\bar{N}-1} + \varepsilon \leq \gamma_{\bar{N}} + \varepsilon \qquad \text{for all } i > \bar{N}$$

holds. For $N \geq \bar{N} + m$, this implies

$$\begin{aligned} \frac{\prod_{i=N-m+1}^{N} (\gamma_i - 1)}{\prod_{i=N-m+1}^{N} \gamma_i - \prod_{i=N-m+1}^{N} (\gamma_i - 1)} &\leq \frac{m(\gamma_{\bar{N}} + \varepsilon - 1)}{m \left[\gamma_{\bar{N}} - (\omega - 1)c_{\bar{N}-1} - (\gamma_{\bar{N}} - (\omega - 1)c_{\bar{N}-1} + \varepsilon - 1) \right]} \\ &= \frac{\gamma_{\bar{N}} + \varepsilon - 1}{1 - \varepsilon} < \infty \end{aligned}$$

which ensures the boundedness of the second quotient in (3.28) for sufficiently large optimization horizons N. Hence, showing that the first quotient in (3.28) convergences to zero for N tending to infinity completes the proof. To this end, for $N > \bar{N}$, we consider the respective reciprocal

$$\begin{aligned} \frac{\prod_{i=m+1}^{N} \gamma_i - (\gamma_{m+1} - \omega) \prod_{i=m+2}^{N} (\gamma_i - 1)}{(\gamma_{m+1} - \omega) \prod_{i=m+2}^{N} (\gamma_i - 1)} &= \frac{\prod_{i=m+1}^{\bar{N}} \gamma_i}{(\gamma_{m+1} - \omega) \prod_{i=m+2}^{\bar{N}} (\gamma_i - 1)} \cdot \frac{\prod_{i=\bar{N}+1}^{N} \gamma_i}{\prod_{i=\bar{N}+1}^{N} (\gamma_i - 1)} - 1 \\ &\geq 1 \cdot \left(\frac{\gamma_{\bar{N}} - (\omega - 1)c_{\bar{N}-1}}{\gamma_{\bar{N}} - (\omega - 1)c_{\bar{N}-1} + \varepsilon - 1} \right)^{N - \bar{N}} - 1. \end{aligned}$$

Since the term in brackets is strictly greater than one, the deduced lower bound grows unboundedly for N approaching infinity. Hence, the first quotient in (3.28) converges to zero for $N \to \infty$ which shows the assertion.

□

In particular, Corollary 3.20 ensures, for sufficiently large optimization horizons N, that the assumptions of Theorem 3.12 hold and, thus, asymptotic stability of the RHC closed loop.

Next, the linear finite dimensional system with quadratic cost function from Examples 1.17, 1.10, 2.7, and 3.3 is considered in order to illustrate the methodology introduced in this chapter. Note that no constraints are present in this example. In particular, the role played by the involved $\mathcal{KL}_0$-functions $\beta(\cdot, \cdot)$ in our controllability Assumption 3.2 is investigated:

- Using exponential controllability according to Example 3.3, i.e. a $\mathcal{KL}_0$-function
$$\beta(r,n) = C\sigma^n \qquad \text{with } C \approx 49.85805 \text{ and } \sigma \approx 0.26288 \tag{3.29}$$
of type (1.11), provides $N = 284$ for $m = 1$. Allowing for larger control horizons reduces this estimate to $N = 94$ for $m = 40$, cf. Section 4.2 for details on implementing more than only the first element of the receding horizon control sequence.

- In contrast to that, already the easily deduced finite time controllability, i.e. a $\mathcal{KL}_0$-function of type (1.12) given by
$$c_0 = 5.04,\ c_1 = 12.96, \text{ and } c_n = 0 \text{ for } n > 1 \tag{3.30}$$
improves the results obtained from Theorem 3.18 significantly, i.e. $N = 52$ $(m = 1)$ and $N = 25$ $(m = 10)$, respectively.

These $\mathcal{KL}_0$-functions were deduced in order to demonstrate the general verifiability of Assumption 3.2 based on asymptotic stability in terms of the used norm. Here, we aim at constructing a $\mathcal{KL}_0$-function which characterizes the stability behavior of the considered system better and, thus, implies tighter performance bounds. To this end, the known feedback F provided by Example 1.10 is employed in order estimate coefficients c_n, $n \in \mathbb{N}_0$, of a $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$ satisfying Assumption 3.2 and Property (1.13):

$$\begin{aligned}
\ell(x(n;x_0), BFx(n;x_0)) &= \|(A+BF)^n x_0\|^2 + \|BF(A+BF)^n x_0\|^2 \\
&\leq (\|(A+BF)^n\|^2 + \|BF(A+BF)^n\|^2)\|x_0\|^2 \\
&= (\|(A+BF)^n\|^2 + \|BF(A+BF)^n\|^2)\ell^\star(x_0).
\end{aligned}$$

Hence, Estimate (3.3) holds with $\mathcal{KL}_0$-function

$$\beta(r,n) = c_n \cdot r \qquad \text{with } c_n := \|(A+BF)^n\|^2 + \|BF(A+BF)^n\|^2,\ n \in \mathbb{N}_0, \tag{3.31}$$

cf. Table 3.1 for numerically computed values. Using this $\mathcal{KL}_0$-function in order to apply Theorem 3.18 yields $\alpha^1_{N,m} > 0$ for $N = 28$ $(m = 1)$ and $N = 16$ $(m = 8)$, respectively. Hence, the performance estimates are considerably improved in contrast to those based on the $\mathcal{KL}_0$-functions $\beta(\cdot,\cdot)$ from (3.29) and (3.30) which shows that the involved bounds c_n, $n \in \mathbb{N}_0$, play an important role for the quality of the horizon estimates. Note that Property (1.13) is not needed in order to deduce suboptimality bounds but ensures that the proposed formula represents the optimal value of Problem 3.8, cf. Remark 3.19.

In order to verify (1.13), the inequality $c_n c_m \geq c_{n+m}$ has to be ensured for all $n, m \in \mathbb{N}_0$. Since $c_0 \geq 1$ holds, this corresponds to checking $c_{n-j}c_j \geq c_n$, $j = 1, 2, \ldots, n-1$, for each $n \in \mathbb{N}_0$. Now, we benefit from computing the horizon estimates first: since solely coefficients c_n, $n < N$, are required in Problem 3.8, Property (1.13) has only to be verified for $n < 28$ — a condition which is satisfied. We point out that the derived function $\beta(\cdot,\cdot)$ is not monotonically decreasing and, thus, does not belong to class $\mathcal{KL}$, cf. Table 3.1.

We emphasize that optimality of the control sequence $u_{x_0}(\cdot)$ is not assumed — a key feature of our approach which simplifies the verification of Assumption 3.2 significantly. This allowed us to employ knowledge on the solution of the algebraic Riccati equation in order to deduce (3.31) and, thus, to tighten the horizon estimates, cf. Section 5.5.1. In Section 5.5.1 this example is considered again and the results are compared with other approaches which can be also used in order to estimate the required horizon length in RHC.

N	$\beta(\cdot,\cdot)$ from (3.29)	$\beta(\cdot,\cdot)$ from (3.30)	$\beta(\cdot,\cdot)$ from (3.31)
0	49.85804850	5.04	3.037786080
1	13.10674606	12.96	5.186783379
2	3.445517772	0.00	2.790245748
3	0.905762015	0.00	0.392116897
4	0.238107850	0.00	0.015203185
5	0.062594089	0.00	0.031327420
6	0.016454812	0.00	0.013169022
7	0.004325662	0.00	0.001422866
8	0.001137135	0.00	0.000105178
9	0.000298932	0.00	0.000179462
10	0.000078584	0.00	0.000059880

Table 3.1: Coefficients of several $\mathcal{KL}_0$-functions $\beta(\cdot,\cdot)$ satisfying Assumption 3.2 for Example 1.17.

3.3.1 Auxiliary Results

In this subsection three lemmata are deduced which are used in order to prove Theorem 3.18. The technical Lemma 3.21 is applied several times in the proof of Theorem 3.18 as well as needed as a preliminary result in order to prove the Lemma 3.22. Lemma 3.22 characterizes the optimal solution of Problem 3.17 which is crucial in order to show Formula (3.21). In conclusion, we present Lemma 3.23 which is based on (1.13).

Lemma 3.21
Let $m, M \in \mathbb{Z}$ with $M \geq m-1$ and constants $\gamma_i \in \mathbb{R}$, $i = m, m+1, \ldots, M$ be given. Furthermore, the conventions $\prod_{i=m}^{m-1} = 1$ and $\sum_{i=m}^{m-1} = 0$ are used. Then, the following formula holds:

$$\prod_{i=m}^{M} \gamma_i = \prod_{i=m}^{M} (\gamma_i - 1) + \sum_{i=m}^{M} \left(\prod_{k=m}^{i-1} \gamma_k \prod_{k=i+1}^{M} (\gamma_k - 1) \right). \tag{3.32}$$

Proof: We carry out an induction over M in order to prove (3.32). Since we have agreed on the conventions with respect to the empty product and empty sum, the assertion holds for $M = m-1$. Hence, we proceed with the induction step:

$$\begin{aligned}
\prod_{i=m}^{M+1} (\gamma_i - 1) &= (\gamma_{M+1} - 1) \prod_{i=m}^{M} (\gamma_i - 1) \\
&\overset{I.A.}{=} (\gamma_{M+1} - 1) \left[\prod_{i=m}^{M} \gamma_i - \sum_{i=m}^{M} \left(\prod_{k=m}^{i-1} \gamma_k \prod_{k=i+1}^{M} (\gamma_k - 1) \right) \right] \\
&= \prod_{i=m}^{M+1} \gamma_i - \prod_{i=m}^{M} \gamma_i - \sum_{i=m}^{M} \left(\prod_{k=m}^{i-1} \gamma_k \prod_{k=i+1}^{M+1} (\gamma_k - 1) \right) \\
&= \prod_{i=m}^{M+1} \gamma_i - \sum_{i=m}^{M+1} \left(\prod_{k=m}^{i-1} \gamma_k \prod_{k=i+1}^{M+1} (\gamma_k - 1) \right).
\end{aligned}$$

$\square$

The following lemma mainly argues with the signs of the respective coefficients of the matrix A and the vector $\bar{b}$. The condition $\gamma_{m+1} - \omega$ is only used in order to ensure that

- $\bar{b}_1 > 0$, $d_i < 0$ for $i \in \{1, 2, \ldots, N-2\}$ and that
- the optimization objective consists of maximizing λ_{N-1}.

Furthermore, we point out that we take the assumptions discussed in Remark 3.15 with respect to the sequence $(c_n)_{n\in\mathbb{N}_0}$ into account in order to conclude the following lemma. Note that these are based on the linearity of $\beta(\cdot,\cdot) \in \mathcal{KL}_0$ in its first argument.

Lemma 3.22
Let $\gamma_{m+1} = \sum_{n=0}^{m-1} c_n + \omega c_m$ be strictly greater than ω. Then the optimal solution λ of Problem 3.17 satisfies $A\lambda = \bar{b}$, $\lambda > 0$ componentwise.

Proof: $\gamma_{m+1} > \omega$ implies that the coefficient of λ_{N-1} in the objective function is negative. As a consequence, maximizing λ_{N-1} subject to given constraints provides the optimum of Problem 3.17, which is denoted by $\lambda^* = (\lambda_1^*, \ldots, \lambda_{N-1}^*)$. In order to prove the assertion, we assume the existence of an index $k \in \{1, \ldots, N-1\}$ such that $A_k\lambda^* = \sum_{n=1}^{N-1} A_{kn}\lambda_n^* < \bar{b}_k$ and deduce a contradiction.

We begin with the case $k = 1$ and define $\varepsilon := \gamma_N - 1 - \sum_{i=1}^{N-2} a_i\lambda_i^* - \omega\lambda_{N-1}^* > 0$, i.e. ε corresponds to the slack in the first inequality, $\delta := -\max_{i=1,\ldots,N-2} d_i$, and $\beta := \max_{i=1,\ldots,N-2} b_i$. Note that $\gamma_{m+1} > \omega$ ensures $\delta > 0$ in view of Remark 3.15 for $\beta(\cdot,\cdot) \in \mathcal{KL}_0$ which is linear in its first argument. Now, we choose $\tilde{\varepsilon} > 0$ such that

$$\tilde{\varepsilon}\left[\omega + \beta \sum_{i=1}^{N-2} a_i \frac{(1+\delta)^{N-2-i}}{\delta^{N-1-i}}\right] \leq \varepsilon.$$

Then, we increase λ_{N-1} by $\tilde{\varepsilon}$ and λ_i, $i = 1, \ldots, N-2$, by $\tilde{\varepsilon}\,\beta(1+\delta)^{N-2-i}/\delta^{N-1-i}$. The choice of $\tilde{\varepsilon}$ ensures the validity of the first inequality. Since inequality $j \in \{2, \ldots, N-1\}$ holds for λ^* the following computation shows that it is still satisfied for the modified λ_i, $i = 1, \ldots, N-1$. Here, we use Lemma 3.21 with $m = 0$, $M = N-2-j$ and $\gamma_i = 1+\delta$ for $i \in \{m, m+1, \ldots, M\}$:

$$\begin{aligned}
& d_{j-1}\tilde{\varepsilon}\beta\frac{(1+\delta)^{N-1-j}}{\delta^{N-j}} + \sum_{i=j}^{N-2} \tilde{\varepsilon}\beta\frac{(1+\delta)^{N-2-i}}{\delta^{N-1-i}} + \tilde{\varepsilon}b_{j-1} \\
\leq\ & \tilde{\varepsilon}\left[-\delta\beta\frac{(1+\delta)^{N-1-j}}{\delta^{N-j}} + \sum_{i=j}^{N-2}\beta\frac{(1+\delta)^{N-2-i}}{\delta^{N-1-i}} + \beta\right] \\
=\ & \frac{\tilde{\varepsilon}\beta}{\delta^{N-1-j}}\left[-(1+\delta)^{N-1-j} + \sum_{i=0}^{N-2-j}(1+\delta)^{N-2-j-i}\delta^i + \delta^{N-1-j}\right] \overset{(3.32)}{=} 0.
\end{aligned}$$

However, this contradicts the assumed optimality of λ^*. Thus, the first inequality holds with equality and $k > 1$ which implies $\lambda_{k-1}^* > 0$. This allows us to reduce λ_{k-1} without violating the non-negativity condition imposed on this variable. As a consequence, the first inequality is not active any more while all other inequalities remain valid. Hence, repeating the above argumentation w.r.t. $k = 1$ leads, again, to a contradiction and, thus, proves $A\lambda^* = \bar{b}$.

It remains to show that $\lambda_i^* > 0$ for all $i \in \{1, 2, \ldots, N-1\}$. Suppose $\lambda_k^* = 0$ for $k \in \{1, \ldots, N-2\}$. Then, the $(k+1)$-st inequality implies $\lambda_i^* = 0$ for $i \in \{k, k+1, \ldots, N-1\}$. Since the k-th inequality is satisfied with equality, we obtain $\lambda_{k-1}^* = 0$. Iterative application of this argument shows $\lambda^* \equiv 0$. However, since $\gamma_{m+1} > \omega$ and Remark 3.15 ensure $\bar{b}_1 = \gamma_N - 1 > 0$, this contradicts $A_1 \lambda^* = \bar{b}_1$. Hence, $\lambda^* > 0$ holds componentwise which completes the proof.

□

The following lemma is only needed for $k = 1$. However, we state the result for all $k \in \mathbb{N}_0$ since this simplifies the induction step significantly. This trick is the main reason for presenting this technical assertion in a separate lemma.

Lemma 3.23
Let $N \in \mathbb{N}_{\geq 2}$, $m \in \{1, \ldots, N-2\}$, and $\omega \geq 1$ be given. Furthermore, let γ_i, $i \in \mathbb{N}_{\geq 2}$, be defined as $\sum_{n=0}^{i-2} c_n + \omega c_{i-1}$, cp. Proposition 3.16. In addition, let the coefficients c_n, $n \in \mathbb{N}_0$, satisfy (1.13) and use the convention $\prod_{m+2}^{m+1} = 1$. Then, for $j = N-2, N-3, \ldots, m$,

$$
\begin{aligned}
&(\gamma_{m+1} - \omega) \prod_{i=m+2}^{N-j+m-1} (\gamma_i - 1) \left[\sum_{n=k}^{N-j+m+k-3} c_n + \omega c_{N-j+m+k-2} \right] \\
&\quad - \prod_{i=m+1}^{N-j+m-1} \gamma_i \left[\sum_{n=N-j+k-2}^{N-j+m+k-3} c_n + \omega c_{N-j+m+k-2} - \omega c_{N-j+k-2} \right] \;\geq\; 0 \qquad \forall\, k \in \mathbb{N}.
\end{aligned}
$$

Proof: We carry out an induction with respect to j. The induction start, $j = N-2$, follows for arbitrary $k \in \mathbb{N}$ from

$$
\begin{aligned}
&(\gamma_{m+1} - \omega) \left[\sum_{n=k}^{m+k-1} c_n + \omega c_{m+k} \right] - \gamma_{m+1} \left[\sum_{n=k}^{m+k-1} c_n + \omega c_{m+k} - \omega c_k \right] \\
&= \omega c_k \gamma_{m+1} - \omega \left[\sum_{n=k}^{m+k-1} c_n + \omega c_{m+k} \right] = \omega \left[\sum_{n=0}^{m-1} (c_k c_n - c_{n+k}) + \omega (c_k c_m - c_{m+k}) \right] \overset{(1.13)}{\geq} 0.
\end{aligned}
$$

In order to perform the induction step from $j+1 \rightsquigarrow j$ we rewrite the considered inequality for arbitrary but fixed $k \in \mathbb{N}$:

$$
\begin{aligned}
&(\gamma_{m+1} - \omega) \prod_{i=m+2}^{N-j+m-2} (\gamma_i - 1) \left[c_k \gamma_{N-j+m-1} - \sum_{n=k}^{N-j+m+k-3} c_n - \omega c_{N-j+m+k-2} \right] \\
&\quad + \gamma_{N-j+m-1} \Bigg((\gamma_{m+1} - \omega) \prod_{i=m+2}^{N-j+m-2} (\gamma_i - 1) \left[\sum_{n=k+1}^{N-j+m+k-3} c_n + \omega c_{N-j+m+k-2} \right] \\
&\qquad\qquad - \prod_{i=m+1}^{N-j+m-2} \gamma_i \left[\sum_{n=N-j+k-2}^{N-j+m+k-3} c_n + \omega c_{N-j+m+k-2} - \omega c_{N-j+k-2} \right] \Bigg) \;\geq\; 0.
\end{aligned}
$$

The positivity of this expression, which consists of two summands, follows from (1.13) and the induction assumption for $j+1$ and $k+1$.

□

3.4 Instantaneous Control of the Linear Wave Equation

In the previous section an analytical formula was deduced which provides the optimal value of Problem 3.8. The key assumption needed in order to apply the respective Theorem 3.18 is the controllability condition introduced in Section 3.1. In this section, Assumption 3.2 is deduced for the linear wave equation which allows to conclude asymptotic stability of the receding horizon closed loop.

The one dimensional linear wave equation with Dirichlet boundary condition and Neumann boundary control is considered, see (2.9) - (2.11). In Example 2.8 we tackled the task of stabilizing this hyperbolic partial differential equation at its unique equilibrium, i.e. the origin, by receding horizon control incorporating a terminal equality constraint. However, the finite propagation speed implied the need for an extremely long optimization horizon in order to satisfy the stabilizing terminal constraint and, thus, to ensure feasibility as well as stability in a sampled-data setting with sampling period $T \ll 2L/c$, cf. Section 2.2. We emphasize that preserving stability properties of a continuous time system typically requires sufficiently fast sampling, cf. [91]. For further results related to terminal constraints or terminal costs for infinite dimensional systems, we refer to [64].

Here, in contrast to Section 2.2, unconstrained RHC is used. Rationale for this approach are provided by numerical results: the linear wave equation is not only stabilizable but also performs well using RHC with the shortest feasible optimization horizon $N = 2$, also termed *instantaneous control*, cf. [62].[1] Our contribution to this problem is the complete theoretical analysis. In particular, we employ Theorem 3.18 in order to prove the observed stability rigorously for suitably chosen stage costs. Exploiting the derived formula allows us to establish this even for the combination of small sampling periods and RHC applied with the shortest feasible optimization horizon.

3.4.1 Constructing Suitable Stage Costs

In Example 2.8 the mathematical problem formulation and the corresponding solution space were already introduced. In addition, this continuous time system was rewritten as a discrete time one and the rough shape of appropriate stage costs was defined, cf. (2.13). Note that the function $\varrho(\cdot,\cdot)$ was not exactly specified, which opens up a certain degree of freedom in order to design the stage costs suitably. Our goal is to steer the system to the origin, which is the unique equilibrium. To this end, we consider the cost functional

$$J_N(y(\cdot,0),u(\cdot)) := \sum_{n=0}^{N-1} \frac{1}{4} \int_0^L \varrho(y_x(x,nT), y_t(x,nT))\,dx + \lambda \int_0^{NT} u(t)^2\,dt$$

which equals its continuous time counterpart (2.12). Since our methodology depends on (3.1), i.e. the relaxed Lyapunov inequality, suitable stage costs, which allow for establishing this estimate, have to be constructed. To this end, (2.9) - (2.11) is numerically investigated with parameters $L = c = 1$, $\lambda = 10^{-3}$, and sampling time $T = 0.025$. Let

[1] In literature, the term “instantaneous control” is also used in a different manner. In [59,60] instantaneous control means that the optimization routine — which is employed in order to compute a sequence of control values $u(\cdot) = (u(n))_{n\in\{0,1,2,\ldots,N-1\}}$ satisfying $J_N(x_0,u(\cdot)) = V_N(x_0)$ — is stopped prematurely in order to reduce the computational effort.

the initial data be specified by

$$y_0(x) := \begin{cases} +2x - 0.5 & \text{for } x \in (0.25, 0.5] \\ -2x + 1.5 & \text{for } x \in (0.5, 0.75] \\ 0 & \text{otherwise} \end{cases}$$

and $y_1(x) \equiv 0$. In order to solve the finite horizon optimal control problems, the spatial domain is discretized with discretization parameter $\Delta x = 0.001$. Furthermore, the canonical weight functions $\omega_1 = \omega_2 \equiv 1$ are chosen, i.e. $\varrho(\cdot, \cdot)$ is set to $\varrho(y_x(\cdot,t), y_t(\cdot,t)) = y_x(\cdot,t)^2 + (y_t(\cdot,t)/c)^2$, which corresponds to measuring the energy of the system at each multiple of the given time parameter T.

Our numerical computations indicate that receding horizon control stabilizes these initial data with this energy based stage costs. Since we aim at employing a relaxed Lyapunov inequality, the respective optimal value function $V_N(\cdot)$ is depicted, cf. the dashed line in Figure 3.1. Here, $V_N(\cdot)$ has plateaus, i.e. areas on which it exhibits constant values. Hence, the cost functional which is based solely on the energy of the system does not provide a strict decrease for the chosen initial data. As a consequence, $V_N(\cdot)$ does not satisfy (3.1), i.e. our key requisite, along the corresponding trajectory. Hence, although the system is asymptotically stable, $V_N(\cdot)$ can not be employed as a Lyapunov function in order to conclude this.

In the observed problem the finite propagation speed of the waves comes into effect. Since the energy of the chosen initial data is located in the middle of our domain Ω it can not be reduced by means of our boundary control during the first few sampling intervals and, in particular, up to time T. This explains why it is impossible to maintain a strict decrease on this time interval. As a remedy, we redesign the stage costs based on the prototype

$$\varrho(y_x(\cdot,t), y_t(\cdot,t)) = \omega_1(\cdot)(y_x(\cdot,t) + (y_t(\cdot,t)/c))^2 + \omega_2(\cdot)(y_x(\cdot,t) - (y_t(\cdot,t)/c))^2,$$

i.e. we split up the energy into two parts. The one weighted by $\omega_1(\cdot)$ represents the waves traveling to the left boundary, whereas the other takes the movement towards the right boundary, at which our control is located, into account, see also Remark 2.9. Using the weight functions

$$\omega_1(x) := 1 + L + x \qquad \text{and} \qquad \omega_2(x) := 1 + L - x \tag{3.33}$$

allows us to employ our cost functional for the desired purpose, i.e. for deducing asymptotic stability. The functions $\omega_i : [0, L] \to \mathbb{R}_0^+$, $i = 1, 2$, weigh the distance to the right boundary taking the direction of movement into consideration, i.e. they measure the time which has to pass until the respective portion of energy can be influenced. Figure 3.1 which depicts the optimal value function $V_2(\cdot)$ along the closed loop trajectories for $\omega_1 = \omega_2 \equiv 1$, i.e. the classical energy norm (dashed line), in comparison to its counterpart based on the weight functions defined above (solid line) puts it in a nutshell. Clearly, each of these two curves is monotonically decreasing, yet only the one corresponding to (3.33) is strictly decreasing.

3.4.2 Verifying Assumption 3.2 and Closed Loop Stability

The goal of this subsection is to deduce stability of the closed loop resulting from instantaneous control, i.e. receding horizon control with optimization horizon $N = 2$. To this end,

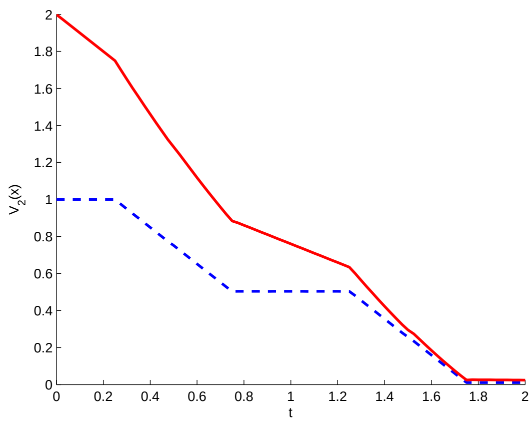


Figure 3.1: Comparison of the optimal value function $V_2(\cdot)$ for stage costs $\ell(\cdot,\cdot)$ based on different weight functions. The dashed curve corresponds to the energy norm, i.e. $\omega_1 = \omega_2 \equiv 1$. Whereas the continuous curve is constructed according to (3.33) and, thus, assesses the distance to the right boundary at which our control comes into effect.

the proposed controllability condition from Assumption 3.2 based on a $\mathcal{KL}_0$-function of type (1.11) is verified. In particular, an appropriate overshoot bound C and a decay rate σ is determined in the following proposition such that Theorem 3.18 ensures the relaxed Lyapunov inequality and, thus, the key assumption of our stability theorem. This allows us to conclude stability of the resulting receding horizon closed loop, i.e. guaranteeing that the RHC feedback steers the system asymptotically to its equilibrium.

Proposition 3.24
Consider the linear wave equation given by (2.9) - (2.11) *with sampling period* $T \leq L/c$. *Let the stage costs* $\ell(\cdot,\cdot)$ *be defined according to* (2.13) *using the weight functions from* (3.33). *Then, the control function* $u^\star(\cdot)$ *from* (2.14) *ensures exponential controllability, i.e. Assumption 3.2 based on a* $\mathcal{KL}_0$*-function of type* (1.11) *with overshoot bound* $C = (1+\lambda/c)$ *and decay rate* $\sigma = 1 - cT/(1+2L)$.

Proof: Before we start by the main part, the solution corresponding to the particular control function $u^\star(\cdot)$ from (2.14) is characterized in a preliminary step. To be more precise, we claim that the respective solution is given by (2.15) and, thus, coincides with the uncontrolled one of the linear wave equation on the semi-infinite interval $[0,\infty)$. Hence, employing $u^\star(\cdot)$ avoids reflections on the right boundary at which the Neumann boundary control takes effect. This noticeably simplifies the calculations involved in the rest of the proof.

In order to prove this identity, we have to show that (2.15) satisfies both the initial and the boundary conditions. The former follows directly by an easy computation whereas the latter is ensured by

$$y_x(L,t) = \frac{1}{2}\left[y_0'(L-ct) - \frac{y_1(L-ct)}{c}\right] = \frac{1}{2}\left[y_x(L-ct,0) - \frac{y_t(L-ct,0)}{c}\right] = u(0)$$

for $L > ct$. Replacing $L-ct$ by $ct-L$ yields the assertion for $ct > L$. Iterative application of this argument shows the assertion on $[0, iT)$ for all $i \in \mathbb{N}$.

We continue with estimating the overshoot constant C from (3.3) for the stage costs based on the specified weight functions. To this end, we show that the control effort caused by $u^\star(\cdot)$ is bounded by a certain portion of the costs induced by the current state. Using $\omega_i(x) \geq 1$, $i = 1, 2$, for all $x \in [0, L]$ yields

$$\begin{aligned} \frac{\lambda}{4} \int_0^T \left[y_x(L - ct, nT) - \frac{y_t(L - ct, nT)}{c} \right]^2 dt &= \frac{\lambda}{4c} \int_{L-cT}^L \left[y_x(x, nT) - \frac{y_t(x, nT)}{c} \right]^2 dx \\ &\leq \lambda/c \cdot \ell^\star(y(nT)) \end{aligned}$$

and, thus, provides

$$\ell(y(n), u(n)) \leq (1 + \lambda/c)\ \ell^\star(y(nT)) = C\ell^\star(y(nT)).$$

Hence, it remains to establish $\ell^\star(y(i + 1)) \leq \sigma\ \ell^\star(y(i))$ which is, in turn, equivalent to $\ell^\star(y(i)) - \ell^\star(y(i + 1)) \geq (1 - \sigma)\ell^\star(y(i))$. The decisive tools in order to verify this inequality are the particular control $u^\star(n)$ from (2.14) and the resulting evolution of the state according to the proven formula. In order to make the ensuing computations easier to follow, the derivatives of $y(x, t)$ from (2.15) are stated, i.e.

$$\begin{aligned} y_x(x, t) &= \begin{cases} \frac{1}{2} \left[y_0'(x + ct) + y_0'(x - ct)\right] + \frac{1}{2c} \left[y_1(x + ct) - y_1(x - ct)\right] & \text{for } x > ct \\ \frac{1}{2} \left[y_0'(ct + x) + y_0'(ct - x)\right] + \frac{1}{2c} \left[y_1(ct + x) + y_1(ct - x)\right] & \text{for } x < ct \end{cases}, \\ y_t(x, t) &= \begin{cases} \frac{c}{2} \left[y_0'(x + ct) - y_0'(x - ct)\right] + \frac{1}{2} \left[y_1(x + ct) + y_1(x - ct)\right] & \text{for } x > ct \\ \frac{c}{2} \left[y_0'(ct + x) - y_0'(ct - x)\right] + \frac{1}{2} \left[y_1(ct + x) - y_1(ct - x)\right] & \text{for } x < ct \end{cases}. \end{aligned}$$

Since $\varrho(\cdot, \cdot)$ is composed of two summands, the respective parts are treated separately. Splitting up the integral from (2.13) and using the calculated derivatives of (2.15) on their respective domains yields

$$\begin{aligned} &\int_0^L \omega_1(x) \left[y_x(x, T) + \frac{y_t(x, T)}{c} \right]^2 dx \\ = &\int_0^{cT} \omega_1(x) \left[y_0'(cT + x) + \frac{y_1(cT + x)}{c} \right]^2 dx + \int_{cT}^L \omega_1(x) \left[y_0'(x + cT) + \frac{y_1(x + cT)}{c} \right]^2 dx \\ = &\int_{cT}^L \omega_1(x - cT) \left[y_0'(x) + y_1(x)/c\right]^2 dx \end{aligned}$$

for the term in $\ell^\star(\cdot)$ containing $\omega_1(\cdot)$. Note that we employed $y_0'(x) = y_1(x) = 0$ for $x > L$ in order to deduce the last equality. Repeating the line of arguments for the other part and taking $\omega_2(cT - x) = \omega_1(x - cT)$ into account, provides

$$\begin{aligned} &\int_0^L \omega_2(x) \left[y_x(x, T) - y_t(x, T)/c\right]^2 dx \\ = &\int_0^{cT} \omega_2(x) \left[y_0'(cT - x) + y_1(cT - x)/c\right]^2 dx + \int_{cT}^L \omega_2(x) \left[y_0'(x - cT) - y_1(x - cT)/c\right]^2 dx \\ = &\int_0^{cT} \omega_1(x - cT) \left[y_0'(x) + y_1(x)/c\right]^2 dx + \int_0^{L-cT} \omega_2(x + cT) \left[y_0'(x) - y_1(x)/c\right]^2 dx. \end{aligned}$$

Combining these equalities leads to

$$\ell^\star(y(1)) = \int_0^L \rho(y_x(x, T), y_t(x, T))\, dx$$

$$= \int_0^L \omega_1(x - cT)\,[y_0'(x) + y_1(x)/c]^2\,dx + \int_0^{L-cT} \omega_2(x + cT)\,[y_0'(x) - y_1(x)/c]^2\,dx.$$

Subtracting this expression from $\ell^\star(y(0))$ and taking $\omega_1(x-cT) = \omega_1(x)-cT$, $\omega_2(x+cT) = \omega_2(x) - cT$, the non-negativity of $\omega_2(\cdot)$ on $[L - cT, L)$ as well as the boundedness of $\omega_i(\cdot)$, $i \in \{1, 2\}$, on Ω into account, allows for deducing the estimate

$$\ell^\star(y(0)) - \ell^\star(y(1)) \geq cT \int_0^L [y_0'(x) + y_1(x)/c]^2 + [y_0'(x) - y(x)/c]^2\,dx \geq \frac{cT}{1+2L}\ell^\star(y(0)).$$

Rearranging this inequality, taking account of $\sigma = 1-cT/(1+2L)$, and iteratively applying the resulting estimate, completes the proof.

□

Remark 3.25
The decrease reflected by the decay rate σ depends only on the chosen weight functions. In addition, an energy loss occurs with amount

$$\int_{L-cT}^L \omega_2(x + cT)[y_0'(x) - y_1(x)/c]^2\,dx$$

which represents the energy which is removed by means of the boundary control.

In order to prove Proposition 3.24, the fact that the control sequence in (3.3) does not have to be optimal is extensively used — a key feature of our approach which allowed us to employ a particular control function in order to simplify the involved computations significantly.

Indeed, Proposition 3.24 ensures exponential controllability in terms of the running costs and, thus, paves the way in order to apply our main stability theorem and, thus, to conclude asymptotic stability of the RHC closed loop with optimization horizon $N = 2$.

Theorem 3.26
Let the assumptions of Proposition 3.24 be satisfied. Furthermore, let the sampling period T satisfy

$$T > \frac{(2 + 4L)\,\lambda}{c\,(c + \lambda)}. \tag{3.34}$$

Then, the receding horizon closed loop with prediction horizon $N = 2$, i.e. instantaneous control, is asymptotically stable.

Proof: Exponential controllability in terms of the stage costs can be ensured by Proposition 3.24, i.e. Assumption 3.2 based on a $\mathcal{KL}_0$-function of type (1.11) with overshoot $C = (1 + \lambda/c)$ and decay rate $\sigma = 1 - cT/(1 + 2L)$. Since we consider instantaneous control, i.e. $N = 2$, Formula (3.21) simplifies to

$$\alpha := \alpha^1_{2,1} = 1 - (C(1 + \sigma) - 1)^2.$$

In order to deduce the desired relaxed Lyapunov inequality from Theorem 3.18, we require $\alpha > 0$ which is, in turn, equivalent to $2 > C(1+\sigma)$. Hence, inserting C and σ and solving the resulting inequality for T leads exactly to Condition (3.34) in order to establish $\alpha > 0$.

It remains to show (1.4) and, thus, Assumption 1.7. Then, Theorem 3.12 can be employed in order to conclude the assertion. To this end, we define the metric $d(y_1, y_2) := \ell^\star(y_1 - y_2)$ which is well defined in view of (2.10), i.e. the Dirichlet boundary condition, cf. [119, Section 2.3]. Hence, choosing the $\mathcal{K}_\infty$-functions $\alpha_1(r) = \alpha_2(r) = r$ ensures (1.4) and, as a consequence, completes the proof.

□

Using the parameters $L = c = 1$, Estimate 3.34 yields the condition $T > 6\lambda/(1+\lambda)$. Hence, the sampling interval has to be sufficiently large in order to allow for compensating the control effort which is reflected by the overshoot bound C. Taking the weight $\lambda = 10^{-3}$ into account, which penalizes the control effort, leads to $T > 6/1001 = 0.005994$. Consequently, stability of the receding horizon closed loop is ensured for $T = 0.006$, which shows that the needed optimization horizon of length $2T = 0.012$ is very short in comparison to $2L/c = 2$, i.e. the optimization horizon required for finite time controllability, cf. [52].

3.4.3 Numerical Results

The example of the linear wave equation from Subsection 3.4.1 is considered again. We observed that using stage costs based on the "classical" energy does not allow to employ $V_2(\cdot)$ as a Lyapunov function which satisfies a relaxed Lyapunov inequality of type (3.1) with $\alpha > 0$. In contrast to that, employing the weight functions defined in (3.33) resolves this problem and, thus, enables us to conclude asymptotic stability of the RHC closed loop. However, the deduced decay rate σ seems to be pessimistic at first glance because it only reflects the weight functions but not the additional energy loss according to Remark 3.25. In this subsection we show that the estimate for the decay rate σ, which was deduced in the previous subsection, is tight.

To this end, the corresponding values for the given initial data are computed. In order to visualize our theoretically calculated estimate, a horizontal line is drawn at $1 - T/3$ on the left hand side of Figure 3.2. The values calculated for the classical energy are arbitrarily close to one and exceed our estimated bound. Contrary to this, the values corresponding to the stage costs which incorporate (3.33) are smaller than but arbitrarily close to $1 - T/3$, which confirms our theoretical results. Hence, a further improvement of the deduced estimate is not possible.

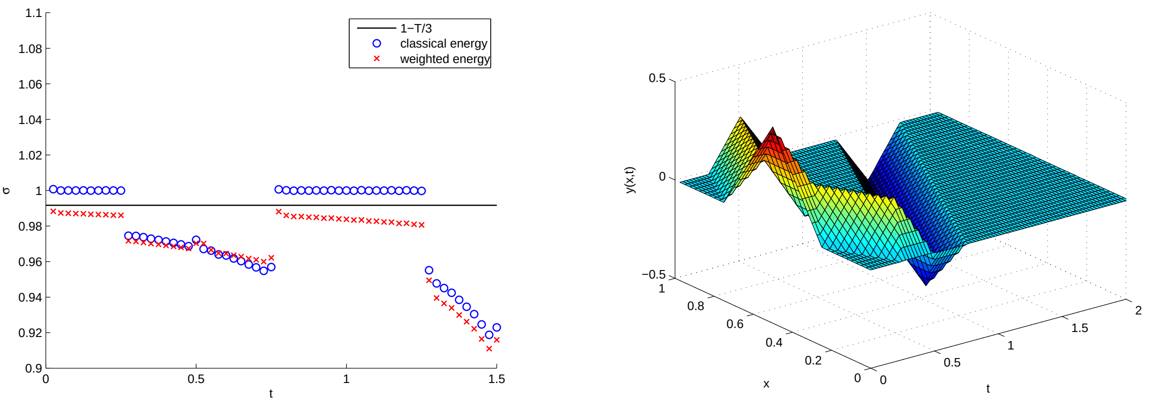


Figure 3.2: The corresponding values for the decay rate σ with respect to the classical ($\circ$) and the weighted energy (3.33, x) are depicted. Furthermore, a horizontal line is drawn at $1 - T/3$ in order to indicate our theoretically deduced bound. On the right, the solution trajectory for the instantaneous controlled wave equation, i.e. RHC with the shortest feasible optimization horizon ($N = 2$), is visualized for the initial data given in Subsection 3.4.1.

The solution trajectory of the instantaneous controlled wave equation is depicted in Figure 3.2 on the right. Indeed, it even coincides with the solution trajectory corresponding to an optimization horizon of length $2L/c = 2$, which is needed in order to show finite time controllability. Hence, model predictive control with $N = 2$ performs very well for the stabilization task in consideration. The overall computing time for solving the instantaneous control problem on the time interval $[0, 2]$ is less than one second — even for a fine spatial discretization, cf. [3].

The analysis of this subsection shows that instantaneous controllability of the one dimensional linear wave equation given by (2.9) - (2.11) can be rigorously proven by Theorem 3.18. Numerical results indicate that RHC also works well for the two dimensional wave equation. Hence, one of our future goals may be to deduce appropriate estimates for this setting as well. Furthermore, we like to point out that RHC based on the "classical" energy also performs well. Hence, a generalization of the proposed technique, which allows for dealing with sampling intervals which may neither improve nor deteriorate the reference quantity $V_N(\cdot)$, is desirable.

Summarizing, Assumption 3.2 is not merely an abstract condition. Rather, in connection with Formula (3.21) it can be used for analyzing differences in the receding horizon closed loop performance for different stage costs $\ell(\cdot, \cdot)$ and, thus, for developing design guidelines for selecting good running costs $\ell(\cdot, \cdot)$. This was also carried out, for instance, for a semi-linear parabolic PDE with distributed and boundary control in [5, 6] (see also [39] for a preliminary study).

Chapter 4

Sensitivity Analysis

We focus on discrete time systems which satisfy Assumption 3.2 with a $\mathcal{KL}_0$-function linear in its first argument. For this class, the nonlinear optimization Problem 3.8 (or its counterpart Problem 3.10 which includes an additional weight on the final term in the respective cost functional) becomes a linear program, cf. Lemma 3.14. Based on this observation, we deduced an explicit formula characterizing the corresponding optimal value $\alpha^{\omega}_{N,m}$ which depends on the optimization horizon $N \in \mathbb{N}_{\geq 2}$, the control horizon $m \in \{1, 2, \ldots, N-1\}$, and the terminal weight $\omega \geq 1$, cf. Theorem 3.18. The receding horizon algorithm yields, in each iteration, a sequence of N control values. The control horizon determines the number of elements of this sequence to be implemented at the plant before the RHC problem is solved again. In this chapter, a sensitivity analysis is carried out with respect to these parameters:

- In Section 3.3 we showed that a positive $\alpha^{\omega}_{N,m}$ is obtained for sufficiently long optimization horizon N which allows — under mild technical conditions, cf. Theorem 3.12 — to conclude asymptotic stability of the receding horizon closed loop. In Section 4.1 the impact of the optimization horizon N is further investigated. In particular, we aim at deducing asymptotic bounds on the required horizon length N in dependence of a given $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$. In this context the term minimal stabilizing horizon is introduced which denotes the smallest horizon N such that Theorem 3.18 guarantees a positive performance index $\alpha^{\omega}_{N,m}$. In addition, results concerning the different influence of the overshoot and the decay rate for exponentially controllable systems are given.

- In the subsequent Section 4.2 the influence of the control horizon m is considered. In particular, Formula (3.21) is exploited in order to establish symmetry and monotonicity properties which show that asymptotic stability of the receding horizon closed loop with time varying control horizons holds under the same conditions as for classical RHC. This result is not only essential in order to deal with networked control systems but also forms the core of the algorithms in the ensuing section.

In Section 4.4 two algorithms are designed — based on the sensitivity analysis carried out in the preceding sections. The first algorithm allows to significantly reduce the required optimization horizon length N in order to ensure a desired closed loop performance by employing control horizons $m > 1$. The second, further developed algorithm deals with the loss of robustness resulting from staying in open loop for longer periods of time while maintaining the stability guarantees of its predecessor.

4.1 Influence of the Optimization Horizon

Corollary 3.20 ensures, for sufficiently large optimization horizon N, asymptotic stability — a result which was already shown in [32] under similar conditions (see also [65] for an analogous result in continuous time). Additionally, Corollary 3.20 generalizes this assertion to arbitrary, but fixed control horizons m. Using the same argumentation as in the proof of Theorem 3.12 allows to conclude asymptotic stability for time varying control horizons $(m_i)_{i\in\mathbb{N}_0} \subseteq M \subseteq \{1, 2, \ldots, m^\star\}$ for an arbitrary, but fixed number $m^\star \in \mathbb{N}$. Combining the inequality $\alpha^1_{N,m} V_\infty^{\mu_{N,m}}(\cdot) \leq V_N(\cdot)$ from Theorem 3.7 and the inequality $V_N(\cdot) \leq V_\infty(\cdot)$, which is ensured by the monotonicity of $V_N(\cdot)$ for $\omega = 1$, implies that the infinite horizon cost $V_\infty^{\mu_{N,m}}(\cdot)$ converges to the optimal value $V_\infty(\cdot)$.

In this section, suppose that a control horizon $m \in \mathbb{N}$ and a terminal weight $\omega \geq 1$ are given. A detailed sensitivity analysis is carried out in order to investigate the impact of the optimization horizon N. We are, in particular, interested in so called stabilizing horizons, i.e. optimization horizons N guaranteeing $\alpha^\omega_{N,m} \geq 0$, and, thus, stability. In this context, two questions are tackled:

(1) Let an optimization horizon N be given. Which class $\mathcal{KL}_0$-functions $\beta(\cdot,\cdot)$ can be employed in Assumption 3.2 in order to conclude $\alpha^\omega_{N,m} \geq 0$ via Theorem 3.18?

Here, $\beta(\cdot,\cdot)$ is assumed to be of type (1.11), i.e. $\beta(r, n) = C\sigma^n r$. Furthermore, we want to elaborate design guidelines. To this end, the interplay of the overshoot C and the decay rate σ is taken into account.

(2) Let Assumption 3.2 be satisfied with a $\mathcal{KL}_0$-function linear in its first argument and γ be defined as the accumulated bound $\sum_{n=0}^\infty \beta(r, n)/r = \sum_{n=0}^\infty c_n$ from the controllability condition (3.3). How does the minimal optimization horizon N ensuring stability via Theorem 3.18 depend on this quantity γ?

Here, our main emphasis is put on the asymptotic growth of the minimal stabilizing horizon with respect to γ. We point out that Formula (3.21) enables us to prove numerical observations from [39] rigorously.

In order to answer the first question, all parameter combinations (C, σ) implying a non-negative suboptimality index $\alpha^\omega_{N,m}$ and, thus, stability for a given optimization horizon N are calculated , cf. Figure 4.1.[1]

As expected, the stability region grows with increasing optimization horizon N. Theorem 3.18 allows us to quantify the observed enlargement, e.g. doubling $N = 2$ increases the considered area by 129.4 percent. Furthermore, we observe that for a given decay rate σ there always exists an overshoot C such that stability is guaranteed. Indeed, Theorem 3.18 enables us to prove this. To this end, we deal with the special case $C = 1$ which yields a significantly simpler expression for $\alpha^\omega_{N,m}$.

Proposition 4.1
Assume exponential controllability without overshoot, i.e. Assumption 3.2 with a $\mathcal{KL}_0$-function of type (1.11) *with* $C = 1$. *Then, the optimal value $\alpha^\omega_{N,m}$ of Problem 3.10 is equal to* $\min\{1,\, 1 - (1 + \sigma\omega - \omega)\sigma^{N-1}\}$ *and strictly positive, i.e.* $\alpha^\omega_{N,m} > 0$.

[1] The idea of visualizing the parameter dependent stability regions in this way goes back to [121].

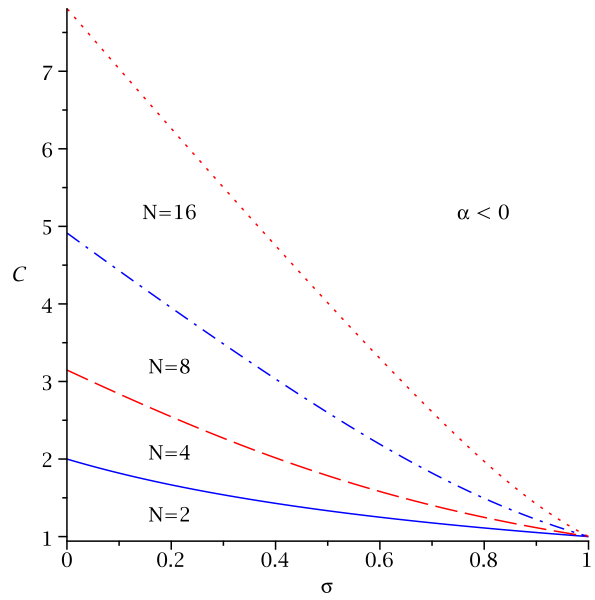


Figure 4.1: Illustration of the stability region guaranteed by Theorem 3.18 for various optimization horizons N given a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ of type (1.11) for RHC with $m=1$.

Proof: Defining the auxiliary quantity $\eta := 1 + \sigma\omega - \omega$, we obtain

$$\gamma_i = \frac{1-\eta\,\sigma^{i-1}}{1-\sigma}, \qquad \gamma_i - 1 = \frac{\sigma(1-\eta\,\sigma^{i-2})}{1-\sigma}, \qquad \gamma_{m+1} - \omega = \frac{\eta(1-\sigma^m)}{1-\sigma}.$$

Hence, the necessary and sufficient condition $(\gamma_{m+1}-\omega) \leq 0$ from Theorem 3.18 holds if and only if the condition $\eta \leq 0$ is satisfied — an equivalence which is reflected by taking the minimum. It remains to consider $\eta > 0$ which ensures that $\alpha^\omega_{N,m}$ is given by (3.21). As a preparatory result, each of the two factors occurring in the respective denominator is investigated separately, i.e.

$$\begin{aligned}
\prod_{i=N-m+1}^{N} \gamma_i - \prod_{i=N-m+1}^{N} (\gamma_i - 1) &= \prod_{i=N-m+1}^{N} \frac{1-\eta\,\sigma^{i-1}}{1-\sigma} - \prod_{i=N-m+1}^{N} \frac{\sigma(1-\eta\,\sigma^{i-2})}{1-\sigma} \\
&= \frac{1-\eta\,\sigma^{N-1} - (1-\eta\,\sigma^{N-m-1})\sigma^m}{1-\sigma} \cdot \prod_{i=N-m+2}^{N} \frac{1-\eta\,\sigma^{i-2}}{1-\sigma} \\
&= \frac{1-\sigma^m}{1-\sigma} \cdot \prod_{i=N-m+2}^{N} \frac{1-\eta\,\sigma^{i-2}}{1-\sigma}
\end{aligned}$$

and, repeating the same line of arguments,

$$\prod_{i=m+1}^{N} \gamma_i - (\gamma_{m+1}-\omega) \prod_{i=m+1}^{N} (\gamma_i - 1) = \frac{1-\eta\,\sigma^{N-m-1}}{1-\sigma} \cdot \prod_{i=m+2}^{N} \frac{1-\eta\,\sigma^{i-2}}{1-\sigma}.$$

Inserting these expressions into Formula (3.21) yields

$$\alpha^\omega_{N,m} \;=\; 1 - \frac{\frac{\eta\,(1-\sigma^m)}{1-\sigma} \prod_{i=m+2}^{N} \frac{\sigma(1-\eta\,\sigma^{i-2})}{1-\sigma} \prod_{i=N-m+1}^{N} \frac{\sigma(1-\eta\,\sigma^{i-2})}{1-\sigma}}{\left(\frac{1-\eta\,\sigma^{N-m-1}}{1-\sigma} \cdot \prod_{i=m+2}^{N} \frac{1-\eta\,\sigma^{i-2}}{1-\sigma}\right)\left(\frac{1-\sigma^m}{1-\sigma} \cdot \prod_{i=N-m+2}^{N} \frac{1-\eta\,\sigma^{i-2}}{1-\sigma}\right)}$$

$$= \quad 1 - \frac{\frac{\eta(1-\sigma^m)}{1-\sigma} \cdot \sigma^{N-m-1} \prod_{i=m+2}^{N} \frac{1-\eta\sigma^{i-2}}{1-\sigma} \cdot \sigma^m \prod_{i=N-m+1}^{N} \frac{1-\eta\sigma^{i-2}}{1-\sigma}}{\frac{1-\sigma^m}{1-\sigma} \cdot \prod_{i=m+2}^{N} \frac{1-\eta\sigma^{i-2}}{1-\sigma} \prod_{i=N-m+1}^{N} \frac{1-\eta\sigma^{i-2}}{1-\sigma}} = 1 - \eta\,\sigma^{N-1}.$$

□

Remark 4.2
Note that the optimal value $\alpha^{\omega}_{N,m}$, i.e. the solution of Problem 3.10, does not depend on the control horizon m for $C = 1$. Consequently, the control horizon m does not play a role for this special case.

Proposition 4.1 states that we always obtain a strictly positive value $\alpha^{\omega}_{N,m}$ for $C = 1$. Due to continuity of the involved expressions this remains true for $C = 1+\varepsilon$ for sufficiently small ε. Hence, for any decay rate $\sigma \in (0,1)$ and sufficiently small $C = C(\sigma) > 1$ (depending on N, m and ω) $\alpha^{\omega}_{N,m} > 0$ is obtained. Recall that a positive performance index $\alpha^{\omega}_{N,m}$ is the key ingredient in Theorem 3.12 in order to deduce asymptotic stability. However, this property does not hold if we exchange the roles of σ and C, i.e. for a given overshoot $C > 1$ stability cannot in general be concluded for a sufficiently small decay rate $\sigma > 0$, cf. Figure 4.1.

Next, the interplay of the optimization horizon N and $\gamma = \sum_{n=0}^{\infty} c_n$ is studied in order to adress question number (2). We aim at determining the asymptotic growth rate of the minimal optimization horizon N guaranteeing stability for a given parameter γ. To this end, we assume finite time controllability in one step, i.e. Assumption 3.2 using a $\mathcal{KL}_0$-function of type (1.12) defined by $c_0 = \gamma$ and $c_n = 0$ for all $n \in \mathbb{N}_{\geq 1}$. For given γ, this represents, as will be seen in the proof of Theorem 4.4, the worst case over all $\mathcal{KL}_0$-functions $\beta(\cdot,\cdot)$ which are linear in their first arguments — at least for the setting without an additional terminal weight, i.e. $\omega = 1$. This fact will be of particular use in order to prove Theorem 4.4. Note that for this problem the additional condition (1.13) is automatically satisfied (since (3.3) ensures $c_0 \geq 1$). Hence, Theorem 3.18 characterizes the optimal value of Problem 3.10 exactly.

We focus on $m = 1$, i.e. the smallest possible control horizon, and $m = \lfloor N/2 \rfloor$, i.e. the control horizon implying – at least in the exponentially controllable and the finite time controllable in a maximum of two steps case the largest $\alpha^{\omega}_{N,m}$ value, cf. Section 4.2 below.

Corollary 4.3
Let $\omega \geq 1$ be given and Assumption 3.2 hold with $\beta(r,n) = r \cdot c_n$, $c_0 = \gamma$ and $c_i = 0$ for $i \in \mathbb{N}$, i.e. finite time controllability in one step. Furthermore, let the minimal stabilizing horizon be defined as

$$\hat{N}(\gamma) := \min\{N \in \mathbb{N}_{\geq 2} : \alpha^{\omega}_{N,m} \geq 0 \textit{ for } \alpha^{\omega}_{N,m} \textit{ given by } (3.21) \textit{ based on } \beta(r,n)\},$$

i.e. the smallest optimization horizon N guaranteeing that the solution $\alpha^{\omega}_{N,m}$ of the linear program given by Problem 3.10 is positive. Then, $\hat{N}(\gamma)$ behaves,

- *for $m = 1$, asymptotically like $\gamma \ln \gamma$, i.e. $\lim_{\gamma\to\infty} \frac{\hat{N}(\gamma)}{\gamma \ln \gamma} = 1$, and*
- *for $m = \lfloor N/2 \rfloor$, asymptotically like $2\ln 2 \cdot \gamma$, i.e. $\lim_{\gamma\to\infty} \frac{\hat{N}(\gamma)}{2\ln 2\cdot\gamma} = 1$.*

Proof: Since Corollary 4.3 deals with the asymptotic behavior with respect to γ, let γ be strictly greater than $\omega \geq 1$. Furthermore, note that, for finite time controllability in one step, $\gamma_i = \gamma$ holds for all $i \in \mathbb{N}_{\geq 2}$ independently of the chosen terminal weight. Hence, Formula (3.21) yields

$$\alpha_{N,m}^{\omega} = 1 - \frac{(\gamma - \omega)(\gamma - 1)^{N-1}}{(\gamma^{N-m} - (\gamma - \omega)(\gamma - 1)^{N-m-1})(\gamma^m - (\gamma - 1)^m)}. \tag{4.1}$$

For $m = 1$, we require a positive optimal value of Problem 3.10 in order to ensure stability. i.e.

$$\alpha_{N,1}^{\omega} = 1 - \frac{(\gamma - \omega)(\gamma - 1)^{N-1}}{\gamma^{N-1} - (\gamma - \omega)(\gamma - 1)^{N-2}} = \frac{\gamma^{N-1} - \gamma(\gamma - \omega)(\gamma - 1)^{N-2}}{\gamma^{N-1} - (\gamma - \omega)(\gamma - 1)^{N-2}} \geq 0$$

This inequality holds if and only if the nominator is positive. Since the logarithm is monotonically increasing, this is, after dividing by γ, equivalent to

$$N \geq 2 + \frac{\ln(\gamma - \omega)}{\ln \gamma - \ln(\gamma - 1)} =: f(\gamma).$$

We show that $f(\gamma)$ tends to $\gamma \ln \gamma$ asymptotically. To this end, we consider

$$\lim_{\gamma \to \infty} \frac{f(\gamma)}{\gamma \ln \gamma} = \underbrace{\lim_{\gamma \to \infty} \frac{2}{\gamma \ln \gamma}}_{=0} + \underbrace{\lim_{\gamma \to \infty} \frac{\ln(\gamma - \omega)}{\ln \gamma}}_{=1} \cdot \lim_{\gamma \to \infty} \frac{\frac{1}{\gamma}}{\ln \gamma - \ln(\gamma - 1)} = \lim_{\gamma \to \infty} \frac{\gamma(\gamma - 1)}{\gamma^2} = 1$$

where we have used l'Hôpital's rule, cf. [124, Subsection 5.4.4]. Clearly, rounding up the derived expression for the optimization horizon N does not change the obtained result.

For $m > 1$, (4.1) and, thus, Theorem 3.18 yields $\alpha_{N,m}^{\omega} > 0$ if and only if

$$\gamma^N \geq \gamma^{N-m}(\gamma - 1)^m + (\gamma - \omega)(\gamma - 1)^{N-m-1}\gamma^m.$$

Hence, for $m = \lfloor N/2 \rfloor$ we obtain analogously the following lower bounds for the optimization horizon N:

$$N \geq \begin{cases} 2 \ln\left(\frac{2\gamma - \omega - 1}{\gamma - 1}\right) / (\ln \gamma - \ln(\gamma - 1)) & \text{for even } N \\ \left(\ln\left(\frac{2\gamma - \omega}{\gamma}\right) + \ln\left(\frac{2\gamma - \omega}{\gamma - 1}\right)\right) / (\ln \gamma - \ln(\gamma - 1)) & \text{for odd } N \end{cases}$$

Again in consideration of L'Hôpital's rule, the investigated expression exhibits asymptotically a behavior like $2 \ln 2 \cdot \gamma$. Since the obtained approximation $2 \ln 2 \cdot \gamma$ holds for both estimates corresponding to even and odd numbers N for $m = \lfloor N/2 \rfloor$, the assertion holds.

$\square$

Figure 4.2 illustrates the resulting horizon lengths for given γ. We like to point out that these estimates coincide with the numerical results derived in [39, Section 6].[2]

Corollary 4.3 deals with the asymptotic growth rate of the minimal stabilizing horizon $\hat{N}$ for arbitrary, but fixed terminal weight $\omega \geq 1$. We point out that, for finite time controllability in one step, γ_i, $i \in \mathbb{N}_{\geq 2}$, is independent of ω. Hence, the sequence $(\gamma_i)_{i \in \mathbb{N}_{\geq 2}}$ is also for $\omega > 1$ non-decreasing — a property which is important for the proof of the following theorem but cannot be assumed for arbitrary $\mathcal{KL}_0$-functions satisfying $\sum_{n=0}^{\infty} c_n = \gamma$ and (1.13). Theorem 4.4 shows that the estimates from Corollary 4.3 carry over to arbitrary $\mathcal{KL}_0$-functions for $\omega = 1$. The assertion for $m = 1$ was also deduced in [120] based on similar assumptions.

[2] Indeed, we determined precisely the constant $2 \ln 2$ for the linear growth estimate in contrast to the numerically observed factor $\sqrt{2}$.

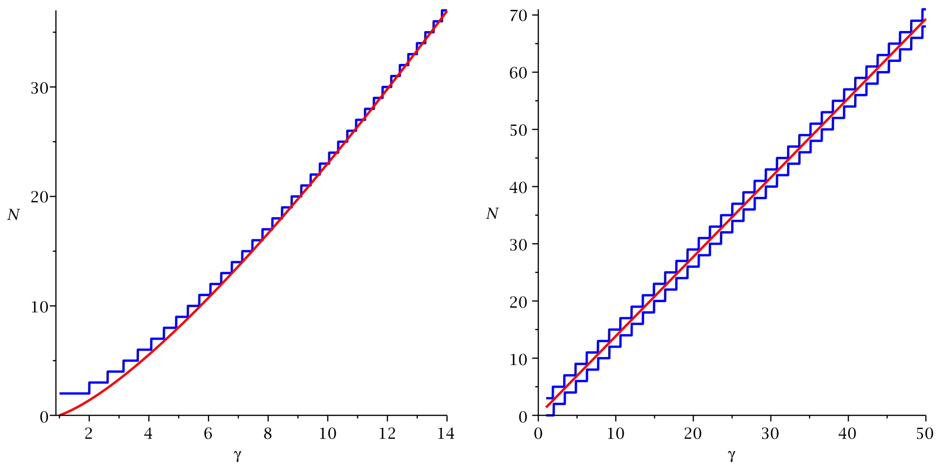


Figure 4.2: Minimal stabilizing optimization horizons for one step finite time controllability for $m = 1$ and $m = \lfloor N/2 \rfloor$ in comparison with their asymptotic approximations.

Theorem 4.4
Let Assumption 3.2 be satisfied with a $\mathcal{KL}_0$-function linear in its first argument, $\omega = 1$, and define $\gamma = \sum_{n=0}^{\infty} c_n$. Then, the asymptotic growth rate of the minimal stabilizing horizon $\hat{N}$ is bounded by $\gamma \ln \gamma$ and $2 \ln 2 \cdot \gamma$ for $m = 1$ and $m = \lfloor N/2 \rfloor$, respectively.

Proof: In order to show the assertion, take a closer look at Problem 3.8. Here, $\beta(\cdot, \cdot)$ from Assumption 3.2 is incorporated in the upper bounds of the constraints. Hence, using finite time controllability in one step relaxes the constraints and, thus, enlarges the feasible set for the posed minimization problem in contrast to every other $\mathcal{KL}_0$-function summing up to γ. Hence, Theorem 4.4 is a direct consequence of Corollary 4.3.

□

Summarizing, the derived estimates provide upper bounds on the growth rate of the minimal stabilizing horizon, e.g. for $c_0 = \gamma = C \sum_{n=0}^{\infty} \sigma^n$ with $C \geq 1$, $\sigma \in (0, 1)$. Moreover, for $m = \lfloor N/2 \rfloor$, Theorem 4.4 exhibits a linear bound. Hence, the corresponding growth rate is linear or even slower. Furthermore, note that the additional property (1.13) is not needed in order to establish Theorem 4.4, cf. Remark 3.19.

In order to conclude this section, the following remark is given which deals with the setting based on a fixed γ but allows to vary the terminal weight ω.

Remark 4.5
Let Assumption 3.2 be satisfied with a $\mathcal{KL}_0$-function of type (1.12) given by $c_0 := \gamma$ and $c_i = 0$, $i \in \mathbb{N}$. Then, choosing the terminal weight large enough always implies $\gamma_{m+1} - \omega = \gamma - \omega \leq 0$ and, as a consequence, $\alpha_{N,m}^{\omega} = 1$. This observation reflects an important property of finite time controllable systems: typically, the optimization horizon has to be sufficiently large in order to ensure that it is preferable to overcome the obstacle despite the needed control effort represented by $c_0 = \gamma$. This dilemma can be resolved by putting more emphasis on the final state of the prediction horizon.

4.2 Characteristics Depending on the Control Horizon

Delays and packet dropouts, which typically occur for networked control systems, motivated the introduction of multistep feedback laws, cf. Definition 1.25. Based on these preliminary considerations Theorem 3.12 was formulated for time varying control horizons $(m_i)_{i \in \mathbb{N}_0}$. In order to check the conditions of Theorem 3.12, appropriate solutions α^{ω}_{N,m_i}, $i \in \mathbb{N}_0$, of Problem 3.10 are needed. At first glance, the conditions of this theorem appear to be more demanding for time varying than for fixed control horizon. However, in this section — based on our standard Assumption 3.2 with a $\mathcal{KL}_0$-function which exhibits linearity in its first argument — we prove that the conditions coincide with those for $m = 1$ for a large subclass of such $\mathcal{KL}_0$-functions including exponentially decaying ones. Summarizing, the described problem of time varying control horizons is resolved.

To this end, we carry out a sensitivity analysis with respect to the control horizon m which determines the number of elements of our computed sequence of control values to be implemented at the plant. Particularly, we establish symmetry and monotonicity properties of $\alpha^{\omega}_{N,m}$ which may be counter-intuitive, e.g. increasing the control horizon in the interval $[1, \lfloor N/2 \rfloor] \subset \mathbb{N}$ improves the performance bounds from Theorem 3.12. This coincides with our observation from the previous section that the upper bounds from Theorem 4.4, which connect the needed control effort on the infinite horizon to the minimal stabilizing horizon length, grow only linearly for $m = \lfloor N/2 \rfloor$ instead of superlinearly ($\gamma \ln \gamma$) for $m = 1$. Furthermore, we deduce a symmetry property which enables us to handle control horizons $m \in \{\lfloor N/2 \rfloor + 1, \ldots, N-2, N-1\}$. Combining this with the derived monotonicity, enables us to show our main result in this section, namely, that stability for RHC control for time varying control horizons via our main tool Theorem 3.18 can be guaranteed under the same conditions as for $m = 1$. The results which are derived in this section form the basis for the algorithm developed in Section 4.4 which allows for significantly reducing the optimization horizon N and, thus, demonstrates the practical use of these theoretical results.

This section is subdivided into two parts. We start, after providing some insight into our motivation, with the main results which are discussed directly afterward. In the following subsections the corresponding proofs, which are rather technical, are presented. Here, we like to point out the elaborate technique thought up in order to deal with the exponentially controllable case.

4.2.1 Presenting the Results

We begin by looking at Figure 4.3 which depicts performance bounds $\alpha^{\omega}_{N,m}$ for control horizons $m \in \{1, 2, \ldots, N-1\}$ for an exponentially decaying function $\beta : \mathbb{R}^+_0 \times \mathbb{N}_0 \to \mathbb{R}^+_0$,

$$\beta(r, n) = C\sigma^n \cdot r \qquad \text{with } C = 2 \text{ and } \sigma = 0.625, \tag{4.2}$$

and a $\mathcal{KL}_0$-function $\beta(\cdot, \cdot)$ characterizing finite time controllability defined by

$$c_0 = 1,\ c_1 = 5/4,\ c_2 = 3/2,\ c_3 = 5/4,\ c_4 = 1/2,\ c_5 = 1/4,\ c_6 = 1/16, \text{ and } c_n = 0,\ n \in \mathbb{N}_{\geq 7}. \tag{4.3}$$

Note that the latter, which is a $\mathcal{KL}_0$-function of type (1.12) satisfying (1.13), is not monotonically decreasing. These examples exhibit the key features with respect to the corresponding suboptimality indices.

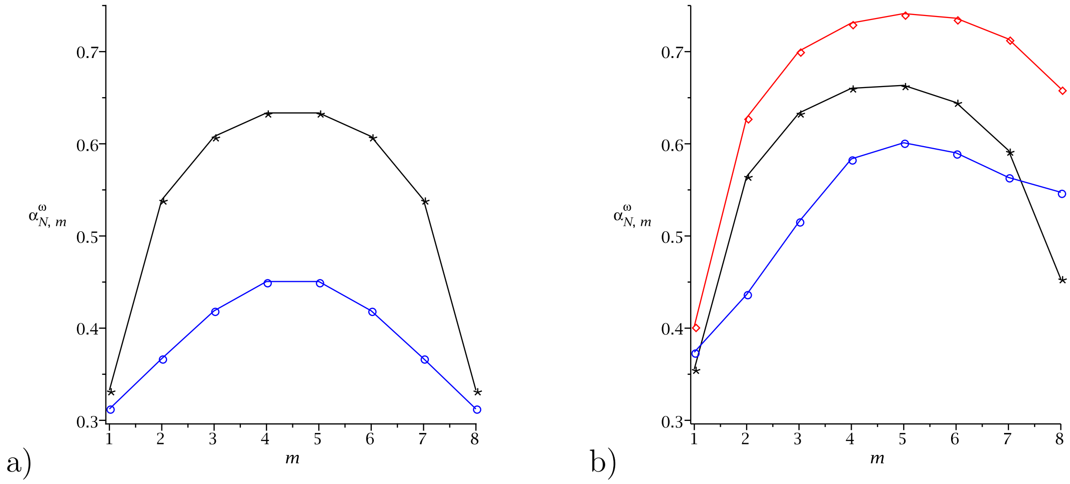


Figure 4.3: In a) the performance bounds $\alpha^1_{9,m}$, $m = 1, 2, \ldots, 8$, from Theorem 3.18 are illustrated for $\mathcal{KL}_0$-functions given by (4.2, $\star$) and (4.3, $\circ$). Whereas in b) a terminal weight was added, i.e. ($\omega = 5/4$, $\star$) and ($\omega = 2$, $\diamond$) for the exponentially controllable (4.2) and ($\omega = 2$, $\circ$) for the finite time controllability case (4.3).

In Figure 4.3 a) two properties can be observed for the setting without an additional weight on the final term:

- monotonicity, i.e. increasing the control horizon in the interval $[1, 2, \ldots, \lfloor N/2 \rfloor]$ improves the optimal value $\alpha^1_{N,m}$ of Problem 3.8, and
- symmetry, i.e. $\alpha^1_{N,m} = \alpha^1_{N,N-m}$, $m = 1, 2, \ldots, \lfloor N/2 \rfloor$, holds for the computed suboptimality estimates.

The interplay of these two properties ensures $\alpha^1_{N,1} \leq \alpha^1_{N,m}$ for each $m \in \{1, 2, \ldots, N-1\}$. This observation will be essential for the proof of Theorem 4.8. Using terminal weights $\omega > 1$ leads — at least in this example — to a further improvement of the guaranteed stability behavior. But instead of symmetry, Figure 4.3 b) exhibits $\alpha^{\omega}_{N,m} \leq \alpha^{\omega}_{N,N-m}$, $m = 1, 2, \ldots, \lfloor N/2 \rfloor$.

Before continuing our study, we state the corresponding results concerning symmetry and monotonicity properties of the optimal value $\alpha^{\omega}_{N,m}$ of Problem 3.10 with respect to the control horizon m. The following two propositions – which are proven in Subsections 4.2.2 and 4.2.3 – do not only pave the way to answer the encountered question for networked control systems and prepare the ground in order to develop an algorithm in Section 4.4 but are also interesting in their own rights.

Proposition 4.6
Let $\beta(\cdot,\cdot) \in \mathcal{KL}_0$ from Assumption 3.2 be either of type (1.11) *or of type* (1.12) *with $c_n = 0$ for $n \geq 3$ satisfying* (1.13). *Then, for $N \in \mathbb{N}_{\geq 2}$ and $\omega \geq 1$, $\alpha^{\omega}_{N,m}$ from Theorem 3.18 satisfies the* symmetric bound

$$\alpha^{\omega}_{N,m} \leq \alpha^{\omega}_{N,N-m} \qquad \textit{for} \qquad m \in \{1, 2, \ldots, \lfloor N/2 \rfloor\}.$$

Proposition 4.7
Suppose $\beta(\cdot,\cdot) \in \mathcal{KL}_0$ from Assumption 3.2 to be either of type (1.11) *with terminal weight*

$\omega \in \{1\} \cup [1/(1-\sigma), \infty)$ or of type (1.12) with $c_n = 0$ for $n \geq 2$ and arbitrary $\omega \geq 1$. Then, for $N \in \mathbb{N}_{\geq 2}$, $\alpha^\omega_{N,m}$ from Theorem 3.18 fulfills

$$\alpha^\omega_{N,m+1} \geq \alpha^\omega_{N,m} \qquad \text{for} \qquad m \in \{1, \ldots, \lfloor N/2 \rfloor - 1\}.$$

Using the symmetric bound from Proposition 4.6 and the monotonicity property from Proposition 4.7 the following noteworthy consequence for our stabilization problem can be concluded.

Theorem 4.8
Let $\beta(\cdot,\cdot) \in \mathcal{KL}_0$ from Assumption 3.2 be either of type (1.11) with terminal weight $\omega \in \{1\} \cup [1/(1-\sigma), \infty)$ or of type (1.12) with $c_n = 0$ for $n \geq 2$ and arbitrary $\omega \geq 1$. Then, for each $N \geq 2$, the stability criterion from Theorem 3.12 is satisfied for $m^\star = N-1$ if and only if it is satisfied for $m^\star = 1$.

Proof: Proposition 4.6 and 4.7 imply $\alpha^\omega_{N,m} \geq \alpha^\omega_{N,1}$ for all $m \in M \subseteq \{1, 2, \ldots, N-1\}$ which yields the assertion.

□

In other words, for exponentially controllable systems without or with sufficiently large terminal weight and for systems which are finite time controllable in at most two steps, we obtain stability for our proposed networked MPC scheme under exactly the same conditions as for MPC with $m^\star = 1$. In this context we recall once again that for $m^\star = 1$ the stability condition of Theorem 3.12 is tight, cf. Remark 3.13.

Similar to our course of action in Section 4.1, we investigate the stability region for exponentially controllable systems with respect to their stage costs, i.e. the set of all parameter combinations of overshoot $C \geq 1$ and decay rate $\sigma \in (0,1)$ such that stability of the underlying discrete time systems is guaranteed by Theorem 3.18.

The investigation of the stability region for exponentially controllable systems in terms of their stage costs is continued. The stability region contains all parameter combinations of overshoot $C \geq 1$ and decay rate $\sigma \in (0,1)$ such that stability of the underlying discrete time systems is guaranteed by Theorem 3.18. Here, the focus is shifted from the optimization horizon N, cf. Section 4.1, to the control horizon m. For simplicity of exposition, the case $\omega = 1$ without an additional weight on the final term is considered. Having in mind the proposed results, in particular Proposition 4.6 which holds with equality for $\omega = 1$, cf. Corollary 4.10, only control horizons $m \in \{1, \ldots, \lfloor N/2 \rfloor\}$ have to be dealt with. For instance, Figure 4.4 shows the stability regions for $N = 7$ and $N = 11$, respectively.

Apparently, increasing the control horizon enlarges the stability region, e.g. allows for larger overshoots C for given decay rates σ. This observation confirms our theoretical results, i.e. the monotonicity property claimed in Proposition 4.7 is reflected. In addition, the growth of the stability region can be quantified, e.g. for optimization horizon $N = 7$: the area containing feasible (C, σ) pairs is scaled up by 21 ($m = 2$) and 30 ($m = 3$) percent. For longer optimization horizons ($N = 11$) increasing the control horizon enhances the attainable gain even further, e.g. $m = 2$ and $m = 5$ enlarge the stability region by 23 and 48 percent, respectively.

In contrast to the exponentially controllable case, restrictions have to be imposed on class $\mathcal{KL}_0$-functions satisfying (1.12) in Theorem 4.8 — although (1.13) is satisfied. Still, we expended the effort to give a complete characterization referring to this setting, cf.

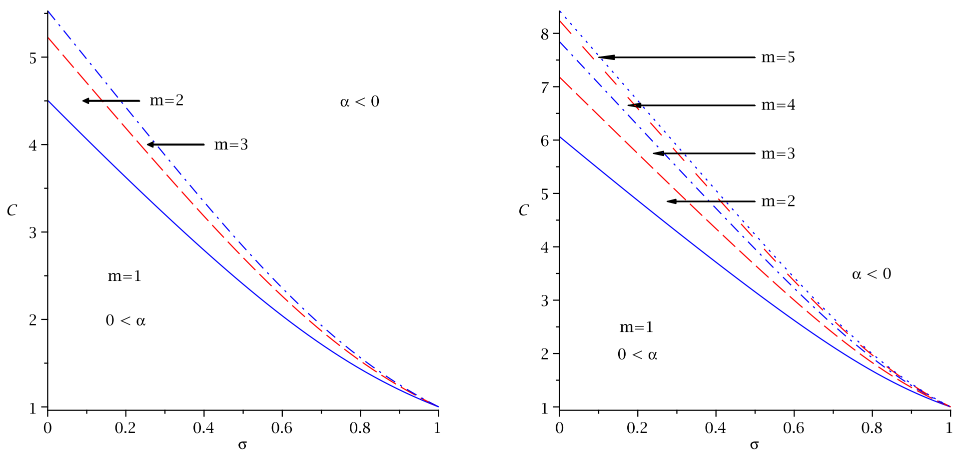


Figure 4.4: Illustration of parameter combinations (C, σ) which ensure stability by Theorem 3.18 depending on the control horizon m for optimization horizons $N = 7$ and $N = 11$, respectively.

Propositions 4.6 and 4.7 as well as Subsections 4.2.2 and 4.2.3. The reason for putting so much emphasis on this is given in the following example.

Example 4.9
Consider the $\mathcal{KL}_0$-function $\beta_1(\cdot,\cdot)$ of type (1.12) defined by $c_0 = 5/2$, $c_1 = 2$, $c_2 = 3/2$, $c_3 = 32/25$, $c_4 = 1$, $c_5 = 1/2$, $c_6 = 1/8$, and $c_i = 0$ for all $i \in \mathbb{N}_{\geq 7}$. An upper bound is constructed by choosing $C = 5/2$ and $\sigma = 4/5$, i.e. a $\mathcal{KL}$-function $\beta_2(\cdot,\cdot)$ of type (1.11), cf. Figure 4.5 on the right. Although this seems to be a good approximation, the corresponding optimal values $\alpha^{\omega}_{N,\cdot}$ of Problem 3.10 are significantly worse, cf. Figure 4.5 on the left. For instance, using the upper bound $\beta_2(\cdot,\cdot)$, stability can not be guaranteed for control horizons $m \in \{2, 3, 4, 12, 13, 14\}$ while $\alpha^1_{N,m} \geq 0$ holds for $\beta_1(\cdot,\cdot)$.

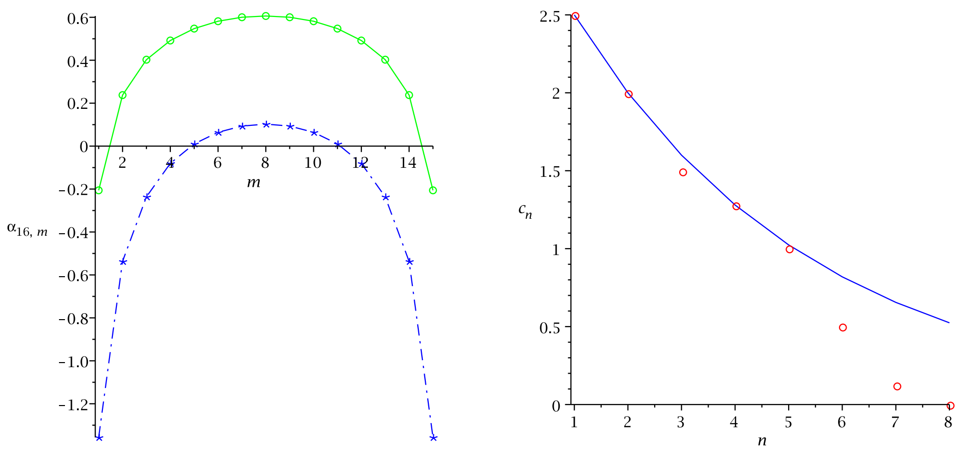


Figure 4.5: On the left a comparison of the optimal values $\alpha^1_{16,\cdot}$ of Problem 3.8 for $\mathcal{KL}_0$-functions of type (1.11,$\circ$) and (1.12,$\star$) is illustrated. On the right we depict the corresponding $\mathcal{KL}_0$-functions.

Hence, it is in general favorable to work with a $\mathcal{KL}_0$-function ensuring finite time controllability in contrast to using an upper bound provided by an estimated $\mathcal{KL}$-function of type (1.11). This conclusion is substantiated by the fact that positivity of the respective estimates is easily checkable by means of Theorem 3.18. Furthermore, we like to point out that even for $\mathcal{KL}_0$-functions that do not satisfy the assumptions of Theorem 4.8, the assertions with respect to symmetry and monotonicity often hold, cf. Figure 4.5.

4.2.2 Symmetry Analysis

In this subsection we carry out a complete symmetry analysis of the optimal value $\alpha^{\omega}_{N,m}$ of Problem 3.10 characterized by Theorem 3.18 with respect to the control horizon m. In particular, Proposition 4.6 is proven. Moreover, for $\omega = 1$, a stronger version of the respective result is shown which, in addition, holds without imposing any restrictions on the $\mathcal{KL}_0$-functions from Assumption 3.2 except linearity in their first arguments.

Corollary 4.10
Let $\beta(\cdot,\cdot) \in \mathcal{KL}_0$ from Assumption 3.2 be linear in its first argument and satisfy (1.13)*. Then, for $N \in \mathbb{N}_{\geq 2}$, the optimal value $\alpha_{N,m} = \alpha^1_{N,m}$ of Problem 3.8 satisfies*

$$\alpha_{N,m} = \alpha_{N,N-m} \qquad \text{for} \qquad m = 1, 2, \ldots, N-1.$$

Proof: Since (3.3) implies $c_0 \geq 1$ for $\beta(\cdot,\cdot)$ linear in its first argument $\gamma_{m+1} \geq \omega = 1$ holds. Hence, the assertion follows immediately from (3.21).

□

This corollary proves Proposition 4.6 for $\omega = 1$. We point out that symmetry holds, i.e. the symmetric bound is tight for $\omega = 1$. Hence, it remains to show Proposition 4.6, i.e. the symmetric bound $\alpha^{\omega}_{N,m} \leq \alpha^{\omega}_{N,N-m}$ for terminal weights $\omega > 1$. At first, $\mathcal{KL}_0$-functions representing finite time controllability are dealt with, cf. Lemma 4.11. Furthermore, a generalization including $\mathcal{KL}_0$-functions of type (1.12) not satisfying the assumptions of Lemma 4.11 is not possible, cf. Remark 4.12.

Lemma 4.11
Suppose $\beta(\cdot,\cdot)$ from Assumption 3.2 to be of type (1.12) *satisfying* (1.13)*. In addition, let $c_n = 0$ for all $n \in \mathbb{N}_{\geq 3}$. Then, for $N \in \mathbb{N}_{\geq 2}$ and $\omega \geq 1$, the optimal value of Problem 3.10 fulfills*

$$\alpha^{\omega}_{N,N-m} - \alpha^{\omega}_{N,m} \geq 0 \qquad \text{for} \qquad m \in \{k \in \mathbb{N} : k < N-k\}, \tag{4.4}$$

i.e. the assertion of Proposition 4.6.

Proof: The proof is based on Theorem 3.18. We have to distinguish whether the respective necessary and sufficient condition $\gamma_{m+1} - \omega \leq 0$ for $\alpha^{\omega}_{N,m} = 1$ holds or not. We begin with supposing that it is satisfied and show that it implies $\gamma_{N-m+1} - \omega \leq 0$ which ensures $\alpha^{\omega}_{N,N-m} = 1 \geq \alpha^{\omega}_{N,m}$. Since γ_{m+1} is defined as $\sum_{n=0}^{m-1} c_n + \omega c_m$ this conclusion holds for $m \geq 2$. For $m = 1$, we obtain $\gamma_{m+1} - \omega = c_0 + c_1\omega - \omega \leq 0$ and, as a consequence,

$$\begin{aligned} \gamma_{N-m+1} - \omega \quad &\leq \quad c_0 + c_1 + c_2\omega - \omega + c_1\omega - c_1\omega \\ &\overset{(1.13)}{\leq} \quad (c_0 + c_1\omega - \omega) + c_1(c_0 + c_1\omega - \omega) \leq 0. \end{aligned}$$

Hence, showing (4.4) for $\gamma_{m+1} - \omega \geq 0$ completes the proof. Since Corollary 4.10 shows the assertion for $\omega = 1$ we restrict ourselves to $\omega > 1$. Without loss of generality we assume

$\gamma_{N-m+1} - \omega > 0$ since otherwise $\alpha^{\omega}_{N,N-m} = 1$ and, thus, the assertion holds. Consequently, showing the desired inequality using the expressions given by Formula (3.21) covers the assertion. Hence, $\alpha^{\omega}_{N,N-m} - \alpha^{\omega}_{N,m} \geq 0$ is equivalent to the following inequality in which the index i is omitted in the product symbols:

$$(\gamma_{N-m+1}-1)(\gamma_{m+1}-\omega)\left[\prod_{N-m+1}^{N}\gamma_i - (\gamma_{N-m+1}-\omega)\prod_{N-m+2}^{N}(\gamma_i-1)\right]\left[\prod_{m+1}^{N}\gamma_i - \prod_{m+1}^{N}(\gamma_i-1)\right]$$
$$\geq (\gamma_{N-m+1}-\omega)(\gamma_{m+1}-1)\left[\prod_{m+1}^{N}\gamma_i - (\gamma_{m+1}-\omega)\prod_{m+2}^{N}(\gamma_i-1)\right]\left[\prod_{N-m+1}^{N}\gamma_i - \prod_{N-m+1}^{N}(\gamma_i-1)\right].$$

Rearranging these terms and dividing by $(\omega-1)\prod_{i=N-m+1}^{N}\gamma_i > 0$ leads to

$$(\gamma_{N-m+1}-\omega)\prod_{i=m+1}^{N-m}\gamma_i \cdot \prod_{i=N-m+1}^{N}(\gamma_i-1)+(\gamma_{m+1}-\gamma_{N-m+1})\prod_{i=m+1}^{N}\gamma_i-(\gamma_{m+1}-\omega)\prod_{i=m+1}^{N}(\gamma_i-1) \geq 0. \tag{4.5}$$

We like to point out that, so far, no assumptions were made on the $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$ except linearity in its first argument and (1.13). Hence, (4.5) also holds for $\beta(\cdot,\cdot)$ of type (1.11). The following cases are distinguished with respect to the control horizon m:

- $m > 2$: since $\gamma_{m+1} = \gamma_{N-m+1}$ holds, (4.5) is equivalent to the following inequality which ensures (4.4):

$$(\gamma_{m+1}-\omega)\prod_{i=N-m+1}^{N}(\gamma_i-1)\left[\prod_{i=m+1}^{N-m}\gamma_i - \prod_{i=m+1}^{N-m}(\gamma_i-1)\right] \geq 0.$$

- $m = 2$: since $\gamma_{m+1} = c_0 + c_1 + \omega c_2$ and $\gamma_{N-m+1} = \sum_{n=0}^{2} c_n$ hold by definition, the equalities $(\gamma_{m+1} - \gamma_{N-m+1}) = (\omega-1)c_2$ and $(\gamma_{m+1}-\omega) = (\omega-1)c_2 + (\gamma_{N-m+1}-\omega)$ are obtained. Using these equalities for the corresponding terms in (4.5) provides $\alpha^{\omega}_{N,N-m} \geq \alpha^{\omega}_{N,m}$.

- $m = 1$: let $N > 2$. Otherwise, i.e. $N = 2$, the equality $m+1 = N-m+1$ and, thus, $\gamma_{m+1} = \gamma_{N-m+1}$ hold which allows to repeat the argumentation used for $m > 2$. Since $\gamma_{m+1} = c_0 + \omega c_1$ and $N - m + 1 = N$ hold for $m = 1$, (4.5) is equivalent to

$$\begin{aligned} 0 &\leq \Big[(\gamma_N-1)(\gamma_N-\omega)+(\gamma_2-\gamma_N)\gamma_N\Big]\gamma_2\prod_{i=3}^{N-1}\gamma_i - (\gamma_2-\omega)\prod_{i=2}^{N}(\gamma_i-1) \\ &= \Big[\gamma_N(\gamma_2-\omega-1)+\omega\Big]\gamma_2\prod_{i=3}^{N-1}\gamma_i - (\gamma_2-\omega)(\gamma_2-1)(\gamma_N-1)\prod_{i=3}^{N-1}(\gamma_i-1). \end{aligned}$$

  We start by showing that the term in square brackets is positive. To this end, considering the case $\gamma_2 - \omega - 1 < 0$ is sufficient. Taking (1.13) into account yields

$$\begin{aligned} \gamma_N(\gamma_2-\omega-1)+\omega &\geq (c_0+c_1+c_1^2\omega)(c_0+c_1\omega-\omega-1)+\omega \\ &= (c_0-1)(c_0+c_1\omega-\omega+c_1)+c_1^2\omega(c_0+c_1\omega-\omega) \geq 0. \end{aligned}$$

  Hence, showing that the coefficient of $\prod_{i=3}^{N-1}\gamma_i$ is greater than or equal to the coefficient of $\prod_{i=3}^{N-1}(\gamma_i-1)$ ensures (4.4) and, thus, completes the proof:

$$[\gamma_N(\gamma_2-\omega-1)+\omega]\gamma_2 - (\gamma_2-\omega)(\gamma_2-1)(\gamma_N-1)$$

$$
\begin{aligned}
&= \gamma_2(\gamma_2 - 1) - \omega(\gamma_N - 1) \\
&\geq (c_0 + \omega c_1)(c_0 + \omega c_1 - 1) - \omega(c_0 + c_1 + \omega c_2 - 1) \\
&= \omega^2(c_1^2 - c_2) + (c_0 - 1)(\gamma_m - \omega + c_1\omega) \overset{(1.13)}{\geq} 0.
\end{aligned}
$$

□

Remark 4.12
Note that Lemma 4.11 does not hold if $c_n \neq 0$ *for some* $n \geq 3$. *Consider, e.g.* $c_0 = 1$, $c_1 = 3/2$, $c_2 = 2/3$, $c_3 = 1$ *and* $c_n = 0$ *for* $n \geq 4$. *For* $N = 5$ *and* $m = 2$ *the necessary and sufficient condition* $\gamma_{m+1} - \omega \leq 0$ *for* $\alpha^\omega_{5,m} = 1$ *is satisfied for* $\omega \geq 15/2$. *However, the inequality* $\gamma_{N-m+1} - \omega = \sum_{n=0}^{2} c_n > 0$ *holds and implies* $\alpha^\omega_{5,3} < 1 = \alpha^\omega_{5,2}$.

In the sense of Remark 4.12 the assumptions of Lemma 4.11 cannot be relaxed. Hence, the deduced results hold only for a subset of the class of finite time controllable systems satisfying (1.13).

In order to complete the proof of Proposition 4.6 we have to deal with $\mathcal{KL}_0$-functions of type (1.11) which characterize, via Assumption 3.2, exponentially controllable systems. In contrast to $\mathcal{KL}_0$-functions of type (1.12) we do not have to impose further restrictions on this subclass of $\mathcal{KL}_0$-functions. We begin our analysis with the special case $\gamma_{m+1} - \omega \leq 0$, i.e. the necessary and sufficient condition from Theorem 3.18 for $\alpha^\omega_{N,m} = 1$. This condition not only guarantees the preservation of the symmetry property stated in Corollary 4.10 but even ensures $\alpha^\omega_{N,\widetilde{m}} = 1$ for all $\widetilde{m} \in \{m+1, \ldots, N-1\}$.

Lemma 4.13
Let the $\mathcal{KL}_0$*-function* $\beta(\cdot,\cdot)$ *from Assumption 3.2 be of type* (1.11). *Then, for* $N \in \mathbb{N}_{\geq 2}$ *and* $\omega \geq 1$, *the inequality* $\gamma_{m+1} - \omega \leq 0$ *implies* $\alpha^\omega_{N,\widetilde{m}} = 1$, $\widetilde{m} \in \{m, m+1, \ldots, N-1\}$, *for* $\alpha^\omega_{N,\widetilde{m}}$ *from Theorem 3.18.*

Proof: In view of Theorem 3.18 it is sufficient to show $\gamma_{\widetilde{m}+1} - \omega \leq 0$ for $\widetilde{m} \in \{m+1, m+2, \ldots, N-1\}$ in order to conclude the assertion. Since $\gamma_{m+1} - \omega = \sum_{n=0}^{m-1} c_n - (1-c_m)\omega \leq 0$, $c_m = C\sigma^m < 1$ holds. Hence, $\gamma_{m+1} \leq \omega$ is equivalent to

$$
\omega \geq \frac{C}{1 - C\sigma^m} \cdot \sum_{n=0}^{m-1} \sigma^n = \frac{C(1-\sigma^m)}{(1-\sigma)(1-C\sigma^m)} > 1.
$$

Using this inequality in order to estimate the terminal weight ω yields

$$
\begin{aligned}
\gamma_{\widetilde{m}+1} - \omega &\leq \frac{C(1-\sigma^{\widetilde{m}})(1-C\sigma^m)}{(1-\sigma)(1-C\sigma^m)} - \frac{C(1-\sigma^m)(1-C\sigma^{\widetilde{m}})}{(1-\sigma)(1-C\sigma^m)} \\
&= C \cdot \frac{\sigma^m + C\sigma^{\widetilde{m}} - \sigma^{\widetilde{m}} - C\sigma^m}{(1-\sigma)(1-C\sigma^m)} = C \cdot \frac{(C-1)(\sigma^{\widetilde{m}} - \sigma^m)}{(1-\sigma)(1-C\sigma^m)} < 0
\end{aligned}
$$

for $\widetilde{m} > m$. Thus, the necessary and sufficient condition from Theorem 3.18 is satisfied which completes the proof.

□

In order to complete the proof of Proposition 4.6 we require the following auxiliary lemma which is an essential tool not only in this section but also in the ensuing one. Originally, the technical Lemma 4.14 was elaborated in order to prove the monotonicity property described by Figure 4.3 and precisely stated in Proposition 4.7 for $\omega = 1$.

Lemma 4.14
Let $p : \mathbb{R} \to \mathbb{R}$ be a monic polynomial[3] *of degree $k > 1$, $k \in \mathbb{N}$, such that*

a) all k roots $z_1, z_2, \ldots, z_k$ are real, exactly one of them is strictly negative, and at most one is equal to zero,

b) the root of the $(k-1)^{st}$ derivative $p^{(k-1)} : \mathbb{R} \to \mathbb{R}$ is strictly smaller than $-c/k$ with $c \in \mathbb{R}_0^+$ and

c) $p(\tilde{z}) = \tilde{z}^{k-1}(\tilde{z} + c)$ for some $\tilde{z} > \max\{z_1, \ldots, z_k\}$.

Then, the polynomial $p(\cdot)$ satisfies

$$p(z) > z^{k-1}(z + c) \qquad \text{for all} \qquad z > \tilde{z}. \tag{4.6}$$

Proof: We prove the assertion via induction with respect to the degree k of the polynomial $p(\cdot)$. For $k = 2$ the polynomial can be written as

$$p(z) = (z - a)(z + b) = z^2 + z(b - a) - ab \qquad \text{with} \qquad b > 0 \text{ and } a \geq 0.$$

Since the assumptions of Lemma 4.14 ensure that the root $(a - b)/2$ of the first derivative is strictly smaller than $-c/2$, $a + c < b$ holds. Furthermore, we deduce $\tilde{z}(b - a - c) - ab = 0$ from $p(\tilde{z}) = \tilde{z}^2 + c\tilde{z}$. Combining the obtained conditions on the coefficients of the polynomial yields $p(z) - z(z + c) = z(b - a - c) - ab > 0$ for $z > \tilde{z}$, i.e. (4.6).

Next, we carry out the induction step from k to $k + 1$. Suppose $p : \mathbb{R} \to \mathbb{R}$ to be a polynomial of degree $k+1$ which satisfies the assumptions of Lemma 4.14 with $k+1$ instead of k. Note that this — in view of Rolle's theorem, cf. [77, Theorem 3.1] — guarantees that all derivatives of $p(\cdot)$ have only strictly positive roots (counted with multiplicities) except for exactly one strictly negative one. Using the definition $z_0 := \max\{z_1, \ldots, z_{k+1}\} \in (0, \tilde{z})$ yields $p(z_0) = 0 < p(\tilde{z}) = \tilde{z}^k(\tilde{z} + c)$. Thus, there exists $\bar{z} \in \,]z_0, \tilde{z}[$ such that

$$\frac{p'(\bar{z})}{k+1} > \bar{z}^{k-1}\left(\bar{z} + \frac{kc}{k+1}\right) \tag{4.7}$$

holds. We define the monic polynomial $q : \mathbb{R} \to \mathbb{R}$ via $q(\cdot) := p'(\cdot)/(k + 1)$ and denote its maximal positive root, which is located in the interval $(0, z_0]$, by $z^\star$. Bearing this definition and (4.7) in mind, the intermediate value theorem, cf. [77, p.218], implies that there exists $\hat{z} \in]z^\star, \bar{z}[$ such that $q(\hat{z}) = \hat{z}^{k-1}(\hat{z} + \frac{kc}{k+1})$. Moreover, note that the condition with respect to the $(k-1)^{st}$ derivative remains unchanged and, thus, satisfied because of our adaptation of c and $q(\cdot)$. Hence, we are able to apply the induction assumption to the polynomial $q(\cdot)$ in order to deduce $q(z) > z^{k-1}(z + \frac{kc}{k+1})$ for all $z > \hat{z}$ and, as a consequence, $p'(z) > (z^k(z + c))'$ for $z \geq \tilde{z}$ ($\tilde{z} > \bar{z} > \hat{z}$) which allows us to conclude the assertion.

$\square$

Proposition 4.1 in combination with Lemmata 4.13 and 4.14 enables us to carry out a complete symmetry analysis of the optimal value of Problem 3.10 for $\mathcal{KL}_0$-functions of type (1.11). Since the part of Proposition 4.6 dealing with $\beta(\cdot, \cdot) \in \mathcal{KL}_0$ of type (1.12) is covered by Lemma 4.11 this completes the respective proof.

[3] A monic polynomial $p(\cdot)$ of degree $k \in \mathbb{N}$ with solely real roots $z_1, z_2, \ldots, z_k$ may be written as $\prod_{i=1}^k (z - z_i)$.

Lemma 4.15
Let the $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$ from Assumption 3.2 be of type (1.11). *Then, for $N \in \mathbb{N}_{\geq 3}$ and $m \in \mathbb{N}$ such that $m < N-m$, the assertion of Proposition 4.6, i.e. $\alpha^{\omega}_{N,N-m} - \alpha^{\omega}_{N,m} \geq 0$, holds.*

Proof: Since Lemma 4.13 covers the assertion for $\gamma_{m+1} - \omega \leq 0$, the line of arguments used in the proof of Lemma 4.11 can be repeated in order show that proving Inequality (4.5) implies the assertion, i.e. we only have to deal with (3.21) in order to establish the symmetric bound and may carry out the first few conversions analogously to those of Lemma 4.11.

Let the control horizon $m \in \mathbb{N}$ be given. Our goal is to prove the assertion by an induction with respect to the optimization horizon N. In the beginning of the induction, the optimization horizon N is chosen as small as possible, i.e. $N = 2m+1$. Then, $N - m = m + 1$ holds and (4.5) can be simplified to

$$0 \leq \Big[\gamma_{m+1}(\gamma_{m+2} - \omega) - (\gamma_{m+1} - \omega)(\gamma_{m+1} - 1)\Big] \prod_{i=m+2}^{N} (\gamma_i - 1) + (\gamma_{m+1} - \gamma_{m+2})\gamma_{m+1} \prod_{i=m+2}^{N} \gamma_i$$

which is, in turn, equivalent to

$$0 \leq \Big[\gamma_{m+1}(\gamma_{m+2} - \gamma_{m+1}) + (\gamma_{m+1} - \omega)\Big] \prod_{i=m+2}^{N} \frac{\gamma_i - 1}{\gamma_i} - (\gamma_{m+2} - \gamma_{m+1})\gamma_{m+1}. \tag{4.8}$$

Before proceeding, we define $\eta := 1 + \sigma\omega - \omega$. Furthermore, note that $\eta \leq 0$ implies $\gamma_{m+2} - \gamma_{m+1} = \sigma^m \eta \leq 0$. Then, since $\prod_{i=m+2}^{N} (\gamma_i - 1)/\gamma_i \in (0,1)$, Inequality (4.8) holds for $\eta \leq 0$. Hence, it remains to consider (4.8) for the case $\eta > 0$. Here, we aim at applying Lemma 4.14 in order to establish this inequality. To this end, we require

$$\gamma_i = C \sum_{n=0}^{i-2} \sigma^n + \omega\, C\sigma^{i-1} = C \cdot \frac{1 - \sigma^{i-1} + \omega\sigma^{i-1} - \omega\sigma^i}{1-\sigma} = C \cdot \frac{1 - \eta\sigma^{i-1}}{1-\sigma} \tag{4.9}$$

and, as a consequence,

$$\frac{\gamma_i - 1}{\gamma_i} = 1 - \frac{1}{\gamma_i} = C^{-1}\left(C - \frac{1-\sigma}{1 - \eta\sigma^{i-1}}\right).$$

In addition, using the representation of γ_i and $\gamma_{m+2} - \gamma_{m+1} = \sigma^m \eta$ yields

$$\begin{aligned}
\frac{\gamma_{m+1}(\gamma_{m+2} - \gamma_{m+1}) + (\gamma_{m+1} - \omega)}{\gamma_{m+1}(\gamma_{m+2} - \gamma_{m+1})} &= C^{-2}\left(C^2 + \frac{C(\frac{1-\sigma^m\eta}{1-\sigma}) - \omega}{\sigma^m\eta\frac{1-\sigma^m\eta}{1-\sigma}}\right) \\
&= C^{-2}\left(C^2 + \frac{C}{\sigma^m\eta} - \frac{\omega(1-\sigma)}{\sigma^m\eta(1-\sigma^m\eta)}\right) \\
&= C^{-2}\left(C + \frac{1}{2\sigma^m\eta} - \sqrt{\xi}\right)\left(C + \frac{1}{2\sigma^m\eta} + \sqrt{\xi}\right)
\end{aligned}$$

with $\xi := \left(\frac{1}{2\sigma^m\eta}\right)^2 + \frac{\omega(1-\sigma)}{\sigma^m\eta(1-\sigma^m\eta)}$. Overall, plugging the obtained expressions into (4.8) provides

$$\underbrace{\left(C + \frac{1}{2\sigma^m\eta} + \sqrt{\xi}\right)\left(C + \frac{1}{2\sigma^m\eta} - \sqrt{\xi}\right) \prod_{i=m+2}^{N}\left(C - \frac{1-\sigma}{1-\sigma^{i-1}\eta}\right)}_{=:p(C)} \geq \underbrace{C^{m+2}}_{=:q(C)}.$$

Since $\eta \in (0,1)$, the polynomial $p(C)$ has clearly $m+1$ strictly positive roots and exactly one negative root. Hence, in order to apply Lemma 4.14, we have to verify the second and third assumption with $c = 0$. Since Proposition 4.1 yields $p(1) = q(1)$ or, in the notation of Lemma 4.14, $\tilde{z} = 1$, we have to show that the positive root $-1/(2\sigma^m\eta) + \sqrt{\xi}$ is located in the interval $(0,1)$, i.e. $\sqrt{\xi} < 1 + 1/(2\sigma^m\eta)$ or, equivalently,

$$\frac{\omega(1-\sigma)}{\sigma^m\eta(1-\sigma^m\eta)} < 1 + \frac{1}{\sigma^m\eta}$$

which holds since $\eta(1-\sigma^m) + \sigma^m\eta(1-\sigma^m\eta) > 0$. Thus, it remains to ensure condition b) of Lemma 4.14 with respect to the $(m+1)^{st}$ derivative. To this end, we calculate $p^{(m+1)}(\cdot)$

$$p^{(m+1)}(C) = (m+2)!\ C + (m+1)! \left(\frac{1}{\sigma^m\eta} - \sum_{i=m+2}^{N} \frac{1-\sigma}{1-\sigma^{i-1}\eta} \right),$$

and show that the only root of this polynomial of degree one is strictly negative. In order to determine the sign of this root, it is sufficient to consider

$$\begin{aligned}
\frac{1}{\sigma^m\eta} - (1-\sigma)\sum_{i=m+2}^{N} \frac{1}{1-\sigma^{i-1}\eta} &> \frac{1}{\sigma^m\eta} - (1-\sigma)\sum_{i=m+2}^{N} \frac{1}{1-\sigma^{m+1}\eta} \\
&= \frac{1-\sigma^{m+1}\eta - m(1-\sigma)\sigma^m\eta}{\sigma^m\eta\,(1-\sigma^{m+1}\eta)} \\
&> \frac{(1-\sigma)}{\sigma^m(1-\sigma^{m+1}\eta)} \left(\sum_{i=0}^{m} \sigma^i - m\sigma^m \right) > 0.
\end{aligned}$$

Hence, Lemma 4.14 applied with $c = 0$ and $\tilde{z} = 1$ ensures (4.8) and, thus, Lemma 4.15 for $N = 2m+1$, i.e. the induction start is carried out. In order to complete the proof we have to perform the induction step.

Suppose that the assertion holds for $N \geq 2m+1$. Again, (4.5) is taken as our starting point. Hence, we have to show

$$(\gamma_{N-m+2}-\omega)\prod_{i=m+1}^{N-m+1}\gamma_i \cdot \prod_{i=N-m+2}^{N+1}(\gamma_i-1) + (\gamma_{m+1}-\gamma_{N-m+2})\prod_{i=m+1}^{N+1}\gamma_i - (\gamma_{m+1}-\omega)\prod_{i=m+1}^{N+1}(\gamma_i-1) \geq 0.$$

Using the induction assumption for $(\gamma_{m+1}-\omega)\prod_{i=m+1}^{N}(\gamma_i - 1)$, reducing the resulting expression by $\prod_{i=m+1}^{N-m}\gamma_i$, and combining the summands which have the factor $\prod_{i=N-m+1}^{N}\gamma_i$ or $\prod_{i=N-m+2}^{N+1}(\gamma_i - 1)$, respectively, yields

$$\begin{aligned}
0 \ \leq\ & \underbrace{\Big[\gamma_{N-m+1}(\gamma_{N-m+2}-\omega) - (\gamma_{N-m+1}-\omega)(\gamma_{N-m+1}-1)\Big]}_{=\gamma_{N-m+1}(\gamma_{N-m+2}-\gamma_{N-m+1})+(\gamma_{N-m+1}-\omega)} \prod_{i=N-m+2}^{N+1}(\gamma_i-1) \\
&+ \underbrace{\Big[\gamma_{N+1}(\gamma_{m+1}-\gamma_{N-m+2}) - (\gamma_{N+1}-1)(\gamma_{m+1}-\gamma_{N-m+1})\Big]}_{=\gamma_{N+1}(\gamma_{N-m+1}-\gamma_{N-m+2})+(\gamma_{m+1}-\gamma_{N-m+1})} \prod_{i=N-m+1}^{N}\gamma_i.
\end{aligned}$$

Since $(\gamma_{m+1} - \gamma_{N-m+1}) = C(\sigma^m - \sigma^{N-m})(-\eta)/(1-\sigma)$ and $\gamma_{N-m+2} - \gamma_{N-m+1} = C\sigma^{N-m}\eta$

are implied by (4.9), this inequality may be rewritten as[4]

$$\left[\gamma_{N-m+1} C \sigma^{N-m} \eta + (\gamma_{N-m+1} - \omega)\right] \prod_{N-m+2}^{N+1} (\gamma_i - 1) \geq \left[\gamma_{N+1} \sigma^{N-m} \eta + \frac{(\sigma^m - \sigma^{N-m})\eta}{1-\sigma}\right] C \prod_{i=N-m+1}^{N} \gamma_i. \tag{4.10}$$

Analogously to the induction start, the case $\eta \leq 0$ is dealt with separately: since the second and forth summand already have the desired signs, showing

$$\gamma_{N-m+1} C \sigma^{N-m} (-\eta) \left[\prod_{i=N-m+2}^{N+1} \gamma_i - \prod_{i=N-m+2}^{N+1} (\gamma_i - 1)\right] \geq 0$$

yields (4.10) for $\eta \leq 0$. Consequently, $\eta > 0$ is supposed from now on. Note that $\eta = 1 - \omega(1-\sigma) < 1$ holds. Reducing (4.10) by $\sigma^{N-m}\eta\gamma_{N+1}/C$ and $\prod_{i=N-m+1}^{N} (\gamma_i/C)$ leads to

$$\underbrace{\left(\frac{\gamma_{N-m+1}}{\gamma_{N+1}} C^2 + \frac{\gamma_{N-m+1} - \omega}{\sigma^{N-m}\eta\gamma_{N+1}/C}\right) \prod_{i=N-m+1}^{N} \frac{\gamma_{i+1} - 1}{\gamma_i/C}}_{=:p(C)} \geq \underbrace{C^{m+1} \left(C + \frac{\sigma^m - \sigma^{N-m}}{\sigma^{N-m}(1 - \sigma^N \eta)}\right)}_{=:q(C)}.$$

Note that both polynomials have degree $m+2$ and the coefficients of C^{m+2} are equal to one, i.e. are monic. We aim at applying Lemma 4.14 in order to conclude this inequality and, thus, the assertion. To this end, we begin by determining the exact location of the respective roots of $p(\cdot)$ and $q(\cdot)$. The polynomial $q(\cdot)$ has exactly one strictly negative root located at $-(\sigma^m - \sigma^{N-m})/((1 - \eta\sigma^N)\sigma^{N-m})$. Next, we consider $p(\cdot)$ and, in particular, the factor $(\gamma_{i+1} - 1)/(\gamma_i/C)$, $i = N-m+1, N-m+2, \ldots, N$, more closely. Using (4.9) provides

$$\frac{\gamma_{i+1} - 1}{\gamma_i/C} = \frac{C(1 - \sigma^i \eta) - (1 - \sigma)}{1 - \sigma^{i-1}\eta} = \frac{1 - \sigma^i \eta}{1 - \sigma^{i-1}\eta} \left(C - \frac{1-\sigma}{1 - \sigma^i \eta}\right),$$

i.e. a polynomial of degree one whose root is located at $(1-\sigma)/(1-\sigma^i\eta)$, i.e. in the interval $(0,1)$ for each $i \in \{N-m+1, N-m+2, \ldots, N\}$. The first factor of $p(\cdot)$ still needs to be investigated. Here, extracting the factor $\gamma_{N-m+1}/\gamma_{N+1}$, which does not depend on C, and using (4.9), yields

$$\frac{1 - \sigma^{N-m}\eta}{1 - \sigma^N \eta} \left(C^2 + \frac{C}{\sigma^{N-m}\eta} - \frac{\omega(1-\sigma)}{(1 - \sigma^{N-m}\eta)\sigma^{N-m}\eta}\right).$$

Setting this expression equal to zero and solving the resulting equation provides the two remaining roots of $p(\cdot)$

$$C = -\frac{1}{2\sigma^{N-m}\eta} \pm \sqrt{\left(\frac{1}{2\sigma^{N-m}\eta}\right)^2 + \frac{\omega(1-\sigma)}{(1 - \sigma^{N-m}\eta)\sigma^{N-m}\eta}}, \tag{4.11}$$

a positive and a negative one. Summarizing, $p(C)$ may be represented by $p(C) = \prod_{i=1}^{m+2} (C - z_i)$ where z_i, $i = 1, 2, \ldots, m+2$, denote the determined roots. Moreover, Proposition 4.1 yields $p(1) = q(1)$. Hence, we have to verify that the positive root from

[4] In fact, this inequality is flawed in [45], i.e. the divisor $(1-\sigma)$ is missing. Nevertheless, the train of thoughts used in order to prove the assertion remains substantially the same.

(4.11) is strictly smaller than one and, thus, contained in the interval $(0,1)$. To this end, the equivalent inequality

$$1 + \frac{1}{2\sigma^{N-m}\eta} > \sqrt{\left(\frac{1}{2\sigma^{N-m}\eta}\right)^2 + \frac{\omega(1-\sigma)}{(1-\sigma^{N-m}\eta)\sigma^{N-m}\eta}}$$

is squared which leads to

$$\frac{\omega(1-\sigma)}{(1-\sigma^{N-m}\eta)\sigma^{N-m}\eta} < 1 + \frac{1}{\sigma^{N-m}\eta} = \frac{1+\sigma^{N-m}\eta}{\sigma^{N-m}\eta}$$

or, equivalently,

$$\omega(1-\sigma) < (1+\sigma^{N-m}\eta)(1-\sigma^{N-m}\eta) = 1 - \sigma^{2(N-m)}\eta^2.$$

Since $\eta > 0$, taking the definition of η into account shows that the respective root is located in the interval $(0,1)$ and, as a consequence, that the third condition of Lemma 4.14 is satisfied with $\tilde{z} = 1$. Hence, the second condition has to be verified in order to apply Lemma 4.14 and deduce that (4.10) holds for $C \geq 1$, which completes the proof. We calculate the $(m+1)^{st}$ derivative of $p(C)$ and $q(C)$

$$\begin{aligned} q^{(m+1)}(C) &= (m+1)!\left((m+2)C + \frac{(\sigma^m - \sigma^{N-m})}{(1-\sigma^N\eta)\sigma^{N-m}}\right), \\ p^{(m+1)}(C) &= (m+1)!\left((m+2)C + \frac{1}{\sigma^{N-m}\eta} - \sum_{i=N-m+1}^{N} \frac{1-\sigma}{1-\sigma^i\eta}\right). \end{aligned}$$

We have to show that the root of $p^{(m+1)}$ is strictly smaller than its counterpart of $q^{(m+1)}$ divided by $m+2$ (the degree of the polynomial $p(\cdot)$), i.e.

$$\frac{m+2}{\sigma^{N-m}\eta} - \frac{(\sigma^m - \sigma^{N-m})}{(1-\sigma^N\eta)\sigma^{N-m}} > (m+2)\sum_{i=N-m+1}^{N} \frac{1-\sigma}{1-\sigma^i\eta}. \tag{4.12}$$

Since $(1-\sigma^i\eta)^{-1} \leq (1-\sigma^{N-m+1})^{-1}$, $i = N-m+1, N-m+2, \ldots, N$, it is sufficient to establish

$$\frac{m+2}{\sigma^{N-m}\eta} - \frac{(\sigma^m - \sigma^{N-m})}{(1-\sigma^{N-m+1}\eta)\sigma^{N-m}} > (m+2)\sum_{i=N-m+1}^{N} \frac{1-\sigma}{1-\sigma^{N-m+1}\eta} = \frac{m(m+2)(1-\sigma)}{1-\sigma^{N-m+1}\eta}$$

or, equivalently,

$$(m+2)(1-\sigma^{N-m+1}\eta) - (\sigma^m - \sigma^{N-m})\eta > m(m+2)(1-\sigma)\sigma^{N-m}\eta$$

in order to deduce (4.12). Since $(1-\sigma^{N-m+1}\eta) - (\sigma^m - \sigma^{N-m})\eta > (1-\sigma)(\sigma^{m-1} - \sigma^{N-m})\eta$ holds, this is ensured by

$$(m+1)\left(\frac{1-\sigma^{N-m+1}\eta}{1-\sigma}\right) \geq (m+1)\sum_{n=0}^{N-m} \sigma^n \overset{N-m \geq m+1}{>} m(m+2)\sigma^{N-m}\eta.$$

$\square$

4.2.3 Monotonicity Properties

Looking at Figure 4.3, one observes — aside from the symmetric bound — a certain monotonicity property, i.e. a monotone growth of the performance bounds $\alpha^{\omega}_{N,m}$ characterizing the optimal value of Problem 3.10 until the control horizon reaches about half the length of the optimization horizon. This feature is precisely stated in Proposition 4.7. The goal of this subsection is to prove this result. Combining the respective assertion with the symmetric bound derived in the preceding subsection, enables us to deduce Theorem 4.8. This theorem, which is based on Theorem 3.18, ensures that using time varying control horizons does not cause additional difficulties in order to verify the assumptions of Theorem 3.12 — at least for a large and important subclass of $\mathcal{KL}_0$-functions linear in their first arguments. For example, time varying horizons are required in networked control systems in order to compensate non-negligible delays or packet dropouts.

The symmetry analysis which was carried out in Subsection 4.2.2 exhibits an especially nice structure for the setting without an additional weight on the final term in the receding horizon cost functional, cf. Corollary 4.10. Furthermore, the restrictions on the class of finite time controllable systems are necessary only for terminal weights $\omega > 1$, cf. Lemma 4.11 and Remark 4.12. In contrast to that, these limitations (indeed, even slightly tighter ones) are necessary for the monotonicity properties dealt with in this subsection — independently of whether a terminal weight is involved or not. Hence, in order to benefit from these theoretical results as demonstrated by Theorem 4.8 and the algorithms to be developed in Section 4.4 there is no escape from these modifications. The necessity of taking these restrictions into account is shown by the following counterexample.

Example 4.16
Let the controllability behavior of a discrete time system be characterized via Assumption 3.2 based on one of the following $\mathcal{KL}_0$-functions of type (1.12)*:*

- $\beta_1(\cdot,\cdot)$ *is defined by* $c_0 = 1.24$, $c_1 = 1.14$, $c_2 = 1.04$, *and* $c_i = 0$ *for* $i \in \mathbb{N}_{\geq 3}$,
- $\beta_2(\cdot,\cdot)$ *is given by* $c_0 = 1$, $c_1 = 1.2$, $c_2 = 1.1$, $c_3 = 1.1$, $c_4 = 1.2$, $c_5 = 1$, $c_6 = 0.75$, $c_7 = 0.25$, *and* $c_i = 0$ *otherwise.*

Note that both functions satisfy (1.13)*. Furthermore,* $\beta_1(\cdot,\cdot)$ *is monotonically decreasing in its second argument. The corresponding optimal values of Problem 3.8 depicted in Fig. 4.6 show that neither for* $\beta_1(,\cdot,\cdot)$ *nor for* $\beta_2(\cdot,\cdot)$ *the desired monotonicity property*

$$\alpha^1_{N,m+1} \geq \alpha^1_{N,m} \qquad \text{for} \qquad m = 1, 2, \ldots, \lfloor N/2 - 1 \rfloor$$

is obtained.

Example 4.16 demonstrates that the desired monotonicity property does not hold for arbitrary $\mathcal{KL}_0$-functions $\beta(\cdot,\cdot)$ of type (1.12). The remaining part of this subsection deals with subclasses of $\mathcal{KL}_0$-functions meeting the assumptions of Proposition 4.7. It is arranged as follows: initially, $\mathcal{KL}_0$-functions of type (1.12) are addressed. Then, for exponentially controllable systems, "sufficiently" large terminal weights are treated separately before we turn our attention to the most delicate situation, i.e. $\omega = 1$. Here, we also indicate problems occurring in extending Proposition 4.7 and, thus, Theorem 4.8 to arbitrary $\omega \geq 1$.

As seen in the previous example, our results can not be generalized to $\beta(\cdot,\cdot)$ describing finite time controllability in more than two steps. Hence, the following lemma gives a complete analysis for $\mathcal{KL}_0$-functions of type (1.12).

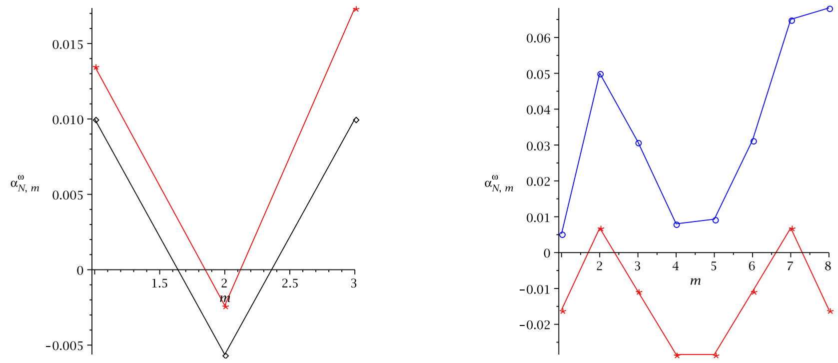


Figure 4.6: On the left, a visualization of $\alpha^1_{4,m}$, $m \in \{1,2,3\}$ for $\beta_1(\cdot,\cdot)$ from Example 4.16 with terminal weights $\omega = 1$ ($\circ$) and $\omega = 1.01$ ($\star$) is shown. On the right, $\alpha^1_{9,m}$, $m = 1,2,\ldots,8$. for $\beta_2(\cdot,\cdot)$ of the same example with $\omega = 1$ ($\star$) and $\omega = 4/3$ ($\circ$) is illustrated.

Lemma 4.17
Let Assumption 3.2 be satisfied with a $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$ of type (1.12) *with $c_n = 0$ for all $n \in \mathbb{N}_{\geq 2}$, i.e. finite time controllability in at most two steps. Then, for $N \in \mathbb{N}_{\geq 4}$, $\alpha^\omega_{N,m}$ from Theorem 3.18 satisfies*

$$\alpha^\omega_{N,m+1} - \alpha^\omega_{N,m} \geq 0 \qquad \text{for} \qquad m \in \{1,2,\ldots,\lfloor N/2\rfloor - 1\}. \tag{4.13}$$

Proof: Since $c_n = 0$ for $n \geq 2$ ensures $\gamma_{m+1} \geq \gamma_i$ for all $i \in \mathbb{N}_{\geq 3}$, the necessary and sufficient condition $\gamma_{m+1} - \omega \leq 0$ for $\alpha^\omega_{N,m} = 1$ implies its validity for every control horizon larger or equal to m and, thus, in particular $\gamma_{m+2} - \omega \leq 0$ holds. Hence, let $\gamma_{m+1} - \omega > 0$ hold. This allows for plugging (3.21) in (4.13) in order to show the assertion which is, as a consequence, equivalent to

$$\begin{aligned}
&(\gamma_{m+1}-\omega)(\gamma_{m+2}-1)\left[\prod_{i=m+2}^{N}\gamma_i - (\gamma_{m+2}-\omega)\prod_{i=m+3}^{N}(\gamma_i-1)\right]\left[\prod_{i=N-m}^{N}\gamma_i - \prod_{i=N-m}^{N}(\gamma_i-1)\right]\\
\geq\; &(\gamma_{m+2}-\omega)(\gamma_{N-m}-1)\left[\prod_{i=m+1}^{N}\gamma_i - (\gamma_{m+1}-\omega)\prod_{i=m+2}^{N}(\gamma_i-1)\right]\left[\prod_{i=N-m+1}^{N}\gamma_i - \prod_{i=N-m+1}^{N}(\gamma_i-1)\right].
\end{aligned} \tag{4.14}$$

Note that for the considered subclass of $\mathcal{KL}_0$-functions of type (1.12) $\gamma_3 = \gamma_i$ holds for all $i \in \mathbb{N}_{\geq 3}$. Consequently, we define $\gamma := \gamma_3 = \gamma_i$, $i \in \mathbb{N}_{\geq 3}$. Furthermore, taking $N - m \geq m + 2 \geq 3$ into account allows us to reduce (4.14) by the factor $(\gamma - 1)$ and leads to

$$\omega(\gamma_{m+1}-\gamma)\prod_{i=m+2}^{N}\gamma\cdot\left[\gamma^m-(\gamma-1)^m\right]+(\gamma_{m+1}-\omega)(\gamma-1)^m\gamma^m\left[\prod_{i=m+2}^{N-m}\gamma-(\gamma-\omega)\prod_{i=m+3}^{N-m}(\gamma-1)\right]\geq 0,$$

which shows the desired inequality and, thus, completes the proof.

$\square$

Next, we aim at deducing the desired monotonicity property assuming exponential controllability. To this end, we make a distinction with respect to the terminal weight ω. We begin with "sufficiently large" ones or, to be more precise, those satisfying $\omega \geq (1-\sigma)^{-1}$. Note that this condition is always achievable but may be demanding for decay rates σ close to one.

Lemma 4.18
Let Assumption 3.2 hold with a $\mathcal{KL}_0$-function of type (1.11)*. Furthermore, let the terminal weight ω be chosen such that $\eta := 1 + \sigma\omega - \omega \leq 0$. Then, for $N \in \mathbb{N}_{\geq 4}$, the assertion from Proposition 4.7, i.e.* (4.13) *holds.*

Proof: Taking the assertion of Lemma 4.13 into account, it suffices to establish (4.14) in order to deduce (4.13), i.e. $\alpha^{\omega}_{N,m+1} - \alpha^{\omega}_{N,m} \geq 0$, $m \in \{1, 2, \ldots, \lfloor N/2 \rfloor - 1\}$. Since $2m + 2 \leq N$, expanding the terms in (4.13) and combining them appropriately yields

$$-a\prod_{i=m+2}^{N}\gamma_i+\Big[a+(\gamma_{m+1}-\omega)(\gamma_{m+2}-1)\Big]\prod_{i=N-m+1}^{N}(\gamma_i-1)\prod_{i=m+2}^{N-m}\gamma_i-(\gamma_{m+1}-\omega)(\gamma_{m+2}-\omega)\prod_{i=m+2}^{N}(\gamma_i-1) \geq 0 \tag{4.15}$$

with

$$\begin{aligned} a &:= -[\omega\gamma_{N-m}(\gamma_{m+1}-\gamma_{m+2}) + \gamma_{m+1}(\gamma_{m+2}-\gamma_{N-m}) + \omega(\gamma_{N-m}-\gamma_{m+1})] \\ &\overset{(4.9)}{=} \frac{C\eta}{1-\sigma}\Big[(C-1)\omega\sigma^m + \sigma^{N-m-1}\omega(1-C\eta\sigma^m) + \gamma_{m+1}(\sigma^{m+1}-\sigma^{N-m-1})\Big] \leq 0. \end{aligned}$$

Hence, the term

$$-a\prod_{i=m+2}^{N-m}\gamma_i\left[\prod_{i=N-m+1}^{N}\gamma_i - \prod_{i=N-m+1}^{N}(\gamma_i-1)\right]$$

from (4.15) as well as the difference of the two remaining summands in this inequality is positive which ensures (4.15) and, consequently, completes the proof.

□

In order to prove the assertion of Proposition 4.7 for $\omega = 1$, let a control horizon m be given. Then, an induction with respect to the optimization horizon N is carried out. The assertion $\alpha^{\omega}_{N,m+1} \geq \alpha^{\omega}_{N,m}$ has to be shown for $m \in \{1, 2, \ldots, \lfloor N/2 \rfloor - 1\}$. Hence, $N = 2m + 2$ is the smallest optimization horizon for given m.

The proof is divided into two parts: the induction assumption, i.e. $N = 2m+2$, is dealt with separately in the following lemma. The induction step is carried out afterward in Lemma 4.20. Splitting up the proof is motivated by the restrictions on the terminal weight ω in Proposition 4.7. Lemma 4.19 covers arbitrary terminal weights, i.e. $\omega \in [1, (1-\sigma)^{-1})$ while the induction step is only shown for $\omega = 1$.[5] This approach allows to indicate problems occurring for terminal weights $\omega \in (1, (1-\sigma)^{-1})$. We conjecture that Proposition 4.7 holds independently of the chosen terminal weight. Furthermore, we point out that the induction is carried out with respect to the optimization horizon N. Proceeding the other way round, i.e. an induction vis-à-vis the control horizon m, has not proven to be fruitful.

Lemma 4.19
Let Assumption 3.2 be satisfied with a $\mathcal{KL}_0$-function of type (1.11)*. Furthermore, let the terminal weight ω be chosen such that $\eta := 1 + \sigma\omega - \omega > 0$. Then, for $m \in \mathbb{N}$ and $N = 2m + 2$,* (4.14) *holds for the optimal value $\alpha^{\omega}_{N,m}$ of Problem 3.10 which is given by Theorem 3.18.*

[5]Note that Lemma 4.18 covers the assertion for $\omega \geq (1-\sigma)^{-1}$, i.e. "sufficiently large" terminal weights.

Proof: Since Lemma 4.13 covers the assertion for $\gamma_{m+1} - \omega \leq 0$, the inequality $\gamma_{m+1} - \omega > 0$ is assumed. Hence, showing (4.15) is sufficient in order to show the assertion. Taking into account $N-m-1 = m+1$, the term a introduced in this inequality simplifies to $C\sigma^m\eta\omega(\gamma_{m+2} - 1)$. Hence, reducing (4.15) by $(\gamma_{m+2} - 1)$ leads to

$$\Big[C\sigma^m\eta\omega\gamma_{m+2} + \omega(\gamma_{m+1} - \omega)\Big] \prod_{i=m+3}^{N} (\gamma_i - 1) \geq C\sigma^m\eta\omega\gamma_{m+2} \prod_{i=m+3}^{N} \gamma_i.$$

Using the representation of γ_{m+2} given by (4.9) and $N-m-2 = m$, proceeding analogously to the proof of Lemma 4.15 yields

$$\underbrace{\left(C^2 + \frac{1-\sigma^m\eta}{\sigma^m\eta(1-\sigma^{m+1}\eta)}C - \frac{\omega(1-\sigma)}{\sigma^m\eta(1-\sigma^{m+1}\eta)}\right) \prod_{i=m+3}^{N} \left(C - \frac{1-\sigma}{1-\sigma^{i-1}\eta}\right)}_{=:p(C)} \geq \underbrace{C^{m+2}}_{=:q(C)}.$$

We aim at applying Lemma 4.14 in order to establish this inequality. Since Proposition 4.1 ensures a point of intersection at $C = 1$ which is supposed to play the part of $\tilde{z}$ in the third assumption of Lemma 4.14, the positive roots of the monic polynomial $p(\cdot) : \mathbb{R} \to \mathbb{R}$ have to be located in $[0, 1)$. Structurally, $p(\cdot)$ consists of two factors. Here, the factor $\prod_{i=m+3}^{N}[C-(1-\sigma)/(1-\sigma^{i-1}\eta)]$ represents a polynomial of degree m which is decomposed in linear factors and, thus, exhibits m real roots located in the open interval $(0, 1)$. Next, we determine the roots of the other factor involved in the definition of $p(\cdot)$ by completing the square:

$$C = \frac{-(1-\sigma^m\eta) \pm \sqrt{(1-\sigma^m\eta)^2 + 4\omega(1-\sigma)\sigma^m\eta(1-\sigma^{m+1}\eta)}}{2\sigma^m\eta(1-\sigma^{m+1}\eta)},$$

i.e. one strictly positive and one strictly negative root. Hence, we complete our assertion with respect to the roots of $p(\cdot)$ by showing that the positive root of this factor is strictly less than one or, equivalently,

$$(1-\sigma^m\eta + 2\sigma^m\eta(1-\sigma^{m+1}\eta))^2 > (1-\sigma^m\eta)^2 + 4\omega(1-\sigma)\sigma^m\eta(1-\sigma^{m+1}\eta).$$

Cancelling out the summand $(1-\sigma^m\eta)^2$, reducing the resulting expression by $4\sigma^m\eta$ $(1-\sigma^{m+1}\eta)$, and using the definition of η leads to

$$\sigma^m\eta(1-\sigma^{m+1}\eta) + (1-\sigma^m\eta) - \omega(1-\sigma) = \eta(1-\sigma^{2m+1}\eta) > 0.$$

Consequently, it remains to establish the second condition of Lemma 4.14 which deals with the $(m+1)^{st}$ derivatives of the polynomials $p(\cdot)$ and $q(\cdot)$. To this end, we calculate

$$p^{(m+1)}(C) = (m+2)!C + (m+1)! \left(\frac{1-\sigma^m\eta}{\sigma^m\eta(1-\sigma^{m+1}\eta)} - \sum_{i=m+3}^{N} \frac{1-\sigma}{1-\sigma^{i-1}\eta}\right).$$

Since $q^{(m+1)}(\cdot)$ has its respective root at the origin, we have to prove that the root of $p^{(m+1)}(C)$ is strictly negative. For this purpose, it suffices to establish

$$\frac{1-\sigma^m\eta}{\sigma^m\eta(1-\sigma^{m+1}\eta)} - \sum_{i=m+3}^{N} \frac{1-\sigma}{1-\sigma^{i-1}\eta} > 0.$$

Taking $N - m - 2 = m$ into account, this is ensured by

$$\frac{1-\sigma^m\eta}{1-\sigma} = \sum_{i=0}^{m-1} \sigma^i + \omega\sigma^m > m\sigma^m > \eta\sigma^m \sum_{i=m+3}^{N} \frac{(1-\sigma^{m+1}\eta)}{1-\sigma^{i-1}\eta}.$$

Hence, all assumptions of Lemma 4.14 are satisfied which enables us to conclude the assertion.

□

Lemma 4.19, which is used as induction start for the proof of the following lemma, holds for $\omega \in [1, (1-\sigma)^{-1})$, i.e. for all terminal weights not covered by Lemma 4.18. In contrast to that, $\omega = 1$ is assumed in the induction step, which is carried out in the proof of Lemma 4.18. However, this restriction is not imposed in the beginning of the induction step in order indicate and briefly discuss problems of extending Lemma 4.20 to terminal weights $\omega \in (1, (1-\sigma)^{-1})$. Furthermore, we like to point out that Lemma 4.14 was originally designed for the following induction step.

Lemma 4.20
Let the $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$ from Assumption 3.2 be of type (1.11) *and* $\omega = 1$. *Then, for $N \in \mathbb{N}_{\geq 4}$, the optimal value $\alpha_{N,m} = \alpha^1_{N,m}$ of Problem 3.8 exhibits the monotonicity property given by Proposition 4.7, i.e. $\alpha^\omega_{N,m+1} - \alpha^\omega_{N,m} \geq 0$ for $m \in \{1, \ldots, \lfloor N/2 \rfloor - 1\}$.*

Proof: Repeating the line of arguments used in Lemma 4.18 yields that it is sufficient to establish (4.15). Let a control horizon m be given. Then, the preceding lemma covers the assertion for the smallest possible choice of the optimization horizon N — our induction assumption. Hence, carrying out the induction step proves the claim.

Suppose that (4.15), i.e.

$$(\gamma_{m+1}-\omega)(\gamma_{m+2}-\omega)\prod_{i=m+2}^{N}(\gamma_i-1) \leq -a\prod_{i=m+2}^{N}\gamma_i + \Big[a+(\gamma_{m+1}-\omega)(\gamma_{m+2}-1)\Big]\prod_{i=m+2}^{N-m}\gamma_i\prod_{i=N-m+1}^{N}(\gamma_i-1),$$

holds for $N \geq 2m+2$. The term a is given by[6]

$$a = \gamma_{N-m}(\omega\gamma_{m+2} - \omega\gamma_{m+1} + \gamma_{m+1} - \omega) - \gamma_{m+1}(\gamma_{m+2} - \omega).$$

Our goal is to show this inequality for $N+1$, i.e. the induction step $N \rightsquigarrow N+1$. To this end, we require the definition

$$\tilde{a} := \gamma_{N+1-m}(\omega\gamma_{m+2} - \omega\gamma_{m+1} + \gamma_{m+1} - \omega) - \gamma_{m+1}(\gamma_{m+2} - \omega),$$

i.e. N is substituted by $N+1$ and, thus, γ_{N-m} by γ_{N-m+1} in a. Then — using the induction assumption — the desired inequality is ensured by

$$\begin{aligned}&(\gamma_{N+1}-1)\left[-a\prod_{i=m+2}^{N}\gamma_i + \Big[a+(\gamma_{m+1}-\omega)(\gamma_{m+2}-1)\Big]\prod_{i=m+2}^{N-m}\gamma_i\prod_{i=N-m+1}^{N}(\gamma_i-1)\right]\\ \leq\ & -\tilde{a}\prod_{i=m+2}^{N+1}\gamma_i + \Big[\tilde{a}+(\gamma_{m+1}-\omega)(\gamma_{m+2}-1)\Big]\prod_{i=m+2}^{N-m+1}\gamma_i\prod_{i=N-m+2}^{N+1}(\gamma_i-1).\end{aligned}$$

[6] Note that, in comparison to the proof of Lemma 4.18, we rearranged only the considered term.

Since $N-m \geq m+2$, dividing this inequality by $\prod_{i=m+2}^{N-m} \gamma_i$ and taking

$$\begin{aligned} \tilde{a} + (\gamma_{m+1}-\omega)(\gamma_{m+2}-1) &= (\gamma_{N-m+1}-1)(\omega\gamma_{m+2}-\omega\gamma_{m+1}+\gamma_{m+1}-\omega), \\ a + (\gamma_{m+1}-\omega)(\gamma_{m+2}-1) &= (\gamma_{N-m} \quad -1)(\omega\gamma_{m+2}-\omega\gamma_{m+1}+\gamma_{m+1}-\omega), \end{aligned}$$

into account, leads to

$$(\omega\gamma_{m+2}-\omega\gamma_{m+1}+\gamma_{m+1}-\omega)\Big[\gamma_{N-m+1}-(\gamma_{N-m}-1)\Big]\prod_{i=N-m+1}^{N+1}(\gamma_i-1) \geq \Big[\tilde{a}\gamma_{N+1}-a(\gamma_{N+1}-1)\Big]\prod_{i=N-m+1}^{N}\gamma_i.$$

Dividing this inequality by $(\omega\gamma_{m+2}-\omega\gamma_{m+1}+\gamma_{m+1}-\omega)$ and using (4.9) yields

$$(C\sigma^{N-m-1}\eta+1)\prod_{i=N-m+1}^{N+1}(\gamma_i-1)\geq\left[C\sigma^{N-m-1}\eta+\frac{\gamma_{N-m}}{\gamma_{N+1}}-\frac{\gamma_{m+1}(\gamma_{m+2}-\omega)}{\gamma_{N+1}(\omega\gamma_{m+2}-\omega\gamma_{m+1}+\gamma_{m+1}-\omega)}\right]\prod_{i=N-m+1}^{N+1}\gamma_i.$$

Since $\eta = 1+\sigma\omega-\omega > 0$ is assumed, the divisor is positive. The quotient consisting of the numerator $\gamma_{m+1}(\gamma_{m+2}-\omega)$ and the denominator $\gamma_{N+1}(\omega\gamma_{m+2}-\omega\gamma_{m+1}+\gamma_{m+1}-\omega)$ is the most difficult to handle. Here, the factor $\gamma_{m+1}/\gamma_{N+1}$ does not contain the overshoot C and, thus, only contributes a constant. The other factor, however, has a polynomial of degree one in the denominator — exactly this prevents the applicability of Lemma 4.14 for $\omega \in (1,(1-\sigma)^{-1})$.

On the contrary, for terminal weight $\omega = 1$, $(\omega\gamma_{m+2}-\omega\gamma_{m+1}+\gamma_{m+1}-\omega) = (\gamma_{m+2}-1)$ holds. Hence, the considered factor cancels out with $(\gamma_{m+2}-\omega) = (\gamma_{m+2}-1)$. Taking $\eta = \sigma$ into account and reducing the inequality in consideration by σ^{N-m} provides

$$p(C) := (C+\sigma^{-(N-m)})\prod_{i=N-m+1}^{N+1}\left(C-\frac{1-\sigma}{1-\sigma^i}\right) \geq C^{m+1}\left(C+\frac{\sigma^{m+1}-\sigma^{N-m}}{(1-\sigma^{N+1})\sigma^{N-m}}\right) =: q(C).$$

A straightforward application of Lemma 4.14 ensures this inequality and, thus, allows for concluding the assertion. Proposition 4.1 yields the point of intersection at $C=1$, i.e. $p(1) = q(1)$. Furthermore, note that $p(\cdot)$ has exactly one negative root and $m+1$ strictly positive roots which are located in the open interval $(0,1)$. Additionally, q can be represented as $C^{m+1}(C+c)$ with $c > 0$. Hence, the only condition which has to be verified is the one with respect to the $(m+1)^{st}$ derivative. To this end, we calculate

$$p^{(m+1)}(C) = (m+2)!C + (m+1)!\left(\sigma^{-(N-m)} - \sum_{i=N-m+1}^{N+1}\frac{1-\sigma}{1-\sigma^i}\right).$$

Consequently, it suffices to establish the following inequality in order to complete the proof:

$$1-\sigma^{N-m}\sum_{i=N-m+1}^{N+1}\frac{1-\sigma}{1-\sigma^i} > \frac{\sigma^{m+1}-\sigma^{N-m}}{(1-\sigma^{N+1})}.$$

Since $(1-\sigma^i)^{-1} < (1-\sigma^{N-m})^{-1}$ for all $i \in \{N-m+1, N-m+2, \ldots, N+1\}$, this is ensured by

$$\frac{1-\sigma^{m+1}}{1-\sigma} = \sum_{n=0}^{m}\sigma^n > (m+1)\sigma^m > (m+1)\sigma^{N-m} = \sigma^{N-m}\sum_{i=N-m+1}^{N+1}\frac{1}{1-\sigma^{N-m}}.$$

□

4.3 Further Results

This section contains miscellaneous results. We begin with commenting on Assumption 3.2 which may seem to be restrictive at first glance. However, since this condition is formulated in terms of the stage costs, it turns out that even systems which are only asymptotically but not exponentially stable satisfy Assumption 3.2 with a $\mathcal{KL}$-function of type (1.11) and, thus, exhibit the desired linearity feature exploited in order to deduce the formula presented in Theorem 3.18. In particular, the results of this and the previous chapter are applicable. Secondly, we deal with the impact of incorporating an additional terminal weight in our setting which significantly complicated deriving results on symmetry and monotonicity. Then, in order to conclude this section, the example of the linearized inverted pendulum on a cart is considered. Here, theoretically observed but presumably astonishing properties like the monotonicity in the control horizon are numerically resembled. This motivates the construction of algorithms which employ these properties in the following Section 4.4.

4.3.1 Comments on Assumption 3.2

Assuming linearity of the $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ from Assumption 3.2 in its first argument seems to be a demanding condition. However, since the stage costs can be used as a design parameter, cf., e.g. [39, Section 7] and [6], this even includes systems which are only asymptotically controllable. For instance, the stage costs were manipulated to respect homogeneity in order to get similar properties for systems that are not exponentially stabilizable in [32].

In order to further substantiate this claim, the control system defined by $x(n+1) = x(n) + u(n)x(n)^3$ is considered which corresponds to the Euler approximation of the differential equation $\dot{x}(t) = u(t)x(t)^3$ with time step $T = 1$. Furthermore, the control $\mathbb{U} = [-1, 1]$ and state constraints $\mathbb{X} = (-1, 1) \subset \mathbb{R}$ are set.[7] This system is asymptotically stabilizable with control function $u(\cdot) \equiv -1$, i.e. $x(n+1) = x(n) - x(n)^3$. But, taking the constraints into account, it is not exponentially stabilizable. In order to show the claimed exponentially controllability in terms of the continuous stage costs, we define

$$\ell(x(n), u(n)) := \begin{cases} e^{-\frac{1}{2x(n)^2}} & \text{for } \|x(n)\| \in \mathbb{X}\backslash\{0\}, \\ 0 & \text{for } x = 0. \end{cases}$$

Note that $\ell^\star(x) = \ell(x,u)$ holds for all admissible control values u because the control effort is not penalized. The $\mathcal{KL}$-function $\beta(r,n) = e^{-n}r$ of type (1.11) with parameters $C = 1$ and $\sigma = e^{-1}$ is chosen. Hence, we have to show the inequality

$$\ell(x(n+1), u(n+1)) = \ell^\star(x(n+1)) = \ell^\star(x(n)(1-x(n)^2)) \leq e^{-1}\ell^\star(x(n))$$

which is, in turn, equivalent to

$$\ell^\star(x(n+1)) = e^{-\frac{1}{2x(n)^2(1-x(n)^2)^2}} \leq e^{-\frac{2x(n)^2+1}{2x(n)^2}} = e^{-1}e^{-\frac{1}{2x(n)^2}} = \sigma\ell^\star(x(n))$$

Using $1 > 1 - 3x(n)^4 + 2x(n)^6 = 2x(n)^2(1-x(n)^2)^2 + (1-x(n)^2)^2 > 0$ for $x \in \mathbb{X} = (0,1)$ ensures this inequality and, thus, inductively implies exponential controllability in terms of $\ell(\cdot,\cdot)$. Summarizing, designing the stage costs $\ell(\cdot,\cdot)$ suitably allows for verifying the needed assumptions in order to apply the deduced results, even for systems which are not exponentially controllable with respect to their norm.

[7] The state and control restrictions are necessary to preserve the characteristics of the continuous time system for the Euler approximation.

4.3.2 Cost Functional Incorporating a Terminal Weight

In order to evaluate the benefit attributed to using an additional weight on the final term in our receding horizon cost functional (2.4), the special case that the coefficient c_m of the $\mathcal{KL}_0$-function contained in Assumption 3.2 is strictly smaller than one is considered first: since $c_m < 1$, the necessary and sufficient condition $\gamma_{m+1} - \omega \leq 0$ of Theorem 3.18 for $\alpha_{N,m} = 1$ and, thus, stability can always be ensured by choosing the terminal weight ω sufficiently large in this case. We point out that the probability of being able to fulfill this condition increases, in general, with longer control horizons, e.g. for $\mathcal{KL}_0$-functions of type (1.11).

However, without this condition being satisfied, analyzing effects resulting from including a terminal weight is much more subtle. Hence, we begin our investigation by looking at the following example which demonstrates the typical positive effects of adding weight on the final term.

Example 4.21
Let Assumption 3.2 hold based on $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$ of type (1.12) *given by*

$$c_0 = 1,\ c_1 = 3/2,\ c_2 = 39/20,\ c_3 = 7/5,\ \text{and } c_i = 0 \text{ for all } i \in \mathbb{N}_{\geq 4},$$

i.e. finite time controllability. Since Property (1.13) *is satisfied, Theorem 3.18 can be applied in order to deduce suboptimality bounds. The resulting values $\alpha^{\omega}_{N,m}$ for optimization horizon $N = 7$, control horizons $m \in \{1, 2, \ldots, 6\}$, and $\omega = 1$ (no terminal weight) as well as $\omega = 3/2$ are illustrated in Figure 4.7 a).*

At first, note that the symmetric bound as well as the monotonicity property of Lemmata 4.11 and 4.17 hold — although the respective assumptions are violated since c_3, c_4 are not equal to zero. The interplay of these two properties and the terminal weight $\omega = 3/2$ implies our stability condition $\alpha^{\omega}_{7,m} \geq 0$ and, thus, asymptotic stability for the receding horizon closed loop for $m = 4$.[8] *Note that this is not the case for $\omega = 1$.*

The next example points out a possible pitfall of large ω.

Example 4.22
As in the previous example, we assume finite time controllability, i.e. Assumption 3.2 with a $\mathcal{KL}_0$-function of type (1.12) *given by $c_0 = 1$, $c_1 = 3/2$, $c_2 = 2/3$, $c_3 = 1$, and $c_i = 0$ for all $i \in \mathbb{N}_{\geq 4}$. Note that these coefficients guarantee* (1.13). *In Figure 4.7 b) the respective performance bounds are depicted for optimization horizon $N = 5$ with several terminal weights.*

Although increasing ω seems to improve, in general, the guaranteed stability behavior significantly, an additional weight on the final term – chosen too big – may even invalidate our stability criterion for $m = 1$. However, in this example shifting to a larger control horizon compensates this drawback.

In conclusion, using terminal weights typically improves the values provided by Theorem 3.18, and, thus, simplifies the verification of asymptotic stability. However, we stress the fact that the performance interpretation of the resulting suboptimality indices does not hold for $\omega > 1$ since $V_N(\cdot) \leq V_\infty(\cdot)$ crucially relies on $\omega = 1$. Furthermore, we emphasize once more that Theorem 3.18 allows for easily calculating the optimal values of Problem 3.10.

[8] We point out that the condition $\gamma_{m+1} - \omega \leq 0$ is not satisfied.

a) b)

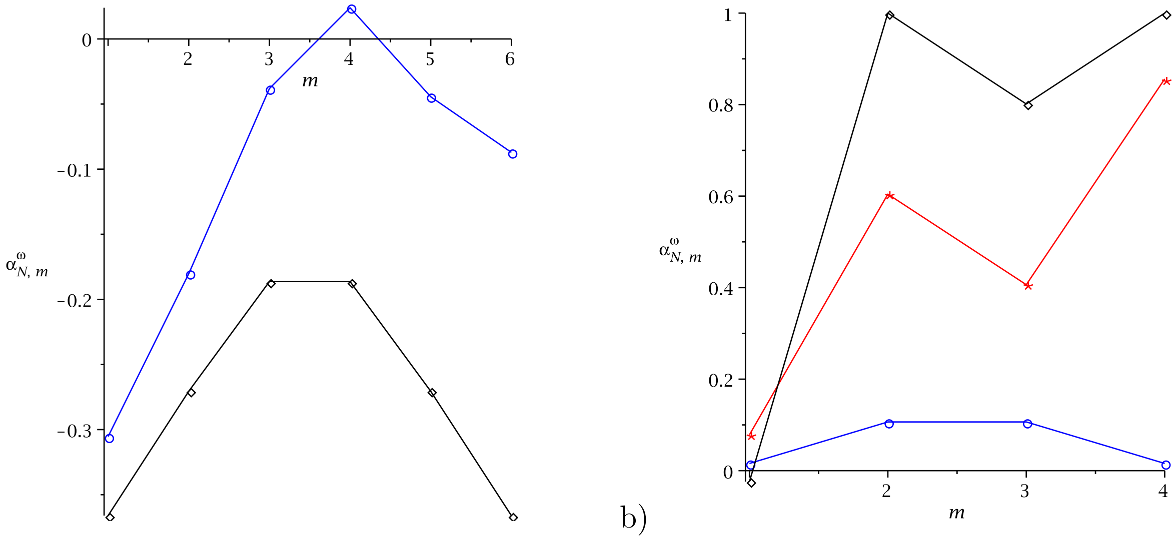


Figure 4.7: In a) the suboptimality bounds $\alpha^{\omega}_{7,m}$ from Theorem 3.18 are visualized for Example 4.21 for $\omega = 1$ ($\diamond$) and $\omega = 3/2$ ($\circ$) and control horizons $m \in \{1, 2, \ldots, 6\}$. In b), the same is done for Example 4.22 with $\omega \in \{1, 3.5, 27.5\}$ ($\circ$, $*$, $\diamond$).

4.3.3 Example: Linear Inverted Pendulum

The key assumption needed in order to deduce asymptotic stability or performance bounds on the receding horizon closed loop in Theorem 3.12 is a relaxed Lyapunov inequality of type (3.1). In this subsection such a Lyapunov inequality is checked numerically for the linearized inverted pendulum on a cart, cf. Examples 1.21 and 1.23. Then, the computed performance bounds are compared with theoretical results from Theorem 4.8, which deals with symmetric bounds and monotonicity properties of the deduced suboptimality estimates from Theorem 3.18 with respect to the control horizon m.

Since we aim at verifying a relaxed Lyapunov inequality for each state contained in the feasible set $\mathbb{X}$, a grid $\mathcal{G}$ on $[-0.375, 0.375]^4$ is considered which is uniformly partitioned in each coordinate direction and consists of $16^4 = 65536$ points. To be more precise, the grid points

$$x_{i_1,i_2,i_3,i_4} = (-0.375, -0.375, -0.375, -0.375)^T + (i_1 h, i_2 h, i_3 h, i_4 h)^T,$$

$i_j \in \{0, 1, \ldots, 15\}$ for each $j \in \{1, 2, 3, 4\}$ and stepsize $h = 0.05$ are used. For each grid point $x_0 \in \mathcal{G}$, the receding horizon Problem (2.4) - (2.6) with optimization horizon $N = 25$ is solved in order to obtain a sequence of open loop control values $u^\star(0; x_0), u^\star(1; x_0), \ldots, u^\star(N-1; x_0)$ satisfying $J_N(x_0, u^\star(\cdot; x_0)) = V_N(x_0)$. Doing so yields, as a by-product, the trajectory $x_{u^\star}(n; x_0)$ and the corresponding stage costs $\ell(x_{u^\star}(n; x_0), u^\star(n; x_0))$ for $n = 0, 1, \ldots, N-1$. Then, the following loop is carried out with respect to the control horizon $m \in \{1, 2, \ldots, N-1\}$:

- Solve the RHC problem (2.4) - (2.6) in order to obtain $V_N(x_{u^\star}(m; x_0))$, i.e. evaluate $V_N(\cdot)$ at $x_{u^\star}(m; x_0)$ and
- compute the suboptimality index $\alpha_{x_0}(m)$ depending on the current grid point x_0 and the control horizon parameter m, i.e.

$$\alpha_{x_0}(m) = \frac{V_N(x_0) - V_N(x_{u^\star}(m; x_0))}{\sum_{n=0}^{m-1} \ell(x_{u^\star}(n; x_0), u^\star(n; x_0))}.$$

In Figure 4.8, we marked $\alpha_{x_0}(m)$, $m \in \{1, 2, \ldots, N-1\}$, for each grid point $x_0 \in \mathcal{G}$. The minima $\alpha(m) := \min_{x_0 \in \mathcal{G}}\{\alpha_{x_0}(m)\}$, $m \in \{1, 2, \ldots, N-1\}$, are connected by the dashed black line. Additionally, we drew a red line in order to indicate whether $\alpha(m) > 0$ and, thus, a relaxed Lyapunov inequality holds for receding horizon control with optimization horizon N and control horizon m.

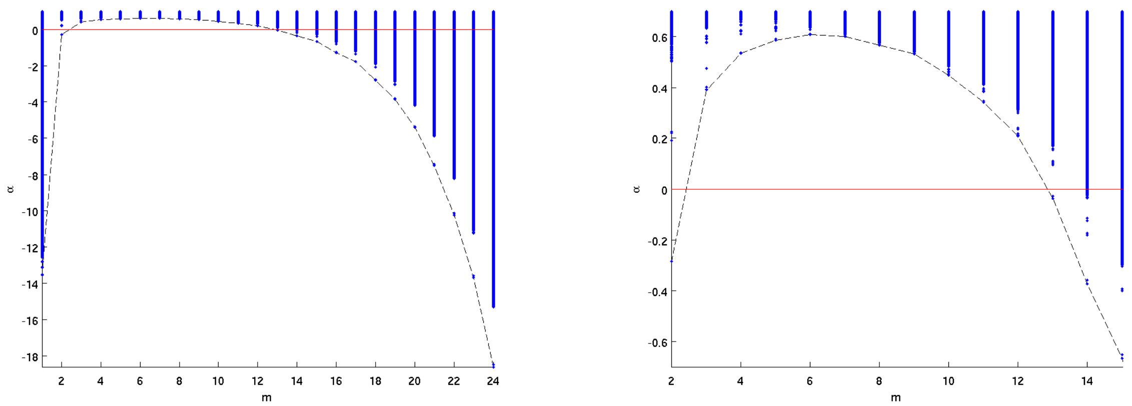


Figure 4.8: Illustration of the numerical computation of the suboptimality index $\alpha_{N,m} = \alpha^1_{N,m}$ from Assumption 3.2 for optimization horizon $N = 25$. For $m \in \{3, 4, \ldots, 12\}$ the relaxed Lyapunov inequality (3.1), i.e. our key condition in order to deduce asymptotic stability, holds.

The key assumption $\alpha(m) > 0$ of our main stability Theorem 3.12 is observed for $m \in \{3, 4, \ldots, 12\}$. For every other control horizon, in particular for $m = 1$ which corresponds to classical receding horizon control, at least one grid point exists which does not satisfy a relaxed Lyapunov inequality and, consequently, is linked with a negative suboptimality bound $\alpha_{x_0}(m)$. While the RHC trajectory may still converge to the origin and exhibit a satisfactory performance, our stability criterion is violated and, thus, stability cannot be ensured anymore.

In the numerical computations, exact symmetry and a maximum at $\lfloor N/2 \rfloor = 12$ are not present. However, the shape of $\alpha(m)$, $m = 1, 2, \ldots, N-1$, resembles the one expected from our derived theoretical results. In particular, increasing the control horizon improves the suboptimality bounds. Here, the best performance specification is guaranteed for $m = 6$. Indeed, many grid point does not satisfy a Lyapunov inequality for $m = 1$. In contrast to that, a slight increase to $m = 3$ ensures stability on the whole grid. These observations play a vital role in developing the algorithm in the ensuing section.

4.4 Algorithms

In Subsection 4.2 qualitative characteristics of the obtained stability bounds $\alpha^{\omega}_{N,m}$ are derived with respect to the control horizon m, i.e. symmetric bounds and monotonicity properties, which can be exploited according to Theorem 4.8. However, from a practitioner's point of view, the most interesting question remains whether a desired performance specification $\overline{\alpha} \geq 0$ is ensured or not, i.e. whether the inequality $\alpha^{\omega}_{N,m} \geq \overline{\alpha}$ and, thus, the desired stability behavior of the receding horizon closed loop holds. Hence, our goal is to deduce conditions implying a sufficiently large optimal value $\alpha^{\omega}_{N,m}$ of Problem

3.10 based on a given $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$. To this end, choosing the optimization horizon N sufficiently large provides an option, cf. Corollary 3.20. On the other hand, the computational cost needed in order to solve the optimal control Problem (2.4) - (2.6) in each receding horizon step grows rapidly with increasing optimization horizon N. In contrast to this, changing the control horizon length m does not affect the finite horizon optimization problem. The symmetry and monotonicity results from our theoretical analysis encourage us to pursue this strategy. Hence, we aim at developing an algorithm which enables us to ensure an a priori specified performance bound using the control horizon m as an tuning parameter in order to reduce the required horizon length N.

In order to illustrate this idea, Example 1.10, for which Assumption 3.2 was deduced in Section 3.3, is considered. Exploiting the corresponding $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$, Formula (3.21) yields $N = 28$ as the minimal horizon length ensuring asymptotic stability, cf. Figure 4.9. Using larger control horizons m leads to $N = 16$ ($m = 6$) and, thus, to a much smaller optimization horizon, cf. Figure 4.9.

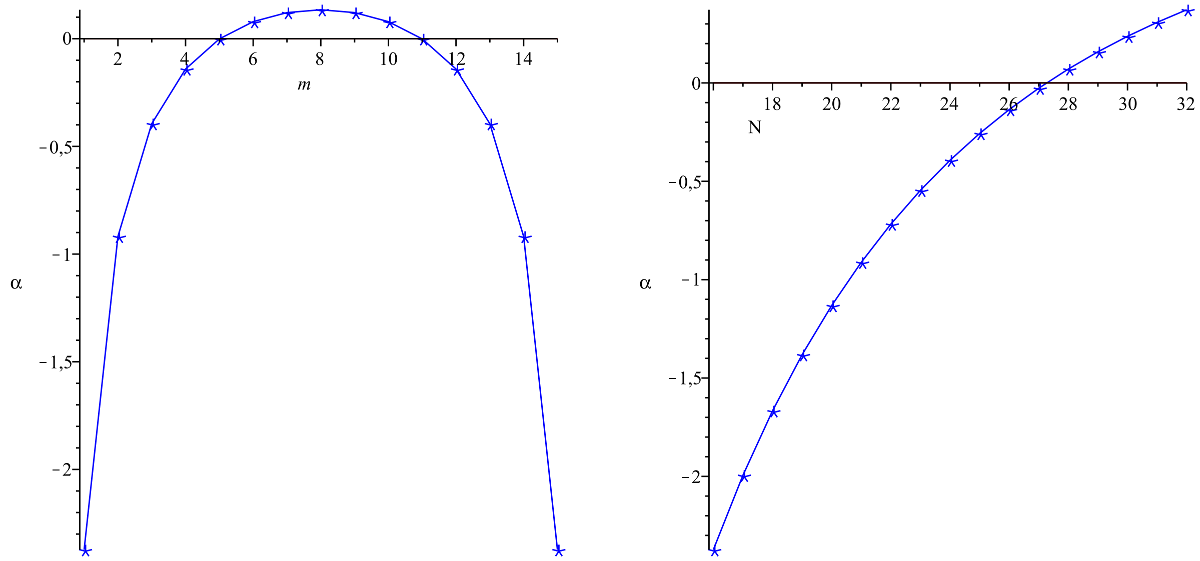


Figure 4.9: On the left, the performance bounds $\alpha^1_{16,m}$, $m \in \{1,2,\ldots,15\}$, are depicted for Example 1.10. The optimization horizon $N = 16$ in combination with $m = 6$ ensures asymptotic stability of the respective RHC closed loop . In contrast to that, RHC with $m = 1$ requires a significantly longer optimization horizon $N = 28$ for $\alpha^1_{N,1} \geq 0$.

Implementing more than only the first element of the resulting sequence of control values postpones the next time instant at which a solution of the updated optimization problem (2.4) - (2.6) has to be found. We point out that this does, in general, not reduce the computational effort because the optimization routine typically solves an auxiliary optimization problem at each time instant in order to get a good initial guess for the successor state. Otherwise, numerical problems are encountered for large control horizons m, cf. [95]. Furthermore, the resulting closed loop stays in open loop for longer periods of time which may affect its robustness, cf. [86]. In conclusion, our goal is to combine the benefits from both perspectives, i.e. using the tighter estimates for longer control horizons while avoiding the loss of robustness. To this end, we proceed in two steps: firstly, we introduce a very simple algorithm which puts the deduced theoretical results directly into practice. Secondly, we employ a more elaborated version which aims at robustifying the previously presented algorithm. A synchronous generator model is considered in order to illustrate the resulting benefits.

4.4.1 Basic Algorithm

For Example 1.10, Theorem 3.18 ensures asymptotic stability for each state x_0 of the entire feasible set $\mathbb{X}$ for optimization horizon $N \geq 28$ and control horizon $m = 1$. Furthermore, the assertion of Theorem 3.18 is only strict for the whole class of systems satisfying the assumed controllability condition, cf. Remark 3.13. Hence, for a particular system and a given initial condition, the corresponding estimates on the required horizon length may be conservative. In addition, the parameter combination $(N, m) = (16, 6)$ guarantees our stability criterion $\alpha^1_{N,m} > 0$, cf. Figure 4.9. The introduced concept of multistep feedbacks allows to employ, depending on the current state, different control horizons. Furthermore, Theorem 3.12 yields $V_N(\cdot)$ as a common Lyapunov function assuming a relaxed Lyapunov inequality. Hence, we aim at maintaining this Lyapunov criterion along the trajectory emanating from x_0 for $(N, m) = (16, 1)$ and employing more than one element of the computed sequence of control values only if needed. The construction of the following algorithm is motivated by this approach: using the enhanced estimates for larger $m > 1$ but not staying in open loop if that is not necessary in order to improve the robustness of the receding horizon closed loop.

Summarizing, ensuring a relaxed Lyapunov inequality for each point contained in the receding horizon trajectory is sufficient. In order to take this into account, Proposition 3.1 is adapted using the terminology of Definition 1.24, cf. [42]. The elementary proof follows the line of argumentation used in order to prove Proposition 3.1, cf. [97].

Proposition 4.23
Let a performance bound $\overline{\alpha} \in (0, 1]$, an initial value $x_0 \in \mathbb{X}$, $m^\star \in \mathbb{N}$, a multistep feedback law $\tilde{\mu} : \mathbb{X} \times \{0, 1, \ldots, m^\star - 1\} \to \mathbb{U}$ according to Definition 1.25, a set $M \subset \{1, 2 \ldots, m^\star\}$, and a control horizon sequence $(m_i)_{i \in \mathbb{N}_0} \subseteq M$ be given and define $\sigma(\cdot)$ accordingly to Definition 1.24. Furthermore, suppose the existence of a function $\widetilde{V} : \mathbb{X} \to \mathbb{R}^+_0$ such that the corresponding solution $x_{\tilde{\mu}}(\cdot)$ with $x_{\tilde{\mu}}(0) = x_{\tilde{\mu}}(\sigma(0)) = x_0$ satisfies $x_{\tilde{\mu}}(\sigma(k)) \in \mathbb{X}$, $k \in \mathbb{N}$, and

$$\widetilde{V}(x_{\tilde{\mu}}(\sigma(k))) \geq \widetilde{V}(x_{\tilde{\mu}}(\sigma(k+1))) + \overline{\alpha} \sum_{n=0}^{m_k - 1} \ell(x_{\tilde{\mu}}(n; x_{\tilde{\mu}}(\sigma(k))), \tilde{\mu}(n, x_{\tilde{\mu}}(\sigma(k)))) \tag{4.16}$$

for all $k \in \mathbb{N}_0$. Then, the following estimate holds:

$$\begin{aligned} V_\infty^{\tilde{\mu},(m_i)}(x_0) &= \sum_{k=0}^{\infty} \sum_{n=0}^{m_k - 1} \ell(x_{\tilde{\mu}}(n; x_{\tilde{\mu}}(\sigma(k))), \tilde{\mu}(n, x_{\tilde{\mu}}(\sigma(k)))) \\ &= \sum_{n=0}^{\infty} \ell(x_{\tilde{\mu}}(n), \tilde{\mu}(x_{\tilde{\mu}}(\varphi(n)), n - \varphi(n))) \leq \widetilde{V}(x_0)/\overline{\alpha}. \end{aligned}$$

Proposition 4.23 ensures suboptimality for the trajectory emanating from x_0 and steered by the chosen feedback.

Now, the announced algorithm is presented. During runtime of the algorithm a list $\mathcal{S}$ is constructed which contains the switching times $\sigma(k)$, $k \in \mathbb{N}_0$. To this end, we make use of the programming notation *back* which allows for fast access to the last element of a list.[9] If (4.16) cannot be ensured for the chosen optimization horizon N, an "exit strategy" is required, cf. Remark 4.25. Furthermore, a stopping criterion may be added. If none is implemented, the algorithm runs forever which fits in with many control task. We point out that the desired relaxed Lyapunov inequality may be violated due to numerical effects in a small neighborhood of the equilibrium $x^\star$, cf. [44] for results concerning practical asymptotic stability, which may motivate a criterion $\|x_{\mu_N}(\sigma(k)) - x^\star\| \leq \varepsilon$ with a sufficiently small $\varepsilon > 0$ for numerical experiments.

Algorithm 4.24
Let an initial state $x_0 \in \mathbb{X}$, a list $\mathcal{S} = (0)$, an optimization horizon $N \in \mathbb{N}_{\geq 2}$, and a performance specification $\overline{\alpha} \in [0, 1)$ be given.
Set $k = 0$. Do

(1) Set $j = 0$, compute $\mu_N(\cdot, x_{\mu_N}(\sigma(k)))$ and $V_N(x_{\mu_N}(\sigma(k)))$. Do

  (a) Set $j = j + 1$, compute $V_N(x_{\mu_N}(j; x_{\mu_N}(\sigma(k))))$.

  (b) Compute the maximal α satisfying (4.16) with α, $V_N(\cdot)$, and $\mu_N(\cdot, \cdot)$ instead of $\overline{\alpha}$, $\widetilde{V}(\cdot)$, and $\tilde{\mu}(\cdot, \cdot)$, i.e.

$$\alpha := \frac{V_N(x_{\mu_N}(\sigma(k))) - V_N(x_{\mu_N}(j; x_{\mu_N}(\sigma(k))))}{\sum_{n=0}^{j-1} \ell(x_{\mu_N}(n; x_{\mu_N}(\sigma(k))), \mu_N(n, x_{\mu_N}(\sigma(k))))} \tag{4.17}$$

  for $x_{\mu_N}(\sigma(k)) \neq x^\star$ and $\alpha = 1$ otherwise.

  (c) If $\alpha \geq \overline{\alpha}$: Set $m_k = j$ and goto (2)

  (d) If $j = N$: use "exit strategy".

  while $\alpha < \overline{\alpha}$

(2) For $j = 1, \ldots, m_k$ do

  Implement $\mu_N(j - 1, x_{\mu_N}(\sigma(k)))$ at the plant.

(3) Set $\mathcal{S} := (\mathcal{S}, \mathrm{back}(\mathcal{S}) + m_k)$, $k := k + 1$, goto (1)

while stopping criterion not satisfied.

Remark 4.25
Algorithm 4.24 checks in step (1)(d) whether the relaxed Lyapunov inequality is satisfied for at least one control horizon $m \in \{1, 2, \ldots, N-1\}$ for the given optimization horizon N. If this verification fails, an "exit strategy" has to be used:

- *From a practitioner's point of view one option is to print a warning, e.g. "solution may diverge", setting $m_k = 1$, and continuing with step (2) of the algorithm, cf. [97]. Then, however, one has to hope, fingers crossed, that everything will turn out to be good in the end, although the desired stability criterion is violated.*

- *If, in addition to asymptotic stability, a performance bound $\overline{\alpha} > 0$ is checked, let $\hat{\alpha}$ denote the maximal value α obtained in (4.17). In case $\hat{\alpha} > 0$ holds, a warning may be issued that the desired specification $\overline{\alpha}$ is not maintained and the performance*

[9] The command back($\mathcal{S}$) returns the last element s_n of the list $\mathcal{S} = (s_0, s_1, s_2, \ldots, s_n)$.

bound may be lowered to $\hat{\alpha}$*. Then, Proposition 4.23 still allows for guaranteeing the reduced bound* $\hat{\alpha}$ *but cannot be employed in order to conclude the original performance index* $\overline{\alpha}$ *any longer.*

In this thesis, the optimization horizon N *is chosen such that condition (1)(c) is ensured for at least one control horizon* m *by our suboptimality analysis and, thus, (1)(d) is excluded a priori.*

Another approach is outlined in [30]: suppose that α *in* (4.17) *is strictly greater than the desired performance bound* $\overline{\alpha}$*. Then, a positive slack variable*

$$s := V_N(x_{\mu_N}(\sigma(k))) - V_N(x_{\mu_N}(j; x_{\mu_N}(\sigma(k)))) - \overline{\alpha} \sum_{n=0}^{j-1} \ell(x_{\mu_N}(n; x_{\mu_N}(\sigma(k))), \mu_N(n, x_{\mu_N}(\sigma(k)))) \tag{4.18}$$

is introduced and added to the numerator of the right hand side in (4.17) *in the next iteration of the algorithm. As a consequence* α *is increased and, thus, condition (1)(c) is weakened.*[10] *This feature can be easily incorporated in our algorithm and may lead to an improvement, cf. [97]. However, a violation in the first iteration of Algorithm 4.24 cannot be dealt with.*

This methodology can be extended such that a negative slack and, thus, a violation of the desired Lyapunov inequality, is allowed. Then, the algorithm may be modified in order to compensate, if possible, such violations in terms of stability or performance a posteriori, cf. [96].

As pointed out in Remark 4.25 we want to exclude step (1)(d). Hence, the algorithm ensures the desired performance a priori — a distinguishing feature in comparison to algorithms which do only verify a suboptimality estimate a posteriori but may run into a dead-end.

Remark 4.26
The introduced list $\mathcal{S}$ *represents a possibility to implement the sequence* $(\sigma(k))_{k\in\mathbb{N}_0}$*. The current time instant is accessible fast via back(*$\mathcal{S}$*). In addition, the corresponding state* $(x_{\mu_N}(\sigma(k)))_{k\in\mathbb{N}_0}$ *may be added to the list* $\mathcal{S}$*, whose entries then consist of two elements. We emphasize that* $V_N(\cdot)$ *is employed as a Lyapunov function at* $x_{\mu_N}(\sigma(k))$*, cf. Remark 3.13 (ii). Another option is to remove the list* $\mathcal{S}$ *and use only the current state* x*. Then, neither the time instances* $\sigma(k)$*,* $k \in \mathbb{N}_0$*, nor the corresponding states* $x_{\mu_N}(\sigma(k))$*,* $k \in \mathbb{N}_0$*, are saved in order to reduce memory usage.*

At first glance, Algorithm 4.24 seems to increase the effort needed in each receding horizon step. However, the a priori computation of $V_N(\cdot)$ at the next switching time corresponds exactly to the evaluation of this function at the ensuing time instant, which has to be done anyway in order to solve the optimal control problem posed in the receding horizon formulation. Hence, the proposed algorithm only produces additional computational cost if needed. In particular, since Algorithm 4.24 allows us to reduce the optimization horizon N significantly and the computational effort grows rapidly with respect to N, this expenditure is, in the majority of cases, more than compensated, cf. [97].

In order to demonstrate benefits of Algorithm 4.24, our investigation of Example 1.10 is carried on. Based on the $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$ from (3.31) the minimal optimization horizon N ensuring a desired performance specification is determined, cf. Table 4.1.

[10] If s was already introduced in a preceding step, the right hand side of (4.18) corresponds to the change of s. Then, s represents the accumulated slack.

For instance, asymptotic stability is ensured for optimization horizon $N = 16$ instead of $N = 28$ ($m = 1$). Hence, for $\overline{\alpha} = 0$, using the developed algorithm with optimization horizon $N = 16$ guarantees that no exit strategy is required. Since the desired relaxed Lyapunov inequality holds for RHC with horizon $N \geq 5$, the algorithm does not employ $m > 1$, cf. [90] and Section 5.5.1 below.

	RHC with $m = 1$	allowing for $m > 1$	
$\overline{\alpha}$	N	N	m
0	28	16	6
0.25	31	17	8
0.5	35	20	7
2/3	39	23	8
0.8	45	26	11
10/11	53	33	11
100/101	77	53	21

Table 4.1: Minimal stabilizing horizon for RHC with $m = 1$ and for RHC with suitably chosen control horizon $m \in \{1, 2, \ldots, N-1\}$ for the suboptimality bounds $\alpha^1_{N,m}$ from Theorem 3.18 based on Assumption 3.2 with $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$ from (3.31).

Hence, the strategy exhibited in the algorithm leads to classical RHC safeguarded by our theoretical results because the estimate deduced via Problem 3.8 provides only conservative bounds for this particular example. However, in many practical and more challenging applications $m > 1$ is indeed a necessary condition which may seem to be counter-intuitive. In this connection we emphasize that the relaxed Lyapunov inequality is checked less often and, thus, weakened by employing larger control horizons: ensuring the inequality

$$V_N(x_{\mu_{N,m}}(n+k+1)) \leq V_N(x_{\mu_{N,m}}(n+k)) + \overline{\alpha}\ell(x_{\mu_{N,m}}(n+k), \mu_{N,m}(x_{\mu_{N,m}}(n+k), 0)),$$

in each step $k \in \{0, 1, \ldots, m-1\}$ implies the desired criterion

$$V_N(x_{\mu_{N,m}}(n+m)) \leq V_N(x_{\mu_{N,m}}(n)) + \overline{\alpha} \sum_{k=0}^{m-1} \ell(x_{\mu_{N,m}}(n+k), \mu_{N,m}(x_{\mu_{N,m}}(n+k), 0))$$

after m steps. But this implication does not hold the other way round which explains why using larger control horizons may ensure the desired criterion independently of whether this is accompanied by an actual performance improvement or not.

In order to investigate the proposed algorithm more thoroughly, we consider the following nonlinear example from [28], which was also examined in [34], numerically.

Example 4.27 (Synchronous generator)
The system dynamics are given by

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -b_1 x_3(t) \sin x_1(t) - b_2 x_2(t) + P \\
\dot{x}_3(t) &= b_3 \cos x_1(t) - b_4 x_3(t) + E + u(t)
\end{aligned}$$

with parameters $b_1 = 34.29$, $b_2 = 0.0$, $b_3 = 0.149$, $b_4 = 0.3341$, $P = 28.22$, *and* $E = 0.2405$. *Choosing* $E = 0.2405$ *matches a stressed and, thus, more challenging, operating* *condition, cf. [28, Subsection 6.1]. This example is reformulated in a sampled-data fashion in order to fit into our discrete time setting. Let $\Phi(\cdot; x_0, \tilde{u})$ denote the solution operator of the differential equation with initial value x_0 and piecewise constant control function $\tilde{u}(\cdot) : [0, T) \to \mathbb{U}$, i.e. $\tilde{u}(t) = u \in \mathbb{U}$ for $t \in [0, T)$. Hence, the state space is given by $\mathbb{X} := \mathbb{R}^3$ and the control value space $\mathbb{U}$ may be identified with $\mathbb{R}$. Control constraints may be easily integrated by adapting $\mathbb{U}$ appropriately.*

For this example, our goal is to steer the system to its stable equilibrium $x^\star \approx (1.124603730, 0.0, 0.9122974248)^T$. In particular, also its unstable counterpart, i.e. the equilibrium $\tilde{x} \approx (1.170838231, 0.0, 0.8934977016)^T$, has to be rendered to $x^\star$, cf. [94]. Receding horizon control is employed for the system constructed with sampling period $T = 0.05$ and running costs

$$\begin{aligned}\ell_0(x,u) &= \int_0^T \|\varphi(t;x,\tilde{u}) - x^\star\|^2 + \lambda\|\tilde{u}(t)\|^2\,dt \qquad \text{or} \\ \ell_1(x,u) &= T\Big(\|\varphi(0;x,\tilde{u}) - x^\star\|^2 + \lambda\|\tilde{u}(0)\|^2\Big) = T\Big(\|x - x^\star\|^2 + \lambda\|u\|^2\Big)\end{aligned}$$

with $\lambda = 10^{-3}$. In addition, the physically motivated state constraints from [28] are taken into account, i.e.

$$\mathbb{X} := \left\{x \in \mathbb{R}^3 : 0 \le x_1 < \pi/2 \quad \text{and} \quad 0 \le x_3\right\}.$$

Let the desired performance bound $\overline{\alpha}_0 := 0$ be specified. Since Assumption 3.2 has to hold on a set of feasible states $x_0 \in \mathbb{X}$, level sets

$$\mathcal{L}_i := \{x_0 : V_6(x_0) = \inf_{u \in \mathcal{U}} \sum_{n=0}^{5} \ell_i(x_u(n), u(n)) \le 0.0196\} \tag{4.19}$$

of $V_N(\cdot) \subseteq \mathbb{X}$ for optimization horizon $N = 6$ are considered. Hence, ensuring our relaxed Lyapunov inequality for each point contained in a level set $\mathcal{L}_i$, $i \in \{1, 2\}$, guarantees to be, after implementing the first m control values, again in this set, i.e. the level sets $\mathcal{L}_i$, $i \in \{1, 2\}$, are receding horizon invariant and, thus, the state constraints are satisfied at each transmission instant $\sigma(k)$, $k \in \mathbb{N}_0$.

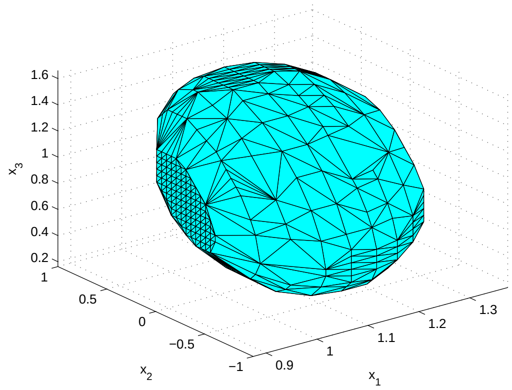
For our numerical investigation, a grid $\mathcal{G}$ contained in the cube

$$[x_1^\star - 0.25, x_1^\star + 0.25] \times [-1, 1] \times [x_3^\star - 0.75, x_3^\star + 0.75] \subset \mathbb{X}$$

is built up with discretization accuracy 0.05 in each coordinate direction. After removing the desired set point $x^\star$, this set consists of $11 \cdot 41 \cdot 31 - 1 = 13980$ grid points. The intersection of this grid $\mathcal{G}$ and the level set $\mathcal{L}_i$, $i \in \{1, 2\}$, is a subset of the introduced cube, cf. Figure 4.10.

Then, we compute, for each $x_0 \in \mathcal{G} \cap \mathcal{L}_i$, $i \in \{0, 1\}$, the corresponding suboptimality index $\alpha_{6,1}(x_0)$ and distinguish whether $\alpha_{6,1} \ge \overline{\alpha}$ is satisfied or not. If this check fails, the control horizon m is increased and the respective performance bound is computed. Indeed, for each considered initial state, a control horizon $m \in \{1, 2, 3, 4, 5\}$ exists such that $\alpha_{6,m}(x_0) \ge \overline{\alpha}$ and, thus, $J_6(x_{\mu_{6,m}}(m, x_0)) \in \mathcal{L}_i$, $i \in \{0, 1\}$ holds. Repeating this line of arguments iteratively shows that the proposed algorithm may be applied without an exit strategy in order to conclude the desired stability behavior.[11]

[11]Since the relaxed Lyapunov inequality is ensured only at each grid point and not necessarily for each point contained in the respective level set $\mathcal{L}_i$, $i \in \{0, 1\}$, the argumentation is not rigorous. Nevertheless, the stated claim is confirmed by our numerical computations.



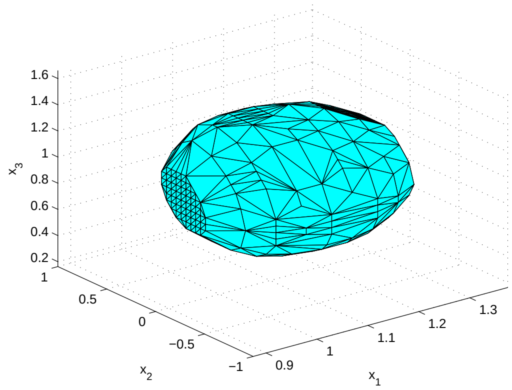


Figure 4.10: Illustration of the level sets $\mathcal{L}_0$ (left) and $\mathcal{L}_1$ (right) from (4.19) by means of the corresponding convex hulls. $\mathcal{L}_0$ contains 3091 grid points, whereas $\mathcal{L}_1$ consists of only 1758. The drawn convex hulls represent 23.1% ($\mathcal{L}_0$) and 12.9% ($\mathcal{L}_1$) of the volume of the cube, respectively.

For $i = 0$, i.e. incorporating the stage costs based on the deviation from $x^\star$ measured along the resulting trajectory, 10 grid points are obtained for which $m > 1$ is necessary in order to ensure the relaxed Lyapunov inequality for optimization horizon $N = 6$, cf. Table 4.2. We point out that, for $m = 1$, three of these points even require an optimization horizon $N = 9$ in order to exhibit the desired performance bound. Hence, allowing for larger control horizons reduces the optimization horizon significantly. Furthermore, the third tabulated point is not stabilized for $N = 5$, $m \in \{1, 2, 3, 4\}$.[12] A further reduction of the optimization horizon is, thus, not possible without incorporating an "exit strategy".

$\ell_0(\cdot,\cdot)$ from (4.19)			$\alpha_{6,m}$			Minimal N:
grid point x_0			$m = 1$	$m = 2$	$m = 3$	$\alpha_{N,1} \geq \overline{\alpha}$
+0.9246	-0.1500	+0.9123	-0.0300	-0.0236	+0.0230	7
+0.9246	-0.1000	+0.9123	-0.0730	-0.0096	+0.0420	9
+0.9246	-0.1000	+0.9623	-0.0819	-0.0440	+0.0103	9
+0.9246	-0.0500	+0.9123	-0.0115	+0.0657	-	7
+0.9246	-0.0500	+0.9623	-0.0807	-0.0034	+0.0455	9
+0.9246	-0.0500	+1.0123	-0.0294	-0.0122	+0.0294	7
+0.9746	-0.1000	+0.9123	-0.0305	-0.0133	+0.0299	8
+0.9746	-0.0500	+0.9123	-0.0355	+0.0335	-	8
+0.9746	-0.0500	+0.9623	-0.0597	-0.0214	+0.0240	8
+1.0246	-0.0500	+0.9123	-0.0410	+0.0018	-	8

Table 4.2: Grid points from $\mathcal{L}_0$ violating $\alpha_{6,1} \geq \overline{\alpha} = 0$. For each of these points $m \in \{1, 2, 3\}$ exists such that $\alpha_{6,m} \geq \overline{\alpha}$ holds. For $m = 1$, the optimization horizon has to be increased to $N = 9$ in order to ensure the desired performance specification.

Similar results are obtained for $\ell_1(\cdot,\cdot)$, cf. Table 4.3. Again, an optimization horizon of $N = 9$ turns out to be the minimal stabilizing horizon for RHC with $m = 1$ in order

[12]Two point not contained in Table 4.2 also require an optimization horizon $N > 5$.

to satisfy the proposed performance specification. Furthermore, we point out that even control horizon $m = 4$ is required in order to ensure the suboptimality bound. The second and sixth point tabulated in Table 4.3 are not stabilizable for $N = 5$.[13]

Note that the generated trajectories may leave the level set. The algorithm applied with $\overline{\alpha} = 0$ ensures a decrease only at the transmission times. However, since the level set is located in the interior of the cube which also exhibits a safety margin away from the boundary of the set of feasible states $\mathbb{X}$ as well as the small sampling time in combination with continuity properties of the considered system a violation of the imposed state constraints seems to be highly unlikely.

$\ell_1(\cdot,\cdot)$ from (4.19)			$\alpha_{6,m}$				Minimal N:
grid point x_0			$m = 1$	$m = 2$	$m = 3$	$m = 4$	$\alpha_{N,1} \geq \overline{\alpha}$
+0.9246	-0.0500	+0.9123	-0.0594	+0.0069	-	-	8
+0.9246	-0.0500	+0.9623	-0.1063	-0.0624	-0.0006	+0.0377	9
+0.9246	+0.0000	+0.9123	-0.0309	+0.1110	-	-	8
+0.9246	+0.0000	+0.9623	-0.1036	+0.0190	-	-	9
+0.9746	-0.0500	+0.9123	-0.0231	-0.0034	+0.0451	-	7
+0.9746	-0.0500	+0.9623	-0.0213	-0.0465	-0.0092	+0.0260	7
+0.9746	+0.0000	+0.9123	-0.0195	+0.1080	-	-	7
+0.9746	+0.0000	+0.9623	-0.0606	+0.0106	-	-	8
+1.0246	+0.0000	+0.9123	-0.0096	+0.1047	-	-	7
+1.0746	+0.0000	+0.9123	-0.0011	+0.1012	-	-	7

Table 4.3: Grid points from $\mathcal{L}_1$ violating the performance specification $\alpha_{6,1} \geq \overline{\alpha} = 0$. The smallest optimization horizon N guaranteeing $\alpha_{N,1} \geq \overline{\alpha}$ for each grid point is $N = 9$.

Concluding, Algorithm 4.24 allows to reduce the optimization horizon significantly, i.e. $N = 6$ instead of $N = 9$ for $\overline{\alpha} = 0$. Similar effects are observable for other performance bounds, e.g. $\overline{\alpha} = 1/3$. Here, applying the proposed algorithm enables us to ensure the desired Lyapunov inequality for $N = 13$ instead of $N = 16$ for classical RHC for $\ell_1(\cdot,\cdot)$ (or $N = 12$ instead of $N = 15$ for $\ell_0(\cdot,\cdot)$). Hence, employing larger control horizons is not only favorable from a theoretical point of view but may also be exploited in practice.

4.4.2 Advanced Algorithm

Although employing $m > 1$ is not needed very often along the closed loop trajectory, it may, nevertheless, be harmful in terms of robustness. Hence, we aim at developing the proposed algorithm further in order to avoid staying in open loop longer than necessary. Here, since $\alpha_{N,1} < \overline{\alpha}$ may occur, step 1 of Algorithm 4.24 seems to be inevitable. But we do not know whether the computed sequence of control values is superior to receding horizon control with $m = 1$. Indeed, also classical RHC may satisfy the performance specification after m steps. Hence, the main idea consists of examining whether the loop can be closed or not without violating the imposed performance specification. Exactly this issue is tackled by the following algorithm.

[13] Two point which are not listed in Table 4.3 also violate our stability criterion for $N = 5$, i.e. $\alpha_{N,m} < 0$ for $m \in \{1, 2, \dots, N-1\}$.

Algorithm 4.28
Let an initial state $x_0 \in \mathbb{X}$, a list $\mathcal{S} = (0)$, an optimization horizon $N \in \mathbb{N}_{\geq 2}$, and a performance specification $\overline{\alpha} \in [0,1)$ be given.
Set $k = 0$. Do

(1) Carry out step (1) from Algorithm 4.24 in order to obtain

- $V_N(x_{\mu_N}(\sigma(k)))$,
- $\mu_N(j, x_{\mu_N}(\sigma(k)))$, $j = 0, 1, \ldots, m_k - 1$ with $m_k \geq 1$

such that

$$\frac{V_N(x_{\mu_N}(\sigma(k))) - V_N(x_{\mu_N}(m_k; x_{\mu_N}(\sigma(k))))}{\sum_{n=0}^{m_k-1} \ell(x_{\mu_N}(n; x_{\mu_N}(\sigma(k))), \mu_N(n, x_{\mu_N}(\sigma(k))))} \geq \overline{\alpha}. \tag{4.20}$$

(2) Set $j = 0$ and define $\hat{u}_N(n) := \mu_N(n, x_{\mu_N}(\sigma(k)))$, $n = 0, 1, \ldots, m_k - 1$. Do

(a) Set $j = j + 1$

(b) Implement $\hat{u}_N(j-1)$ at the plant $\rightsquigarrow \hat{x}(j) := x_{\hat{u}_N}(j; x_{\mu_N}(\sigma(k)))$.

(c) If $j < m_k$: compute $\mu_N(\cdot, \hat{x}(j))$ and $V_N(x_{\mu_N}(m_k - j; \hat{x}(j)))$. Check whether

$$\frac{V_N(x_{\mu_N}(\sigma(k))) - V_N(x_{\mu_N}(m_k - j; \hat{x}(j)))}{\sum_{n=0}^{j-1} \ell(\hat{x}(n), \hat{u}_N(n)) + \sum_{n=j}^{m_k-1} \ell(x_{\mu_N}(n-j; \hat{x}(j)), \mu_N(n-j, \hat{x}(j)))} \geq \overline{\alpha} \tag{4.21}$$

holds. In case it does: exchange the remaining tail of $\hat{u}$, i.e.

$$\hat{u}_N(n) := \begin{cases} \hat{u}_N(n) & n < j \\ \mu_N(n-j, \hat{x}(j)) & n \geq j \end{cases}.$$

while $j < m_k$

(3) Set $\mathcal{S} := (\mathcal{S}, \operatorname{back}(\mathcal{S}) + m_k)$, $k := k + 1$, goto (1)

while stopping criteria not satisfied.

In Algorithm 4.24 we ensured the relaxed Lyapunov inequality – the key element of our approach. However, guaranteeing (4.20) may require the implementation of more than only the first element of the computed sequence of control values $\mu_N(n, x_{\mu_N}(\sigma(k)))$, $n = 0, 1, \ldots, N-1$, i.e. staying in open loop for a longer period of time. Algorithm 4.28 proposes a strategy to close the resulting control loop more often. In this context, we have to distinguish between switching instances, which coincide with the transmission times, cf. Section 1.4, and time instances not contained in the sequence $(\sigma(k))_{k \in \mathbb{N}_0}$. According to Definition 1.24, the existence of such sampling instances implies an element $m_k = \sigma(k+1) - \sigma(k) > 1$. Here, we decouple the update times, i.e. the sampling instants at which the sequence of control values to be implemented is modified, and the transmission times, i.e. sampling instances at which the relaxed Lyapunov inequality has to hold. This leads to the question, which condition allows us to update the sequence of control values more often and, thus, robustifies the control loop. To this end, Algorithm 4.28 employs (4.21) which ensures that the sequence assembled from the previously used and the newly computed one also satisfies the relaxed Lyapunov inequality. To be more precise, the condition checks whether a sequence for which we have ensured that the relaxed Lyapunov inequality holds at the next transmission time may be updated at a sampling instant preceding that time instant. The candidate is concatenated by the sequence of which at least one control value was implemented at the plant and the sequence resulting from applying RHC at the current time instant. If the concatenated sequence also satisfies the relaxed Lyapunov inequality at the forthcoming transmission instant, the old sequence is replaced by the newly computed one in order to improve robustness by closing the control loop once more.

Indeed, for trajectories emanating from the points violating the desired Lyapunov inequality for optimization horizon $N = 6$ for the synchronous generator this condition is fulfilled each time. Hence, Algorithm 4.28 indeed performs classical RHC but ensures — a priori — asymptotic stability. Here, verifying the relaxed Lyapunov inequality for larger control horizons enabled us to check our stability criterion in advance. Although (4.21) holds for these trajectories there is no guarantee that it always does, i.e. only being able to stick, if necessary, to a computed control sequence for more than one sampling instant yields the desired stability guarantee. Hence, Algorithm 4.28 robustifies the applied RHC strategy. Furthermore, it smoothes the resulting trajectories, cf. Figures 4.11 and 4.12.

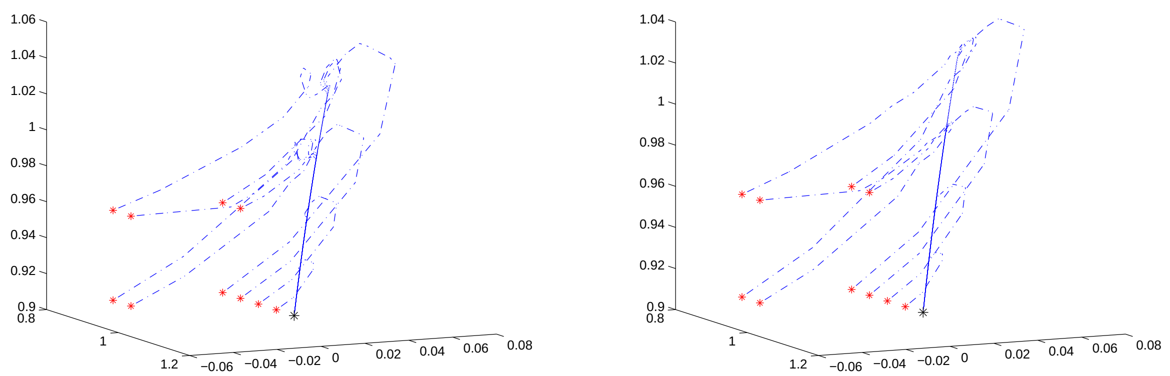


Figure 4.11: Trajectories emanating from the crucial points from Table 4.3 computed with the basic (on the left) and the advanced algorithm (on the right) based on the stage costs $\ell_1(\cdot, \cdot)$. The more elaborated Algorithm 4.28 updates each time and, thus, smoothes the corresponding trajectories, cf. also Figure 4.12.

In order to conclude this subsection we comment on some effects observed for the considered example. To this end, we focus on the interval between $\sigma(0)$ and $\sigma(1)$.

- For stage costs $\ell_1(\cdot, \cdot)$, the performance estimate is improved by Algorithm 4.28 for each update, i.e. the left hand side of (4.21) is larger than the one from (4.20). While the corresponding change in the optimal value function is non-monotone.

- For stage costs $\ell_0(\cdot, \cdot)$, the last update, e.g. the second for $m = 3$, deteriorates the performance estimate whereas the preceding one contributes positively. Summing up these effects yields increased suboptimality bounds for $m = 3$ and decreased estimates for $m = 2$. The optimal value function evaluated at the next transmission time increases by updating.

Hence, the main benefit of applying the advanced version of the proposed algorithm is the concomitant robustification. The overall performance of the receding horizon closed loop is investigated in [96]. Next, the control values actually implemented at the plant during runtime of Algorithm 4.28 are considered. The next control value to be applied after updating the sequence of control values decreases in norm for each point from Tables 4.2 and 4.3, cf. Table 4.4 for a typical course. For another update criterion, which is preferable from a computational point of view, we refer to [97].

	$\hat{u}(0)$	$\hat{u}(1)$	$\hat{u}(2)$	$\hat{u}(3)$	$\hat{u}(4)$	$\hat{u}(5)$	$\hat{u}(6)$	$\hat{u}(7)$	$\hat{u}(8)$
$\hat{x}(0)$	+0.755	+0.596	+0.110	-0.318	-0.775	-0.000	—	—	—
$\hat{x}(1)$	—	+0.458	+0.257	-0.014	-0.336	-0.762	-0.000	—	—
$\hat{x}(2)$	—	—	+0.121	+0.131	-0.038	-0.330	-0.747	-0.000	—
$\hat{x}(3)$	—	—	—	-0.004	+0.104	-0.036	-0.322	-0.735	-0.000

Table 4.4: Computed sequences of control values at state $\hat{x}(j)$, $j = 0, 1, 2, 3$, for the second point from Table 4.3. An update is carried out at each sampling instant. The applied control values are written in red.

Our focus was put on the control horizon. In particular, we pointed out that the theoretically deduced results with respect to symmetry and monotonicity properties may be exploited in such a way that a positive impact not only on networked control systems but also on RHC in general is attainable. In particular, the proposed algorithms represent a methodology to reduce the optimization horizon N which predominantly determines the computational effort associated with solving the optimal control problem in each receding horizon step. Furthermore, the more elaborated Algorithm 4.28 ensures that the robustness properties of RHC are preserved. We emphasize once more that both algorithms verify the desired performance estimates a priori.

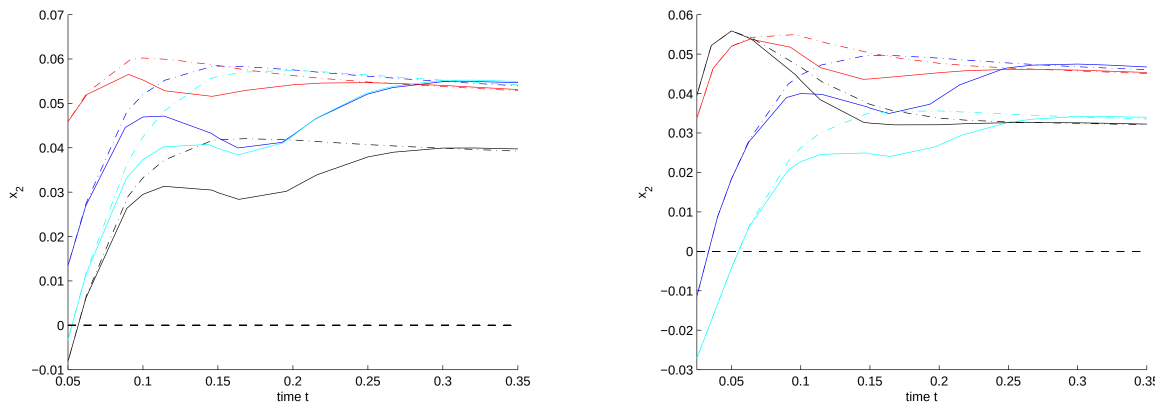


Figure 4.12: The x_2 component of the trajectories which are generated by Algorithm 4.24 (solid line) and Algorithm 4.28 (dashed-dotted line) and emanate from the second, forth, sixth, and seventh point from Table 4.2 ($\ell_0(\cdot,\cdot)$, on the left) and Table 4.3 ($\ell_1(\cdot,\cdot)$, on the right), are drawn. By construction the trajectories cannot deviate up to the first sampling instant $T = 0.05$.

Chapter 5

Sampled-Data Systems and Growth Condition

In Chapter 3 we deduced our key results in order to ensure asymptotic stability and, in addition, gave estimates on the performance of the receding horizon closed loop provided Assumption 3.2 holds. Furthermore, in Section 1.3, sampled-data systems were introduced in order to incorporate systems originally defined continuously in time in our discrete time setting. Typically, sampled-data systems are induced by an ordinary or partial differential equation and, thus, the corresponding control input, which is actually implemented at the plant, has to be specified on the entire sampling interval. Allowing for arbitrary metric spaces in the definition of the admissible set of control values enables us to deal with this fact. However, our standing Assumption 3.2 imposes bounds only at the sampling instances which fits well to the discrete time setting but may not fully reflect the stability behavior of a sampled-data system, cf. the following example of a reaction diffusion equation taken from [5].

Example 5.1 (Semi-linear reaction diffusion equation)
In this example we change the notation to be consistent with the usual PDE notation: $x \in \Omega \subset \mathbb{R}^d$ *is the independent space variable while the unknown function* $y(\cdot,t) : \Omega \to \mathbb{R}$ *denotes the state. Let the open and connected set* Ω *be a Lipschitz-domain in order to ensure well-posedness of the following semi-linear parabolic partial differential equation (PDE), cf. [119, Subsection 2.2.2]. We consider a reaction diffusion equation*

$$\begin{aligned} y_t(x,t) &= \Delta y(x,t) - f(y(x,t)) + u(x,t) && \text{on} \quad \Omega \times (0,\infty) && (5.1)\\ y(x,t) &= 0 && \text{on} \quad \partial\Omega \times (0,\infty) && (5.2) \end{aligned}$$

with homogeneous Dirichlet boundary conditions, initial data $y(x,0) = y_0$*, distributed control* $u(\cdot,t) : \Omega \to \mathbb{R}$*, and continuously differentiable non-linearity* $f : \mathbb{R} \to \mathbb{R}$*. In addition, let* $f(0) = 0$ *in order to ensure that the origin is an equilibrium. For existence and regularity results we refer to [16].*

Before we continue our investigation of Example 5.1, we present the following theorem concerning the local stability behavior of the uncontrolled version of this semi-linear parabolic equation which is proven in [16].

Theorem 5.2
For each $\gamma \in (0, \lambda_1 + f'(0))$*, a constant* $R = R(\gamma)$ *exists such that for all* $y_0 \in \mathcal{C}_0(\Omega)$ *with* $\|y_0\| \leq R$ *the solution* $y(\cdot,t)$ *of* (5.1), (5.2) *with* $u(\cdot,t) \equiv 0$ *for all* $t \in [0,\infty)$ *satisfies*

$$\|y(\cdot,t)\| \leq M\|y_0\|e^{-\gamma t} \qquad \forall\, t \geq 0. \tag{5.3}$$

Here, $\lambda_1 = \lambda_1(\Omega)$ denotes the smallest eigenvalue of the differential operator $-\Delta$ in $H_0^1(\Omega)$.

The constant M from (5.3) is independent of γ and the given initial state y_0. Indeed, it depends only on the used norm, e.g. $M = 1$ for $\|\cdot\|_{L^2(\Omega)}$. Furthermore, the origin is unstable for $\lambda_1 < -f'(0)$, cf. [16] for details. In the ensuing example, we show that Theorem 5.2 allows us to establish a continuous time counterpart to our standing Assumption 3.2. We point out that the following line of arguments is crucially based on the fact that we do not require optimality of the involved control law.

Example 5.3 (Chaffee-Infante equation)
An important representative of the class of reaction diffusion equations considered in Example 5.1 is the one dimensional Chaffee-Infante *equation*

$$y_t(x,t) = y_{xx}(x,t) + \mu(y(x,t) - y(x,t)^3) + u(x,t), \tag{5.4}$$

i.e. $f(y) = -\mu(y - y^3)$. For domain $\Omega = (0,1)$, parameter $\mu = 11$ and the initial data $y(x,0) = 0.2\sin(\pi x)$, the origin is unstable because $\lambda_1 = \pi^2 < 11 = -f'(0)$ holds, cf. Figure 5.1.

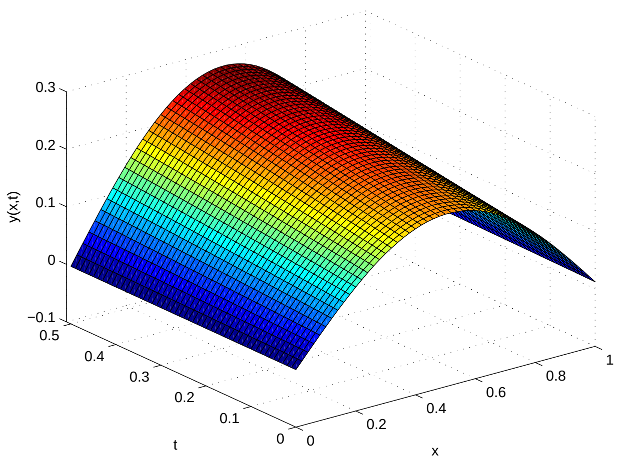


Figure 5.1: Solution of the uncontrolled Chaffee-Infante equation (5.4) with $\mu = 11$ and initial condition $y(x,0) = 0.2\ \sin(\pi x)$.

Our goal is to stabilize the system governed by Equations (5.1) *and* (5.2) *to the origin by RHC. In addition, we want to give explicit estimates for minimal stabilizing horizons. To this end, this evolution equation is interpreted as a discrete time system, cf. Section 1.3, and the standard L^2-cost functional*

$$\ell(y(n), u(n)) = \|y(\cdot, nT)\|^2_{L^2(\Omega)} + \lambda\|u(\cdot, nT)\|^2_{L^2(\Omega)}$$

is used. Existence results for the solution of this optimal control problem can be found in [89]. At this moment, Theorem 5.2 is applied in order to establish the (exponential) controllability condition given in Assumption 3.2. Then, Theorems 3.18 and 3.12 are used in order to conclude stability and to guarantee performance bounds. However, we make a small detour which turns out to be fruitful in order to deduce tighter suboptimality estimates for this example. Doing so motivates the further analysis in this chapter. Nevertheless, the main ideas remain completely the same as in [5].

In order to deduce exponential controllability in terms of the running costs, the feedback control $u(x,t) := -Ky(x,t)$ *with a real constant* K *is chosen. Then, Theorem 5.1 is applied with* $F(y) := f(y) + Ky$ *in order to obtain*

$$\ell^\star(y(\cdot,t)) = \|y(\cdot,t)\|^2_{L^2(\Omega)} \leq M^2 e^{-2\gamma t}\ell^\star(y_0(\cdot)) \tag{5.5}$$

with $\gamma = \lambda_1 + f'(0) + K$*. Furthermore, we get*

$$\begin{aligned} \ell(y(\cdot,t),u(\cdot,t)) &= \|y(\cdot,t)\|^2_{L^2(\Omega)} + \lambda\|u(\cdot,t)\|^2_{L^2(\Omega)} \\ &= \|y(\cdot,t)\|^2_{L^2(\Omega)} + \lambda K^2\|y(\cdot,t)\|^2_{L^2(\Omega)} = (1+\lambda K^2)\ell^\star(y(\cdot,t)). \end{aligned}$$

Combining this equation with (5.5) *yields exponential controllability in terms of the stage costs, i.e.*

$$\ell(y(\cdot,t),u(\cdot,t)) = (1+\lambda K^2)\,\ell^\star(y(\cdot,t)) \leq Ce^{-\mu t}\ell^\star(y_0(\cdot)) \tag{5.6}$$

with overshoot $C := (1+\lambda K^2)M^2$ *and decay rate* $\mu = 2\gamma$*. Indeed,* (3.3) *is ensured for each* $t \in \mathbb{R}_0^+$ *and not only at the sampling instants, i.e.* (5.6) *represents a continuous time counterpart to* (3.3)*. Of course, we may return to the discrete time version, i.e. interpret* $y(\cdot,nT)$ *as the* n^{th} *state* $y(n)$ *for sampling period* T*. Then, we conclude*

$$\ell(y(n),u(n)) \leq C\sigma^n\ell^\star(y(0)) \qquad \forall\, n \in \mathbb{N}$$

with overshoot $C := (1+\lambda K^2)M^2$*, decay rate* $\sigma = e^{-2\gamma T}$*, and* $\gamma = \lambda_1 + f'(0) + K$*. Note that only the decay rate* σ *was adapted. Here, we like to point out the ambiguous role of the constant* K*. Picking larger values for* K *leads to faster decay but also to an increased overshoot. Hence, there is a trade-off between advantages and disadvantages. At this point, having an explicit formula for the optimal value of Problem 3.8 turns out to be extremely beneficial because it allows to easily solve the following optimization problem in order to choose* K *appropriately.*

Problem 5.4
Minimize the optimization horizon N *subject to* $\mathbb{N}_{\geq 2}$*,* $K \in \mathbb{R}$ *and*

$$\begin{aligned} &\gamma = \lambda_1 + f'(0) + K = \pi^2 - 11 + K \geq 0, \\ &\alpha^1_{N,1} = 1 - \frac{(\gamma_N - 1)\prod_{i=2}^N(\gamma_i - 1)}{\prod_{i=2}^N \gamma_i - \prod_{i=2}^N(\gamma_i - 1)} \geq 0 \qquad \text{with} \\ &\gamma_i = C\sum_{n=0}^{i-1}\sigma^n = (1+\lambda K^2)M^2\sum_{n=0}^{i-1}\left(e^{-2\gamma T}\right)^n,\ i \in \{2,3,\ldots,N\}. \end{aligned}$$

This is a mixed integer problem. The maximal $\alpha^1_{N,1}$ *with respect to the parameter* K *is computed for given optimization horizon* N*, cf. Figure 5.2. Here, we use the regularization parameter* $\lambda = 0.01$ *and the sampling period* $T = 0.01$*. Note that the choice of* K *influences the resulting estimates only on the overshoot* C *and the decay rate* σ *in Assumption 3.2, but does not appear in the actual receding horizon algorithm. Taking a closer look at the corresponding optimal values for* K *provides interesting information on its own. For example, the optimization horizon has to be sufficiently large in order to ensure that fast decay, which is implied by choosing* K *large, is preferable to small overshoot bounds. Further results linked with the choice* K *in this example are given in Remark 5.5.*

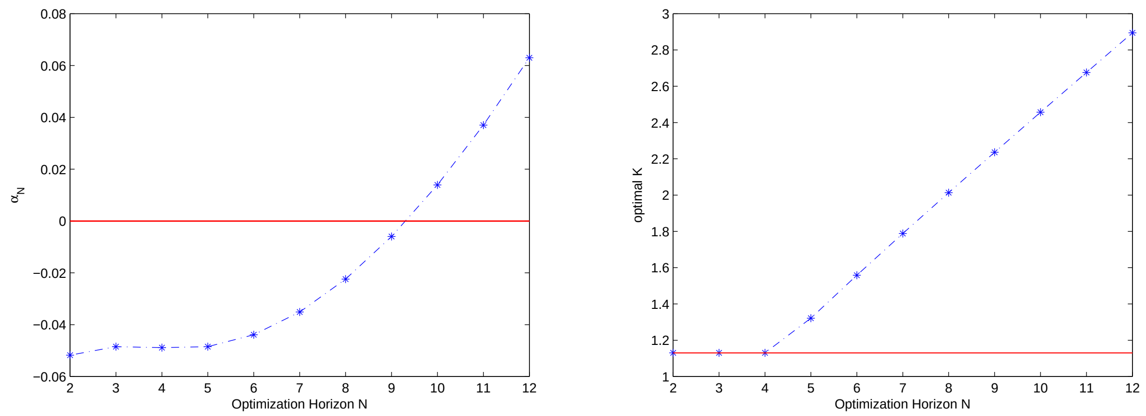


Figure 5.2: On the left we depict the maximal achievable $\alpha^1_{N,1}$ for a given optimization horizon N. According to this, the minimal stabilizing horizon, i.e. the smallest N ensuring asymptotic stability, is obtained for $N = 10$. On the right, the corresponding optimal choices for the parameter K are illustrated.

In this example, Theorem 3.18 guarantees a (strictly) positive optimal value of Problem 3.8 for optimization horizon $N = 10$, cf. Figure 5.2. In addition, since $\|y(\cdot,t)\|^2_{L^2(\Omega)} = \ell^\star(y(\cdot,t))$ holds, Assumption 1.7 is satisfied for $\alpha_1(r) = \alpha_2(r) = r^2$ for the discrete as well as for the continuous time setting. Indeed, using Theorem 5.2, even enables us to easily deduce the estimate $c\alpha_2(\|y_0\|_2) = c\ell^\star(y_0) \geq V_N(y_0)$ for a suitably chosen constant $c \in \mathbb{R}_{>0}$, cf. [5, Proof of Theorem 3]. Hence, Theorem 3.12 is applicable and, thus, ensures asymptotic stability for the RHC feedback with optimization horizon $N = 10$.

Before proceeding with a comparison of theoretical results and numerical experiments for the considered example, we state the following remark which is based on Example 5.3 but makes an important contribution to the sensitivity analysis carried out in Chapter 4.

Remark 5.5
We look at the optimal values of Problem 3.8 obtained in Example 5.3 for $N = 2, 3, 4$ in detail, cf. Table 5.1.

Horizon N	Value $\alpha^1_{N,1}$	Parameter K
2	-0.0518	1.1304
3	-0.0485	1.1304
4	-0.0489	1.1304

Table 5.1: Optimal values $\alpha^1_{N,1}$ of Problem 3.8 resulting from Example 5.3 for $N = 2, 3, 4$.

Since the optimal parameter K does not change on this interval, the overshoot C and the decay rate σ incorporated in Assumption 3.2 and, consequently, in the calculation of $\alpha^1_{N,1}$ do not change for $N = 2, 3, 4$. However, counter-intuitively the corresponding values are not monotonically increasing. Hence, we conclude that the optimal values of Problem 3.8 and, thus, Problem 3.10 are not monotonically increasing with respect to the optimization horizon N which fills a gap in our sensitivity analysis.

We continue with a comparison of the obtained estimate with numerical computations.

Example 5.6
In Figure 5.3 one observes that receding horizon control with optimization horizon $N = 4$ and spatial discretization parameter $\Delta x = 0.01$ stabilizes the system. For larger optimization horizons, e.g. $N = 8$, the stabilization is achieved much faster, i.e. the receding horizon algorithm exhibits a superior performance. Numerically, $N = 2$ turns out to be the "minimal stabilizing horizon".[1] *Here, we emphasize that the numerical experiment is based on a single initial condition. In contrast to that, our theoretical estimates hold for a set of initial conditions. This explains why the deduced estimates for the minimal stabilizing horizon seem to be, in general, conservative for a concrete example. The second reason may be the estimate from Theorem 5.2 used in* (5.5) *which is, in general, conservative. Choosing a more elaborate control function instead of our simple feedback control may tighten the deduced results.*

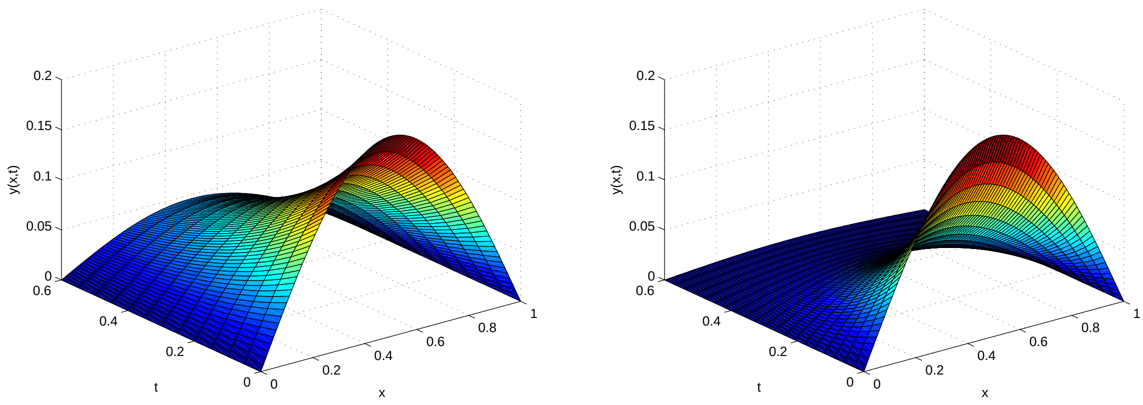


Figure 5.3: Receding horizon trajectories for optimization horizon $N = 4$ (left) and $N = 8$ (right).

Remark 5.7
Since, for optimization horizon $N = 7$ and control horizon $m = 3$, Formula (3.21) *yields* $\alpha = 0.0020 > 0$ *($K = 2.3223$), employing Algorithm 4.24 reduces the optimization horizon significantly and, thus, tightens the deduced estimate. Indeed, the numerical verification of the relaxed Lyapunov inequality never fails for this example. Hence, 4.24 applies "classical" receding horizon control.*[2]

Remark 5.8
In [6, 39] the reaction diffusion equation from Example 5.1 is extended to a reaction advection diffusion equation. In these references, it was shown how the developed theory from Chapter 3 can be applied in order to derive design guidelines for the running costs which allow for reducing the optimization horizon in the control strategy. In contrast to the heuristic arguments in these references, we deduced rigorous estimates on the required horizon length.

[1] Note that the numerical computations deviate from [5]. In particular, the required optimization horizon in order to stabilize the system by RHC is determined more carefully and, thus, corrected.

[2] As a consequence, the numerical results of Algorithm 4.24 and 4.28 coincide for this particular example.

Besides being a nice application of our methodology deduced in Chapter 3, the previous examples exhibit a continuous time counterpart to Assumption 3.2 satisfied with $\beta(\cdot,\cdot) \in \mathcal{KL}$ of type (1.11).

Assumption 5.9
Let a function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be given by $\beta(r,t) = r \cdot Ce^{-\mu t}$ with overshoot $C \geq 1$ and decay rate $\mu > 0$ and consider a continuous time system governed by $\dot{x}(t) = f(x(t), \tilde{u}(t))$ for all $t \in [0,\infty)$. Suppose that, for each $x_0 \in \mathbb{X}$, an admissible control function $\tilde{u}_{x_0} : \mathbb{R}_0^+ \to \mathbb{U}$ exists such that

$$\ell(\Phi(t;x_0,\tilde{u}),\tilde{u}(t)) \leq Ce^{-\mu t}\ell^\star(x_0) \qquad \text{holds for all } t \geq 0, \tag{5.7}$$

i.e. exponential controllability in terms of the stage costs holds.[3]

Clearly, Assumption 5.9 implies Assumption 3.2 with suitably adapted discrete time decay rate $\sigma = e^{-\mu T}$, cf. Theorem 5.12 below. Hence, for a chosen (discretization) parameter T, Inequality (5.7) is reflected at time instants $0, T, 2T, \ldots$. Consequently, for a fraction T/k, $k \in \mathbb{N}_{\geq 2}$, of the parameter T, Inequality (5.7) is taken into account at k times as many time instants. What effects does such a refinement of the (discretization) parameter T have on our performance bounds? Intuitively, we expect that the behavior of a continuous time system is characterized more precisely. This question is tackled in the first section of this chapter. And does repeating this refinement iteratively improve our suboptimality estimates? An alternative approach based on a purely continuous time setting is given in [104]. Here, we show that the results coincide in the limit, cf. Section 5.2. Furthermore, a problem connected with applying Theorem 3.18 for arbitrarily fast sampling is observed.

In the ensuing Section 5.3, a growth condition is introduced which does not only tighten our suboptimality estimates but also solves the problem observed for very fast sampling by reflecting, e.g. continuity properties of an underlying continuous time system. In order to incorporate this additional assumption in our setting, the concept of equivalent sequences is used. These equivalent sequences turn out to be the key ingredient in order to generalize the methodology of Chapter 3 to a weaker controllability assumption in Section 5.4 which is also used in [120]. In order to conclude this thesis, our results are compared to those from [90, 120].

5.1 Discretization and Sampled-Data Systems

The example of the reaction diffusion equation considered in Examples 5.1, 5.3, and 5.6 motivated the extension of Assumption 3.2 to its continuous time counterpart Assumption 5.9. Since the continuous time version implies the discrete one for an arbitrary sampling period T (and suitably adapted spaces $\mathbb{X}$ and $\mathbb{U}$ of feasible states and controls, respectively) with adjusted decay rates $\sigma = e^{-\mu T}$, implications of using this assumption at additional time instants on our suboptimality estimate from Theorem 3.18 are investigated. To this end, the following definition is needed.

Definition 5.10
Suppose that Assumption 5.9 is satisfied with overshoot $C \geq 1$ and decay rate $\mu > 0$.

[3] A control function $\tilde{u}_{x_0} : \mathbb{R}_0^+ \to \mathbb{U}$ is admissible if and only if the corresponding solution $\Phi(\cdot;x_0,\tilde{u})$ satisfies $\Phi(t;x_0,\tilde{u}) \in \mathbb{X}$ for each $t \geq 0$.

Let a sampling period $T > 0$ and a discrete time optimization horizon $N \in \mathbb{N}_{\geq 2}$ be given which determines the corresponding continuous time optimization horizon of length $[0, NT)$. Furthermore, let a sequence $(k_j)_{j \in \mathbb{N}_0} \subseteq \mathbb{N}$ be given. We distinguish between the following two cases.

- *A constant continuous time control horizon. Let $m \in \{1, 2, \ldots, N-1\}$ be given. Then the elements of the discretization sequence $(\mathcal{D}_j)_{j \in \mathbb{N}_0}$ are defined by the quadruplets*
$$(T_j, N_j, \sigma_j, m_j) = (k_j^{-1} T, k_j N, e^{-\mu k_j^{-1} T}, k_j m). \tag{5.8}$$
- *A constant discrete time control horizon $m_k = m = 1$. Then, the elements of the discretization sequence $(\mathcal{D}_j)_{j \in \mathbb{N}_0}$ are defined by the triples*
$$(T_j, N_j, \sigma_j) = (k_j^{-1} T, k_j N, e^{-\mu k_j^{-1} T}). \tag{5.9}$$

T_j denotes the sampling period for the j-th discretization. While N_j and σ_j represent the discrete time optimization horizon and the corresponding decay rate, respectively, which have to be suitably adapted in order to keep their continuous time counterparts unchanged, e.g. $NT = N_j T_j$ holds for each $j \in \mathbb{N}_0$ and, thus, the continuous time optimization horizon is constant. The control horizon is either adapted as well, cf. (5.8), *or kept constant which implies that the continuous time control horizon changes depending on the discretization parameter k_j. If $k_{j+1} = n k_j$ holds with $n \in \mathbb{N}_{\geq 2}$, the $(j+1)$-th discretization is called a refinement of the j-th discretization and, thus, a more accurate or finer discretization.*

We point out that the continuous time optimization horizon remains constant independently of whether (5.8) or (5.9) is chosen in order to deal with the control horizon. In order to illustrate Definition 5.10, the following example is given which seems to be the prototype of a discretization and, thus, will be investigated in detail in this section.

Example 5.11
Let $T > 0$, $N \in \mathbb{N}_{\geq 2}$, $C \geq 1$, and $\mu < 0$ be given. Then, the sequence $(k_j)_{j \in \mathbb{N}_0} \subseteq \mathbb{N}$ of discretization parameters defined by $k_{j+1} = 2k_j$ and $k_0 = 1$ is chosen. Hence, the sampling period T is halved and the discrete time optimization horizon N is doubled in each refinement step. This leads to the sequence $(\mathcal{D}_j)_{j \in \mathbb{N}_0} = (T_j, N_j, \sigma_j)_{j \in \mathbb{N}_0} = (2^{-j} T, 2^j N, \sigma^{2^{-j}})_{j \in \mathbb{N}_0}$. Figure 5.4 illustrates the resulting tighter bounds on the stability behavior of the underlying system.

The construction described in Example 5.11 iteratively leads to more accurate discretizations and, thus, reflects Assumption 5.9 in the discrete time setting via Assumption 3.2 better after each refinement step. We are interested in the resulting effects on our suboptimality estimates. The investigation is subdivided into two parts:

- on the one hand, also the control horizon is adjusted which corresponds to $m_j := 2^j m$ in Example 5.11, cf. (5.8). This yields — as intuitively expected — improved performance bounds.
- on the other hand, the control horizon is fixed, cf. (5.9). In practical applications sampled-data systems often use piecewise constant control functions. Hence, in general, sufficiently fast sampling is required in order to preserve stability properties for the sampled-data system, cf. [91]. Here, possible pitfalls in this setting are pointed out.

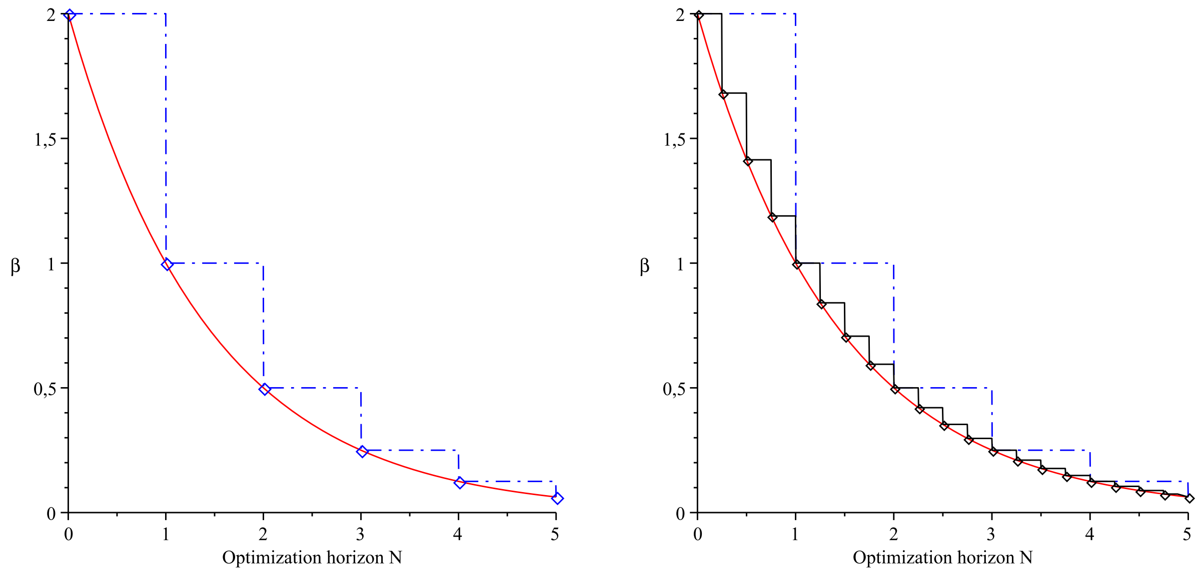


Figure 5.4: The continuous curve on the left depicts the bounds induced by Assumption 5.9 with overshoot $C = 2$, decay rate $\mu = \ln(2)$, and sampling period $T = 1$. The blue line indicates the implied bounds taken into account by Assumption 3.2 with $\sigma = e^{-\mu T} = 1/2$ ($\diamond$). On the right, we added the respective bounds for a more accurate discretization corresponding to $(T_2, N_2, \sigma_2) = (0.25, 20, \sqrt[4]{\frac{1}{2}})$ (black line).

In order to be able to apply Formula (3.21) for a given discretization parameter $k \in \mathbb{N}$, the definition of γ_i from Theorem 3.18 is combined with the setting given in this section.

Theorem 5.12
Let Assumption 5.9 be satisfied with decay rate $\mu > 0$ and overshoot $C \geq 1$. In addition, let an optimization horizon $N \in \mathbb{N}_{\geq 2}$, a control horizon $m \in \{1, 2, \ldots, N-1\}$, and a sampling period $T > 0$ be given. Furthermore, we define $\sigma := e^{-\mu T} \in (0,1)$, $\beta_k(r,n) := r \cdot C \sqrt[k]{\sigma}^{\,n}$, and

$$\gamma_{i,k} := \sum_{n=0}^{i-1} r^{-1} \cdot \beta_k(r,n) = C \sum_{n=0}^{i-1} \left(\sigma^{1/k}\right)^n = \frac{C(1-\sigma^{i/k})}{1-\sigma^{1/k}}. \tag{5.10}$$

Then, for each $k \in \mathbb{N}$, Assumption 3.2 holds for $\beta_k(r,n)$, i.e. a $\mathcal{KL}$-function of type (1.11) with overshoot C and decay rate $\sqrt[k]{\sigma}$. Furthermore, the optimal value $\alpha_{kN,km}(k)$ of the corresponding optimization problem $(\mathcal{P}_k)$, i.e. Problem 3.8 based on $\beta_k(\cdot,\cdot)$ with optimization horizon kN and control horizon km, is given by Formula (3.21) based on $\gamma_{i,k}$ instead of γ_i, i.e.

$$\alpha_{kN,km}(k) = 1 - \frac{\prod_{i=km+1}^{kN}(\gamma_{i,k}-1)}{\prod_{i=km+1}^{kN}\gamma_{i,k} - \prod_{i=km+1}^{kN}(\gamma_{i,k}-1)} \cdot \frac{\prod_{i=k(N-m)+1}^{kN}(\gamma_{i,k}-1)}{\prod_{i=k(N-m)+1}^{kN}\gamma_{i,k} - \prod_{i=k(N-m)+1}^{kN}(\gamma_{i,k}-1)}. \tag{5.11}$$

Proof: For $k = 1$ verifying Assumption 3.2 and showing (5.11) follows directly from Assumption 5.9. For $k \in \mathbb{N}_{\geq 2}$, we adapt the sampling period, i.e. $T_k := T/k$. Hence, the corresponding decay rate $e^{-\mu T/k}$ equals $\sqrt[k]{\sigma}$ and, thus, the assertion is ensured by taking the introduced notation into account.

□

Note that (5.10) has to be interpreted in the sense that $\gamma_{i,k}$ does not depend on the first argument of $\beta_k(r,n)$. Indeed, $\gamma_{i,k}$ may have been defined directly by the expression given by the right hand side of (5.10) but using the involved $\mathcal{KL}$-function emphasizes its original background, cf. Remark 3.15. Furthermore, the additional index k in (5.10) and the additional argument k in (5.11) clearly indicate the involved discretization parameter.

Theorem 5.12 allows us to begin our study of more accurate discretizations. To this end, the continuous time optimization and control horizon are fixed.

Proposition 5.13
Let the assumptions of Theorem 5.12 be satisfied. Furthermore, we define σ, $\gamma_{i,k}$, and $(\mathcal{P}_k)$ according to Theorem 5.12. Then, for the sequence $(k_j)_{j\in\mathbb{N}_0}$ with $k_j := 2^j$, the optimal values $\alpha_{k_jN,k_jm}(k_j)$ of $(\mathcal{P}_{k_j})$ satisfy

$$\alpha_{N,m} = \alpha_{k_0N,k_0m}(k_0) \qquad \text{and} \qquad \alpha_{k_jN,k_jm}(k_j) \leq \alpha_{k_{j+1}m,k_{j+1}N}(k_{j+1}) \leq 1-\sigma^N \qquad \forall\, j\in\mathbb{N}_0, \tag{5.12}$$

i.e. using an iterative refinement as specified by $(k_j)_{j\in\mathbb{N}_0}$ of the control and the optimization horizon ensures monotonically increasing suboptimality estimates.

Proof: The proof is subdivided into two parts. Firstly, we show the monotonicity of the sequence $(\alpha_{k_jN,k_jm}(k_j))_{j\in\mathbb{N}_0}$. In the second portion of the proof we deduce the upper bound which is independent of the index j.

Let k be an element of $(k_j)_{j\in\mathbb{N}_0}$. Then, using the representation given by (5.11) yields

$$\alpha_{kN,km}(k) = 1 - \left[\prod_{i=km+1}^{kN} \frac{\gamma_{i,k}}{\gamma_{i,k}-1} - 1\right]^{-1} \left[\prod_{i=k(N-m)+1}^{kN} \frac{\gamma_{i,k}}{\gamma_{i,k}-1} - 1\right]^{-1}.$$

Hence, taking $k_{j+1} = 2k_j$ into account, it is sufficient to establish

$$\prod_{i=k\varrho+1}^{kN} \frac{\gamma_{i,k}}{\gamma_{i,k}-1} \leq \prod_{i=k\varrho+1}^{kN} \left(\frac{\gamma_{2i,2k}}{\gamma_{2i,2k}-1}\cdot\frac{\gamma_{2i-1,2k}}{\gamma_{2i-1,2k}-1}\right) = \prod_{i=2k\varrho+1}^{2kN} \frac{\gamma_{i,2k}}{\gamma_{2i,k}-1}$$

for $\varrho\in\{m, N-m\}$. Since the products of either side of the inequality sign consist of the same number of factors, showing the desired inequality componentwise, i.e.

$$\gamma_{i,k}(\gamma_{2i,2k}-1)(\gamma_{2i-1,2k}-1) \leq \gamma_{2i,2k}\,\gamma_{2i-1,2k}(\gamma_{i,k}-1)$$

or, equivalently, $\gamma_{2i,2k}\,\gamma_{2i-1,2k} \leq \gamma_{i,k}(\gamma_{2i,2k}+\gamma_{2i-1,2k}-1)$, suffices. Note that the concrete value of ϱ does not play a role. In order to verify this inequality, we establish $\gamma_{2i,2k}\,\gamma_{2i-1,2k} \leq \gamma_{i,k}(\gamma_{2i,2k}+\gamma_{2i-1,2k}-C)$ reduced by C^2, i.e. bearing (5.10) in mind

$$\frac{1-\sigma^{\frac{i}{k}}}{1-\sigma^{\frac{1}{2k}}}\cdot\frac{1-\sigma^{\frac{2i-1}{2k}}}{1-\sigma^{\frac{1}{2k}}} \leq \frac{1-\sigma^{\frac{i}{k}}}{1-\sigma^{\frac{i}{k}}}\left(\frac{1-\sigma^{\frac{i}{k}}+1-\sigma^{\frac{2i-1}{2k}}-1+\sigma^{\frac{1}{2k}}}{1-\sigma^{\frac{1}{2k}}}\right)$$

which is, in turn, equivalent to

$$\left[(1-\sigma^{\frac{1}{k}})-(1-\sigma^{\frac{1}{2k}})\right](1-\sigma^{\frac{2i-1}{2k}}) \leq (1-\sigma^{\frac{1}{2k}})(\sigma^{\frac{1}{2k}}-\sigma^{\frac{i}{k}}).$$

Since the left and the right hand side of this inequality are equal to $\sigma^{\frac{1}{2k}}(1-\sigma^{\frac{1}{2k}})(1-\sigma^{\frac{2i-1}{2k}})$, this turns out to be an equality. Indeed, strict growth is shown with respect to i for $C > 1$. For $C = 1$, the value is constant.

It remains to establish the upper bound stated in (5.12). To this end, we require the estimate

$$\prod_{i=k\varrho+1}^{kN} \frac{\gamma_{i,k}}{\gamma_{i,k}-1} \overset{C\geq 1}{\leq} \prod_{i=k\varrho+1}^{kN} \frac{\gamma_{i,k}}{\gamma_{i,k}-C} \overset{(5.10)}{=} \prod_{i=k\varrho+1}^{kN} \frac{1-\sigma^{\frac{i}{k}}}{(1-\sigma^{\frac{i-1}{k}})\sigma^{\frac{1}{k}}} = \frac{1-\sigma^N}{(1-\sigma^\varrho)\sigma^{N-\varrho}}.$$

which is independent of the chosen k. Then, combining this bound with the above representation of the optimal value $\alpha_{kN,km}(k)$ provides

$$\begin{aligned} \alpha^k_{2^kN,2^km} &\leq 1 - \left(\frac{1-\sigma^N}{(1-\sigma^m)\sigma^{N-m}} - 1 \right)^{-1} \left(\frac{1-\sigma^N}{(1-\sigma^{N-m})\sigma^m} - 1 \right)^{-1} \\ &= 1 - \left(\frac{(1-\sigma^m)\sigma^{N-m}}{1-\sigma^{N-m}} \right) \left(\frac{(1-\sigma^{N-m})\sigma^m}{1-\sigma^m} \right) = 1 - \sigma^N, \end{aligned}$$

i.e. the desired upper bound which is tight for $C = 1$.

□

In Proposition 5.13 we adapted both the control and the optimization horizon. Hence, refining the discretization and, thus, increasing the discrete time optimization horizon in order to keep the continuous one constant also implied that the discrete time control horizon grows, which is manageable because of our multistep feedback approach introduced in Section 1.4. Indeed, Proposition 5.13 ensures enhanced performance estimates.

Example 5.14
We consider the reaction diffusion equation from Examples 5.1, 5.3, and 5.6 and investigate effects of an iterative refinement. To this end, the sequence $(2^j)_{j\in N_0}$ of discretization parameters is employed, cf. Example 5.11 and Figure 5.4. As shown in Figure 5.5, the first refinement step allows to decrease the minimal stabilizing horizon by one for $m = 1$, i.e. $\alpha_{N,1} > 0$ holds for $N = 9$ instead of $N = 10$, cf. Figure 5.2. Carrying out a second refinement step yields $\alpha_{N,1} > 0$ for $N = 8$. A further reduction is not possible, cf. Section 5.2 below. The improvement associated to the respective refinement step seems to decline such that the first refinement steps should seem to be the most important ones. Employing larger control horizons, i.e. $m > 1$ in combination with a more accurate discretization does not allow for using smaller optimization horizons N in comparison to the previously derived results, cf. Remark 5.7. Nevertheless, enhanced estimates are obtained.

Hence, the question arises whether similar results are obtainable for classical RHC, i.e. $m = 1$. This corresponds to shortening the continuous time control horizon, re-optimizing more often and, thus, robustifying the resulting closed loop. However, in Chapter 4 we observed that using longer control horizons improves the deduced suboptimality estimates, cf. Theorem 4.8. Here, it turns out that iterating the refinement process too often causes negative suboptimality bounds and, thus, makes our estimates useless, cf. Figure 5.6.

This claim is shown in Theorem 5.15. In order to proof this theorem, we need Lemma 5.20 which is based on results concerning the Gamma $\Gamma(\cdot)$ as well as the Beta $B(\cdot,\cdot)$ function, i.e. the functional equation of the Gamma function, a formula which allows for a transition from the one to the other, and, in particular, a more sophisticated result which goes back to Binet. However, since this integral part of the following proof is rather technical, it is postponed until Subsection 5.1.1 in order to streamline the presentation.

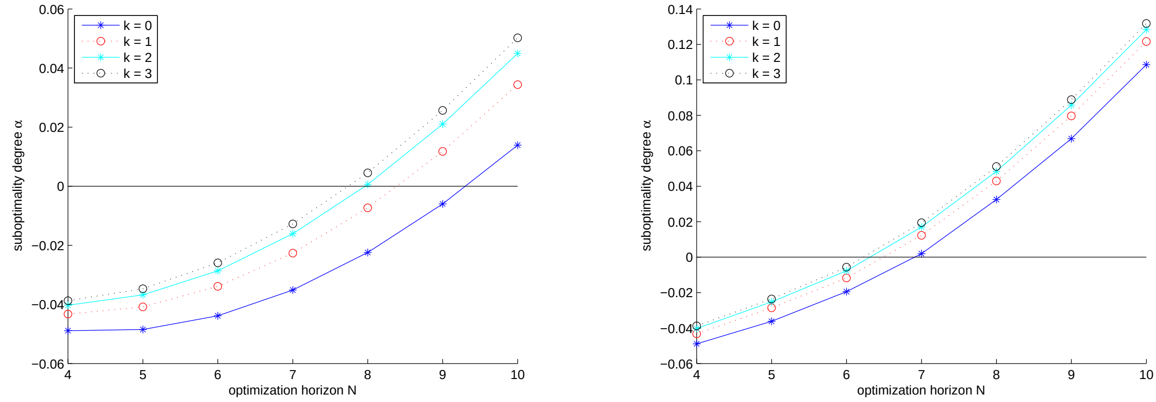


Figure 5.5: Impact of using more accurate discretizations for the reaction diffusion equation. For classical RHC one refinement step decreases the required optimization horizon to $N = 9$. The second refinement step leads to a further improvement ($N = 8$). On the right, one observes improved estimates also for control horizon $m = 3$. Here, the minimal stabilizing horizon N, however, remains the same.

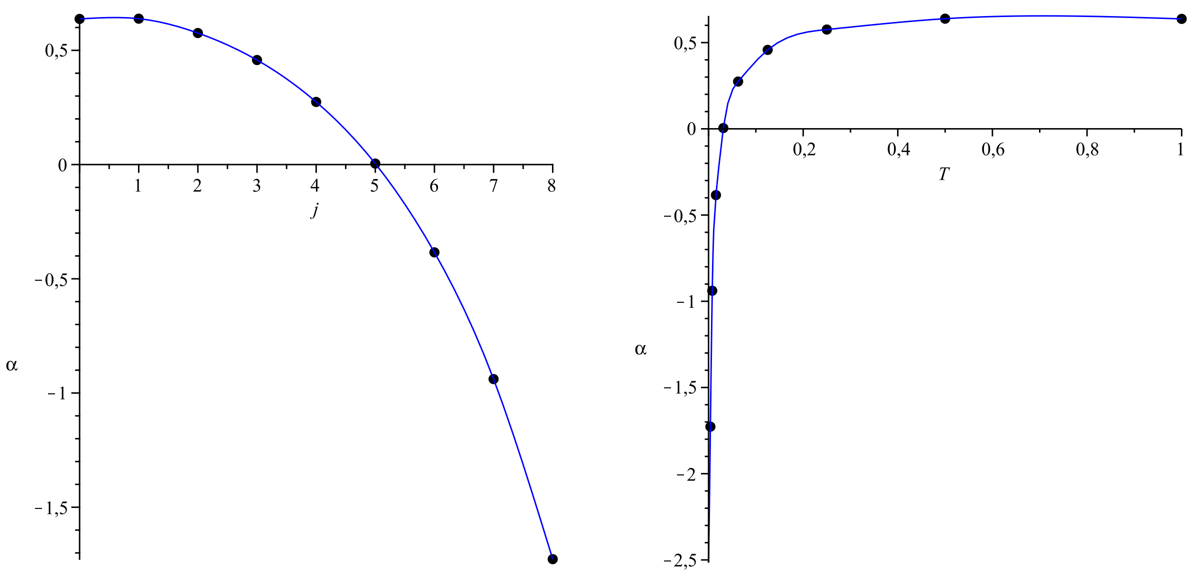


Figure 5.6: The assertion of Theorem 5.15 is illustrated for $N = 8$, $C = 2$, and $\sigma = 0.5$: for arbitrarily fast sampling — which corresponds to using a very fine discretization — our suboptimality estimates become negative for $m = 1$. Hence, neither asymptotic stability nor a performance bound is ensured.

Theorem 5.15
Let the assumptions of Theorem 5.12 be satisfied. Furthermore, we define σ, $\gamma_{i,k}$, and $(\mathcal{P}_k)$ according to Theorem 5.12. Then, for $C > 1$ and the sequence $(k_j)_{j \in \mathbb{N}_0}$ with $k_j := 2^j$, the corresponding sequence of optimal values $(\alpha_{k_j N,1}(k_j))_{j \in \mathbb{N}_0}$ diverges to minus infinity, i.e.

$$\alpha_{k_j N,1}(k_j) = 1 - \frac{(\gamma_{k_j N, k_j} - 1) \prod_{i=2}^{k_j N} (\gamma_{i,k_j} - 1)}{\prod_{i=2}^{k_j N} \gamma_{i,k_j} - \prod_{i=2}^{k_j N} (\gamma_{i,k_j} - 1)} \longrightarrow -\infty \qquad \textit{for } j \to \infty.$$

Proof: Since $\prod_{i=2}^{k_jN}\gamma_{i,k_j} \geq \prod_{i=2}^{k_jN}(\gamma_{i,k_j}-1) \geq 0$ holds, the assertion follows from

$$0 \leq \frac{1}{\gamma_{k_jN,k_j}-1}\cdot\prod_{i=2}^{k_jN}\frac{\gamma_{i,k_j}}{\gamma_{i,k_j}-1} \longrightarrow 0 \qquad \text{for } j\to\infty. \tag{5.13}$$

In order to deal with (5.13), we first prove the auxiliary inequalities

$$\frac{1}{\gamma_{k_jN,k_j}-1} \leq \frac{1-\sigma^{1/k_j}}{C_1} \qquad \text{and} \qquad \prod_{i=2}^{k_jN}\frac{\gamma_{i,k_j}}{\gamma_{i,k_j}-1} \leq C_0(2^{1/C})^j \tag{5.14}$$

with $C_0 := \sigma^{-N/C}\prod_{i=2}^{N}\frac{iC}{iC-1}$ and $C_1 := C(1-\sigma^N)-1+\sigma$. Note that the constants C_0 and C_1 do not depend on k_j. The first inequality is directly ensured by using (5.10) because $\sigma \leq \sigma^{1/k_j}$ and, thus, the denominator of the right hand side is smaller while the nominators are the same. In order to establish the second claim in (5.14), we require the auxiliary estimate

$$\frac{\gamma_{i,k_j}}{\gamma_{i,k_j}-1} = \frac{C}{C-1+\sigma^{1/k_j}}\frac{(1-\sigma^{i/k_j})(C-1+\sigma^{1/k_j})}{(C-1+\sigma^{1/k_j})-C\sigma^{i/k_j}} \leq \frac{C}{C-1+\sigma^{1/k_j}}\cdot\frac{iC}{iC-1} \tag{5.15}$$

which is equivalent to $iC\sigma^{i/k_j}(1-\sigma^{1/k_j}) \leq (C-1+\sigma^{1/k_j})(1-\sigma^{i/k_j})$ for $i\in\{2,3,\ldots,k_jN\}$. Dividing this inequality by $(1-\sigma^{1/k_j})$, splitting up the resulting left hand side into the two factors $C\sigma^{1/k_j}$ and $i\,\sigma^{(i-1)/k_j}$, and applying the estimates $C\sigma^{1/k_j} < (C-1+\sigma^{1/k_j})$ and $i\,\sigma^{(i-1)/k_j} \leq \sum_{n=0}^{i-1}\sigma^{n/k_j} = (1-\sigma^{i/k_j})/(1-\sigma^{1/k_j})$ ensures (5.15). In addition, we require another preliminary result, i.e.

$$\left(\frac{C}{C-1+\sigma^{1/k_j}}\right)^{k_jN} \leq \sigma^{-N/C}, \tag{5.16}$$

in order to conclude (5.14). Taking the (k_jN)-th root, (5.16) is equivalent to $C\sigma^{1/(k_jC)} \leq (C-1+\sigma^{1/k_j})$. Since $\sigma^{1/k_j}\in(0,1)$, defining $f(x) := C - Cx^{1/C} - 1 + x$ and showing $f(x)\geq 0$ for all $x\in[0,1]$ guarantees the desired inequality. Since $f(0)=C-1\geq 0$ and $f(1)=0$, verifying that $f(\cdot)$ is monotonically decreasing suffices. However, this is ensured because $f(\cdot)$ is continuous on the interval $[0,1]$, continuously differentiable on $(0,1)$, and $f'(x) = 1-(\sigma^{1/k_j})^{-(C-1)/C} \leq 0$ for all $x\in(0,1)$.

Hence, bearing in mind that the factor $C/(C-1+\sigma^{1/k_j})$ is independent of the control variable i, taking (5.15) and (5.16) into account, using $k_j = 2^j$, and applying Lemma 5.20 yields

$$\prod_{i=2}^{k_jN}\frac{\gamma_{i,k_j}}{\gamma_{i,k_j}-1} < \sigma^{-N/C}\cdot\prod_{i=2}^{2^jN}\frac{iC}{iC-1} = C_0\prod_{\nu=0}^{j-1}\left(\prod_{i=2^\nu N+1}^{2^{\nu+1}N}\frac{iC}{iC-1}\right) \leq C_0(2^{1/C})^j$$

for $j\in\mathbb{N}_0$, i.e. (5.14). Now, showing $(2^{1/C})^j(1-\sigma^{1/k_j})\to 0$ as j approaches infinity is sufficient in order to complete the proof. For this purpose, we define $\eta_j := (2^{1/C})^j(1-\sigma^{1/k_j})$ and show that the quotient η_{j+1}/η_j converges to $2^{1/C}/2$ for $j\to\infty$:

$$\frac{\eta_{j+1}}{\eta_j} = \frac{1-\sigma^{1/2^{(j+1)}}}{1-\sigma^{1/2^j}}2^{1/C} = \frac{(1-\sigma^{1/2^{(j+1)}})2^{1/C}}{(1-\sigma^{1/2^{(j+1)}})(1+\sigma^{1/2^{(j+1)}})} = \frac{2^{1/C}}{1+\sigma^{1/2^{(j+1)}}} \overset{j\to\infty}{\longrightarrow} 2^{1/C}/2.$$

Hence, there exists $j^\star$ such that the considered quotient η_{j+1}/η_j is less or equal $\theta := (2+2^{1/C})/4 < 1$ for all $j\geq j^\star$, i.e. the quotient is bounded from above by $2^{1/C}/2+\varepsilon$ with $\varepsilon := (2-2^{1/C})/4 > 0$. This implies the convergence of $2^{1/C}(1-\sigma^{1/k_j})$ to zero for j approaching infinity, i.e. (5.13) and, thus, the assertion.

□

Often, solutions of a control system generated by a differential equation are continuous which can be exploited, e.g. in order to preserve stability for the corresponding sampled-data system by sufficiently fast sampling, cf. [91]. Inherent properties like the mentioned continuity yield, in particular for (sufficiently) small intervals, tighter bounds on the transient behavior of the considered system than those provided by our controllability Assumption 3.2. However, they are not taken into account in the derivation of Problem 3.8 and, thus, in the suboptimality estimates from Theorem 3.18. Hence, a growth condition is incorporated in our setting in order to reflect, e.g. continuity properties in the deduced performance bounds. The combination of our controllability assumption and a growth condition will resolve the problem resulting from Theorem 5.15 for very fine discretization and $m = 1$, cf. Section 5.3. Furthermore, the growth condition to be introduced will allow to tighten our performance bounds, cf. Section 5.4.

5.1.1 Auxiliary Results for the Proof of Theorem 5.15

In this subsection, the auxiliary Lemma 5.20 is presented which is needed in order to show the second inequality of (5.14), i.e.

$$\prod_{i=2}^{k_j N} \frac{\gamma_{i,k_j}}{\gamma_{i,k_j} - 1} \leq C_0 (2^{1/C})^j \tag{5.17}$$

and, thus, Theorem 5.15. The proof of this lemma is, as already indicated in Section 5.1, essentially based on a result going back to Binet which provides a suitable series expansion of the beta function $B(\cdot,\cdot)$, cf. Lemma 5.19.

Furthermore, two alternative proofs of (5.17) and, thus, Theorem 5.15 for the special case $C = 2$ are presented afterward which are interesting from a mathematical point of view. In both approaches a representation of the analytic function $f(z) = \cos(z)$, which is given in Lemma 5.21, is applied:

- in the first proof the assertion of Lemma 5.20 is deduced without applying Lemma 5.19 which allows to avoid the use of the Beta function $B(\cdot,\cdot)$ entirely.
- While in the second, elementary proof (5.17) is shown independently of the auxiliary Lemmata 5.19 and 5.20. In particular, neither the Beta $B(\cdot,\cdot)$ nor the Gamma function $\Gamma(\cdot)$ are employed.

At first, the Gamma $\Gamma(\cdot)$ and the beta function $B(\cdot,\cdot)$ are defined. Then, some basic properties of these two functions are given.

Definition 5.16
Let x, $y \in \mathbb{R}_{>0}$. Then, we define the Eulerian integrals of first and second kind by

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad \text{and} \qquad \Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt.$$

$B(\cdot,\cdot)$ is known as the Beta function, $\Gamma(\cdot)$ is called Gamma function.

Remark 5.17
The Gamma function $\Gamma(\cdot)$ is well-defined on $]0,\infty)$, cf. [88, Theorem 6.4.1]. In addition,

$$\Gamma(1) = 1 \qquad \text{and} \qquad \Gamma(x+1) = x\Gamma(x) \tag{5.18}$$

hold, cf. [88, Theorem 6.4.4]. This identity is said to be the functional equation of the Gamma function $\Gamma(\cdot)$ *and also known as the reduction formula or the difference equation, cf. [123, p.237]. For non-negative integers* $n \in \mathbb{N}$*, the Gamma function* $\Gamma(\cdot)$ *represents the factorial, i.e.* $\Gamma(n+1) = n!$ *as well. However,* (5.18) *holds for arbitrary real numbers.*

Remark 5.18

The Beta function $B(\cdot, \cdot)$ *is well-defined on* $]0, \infty) \times]0, \infty)$*, cf. [124, p.437] and connected to the Gamma function* $\Gamma(\cdot)$ *via the formula*

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{5.19}$$

cf. [124, p.442].

Lemma 5.19

Let $p > 0$*,* $p + s > 0$*. Then the following equation holds for the Beta function* $B(\cdot, \cdot)$

$$B(p, p+s) = \frac{B(p,p)}{2^s} \left(1 + \frac{s(s-1)}{2(2p+1)} + \frac{s(s-1)(s-2)(s-3)}{2 \cdot 4 \cdot (2p+1) \cdot (2p+3)} + \ldots \right).$$

Proof: We prove only the special case $s = 2$. For $s \neq 2$ we refer to [123, p.262]. Using (5.18) and (5.19) yields

$$B(p, p+2) = \frac{\Gamma(p)\Gamma(p+2)}{\Gamma(2p+2)} = \frac{p(p+1)\Gamma(p)\Gamma(p)}{2p(2p+1)\Gamma(2p)} = 2^{-2} \frac{(2p+1)+1}{2p+1} = \frac{1}{2^2} \left(1 + \frac{2(2-1)}{2(2p+1)} \right)$$

and, thus, the assertion.

□

Bearing these preliminary results in mind allows for tackling Lemma 5.20 which paves the way in order to prove Theorem 5.15.

Lemma 5.20

Let $N \in \mathbb{N}_{\geq 2}$*,* $C \geq 1$*, and* $\nu \in \mathbb{N}$ *be given. Then, we get*

$$\prod_{i=2^\nu N+1}^{2^{\nu+1}N} \frac{iC}{iC-1} \leq 2^{1/C} = \sqrt[C]{2}.$$

Proof: In the following, the functional equation (5.18) of the Gamma function $\Gamma(\cdot)$, its interplay with the Beta function $B(\cdot, \cdot)$ via (5.19) and Lemma (5.19) applied with $s = (C-1)/C \in [0, 1)$ and $p = 2^\nu N$ are used in order to rewrite the term to be estimated

$$\begin{aligned}
\prod_{i=2^\nu N+1}^{2^{\nu+1}N} \frac{iC}{iC-1} &= \prod_{i=2^\nu N+1}^{2^{\nu+1}N} \frac{i}{i - \frac{1}{C}} = \frac{(2^{\nu+1}N)!}{(2^\nu N)!} \left(\prod_{i=2^\nu N+1}^{2^{\nu+1}N} i - \frac{1}{C} \right)^{-1} \\
&= \frac{\Gamma(2^{\nu+1}N+1)}{\Gamma(2^\nu N+1)} \cdot \frac{\Gamma(2^\nu N + 1 - \frac{1}{C})}{\Gamma(2^{\nu+1}N + 1 - \frac{1}{C})} \\
&= \frac{B(2^\nu N, 2^\nu N + \frac{C-1}{C})}{B(2^\nu N, 2^\nu N + 1)} \\
&= 2^{1/C} \left(1 + \frac{s(s-1)}{2(2p+1)} + \frac{s(s-1)(s-2)(s-3)}{2 \cdot 4 \cdot (2p+1) \cdot (2p+3)} + \ldots \right).
\end{aligned}$$

Since $s \in [0, 1)$, the term in brackets is less or equal to one. Hence, the desired inequality is obtained.

□

For the special case $C = 2$, Lemma 5.19 can be replaced by the following lemma from [123, § 7.5] which allows us to establish Lemma 5.20 and, thus, to conclude the assertion of Theorem 5.15 without employing the Beta function $B(\cdot,\cdot)$.

Lemma 5.21
Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function having simple zeros at each element of the sequence $(a_i)_{i\in\mathbb{N}} \subset \mathbb{R}\backslash\{0\}$ which satisfies $\lim_{n\to\infty} |a_n| = \infty$. Furthermore, there exists a sequence of circles $(C_m)_{m\in\mathbb{N}}$ satisfying the conditions described in [123, § 7.4]. Then, $f(z)$ may be written as an infinite product of the form

$$f(z) = f(0)e^{f'(0)z/f(0)} \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{a_n}\right) e^{z/a_n} \right].$$

Lemma 5.21 is, e.g. applicable for $\sin(z)/z$, cf. [123, p.137].

Proof: [Alternative proof of Lemma 5.20 for $C = 2$] Using $C = 2$ and proceeding analogously to the proof of Lemma 5.20 yields

$$\prod_{i=2^\nu N+1}^{2^{\nu+1}N} \frac{2i}{2i-1} = \frac{\Gamma(2^{\nu+1}N+1)}{\Gamma(2^\nu N+1)} \cdot \frac{\Gamma(2^\nu N+\frac{1}{2})}{\Gamma(2^{\nu+1}N+\frac{1}{2})}.$$

Next, we require the duplication formula

$$2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\,\Gamma(2z) \tag{5.20}$$

which holds for the Gamma function $\Gamma(\cdot)$ according to [123, p.240] and goes back to Legendre.[4] Using the identity given by (5.20) for $z = 2^\nu N + \frac{1}{2}$ in the first and for $z = 2^{\nu+1}N + \frac{1}{2}$ in the second equation, leads to

$$\begin{aligned} \frac{\Gamma(2^{\nu+1}N+1)}{\Gamma(2^\nu N+1)} \cdot \frac{\Gamma(2^\nu N+\frac{1}{2})}{\Gamma(2^{\nu+1}N+\frac{1}{2})} &= \left(\frac{\Gamma(2^{\nu+1}N+1)}{\Gamma(2^\nu N+1)}\right)^2 \frac{\sqrt{\pi}}{2^{2\cdot(2^\nu N+\frac{1}{2})-1}\,\Gamma(2^{\nu+1}N+\frac{1}{2})} \\ &= \frac{\Gamma(2^{\nu+1}N+1)^3}{\Gamma(2^\nu N+1)^2\,\Gamma(2^{\nu+2}N+1)} \frac{4^{2^{\nu+1}N}}{4^{2^\nu N}} \\ &= 4^{2^\nu N} \cdot \frac{(2^{\nu+1}N)!\,(2^{\nu+1}N)!\,(2^{\nu+1}N)!}{(2^\nu N)!\,(2^\nu N)!\,(2^{\nu+2}N)!}. \end{aligned}$$

Using the definition of the factorial, enables us to expand this expression as a product

$$\begin{aligned} 4^{2^\nu N} \cdot \frac{(2^{\nu+1}N)!\,(2^{\nu+1}N)!\,(2^{\nu+1}N)!}{(2^\nu N)!\,(2^\nu N)!\,(2^{\nu+2}N)!} &= \prod_{n=1}^{2^\nu N} \frac{4(2n)^3\,(2n-1)^3}{n^2\,4n\,(4n-1)\,(4n-2)\,(4n-3)} \\ &= \prod_{n=1}^{2^\nu N} \frac{4(2n-1)^2}{(4n-1)(4n-3)}. \end{aligned}$$

Since each factor of this product is strictly greater than one, this term is strictly monotonically increasing in ν. In addition, we are interested in deducing an uniformly upper

[4] Indeed, (5.20) may be concluded as a corollary of the multiplication-theorem of Gauss, cf. [123, p.240].

bound, i.e. an estimate which does not depend on ν. Hence, our goal consists of calculating the following infinite product

$$\prod_{n=1}^{2^\nu N} \frac{4(2n-1)^2}{(4n-1)(4n-3)} < \prod_{n=1}^{\infty} \frac{(4n-2)^2}{(4n-2)^2-1} = \left(\prod_{n=1}^{\infty} 1 - \frac{(\pi/4)^2}{\pi^2(n-\frac{1}{2})^2} \right)^{-1}. \tag{5.21}$$

Applying Lemma 5.21 to $f(z) = \cos(z)$ yields (the respective assumptions may easily be checked)

$$\begin{aligned} \cos(z) &= \cos(0) e^{-\sin(0)z/\cos(0)} \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{\pi(n-\frac{1}{2})}\right) e^{z/[\pi(n-\frac{1}{2})]} \right] \left[\left(1 - \frac{z}{\pi(\frac{1}{2}-n)}\right) e^{z/[\pi(\frac{1}{2}-n)]} \right] \\ &= \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2(n-\frac{1}{2})^2}\right). \end{aligned}$$

Plugging $\pi/4$ in this representation of $\cos(\cdot)$ ensures, since $\cos(\pi/4)^{-1} = \sqrt{2} = 2^{1/C}$, the desired estimate for $C = 2$.

□

In order to conclude this subsection, an elementary proof of (5.17) and, thus, Theorem 5.15 is, again for the special case $C = 2$, given, which does not make use of the Gamma $\Gamma(\cdot)$ or the Beta function $B(\cdot, \cdot)$. In particular, positivity of an auxiliary function is shown by using arguments with respect to its derivatives — presenting the respective technique further motivates including the following lemma. In order to avoid technical difficulties we stick to the notation introduced in Theorem 5.15 and the respective proof.

Lemma 5.22
Let $N \in \mathbb{N}_{\geq 2}$, $C = 2$, $\sigma \in (0,1)$, and the sequence $(k_j)_{j \in \mathbb{N}_0} \subset \mathbb{N}$ with $k_j := 2^j$ be given. In addition, let $\gamma_{i,k}$ be given by (5.10) and C_0 be defined as $\sigma^{-N/2} \prod_{i=2}^{N} \frac{2i}{2i-1}$. Then, the following inequality holds

$$\prod_{i=2}^{k_j N} \frac{\gamma_{i,k_j}}{\gamma_{i,k_j} - 1} \leq C_0 (2^{1/2})^j = C_0 \sqrt{2}^j. \tag{5.22}$$

Proof: Since $2\sigma^{i/k_j} = 2\sigma^{1/2k_j} \sigma^{(2i-1)/2k_j} \leq (1 + \sigma^{1/k_j}) \sigma^{(2i-1)/2k_j}$ holds, the factors of the product from the left hand side of (5.22) can be rewritten as

$$\frac{\gamma_{i,k_j}}{\gamma_{i,k_j} - 1} = \frac{2(1 - \sigma^{i/k_j})}{(1 + \sigma^{1/k_j}) - 2\sigma^{i/k_j}} \leq \frac{2(1 - \sigma^{i/k_j})}{(1 + \sigma^{1/k_j})(1 - \sigma^{(2i-1)/2k_j})}.$$

Hence, taking $C = 2$ into account, we obtain analogously to the proof of (5.16)

$$\prod_{i=2}^{2^j N} \frac{\gamma_{i,k_j}}{\gamma_{i,k_j} - 1} \leq \sigma^{-N/2} \prod_{i=2}^{2^j N} \frac{1 - \sigma^{i/k_j}}{1 - \sigma^{(2i-1)/(2k_j)}}. \tag{5.23}$$

The remaining portion of this proof is subdivided into two parts:

- Firstly, the estimate

$$\sigma^{-N/2} \cdot \prod_{i=2}^{2^0 N} \frac{1 - \sigma^i}{1 - \sigma^{(2i-1)/2}} = \sigma^{-N/2} \cdot \prod_{i=2}^{N} \frac{1 - \sigma^i}{1 - \sigma^{i - \frac{1}{2}}} \leq \sigma^{-N/2} \cdot \prod_{i=2}^{N} \frac{2i}{2i-1} = C_0 \tag{5.24}$$

  is shown which covers the assertion for $j = 0$.

- Secondly, the growth bound

$$\prod_{i=2}^{2^{j+1}N} \left(\frac{1-\sigma^{i/k_{j+1}}}{1-\sigma^{(2i-1)/2k_{j+1}}} \right) \Big/ \prod_{i=2}^{2^j N} \left(\frac{1-\sigma^{i/k_j}}{1-\sigma^{(2i-1)/2k_j}} \right) \leq \sqrt{2}. \tag{5.25}$$

  is deduced which ensures that incrementing the index j in (5.23) leads at most to a multiplication of the estimate for j by a factor of $\sqrt{2}$. Combining this growth bound with the estimate for $j=0$ in order to estimate the term on the right hand of (5.23) implies the assertion.

In order to ensure (5.24), we prove

$$f(x) := 1 - 2ix^{i-\frac{1}{2}} + (2i-1)x^i \geq 0 \qquad \forall\, x \in (0,1) \tag{5.26}$$

for $i \in \{2,3,\ldots,N\}$, which implies, since $\sigma \in (0,1)$, the inequality

$$(1-\sigma^i)/(1-\sigma^{i-\frac{1}{2}}) \leq 2i/(2i-1) \tag{5.27}$$

for $i \in \{2,3,\ldots,N\}$. Since $f(0)=1$, $f(1)=0$, and $f \in \mathcal{C}^1([0,1])$ showing $f'(x) \leq 0$, $i = 2,3,\ldots,N$, for $x \in (0,1)$ implies $f(x) \geq 0$ for all $x \in [0,1]$:

$$f'(x) = x^{i-\frac{3}{2}} \left[i(2i-1)\sqrt{x} - 2i(i-1/2) \right] \leq x^{i-\frac{3}{2}} [i(2i-1) - 2i(i-1/2)] = 0 \quad \forall\, x \in (0,1).$$

Hence, it remains to verify the claimed growth property (5.25) in order conclude the assertion, i.e. for $j \in \mathbb{N}_0$,

$$\begin{aligned}
&\prod_{i=2}^{2^{j+1}N} \left(\frac{1-\sigma^{i/k_{j+1}}}{1-\sigma^{(2i-1)/2k_{j+1}}} \right) \Big/ \prod_{i=2}^{2^j N} \left(\frac{1-\sigma^{i/k_j}}{1-\sigma^{(2i-1)/2k_j}} \right) \\
= \quad & \frac{1-\sigma^{2/(k_{j+1})}}{1-\sigma^{3/(2k_{j+1})}} \cdot \prod_{i=2}^{2^j N} \frac{(1-\sigma^{2i/k_{j+1}})(1-\sigma^{(2i-1)/k_{j+1}})(1-\sigma^{(2i-1)/2k_j})}{(1-\sigma^{(4i-1)/2k_{j+1}})(1-\sigma^{(4i-3)/2k_{j+1}})(1-\sigma^{i/k_j})} \\
\overset{k_{j+1}=2k_j}{=} \quad & \frac{1-\sigma^{2/(k_{j+1})}}{1-\sigma^{3/(2k_{j+1})}} \cdot \prod_{i=2}^{2^j N} \frac{(1-\sigma^{(2i-1)/k_{j+1}})^2}{(1-\sigma^{(4i-1)/2k_{j+1}})(1-\sigma^{(4i-3)/2k_{j+1}})} \leq \sqrt{2}.
\end{aligned}$$

Using (5.26) for $i=2$ and $x = \sigma^{1/k_{j+1}} \in (0,1)$ ensures, in analogy to (5.27) with $\sigma^{1/k_{j+1}}$ instead of σ, $(1-\sigma^{2/(k_{j+1})})/(1-\sigma^{3/(2k_{j+1})}) \leq 4/3 = (4\nu-2)^2/[(4\nu-2)^2-1]$ for $\nu = 1$. As a preliminary goal, we want to establish this inequality also for the other factors involved in the product in consideration, i.e.

$$\frac{(1-\sigma^{(2i-1)/k_{j+1}})^2}{(1-\sigma^{(4i-1)/2k_{j+1}})(1-\sigma^{(4i-3)/2k_{j+1}})} \leq \frac{(4i-2)^2}{(4i-2)^2-1} \qquad \text{for } i \in \{2,3,\ldots,2^j N\}. \tag{5.28}$$

Using $2k_{j+1} = k_{j+2}$ and substituting $(2i-1)$ by ν, (5.28) is equivalent to

$$1 - 4\nu^2\sigma^{(2\nu-1)/k_{j+2}} + 2(4\nu^2-1)\sigma^{2\nu/k_{j+2}} - 4\nu^2\sigma^{(2\nu+1)/k_{j+2}} + \sigma^{4\nu/k_{j+2}} \geq 0$$

for $\nu = 3,5,\ldots,2^{j+1}N-1$. Instead of deducing this inequality directly, we subtract the positive term $\nu^2\sigma^{2(\nu-1)/k_{j+2}}(1-\sigma^{1/k_{j+2}})^4$ from the left hand side and prove that the resulting expression, i.e.

$$1 - \nu^2\sigma^{(\nu-1)/k_{j+1}} + 2(\nu^2-1)\sigma^{\nu/k_{j+1}} - \nu^2\sigma^{(\nu+1)/k_{j+1}} + \sigma^{2\nu/k_{j+1}} \qquad \text{for } \nu \in \mathbb{N}_{\geq 3}, \tag{5.29}$$

is still positive. Note that the range of feasible indices ν is extended. For $\nu = 3$, (5.29) equals $(1-\sigma^{1/k_{j+1}})^6 + 6\sigma^{1/k_{j+1}}(1-\sigma^{1/k_{j+1}})^4$ which covers the assertion. In order to conclude the assertion for $\nu \in \mathbb{N}_{>3}$, (5.29) is shown to be monotonically increasing with respect to ν. To this end, (5.29) is subtracted from the respective expression for $\nu+1$ and the resulting expression is reduced by $\sigma^{(\nu-1)/k_{j+1}}$. Then, since $\sigma^{1/k_{j+1}} \in (0,1)$, verifying the inequality

$$f(x) := \nu^2 - [3\nu^2 + 2\nu - 1]x + [3\nu^2 + 4\nu]x^2 - (\nu+1)^2 x^3 - x^{\nu+1} + x^{\nu+3} \geq 0 \qquad \forall\, x \in [0,1],$$

$\nu \in \mathbb{N}_{\geq 3}$, for the polynomial $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{C}^\infty(\mathbb{R})$, ensures the claimed monotonicity and, thus, positivity of (5.29) which implies (5.28). Since $f(0) = \nu^2 > 0$ and $f(1) = 0$, showing $f'(x) \leq 0$ for all $x \in [0,1]$ suffices. For this purpose, we calculate

$$\begin{aligned}
f'(x) &= -[3\nu^2 + 2\nu - 1] + [6\nu^2 + 8\nu]x - 3(\nu+1)^2 x^2 - (\nu+1)x^\nu + (\nu+3)x^{\nu+2},\\
f^{(2)}(x) &= [6\nu^2 + 8\nu] - 6(\nu+1)^2 x - \nu(\nu+1)x^{\nu-1} + (\nu+2)(\nu+3)x^{\nu+1},\\
f^{(3)}(x) &= -6(\nu+1)^2 - (\nu-1)\nu(\nu+1)x^{\nu-2} + (\nu+1)(\nu+2)(\nu+3)x^\nu,\\
f^{(4)}(x) &= \nu(\nu+1)[(\nu+2)(\nu+3)x^2 - (\nu-2)(\nu-1)]x^{\nu-3}.
\end{aligned}$$

Taking $f'(0) < 0$ and $f'(1) = 0$ into account enables us to repeat the above line of argumentation. Hence, it remains to establish $f^{(2)}(x) \geq 0$ for all $x \in [0,1]$. Since $f^{(2)}(0) > 0$ and $f^{(2)}(1) = 0$ hold, applying this argument once more shows that the condition $f^{(3)}(x) \leq 0$ for all $x \in [0,1]$ is sufficient in order to ensure the desired inequality.

However, establishing this claim requires a sophisticated argument. The sign of the fourth derivative $f^{(4)}(\cdot)$ is determined in the interval $[0,1]$ which evolves like the one of $(\nu+2)(\nu+3)x^2 - (\nu-2)(\nu-1)$. Hence, $f^{(4)}(\cdot)$ is negative on $[0,\bar{x}]$ and strictly positive for $x \in (\bar{x}, 1]$ with $\bar{x} := \sqrt{[(\nu-2)(\nu-1)]/[(\nu+2)(\nu+3)]} \in (0,1)$ which guarantees in combination with $f^{(3)}(0) < 0$ and $f^{(3)}(1) = 0$ the assertion for $\nu \in \mathbb{N}_{\geq 3}$.

Collecting the deduced inequalities yields the following estimate. Since the right hand side coincides with the one from (5.21), the proof is completed by using the representation of $f(x) = \cos(x)$ provided by the Euler product formula given in Lemma 5.21 analogously

$$\begin{aligned}
\frac{1-\sigma^{2/(k_{j+1})}}{1-\sigma^{3/(2k_{j+1})}} \prod_{i=2}^{2^j N} \frac{(1-\sigma^{(2i-1)/k_{j+1}})^2}{(1-\sigma^{(4i-1)/2k_{j+1}})(1-\sigma^{(4i-3)/2k_{j+1}})} &\leq \left(\prod_{i=1}^{\infty} \frac{(4i-2)^2 - 1}{(4i-2)^2} \right)^{-1}\\
&= \left(\prod_{i=1}^{\infty} \left(1 - \frac{\pi^2/16}{\pi^2 (i - \frac{1}{2})^2} \right) \right)^{-1}\\
&= \cos(\pi/4)^{-1} = \sqrt{2}.
\end{aligned}$$

□

5.2 Continuous Time Counterpart

This section continues the investigation of using more accurate discretizations, cf. Definition 5.10. To this end, also the discrete time control horizon is adapted and, thus, the discretization sequence $(\mathcal{D}_j)_{j\in\mathbb{N}_0}$ is given by (5.8), cf. also Proposition 5.13 in which an iterative refinement process was carried out. Here, the limit of this refinement process is calculated, cf. (5.30). Indeed, (5.30) is independent of the exact shape of the employed sequence of discretization parameters $(k_j)_{j\in\mathbb{N}_0}$. This limit, which represents a performance bound reflecting Assumption 5.9 for all $t \geq 0$, may be rewritten, cf. (5.31), and, thus, coincides with a suboptimality estimate from [104] in the exponentially controllable case.[5] The methodology introduced in [104] and further developed [103] is based on Assumption 5.9 and composes a linear program analogously to Section 3.1 — however, based on a solely continuous time setting.[6] Solving this continuous time linear program (LP) yields the corresponding performance bound given in (5.31), cf. Figure 5.7.[7] Although [103,104] are dealing with finite dimensional spaces, a generalization to time-delay systems was already carried out, cf. [105].

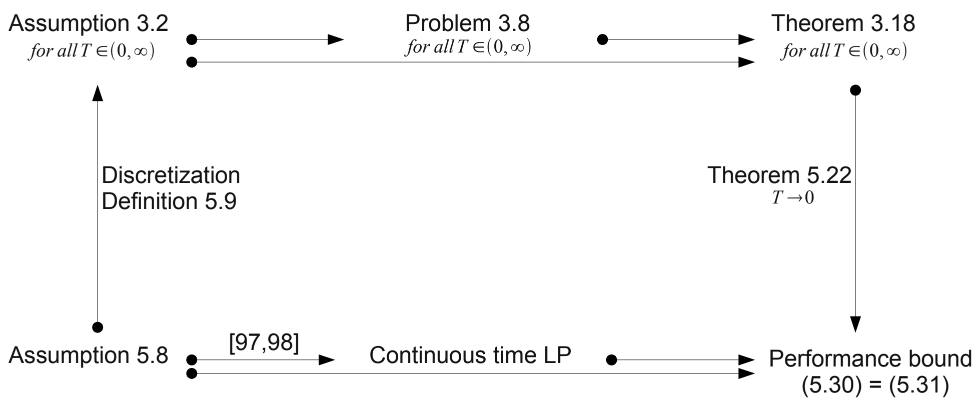


Figure 5.7: Schematic illustration of the assertion of Theorem 5.23

Hence, the contribution of Theorem 5.23 is twofold: on the one hand the limit of the discretization process from Definition 5.10 is provided which can be approximated by employing a sufficiently fine discretization. On the other hand, Theorem 5.23 clarifies the connection between the continuous time approach from [104] and the previously introduced discrete time one from [39] by proving that the suboptimality estimates coincide in the limit for discretization parameter $T \to 0$.

Theorem 5.23
Let Assumption 5.9 be satisfied with decay rate $\mu > 0$ and overshoot $C \geq 1$. In addition, let an optimization horizon $N \in \mathbb{N}_{\geq 2}$, a control horizon $m \in \{1, 2, \ldots, N-1\}$ and a sampling period $T > 0$ be given and define $\sigma := e^{-\mu T} \in (0,1)$ and $\gamma_{i,k}$ according to (5.10). Then, for every sequence $(k_j)_{j \in \mathbb{N}_0} \subseteq \mathbb{N}$ satisfying $k_j \to \infty$ for $j \to \infty$, we get

$$\alpha_{k_j N, k_j m}(k_j) \longrightarrow 1 - \frac{\left(\frac{1-\sigma^m}{\sigma^m}\right)^{1/C}}{\left(\frac{1-\sigma^N}{\sigma^N}\right)^{1/C} - \left(\frac{1-\sigma^m}{\sigma^m}\right)^{1/C}} \cdot \frac{\left(\frac{1-\sigma^{N-m}}{\sigma^{N-m}}\right)^{1/C}}{\left(\frac{1-\sigma^N}{\sigma^N}\right)^{1/C} - \left(\frac{1-\sigma^{N-m}}{\sigma^{N-m}}\right)^{1/C}} \qquad \textit{for } j \to \infty, \tag{5.30}$$

[5]Indeed, improved estimates from [103] are needed in order to show (5.31). We thank the authors for sending us this paper in a preliminary stage. In particular, we like to point out that knowing the expected limit facilitated its proof.

[6]Note that Section 3.1 only summarizes results from [39].

[7]More precisely, (5.31) is shown to be the solution of a relaxed problem in [103,104] and, thus, only to be a lower bound on the optimal value of the continuous time LP, cp. the connection between Problems 3.8 and 3.17. Theorem 5.23 shows that this lower bound also solves the original continuous time problem in the exponentially controllable case.

or, introducing the abbreviations $\delta := mT$ and $\Upsilon := NT$ for the continuous time control and optimization horizon, respectively,

$$\alpha_{k_jN,k_jm}(k_j) \stackrel{j\to\infty}{\longrightarrow} 1 - \frac{\left(e^{\mu\delta}-1\right)^{1/C}}{\left(e^{\mu\Upsilon}-1\right)^{1/C} - \left(e^{\mu\delta}-1\right)^{1/C}} \cdot \frac{\left(e^{\mu(\Upsilon-\delta)}-1\right)^{1/C}}{\left(e^{\mu\Upsilon}-1\right)^{1/C} - \left(e^{\mu(\Upsilon-\delta)}-1\right)^{1/C}} \tag{5.31}$$

for the corresponding sequence $(\alpha_{k_jN,k_jm}(k_j))_{j\in\mathbb{N}}$ of optimal values of the suitably adapted Problems 3.8 given by (5.11)*.*

Proof: Taking account of (5.11), the subtrahend of $(\alpha_{k_jN,k_jm}(k_j))_{j\in\mathbb{N}}$ consists of two factors. For $k \in (k_j)_{j\in\mathbb{N}_0}$, the first of these is rewritten as

$$\frac{\prod_{i=km+1}^{kN}(\gamma_{i,k}-1)}{\prod_{i=km+1}^{kN}\gamma_{i,k} - \prod_{i=km+1}^{kN}(\gamma_{i,k}-1)} = \left[\prod_{i=km+1}^{kN}\frac{\gamma_{i,k}}{\gamma_{i,k}-1} - 1\right]^{-1}. \tag{5.32}$$

Considering the limit of the right hand side w.r.t. j and applying Lemma 5.26, yields

$$\lim_{j\to\infty}\frac{\prod_{i=k_jm+1}^{k_jN}(\gamma_{i,k_j}-1)}{\prod_{i=k_jm+1}^{k_jN}\gamma_{i,k_j} - \prod_{i=k_jm+1}^{k_jN}(\gamma_{i,k_j}-1)} = \left[\frac{\left(\frac{1-\sigma^N}{\sigma^N}\right)^{1/C}}{\left(\frac{1-\sigma^m}{\sigma^m}\right)^{1/C}} - 1\right]^{-1} = \frac{\left(\frac{1-\sigma^m}{\sigma^m}\right)^{1/C}}{\left(\frac{1-\sigma^N}{\sigma^N}\right)^{1/C} - \left(\frac{1-\sigma^m}{\sigma^m}\right)^{1/C}}. \tag{5.33}$$

Repeating this argument for the second factor of the subtrahend and combining the result with (5.33) shows (5.30). In order to complete the proof, we have to establish equality of the right hand sides of (5.30) and (5.31). Using the definitions of σ and δ we obtain

$$\left(\frac{1-\sigma^m}{\sigma^m}\right)^{1/C} = \left(\frac{1-e^{-\mu mT}}{e^{-\mu mT}}\right)^{1/C} = \left(\frac{1-e^{-\mu\delta}}{e^{-\mu\delta}}\right)^{1/C}.$$

Hence, taking the definition of Υ into account, repeating this argumentation and plugging the resulting expressions in (5.30) allows for concluding the assertion.

□

Remark 5.24

A sequence $(k_j)_{j\in\mathbb{N}_0} \subset \mathbb{N}$ of discretization parameters satisfying that k_{j+1} is a multiple of k_j and $k_{j+1} > k_j$ corresponds to an iterative refinement process. For instance, let us define $k_j := 2^j$ which implies $k_{j+1} = 2k_j$. Then, Proposition 5.13 ensures that the performance bounds $(\alpha_{k_jN,k_jm}(k_j))_{n\in\mathbb{N}_0}$ are increasing in j while Theorem 5.23 yields the respective limit, cf. Figure 5.8.

5.2.1 Auxiliary Results

The goal of this subsection is to deduce Lemma 5.26 which plays an important role in the proof of Theorem 5.23. To this end, the technical auxiliary Lemma 5.25 is required, whose proof is based on a simple Taylor series expansion.

Lemma 5.25

Let a parameter $s \in (0,1)$ and a real constant c be given such that $c - 1 + s > 0$ holds. Then, for each sequence $(n_i)_{i\in\mathbb{N}_0} \subset \mathbb{N}$ satisfying $n_i \to \infty$ for i approaching infinity, we get the convergence

$$\left(1 + \frac{1 - s^{\frac{1}{n_i}}}{c - (1 - s^{\frac{1}{n_i}})}\right)^{n_i} \longrightarrow \left(s^{-1}\right)^{\frac{1}{c}} \qquad \text{for } i \to \infty.$$

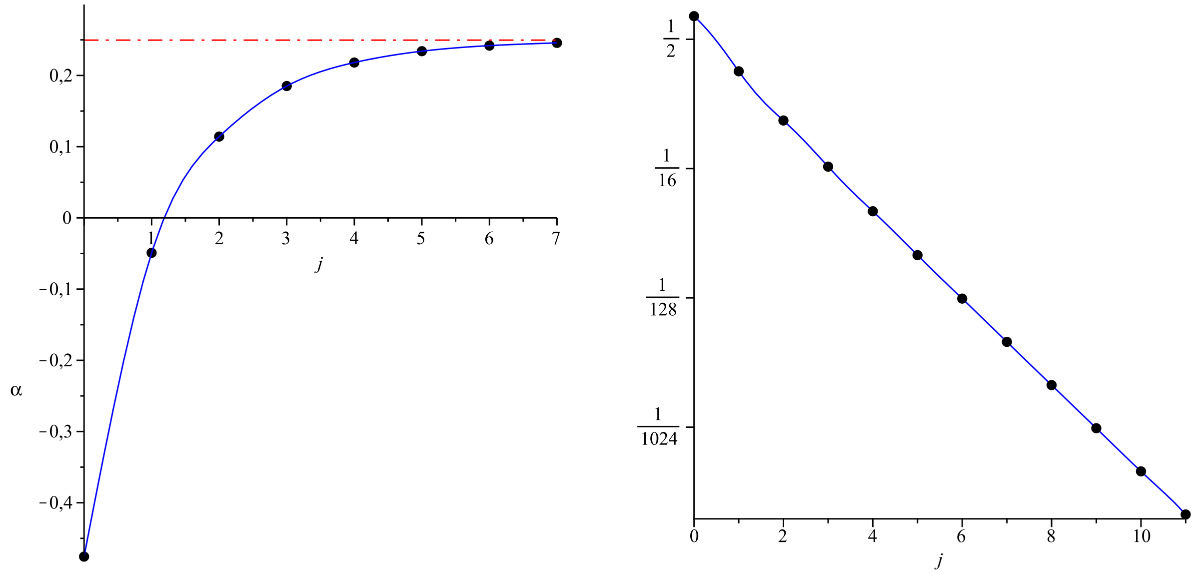


Figure 5.8: The figure on the left depicts the monotone convergence of the sequence of suboptimality estimates $(\alpha_{k_jN,k_jm}(k_j))_{j\in\mathbb{N}_0}$ for $k_j = 2^j$ to the limit computed in Theorem 5.23. Indeed, one observes that the discrete time estimates approximate their continuous time counterpart already after very few refinements astonishingly well. This observation is supported by the illustration drawn on the right which presents the deviations from the theoretically obtained upper bound: the error decays exponentially in the iteration index j. Here, the parameters $N = 4$, $m = 1$, $T = 1$, and $\sigma = 0.5$ were chosen.

Proof: We define the function $f : \mathbb{R}_{\leq 1} \to \mathbb{R}$

$$f(x) := \frac{1 - s^x}{c - (1 - s^x)} = \frac{1 - e^{x \ln s}}{c - (1 - e^{x \ln s})}$$

and calculate its first derivative

$$f'(x) = \frac{-c \ln s \cdot e^{x \ln s}}{(c - (1 - e^{x \ln s}))^2}.$$

We point out that the norm of the second derivative $f''(\cdot)$ is uniformly bounded on the interval $[0, 1]$, i.e. a constant $M \in (0, \infty)$ exists such that $\sup_{x\in[0,1]} |f''(x)| \leq M$ holds. In addition, $f(0) = 0$ and $f'(0) = -(\ln s)/c$ hold. Hence, for each element $n \in (n_i)_{i\in\mathbb{N}}$, using the Taylor series expansion of $f(1/n)$ at $x = 0$, cf. [77, chapter XIII], yields the existence of a real number $\xi_n \in (0, 1/n)$ satisfying

$$f(1/n) = f(0) + \frac{1}{n} f'(0) + \sum_{i=2}^{\infty} \left(\frac{1}{n}\right)^i \frac{f^{(i)}(0)}{i!} = -\frac{\ln s}{cn} + \frac{f''(\xi_n)}{2n^2}. \tag{5.34}$$

Since $\xi_n \in (0, 1/n) \subseteq (0, 1]$, $|f''(\xi_n)| \leq M$ independently of n. Moreover, for an arbitrarily chosen constant $\varepsilon > 0$ and sufficiently large n (which holds for all $n_i \in (n_i)_{i\in\mathbb{N}}$ with sufficiently large control index i), $(1+f'(0)/n)^n \leq e^{f'(0)}+\varepsilon$ holds because $(1+f'(0)/n_i)^{n_i} \to e^{f'(0)} \in [1, \infty)$ for i tending to infinity. These preliminary considerations enable us to deduce the following estimate which is essential in order to conclude the assertion. Since

$1+\frac{1}{n}f'(0)\geq 1$, we get

$$
\begin{aligned}
0 &\leq \sum_{j=1}^{n}\binom{n}{j}\left(1+\frac{f'(0)}{n}\right)^{n-j}\left(\frac{|f''(\xi_n)|}{2n^2}\right)^{j} \\
&= \sum_{j=1}^{n}\left(\frac{|f''(\xi_n)|}{2}\right)^{j}\frac{1}{j!}\underbrace{\frac{n\,(n-1)(n-2)\cdots(n-j+1)}{n\quad n\quad\quad n\quad \cdots\quad n}}_{\leq 1}\underbrace{\left(1+\frac{f'(0)}{n}\right)^{n-j}}_{\leq\left(1+\frac{1}{n}f'(0)\right)^{n}}\underbrace{\left(\frac{1}{n}\right)^{j}}_{\leq\frac{1}{n}} \\
&< \left(e^{f'(0)}+\varepsilon\right)\frac{1}{n}\sum_{j=0}^{n}\frac{(M/2)^j}{j!} < \left(e^{f'(0)}+\varepsilon\right)\frac{e^{\frac{M}{2}}}{n}\overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Taking this into account and carrying out a binomial expansion, cf. [77, p.466], we obtain

$$
(1+f(1/n_i))^{n_i}\overset{(5.34)}{=}(1+f'(0)/n_i)^{n_i}+\sum_{j=1}^{n_i}\binom{n_i}{j}\left(1+\frac{f'(0)}{n_i}\right)^{n_i-j}\left(\frac{f''(\xi_{n_i})}{2n_i^2}\right)^{j}\overset{i\to\infty}{\longrightarrow}e^{f'(0)}.
$$

In view of the definition of $f(\cdot)$, $e^{f'(0)}=e^{-(\ln s)/c}=(s^{-1})^{\frac{1}{c}}$ completes the proof.

□

The next lemma is the cornerstone needed in order to prove Theorem 5.23.

Lemma 5.26
Let $\sigma\in(0,1)$, $C\geq 1$, $N\in\mathbb{N}_{\geq 2}$, and $m\in\{1,2,\ldots,N-1\}$ be given and define $\sigma_k:=\sqrt[k]{\sigma}$. Then, for $\gamma_{i,k}=C\sum_{n=0}^{i-1}\sigma_k^n$ and a sequence $(k_j)_{j\in\mathbb{N}_0}\subset\mathbb{N}$ satisfying $k_j\to\infty$ for j tending to infinity, the following convergence holds

$$
\prod_{i=mk_j+1}^{k_jN}\frac{\gamma_{i,k_j}}{\gamma_{i,k_j}-1}\longrightarrow\left(\sigma^{-(N-m)}\cdot\frac{1-\sigma^N}{1-\sigma^m}\right)^{\frac{1}{C}}\qquad \textit{for } j\to\infty. \tag{5.35}
$$

Proof: In order to establish the desired convergence, we introduce a discretization parameter $\mu\in\mathbb{N}_{\geq 2}$ which is chosen arbitrarily but fixed. Furthermore, we only permit sequences $(k_j)_{j\in\mathbb{N}_0}\subset\mathbb{N}$ with $k_j\to\infty$ for j tending to infinity such that a sequence $(\tilde{k}_j)_{j\in\mathbb{N}_0}\subset\mathbb{N}$ exists which satisfies $k_j=\mu\tilde{k}_j$ for sufficiently large j, cf. Remark 5.27. This ensures, in particular, that $k_j/\mu\in\mathbb{N}$ holds for sufficiently large index j. Let k denote such a sufficiently large element k_j. Then, we rewrite the term in consideration

$$
\begin{aligned}
\prod_{i=mk+1}^{Nk}\frac{\gamma_{i,k}}{\gamma_{i,k}-1} &= \prod_{\nu=m}^{N-1}\prod_{i=\nu k+1}^{(\nu+1)k}\left(1+\frac{1}{\gamma_{i,k}-1}\right)=\prod_{\nu=m}^{N-1}\prod_{i=1}^{k}\left(1+\frac{1-\sigma^{\frac{1}{k}}}{C\left[1-\sigma^{\nu}\sigma^{\frac{i}{k}}\right]-1+\sigma^{\frac{1}{k}}}\right) \\
&= \prod_{\nu=m}^{N-1}\prod_{l=0}^{\mu-1}\prod_{i=1}^{k/\mu}\left(1+\frac{1-\sigma^{\frac{1}{k}}}{C\left[1-\sigma^{\nu+\frac{l}{\mu}}\sigma^{\frac{i}{k}}\right]-1+\sigma^{\frac{1}{k}}}\right).
\end{aligned}
\tag{5.36}
$$

Since the denominator of the involved factors, i.e. $C(1-\sigma^{\nu+\frac{l}{\mu}}\sigma^{\frac{i}{k}})-(1-\sigma^{\frac{1}{k}})$, is (strictly) greater than $C(1-\sigma^{\nu})-(1-\sigma)\geq(C-1)(1-\sigma)$ and, thus, (strictly) positive, reducing the respective denominators increases the corresponding fractions, i.e. leads to an estimate from above. Bearing this in mind, yields the following chain of inequalities

$$\left(1+\frac{1-\sigma^{\frac{1}{k}}}{C\big[1-\sigma^{\nu+\frac{l+1}{\mu}}\big]-1+\sigma^{\frac{1}{k}}}\right)\leq\left(1+\frac{1-\sigma^{\frac{1}{k}}}{C\big[1-\sigma^{\nu+\frac{l}{\mu}}\sigma^{\frac{i}{k}}\big]-1+\sigma^{\frac{1}{k}}}\right)<\left(1+\frac{1-\sigma^{\frac{1}{k}}}{C\big[1-\sigma^{\nu+\frac{l}{\mu}}\big]-1+\sigma^{\frac{1}{k}}}\right) \tag{5.37}$$

for $i \in \{1, 2, \ldots, k/\mu\}$. Note that the lower and upper estimate does not depend on the control index i which motivates, for $\nu \in \{m, m+1, \ldots, N-1\}$ and $l \in \{0, 1, \ldots, \mu-1\}$, the definitions

$$\begin{aligned}
\overline{g}_{\nu,l}(k) &:= \prod_{i=1}^{k/\mu}\left(1+\frac{1-\sigma^{\frac{1}{k}}}{C\big[1-\sigma^{\nu+\frac{l}{\mu}}\big]-1+\sigma^{\frac{1}{k}}}\right) = \left(1+\frac{1-\sigma^{\frac{1}{k}}}{C\big[1-\sigma^{\nu+\frac{l}{\mu}}\big]-1+\sigma^{\frac{1}{k}}}\right)^{\frac{k}{\mu}}, \\
\underline{g}_{\nu,l}(k) &:= \prod_{i=1}^{k/\mu}\left(1+\frac{1-\sigma^{\frac{1}{k}}}{C\big[1-\sigma^{\nu+\frac{l+1}{\mu}}\big]-1+\sigma^{\frac{1}{k}}}\right) = \left(1+\frac{1-\sigma^{\frac{1}{k}}}{C\big[1-\sigma^{\nu+\frac{l+1}{\mu}}\big]-1+\sigma^{\frac{1}{k}}}\right)^{\frac{k}{\mu}}.
\end{aligned}$$

Carrying out the transformation $k_j = \mu\tilde{k}_j$, i.e. $k_j/\mu = \tilde{k}_j$, for sufficiently large control indices j enables us to apply Lemma 5.25 with $s := \sigma^{1/\mu}$ and $c := C[1-\sigma^{\nu+l/\mu}] \geq 1-\sigma^{\nu} > 1-s$ $(\mu \geq 1)$ to $\overline{g}_{\nu,l}(\cdot)$ which provides

$$\overline{g}_{\nu,l}(k_j) \overset{j\to\infty}{\longrightarrow} \left(s^{-1}\right)^{\frac{1}{c}} = \left(\sigma^{-\frac{1}{\mu}}\right)^{\frac{1}{c}} = \left(\sigma^{-\frac{1}{\mu}}\right)^{\frac{1}{C[1-\sigma^{\nu+l/\mu}]}} =: \overline{g}^{\star}_{\nu,l}.$$

Analogously, the convergence $\underline{g}_{\nu,l}(k_j) \to \left(\sigma^{-\frac{1}{\mu}}\right)^{\frac{1}{C[1-\sigma^{\nu+(l+1)\mu}]}} =: \underline{g}^{\star}_{\nu,l}$ for j approaching infinity follows. Note that these formulas confirm $0 < \underline{g}^{\star}_{\nu,l} < \overline{g}^{\star}_{\nu,l} < \infty$. We emphasize that $\underline{g}^{\star}_{\nu,l}$ as well as $\overline{g}^{\star}_{\nu,l}$ depend explicitly on μ — although this is not reflected by the respective notation. We continue our examination of the auxiliary approximations. To this end, we define

$$\overline{\mathcal{G}}(\mu) := \lim_{j\to\infty}\prod_{\nu=m}^{N-1}\prod_{l=0}^{\mu-1}\overline{g}_{\nu,l}(k_j) \qquad \text{and} \qquad \underline{\mathcal{G}}_{\mu} := \lim_{j\to\infty}\prod_{\nu=m}^{N-1}\prod_{l=0}^{\mu-1}\underline{g}_{\nu,l}(k_j).$$

Note that the above computations ensure that $\overline{\mathcal{G}}(\mu)$ and $\underline{\mathcal{G}}(\mu)$ are located in the open interval $(0, \infty)$. Clearly, choosing a larger discretization parameter μ refines the approximation. Hence, $\overline{\mathcal{G}}(\mu)$ is monotonically decreasing and $\underline{\mathcal{G}}(\mu)$ monotonically increasing with respect to the discretization accuracy μ. Since $\underline{\mathcal{G}}(\mu) < \overline{\mathcal{G}}(\mu)$ holds, this guarantees both existence of the respective limits for μ approaching infinity and the inequality $\lim_{\mu\to\infty}\underline{\mathcal{G}}(\mu) \leq \lim_{\mu\to\infty}\overline{\mathcal{G}}(\mu)$. Indeed, this inequality turns out to be an equality:

$$\frac{\overline{\mathcal{G}}(\mu)}{\underline{\mathcal{G}}(\mu)} = \prod_{\nu=m}^{N-1}\prod_{l=0}^{\mu-1}\frac{\lim_{j\to\infty}\overline{g}_{\nu,l}(k_j)}{\lim_{j\to\infty}\underline{g}_{\nu,l}(k_j)} = \prod_{\nu=m}^{N-1}\prod_{l=0}^{\mu-1}\frac{\left(\sigma^{-\frac{1}{\mu}}\right)^{\frac{1}{C[1-\sigma^{\nu+l/\mu}]}}}{\left(\sigma^{-\frac{1}{\mu}}\right)^{\frac{1}{C[1-\sigma^{\nu+(l+1)\mu}]}}} = \frac{\left(\sigma^{-\frac{1}{\mu}}\right)^{\frac{1}{C(1-\sigma^{m})}}}{\left(\sigma^{-\frac{1}{\mu}}\right)^{\frac{1}{C(1-\sigma^{N})}}} \overset{\mu\to\infty}{\longrightarrow} 1.$$

This allows for drawing conclusions on the limit of the left hand side of (5.35). For μ tending to infinity, each of the two introduced approximations converges to the same finite limit. Hence, it remains to verify that the expression specified in the right hand side of (5.35) equals the limit of the approximations.

To this end, we exploit the specific form of the limits $\overline{g}^{\star}_{\nu,l}$ and $\underline{g}^{\star}_{\nu,l}$. In particular, Lemma 5.25 provides

$$\begin{aligned}
\left(\overline{g}^{\star}_{\nu,l}\right)^{C} &= \left(\sigma^{-\frac{1}{\mu}}\right)^{\frac{1}{1-\sigma^{\nu+l/\mu}}} = \lim_{j\to\infty}\prod_{i=1}^{k_j/\mu}\left(1+\frac{1-\sigma^{1/k_j}}{1-\sigma^{j+l/\mu}-1+\sigma^{1/k_j}}\right) \qquad \text{and} \\
\left(\underline{g}^{\star}_{\nu,l}\right)^{C} &= \left(\sigma^{-\frac{1}{\mu}}\right)^{\frac{1}{1-\sigma^{\nu+(l+1)/\mu}}} = \lim_{j\to\infty}\prod_{i=1}^{k_j/\mu}\left(1+\frac{1-\sigma^{1/k_j}}{1-\sigma^{j+(l+1)/\mu}-1+\sigma^{1/k_j}}\right).
\end{aligned}$$

This allows for elaborating the following chain of inequalities, which resembles the structure of (5.37). Again, we use $k \in (k_j)_{j\in\mathbb{N}_0}$ for a sufficiently large control index j in order to avoid technical difficulties

$$\prod_{i=1}^{k/\mu}\left(1+\frac{1-\sigma^{1/k}}{\sigma^{1/k}-\sigma^{\nu+(l+1)/\mu}}\right) \leq \prod_{i=1}^{k/\mu}\left(1+\frac{1-\sigma^{1/k}}{\sigma^{1/k}-\sigma^{\nu+l/\mu}\sigma^{i/k}}\right) < \prod_{i=1}^{k/\mu}\left(1+\frac{1-\sigma^{1/k}}{\sigma^{1/k}-\sigma^{\nu+l/\mu}}\right).$$

However, in contrast to (5.37), we are able to deal with the term representing the core of this expression using an argument similar to those applied to telescoping series

$$\prod_{i=1}^{k/\mu}\left(1+\frac{1-\sigma^{1/k}}{\sigma^{1/k}-\sigma^{\nu+l/\mu}\sigma^{i/k}}\right) = \prod_{i=1}^{k/\mu}\frac{1-\sigma^{\nu+l/\mu}\sigma^{i/k}}{\sigma^{1/k}\left(1-\sigma^{\nu+l/\mu}\sigma^{(i-1)/k}\right)} = \sigma^{-1/\mu}\cdot\frac{1-\sigma^{\nu+(l+1)/\mu}}{1-\sigma^{\nu+l/\mu}}.$$

Hence, using these preliminary considerations yields

$$\underline{\mathcal{G}}(\mu) \leq \prod_{\nu=m}^{N-1}\prod_{l=0}^{\mu-1}\left[\sigma^{-\frac{1}{\mu}}\frac{1-\sigma^{\nu+\frac{l+1}{\mu}}}{1-\sigma^{\nu+\frac{l}{\mu}}}\right]^{\frac{1}{C}} = \prod_{\nu=m}^{N-1}\left[\sigma^{-1}\frac{1-\sigma^{\nu+1}}{1-\sigma^{\nu}}\right]^{\frac{1}{C}} = \left[\sigma^{-(N-m)}\frac{1-\sigma^{N}}{1-\sigma^{m}}\right]^{\frac{1}{C}} \leq \overline{\mathcal{G}}(\mu).$$

Since $\underline{\mathcal{G}}(\mu) \leq \lim_{\mu\to\infty}\underline{\mathcal{G}}(\mu) = \lim_{\mu\to\infty}\overline{\mathcal{G}}(\mu) \leq \overline{\mathcal{G}}(\mu)$, the respective limits coincide with the deduced bound. Summarizing these computations provide

$$\left[\sigma^{-(N-m)}\frac{1-\sigma^{N}}{1-\sigma^{m}}\right]^{\frac{1}{C}} = \lim_{\mu\to\infty}\underline{\mathcal{G}}(\mu) \leq \lim_{j\to\infty}\prod_{i=k_jm+1}^{k_jN}\frac{\gamma_{i,k_j}}{\gamma_{i,k_j}-1} \leq \lim_{\mu\to\infty}\overline{\mathcal{G}}(\mu) = \left[\sigma^{-(N-m)}\frac{1-\sigma^{N}}{1-\sigma^{m}}\right]^{\frac{1}{C}}$$

and, thus, concludes (5.35), i.e. the assertion.

□

The following remark justifies the simplification which was made in the proof of Lemma 5.26 in order to streamline the presentation.

Remark 5.27
In the proof of Lemma 5.26, the sequence $(k_j)_{j\in\mathbb{N}_0} \subset \mathbb{N}$ was chosen such that the condition $k_j/\mu \in \mathbb{N}$ holds for sufficiently large index j which can be assumed for an iterative refinement process without loss of generality. We emphasize that this assumption is not necessary in order to prove Lemma 5.26 but allows the reader to concentrate on the essential steps without being distracted by technical details. If this condition is violated, the switching index $\mu^\star := k \bmod \mu$ is defined. Then, the product

$$\prod_{l=0}^{\mu-1}\prod_{i=1}^{k/\mu}\left(1+\frac{1-\sigma^{\frac{1}{k}}}{C\left[1-\sigma^{\nu+\frac{l}{\mu}}\sigma^{\frac{i}{k}}\right]-1+\sigma^{\frac{1}{k}}}\right)$$

from (5.36) *is replaced by*

$$\prod_{l=0}^{\mu^\star-1}\prod_{i=1}^{\lceil k/\mu\rceil}\left(1+\frac{1-\sigma^{\frac{1}{k}}}{C\big[1-\sigma^{\nu+\frac{l}{\mu}}\sigma^{\frac{i}{k}}\big]-1+\sigma^{\frac{1}{k}}}\right)\cdot\prod_{l=\mu^\star}^{\mu-1}\prod_{i=1}^{\lfloor k/\mu\rfloor}\left(1+\frac{1-\sigma^{\frac{1}{k}}}{C\big[1-\sigma^{\nu+\frac{l}{\mu}}\sigma^{\frac{i}{k}}\big]-1+\sigma^{\frac{1}{k}}}\right),$$

i.e., the involved factors are distributed such that the number of factors is either $\lfloor k/\mu\rfloor$ *or* $\lceil k/\mu\rceil$ *and the total number of factors is equal to* k*. The following chain of inequalities remains unchanged, only the corresponding index range has to be adapted to the set* $\{1,2,\ldots,\lceil k/\mu\rceil\}$*. The upper index of the product in the definitions of* $\overline{g}_{\nu,l}(k)$ *and* $\underline{g}_{\nu,l}(k)$ *depends on whether or not the index* l *is contained in* $[0,\mu^\star-1]$ *for the considered argument* k*. However, since we are only interested in the limit for* k *approaching infinity, this distinction does not play a role: looking at the proof of Lemma 5.25 shows that the assertion also holds for a sequence* $(n_i)_{i\in\mathbb{N}_0}\subset\mathbb{R}_0^+$ *satisfying* $n_i\to\infty$*, if the exponent is, for each index* i*, randomly substituted by either* $\lfloor n_i\rfloor$ *or* $\lceil n_i\rceil$*. The remaining part of the proof of Lemma 5.26 does not require further modifications.*

5.3 Growth Condition

Although the estimate stated in Theorem 3.18 is strict for the whole class of systems satisfying the assumed controllability condition, cf. Remark 3.13 (i), it may be conservative for subsets of this class. For instance, for sampled-data systems governed by an ordinary differential equation $\dot{x}(t)=g(x(t),\tilde{u}(t))$ the difference between $x(n+1)$ and $x(n)$ is usually of order $\mathcal{O}(T)$ — a continuity property which is not reflected in Assumption 3.2 and, thus, in the optimization problem characterizing our suboptimality bounds. Neglecting this specific characteristic leads to very pessimistic estimates for sampling periods T tending to zero, cf. Theorem 5.15. In order to exploit the mentioned continuity properties, the following growth condition is introduced.

Assumption 5.28 (Growth Condition)
For each $x_0\in\mathbb{X}$ *there exists an admissible control function* $u_{x_0}\in\mathcal{U}=\mathcal{U}^\infty(x_0)$ *satisfying*

$$\ell(x_{u_{x_0}}(n),u_{x_0}(n))\leq L^n\ell^\star(x_0)\qquad\forall\, n\in\mathbb{N}_0. \tag{5.38}$$

Here $L\geq 1$ *denotes the growth bound which typically depends on the sampling period* T*.*

This section is subdivided into two parts:

- Firstly, the growth condition Assumption 5.28 is incorporated in Problem 3.8 and Theorem 3.18 is generalized accordingly. The impact on our performance bounds is investigated for an analytical example.
- Secondly, in Subsection 5.3.4, Assumption 5.28 will be verified for sampled-data systems governed by ordinary differential equations. In particular, estimates on the involved growth bound L are deduced which depend explicitly on the sampling period T and, thus, allow for a refinement process analogously to Section 5.1. We show that Assumption 5.28 provides remedy for the problem which occurred for very fast sampling: in Section 5.1 an iterative refinement process was carried out without adapting the discrete time control horizon $m=1$. The corresponding sequence of suboptimality estimates diverged to minus infinity and, thus, was not applicable in order to guarantee a performance bound or stability. The introduced growth bound counteracts this phenomenon, cf. Theorem 5.37 and the ensuing comments.

5.3.1 Exponential Controllability

At first, the exponentially controllable case without an additional terminal weight is considered, i.e. Assumption 3.2 is supposed to hold with a $\mathcal{KL}_0$-function of type (1.11). To this end, γ_i is defined as

$$\gamma_i := \min\left\{C \cdot \sum_{n=0}^{i-1} \sigma^n, \sum_{n=0}^{i-1} L^n\right\} = \min\left\{\frac{C(1-\sigma^i)}{1-\sigma}, \frac{1-L^i}{1-L}\right\}. \tag{5.39}$$

Definition (5.39) reflects both, i.e. the exponential controllability and the growth condition. Hence, being able to satisfy Assumption 5.28 and, thus, using (5.39) instead of (3.17) yields tighter bounds on the stage costs in Problem 3.8 and, thus, allows to characterize the behavior of the system to be investigated better, cf. Figure 5.9.

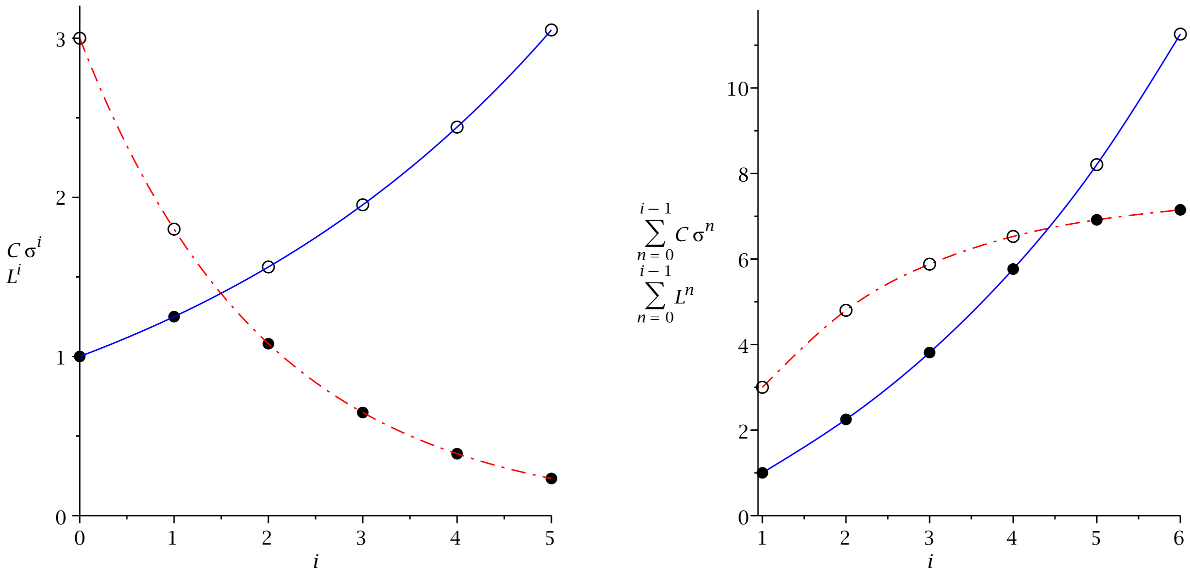


Figure 5.9: Visualization of the bounds induced by our controllability assumption (dashed-dotted line) and our growth condition (solid line) for $C = 3$, $\sigma = 3/5$, and $L = 5/4$. The minimum is marked with solid circles.

Although the observation pointed out in the following lemma does not seem to be exceptionally remarkable, Lemma 5.29 is very useful in order to prove Theorems 5.31 and 5.37.

Lemma 5.29 (Switching index)
Let Assumptions 3.2 and 5.28 based on a $\mathcal{KL}_0$-function of type (1.11) *with overshoot $C \geq 1$, decay rate $\sigma \in (0,1)$, and growth constant $L \geq 1$ hold. If the condition*

$$1 + L \leq C(1+\sigma) \tag{5.40}$$

is satisfied, exactly one switching index $i^\star \in \mathbb{N}_{\geq 2}$ exists such that

$$\gamma_i = \begin{cases} \sum_{n=0}^{i-1} L^n \leq C\sum_{n=0}^{i-1} \sigma^n & \text{for } i \leq i^\star, \\ C\sum_{n=0}^{i-1} \sigma^n < \sum_{n=0}^{i-1} L^n & \text{for } i > i^\star. \end{cases}$$

If Condition (5.40) *is violated, no such switching index $i^\star \in \mathbb{N}_{\geq 2}$ exists.*

Proof: If Condition (5.40) is violated, $\gamma_2 = C(1+\sigma)$ holds which implies

$$L^n \geq L \geq C + C\sigma - 1 \geq C\sigma > C\sigma^n \qquad \text{for } n \in \mathbb{N}_{\geq 2}$$

and, thus, $\sum_{n=0}^{i-1} L^n \leq C \sum_{n=0}^{i-1} \sigma^n$. Hence, no switching index exists.

Suppose that Condition (5.40) is satisfied. Then, $\gamma_2 = 1 + L$ holds. For each index i satisfying the inequality $i \geq C/(1-\sigma)$

$$\sum_{n=0}^{i-1} L^n \geq \sum_{n=0}^{i-1} 1 = i \geq C/(1-\sigma) = C \sum_{n=0}^{\infty} \sigma^n \geq C \sum_{n=0}^{i-1} \sigma^n$$

holds. Hence, $\gamma_i = C \sum_{n=0}^{i-1} \sigma^n$ for each $i \geq C/(1-\sigma)$ which implies the existence of a switching index. Let $i^\star \in \mathbb{N}_{\geq 2}$ denote the smallest switching index. Then, $L^{i^\star} > C\sigma^{i^\star}$ and, thus, $L^i > C\sigma^i$ for all $i \geq i^\star$ hold, i.e. the increments L^i are larger than their counterparts $C\sigma^i$ for $i \geq i^\star$ which shows that no further switching index exists and, thus, that the assertion holds.

□

For instance, the switching index $i^\star$ equals 4 for the parameters $C = 3$, $\sigma = 0.6$, and $L = 1.25$, cf. Figure 5.9. In order to generalize Theorem 3.18 to the setting incorporating the growth condition, the following definition is required.

Definition 5.30 (Equivalent sequence or equivalent $\mathcal{KL}_0$-function)
Let $(\gamma_i)_{i \in \mathbb{N}_{\geq 2}} \subset \mathbb{R}_{\geq 1}$ be a monotone sequence and define $\gamma_1 := 1$. Then, for given optimization horizon $N \geq \mathbb{N}_{\geq 2}$, a $\mathcal{KL}_0$-function $\beta : \mathbb{R}_0^+ \times \mathbb{N}_0 \to \mathbb{R}_0^+$ of type (1.12) *given by*

$$c_0 = 1, \qquad c_n := \gamma_{n+1} - \gamma_n,\ n \in \{1, 2, \ldots, N-1\}, \qquad \text{and} \qquad c_n = 0 \tag{5.41}$$

is called equivalent sequence or equivalent $\mathcal{KL}_0$-function for $(\gamma_i)_{i \in \mathbb{N}_{\geq 2}}$.

Equation (5.39), Lemma 5.29, and Definition 5.30 enable us to extend Theorem 3.18 to the setting incorporating the growth condition. Indeed, except for adapting γ_i, $i = 2, 3, \ldots, N$, Theorem 3.18 maintains exactly its shape.

Theorem 5.31
Let Assumption 3.2 with $\mathcal{KL}_0$-function of type (1.11) *and Assumption 5.28 hold. Furthermore, let an optimization horizon $N \in \mathbb{N}_{\geq 2}$ and a control horizon $m \in \{1, 2, \ldots, N-1\}$ be given. Then, the optimal value $\alpha_{N,m} = \alpha_{N,m}^1$ of Problem 3.8 with γ_i, $i \in \{2, 3, \ldots, N\}$, defined according to* (5.39) *is given by Formula* (3.21).

Proof: If Condition (5.40) is not satisfied, the growth condition does not change γ_i, $i \in \{2, 3, \ldots, N\}$ and the assertion is ensured by Theorem 3.18. Hence, Condition (5.40) is assumed which implies exactly one switching index $i^\star \in \mathbb{N}_{\geq 2}$.

Note that the sequence $(\gamma_i)_{i \in \mathbb{N}_{\geq 2}}$ satisfies the assumptions of Definition 5.30. Since the values c_n, $n \geq N$, do not contribute to γ_i, $i = 2, 3, \ldots, N$, Condition (3.3) from Assumption 3.2 is not needed for $n \geq N$ in order to deduce Problem 3.8 and, consequently, Theorem 3.18, cf. [51]. Hence, using the equivalent $\mathcal{KL}_0$-function from Definition 5.30 and, thus, setting $c_n = 0$ for all $n \geq N$ does not change Problem 3.8 in the setting without an additional terminal weight. As a consequence, our goal is to ensure Condition (1.13) for c_n, $n = 0, 1, \ldots, N-1$, from (5.41) which is sufficient in order to guarantee that Theorem

3.18 provides the optimal value of Problem 3.8 with γ_i, $i \in \{2,3,\ldots,N\}$, from (5.39) and, thus, the assertion.

An equivalent $\mathcal{KL}_0$-function is constructed for the given optimization horizon N. If the switching index $i^\star$ satisfies $i^\star \geq N$, $c_n = L^n$ holds for all $n < N$ and $c_n = 0$ otherwise which corresponds to a $\mathcal{KL}_0$-function of type (1.12) satisfying (1.13). Hence, the assertion is ensured by Theorem 3.18. Consequently, $i^\star < N$ is assumed. Taking Lemma 5.29, its proof, and (5.41) into account yields

$$c_n = \begin{cases} L^n & , n \in \{0,1,\ldots,i^\star - 1\}, \\ C\sum_{i=0}^{i^\star} \sigma^i - \sum_{i=0}^{i^\star - 1} L^i & , n = i^\star, \\ C\sigma^n & , n \in \{i^\star + 1, i^\star + 2, \ldots, N\}. \end{cases}$$

In order to show $c_{n+m} \leq c_n c_m$, three cases are distinguished. Since $c_0 = 1$ holds, $n, m > 0$ can be assumed.

- $n + m < i^\star$: Since $\max\{n,m\} \leq n + m$ holds, $c_{n+m} = L^{n+m} = L^n L^m = c_n c_m$ is implied.
- $n + m = i^\star$: Since $n, m > 0$ holds, $c_{i^\star} \leq L^{n+m} = L^n L^m = c_n c_m$ is ensured by the definition of the switching index $i^\star$.
- $n + m > i^\star$: Taking the proof of Lemma 5.29 into account leads to the inequality $c_{i^\star + j} = C\sigma^{i^\star + j} < L^{i^\star + j}$ for all $j \in \mathbb{N}$. Hence, $n^\star := \max\{n,m\} \geq i^\star$ can be assumed. Furthermore, $m^\star := \min\{n,m\}$ is defined. Taking the definition of the switching index $i^\star$ into account yields $c_{i^\star} \geq C\sigma^{i^\star}$ and, thus, $c_{n^\star} \geq C\sigma^{n^\star}$. Combining this inequality with $c_{m^\star} \geq \sigma^{m^\star}$ implies $c_{n+m} = c_{n^\star + m^\star} = C\sigma^{n^\star + m^\star} \leq c_{m^\star} c_{n^\star} = c_m c_n$, i.e. the assertion.

Hence, Condition (1.13) is ensured for the equivalent sequence of Definition 5.30 which completes the proof.

□

Theorem 5.31 generalizes the key result given in Theorem 3.18 to the setting incorporating the growth condition which allows for reflecting continuity properties of a considered system. The proof of this theorem shows that one may easily check whether Theorem 3.18 is applicable in order to determine the optimal value of Problem 3.8 by constructing an equivalent $\mathcal{KL}_0$-function of type (1.12) and verifying Condition (1.13). This application of finite time controllability allows us to transfer the results with respect to $\mathcal{KL}_0$-functions of type (1.12) satisfying (1.13) deduced in the previous chapters to exponentially controllable ones which gives further reason for the performed, complete symmetry and monotonicity analysis for this setting. Furthermore, the construction of equivalent sequences is not only a theoretical concept but may be used in order to incorporate further estimates in our controllability assumption. This will be shown in the ensuing Section 5.4 in detail.

An alternative proof of Theorem 5.31 is given in Subsection 5.3.5.

5.3.2 Finite Time Controllability

The contribution of this subsection is twofold. On the one hand, the counterpart to Theorem 5.31 is established for control systems satisfying Assumption 3.2 based on a $\mathcal{KL}_0$-function of type (1.12) with $c_0 \geq 1$, $c_1^2 \geq c_2$, and $c_n = 0$ for all $n \in \mathbb{N}_{\geq 3}$, i.e. finite time controllability in at most three steps such that (1.13) holds. On the other hand, we show that a further generalization to arbitrary $\mathcal{KL}_0$-functions satisfying (1.13) is not possible and comment on a remedy which works in a majority of cases. An example dealing with finite time controllability will be investigated in the ensuing subsection.

We begin with extending our results concerning the growth condition to an important subclass of finite time controllable systems.

Theorem 5.32
Let Assumption 3.2 based on a $\mathcal{KL}_0$-function of type (1.12) satisfying (1.13) with $c_n = 0$ for all $n \in \mathbb{N}_{\geq 3}$ and the growth condition, i.e. Assumption 5.28, hold. Furthermore, let an optimization horizon $N \in \mathbb{N}_{\geq 2}$ and a control horizon $m \in \{1, 2, \ldots, N-1\}$ be given. Then, the optimal value $\alpha_{N,m} = \alpha^1_{N,m}$ of Problem 3.8 with γ_i, $i \in \{2, 3, \ldots, N-1\}$, defined according to (5.39) is given by Formula (3.18).

Proof: Since the setting without incorporating an additional weight ω on the final term in the receding horizon cost Functional (2.4) is considered, the distribution of γ_2 on c_0 and c_1 does not play a role. Hence, $c_0 = 1$ and $c_1 = \gamma_2 - 1$ can be assumed because this choice maximizes the range in which c_2 has to be located according to (1.13) ($c_2 \leq c_1^2$). Furthermore, let, without loss of generality, $\gamma_2 = 1 + L < c_0 + c_1$ hold. Otherwise, the growth condition has no impact on γ_i, $i = 2, 3, \ldots, N-1$, and, as a consequence, Theorem 3.18 ensures the assertion.

Then, since $c_n = 0$ for $n \in \mathbb{N}_{\geq 3}$, exactly one switching index $i^\star \in \mathbb{N}_{\geq 2}$ exists such that

$$\gamma_i = \begin{cases} \sum_{n=0}^{i-1} L^n \leq \sum_{n=0}^{i-1} c_n & \text{for } i \leq i^\star, \\ \sum_{n=0}^{i-1} c_n < \sum_{n=0}^{i-1} L^n & \text{for } i > i^\star \end{cases}$$

holds, cf. Lemma 5.29. This enables us to define an equivalent $\mathcal{KL}_0$-function of type (1.12) which exhibits precisely the same γ_i, $i = 2, 3, \ldots, N$, i.e. $c_n := L^n$ for $n \leq i^\star - 1$, $c_{i^\star} = \gamma_{i^\star+1} - \gamma_{i^\star}$, and $c_n = 0$ for $n > i^\star$. Hence, verifying (1.13) for the respective sequence $(c_n)_{n \in \mathbb{N}_0}$ and applying Theorem 3.18 completes the proof. However, since $c_{i^\star} \leq L^{i^\star}$ holds, (1.13) is guaranteed.

□

We continue with the mentioned negative result, which shows that the assertion of the previous theorem is strict with respect to the class of $\mathcal{KL}_0$-functions considered.

Example 5.33
Let the $\mathcal{KL}_0$-function $\beta(\cdot, \cdot)$ of type (1.12) be defined by $c_0 := 1$, $c_1 := 10$, $c_2 := 10$, $c_3 := 100$, and $c_n = 0$ for all $n \in \mathbb{N}_{\geq 4}$. Note that $\beta(\cdot, \cdot)$ exhibits linearity in its first argument and satisfies (1.13). Furthermore, let Assumption 3.2 with $\beta(\cdot, \cdot)$ and the growth condition with growth bound $L = 5$ hold. Then, we obtain

$$\begin{aligned} \gamma_2 &= 1 + L &&= 6 < 11 = c_0 + c_1, \\ \gamma_3 &= c_0 + c_1 + c_2 &&= 21 < 31 = 1 + L + L^2, \\ \gamma_4 &= c_0 + c_1 + c_2 + c_3 &&= 121 < 156 = 1 + L + L^2 + L^3, \end{aligned}$$

and $\gamma_i = \gamma_4$ for $i \geq 5$. We want to establish Theorem 5.31 for optimization horizon $N = 5$ and $m = 1$. To this end, we construct the equivalent $\mathcal{KL}_0$-function of type (1.12) according to Definition 5.30 which is given by $c_0 := 1$, $c_1 := 5$, $c_2 := 15$, $c_3 := 100$, and

$c_n = 0$ for $n \in \mathbb{N}_{\geq 4}$. But, since $c_1 c_2 = 75 < 100 = c_3$ holds, Condition (1.13) *is violated. Hence, an assumption of Theorem 3.18 is not satisfied and, thus, this theorem cannot be applied in order to conclude that the optimal value of Problem 3.8 is given by Formula* (3.21)*.*

In order to further investigate this issue, the alternative proof of Theorem 5.31 is considered, cf. Section 5.3.5 below. This proof shows that Condition (5.48) *has to be satisfied, i.e.*

$$(\gamma_{6-j} - 1) \prod_{i=2}^{5-j} (\gamma_i - 1) - (\gamma_{6-j} - \gamma_{5-j}) \prod_{i=2}^{5-j} \gamma_i \geq 0 \qquad \text{for } j = 1, 2, 3.$$

Evaluating the left hand side yields 1.440.000 *($j = 1$),* -600 *($j = 2$), and* 10 *($j = 3$). Hence, this condition is violated for $j = 2$. Consequently, the solution of the original Problem 3.8 and its relaxed counterpart Problem 3.17 do not coincide.*

In conclusion, Condition (5.48) *has to be checked in order to decide which constraints have to be taken into account in the corresponding optimization Problem 3.8. We point out that Theorem 3.18 nevertheless provides valuable information because the respective formula may still be used as a lower bound for the suboptimality index of Problem 3.8.*

In order to conclude this subsection, another example violating (1.13) and (5.48) for $N = 6$ and $m = 1$ is given in Figure 5.10. In particular, this example which is based on a non monotone $\mathcal{KL}_0$-function satisfying (1.13) exhibits more than one switching index $i^\star$, cf. Lemma 5.29.

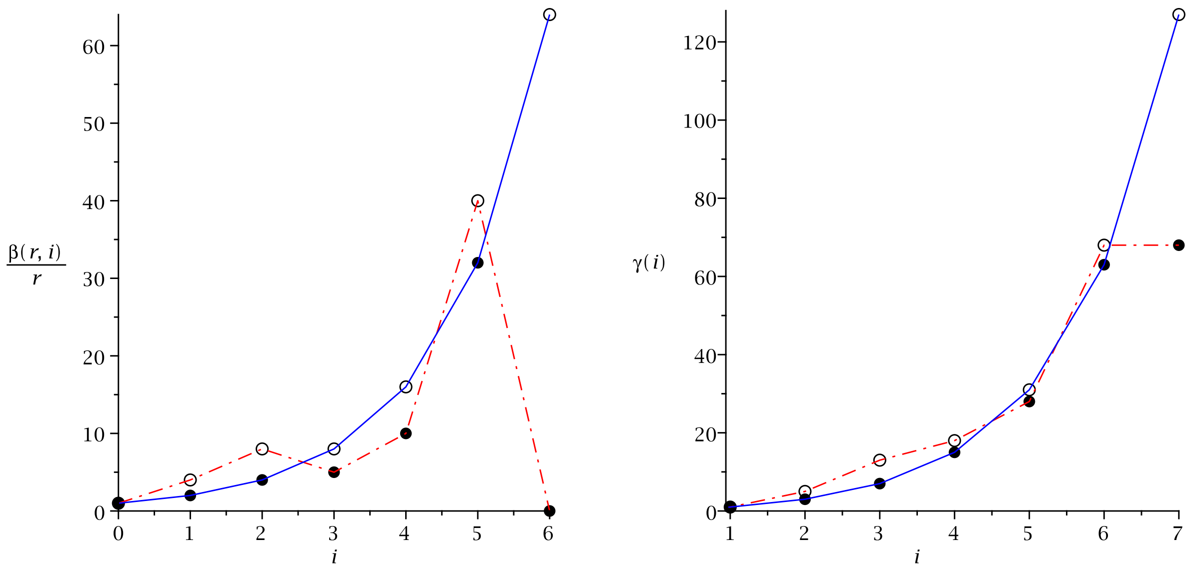


Figure 5.10: Visualization of the bounds induced by Assumption 3.2 (dash-dotted line) and the growth condition (solid line) for $c_0 = 1$, $c_1 = 4$, $c_2 = 8$, $c_3 = 5$, $c_4 = 10$, $c_5 = 40$, $c_i = 0$ for $i \in \mathbb{N}_{\geq 6}$, and growth bound $L = 2$. The minimum is marked with a solid circle each time.

5.3.3 Analytical Example

Here, we focus on quantitative effects caused by incorporating Assumption 5.28 in Problem 3.10 and, thus, in our suboptimality analysis. Since the overshoot C has been proven to be the decisive parameter in this context, cf. Section 4.1 and [39, section 6], we investigate its sensitivity to changes in the growth bound L. To this end, we fix the decay rate $\sigma = 0.7$. Then, our goal is to determine the maximal overshoot C which allows to guarantee our stability condition $\alpha_{N,1} \geq 0$ for the whole class of systems satisfying Assumption 3.2 for a given optimization horizon N. Table 5.2 shows results for two extremal values of L, i.e. completely neglecting our growth condition in comparison to incorporating it with growth constant $L = 1$.

N	C such that $\alpha_{N,1} \geq 0$ $(L = \infty)$	C such that $\alpha_{N,1} \geq 0$ $(L = 1)$	increase (%)
4	1.4028	1.5790	12.56
6	1.6130	2.0397	26.45
8	1.8189	2.5462	39.98
12	2.2208	3.6489	64.30
16	2.6081	4.8128	84.53
24	3.3409	7.1938	115.33

Table 5.2: In this table we give the maximal overshoot C such that the optimal value $\alpha_{N,1}$ of Problem 3.8 is ensured to be positive in dependence on the optimization horizon N for the setting with and without our growth condition. We chose $L = 1$ in order to determine the maximal increase realizable by Assumption 5.28.

Figure 5.11 illustrates that using Assumption 5.28 allows for significantly larger values for C. Furthermore, this figure shows that these findings remain basically the same for suboptimality estimates $\alpha_{N,1} > \overline{\alpha} > 0$, i.e. if we aim at ensuring certain performance specifications for our receding horizon feedback. Hence, Assumption 5.28 allows us to calculate tighter bounds and, thus, characterizes the behavior of the closed loop more accurately. In particular, we like to point out the curve for $N = 8$ in Figure 5.11. Here, the kink marks the upper boundary of the range into which incorporating the growth condition contributes positively to posing the optimization problem and, thus, to deducing stability margins.

The next example demonstrates the interplay between the growth condition and terminal weights. To this end, the following proposition is needed.

Proposition 5.34
Let Assumption 3.2 based on a $\mathcal{KL}_0$-function of type (1.12) *satisfying* (1.13) *with $c_n = 0$ for all $n \in \mathbb{N}_{\geq 2}$ and the growth condition, i.e. Assumption 5.28, hold. Furthermore, let an optimization horizon $N \in \mathbb{N}_{\geq 2}$, a control horizon $m \in \{1, 2, \ldots, N-1\}$, and a terminal weight $\omega \geq 1$ be given. Then, the optimal value $\alpha^{\omega}_{N,m}$ of Problem 3.8 with γ_i, $i \in \{2, 3, \ldots, N-1\}$, defined according to* (5.39) *is given by Formula* (3.18).

Proof: Without loss of generality, $1 + \omega L < C + \omega C \sigma$ is assumed. Otherwise the growth condition does not change γ_i, $i \in \{2, 3, \ldots, N\}$, and, thus, Problem 3.8. Then, an equivalent $\mathcal{KL}_0$-function can be constructed such that γ_i, $i \in \{2, 3, \ldots, N\}$, remain unchanged in comparison to those resulting from Assumptions 3.2 and 5.28. Since $c_n = 0$, $n \in \mathbb{N}_{\geq 2}$, the assertion can be concluded analogously to the proof of Theorem 5.32.

□

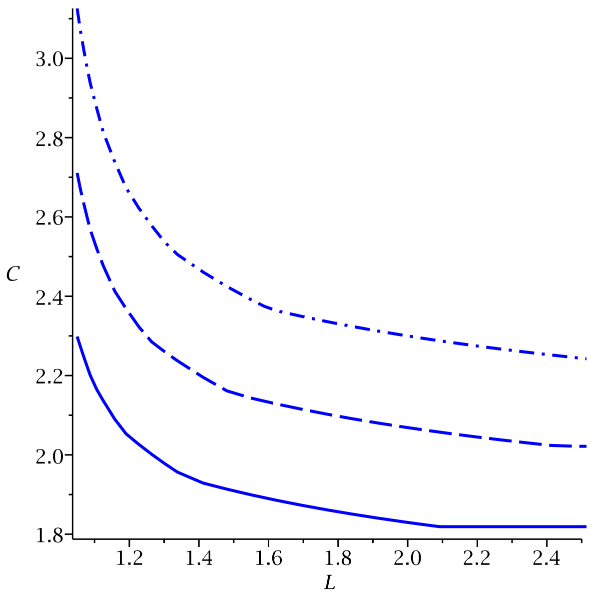


Figure 5.11: Illustration of the corresponding maximal feasible overshoot C which ensures asymptotic stability in dependence on the growth bound L. The larger the optimization horizon N, the larger are the resulting bounds. The solid line stands for $N = 8$, $N = 10$ is represented by the dashed line, and the dash-dotted line illustrates the interplay of the considered parameters for $N = 12$.

In order to simplify the following calculations, we focus on RHC with $m = 1$. Neglecting the growth condition leads to $\gamma_2 = c_0 + \omega c_1$ and $\gamma_i = c_0 + c_1$ for all $i \geq 3$. Hence, for optimization horizon $N \geq 3$, Theorem 3.18 yields

$$\begin{aligned}
\alpha_{2,1}^{\omega} &= (c_0 + \omega c_1)(1 + \omega - c_0 - c_1\omega)/\omega, \\
\alpha_{N,1}^{\omega} &= \frac{\prod_{i=2}^{N} \gamma_i - (\gamma_2 - \omega) \prod_{i=3}^{N} (\gamma_i - 1)\gamma_N}{\prod_{i=2}^{N} \gamma_i - (\gamma_2 - \omega) \prod_{i=3}^{N} (\gamma_i - 1)} \\
&\overset{N \geq 2}{=} \frac{(c_0 + \omega c_1)(c_0 + c_1)^{N-2} - (c_0 + \omega c_1 - \omega)(c_0 + c_1 - 1)^{N-2}(c_0 + c_1)}{(c_0 + \omega c_1)(c_0 + c_1)^{N-2} - (c_0 + \omega c_1 - \omega)(c_0 + c_1 - 1)^{N-2}}.
\end{aligned}$$

We choose $c_0 := 3$ and $c_1 := 2$ and determine the minimal horizon which guarantees our stability condition $\alpha_{N,1}^{\omega} \geq 0$ for an appropriately chosen final weight ω. This is, in turn, equivalent to

$$(c_0 + \omega c_1)(c_0 + c_1)^{N-3} \geq (c_0 + \omega c_1 - \omega)(c_0 + c_1 - 1)^{N-2}$$

for $N \geq 3$ and not possible for $N = 2$ because $c_0 \geq 1$, $c_1 \geq 1$, and $c_0 + c_1 > 2$. Inserting the coefficients c_0, c_1 in the considered inequality yields the necessary condition

$$N \geq 3 + \ln\left(\frac{12 + 4\omega}{3 + 2\omega}\right) / \ln\left(\frac{5}{4}\right) \geq 3 + \ln 2 / \ln(5/4) \approx 6.106 \qquad \text{for all } \omega \geq 1.$$

Hence, the minimal stabilizing horizon resulting from Theorem 3.18 has to necessarily satisfy $N \geq 7$. Without adding a final weight we obtain $N \geq 9$, cf. Figure 5.12 on the left.

Furthermore, the deduced inequality allows us to calculate the minimal additional weight on the final term needed in order to reduce this bound on the optimization horizon

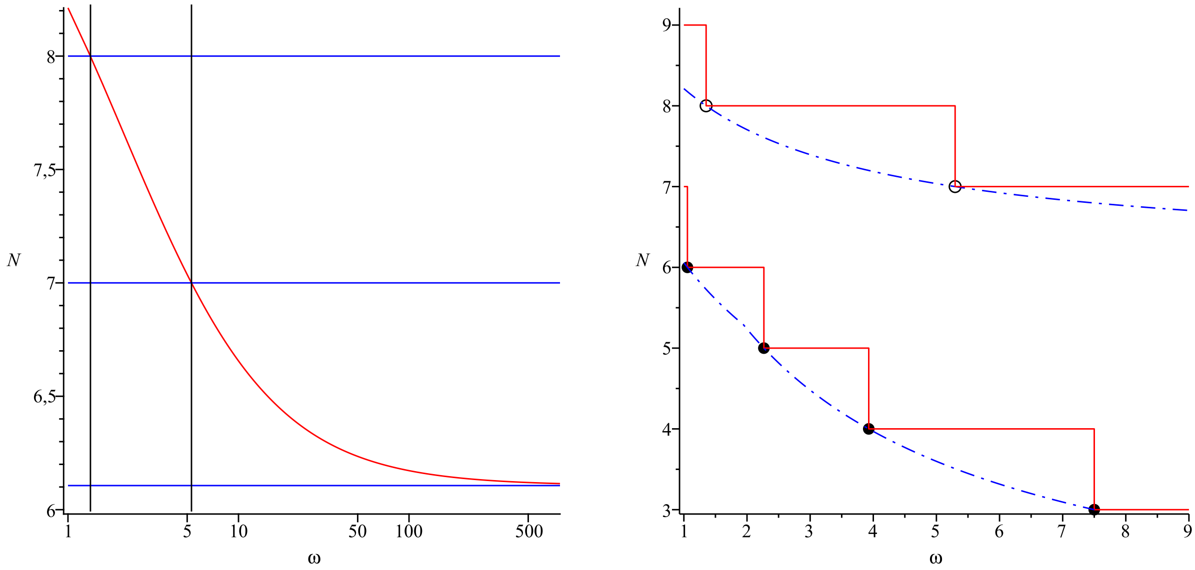


Figure 5.12: Investigation of the influence of incorporating an additional weight on the final term in (2.4) on the minimal stabilizing horizon length N. On the right, the impact of our growth condition on this example is depicted, in addition.

N. Let N be equal to 7. Then, using a final weight $\omega \geq 1197/226 \approx 5.296$ leads to $\alpha^{\omega}_{7,1} \geq 0$ and, thus, ensures that the desired relaxed Lyapunov inequality follows from Theorem 3.18, cf. Table 5.3. As already mentioned, guaranteeing stability via Theorem 3.18 is not possible for smaller N.

	minimal ω such that $\alpha^{\omega}_{N,1} \geq 0$			
N	without growth condition		with growth condition	
3	-	-	15/2	7.500
4	-	-	55/14	3.929
5	-	-	195/86	2.267
6	-	-	545/516	1.056
7	1197/226	5.296	1	1.000
8	971/718	1.352	1	1.000
9	1	1.000	1	1.000

Table 5.3: The table shows the final weights needed in order to ensure $\alpha^{\omega}_{N,1}$ from Theorem 3.18 based on Assumption 3.2 with $\mathcal{KL}_0$-function of type (1.12) given by $c_0 := 3$, $c_1 := 2$, and $c_i = 0$ for all $i \in \mathbb{N}_{\geq 2}$ in its first two columns, which contain the exact and the approximated values for ω. $N = 7$ has turned out to be the minimal stabilizing horizon. Taking, in addition, Assumption 5.28 into account and, thus, using Theorem 5.31 allows for guaranteeing stability for significantly smaller optimization horizons N, e.g. choosing the final weight $\omega = 7.5$ allows us to reduce the horizon to $N = 3$, cf. the third and forth column.

We continue with incorporating our growth condition in the considered setting. To this end, suppose that Assumption 5.28 holds with growth bound $L = 1.2$. Since the assertion of Theorem 5.32 also holds for finite time controllability in at most two steps in combination with terminal weights, a minimal optimization horizon guaranteeing $\alpha_{N,1}^{\omega} \geq 0$ can be computed by Formula (3.21) with γ_i, $i \in \{2, 3, \ldots, N\}$ from (5.39), i.e.

$$\gamma_2 = 1 + 1.2\omega, \qquad \gamma_3 = \min\{5, 2.2 + 1.44\omega\}, \qquad \text{and} \qquad \gamma_i = 5 \quad \text{for } i \geq 4,$$

cf. the third and forth column of Table 5.3. Hence, stability of the resulting receding horizon closed loop is ensured for optimization horizons $N < 7$. In particular, for $\omega \geq 7.5$, using $N = 3$ is sufficient in order to conclude stability. Summarizing, we see that using the growth condition leads to significantly better suboptimality estimates.

5.3.4 Growth Condition and Discretizations

In the introduction of this section the growth condition Assumption 5.28 was motivated by continuity properties of sampled-data systems governed by ordinary differential equations

$$\dot{x}(t) = g(x(t), \tilde{u}(t)) \qquad \text{with} \qquad x(0) = x_0$$

with sampling period $T > 0$. In this subsection Assumption 5.28 is verified for this class of systems. In particular, for given sampling period T, estimates on the growth bound L are deduced. This allows to analyze the impact of our growth condition on an iterative refinement process, cf. Section 5.1. Theorem 5.15 showed that arbitrarily fast sampling leads to negative and, thus, useless performance bounds $\alpha_{N,m} = \alpha_{N,m}^1$ for $m = 1$ — a problem which can be resolved by taking the growth condition into account.

In order to avoid technical difficulties, state constraints are not considered in this subsection, i.e. $\mathbb{X} = X = \mathbb{R}^n$. Furthermore, the control constraints are modeled by a compact connected set $\mathbb{W} \subseteq \mathbb{R}^m$ containing the origin in its interior. Hence, the space $\mathbb{U}$ of control values for the discrete time system is the set $\{\tilde{u}(\cdot) \in \mathcal{L}^1([0, T), \mathbb{W})\}$ and each element $u \in \mathbb{U}$ is admissible — independently of the given state. Then, w.l.o.g., $g(x^\star, 0) = 0$ and $\ell(x^\star, 0_{\mathbb{U}}) = 0$ are assumed. Here $0_{\mathbb{U}}$ denotes the $\mathcal{L}^1([0, T), \mathbb{W})$-function satisfying $\tilde{u}(t) = 0$ for all $t \in [0, T)$. The following two types of stage costs $\ell : X \times U \to \mathbb{R}_0^+$ are considered:

(1) stage costs which evaluate state and control separately, i.e.

$$\ell(x, u) = T\ell_x(\|x - x^\star\|) + \ell_u(u) \tag{5.42}$$

with continuous functions $\ell_x : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $\ell_u : U \to \mathbb{R}_0^+$. Furthermore, let $\ell_x(\cdot)$ satisfy the property

$$\ell_x(ar) \leq a^d \ell_x(r) \qquad \text{for all } a \in \mathbb{R}_{\geq 1} \text{ with } d \in \mathbb{R}_{>0}. \tag{5.43}$$

Since solely the initial state is taken into account, the computation of $\ell^\star(x_0)$ corresponds to minimizing the control effort. Taking $\ell(x^\star, 0_{\mathbb{U}}) = 0$ into account leads to $\ell_u(0_{\mathbb{U}}) = 0$ and, thus, implies $\ell^\star(x_0) = \ell(x_0, 0_{\mathbb{U}})$ — an important property for the upcoming investigation.

(2) stage costs which are defined by

$$\ell(x, u) := \int_0^T \|\varphi(t; x, \tilde{u}) - x^\star\|^2 dt + \lambda \int_0^T \|\tilde{u}(t)\|^2 dt. \tag{5.44}$$

with $u = \tilde{u}(\cdot)|_{[0,T)}$. The state trajectory, which is influenced by the chosen control value u, on the whole sampling interval is considered. Hence, in contrast to (1), $\ell^\star(x_0) < \ell(x_0, 0_{\mathbb{U}})$ is not excluded and $\ell^\star(x_0) = V_1(x_0)$ has to be computed.

The standard assumption in order to guarantee existence and uniqueness of the solution $\Phi(\cdot; x, \tilde{u})$ is a Lipschitz condition with Lipschitz constant $L_g > 0$ on $g(\cdot, \cdot)$ with respect to its first argument, i.e.

$$\|g(x_1, u) - g(x_2, u)\| \leq L_g \|x_1 - x_2\| \qquad \text{for all } x_i \in \mathbb{X},\ i \in \{1, 2\}, \tag{5.45}$$

cf. [77, chapter XIV, §3].[8]

Our first goal in this subsection is to deduce Assumption 5.28 for stage cost $\ell(\cdot, \cdot)$ given in (1). Since $\ell^\star(x_0) = \ell(x_0, 0_\mathbb{U})$ holds, i.e. the minimum is attained in $u = 0_\mathbb{U}$, $\tilde{u} \equiv 0$ is defined. Then, taking (5.45) and $g(x^\star, 0) = 0$ into account, using the triangle inequality yields

$$\begin{aligned} \|\Phi(t; x_0, \tilde{u}) - x^\star\| &\leq \|x_0 - x^\star\| + \int_0^t \|g(\Phi(s; x_0, \tilde{u}), 0) - g(x^\star, 0)\|\, ds \\ &\leq \|x_0 - x^\star\| + \int_0^t L_g \|\Phi(s; x_0, \tilde{u}) - x^\star\| ds. \end{aligned}$$

Hence, using Gronwall's inequality implies the estimate

$$\|\Phi(t; x_0, \tilde{u}) - x^\star\| \leq e^{L_g t} \|x_0 - x^\star\|,$$

cf. [19], which enables us to conclude

$$\begin{aligned} \ell(\Phi(nT; x_0, \tilde{u}), \tilde{u}(\cdot)|_{[nT,(n+1)T)}) &= T\ell_x(\|\Phi(nT; x_0, \tilde{u}) - x^\star\|) + f_u(\tilde{u}(\cdot)_{[nT,(n+1)T)}) \\ &\overset{(5.43)}{\leq} T(e^{dL_g T})^n \|x_0 - x^\star\| + 0 = L^n \ell^\star(x_0) \end{aligned}$$

with $L := e^{dL_g T}$.

Remark 5.35
The growth bound $L = e^{dL_g T}$ converges to one as the sampling period T tends to zero. Hence, for sufficiently fast sampling and $C > 1$, Condition (5.40) is ensured which implies that γ_i from (5.39) coincides with $\sum_{n=0}^{i-1} L^n$ — at least for small i, cf. Figure 5.9. In conclusion, the introduced growth condition provides tighter estimates for the behavior of the considered system and, thus, compensates conservatism caused by the overshoot constant C for exponentially controllable systems.

Remark 5.35 points out a key observation which explains why our growth condition will resolve problems occurring for fast sampling — independently of the exact growth bound L, cf. Theorem 5.37.

For instance, the growth condition is ensured for cost functions with $f_x(r) := \sum_{i=0}^{k} c_i r^{d_i}$ with coefficients $c_i > 0$ and exponents $d_i \in \mathbb{R}_{>0}$, $i = 0, 1, \ldots, k$, as well as arbitrary functions $f_u : \mathbb{U} \to \mathbb{R}_0^+$ with $f(0_\mathbb{U}) = 0$. This includes cost functions which are not differentiable at the origin, e.g. by choosing $c_0 = 1$ and $e_0 = 1/2$. Furthermore, note that the involved norm $\|\cdot\|$ with respect to the state was not specified. Hence, employing an arbitrary positive definite weighting matrix Q in order to define, e.g. $\ell(x, u) = Tx^T Qx + \ell_u(u)$ is possible. A typical representative of cost functions of type (5.42) satisfying (5.43) is

$$\ell(x, u) = T\|x\|^2 + \lambda \int_0^T \|\tilde{u}(t)\|^2\, dt = T\|x\|^2 + \lambda T \|u\|^2$$

[8] Here, global Lipschitz continuity is assumed in order to avoid technical difficulties. The results of this subsection can be deduced for local Lipschitz continuity analogously.

with regularization parameter $\lambda \geq 0$.

Next, we want to establish the growth condition for cost functions of type (2). Here, a control value satisfying $\ell(x_0, u^\star) = \ell^\star(x_0)$ is denoted by $u^\star$. Existence of $u^\star$ is ensured by Assumption 2.4. In contrast to that, uniqueness of $u^\star$ is not required. Note that, in general, $u^\star \neq 0$ holds. Here, the sequence of control values $(u(n))_{n\in\mathbb{N}_0}$ defined by $u(0) = u^\star$ and $u(n) = 0_{\mathbb{U}}$, $n \in \mathbb{N}$ is employed which is, in general, not optimal. The first element of this sequence has to be chosen in this manner, since otherwise a verification of Assumption 5.28 for $n = 0$ is impossible.

Here, sampled-data systems with zero order hold are considered, cf. Definition 1.20. The following proposition may be, however, analogously shown without the zero order hold assumption. But using zero order hold allows to calculate some integrals exactly and, thus, to deduce smaller growth bounds L. The characteristic property $L \to 1$ for sampling periods T tending to zero does not depend on the zero order hold assumption.

Proposition 5.36
Let a sampled-data system with zero order hold induced by the ordinary differential equation $\dot{x}(t) = g(x(t), \tilde{u}(t))$ and a sampling period $T > 0$ be given. In addition, suppose that $g(\cdot,\cdot)$ satisfies the Lipschitz condition (5.45) *with Lipschitz constants L_g and L_u with respect to its first and second argument, respectively. Furthermore, let the cost function $\ell : X \times \mathbb{U} \to \mathbb{R}_0^+$ be given by* (5.44) *and $\ell(x_0, u^\star) = \ell^\star(x_0)$ hold. Then, the growth condition, i.e. Assumption 5.28 is satisfied with $L := \max\{c_s, c_u\}$ with*

$$\begin{aligned}
c_s &= e^{2L_gT} + e^{L_gT} L_u L_g^{-1}(e^{L_gT} - 1),\\
c_u &= \lambda^{-1}\left(\left[\frac{2L_gTe^{2L_gT} - 3e^{2L_gT} + 4e^{L_gT} - 1}{2L_gT}\right] + \left[\frac{L_gTe^{L_gT} - e^{L_gT} + 1}{L_gT}\right]\right).
\end{aligned}$$

Proof: The following calculation is carried out in order to apply a more sophisticated version of Gronwall's inequality which takes the impact of the chosen control into account. Using the triangle inequality and the Lipschitz condition of $g(\cdot,\cdot)$ in its first argument yields

$$\begin{aligned}
\|\Phi(t; x_0, \tilde{u}) - x^\star\| &\leq \|x_0 - x^\star\| + \int_0^t \|g(x^\star, \tilde{u}(s)) + g(\Phi(s; x_0, \tilde{u}), \tilde{u}(s)) - g(x^\star, \tilde{u}(s))\|\, ds\\
&\leq \|x_0 - x^\star\| + \int_0^t \|g(x^\star, \tilde{u}(s))\|\, ds + L_g \int_0^t \|\Phi(s; x_0, \tilde{u}) - x^\star\|\, ds.
\end{aligned}$$

Defining $K := L_g$ as well as the functions $f(t) := \|\Phi(t; x_0, \tilde{u}) - x^\star\| \in \mathcal{C}([0,a], \mathbb{R}_0^+)$ and $g(t) := \|x_0 - x^\star\| + \int_0^t \|g(x^\star, \tilde{u}(s))\|\, ds \in \mathcal{C}([0,a], \mathbb{R}_0^+)$ for a sufficiently large $a > 0$, allows to apply [57, Corollary 1 of Theorem 1.5.7] providing

$$\|\Phi(t; x_0, \tilde{u}) - x^\star\| = f(t) \leq g(t) + L_g \int_0^t e^{L_g(t-s)} g(s)\, ds.$$

Suppose that $\tilde{u}(\cdot)$ is constant on the interval $[0, t)$, i.e. $\tilde{u}(s) = \bar{u}$ for all $s \in [0, t)$. Then, $\int_0^t \|g(x^\star, \tilde{u}(s))\|\, ds$ equals $t\,\|g(x^\star, \bar{u})\|$. Since $g(\cdot,\cdot)$ fulfills the Lipschitz condition w.r.t. its second argument, we obtain $\|g(x^\star, \bar{u})\| = \|g(x^\star, \bar{u}) - g(x^\star, 0) + 0\| \leq L_u\|\bar{u}\|$. Taking these computations, $\int_0^t e^{L_g(t-s)}\, ds = (e^{L_gt} - 1)/L_g$, and $\int_0^t s\, e^{L_g(t-s)}\, ds = (e^{L_gt} - tL_g - 1)/L_g^2$ into account yields

$$\|\Phi(t; x_0, \tilde{u}) - x^\star\| \leq \|x_0 - x^\star\| + tL_u\bar{u} + L_g \int_0^t e^{L_g(t-s)} \Big[\|x_0 - x^\star\| + sL_u\|\bar{u}\|\Big]\, ds$$

$$= \quad \|x_0 - x^\star\| \, e^{L_g t} + L_u L_g^{-1}(e^{L_g t} - 1)\|\bar{u}\|. \tag{5.46}$$

Having completed these preliminary calculations, the sequence of control values $u_{x_0} = (u(n))_{n\in\mathbb{N}_0}$ to be applied is defined by $u(0) = u^\star$ and $u(n) = 0$ for $n \in \mathbb{N}$. Here, since $\tilde{u}(\cdot) : \mathbb{R}_0^+ \to \mathbb{U}$ is constant on each sampling period, i.e. $\tilde{u}(t) = u(n)$ for all $t \in [nT, (n+1)T)$, $u(n) = 0$ has to be interpreted as $\tilde{u}(t) = 0$ for all $t \in [nT, (n+1)T)$. Now, we focus on deducing an estimate for $\ell(x_{u_{x_0}}(1), u(1)) = \ell(x_{u_{x_0}}(1), 0) = \int_T^{2T} \|\varphi(t; x_0, \tilde{u}) - x^\star\|^2 \, dt$. Using Gronwall's-inequality (5.46), at first with $\bar{u} = 0$ and, then, with $\bar{u} = u^\star$ yields

$$\begin{aligned}
\ell(x_{u_{x_0}}(1), u(1)) &= \int_0^T \|\Phi(T + t; x_0, \tilde{u}) - x^\star\| \\
&\le \int_0^T e^{2L_g t}\|\Phi(T; x_0, \tilde{u}) - x^\star\|^2 \, dt \\
&= \int_0^T e^{2L_g t}\|\Phi(T - t; \Phi(t; x_0, \tilde{u}), \tilde{u}(t + \cdot)) - x^\star\|^2 \, dt \\
&\le \int_0^T e^{2L_g t}\Big[e^{L_g(T-t)}\|\Phi(t; x_0, \tilde{u}) - x^\star\| + L_u L_g^{-1}(e^{L_g(T-t)} - 1)\|u^\star\|\Big]^2 \, dt.
\end{aligned}$$

Using the Cauchy-Schwarz inequality $2\|\varphi(t; x_0, \tilde{u}) - x^\star\|\|u^\star\| \le \|\varphi(t; x_0, \tilde{u}) - x^\star\|^2 + \|u^\star\|^2$ in order to resolve the term in brackets and, then, applying

$$\begin{aligned}
\int_0^T (e^{L_g T} - e^{L_g t})^2 dt &= T\left[\frac{2L_g T e^{2L_g T} - 3e^{2L_g T} + 4e^{L_g T} - 1}{2L_g T}\right] \qquad \text{and} \\
\int_0^T (e^{L_g T} - e^{L_g t}) dt &= T\left[\frac{L_g T e^{L_g T} - e^{L_g T} + 1}{L_g T}\right],
\end{aligned}$$

leads to

$$\begin{aligned}
\ell(x_{u_{x_0}}(1), u(1)) &\le \int_0^T \Big[e^{2L_g T} + e^{L_g T} L_u L_g^{-1}(e^{L_g T} - e^{L_g t})\Big]\|\Phi(t; x_0, \tilde{u}) - x^\star\|^2 \\
&\qquad + \Big[e^{L_g T} L_u L_g^{-1}(e^{L_g T} - e^{L_g t}) + L_u^2 L_g^{-2}(e^{L_g T} - e^{L_g t})^2\Big]\|u^\star\|^2 \, dt \\
&\le \max\{c_s, c_u\}\left(\int_0^T \|\Phi(t; x_0, \tilde{u}) - x^\star\|^2 dt + \lambda \int_0^T \|u^\star\|^2 dt\right).
\end{aligned}$$

Since $e^{2L_g T} \le c_s \le \max\{c_s, c_u\} = L$ and $u(n) = 0$ for all $n \in \mathbb{N}$, this ensures

$$\ell(x_{u_{x_0}}(n), u(n)) = \ell(x_{u_{x_0}}(n), 0) \le L^{n-1}\ell(x_{u_{x_0}}(1), u(1)) \le L^n \ell^\star(x_0)$$

and, thus, the growth condition for cost functions defined according to (5.44), i.e. the assertion.

$\square$

The convergence of the growth bound L to one for $T \to 0$ is ensured by Proposition 5.36.[9] Hence, the growth bound is close to one for sufficiently fast sampling. Assuming the Lipschitz condition for $g(\cdot, \cdot)$ in its second argument is, e.g. for control affine systems, automatically satisfied and typically not a restrictive assumption. Furthermore, note that

[9] Applying l'Hôpital's rule shows that $c_u \to 1$ for T tending to zero, cf. [124, Subsection 5.4.4].

using the sequence of control values specified in Proposition 5.36 has led to an improved growth constant in contrast to the earlier version of this result published in [50].

Summarizing, Assumption 5.28 is ensured for sampled-data systems governed by ordinary differential equations. The deduced estimates are in particular useful for small sampling periods.

Next, infinite dimensional systems are considered. This is motivated by sampled-data systems induced by partial differential equations with linear operators which allow for a wide range of applications, cf. Section 3.4 and the Chaffee-Infante equation from the introduction of this chapter. Typically, these operators are — in contrast to the finite dimensional case — unbounded, cf. [98].

In order to establish Assumption 5.28 for this setting we choose $\tilde{u} \equiv 0$. Then [98, Theorem 1.2.2] provides the estimate

$$\|S(t)\| \leq Me^{\omega t}, \qquad 0 \leq t < \infty$$

with $\omega \geq 0$, $M \geq 1$ for the $\mathcal{C}_0$-semigroup $S(\cdot)$ whose infinitesimal generator is the linear operator corresponding to the considered PDE. For the corresponding sampled-data system with sampling period $T > 0$ and cost function $\frac{1}{2}\|x\|^2 + \lambda\|u\|^2$, this yields Assumption 5.28 with $M^2 e^{2\omega T}$. Note that this constant does not necessarily converge to one for a sampling period tending to zero, cf. [24, Example 5.7 (iii), p.40]. Nevertheless, the resulting growth condition may tighten the estimate from Theorem 3.18.

In Chapter 4 and Sections 5.1, 5.2 Theorem 3.18 was exploited in order to investigate the performance bounds deduced in Section 3.1. The performed analysis has proven to be fruitful in order to recognize patterns which, e.g., motivated the development of enhanced algorithms, cf. Section 4.4. However, for receding horizon control with $m = 1$ a problem with very fast sampling was observed, cf. Theorem 5.15. In the following theorem, we demonstrate that the growth condition resolves this problem, cf. Figure 5.13.

Theorem 5.37
Let Assumption 3.2 with $\mathcal{KL}$-function of type (1.11) *and parameters $C = 2$, $\sigma = 0.5$ and Assumption 5.28 with growth bound $L = 2$ be satisfied. Furthermore, let the optimization horizon $N = 8$ and the discretization sequence $(k_j)_{j \in \mathbb{N}_0}$, $k_j := 2^j$, be given. Then, combining* (5.10) *and* (5.39) *in order to define $\gamma_{i,k}$ appropriately, i.e.*

$$\gamma_{i,k} := \min \left\{ C \sum_{n=0}^{i-1} \left(\sigma^{1/k}\right)^n, \ \sum_{n=0}^{i-1} \left(L^{1/k}\right)^n \right\} = \min \left\{ \frac{C(1-\sigma^{i/k})}{1-\sigma^{1/k}}, \ \frac{1-L^{i/k}}{1-L^{1/k}} \right\},$$

the corresponding sequence $(\alpha_{k_j N,1}(k_j))_{j \in \mathbb{N}_0}$ of optimal values satisfies

$$\alpha_{k_j N,1}(k_j) = 1 - \frac{(\gamma_{k_j N, k_j} - 1) \prod_{i=2}^{k_j N} (\gamma_{i,k_j} - 1)}{\prod_{i=2}^{k_j N} \gamma_{i,k_j} - \prod_{i=2}^{k_j N} (\gamma_{i,k_j} - 1)} \geq 0, \qquad \forall\ k \in \mathbb{N}_0, \tag{5.47}$$

i.e. Theorem 5.31 ensures a positive performance index for arbitrary fast sampling with $m = 1$ and, thus, enables us to apply our results in the sampled-data setting.

Proof: At first, we show that the switching index $i_j^\star$ from Lemma 5.29, which depends on the discretization parameter j, fulfills $i_j^\star = k_j + 1$. Since $(\sigma^{1/k_j})^{-1} = L^{1/k_j}$ holds for all $j \in \mathbb{N}_0$, we obtain $L^{n/k_j}\sigma^{-(i_j^\star - 1 - n)/k_j} = L^{n/k_j} L^{(k_j - n)/k_j} = L = C$ for $n = 0, \ldots, k_j$ and, thus, $C\sigma^{(i_j^\star - 1 - n)/k_j} = L^{n/k_j}$. Taking this equation into account yields $C \sum_{n=0}^{i_j^\star - 1} \sigma^{n/k_j} =$

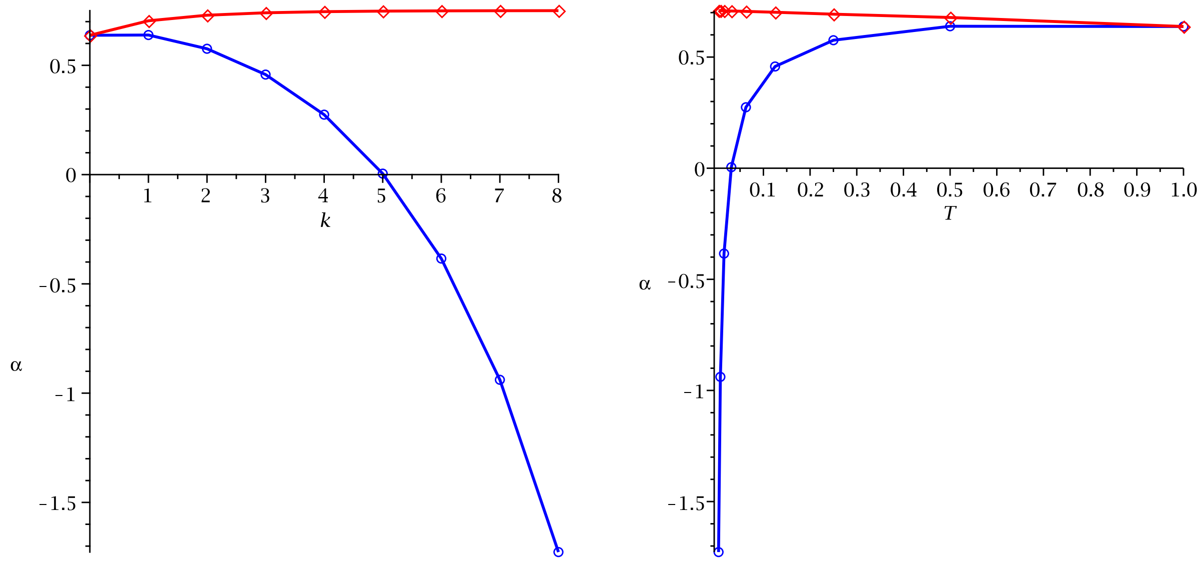


Figure 5.13: The figure on the left illustrates the sequence of suboptimality estimates $(\alpha_{k_jN,1}(k_j))_{j\in\mathbb{N}_0}$ corresponding to the discretization sequence $(k_j)_{j\in\mathbb{N}_0}$, $k_j := 2^j$, cf. Theorems 5.15 and 5.31. The blue line ($\circ$) corresponds to the one depicted in Figure 5.6 which is assigned to the setting based solely on our controllability condition Assumption 3.2. In contrast to that, the other trajectory ($\diamond$) takes the growth condition Assumption 5.28 with growth constant $L = 2$ into account. On the right we visualized exactly the same situation, however, in dependence on the sampling period T.

$C\sum_{n=0}^{k_j}\sigma^{(i_j^\star-1-n)/k_j} = \sum_{n=0}^{i_j^\star} L_k^n$ and the assertion is, consequently, verified for $i_j^\star$. Hence, (5.47) is equivalent to

$$\gamma_{k_jN,k_j}\prod_{i=2}^{i_j^\star}\gamma_{i,k_j}\prod_{i=i_j^\star+1}^{k_jN-1}\gamma_{i,k_j} = \prod_{i=2}^{k_jN}\gamma_{i,k_j} \geq \gamma_{k_jN,k_j}\prod_{i=2}^{k_jN}(\gamma_{i,k_j}-1) = \gamma_{k_jN,k_j}\prod_{i=2}^{i_j^\star}(\gamma_{i,k_j}-1)\prod_{i=i_j^\star+1}^{k_jN}(\gamma_{i,k_j}-1).$$

Dividing this inequality by γ_{k_jN,k_j}, inserting the definition of γ_i depending on the switching index $i_j^\star$, and noting that the divisors $(L^{1/k_j}-1)^{i_j^\star-1}$ may be canceled out, yields

$$\prod_{i=2}^{i_j^\star}(L^{i/k_j}-1)\prod_{i=i_j^\star+1}^{k_jN-1}\frac{C(1-\sigma^{i/k_j})}{1-\sigma^{1/k_j}} \geq \prod_{i=2}^{i_j^\star}L^{1/k_j}(L^{(i-1)/k_j}-1)\prod_{i=i_j^\star+1}^{k_jN}\frac{C(1-\sigma^{i/k_j})-(1-\sigma^{1/k_j})}{1-\sigma^{1/k_j}}.$$

Plugging in $C = 2$ and using the estimate $1-2\sigma^{i/k_j}+\sigma^{1/k_j} \leq (1+\sigma^{1/k_j})(1-\sigma^{i/k_j})$, establishing the following inequality suffices in order to ensure the assertion

$$(L^{i_j^\star/k_j}-1)\prod_{i=i_j^\star+1}^{k_jN-1}2(1-\sigma^{i/k_j}) \geq L^{(i_j^\star-1)/k_j}\frac{(L^{1/k_j}-1)}{1-\sigma^{1/k_j}}\prod_{i=i_j^\star+1}^{k_jN}(1+\sigma^{1/k_j})(1-\sigma^{i/k_j}).$$

In consideration of $(L^{1/k_j}-1)/(1-\sigma^{1/k_j}) = L^{1/k_j} = \sigma^{-1/k_j}$ and $i_j^\star = k_j+1$, this condition is transformable to

$$(LL^{1/k_j}-1)\sigma^{1/k_j}\prod_{i=k_j+2}^{k_jN}\frac{2}{1+\sigma^{1/k_j}} \geq 2L(1-\sigma^N)$$

which, in turn, using the estimates $(1-\sigma^N) < 1$ and $k_jN - i_j^\star = 7k_j - 1 > 6k_j$, $LC = 4$, may be ensured by proving

$$(2-\sigma^{1/k_j})\left(\frac{2}{1+\sigma^{1/k_j}}\right)^{6k_j} > \left(\frac{2}{1+\sigma^{1/k_j}}\right)^{6k_j} \geq 4.$$

To this end, showing the second inequality for $j = 0$ and deducing monotonicity of $\left(2/(1+\sigma^{1/k_j})\right)^{k_j}$ with respect to j completes the proof. Firstly, we deal with $j = 0$. Here, $k_j = 2^j = 1$ implies $\left(2/(1+\sigma^{1/k_j})\right)^{6k_j} = (4/3)^6 = 4(4/3)(4^4/3^5) > 4(4/3) > 4$ and, thus, ensures the assertion. Taking $k_{j+1} = 2k_j$ into account, establishing

$$\left(\frac{2}{1+\sqrt{\sigma^{1/k_j}}}\right)^2 \geq \frac{2}{1+\sigma^{1/k_j}}$$

or, equivalently, $2 + 2\sigma^{1/k_j} \geq (1+\sqrt{\sigma^{1/k_j}})^2 = 1 + 2\sqrt{\sigma^{1/k_j}} + \sigma^{1/k_j}$ is sufficient in order to verify the claimed monotonicity. Hence, completing the square provides the assertion.

□

Arbitrarily fast sampling and, thus, employing a very fine discretization led, as observed in Figure 5.6 and rigorously proven in Theorem 5.15, to negative suboptimality bounds. Theorem 5.37 ensures positive performance estimates and, consequently, resolves this problem by incorporating the introduced growth condition Assumption 5.28 — but only for very special parameters.

Indeed, Theorem 5.15 shows that the sequence of suboptimality estimates corresponding to an iterative refinement process decreases unboundedly. Hence, the assertion of Theorem 5.37 consists of two parts: firstly the existence of a lower bound is ensured and, secondly, positivity of this bound is shown. In order to further investigate this issue, Figure 5.14 is considered which depicts performance bounds for very large growth bounds L. All curves reflecting the growth condition exhibit a lower bound but whether a positive suboptimality index and, thus, stability can be ensured depends on the chosen parameters — like in the setting solely based on our controllability Assumption 3.2.

5.3.5 Alternative Proof of Theorem 5.31

The presented proof of Theorem 5.31 is essentially based on the construction of an equivalent $\mathcal{KL}_0$-function according to Definition 5.30. A generalization of this proof technique to the setting with a terminal weight $\omega > 1$ seems to be difficult. To this end, the following proof, which generalizes the technique employed in order to prove Theorem 3.18 to γ_i, $i \in \{2, 3, \ldots, N\}$ defined by (5.39), may be helpful.

Proof: [Alternative proof of Theorem 5.31] The starting point of this proof is the same as the one of Theorem 3.18 except for the changed definition of γ_i, $i = 2, 3, \ldots, N$. This modification does not affect the part of the corresponding proof in which (3.21) is established as optimal value of the relaxed Problem 3.17. However, we still have to validate the counterpart to (3.27), i.e. the same inequality based on the adapted definition of γ_i, in order to ensure that (3.21) yields the solution of Problem 3.8. Repeating the arguments applied to (3.27) in the proof of Theorem 3.18 shows that checking the following

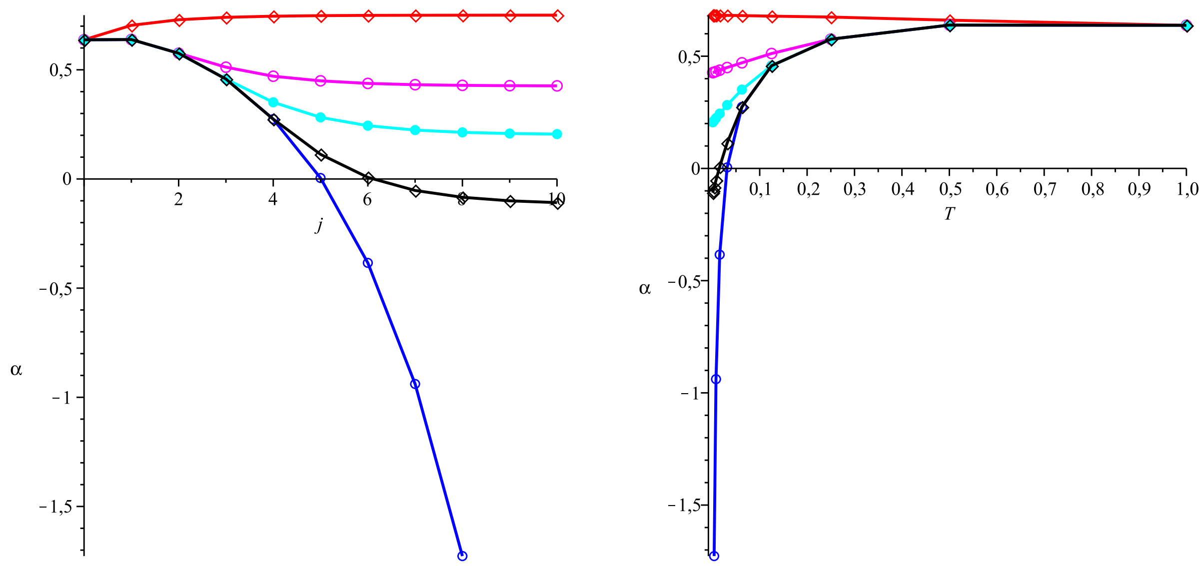


Figure 5.14: The figures illustrate sequences of suboptimality estimates $(\alpha_{k_j N,1}(k_j))_{j\in\mathbb{N}_0}$ corresponding to the discretization sequence $(k_j)_{j\in\mathbb{N}_0}$, $k_j := 2^j$, cf. Theorems 5.15 and 5.31. The blue line corresponds to the setting without the growth condition. The other curves are obtained by using the growth bounds $L = 2$ (red, $\diamond$), $L = 2^8 = 256$ (magenta, $\circ$), $L = 2^{16} = 65536$ (cyan, $\bullet$), and $L = 2^{32} = 4294967296$ (black, $\diamond$).

inequalities is sufficient

$$(\gamma_{N-j+m} - 1) \prod_{i=m+1}^{N-j+m-1} (\gamma_i - 1) \geq (\gamma_{N-j+m} - \gamma_{N-j}) \prod_{i=m+1}^{N-j+m-1} \gamma_i, \qquad j = m, m+1, \ldots, N-2. \tag{5.48}$$

To this end, $\gamma_{N-j+m} = C\sum_{n=0}^{N-j+m-1} \sigma^n$ is assumed. Otherwise, the involved γ_i, $i \in \{2, 3, \ldots, N-j+m-1\}$, may be defined by $c_n := L^n$, $n \in \{0, 1, \ldots, N-j+m-2\}$. Since these coefficients satisfy Condition (1.13) the respective inequality in (5.48) can be concluded analogously to the one in the proof of Theorem 3.18.

In the following $\gamma_{N-j} = C\sum_{n=0}^{N-j-1} \sigma^n$ is assumed. The case $\gamma_{N-j} = \sum_{n=0}^{N-j-1} L^n$ will be dealt with afterward. Using the representations of γ_{N-j+m} and γ_{N-j} yields $\gamma_{N-j+m} - 1 = (C-1) + C\sum_{n=1}^{N-j+m-1} \sigma^n$ as well as $\gamma_{N-j+m} - \gamma_{N-j} = C\sigma^{N-j}\sum_{n=0}^{m-1} \sigma^n$. Taking this into account allows us to rewrite (5.48) as

$$(C-1) \prod_{i=m+1}^{N-j+m-1} (\gamma_i - 1) + C\left[\sum_{n=1}^{N-j+m-1} \sigma^n \cdot \prod_{i=m+1}^{N-j+m-1} (\gamma_i - 1) - \sigma^{N-j} \sum_{n=0}^{m-1} \sigma^n \cdot \prod_{i=m+1}^{N-j+m-1} \gamma_i\right] \geq 0.$$

Clearly, the first summand is positive. Hence, showing positivity of the term in square brackets suffices in order to deduce the desired assertion. We point out that the corresponding inequality resembles the one dealt with in Lemma 3.23. However, we do not know whether γ_i is given by our controllability assumption or the growth condition for $i \in \{m+1, m+2, \ldots, N-j+m-1\}$ — except for γ_{N-j}. Hence, the respective steps of the proof need to be redone for this our setting. An induction with respect to j is carried out. The induction start $j = N-2$ is implied by

$$\sum_{n=1}^{m+1} \sigma^n \cdot (\gamma_{m+1} - 1) - \sigma^2 \sum_{n=0}^{m-1} \sigma^n \cdot \gamma_{m+1} = \sigma \cdot \min\left\{C\sum_{n=0}^{m} \sigma^n, \sum_{n=0}^{m} L^n\right\} - \sigma \sum_{n=0}^{m} \sigma^n \geq 0.$$

The induction step is performed from $j+1 \rightsquigarrow j$. For this purpose, the factor $(\gamma_{N-j+m-1} - 1)\sum_{n=1}^{N-j+m-1} \sigma^n$ of the minuend is decomposed in order to rewrite the inequality to be established as

$$\left[\sigma\gamma_{N-j+m-1} - \sum_{n=1}^{N-j+m-1} \sigma^n\right] \prod_{i=m+1}^{N-(j+1)+m-1} (\gamma_i - 1)$$
$$+ \left[\sum_{n=1}^{N-(j+1)+m-1} \sigma^n \cdot \prod_{i=m+1}^{N-(j+1)+m-1} (\gamma_i - 1) - \sigma^{N-(j+1)} \sum_{n=0}^{m-1} \sigma^n \prod_{i=m+1}^{N-(j+1)+m-1} \gamma_i\right] \sigma\,\gamma_{N-j+m-1} \geq 0.$$

Since (5.39) ensures positivity for the term contained in the first bracket, applying the induction assumption to the second yields the assertion and, thus, guarantees (5.48) for $\gamma_{N-j} = C\sum_{n=0}^{N-j-1} \sigma^n$.

In order to complete the proof, The more complicated case $\gamma_{N-j} = \sum_{n=0}^{N-j-1} L^n < C\sum_{n=0}^{N-j-1} \sigma^n$ is considered. Since $\gamma_{N-j+m} = C\sum_{n=0}^{N-j+m-1} \sigma^n < \sum_{n=0}^{N-j+m-1} L^n$, the switching index $i^\star$ defined in Lemma 5.29 satisfies $i^\star < N-j+m$. Hence, taking

$$\gamma_{N-j+m} = C \sum_{n=0}^{N-j+m-1} \sigma^n \leq \sum_{n=0}^{i^\star} L^n + C \sum_{n=i^\star+1}^{N-j+m-1} \sigma^n$$

into account yields that the inequality

$$\prod_{i=m+1}^{N-j+m-1}(\gamma_i - 1)\left[\sum_{n=1+k}^{\max\{i^\star,k\}} L^n + C \sum_{n=\max\{i^\star,k\}+1}^{N-j+m-1+k} \sigma^n\right] \geq \left[\sum_{n=N-j+k}^{\max\{i^\star,N-j+k-1\}} L^n + C \sum_{n=\max\{i^\star,N-j+k-1\}+1}^{N-j+m-1+k} \sigma^n\right] \prod_{i=m+1}^{N-j+m-1} \gamma_i \tag{5.49}$$

for $k = 0$ is a sufficient condition for the desired inequality. However, in order to deal with technical difficulties to be encountered in the upcoming induction, our goal is to show (5.49) for all $k \in \mathbb{N}_0$. Again, we perform an induction starting with $j = N-2$, i.e.

$$(\gamma_{m+1} - 1)\left[\sum_{n=1+k}^{\max\{i^\star,k\}} L^n + C \sum_{n=\max\{i^\star,k\}+1}^{m+1+k} \sigma^n\right] \geq \left[\sum_{n=2+k}^{\max\{i^\star,k+1\}} L^n + C \sum_{n=\max\{i^\star,k+1\}+1}^{m+1+k} \sigma^n\right] \gamma_{m+1}.$$

If $k \geq i^\star$ this inequality simplifies to $C\sigma^{1+k}(\gamma_{m+1} - \sum_{n=0}^{m} \sigma^n) \geq 0$ which is satisfied in view of (5.39). Otherwise, i.e. for $k < i^\star$, the inequality above is implied by

$$L^{1+k}\gamma_{m+1} - \left[\sum_{n=1+k}^{i^\star} L^n + C \sum_{n=i^\star+1}^{m+1+k} \sigma^n\right] \geq L\left(L^k\gamma_{m+1} - \left[\sum_{n=k}^{i^\star-1} L^n + C \sum_{n=i^\star}^{m+k} \sigma^n\right]\right) \geq 0.$$

Let $\gamma_{m+1} = C\sum_{n=0}^{m} \sigma^m$ hold. Then, applying $\sigma^k \leq L^k$ for the second subtrahend and $\sum_{n=k}^{i^\star-1} L^n = L^k \sum_{n=0}^{i^\star-1-k} L^n \leq L^k C \sum_{n=0}^{i^\star-1-k} \sigma^n$ for the first, ensures this inequality. Otherwise, i.e. if $\gamma_{m+1} = \sum_{n=0}^{m} L^n$ holds, using $C\sigma^{i^\star+n} \leq L^{i^\star+n}$ for $n \in \{0, 1, \ldots, m+k\}$ provides the assertion.

In order to complete the proof, the induction step is carried out from $j+1 \rightsquigarrow j$. To this end, the left hand side of (5.49) is considered. Leaving the factor $\prod_{i=m+1}^{N-j+m-2}(\gamma_i - 1)$ aside allows for rewriting the remaining term as

$$\gamma_{N-j+m-1}\left[\sum_{n=1+(k+1)}^{\max\{i^\star,k+1\}} L^n + C \sum_{n=\max\{i^\star,k+1\}+1}^{N-j+m-1+k} \sigma^n\right] + \left(c_{k,i^\star}\gamma_{N-j+m-1} - \left[\sum_{n=1+k}^{\max\{i^\star,k\}} L^n + C \sum_{n=\max\{i^\star,k\}+1}^{N-j+m-1+k} \sigma^n\right]\right)$$

with $c_{k,i^\star} = L^{k+1}$ for $k < i^\star$ and $c_{k,i^\star} = C\sigma^{k+1}$ otherwise. Since the ignored factor $\prod_{i=m+1}^{N-j+m-2}(\gamma_i - 1)$ is positive, positivity of the second summand can be shown analogously to the induction start. Hence, it remains to show that the difference of the first summand multiplied with $\prod_{i=m+1}^{N-j+m-2}(\gamma_i - 1)$ and the subtrahend of (5.49) is positive. To this end, dividing the respective inequality by $\gamma_{N-j+m-1}$ we have to establish

$$\prod_{i=m+1}^{N-(j+1)+m-1}[(\gamma_i - 1)/\gamma_i]\left[\sum_{n=1+(k+1)}^{\max\{i^\star,k+1\}} L^n + C\sum_{n=\max\{i^\star,k+1\}+1}^{N-j+m-1+k}\sigma^n\right] \geq \left[\sum_{n=N-j+k}^{\max\{i^\star,N-j+k-1\}} L^n + C\sum_{n=\max\{i^\star,N-j+k-1\}+1}^{N-j+m-1+k}\sigma^n\right].$$

Noting that $k - j = (k+1) - (j+1)$ enables us to apply the induction assumption applied for $k+1$ and $j+1$ and, thus, to conclude the assertion.

□

5.4 Accumulated Bounds

In the last section the growth condition Assumption 5.28 was introduced and incorporated in Problem 3.8 by appropriately modifying the definition of γ_i, $i = 2, 3, \ldots, N$. In order to solve the corresponding optimization problem, Theorem 3.18 was generalized, cf. Theorems 5.31 and 5.32. To this end, the concept of equivalent $\mathcal{KL}_0$-functions was employed which exploits that only the accumulated bounds γ_i, $i = 2, 3, \ldots, N$, resulting from Assumptions 3.2 and 5.28 are used while the exact shape of the sequences $(c_n)_{n\in\mathbb{N}_0}$ and $(L^n)_{n\in\mathbb{N}_0}$ does not play a role, cf. Definition 5.30.[10] Hence, the controllability and the growth condition may be replaced by the following weaker assumption taken from [120].

Assumption 5.38
Let a monotone, bounded sequence $(M_i)_{i\in\mathbb{N}_{\geq 2}}$ and an upper bound $M \in [1,\infty)$ exist such that $1 \leq M_i \leq M$ holds and, for each $x_0 \in \mathbb{X}$, the following inequality is satisfied

$$V_i(x_0) \leq M_i \ell^\star(x_0) \qquad \textit{for all } i \in \mathbb{N}. \tag{5.50}$$

Note that supposing linearity of the $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$ from our controllability Condition 3.2 in its first argument fits well to the structure of this assumption. In [120] Assumption 5.38 is formulated with a function $\sigma : \mathbb{X} \to \mathbb{R}_0^+$ which has to satisfy the condition $\sigma(x) \leq \ell(x,u)$ for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$. Hence, $\ell^\star(\cdot)$ is the maximal feasible function $\sigma(\cdot)$ and, thus, allows to employ smaller elements M_i, $i \in \mathbb{N}_{\geq 2}$, in comparison to other choices. Furthermore, since $N = 2$ corresponds to the smallest possible optimization horizon in our setting, the smallest index of the sequence $(M_i)_{i\in\mathbb{N}_{\geq 2}}$ is two. The upper bound M corresponds to our assumption that the sequence $(c_n)_{n\in\mathbb{N}_0}$ from our controllability condition is summable. The supposed monotonicity results directly from unconstrained receding horizon control and reflects that the optimal value function $V_N(\cdot)$ is monotone in the optimization horizon N.

Supposing Assumption 3.2 based on a $\mathcal{KL}_0$-function $\beta(\cdot,\cdot)$ linear in its first argument enables us to easily construct a sequence $(M_i)_{i\in\mathbb{N}_{\geq 2}}$ satisfying Assumption 5.38. For instance, this can be done by the definitions

$$M_i := \gamma_i = \sum_{n=0}^{i-1} c_n \qquad \text{or} \qquad M_i := \gamma_i = \sum_{n=0}^{i-1} C\sigma^n = \frac{C(1-\sigma^i)}{(1-\sigma)}$$

[10] Note that, for $x_0 \in \mathbb{X}$, existence of an admissible sequence $(u_{x_0}(n))_{n\in\mathbb{N}_0}$ of control values satisfying Condition (3.3) cannot be guaranteed for such an equivalent $\mathcal{KL}_0$-function.

in the exponentially controllable case, cf. [120, Section V]. The other way round, the concept of equivalent $\mathcal{KL}_0$-functions can be employed in order to obtain a sequence $(c_n)_{n\in\mathbb{N}_0}$ from $(M_i)_{i\in\mathbb{N}_{\geq 2}}$, cf. Definition 5.30. Then, Property (1.13) may be checked in order to decide whether the estimate from Theorem 3.18 characterizes the optimal value of the corresponding optimization Problem 3.8 exactly or provides a lower bound.

In summary, also Assumption 5.38 implies the presented results. Based on Assumptions 3.2 and 5.28, which were rigorously verified for, e.g., the linear wave equation in Section 3.4 or the reaction diffusion equation considered in the introduction of this chapter, suitable bounds γ_i, $i = 2, 3, \ldots, N$, can easily be computed. Using Assumption 5.38 yields improved performance estimates resulting from Theorem 3.18 as shown below for the reaction diffusion equation and the example of the synchronous generator, cf. Subsections 5.4.1 and 5.4.2. First a theoretical example is investigated in order to illustrate the technique to be applied.

A discrete time system whose dynamics are given in Figure 5.15 is considered.

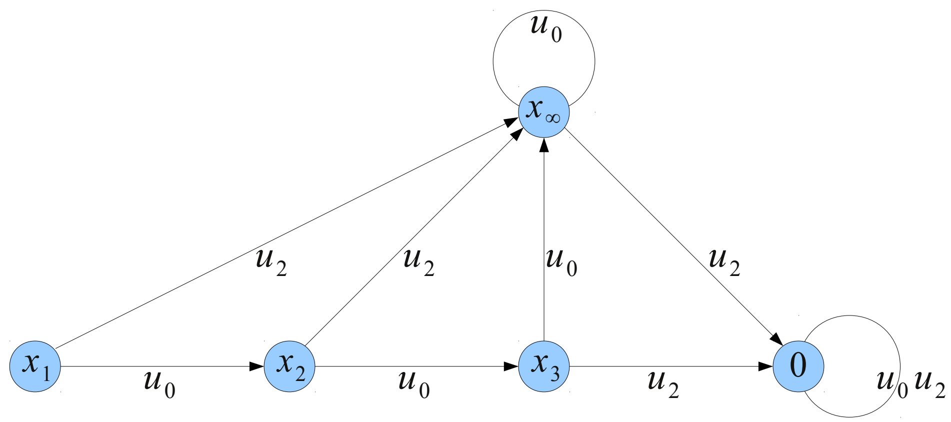


Figure 5.15: The system dynamics of a theoretical example are illustrated. $x^\star = 0$ is the desired equilibrium whereas x_∞ is an expensive state with respect to the stage costs given by (5.51).

The stage costs depending on the current state and the applied control are given by

$$\ell(x, u_i) := \begin{cases} 1+i & \text{for } x \in \{x_1, x_2, x_3\}, \\ 100+i & \text{for } x = x_\infty, \\ 0+i & \text{for } x = 0. \end{cases} \tag{5.51}$$

Consequently, the system to be investigated is finite time controllable. We want to deduce a sequence $(c_n)_{n\in\mathbb{N}_0}$ satisfying Assumption 3.2. To this end, x_3 is considered first for which choosing $u_{x_3}(0) = u_2$ minimizes $\gamma_2 = c_0 + c_1$. Then, $x_{u_{x_3}}(n; x_3) = x^\star$, $n \geq 1$, is ensured by $u_{x_0}(n) = u_0$, $n \in \mathbb{N}$, without creating further costs. As a consequence, $c_0 \geq 3$ is required. For x_2, the choice $u_{x_2}(0) = u_{x_2}(1) = u_0$ ensures $c_0 \leq 3$ and minimizes c_1 ($c_1 = 1$ instead of $c_1 \geq 3$). However, then $c_2 \geq 100$ follows and, thus, the minimal stabilizing horizon $\hat{N}$ is at least 453 for $m = 1$ or 144 for $m = \lfloor N/2 \rfloor$.[11] Hence, avoiding x_∞ seems to be clearly

[11] For the horizon estimates the sequence $c_0 = 3$, $c_1 = 1$, $c_2 = 100$, and $c_n = 0$, $n \in \mathbb{N}_{\geq 3}$ was used. The asymptotic estimates on the minimal required horizon length from Section 4.1 are $\gamma \ln \gamma \approx 483$ for $m = 1$ and $2\ln(2) \cdot \gamma \approx 144$ for $m = \lfloor N/2 \rfloor$ (for $c_0 = 104$ and $c_n = 0$, $n \in \mathbb{N}$, the minimal stabilizing horizon is 482 for $m = 1$).

favorable in view of (5.51). Furthermore, staying in the state x_∞ longer than necessary is not useful. Suitable sequences $(c_n(x_0))_{n\in\mathbb{N}_0}$, $x_0 \in \{x_1, x_2, x_3, x_\infty, 0\}$, satisfying

$$\ell(x_{u_{x_0}}(n, u_{x_0}(n)) \leq c_n(x_0)\ell^\star(x_0) \tag{5.52}$$

are given in Table 5.4. In order to guarantee Assumption 3.2, the sequence defined by $c_i := \max_{x_0\in\{0,x_\infty,x_1,x_2,x_3\}} c_i(x_0)$, i.e. $c_0 = c_1 = c_2 = 3$ and $c_n = 0$ for $n \in \mathbb{N}_{\geq 3}$, has to be employed which yields $\gamma_2 = 6$ and $\gamma_i = 9$, $i \in \mathbb{N}_{\geq 3}$. Applying Theorem 3.18 yields $N = 20$ and, taking Algorithms 4.24 and 4.28 and, thus, larger control horizons $m > 1$ into account, $N = 12$.

x_0	c_0	c_1	c_2	c_3
x_1	1	1	3	0
x_2	1	3	0	0
x_3	3	0	0	0
x_∞	1.02	0	0	0
0	0	0	0	0

Table 5.4: Sequences $(c_n(x_0))_{n\in\mathbb{N}_0}$ depending on the initial state x_0 are deduced for a theoretical example which are used in order to illustrate the ramifications of using Assumption 5.38 instead of Assumption 3.2 in order to compute suboptimality bounds.

In contrast to that, Assumption 5.38 is satisfied with $M_2 = 4$ and $M_i = 5$, $i \in \mathbb{N}_{\geq 3}$, cf. Table 5.4. The equivalent sequence is defined by $c_0 = 3$, $c_1 = 1$, $c_2 = 1$, and $c_n = 0$ for all $n \in \mathbb{N}_{\geq 3}$ and satisfies Property (1.13), cf. (5.41). Employing Theorem 3.18 in order to compute the respective performance bounds yields $N = 8$ (or $N = 7$ for $m > 1$) as minimal stabilizing horizon — a significant improvement in comparison to the prior estimates.

In order to fathom out this observation, a closer look is taken at the involved accumulated bounds. Denoting the accumulated bounds from Assumptions 3.2 with γ_i and their counterparts from the newly introduced Assumption 5.38 with M_i leads to

$$\gamma_i = \sum_{n=0}^{i-1} c_n = \sum_{n=0}^{i-1} \max_{x_0\in\mathbb{X}} c_n(x_0) \geq \max_{x_0\in\mathbb{X}} \sum_{n=0}^{i-1} c_n(x_0) = M_i.$$

Hence, using Assumption 5.38 allows for maximizing the accumulated bounds instead of accumulating the maximized bounds.

5.4.1 Reaction Diffusion Equation: Impact of Assumption 5.38

In this subsection, first Assumption 5.38 is verified for the example of the reaction diffusion equation which was considered in the introduction of this chapter. Then, Theorem 3.18 is employed in order to compute suboptimality estimates $\alpha_{N,m} = \alpha^1_{N,m}$ based on the corresponding equivalent sequence, cf. (5.41).

In the mixed integer optimization Problem 5.4 the formula

$$\gamma_i = C\sum_{n=0}^{i-1} \sigma^n = (1+\lambda K^2)M^2 \sum_{n=0}^{i-1} \left(e^{-2\gamma T}\right)^n, \qquad i \in \{2, 3, \ldots, N\} \tag{5.53}$$

was used. Then, for each optimization horizon N, the involved feedback gain K was suitably chosen in order to maximize the performance bound $\alpha^1_{N,1}$. However, for each γ_i, $i \in \{2,3,\ldots,N\}$, the same parameter K was used. Here, each $M_i = \gamma_i$ is minimized with respect to K individually. Then, for each optimization horizon N, Theorem 3.18 is applied with the resulting γ_i, $i \in \{2,3,\ldots,N\}$, in order to compute $\alpha^1_{N,1}$, cf. Figure 5.16 on the left. The minimal stabilizing horizon decreases to $N = 7$ from $N = 10$. The minimal horizon ensuring $\alpha^1_{N,m} \geq 0.5$ is reduced to $N = 19$ — in contrast to $N = 25$. Combining this approach with the discretization technique from Section 5.1 even reduces the minimal stabilizing horizon to $N = 6$.

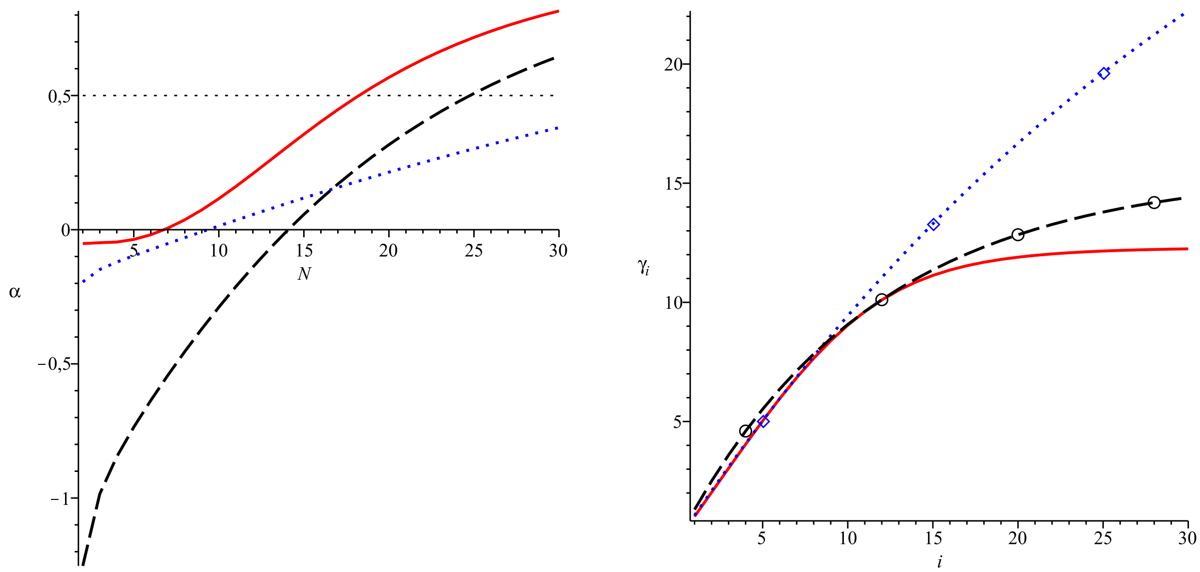


Figure 5.16: Impact of the growth condition on our suboptimality estimates for the reaction diffusion equation. On the left the performance bounds $\alpha^1_{N,1}$ are illustrated for the $(\gamma_i)_{i\in\mathbb{N}_{\geq 2}}$ shown on the right. The red curves correspond to the equivalent sequence induced by Assumption 5.38, the others are based on the optimal choice of the feedback gain K for given horizon N ($N = 10$ is marked with $\diamond$, $N = 25$ with $\circ$).

Summarizing, using Assumption 3.2 for different parameters K led to improved accumulated bounds γ_i, $i = 2,3,\ldots,N$. Then, an equivalent sequence was constructed in order to apply Theorem 3.18 and, thus, to deduce tighter performance estimates. To this end, the observation that only γ_i, $i = 2,3,\ldots,N$, are used in Problem 3.8 is essential. Hence, guaranteeing Assumption 5.38 is sufficient — although this was done by first ensuring Assumption 3.2 depending on the feedback gain K.

5.4.2 Synchronous Generator: A Case Study

In Section 4.4 Example 4.27 was investigated numerically for sampling period $T = 0.05$ in order to illustrate the proposed Algorithms 4.24 and 4.28. Here, this example is considered again. We want to determine a sequence $(c_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{R}^+_0$ numerically such that, for each state x_0 from a given set, a control $u_{x_0}(\cdot)$ exists satisfying Condition (3.3)

$$\ell(x_{u_{x_0}}(n), u_{x_0}(n)) \leq c_n \ell^\star(x_0) = c_n \min_{u\in\mathbb{U}} \ell(x_0,u) \qquad \text{for all } n \in \mathbb{N}_0. \tag{5.54}$$

Since the proposed methodology in order to estimate the required optimization horizon length becomes, in general, more conservative for large sets, level sets $\mathcal{L}_i$, $i \in \{0,1\}$, of the optimal value function given by (4.19) are considered. The corresponding stage costs $\ell_i(\cdot,\cdot)$ take – in addition to the control effort – either the whole trajectory or only its states at the sampling instants into account, cf. Section 4.4. Again, the level sets are intersected with a grid $\mathcal{G}$ defined on the cube

$$[x_1^\star - 0.25, x_1^\star + 0.25] \times [-1, 1] \times [x_3^\star - 0.75, x_3^\star + 0.75] \subset \mathbb{X}.$$

To this end, an equidistant discretization is used in each coordinate direction with stepsize $\Delta x = 0.05$. This construction yields 13981 grid points.[12] For each grid point $x_0 \in \mathcal{L}_i \cap \mathcal{G}$, $i \in \{0,1\}$, $\ell^\star(x_0)$ as well as the first 80 RHC steps are computed. The latter generates a trajectory $x_{\mu_{\widetilde{N},1}}(\cdot)$ and, thus, allows to evaluate the stage costs $\ell(x_{\mu_{\widetilde{N},1}}(n), \mu_{\widetilde{N},1}(0; x_{\mu_{\widetilde{N},1}}(n)))$, $n = 0, 1, \ldots, 80$ along the closed loop trajectory. Hence, a sequence $(c_n(x_0))_{n \in \mathbb{N}_0 \cap [0,80]}$ can be defined by

$$c_n(x_0) := \frac{\ell(x_{\mu_{\widetilde{N},1}}(n), \mu_{\widetilde{N},1}(0; x_{\mu_{\widetilde{N},1}}(n)))}{\ell^\star(x_0)}, \qquad n = 0, 1, \ldots, 80. \tag{5.55}$$

Taking the maximum with respect to x_0, i.e. setting $c_n := \max_{x_0 \in \mathcal{L}_i \cap \mathcal{G}} c_n(x_0)$, yields a sequence $(c_n)_{n \in \mathbb{N}_0}$ satisfying Condition (5.54). Then, Formula (3.21) provides a lower bound for the suboptimality index $\alpha_{N,m}$ depending on the optimization and the control horizon.

Note that (5.54) is not checked rigorously because the maximization in order to compute c_n was performed only on $\mathcal{L}_i \cap \mathcal{G}$ instead of the whole level set $\mathcal{L}_i$, $i \in \{0,1\}$. However, our numerical experiments confirm that the used grid is sufficiently fine in order to allow for a reliable estimate.

The procedure is repeated for various optimization horizons $\widetilde{N}$, i.e.

$$\widetilde{N} \in \{6, 8, 10, \ldots, 24\} \quad \text{for} \quad \ell_0(\cdot,\cdot) \qquad \text{and} \qquad \widetilde{N} \in \{10, 12, 14, \ldots, 28\} \quad \text{for} \quad \ell_1(\cdot,\cdot)$$

in order to enhance the deduced performance bounds.[13] Subsequently, we check whether the respective suboptimality estimates are improved or not which is facilitated by Formula (3.21). The resulting horizon estimates for selected suboptimality bounds $\overline{\alpha}$ are given in Table 5.5 and Figure 5.17.

The improvement for larger control horizons m is significant which, once more, shows the advantages of employing Algorithms 4.24 and 4.28 which only use $m > 1$ if necessary. This observation fits well to Corollary 4.3 and Theorem 4.4, i.e. to the fact that the asymptotic growth of the minimal stabilizing horizon declines from $\gamma \ln \gamma$ for $m = 1$ to a linear one for $m = \lfloor N/2 \rfloor$.

Comparing these theoretically calculated bounds with the numerical results from Section 4.4 shows that the deduced estimates are conservative. In this context, we emphasize that the quality of the derived performance estimates crucially depends on the provided controllability and growth bounds. Hence, our goal is to deduce tighter estimates by using Assumption 5.38. To this end, the sequences $(c_n^{\widetilde{N}}(x_0))_{n \in \mathbb{N}_0}$ with $x_0 \in \mathcal{G} \cap \mathcal{L}_i$, $i \in \{0,1\}$, are used once more. Each of these sequences is converted to a sequence $(M_i^{\widetilde{N}}(x_0))_{i \in \mathbb{N}_{\geq 2}}$

[12] Since the zero sequence is admissible for $x_0 = x^\star$, the equilibrium point is not treated separately — in contrast to Section 4.4.

[13] The optimization horizon which is used in RHC in order to generate coefficient sequences $(c_n(x_0))_{n \in \mathbb{N}_0} = (c_n^{\widetilde{N}}(x_0))_{n \in \mathbb{N}_0}$ in (5.55) and, thus, the sequence $(c_n)_{n \in \mathbb{N}_0} = (c_n^{\widetilde{N}})_{n \in \mathbb{N}_0}$ is denoted by $\widetilde{N}$. Based on this sequence performance bounds $\alpha_{N,m}(\widetilde{N})$ are computed by applying Theorem 3.18 depending on the horizon length N.

	$x_0 \in \mathcal{L}_0 \cap \mathcal{G}$, $\ell_0(\cdot,\cdot)$		$x_0 \in \mathcal{L}_1 \cap \mathcal{G}$, $\ell_1(\cdot,\cdot)$	
$\overline{\alpha}$	$m=1$	$m=\lfloor N/2 \rfloor$	$m=1$	$m=\lfloor N/2 \rfloor$
0	41 (14)	25 (12)	58 (20)	30 (14)
1/5	47 (14)	30 (14)	63 (22)	34 (18)
1/3	52 (16)	33 (16)	68 (24)	38 (22)
1/2	59 (18)	38 (22)	75 (26)	42 (28)

Table 5.5: RHC performance estimates for the synchronous generator based on Theorem 3.18 and a numerically computed sequence $(c_n)_{n\in\mathbb{N}_0} = (c_n^{\widetilde{N}})_{n\in\mathbb{N}_0}$ satisfying (5.54) are given. The respective horizon $\widetilde{N}$ is denoted in brackets.

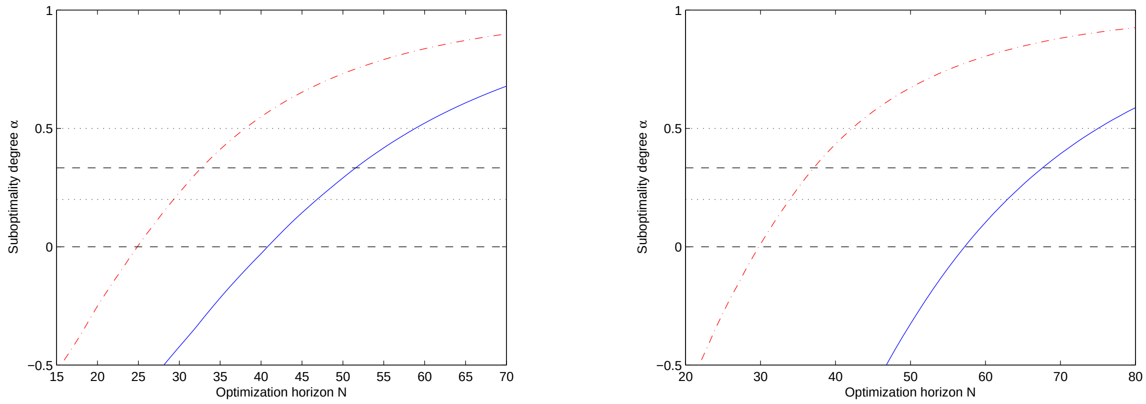


Figure 5.17: Numerically computed suboptimality bounds for the synchronous generator deduced from Theorem 3.18 supposing Assumption 3.2 in dependence on different stage costs ($\ell_0(\cdot,\cdot)$ on the left and $\ell_1(\cdot,\cdot)$ on the right). The horizontal lines are located at 0, 0.2, 0.33, and 0.5. The solid blue line marks the results concerning $m = 1$ whereas the other stands for $m = \lfloor N/2 \rfloor$.

by $M_i^{\widetilde{N}}(x_0) := \sum_{n=0}^{i-1} c_n^{\widetilde{N}}(x_0)$. In contrast to the approach based on Assumption 3.2, now a sequence is constructed which reflects the best estimates depending on the state x_0, i.e.

$$
\begin{aligned}
M_i(x_0) &:= \min_{\widetilde{N}\in\{\ 6,\ 8,10,\ldots,24\}} M_i^{\widetilde{N}}(x_0) \qquad \text{for RHC based on } \ell_0(\cdot,\cdot),\\
M_i(x_0) &:= \min_{\widetilde{N}\in\{10,12,14,\ldots,28\}} M_i^{\widetilde{N}}(x_0) \qquad \text{for RHC based on } \ell_1(\cdot,\cdot).
\end{aligned}
$$

Then, the maximum is taken with respect to x_0 which yields $M_i := \max_{x_0\in\mathcal{G}\cap\mathcal{L}_j} M_i(x_0)$, $j \in \{0, 1\}$. For the considered example, in doing so the suboptimality estimates from Theorem 3.18 are significantly improved, in particular for RHC with $m = 1$, cf. Table 5.6.

In conclusion, using Assumption 5.38 leads to better performance bounds which shows that the accumulated bounds are the decisive ingredient in order to deduce good suboptimality estimates.

	$x_0 \in \mathcal{L}_0 \cap \mathcal{G}$, $\ell_0(\cdot,\cdot)$				$x_0 \in \mathcal{L}_1 \cap \mathcal{G}$, $\ell_1(\cdot,\cdot)$			
	$m = 1$		$m = \lfloor N/2 \rfloor$		$m = 1$		$m = \lfloor N/2 \rfloor$	
$\overline{\alpha}$	N	ΔN	N	ΔN	N	ΔN	N	ΔN
0	32	09	24	1	44	14	27	3
1/5	37	10	28	2	49	14	31	3
1/3	41	11	31	2	53	15	34	4
1/2	48	11	36	2	59	16	39	3

Table 5.6: Minimal horizon N such that a performance bound $\alpha_{N,m} \geq \overline{\alpha}$ is ensured by Theorem 3.18 applied with a numerically computed sequence $(\gamma_i)_{i\in\mathbb{N}_{\geq 2}}$ satisfying Assumption 5.38 for the synchronous generator. In addition, the improvement ΔN in comparison with estimates deduced from Assumption 3.2 is shown, cf. Table 5.5.

5.5 Comparison with Other Approaches

In this section Assumption 5.38 is supposed to be given with a sequence $(M_i)_{i\in\mathbb{N}_{\geq 2}} = (\gamma_i)_{i\in\mathbb{N}_{\geq 2}}$ which is determined either analytically or numerically. Then, the performance bounds from Theorem 3.18, which are based on the methodology from [39] introduced in Section 3.1, are compared with their counterparts from two other approaches [90, 120]. All three approaches have in common that $V_N(\cdot)$ is employed as a Lyapunov function. Since the technique presented in [90] is restricted to linear finite dimensional systems, an example suitable for this setting is considered, cf. Subsection 5.5.1. Afterward, we return to the nonlinear example of the synchronous generator in order to further investigate the other approaches, cf. Subsection 5.5.2. At first the methodologies [90, 120] are briefly sketched.

The technique from [90], which is developed by V. Nevistić and J. Primbs, is limited to linear finite dimensional systems governed by $x(n+1) = Ax(n) + Bu(n)$ with quadratic stage costs $\ell(x,u) = x^TQx + u^TRu$ with positive definite matrices Q, R.[14] The goal of this approach is to employ the cost functional $V_N(\cdot)$ as a Lyapunov function and, thus, to ensure $V_N(x_{\mu_N}(k;x_0)) - V_N(x_{\mu_N}(k+1;x_0)) > 0$ for $x \neq x^\star = 0$. The main idea is to rewrite this inequality by means of Bellman's principle of optimality, i.e.

$$\begin{aligned} & V_N(x_{\mu_N}(k;x_0)) - V_N(x_{\mu_N}(k+1;x_0)) \qquad (5.56)\\ = \; & \ell(x_{\mu_N}(k;x_0), \mu_N(x_{\mu_N}(k;x_0),0)) + [V_{N-1}(x_{\mu_N}(k+1;x_0)) - V_N(x_{\mu_N}(k+1;x_0))]. \end{aligned}$$

Then, the optimization horizon is chosen sufficiently large in order to ensure that the difference contained in the second line is negligibly small compared to the stage costs evaluated at time instant k. In particular, a methodology in order to estimate the required horizon length is proposed.

Let P_N denote the solution of the Riccati difference equation (RDE) and P its counterpart of the algebraic Riccati equation (ARE), cf. Examples 1.23 and 1.10, which satisfy $P_N \geq P_{N-1} \geq P_0 = Q > 0$ for $N \in \mathbb{N}$ and $P_N \to P$ for N tending to infinity.[15] In addition, let $\overline{\lambda}_N$ and $\underline{\lambda}_N$ be the largest and the smallest eigenvalue of P_N, respectively.

[14] In [90, Section 5] the authors state that their approach is "based on the ideas found in [112] for non-quadratic finite horizon based receding horizon control" which clarifies the relation of these two references.

[15] Note that terminal costs are not taken into account.

Then, we obtain $0 < \underline{\lambda}_0 \leq \underline{\lambda}_N \leq \overline{\lambda}_N \leq \overline{\lambda}$ and, for each $x_0 \in \mathbb{R}^n$,

$$\underline{\lambda}_0 \|x_0\|_2^2 \leq \ell^\star(x_0) = V_1(x_0) \leq x_0^T Q x_0 + u^T(0) R u(0) \qquad \forall\, u(0) \in \mathbb{R}^m.$$

Defining $\theta_N := \min\{\theta : \theta P_N \geq P_{N+1}\}$ yields $\lim_{N\to\infty} \theta_N = 1$, cf. [90, Proposition 5.2], and, thus, $V_{N+1}(x) \geq V_N(x) \geq \frac{1}{\theta_N} V_{N+1}(x)$. These preliminary considerations enable us to state the main result, cf. [90, Theorems 5.1 and 5.2].

Theorem 5.39
Let the pair $[A, B]$ be controllable. Furthermore, let the optimization horizon N be such that $\underline{\lambda}_0 - (\theta_{N-1} - 1)\overline{\lambda}_N > 0$ holds. Then, RHC stabilizes the system governed by the linear dynamics $x(n+1) = Ax(n) + Bu(n)$ with stage costs $\ell(x,u) = x^T Q x + u^T R u$, Q, R positive definite. The cost functional $V_N(\cdot)$ is a Lyapunov function for the receding horizon closed loop satisfying

$$V_N(x_{\mu_N}(k+1; x_0)) \leq \left(1 - \frac{\underline{\lambda}_0 - (\theta_{N-1} - 1)\overline{\lambda}_N}{\overline{\lambda}_N}\right) V_N(x_{\mu_N}(k; x_0)) =: \varrho_N V_N(x_{\mu_N}(k; x_0)).$$

Moreover, the performance on the infinite horizon is bounded by

$$\sum_{k=0}^{\infty} \ell(x_{\mu_N}(k; x_0), \mu_N(x_{\mu_N}(k; x_0), 0)) \leq \left(1 + \left(\frac{\theta_{N-1} - 1}{\theta_{N-1}}\right) \frac{\varrho_N}{1 - \varrho_N}\right) V_N(x_0). \tag{5.57}$$

Proof: Taking account of the estimate

$$\theta_{N-1} V_N(x_{\mu_N}(k; x_0)) \geq \theta_{N-1} V_{N-1}(x_{\mu_N}(k+1; x_0)) \geq V_N(x_{\mu_N}(k+1; x_0))$$

and Equality (5.56) allows to deduce the following inequality

$$\begin{aligned} V_N(x_{\mu_N}(k;x_0)) - V_N(x_{\mu_N}(k+1;x_0)) &\geq \ell(x_{\mu_N}(k;x_0), \mu_N(x_{\mu_N}(k;x_0),0)) - (\theta_{N-1} - 1) V_N(x_{\mu_N}(k;x_0)) \\ &\geq \left(\underline{\lambda}_0 - (\theta_{N-1} - 1)\overline{\lambda}_N\right) \|x_{\mu_N}(k;x_0)\|_2^2 \\ &\geq \left(\frac{\underline{\lambda}_0 - (\theta_{N-1} - 1)\overline{\lambda}_N}{\overline{\lambda}_N}\right) V_N(x_{\mu_N}(k;x_0)). \end{aligned}$$

Then, using the definition of ϱ_N shows the first of the two claimed estimates. In order to deduce the second, the following bound for the stage cost is established

$$\begin{aligned} \ell(x_{\mu_N}(k; x_0), u_{\mu_N}(x_{\mu_N}(k; x_0), 0)) &= V_N(x_{\mu_N}(k; x_0)) - V_{N-1}(x_{\mu_N}(k+1; x_0)) \\ &\leq V_N(x_{\mu_N}(k; x_0)) - \frac{V_N(x_{\mu_N}(k+1; x_0))}{\theta_{N-1}}. \end{aligned}$$

Rewriting the subtrahend of the difference on the right hand side as

$$\frac{V_N(x_{\mu_N}(k+1; x_0))}{\theta_{N-1}} = V_N(x_{\mu_N}(k+1; x_0)) - \frac{\theta_{N-1} - 1}{\theta_{N-1}} V_N(x_{\mu_N}(k+1; x_0))$$

enables us to proceed analogously to the proof of Proposition 3.1, i.e.

$$\begin{aligned} \sum_{k=0}^{\infty} \ell(x_{\mu_N}(k; x_0), u_{\mu_N}(x_{\mu_N}(k; x_0), 0)) &\leq V_N(x_0) + \frac{\theta_{N-1} - 1}{\theta_{N-1}} \sum_{k=0}^{\infty} V_N(x_{\mu_N}(k+1; x_0)) \\ &\leq \left(1 + \frac{\theta_{N-1} - 1}{\theta_{N-1}} \varrho_N \sum_{k=0}^{\infty} \varrho_N^k\right) V_N(x_0) \\ &= \left(1 + \frac{\theta_{N-1} - 1}{\theta_{N-1}} \frac{\varrho_N}{1 - \varrho_N}\right) V_N(x_0). \end{aligned}$$

□

Since the positive definiteness of Q implies $\underline{\lambda_0} > 0$, the convergence of θ_N to one for N approaching infinity, and the uniform boundedness of the sequence $(\overline{\lambda}_i)_{i\in\mathbb{N}_0}$, the following condition is always satisfied for sufficiently large optimization horizons N

$$\underline{\lambda_0} - (\theta_{N-1} - 1)\overline{\lambda}_N > 0. \tag{5.58}$$

Inequality (5.57) yields a suboptimality estimate. Here, since $V_N(x_0) \leq V_\infty(x_0)$ holds for unconstrained RHC, the respective performance index is given by the inverse of the term in brackets in front of $V_N(x_0)$, i.e.

$$\alpha_N = \left(1 + \left(\frac{\theta_{N-1} - 1}{\theta_{N-1}}\right) \frac{\varrho_N}{1 - \varrho_N}\right)^{-1}$$

and, thus, RHC performs "within a specified tolerance of the optimal infinite horizon policy", cf. [90, Section 5]. The convergence $\theta_N \to 1$ for $N \to \infty$ implies that this bound tends to one — an assertion which is also guaranteed by Corollary 3.20.

In order to apply Theorem 5.39, Condition (5.58) has to be verified. In the described linear setting without control or state constraints, the matrices P_0, P_{N-1}, and P_N can be computed by solving the RDE for finite N. Hence, the eigenvalues $\underline{\lambda_0}$, $\overline{\lambda}_N$, and, since P_N is invertible, θ_{N-1} can be determined. This raises the question whether Condition (5.58) can also be ensured in a more general setting solely based on Assumption 5.38? Here, the maximal eigenvalue $\overline{\lambda}_i$ corresponds to γ_i and $\underline{\lambda_0}$ is provided the $\mathcal{K}_\infty$-function from (1.4). However, estimating θ_{N-1} is more difficult. Using Assumption 5.38 and the inherent monotonicity of $V_N(\cdot)$ for unconstrained RHC leads to

$$V_N(x_0) \leq \gamma_N \ell^\star(x_0) \leq \gamma_N V_{N-1}(x_0). \tag{5.59}$$

Since γ_N does, in general, not converge to one, this estimate is too coarse in order to be applied in Theorem 5.39. In conclusion, Condition (5.58) seems to be too restrictive in order to extend this approach to a more general setting which, e.g., includes constraints.

The second methodology, which is designed by S. E. Tuna, M. J. Messina, and A. R. Teel and was introduced in [120], is based on Assumption 5.38 formulated with a function $\sigma : \mathbb{X} \to \mathbb{R}_0^+$ which we replace — as in the preceding section — by $\ell^\star : \mathbb{X} \to \mathbb{R}_0^+$. In addition, the following assumption is needed in order to apply this approach.[16]

Assumption 5.40
Let $\gamma \geq 0$ be given and define $g(x) := \ell^\star(x) = \min_{u\in\mathcal{U}^1(x_0)} \ell(x,u)$. Then, for each $x_0 \in \mathbb{X}$ there exists $u \in \mathbb{U}$ such that $f(x_0,u) \in \mathbb{X}$ and the following inequality holds

$$\ell^\star(f(x_0,u)) + \ell(x_0,u) = g(f(x_0,u)) + \ell(x_0,u) \leq (1+\kappa)g(x_0) = (1+\kappa)\ell^\star(x_0).$$

This choice of the function $g : \mathbb{X} \to \mathbb{R}_0^+$ allows to easily compare the suboptimality bounds from Problem 3.8 with those obtained in [120] because our cost functional $V_N(\cdot)$ may be rewritten as

$$V_N(x_0) = \min_{u\in\mathcal{U}} J_N(x_0,u) = \min_{u\in\mathcal{U}} \sum_{n=0}^{N-1} \ell(x_u(n),u(n)) = \min_{u\in\mathcal{U}} \sum_{n=0}^{N-2} \ell(x_u(n),u(n)) + \ell^\star(x_u(N-1))$$

[16] This assumption is similar to the first part of [49, Assumption 4.2].

by Bellman's principle of optimality. Hence, the optimization horizon N from [120] corresponds to $N+1$ in our setting.[17] Since terminal weights are not considered in this section, $\kappa \geq 0$ is given by $c_1 = \gamma_2 - 1$ from the equivalent sequence corresponding to $(\gamma_i)_{i\in\mathbb{N}_{\geq 2}}$ from Assumption 5.38.[18] Based on Assumptions 5.38 and 5.40 the estimate

$$V_N(f(x,\mu_N(x)) - V_N(x) \leq -(1-\eta(N))\ell(x,\mu_N(x)) \qquad \text{with } \eta(N) := \kappa \prod_{i=1}^{N-1} \frac{M_i - 1}{M_{i-1}}, \tag{5.60}$$

which can be interpreted as a relaxed Lyapunov inequality with $\alpha = 1 - \eta(N)$, is shown, cf. [120, Theorem 1]. To this end, a similar approach to the one from Section 3.1 is pursued. However, only Inequalities (3.5) and (3.6), $j = 1, 2, \ldots, N-2$, are used.

Theorem 3.18 showed that the solution of Problem 3.8, which additionally takes the Inequalities (3.7), $j = 0, 1, \ldots, N-m-1$, into account, coincides with its counterpart from the relaxed Problem 3.17 — assuming Condition (1.13). This "relaxed" Problem is, for $m = 1$, based on (3.5) and (3.7), $j = 0, 1, \ldots, N-2$, but does not reflect (3.6), $j = 1, 2, \ldots, N-2$. The proof of Theorem 3.18 shows that these inequalities represent the tighter bounds in order to estimate the desired performance index $\alpha^1_{N,1}$. Hence, we expect that the performance estimates resulting from Problem 3.8 are better than their counterparts from [120].

5.5.1 A Linear Finite Dimensional Example

In order to illustrate and compare the techniques mentioned in the introduction of this section, the linear finite dimensional system with quadratic cost function from Examples 1.10, 1.17, 2.7, and 3.3 is considered. In particular, we are interested in the performance loss of RHC compared to the infinite horizon optimal solution. For the approach in [90] the performance loss is given by $(\theta_{N-1}-1)\varrho_N/(\theta_{N-1}(1-\varrho_N))$ which corresponds to a suboptimality estimate with relaxation parameter $1/\alpha - 1$ for the other two settings. Employing the approach from [90] provides the parameters shown in Table 5.7 and, thus, ensures stability for $N \geq 5$.[19]

N	θ_{N-1}	$\overline{\lambda}_N$	$\underline{\lambda}_0 - (\theta_{N-1}-1)\overline{\lambda}_N$	Performance
3	2.0681	+6.6038	−6.0534	-
4	1.3330	8.7401	−1.9106	-
5	1.0741	9.3781	+0.3048	2.0548
6	1.0214	9.5784	+0.7954	0.2310
7	1.0068	9.6437	+0.9341	0.0633
8	1.0017	9.6593	+0.9840	0.0146

Table 5.7: Performance estimates according to Theorem 5.39 from [90, Table 1].

The computed maximal eigenvalues λ_i, $i = 2, 3, \ldots, N$, are used as accumulated bounds γ_i, $i = 2, 3, \ldots, N$, in Assumption 5.38. The parameter κ in Assumption 5.40 is approximately 2.21. Then, the minimal stabilizing horizon decreases from $N = 25$ for the

[17]This explains the index $N-1$ in the definition of $\eta(N)$ in (5.60).

[18]Otherwise, $g(x) := \omega\ell^\star(x)$ is a suitable definition.

[19]Note that our notation deviates from the one used in [90], i.e. the cost functional $V_N(\cdot)$ used in the reference sums up from $n = 0$ to N instead of $N-1$, cf. (2.4). Hence, the cited results are adapted with respect to this.

approach based on (5.60) to $N = 12$ for $m = 1$ and $N = 6$ for $m = \lfloor N/2 \rfloor$ for our approach to $N = 5$ for the methodology from [90], cf. Figure 5.18.[20] Taking a look at the corresponding equivalent $\mathcal{KL}_0$-function shows that $c_i > 1$ holds for $i \in \{1, 2, 3\}$. We emphasize that the performance bounds get worse for $N \leq 4$ for increasing optimization horizons N. Hence, $c_n < 1$ seems to be the appropriate criterion in order to decide whether prolonging the horizon contributes positively or not.

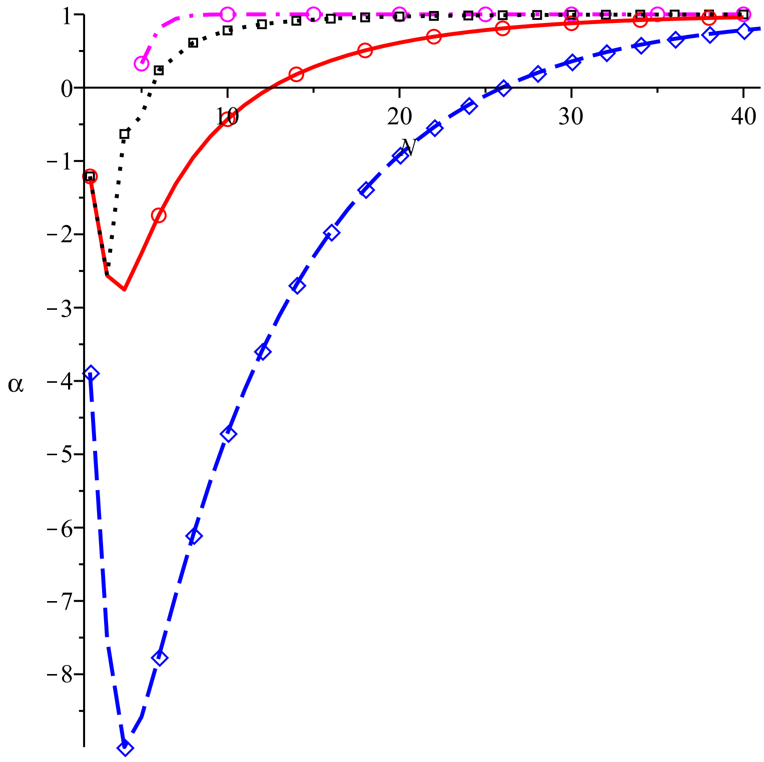


Figure 5.18: Comparison of the different approaches in order to estimate the required length of the optimization horizon for the example considered in Subsection 5.5.1. The blue curves correspond to [120], the dotted black one to [90]. The other two represent the performance bounds from Theorem 3.18 for $m = 1$ and $m = \lfloor N/2 \rfloor$

In conclusion, when dealing with linear finite dimensional systems whose Riccati difference equations are solvable with a tenable expenditure the methodology from [90] is superior. However, the additional Condition (5.58) is required which seems to be too restrictive for most applications — in particular nonlinear ones or systems with input or state constraints. The other approaches use only the information aggregated in the respective bounds γ_i, $i = 2, 3, \ldots, N$. Hence, their assumptions are less demanding and significantly easier to verify, especially for nonlinear or infinite dimensional problems which are not covered by the methodology from [90]. This is due to the fact that the employed bounds in Assumption 5.38 do not have to be optimal.

5.5.2 Synchronous Generator

In this subsection the nonlinear example of the synchronous generator is considered in order to compare the suboptimality estimates from Problem 3.8 with those given in [120].

[20]Even for $\kappa = 1$, a horizon length of $N = 18$ is obtained by the technique from [120].

To this end, the accumulated bounds $M_i = \gamma_i$, $i = 2, 3, \ldots, N$, from Subsection 5.4.2 are employed. In order to estimate κ from Assumption 5.40, we compute

$$\kappa = \max_{x_0 \in \mathcal{L}_i \cap \mathcal{G} \backslash x^\star} \frac{V_2(x_0)}{\ell^\star(x_0)} - 1 = \max_{x_0 \in \mathcal{L}_i \cap \mathcal{G} \backslash x^\star} \frac{\min_{u \in \mathbb{U}} \left(\ell(x_0, u) + \ell^\star(f(x_0, u))\right)}{\ell^\star(x_0)} - 1$$

which yields $\kappa = 1.29963597$ or $\kappa = 2.12788813$ for $\ell_0(\cdot,\cdot)$ and $\ell_1(\cdot,\cdot)$, respectively. This leads to the estimates given in Table 5.8 and, thus, shows that the performance bounds computed in Subsection 5.4.2 are significantly tighter.

	$x_0 \in \mathcal{L}_0 \cap \mathcal{G}$, $\ell_0(\cdot,\cdot)$				$x_0 \in \mathcal{L}_1 \cap \mathcal{G}$, $\ell_1(\cdot,\cdot)$			
	$\gamma = 1.29963597$		$\gamma = 1$		$\gamma = 2.12788813$		$\gamma = 1$	
$\overline{\alpha}$	N	ΔN	N	ΔN	N	ΔN	N	ΔN
0	51	19	45	13	70	26	53	09
1/5	56	19	50	13	76	27	58	09
1/3	60	19	54	13	80	27	62	09
1/2	66	18	61	13	87	28	69	10

Table 5.8: Minimal horizon N such that a performance bound $(1-\eta(N)) \geq \overline{\alpha}$ is ensured by Formula (5.60) from [120] for the synchronous generator based on a numerically computed sequence $(M_i)_{i \in \mathbb{N}_0} = (\gamma_i)_{i \in \mathbb{N}_0}$. In addition, the needed prolongation of the optimization horizon N in comparison to the estimates resulting from our approach is given by ΔN, cf. Table 5.6.

For the reaction diffusion equation the suboptimality estimate $1 - \eta(N)$ from (5.60) yields the minimal stabilizing horizon $N = 8$ instead of $N = 7$ for the technique applied in Subsection 5.4.1. Furthermore, only a marginal improvement of the performance bounds can be observed for the examples considered in [120]. However, for these examples the inequality $\max_{n \in \mathbb{N}_0} c_n \leq 1$ holds for the equivalent sequence $(c_n)_{n \in \mathbb{N}_0}$ from Definition 5.30.[21] In contrast to that, the sequence corresponding to the linear example from the previous Subsection 5.5.1 exhibits $c_0 = 1$, $c_1 = 2.21$, $c_2 \approx 3.3938$, $c_4 \approx 2.1364$, and $c_n < 1$ for $n \geq 5$ and, thus, larger increments for the sequence $(M_i)_{i \geq 2} = (\gamma_i)_{i \geq 2}$. Here, a considerable reduction in terms of the required horizon length was obtained by applying Theorem 3.18 in comparison to [120].

In conclusion, the methodology developed in this thesis, which is based on [39], yields significantly better estimates than the prior approach from [120] and, thus, turns out to be superior. In addition, the technique presented in [120] does not allow for larger control horizons — a concept which led to a further improvement in view of Algorithms 4.24 and 4.28. Additionally, the concept of a multistep feedback was essential for the discretization carried out in Sections 5.1 and 5.2.

[21]For the reaction diffusion equation, this maximum is bounded by 1.0128.

Appendix A

Supplementary Results

This chapter is composed of two independent sections which exhibit a supplementary nature to this thesis. Section A.1 is concerned with the phenomenon of finite escape times and its ramifications on constructing a discrete time system from a continuous time one which is governed by a differential equation. In the ensuing section a model of the inverted pendulum on a cart is derived which serves as one of our main examples in order to explain the basic ideas of receding horizon control and the developed theory which can be employed in order to deduce asymptotic stability of the resulting receding horizon closed loop.

A.1 Finite Escape Times

In Section 1.3 continuous time systems governed by differential equations

$$\dot{x}(t) = g(x(t), \tilde{u}(t)),$$

were represented in our discrete time setting. Here, the meaning of Remark 1.18, in which attention was paid to existence of solutions, is explained in more detail. To be more precise, the phenomenon of finite escape times is dealt with. In order to avoid technical difficulties, we focus on systems with a finite dimensional state space $X \subseteq \mathbb{R}^n$.

Since time invariant differential equations are considered, being at time instant $t = 0$ and initial state x_0 is assumed without loss of generality. In addition, the control input is removed by plugging in a feedback $\mu : X \to U$, which corresponds to the most frequently employed approach in this thesis. Hence, the resulting dynamical system generated by the system dynamics $\dot{x}(t) = g(x(t), \mu(x(t))) = \tilde{g}(x(t))$ is considered. Let a parameter $T \in \mathbb{R}_{>0}$ be given and construct a discrete time system according to (1.18), i.e. the next state is given as the solution of the differential equation at time T with initial condition $\Phi(0; x_0) = x_0$. Consequently, existence of the solution at time T has to be ensured. However, theorems concerned with existence typically guarantee this only on an interval $[0, \delta]$ and require continuity of $\tilde{g}$ and, thus, implicitly of the involved control function.[1] Here, δ might be very small, cf. [70, Theorem 3.1 and p. 92]. Then, trying to extend the solution, i.e. applying the same theorem again, provides existence on $[\delta, \delta_2]$ and, by concatenating the obtained trajectories, a solution on $[0, \delta_2]$. Iterating this continuation process yields a sequence $(\delta_i)_{i \in \mathbb{N}}$ with $\delta_1 := \delta$ and, thus, existence of the solution $\Phi(\cdot; x_0)$ at time $\delta_{i+1} = \sum_{i=0}^{n} t_i$, $t_i := \delta_{i+1} - \delta_i$, $\delta_0 := 0$, but not necessarily at time T. For

[1] We refer to [115] for details concerning details related to discontinuous control functions.

example, $t_i := 2^{-(i+2)}T$ leads to $\sum_{i=0}^{\infty} t_i = 0.5T < T$. In order to illustrate this, we consider [70, Example 3.3], i.e. $\dot{x}(t) = -x(t)^2$ with $x_0 := -1$. Albeit the right hand side is locally Lipschitz for all $x \in \mathbb{R}$, the unique solution $x(t) = (t-1)^{-1}$ only exists on $[0, 1)$ and "explodes" at time $t = 1$, i.e. leaves any compact set. This explains the terminology finite escape time. We point out that this is closely linked to nonlinearity — at least in a finite dimensional state space.

In order to tackle this problem one may impose more regularity on the right hand side, e.g. a global Lipschitz condition, cf. [70, Theorem 3.2]. However, even simple systems like the considered example do not exhibit this. Hence, another remedy is proposed, which fits well to our receding horizon control strategy, centered at a (relaxed) Lyapunov inequality, cf. Section 3.1. To this end, we state the following theorem for time varying ordinary differential equations from [70]. Allowing for a time varying right hand side enables us to employ this theorem for controlled system: for a given control function $u(\cdot)$ one may rewrite $\dot{x}(t) = f(x(t), u(t))$ as $\dot{x}(t) = f(t, x(t))$.

Theorem A.1
Let an ordinary differential equation $\dot{x}(t) = f(t, x(t))$, $x(t_0) = x_0$ be given. Furthermore, suppose that the right hand side is piecewise continuous in t and locally Lipschitz for all $t \geq t_0$ in x in a domain $\mathbb{X} \subseteq X = \mathbb{R}^n$. Let $\mathcal{L} \subset \mathbb{X}$ be compact and, in addition, for $x_0 \in \mathcal{L}$ every solution $\Phi(\cdot; x_0, t_0)$ is contained in $\mathcal{L}$, i.e., $\bigcup_{t \geq t_0} \Phi(t; x_0, t_0) \subseteq \mathcal{L}$. Then, existence of a unique solution is guaranteed for all $t \geq t_0$.

Typically, we follow the path preordained by Theorem A.1 in order to ensure existence and uniqueness, cf. Section 4.4: for given optimization horizon N, a level set of the optimal value function $V_N(\cdot)$ is determined. Then, a relaxed Lyapunov inequality is deduced for $V_N(\cdot)$ and $\alpha \in [0, 1]$ which ensures that the state is, again, contained in this level set after implementing m control signals. Iterative application of this line of arguments allows to conclude existence and uniqueness and, thus, to exclude finite escape times.

Note that the setting of control systems allows to pick a control from the set of admissible input signals. Existence and uniqueness have only to be verified for this particular control input — an additional degree of freedom. The assumption of Theorem A.1 that the chosen control is at least piecewise continuous seems not to impose severe restrictions — particularly from a practitioner's point of view. Furthermore, note that this phenomenon, which is typical for nonlinear systems, is excluded for linear finite dimensional differential equations.

Another phenomenon occurring for nonlinear finite dimensional systems are multiple isolated equilibrium points, cf. the synchronous generator example from Section 4.4. Here, we emphasize that both equilibrium points are contained in the considered level set.

A.2 Inverted Pendulum

In this section a model for the inverted pendulum on a cart is motivated and derived, which can be done in various ways. In [15, pp.703–710] this is done exemplarily in order to illustrate the control of an unstable mechanical system. To this end, the process is subdivided into three stages: the physical model, the equations of motion, and the dynamic behaviour of the inverted pendulum which is termed stick balancer. In particular, the emphasis is put on the second and third stage, e.g. the equations of motion are deduced in three steps. First, the geometry of the physical model is taken into account using D'Alembert's method, then the force equilibrium is calculated, and in the third step the physical force-geometry relations are incorporated. In the third stage a Laplace transform of the equations of motions is carried out, cf. [111] for details on the Laplace transform. The resulting algebraic expressions are manipulated in order to obtain transfer and response functions which can be used to study the so called natural characteristics. However, since the analysis is based on transfer and response functions the study is confined to small angles in order to allow for linearizing and, as a consequence, for applying the Laplace transform.

In contrast to that, we aim at deriving a nonlinear model. To this end, we rely on the approach given in [58, pp.13–27] which is based on mechanics. In contrast to the model presented in [111], viscous friction at the pivot is incorporated. Simplified models can be found, e.g. in [69, 115]. In this approach translational mechanical systems with rotational elements are considered. In a preliminary stage the dynamics of the cart (trolley) and the pendulum are deduced separately. The cart is treated as a point mass M which is located at $r(t)$ and accelerated by a driving force $\beta u(t)$. Here the parameter β denotes a constant which transforms the control variable u, e.g. a voltage, into a force. Moreover, we allow for viscous friction $c\dot{r}(t)$ between the wheels and the rails, whose influence is assumed to be proportial to the speed of the cart and neglect drag friction as well as the friction in the wheel bearings. Hence, we obtain the equation

$$M\ddot{r}(t) = \beta u(t) - c\dot{r}(t) + H(t). \tag{A.1}$$

$H(t)$ stands for the horizontal component of the contact force. The vertical forces on the cart are assumed to be in balance. Furthermore, a rest position is fixed at $r = 0$ as the set point.

Since our goal is to steer the pendulum to the upright position, the position of the pendulum is measured by the angular displacement φ of the line joining its centre of mass with the pivot from the upward vertical. φ is measured, in contrast to [58], in a clockwise direction. Taking results from [74] into account, free-body diagrams are used for each element in order to deduce the desired model. Let $(x(t), y(t))$ denote the coordinates of the centre of mass at time t. Then the following equations describe the planar motion of the pendulum

$$\begin{aligned} m\ddot{x}(t) &= m\,\frac{d^2}{dt^2}\Big(r(t) + l\sin(\varphi(t))\Big) \\ &= m\ddot{r}(t) + ml\ddot{\varphi}(t)\cos(\varphi(t)) - ml\dot{\varphi}(t)^2\sin(\varphi(t)) = -H(t), \qquad \text{(A.2)} \\ m\ddot{y}(t) &= m\,\frac{d^2}{dt^2}\Big(l\cos\varphi(t)\Big) \\ &= -ml\ddot{\varphi}(t)\sin(\varphi(t)) - ml\dot{\varphi}(t)^2\cos(\varphi(t)) = V(t) - mg. \qquad \text{(A.3)} \end{aligned}$$

Taking account of viscous friction, the pendulum rotates about a pivot point which is mounted on a cart. The equation

$$J\ddot{\varphi} = l\sin(\varphi(t))V(t) + l\cos(\varphi(t))H(t) - c_P\dot{\varphi}(t) \tag{A.4}$$

determines the rotational movement of the pendulum. Here J denotes the moment of inertia of the pendulum. Inserting (A.2), (A.3) into (A.1) and (A.4) yields

$$\begin{aligned} (J + ml^2)\ddot{\varphi}(t) &= mgl\sin(\varphi(t)) - ml\cos(\varphi(t))\ddot{r}(t) - c_P\dot{\varphi}(t), \qquad \text{(A.5)} \\ (M + m)\ddot{r}(t) &= \beta u(t) - c\dot{r}(t) - ml\cos(\varphi(t))\ddot{\varphi}(t) + ml\dot{\varphi}(t)^2\sin(\varphi(t)). \qquad \text{(A.6)} \end{aligned}$$

We solve these equations for $\ddot{r}(t)$, $\ddot{\varphi}(t)$. To this end, (A.5) is plugged into (A.6) which provides

$$\begin{aligned}(M+n)(J+ml^2)\ddot{r}(t) &= (J+ml^2)\left(\beta u(t) - c\dot{r}(t) + ml\dot{\varphi}(t)^2\sin(\varphi(t))\right)\\ &\quad - ml\cos(\varphi(t))\left(mgl\sin(\varphi(t)) - ml\cos(\varphi(t))\ddot{r}(t) - c_P\dot{\varphi}(t)\right).\end{aligned}$$

Dropping the time variable, this equation is equivalent to

$$M(\varphi)\ddot{r} = (J+ml^2)\left(\beta u - c\dot{r} + ml\dot{\varphi}^2\sin(\varphi)\right) - ml\cos(\varphi)\left(mgl\sin(\varphi) - c_P\dot{\varphi}\right)$$

with $M(\varphi) := \left[(M+m)J + Mml^2 + m^2l^2\sin^2(\varphi)\right]$. Consequently, using the derived equation for (A.6) yields

$$M(\varphi)\ddot{\varphi} = ml\cos(\varphi)\left(c\dot{r} - \beta u - ml\dot{\varphi}^2\sin(\varphi)\right) - (M+m)\left(c_P\dot{\varphi} - mgl\sin(\varphi)\right).$$

Substituting φ by $-\theta$, i.e. $\sin(\varphi) = -\sin(\theta)$, $\dot{\varphi} = -\dot{\theta}$, and $\ddot{\varphi} = -\ddot{\theta}$, in order to orientate the considered system in a mathematically positive way, i.e. measured in an anti-clockwise direction, leads to

$$\begin{aligned}M(\theta)\ddot{r} &= (J+ml^2)\Big(\beta u - c\dot{r} - ml\dot{\theta}^2\sin(\theta)\Big) - ml\cos(\theta)\Big(c_P\dot{\theta} - mgl\sin(\theta)\Big),\\ M(\theta)\ddot{\theta} &= ml\cos(\theta)\Big(\beta u - c\dot{r} - ml\dot{\theta}^2\sin(\theta)\Big) - (M+m)\Big(c_P\dot{\theta} - mgl\sin(\theta)\Big).\end{aligned}$$

Note that these equations coincide with [58, p.26, eq.(26)]. Defining $x_1(t) := r(t)$ and $x_3(t) := \theta(t)$ yields the system of first order ordinary differential equations

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{1}{M(x_3)}\left[(J+ml^2)\left(\beta u - cx_2 - mlx_4^2\sin(x_3)\right) - ml\cos(x_3)\left(c_Px_4 - mgl\sin(x_3)\right)\right]\\ \dot{x}_3 &= x_4\\ \dot{x}_2 &= \frac{1}{M(x_3)}\left[ml\cos(x_3)\left(\beta u - cx_2 - mlx_4^2\sin(x_3)\right) - (M+m)\left(c_Px_4 - mgl\sin(x_3)\right)\right].\end{aligned} \tag{A.7}$$

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Bibliography

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Ehrenwörtliche Erklärung

Hiermit versichere ich an Eides statt, dass ich die von mir vorgelegte Dissertation mit dem Thema

"Stability Analysis of Unconstrained Receding Horizon Control Schemes"

selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Zudem erkläre ich, dass

- ich diese Arbeit in gleicher oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegt habe und
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Bayreuth, den 15. Dezember 2011

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