TABLES OF SUBSPACE CODES

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ABSTRACT. The main problem of subspace coding asks for the maximum possible cardinality of a subspace code with minimum distance at least d over $\mathbb{F}_q^n$, where the dimensions of the codewords, which are vector spaces, are contained in $K \subseteq \{0, 1, \ldots, n\}$. In the special case of $K = \{k\}$ one speaks of constant dimension codes. Since this emerging field is very prosperous on the one hand side and there are a lot of connections to classical objects from Galois geometry it is a bit difficult to keep or to obtain an overview about the current state of knowledge. To this end we have implemented an on-line database of the (at least to us) known results at `subspacecodes.uni-bayreuth.de`. The aim of this recurrently updated technical report is to provide a user guide how this technical tool can be used in research projects and to describe the so far implemented theoretic and algorithmic knowledge.

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1. INTRODUCTION

The seminal paper by Kötter and Kschischang [15] started the interest in subspace codes which are sets of subspaces of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^n$. Two widely used distance measures for subspace codes (motivated by an information-theoretic analysis of the Koetter-Kschischang-Silva model, see e.g. [21]) are the *subspace distance*

$$d_S(U, W) := \dim(U + W) - \dim(U \cap W) = 2 \cdot \dim(U + W) - \dim(U) - \dim(W)$$

and the *injection distance*

$$d_I(U, W) := \max\{\dim(U), \dim(W)\} - \dim(U \cap W),$$

where U and W are subspaces of $\mathbb{F}_q^n$. The two metrics are equivalent, i.e. it is known that $d_I(U, W) \le d_S(U, W) \le 2d_I(U, W)$.

The set of all k-dimensional subspaces of an $\mathbb{F}_q$-vector space V will be denoted by $\begin{bmatrix} V \\ k \end{bmatrix}_q$. For $n = \dim(V)$, its cardinality is given by the Gaussian binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} & \text{if } 0 \le k \le n; \\ 0 & \text{otherwise.} \end{cases}$$

A set $\mathcal{C}$ of subspaces of V is called a *subspace code*. The *minimum distance* of $\mathcal{C}$ is given by $d = \min\{d_S(U, W) \mid U, W \in \mathcal{C}, U \ne W\}$. If the dimensions of the *codewords*, i.e. the elements of $\mathcal{C}$ are contained in some set $K \subseteq \{1, \ldots, n\}$, $\mathcal{C}$ is called a $(n, \#\mathcal{C}, d; K)_q$ subspace code. In the unrestricted case $K = \{0, \ldots, n\}$ we use the notation $(n, \#\mathcal{C}, d)_q$ subspace code. In the other extreme case $K = \{k\}$, we use the notation $(n, \#\mathcal{C}, d)_q$ and call $\mathcal{C}$ a *constant dimension code*.

For fixed ambient parameters q, K and d, the *main problem of subspace coding* asks for the determination of the maximum possible size $A_q^S(n, d; K) := M$ of an $(n, M, d)_q$ subspace code and – as a refinement – the classification of all corresponding optimal codes up to isomorphism. Again, the simplified notations $A_q^S(n, d)$ and $A_q^S(n, d; k)$ are used for the unrestricted case $K = \{0, \ldots, n\}$ and the constant

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dimension case $K = \{k\}$, respectively. Note that in the latter case $d_S(U, W) = 2 \cdot d_I(U, W) \in 2 \cdot \mathbb{N}$ is an even number.

In general, the exact determination of $A_q^S(n, d; K)$ is a hard problem, both on the theoretic and the algorithmic side. Therefore, lower and upper bounds on $A_q^S(n, d; K)$ have been intensively studied in the last years, see e.g. [7]. Since the underlying discrete structures arose under different names in different fields of discrete mathematics, it is even more difficult to get an overview of the state of the art. For example, geometers are interested in so-called partial $(k-1)$-spreads of $\mathrm{PG}(n-1, q)$. Following the track of partial spreads, one can end up with orthogonal arrays or (s, r, μ)-nets. Furthermore, q-analogs of Steiner systems provide optimal constant dimension codes. For some sets of parameters constant dimension codes are in one-to-one correspondence with so-called vector space partitions.

The aim of this report is to describe the underlying theoretical base of an on-line database[1] maintained by the authors that tries to collect up-to-date information on the best lower and upper bounds for subspace codes. Whenever the exact value $A_q^S(n, d; K)$ could be determined, we ask for a complete classification of all optimal codes up to isomorphism. Since the overall task is rather comprehensive, we start by focusing on the special cases of constant dimension codes, $A_q^S(n, d; k)$, and (unrestricted) subspace codes, $A_q^S(n, d)$, using the subspace distance as metric. For a more comprehensive survey on network coding we refer the interested reader e.g. to [2]. For algorithmic aspects we refer the interested reader e.g. to [16].

The remaining part of this report is structured as follows. In Section 2 we outline how to use the tables. The currently implemented constructions and upper bounds are described in Section 3 and Section 4, respectively. The still rather unsteady Application programming interface (API) is the topic of Section 5. Finally we draw a conclusion in Section 6 and list some explicit tables on upper and lower bounds in an appendix.

2. HOW TO USE THE TABLES

On the website the two special cases $A_q^S(n, d; k)$ and $A_q^S(n, d)$ can be accessed via the menu items `CDC` (constant dimension code) and `MDC` (mixed dimension code), see Figure 1. Selecting the item `Table` yields the rough data that we will outline in this section. Selecting the item `Constraints` yields information abound the so far implemented general-purpose lower bounds, see Section 3, and upper bounds, see Section 4.

2.1. Constant dimension codes – CDC.

For a constant dimension code the dimension n of the ambient space (first *selection row*) and the field size q (second *selection row*) can be chosen. The current limits are $4 \le n \le 19$ and $2 \le q \le 9$. For each chosen pair of those parameters a table with the information on lower and upper bounds on constant dimension codes over $\mathbb{F}_q^n$ is displayed.

The rows of those tables are labeled by the minimum distance $d = d_S(\star)$ and the columns are labeled by the dimension k of the codewords. In the third *selection row* several *views* can be picked. The first three options, `short`, `normal`, and `large`, specify the subset of possible values for the parameters d and k. In the most extensive view `large`, k can take all integers between 0 and n. For d the integers between 1 and n are considered. As

- $A_q^S(n, d; 0) = 1$ for all $1 \le d \le n$;
- $A_q^S(n, d; k) = A_q^S(n, d; n-k)$;
- $A_q^S(n, 2d'+1; k) = A_q^S(n, 2d'+2; k)$ for all $d' \in \mathbb{N}_{>0}$;

one may assume $1 \le k \le \lfloor n/2 \rfloor$, $2 \le d \le n$, and $d \in 2\mathbb{N}$. These assumptions are implemented in the view `normal`. However, some exact values $of A_q^S(n, d; k)$ are rather easy to determine

- $A_q^S(n, 2; k) = \begin{bmatrix} n \\ k \end{bmatrix}_q$, since any two different k-dimensional subspaces of $\mathbb{F}_q^n$ have a subspace distance of at least 2;

[1] http://subspacecodes.uni-bayreuth.de

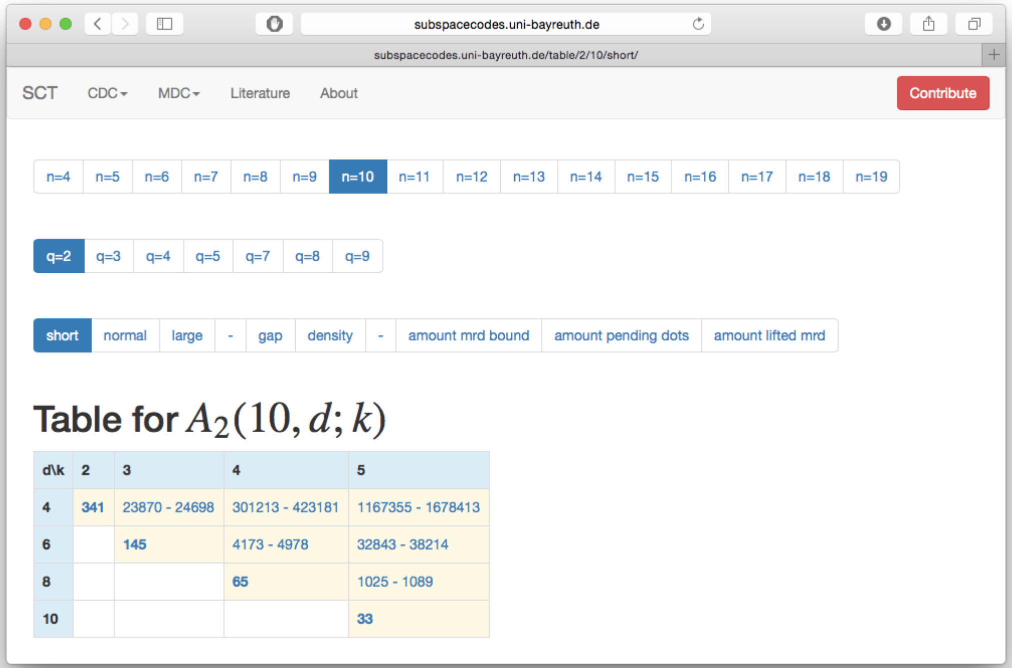


FIGURE 1. Tables of constant dimension codes

- if $d > 2k$, then we can have at most one codeword, i.e., $A_q^S(n, d; k) = 1$.

Thus, we may assume $2 \leq k \leq \lfloor n/2 \rfloor$, $4 \leq d \leq 2k$, and $d \in 2\mathbb{N}$. These assumptions are implemented in the view `short`. The standard selection is given by $n = 4$, $q = 2$ and view `short`.

Given one of these three views, a table entry may consist of

- a range l–u: An example is given by the parameters $q = 2$, $n = 7$, $d = 4$, $k = 3$, where $l = 329$ and $u = 381$. The meaning is that for the corresponding maximum cardinality of a constant dimension code only the lower bound l and the upper bound u is known, i.e., $329 \leq A_2^S(7, 4; 3) \leq 381$ in the example.
- a **bold** number m: An example is given by the parameters $q = 2$, $n = 10$, $d = 8$, $k = 4$, where $m = 65$. The meaning is that the corresponding maximum cardinality of a constant dimension code is exactly determined, i.e., $A_2^S(10, 8; 4) = 65$ in the example.
- a **bold** number m with an asterisk and a number l in brackets: An example is given by the parameters $q = 2$, $n = 6$, $d = 4$, $k = 3$, where $m = 77$ and $l = 5$. The meaning is that the corresponding maximum cardinality of a constant dimension code is exactly determined and all optimal codes have been classified up to isomorphism, i.e., $A_2^S(6, 4; 3) = 77$ and there are exactly 5 isomorphism types in the example [14]. Another example is given for the parameters $q = 2$, $n = 6$, $d = 4$, and $k = 2$, where there are exactly $131,044$ isomorphism types of constant dimension codes attaining cardinality $A_2^S(6, 4; 2) = 21$ [18].

Each nontrivial table entry is clickable and then yields further information on several lower and upper bounds, see Section 3 and Section 4 for the details.

In some cases, e.g., for the parameters $q = 2$, $n = 6$, $d = 4$, and $k = 3$, the corresponding codes are also available for download using the button called "file". The format of these codes is GAP[2].

Besides the views `short`, `normal`, and `large` for the selection of ranges for the parameters d and k, there are some additional views. The views `relative gap` and `ratio of bounds` condense the current lack of knowledge on the exact value of $A_q^S(n, d; k)$ to a single number. For the view `relative gap` this number is given by the formula

$$\frac{\text{upper bound} - \text{lower bound}}{\text{lower bound}},$$

i.e., we obtain a non-negative real number. While principally any number in $\mathbb{R}_{\geq 0}$ can be obtained, the largest relative gap in our database is currently given by about 0.728 for the parameters $q = 2, n = 19, d = 4, k = 9$. A gap of 0.0 corresponds to the determination of the exact value $A_q^S(n, d; k)$. The mentioned formula is also displayed on the webpage, when you move your mouse over the word `relative gap`. For the view `ratio of bounds` the corresponding number is given by the formula

$$\frac{\text{lower bound}}{\text{upper bound}},$$

which may take any real number in $(0, 1]$. The smallest ratio of bounds in our database is given by about 0.578 for the same parameters as above. Clearly, the largest relative gap yields the smallest ratio of bounds and vice versa as the function $x \mapsto \frac{1}{x} - 1$ is strictly decreasing in $(0, 1]$. A ratio of bounds of 1.0 corresponds to the determination of the exact value $A_q^S(n, d; k)$. The mouse-over effect is also implemented in that case.

Another type of views arose from some of the various constructions described in Section 3. They are all labeled as `amount pending dots` and `amount lifted mrd` and condense the *strength* of a certain construction to a single number in $\mathbb{R}_{\geq 1}$. This number is always given as the quotient between the currently best known lower bound and the value obtained by the respective construction. Here, a value of one means that the currently best known code can be obtained by the respective construction. A value larger than 1 measures how much better a more tailored construction is for this specific set of parameters compared to the respective general construction method. We remark that `amount pending dots` is still experimental and in some cases there may still be better codes obtained from the underlying very general construction technique, which has quite some degrees of freedom. With respect to upper bounds the additional view `amount mrd bound` is introduced. Here the displayed single number is given by the currently best known lower bound divided by the so-called MRD bound, see Subsection 4.2.

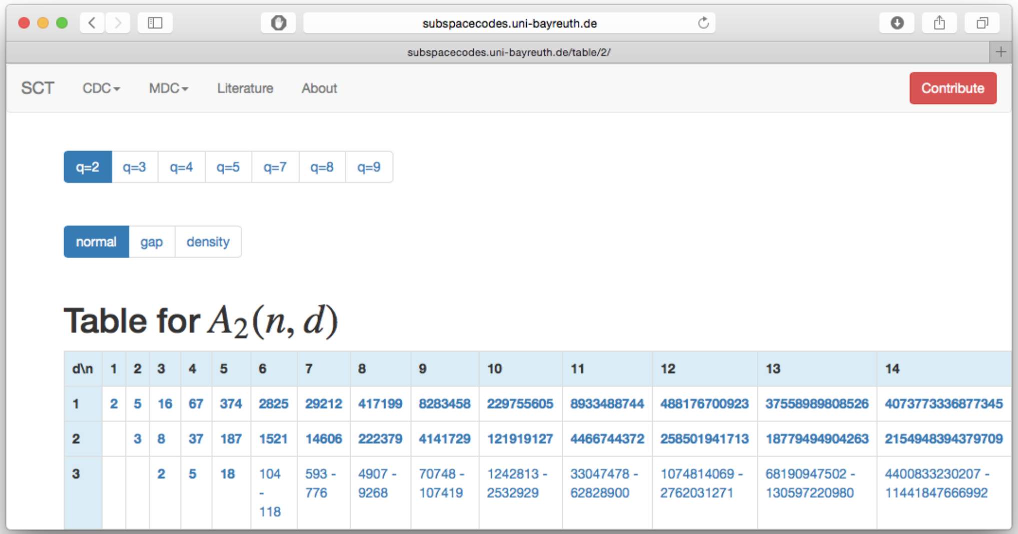
2.2. **Mixed dimension codes – `MDC`.** For a subspace code with mixed dimensions the field size q (*selection row* number one) can be chosen. The current limits are given by $2 \leq q \leq 9$. For each chosen parameter a table with the information on lower and upper bounds on mixed dimension codes over $\mathbb{F}_q^n$ is displayed, see Figure 2.

The rows of those tables are labeled by the distance $d = d_S(\star)$ and the columns are label by the dimension n of the ambient space $\mathbb{F}_q^n$. In the second *selection row* several *views* can be picked. The view `normal`, c.f. Subsection 2.1, already incorporates the restriction to $1 \leq d \leq n \leq 19$. The views `relative gap` and `ratio of bounds` condense the current lack of knowledge on the exact value of $A_q^S(n, d)$ to a single number. For the view `relative gap` this number is given by the formula

$$\frac{\text{upper bound} - \text{lower bound}}{\text{lower bound}},$$

i.e. we obtain a non-negative real number. While principally any number in $\mathbb{R}_{\geq 0}$ can be obtained, the largest relative gap in our database is currently given by about 2.899 for the parameters $q = 2, n =$

[2]`http://www.gap-system.org`



d\n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	2	5	16	67	374	2825	29212	417199	8283458	229755605	8933488744	488176700923	37558989808526	4073773336877345
2		3	8	37	187	1521	14606	222379	4141729	121919127	4466744372	258501941713	18779494904263	2154948394379709
3			2	5	18	104 - 118	593 - 776	4907 - 9268	70748 - 107419	1242813 - 2532929	33047478 - 62828900	1074814069 - 2762031271	68190947502 - 130597220980	4400833230207 - 11441847666992

FIGURE 2. Tables of subspace codes

$19, d = 4$. A relative gap of 0.0 corresponds to the determination of the exact value $A_q^S(n, d)$. The mentioned formula is also displayed on the webpage, when you move your mouse over the word `relative gap`. For the view `ratio of bounds` the corresponding number is given by the formula

$$\frac{\text{lower bound}}{\text{upper bound}},$$

which may take any real number in $(0, 1]$. The smallest ratio of bounds in our database is given by about 0.256 for the same parameters as above. Clearly the largest relative gap yields the smallest ratio of bounds and vice versa as the function $x \mapsto \frac{1}{x} - 1$ is strictly decreasing in $(0, 1]$. A ratio of bounds of 1.0 corresponds to the determination of the exact value $A_q^S(n, d)$. The mouse-over effect is also implemented in that case.

3. IMPLEMENTED CONSTRUCTIONS – LOWER BOUNDS

3.1. **Lifted MRD codes.** For matrices $A, B \in \mathbb{F}_q^{m \times n}$ the rank distance is defined via $d_R(A, B) := \operatorname{rk}(A - B)$. It is indeed a metric, as observed in [12].

Theorem 3.1. *(see [12]) Let $m, n \geq d$ be positive integers, q a prime power, and $\mathcal{C} \subseteq \mathbb{F}_q^{m \times n}$ be a rank-metric code with minimum rank distance d. Then, $|\mathcal{C}| \leq q^{\max(n,m) \cdot (\min(n,m) - d + 1)}$.*

Codes attaining this upper bound are called maximum rank distance (MRD) codes. They exist for all (suitable) choices of parameters. If $m < d$ or $n < d$, then only $|\mathcal{C}| = 1$ is possible, which may be summarized to the single upper bound $|\mathcal{C}| \leq \lceil q^{\max(n,m) \cdot (\min(n,m) - d + 1)} \rceil$. Using an $m \times m$ identity matrix as a prefix one obtains the so-called lifted MRD codes.

Theorem 3.2. *(see* [21]*) For positive integers k, d, n with $k \le n$, $d \le 2\min(k, n-k)$, and $d \equiv 0 \pmod 2$, the size of a lifted MRD code in $G_q(n,k)$ with subspace distance d is given by*

$$M(q,k,n,d) := q^{\max(k,n-k)\cdot(\min(k,n-k)-d/2+1)}.$$

If $d > 2\min(k, n-k)$, then we have $M(q,k,n,d) = 1$.

3.2. **Echelon-Ferrers or multilevel construction.** In [8] a generalization, the so-called multi-level construction, based on lifted MRD codes was presented. Let $1 \le k \le n$ be integers and $v \in \mathbb{F}_2^n$ a binary vector of weight k. By $\mathrm{EF}_q(v)$ we denote the set of all $k \times n$ matrices over $\mathbb{F}_q$ that are in row-reduced echelon form, i.e. the Gaussian algorithm had been applied, and the pivot columns coincide with the positions where v has a 1-entry.

Theorem 3.3. *(see* [8]*) For integers k, n, δ with $1 \le k \le n$ and $1 \le \delta \le \min(k, n-k)$, let $\mathcal{B}$ be a binary constant weight code of length n, weight k, and minimum Hamming distance 2δ.*

For each $b \in \mathcal{B}$ let $\mathcal{C}_b$ be a code in $\mathrm{EF}(b)$ with minimum rank distance at least δ. Then, $\cup_{b\in\mathcal{B}} \mathcal{C}_b$ is a constant dimension code of dimension k having a subspace distance of at least 2δ.

Theorem 3.4. *(see* [8]*) Let $\mathcal{F}$ be the Ferrers diagram of $\mathrm{EF}_q(v)$ and $\mathcal{C} \subseteq \mathrm{EF}_q(v)$ be a subspace code having a subspace distance of at least 2δ, then*

$$|\mathcal{C}| \le q^{\min_{0\le i\le \delta-1} \nu_i},$$

where ν_i is the number of dots in $\mathcal{F}$, which are neither contained in the first i rows nor contained in the rightmost $\delta - 1 - i$ columns.

The authors of [8] conjecture that Theorem 3.4 is tight for all parameters q, $\mathcal{F}$, and δ, which is still unrebutted.

Taking binary vectors with k consecutive ones we are in the classical MRD case. So, taking binary vectors v_i, where the ones are located in positions $(i-1)k+1$ to ik for all $1 \le i \le \lfloor n/k \rfloor$, clearly gives a binary constant weight code of length n, weight k, and minimum Hamming distance $2k$.

Observation 3.5. *(see e.g.* [17]*) For positive integers k, n with $n > 2k$ and $n \not\equiv 0 \pmod k$, there exists a constant dimension code in $G_q(n,k)$ with subspace distance $2k$ having cardinality*

$$1 + \sum_{i=1}^{\lfloor n/k \rfloor - 1} q^{n-ik} = 1 + q^{k+(n \bmod k)} \cdot \frac{q^{n-k-(n \bmod k)} - 1}{q^k - 1} = \frac{q^n - q^{k+(n \bmod k)} + q^k - 1}{q^k - 1}.$$

We remark that a more general construction, among similar lines and including explicit formulas for the respective cardinalities, has been presented in [22].

We remark that the general Echelon-Ferrers or multilevel construction contains the mentioned observation as a very easy special case. However, our knowledge on the size of an MRD code over $\mathrm{EF}(v)$ is still very limited. As mentioned, there is an explicit conjecture, which so far is neither proven nor disproven. The construction has even been fine-tuned to the so-called pending dots [9, 19] and the so-called pending blocks [20] constructions.

3.3. **Nonconstructive bounds.**

Theorem 3.6. *(Sphere covering, see* [15]*)*

$$A_q^S(n,d;k) \ge \begin{bmatrix} n \\ k \end{bmatrix}_q \Big/ \left(\sum_{i=0}^{(d/2-1)+1} \begin{bmatrix} k \\ i \end{bmatrix}_q \cdot \begin{bmatrix} n-k \\ i \end{bmatrix}_q \cdot q^{i^2} \right)$$

This lower bound is implemented as `sphere_covering`.

3.4. **Explicit, nonrecurring constructions.**

Theorem 3.7. *(Graham, Sloane, see* [23]*)*

$$A_q^S(n,d;k) \geq \frac{(q-1)\begin{bmatrix} n \\ k \end{bmatrix}_q}{(q^n-1)q^{n(d/2-2)}}$$

This lower bound is implemented as `graham_sloane`.

Theorem 3.8. *(Linearized polynomials, see* [15]*)*

$$A_q^S(n,d;k) \geq q^{(n-k)(k-d/2+1)}$$

This lower bound is implemented as `lin_poly`.

Theorem 3.9. *(Partial spreads, see* [11]*) If $d = 2k$ then:*

$$A_q^S(n,d;k) \geq \frac{q^n - q^k(q^{(n \bmod k)} - 1) - 1}{q^k - 1}$$

This lower bound is implemented as `partial_spread_3`.

4. Implemented upper bounds

Assuming $0 \leq k \leq n$ we always have $A_q^S(n,d;k) \geq 1$. Since we can take no more than all subspaces of a given dimension, we obtain the trivial upper bound $A_q^S(n,d;k) \leq \begin{bmatrix} n \\ k \end{bmatrix}_q$ which is implemented as `all_subs`.

4.1. ***Classical* coding theory bounds.**

Theorem 4.1. *(Singleton bound, see* [15]*)*

$$A_q^S(n,d;k) \leq \begin{bmatrix} n-d/2+1 \\ k-d/2+1 \end{bmatrix}_q$$

This upper bound is implemented as `singleton`.

Theorem 4.2. *(Sphere packing bound, see* [15]*)*

$$A_q^S(n,d;k) \leq \begin{bmatrix} n \\ k \end{bmatrix}_q \Big/ \left(\sum_{i=0}^{(d/2-1)/2+1} \begin{bmatrix} k \\ i \end{bmatrix}_q \cdot \begin{bmatrix} n-k \\ i \end{bmatrix}_q \cdot q^{i^2} \right)$$

This upper bound is implemented as `sphere_packing`.

Theorem 4.3. *(Johnson bounds, see* [11]*)*

$$A_q^S(n,d;k) \leq \left\lfloor \frac{(q^n-1) \cdot A_q^S(n-1,d;k-1)}{q^k-1} \right\rfloor$$

and

$$A_q^S(n,d;k) \leq \left\lfloor \frac{(q^n-1) \cdot A_q^S(n-1,d;k)}{q^{n-k}-1} \right\rfloor$$

These upper bounds are implemented as `johnson_1` and `johnson_2`.

Theorem 4.4. *(Anticode bounds, see* [11]*)*

$$A_q^S(n,d;k) \leq \left\lfloor \begin{bmatrix} n \\ k \end{bmatrix}_q \Big/ \begin{bmatrix} n-k+d/2-1 \\ d/2-1 \end{bmatrix}_q \right\rfloor$$

This upper bound is implemented as `anticode`.

4.2. **MRD bound.** Since the size of the lifted MRD code, see Theorem 3.1, is quite competitive it is interesting to compare the best known constructions with this very general explicit construction. Even more, lifted MRD codes are the basis for more involved constructions, see Subsection 3.2. From this point of view it is very interesting that an upper bound for the cardinality of constant dimension codes containing the lifted MRD code (of shape $k \times (n-k)$ and rank distance $d/2$) can be stated that is tighter than the best known general upper bounds:

Theorem 4.5. *(see* [9, Theorem 10 and 11]*) Let $\mathcal{C} \subseteq G_n(k,q)$, where $n \geq 2k$, with minimum subspace distance d that contains the lifted MRD code.*

- *If $d = 2(k-1)$ and $k \geq 3$, then $|\mathcal{C}| \leq q^{2(n-k)} + A_q(n-k, 2(k-2); k-1)$;*
- *if $d = k$, where k is even, then $|\mathcal{C}| \leq q^{(n-k)(k/2+1)} + \begin{bmatrix} n-k \\ k/2 \end{bmatrix}_q \frac{q^n - q^{n-k}}{q^k - q^{k/2}} + A_q(n-k, k; k)$.*

4.3. **Bounds for spreads.**

Theorem 4.6. *(*[1]*; see also* [4, p. 29]*, Result 2.1 in* [3]*) $\mathbb{F}_q^n$ contains a k-spread if and only if k divides n, where we assume $1 \leq k \leq n$ and $k, n \in \mathbb{N}$.*

The corresponding exact value is implemented as upper bound `spread`.

Theorem 4.7. *(*[3]*; see also* [13] *for the special case $q = 2$) For positive integers $1 \leq k \leq n$ be positive integers with $n \equiv 1 \pmod{k}$ we have $A_q^S(n, 2k; k) = \frac{q^n - q}{q^k - 1} - q + 1 = q \cdot \frac{q^{n-1} - 1}{q^k - 1} - q + 1 = \frac{q^n - q^{k+1} + q^k - 1}{q^k - 1}$.*

The corresponding exact value is implemented as upper bound `partial_spread_2`.

Theorem 4.8. *(Corollary 8 in* [5]*) If $n = k(t+1) + r$ with $0 < r < k$, then*

$$A_q^S(n, 2k; k) \leq \sum_{i=0}^{t} q^{ik+r} - \lfloor \theta \rfloor - 1 = q^r \cdot \frac{q^{k(t+1)} - 1}{q^k - 1} - \lfloor \theta \rfloor - 1,$$

where $2\theta = \sqrt{1 + 4q^k(q^k - q^r)} - (2q^k - 2q^r + 1)$.

We remark that this theorem is also restated as Theorem 13 in [7] and as Theorem 44 in [10] with the small typo of not rounding down θ (Ω in their notation). The corresponding upper bound is implemented as `partial_spread_4`.

Theorem 4.9. *(see* [6]*) For each integer $m \geq 2$ we have*

(a) $A_2(3m, 6; 3) = \frac{2^{3m} - 1}{7}$;
(b) $A_2(3m+1, 6; 3) = \frac{2^{3m+1} - 9}{7}$;
(c) $A_2(3m+2, 6; 3) = \frac{2^{3m+2} - 18}{7}$.

The corresponding upper bound is implemented as `partial_spread_1`.

Theorem 4.10. *(Theorem 4.3 in* [17]*) For each pair of integers $t \geq 1$ and $k \geq 4$ we have $A_2^S(k(t+1) + 2, 2k; k) = \frac{2^{k(t+1)+2} - 3 \cdot 2^k - 1}{2^k - 1}$.*

The corresponding upper bound is implemented as `partial_spread_kurz_q2`.

Lemma 4.11. *(Lemma 4.6 in* [17]*) For integers $t \geq 1$ and $k \geq 4$ we have $A_3^S(k(t+1) + 2, 2k; k) \leq \frac{3^{k(t+1)+2} - 3^2}{3^k - 1} - \frac{3^2 + 1}{2}$.*

The corresponding upper bound is implemented as `partial_spread_kurz_q3`.

5. Application programming interface

There is also an API available to access most data of the database. It is inspired by the REST (representational state transfer) style and only GET querys are supported. In order to access the data for the constant dimension case with parameters q, n, d and k, you query the URL

http://subspacecodes.uni-bayreuth.de/api/q/n/d/k/

Similarly in the mixed dimension case, the URL is

http://subspacecodes.uni-bayreuth.de/api/q/n/d/

The result is a JSON file which contains a subset of the following attributes:

- request = contains your specified q, n, d and k
- $\{$lower,upper$\}$_bound = lower or upper bound for the value $A_q(n, d; k)$
- comments = commentaries to this entry
- nondeduced = if the parameters are no parameters that are also viewable in the "short" mode, then they are trivial or computed using other parameters. nondeduced lists these other parameters.
- $\{$lower,equal,upper$\}$_bound_constraints = list of tuples which contain name, parameter and value of the applied constraints
- classified = boolean that is true if $A_q(n, d; k)$ is classified up to isomorphism
- known codes = list of tuples of size, details, file (to enable automatic downloads) and nrisotypes (the number of isomorphism types of this entry)
- liftedmrdsizebound = the bound for codes that contains the lifted MRD code as described in 4.2

In order to download the codes, you have to use the attribute file above and the URL

http://subspacecodes.uni-bayreuth.de/codes/*file*

We want to remark that the API (as well as the whole homepage) is still in an early evolutionary phase and therefore changes are likely to occur. As an example, the URL

`http://subspacecodes.uni-bayreuth.de/api/2/6/4/3/`

yields the output:

{"upper_bound_constraints": [{"parameter": "", "name": "all_subs", "value": 1395},
{"parameter": "", "name": "anticode", "value": 93}, {"parameter": "2", "name": "ilp_2", "value": 93},
{"parameter": "4", "name": "ilp_3", "value": 93}, {"parameter": "", "name": "singleton", "value": 155},
{"parameter": "", "name": "sphere_packing", "value": 1395}, {"parameter": "1", "name": "ilp_1", "value": 81},
{"parameter": "5", "name": "ilp_4", "value": 81}, {"parameter": "", "name": "johnson_1", "value": 81},
{"parameter": "", "name": "johnson_2", "value": 81}],
"known_codes": [{"nrisotypes": 5, "details": "", "file": "code_2_6_4_3_optimal_size_77.zip", "size": 77}], "upper_bound": 77,
"classified": true, "lower_bound": 77, "lower_bound_constraints": [{"parameter": "", "name": "HonoldKiermaierKurz_n6_d4_k3",
"value": 77}, {"parameter": "", "name": "construction_1", "value": 71}, {"parameter": "", "name": "sphere_covering",
"value": 15}, {"parameter": "", "name": "trivial_1", "value": 0}, {"parameter": "3",], "request": [2, 6, 4, 3], "liftedmrdsize-
bound": 71, "comments": "", "equal_bound_constraints": []}

6. Conclusion

The collection of the known results on lower and upper bounds for subspace codes is an ongoing project. So far we have merely implemented the tip of the iceberg of the available knowledge. We still hope that the emerging on-line data base and the accompanying user's guide is already valuable for researcher in the field at this early stage. If you want to support us in this task, please let us know any known constructions, bounds or papers that we have missed so far via daniel.heinlein@uni-bayreuth.de or the *contribute*-button in the upper right corner of the webpage `subspacecodes.uni-bayreuth.de`.

Tracing back results to their original source is a task on its own. We want to work on that issue more intensively in the future. If you observe possible enhancements in that direction, please let us know. Critique, suggestions for improvements and feature requests are also highly welcome.

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APPENDIX A. TABLES FOR BINARY CONSTANT DIMENSION CODES

$n=6$	2	3
4	21 * (131044)	77 * (5)
6		9

$n=7$	2	3
4	41	329–381
6		17

$n=8$	2	3	4
4	85	1326–1493	4801–6477
6		34	257–289
8			17

$n=9$	2	3	4
4	169	5986–6205	37265–50861
6		73	1033–1158
8			33

$n=10$	2	3	4	5
4	341	23870–24698	301213–423181	1167355–1678413
6		145	4173–4978	32843–38214
8			65	1025–1089
10				33

$n=11$	2	3	4	5
4	681	97526–99718	2134417–3370453	16814481–27943597
6		290	16641–19787	262780–328708
8			129–133	4097–4292
10				65

$n=12$	2	3	4	5	6
4	1365	385515–398385	19664917–27223014	269602811–445225968	1074043037–1816333805
6		585	66569–79170	2098185–2613798	16777289–21366020
8			273	16385–17568	262145–278980
10				129	4097–4225
12					65

$n=13$	2	3	4	5	6
4	2729	1597245	157319497–217544769	4311781777–7193022828	34376552849–57886442918
6		1169	266891–319449	16810059–20918757	268439629–339835228
8			545	65569–72133	2097153–2284118
10				257–260	16385–16772
12					129

APPENDIX B. TABLES FOR TERNARY CONSTANT DIMENSION CODES

$n=6$	2	3
4	91	754–784
6		28

$n=7$	2	3
4	271	6977–7651
6		82

$n=8$	2	3	4
4	820	60259–68375	543142–627382
6		244–248	6562–6724
8			82

$n=9$	2	3	4
4	2458	549667–620740	14585908–16821959
6		757	59059–61014
8			244

$n=10$	2	3	4	5
4	7381	5086234–5582307	394218370–458168194	3539131096–4104557996
6		2269	531689–558741	14349658–14887416
8			730–733	59050–59536
10				244

$n=11$	2	3	4	5
4	22141	45776125–50289024	10474939111–12361041947	282444122389–335382904522
6		6805–6809	4789531–5024303	387441117–409003029
8			2188–2202	531442–536562
10				730

APPENDIX C. TABLES FOR QUATERNARY CONSTANT DIMENSION CODES

$n=6$	2	3
4	273	4137–4225
6		65

$n=7$	2	3
4	1089	66828–70993
6		257

$n=8$	2	3	4
4	4369	1054373–1132819	16874321–18245201
6		1025–1033	65537–66049
8			257

$n=9$	2	3	4
4	17473	16945153–18179409	1078557605–1164551259
6		4161	1048593–1061936
8			1025

$n=10$	2	3	4	5
4	69905	273727489–290821444	69040760145–74754799185	1104214839637–1193665040475
6		16641	16778246–17110276	1073745957–1088484400
8			4097–4104	1048577–1050625
10				1025

$n=11$	2	3	4	5
4	279617	4379639873–4654011924	4399125068709–4783502960915	281476055268261–306494895880785
6		66561–66569	268500993–273715279	68719743536–70152181776
8			16385–16418	16777217–16826412
10				4097

APPENDIX D. TABLE FOR (UNRESTRICTED) BINARY SUBSPACE CODES

$q=2$	1	2	3	4	5	6	7	8	9	10
1	2	5	16	67	374	2825	29212	417199	8283458	229755605
2		3	8	37	187	1521	14606	222379	4141729	121919127
3			2	5	18	104–118	593–776	4907–9268	70748–107419	1242813–2532929
4				5	9	77	330–463	4803–9268	37267–107419	1167613–2532929
5					2	9	34	261–359	1990–2462	32971–48402
6						9	17	257–357	1033–2462	32843–48402
7							2	17	62–66	1025–1219
8								17	33	1025–1219
9									2	33
10										33

APPENDIX E. TABLE FOR (UNRESTRICTED) TERNARY SUBSPACE CODES

$q=3$	1	2	3	4	5	6	7	8	9
1	2	6	28	212	2664	56632	2052656	127902864	13721229088
2		4	14	132	1332	34608	1026328	77705744	6860614544
3			2	10	53–56	764–966	13247–15802	544431–762424	29009271–34890316
4				10	28	754–966	6978–15802	543144–762424	14585910–34890314
5					2	28	160–164	6568–7222	117620–123544
6						28	82	6562–7220	59060–123542
7							2	82	484–488
8								82	244
9									2

APPENDIX F. TABLE FOR (UNRESTRICTED) QUATERNARY SUBSPACE CODES

$q=4$	1	2	3	4	5	6	7	8
1	2	7	44	529	12278	565723	51409856	9371059621
2		5	22	359	6139	379535	25704928	6269331761
3			2	17	127–130	4154–4771	131317–144060	16881731–20519579
4				17	65	4137–4771	66829–144060	16874323–20519577
5					2	65	510–514	65545–68117
6						65	257	65537–68115
7							2	257
8								257

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