Limitations of Stabilizing Effects of Fundamentalists Facing Positive Feedback Traders

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Abstract

The behavior of different types of investors plays an important role in determining asset price fluctuations. We analyze financial interactions between fundamentalists and chartists within a heterogeneous agent model. In particular, we ask whether the presence of fundamentalists is enough to stabilize prices. In contrast to related work, which is based on simulations, we analytically prove that the presence of fundamentalists is not sufficient to avoid asset price bubbles. The behavior of trend followers might lead to exploding price processes irrespective of fundamentalists' investment decisions. Upper boundaries for positive feedback traders' investments are derived which are necessary to avoid exploding prices. In this situation, intervention measures might be necessary in order to stabilize stock/asset markets.

Keywords

Heterogeneous Agent Model; Feedback-based Trading Strategies; Fundamentalists; Chartists; Trend Followers; Financial Bubbles

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JEL classification codes

D84, G01, G11

1 Motivation

Bubbles in financial markets have over and over again caused major macroeconomic problems, e.g., very prominently in the subprime havoc of 2007/8 [Reinhart and Rogoff, 2009]. While misguided macroeconomic policies have been chief among the usual suspects in the search for the causes of such aberrations, an important strand of the literature is focusing on the question whether the specific behavior of market participants is responsible for price bubbles. In particular, heterogeneous agent models (HAMs) analyze how both chartists and fundamentalists are able to determine asset price movements [Hommes, 2006a].

Chartists, e.g., trend followers, only trade based on information of the price process, i.e., they assume that all important information is present in the asset price [Graham et al., 1934]. In contrast, fundamentalists have some fundamental value in mind and trade according to the perceived over- or undervaluation of the underlying asset. Trend followers magnify the current trend, either positively or negatively, as they trade according to the rule: the greater the absolute value of the slope of the price process, the greater the investment or disinvestment [Covel, 2004]. Fundamentalists, in contrast, invest or disinvest if the price is below or above the fundamental value, thereby pushing the asset price towards its fundamental value. Traders act self-interested[1], with the intention of making profit, and do not ponder the consequences on prices. As a consequence of the two different investment strategies, the presence of chartists can cause exploding prices (see also De Long et al. [1990b]) while fundamentalists are associated with a stabilizing influence on assets. Out of this, the following question arises:

> *Are the balancing effects of fundamentalists strong enough to stabilize prices or are chartists' destabilizing impacts able to surpass the fundamentalists' compensatory behavior?*

To address this research question, in recent years HAMs have been increasingly applied in respective analyses [Gaunersdorfer and Hommes, 2005, Hommes, 2002, Lux, 1995, 1998, Lux and Marchesi, 1999, 2000].[2] The models typically use bounded rational agents, (imperfect) heuristics or rules of thumb, and nonlinear dynamics (which might be chaotic). Some studies state that fundamentalists' stabilizing effects are not necessarily strong enough to stabilize markets as summarized in Hommes [2006a]. However,

[1] If they were alone on the market both fundamentalists' and trend followers' strategies are self-fulfilling prophecies and would in general make money.

[2] The results of theses studies in principle provide useful explanations for many stylized facts, like excess volatility, high trading volume, temporary bubbles, trend following, sudden crashes, mean reversion, clustered volatility and fat tailed distribution returns. For an excellent overview regarding HAM see, among others, Hommes [2006a].

the results are mostly obtained through simulations and are not analytically proven (see, e.g., Hommes [2006a]).[3] A second question arises:

> *Is it possible to analytically prove that chartists' behavior can lead to exploding prices irrespective of fundamentalists' compensatory effects?*

The main contribution of our paper is a mathematically rigorous proof that chartists' behaviour – in particular the behavior of linear feedback traders without rational expectations and without information about the market (fundamental value, trading volume, or even prices) – can surpass the stabilizing effects of fundamentalists, i.e., traders with rational expectations of the fundamental value. Put differently, prices can explode as the stabilizing effects of fundamentalists can be outweighed by linear feedback traders. Unstable price developments are the result, which in turn increase the likelihood of the occurrence of a financial bubble. As shown in the proof, thresholds for model inherent values can be specified for which a bubble will occur with certainty. Furthermore, an assignment for external parameters exists which means that the thresholds of the inherent values can be reached at all. The analysis shows that even fundamentalists without any liquidity constraints and with perfect information about the price, the fundamental value, and the market's characteristics are not sufficient to stabilize a very simply constructed market based on (excess) demand if the feedback trader's initial investment is large enough.

In applied mathematics resp. control engineering there are many new results concerning technical trading strategies [Barmish and Primbs, 2011, 2015]. Performance properties of chartist strategies are proven and it is explained why it is reasonable to trade according to a feedback strategy. In contrast to the feedback trading literature where the price taker property is usually presumed, we study the effects of trading strategies in an HAM that displays phenomena caused by (excess) demand [Baumann, 2015a].

The paper is organized as follows: Section 2 explains the price model as well as the investment strategies of feedback traders and fundamentalists. Section 3 answers the main question of the paper: whether the presence of fundamentalists is sufficient to stabilize the market. Section 4 presents further results based on calculations of Section 3 and Section 5 discusses limitations of the work at hand, gives ideas for future work, and concludes the paper.

[3] Among the few exception is, e.g., De Long et al. [1990b], where the effect of positive feedback traders and informed speculators, who evaluate and consider the needs of the other market participants, especially the growing needs of the positive feedback traders, in a three period market model facing fundamantalists is investigated. De Long et al. [1990b] show that the interaction of these two trader types pushes the price away from the fundamental value under specific assumptions and despite the fundamentalists' stabilizing behavior. The present work differs from De Long et al. [1990b] in that we do not investigate how two types of traders, namely positive feedback traders and informed speculators, jointly push up the price but rather only trend followers. Positive feedback traders have, compared to long-sighted informed speculators, less information about the market and are short-sighted. We further do not assume a predetermined end of the market.

2 Model structure

The model consists of a one asset market and is populated with two types of heterogeneous agents, namely fundamentalists and chartists. How their interaction with the market maker takes place is illustrated in Section 2.1. Section 2.2 presents the price process in the interactive market model. Sections 2.3, 2.4, and 2.5 introduce the traders and their expectations. For simplification of the analysis we assume that there is only one feedback trader and one (or exceptionally none) fundamentalist. One can think of many, coordinated traders that are summarized to one big trader, i.e., we treat, e.g., all existing feeadback traders as one average feedback trader.[4]

2.1 Timeline

At the beginning of every period $t \in \{0, 1, \ldots, T\}$, each agent $\ell \in \{FT, FU\}$, where FT is the feedback trader (chartist) and FU the fundamentalist, decides on how to invest based on his investment strategy, where T is unknown or even ∞. Each investment strategy I_t^ℓ is characterized by a different heuristic (rule of thumb) in order to make a decision. According to the strategy, each agent then allocates his financial resources among the asset market. In doing so, the trader knows the past market data. The resulting changes in the investments, denoted by ΔI_t^ℓ, are cleared by a market maker who adjusts asset prices according to (excess) demand. After the traders have observed the price change Δp_t and hence their own gains or losses Δg_t^ℓ in the recent period, they use this information to derive their next investment decision.[5] One can imagine, that at the end of each period the investment is capitalized and reinvested. The timeline of the traders' and the market maker's decisions and interactions is shown in the diagram of Figure 1.

2.2 Price Process for the Interactive Market Model

In feedback trading literature, price is usually determined through a certain price process, like geometric Brownian motion (GBM), which is exogenously given [Barmish and Primbs, 2015]. This implies that the traders are not able to influence the price. To avoid this price taker property, which is a strong restriction of every market model, so-called agent based price models have evolved in academically economic literature (see, e.g., Hommes [2006a]). According to these models, the price is a function of traders' investment decisions. We denote the sum of all traders' changes of investment at time t with $\Delta I_t = \sum_\ell \Delta I_t^\ell$. Based

[4] There is indeed no difference between one feedback trader with an initial investment I_0^{FT} and fixed K, see equation (3), and n feedback traders with initial investments $\frac{I_0^{FT}}{n}$ and the same K. I.e., for the feedback traders this summarization is without loss of generality (WLOG). Whether this assumption is WLOG for fundamentalists, too, has to be examined in future work.

[5] For all processes α_t we set $\Delta\alpha_t = \alpha_t - \alpha_{t-1}$ as the change of the underlying process, e.g., Δg_t^ℓ is the period profit while g_t^ℓ is the overall gain/loss of trader ℓ.

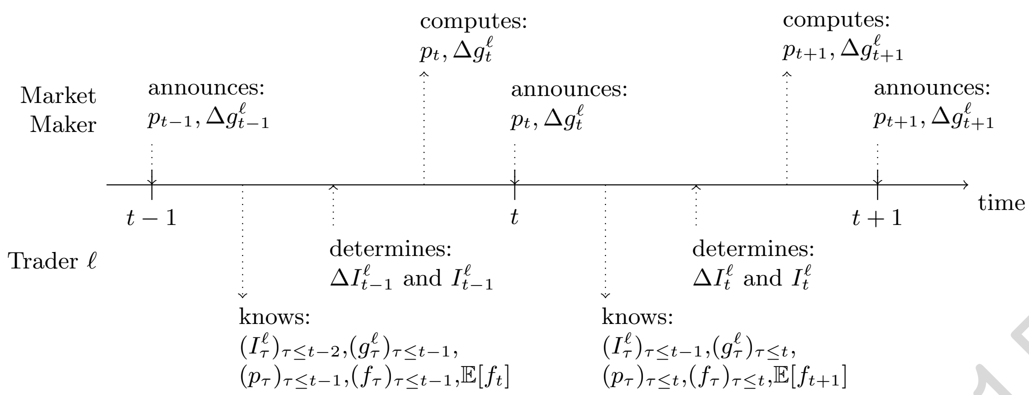


Figure 1: Timeline of the traders' and the market maker's decisions and interactions with $\Delta g_t^\ell = I_{t-1}^\ell \cdot \frac{\Delta p_t}{p_{t-1}}$

on the idea of interacting agents, in Baumann [2015a] a pricing model that fulfills the law of (excess) demand, namely

(I1) $p_{t+1} = p_t$, if $\Delta I_t = 0$

(I2) $p_{t+1} \to \infty$, if $\Delta I_t \to \infty$

(I3) $p_{t+1} \to 0$, if $\Delta I_t \to -\infty$

(I4) p_{t+1} strictly monotonous increasing in ΔI_t

is constructed. Thereby, for the purpose of simplification, we assume a infinite supply. That means, the law of supply and demand reduces to a law of (excess) demand.[6] This model, which is in a sense a natural generalization of the GBM (proven in Baumann [2015a]), in its general form is given by

$$p_{t+1} = p_t \cdot e^{M^{-1}\Delta I_t} \tag{1}$$

$$= p_0 \cdot e^{M^{-1} I_t} \tag{2}$$

where $M > 0$ is a scaling factor expressing the trading volume of the underlying stock[7]. This pricing model is closed through a market maker (see Drescher and Herz [2012]). As common practice, the market maker acts as a privileged trader that sets prices according to (excess) demand (see Figure 2) and hence ensures market clearing (cf. the role of a broker in stock markets) (see, e.g., Hommes [2006a]).[8] In Baumann and Baumann [2015] it is shown that this market model meets several stylized facts formulated in Hommes [2006b].

[6]Alternatively, one can define (I1)-(I4) by use of the buying/selling decision $B_t := I_t - \frac{p_t}{p_{t-1}} \cdot I_{t-1}$ instead of the change of investment $I_t - I_{t-1}$. Then, a change of investment caused by price increase would not affect the price. The use of the buying/selling decision instead of the change of investment would, according to simulations, affect the proposition of this paper only quantitatively but not qualitatively. However, a finite supply would make the analysis much more complicated.

[7]If nothing else is given, for simplicity M is set to $M = 1$.

[8]A possible profit making and the survival of the market maker will not be discussed in the work at hand but is an interesting topic for future work.

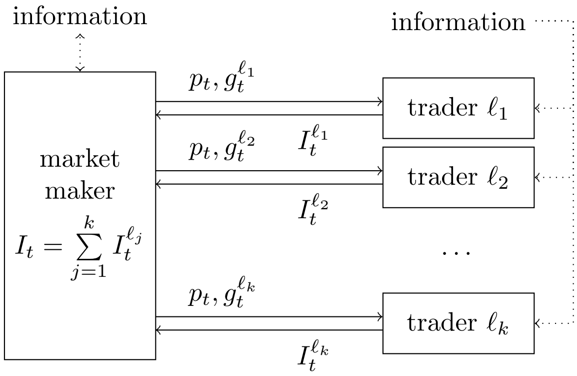


Figure 2: Schematical representation of the role of the market maker with k traders

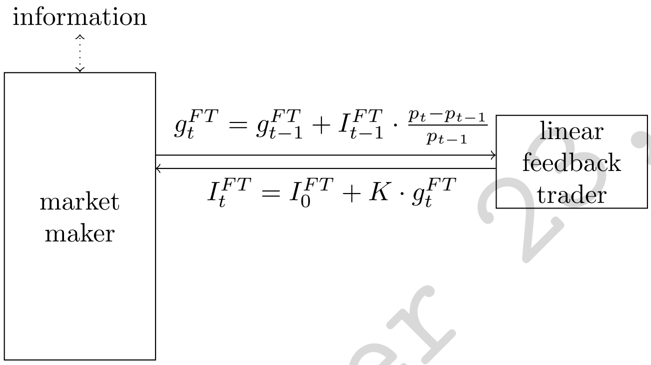


Figure 3: Schematical interaction between market maker and linear feedback trader

2.3 Feedback Traders

In Barmish [2011], Barmish and Primbs [2011, 2015], and Baumann [2015a] a special class of trading strategies based on control techniques, namely feedback trading, is outlined. The traders using such a trading strategy are called feedback traders and utilize neither fundamentals nor the absolute asset value to determine their investment. They only take into account their own gains and losses for specifying their investment decision. For that reason, their strategy depends on prices relative to their previous investments, i.e., feedback traders are chartists because gains or losses, respectively, are a function of the price but not of any fundamental value. From a control theoretic point of view, feedback traders treat the price like a disturbance variable and their strategy needs to be robust against this disturbing influence. For calculating the gain of a certain trader, the market maker takes into account the traders' investment and the asset price.[9] Not only for feedback traders it is true that the investment affects the gain, but for feedback traders it additionally holds that the gain determines the investment.[10]

One specific feedback strategy, for example discussed in Barmish and Primbs [2011, 2012] and Bau-

[9]The price is a function of all traders' investment, see section 2.2, and especially Figure 2.

[10]In the listed literature, mostly continuous time models are applied whereas this analysis uses a discrete time model because this is, as for example stated by Barmish and Primbs [2011], the weaker, more general assumption.

mann [2015b], is the so-called (positive) linear feedback strategy

$$I_t^{FT} := I_0^{FT} + K \cdot g_t^{FT} \tag{3}$$

performed by the linear feedback trader who calculates his investment I_t^{FT} at time t as a linear function of his gain/loss function g_t^{FT} using the initial investment $I_0^{FT} > 0$ and a feedback parameter $K > 0$. Figure 3 shows a feedback loop between the gain or loss g^{FT} of a linear feedback trader and his investment I^{FT}. By calculating the gain or loss of a specific trader (or group of traders) ℓ through

$$g_t^{\ell} = \sum_{i=1}^{t} I_{i-1}^{\ell} \cdot \frac{p_i - p_{i-1}}{p_{i-1}} \tag{4}$$

where p_t denotes the price process[11] and I_t^{ℓ} the trader's investment at time t, it follows that linear feedback traders are trend followers given $I_t^{FT} > 0$ (see also (7)). A trader is called trend follower (cf. Covel [2004]) if his investment increases when prices are rising and decreases when prices are falling. It should be mentioned that the particular bought/sold investment amount at time $t \geq 1$ is given through

$$\Delta I_t^{FT} = I_t^{FT} - I_{t-1}^{FT} \tag{5}$$

$$= K \cdot (g_t^{FT} - g_{t-1}^{FT}) \tag{6}$$

$$= K \cdot I_{t-1}^{FT} \cdot \frac{p_i - p_{i-1}}{p_{i-1}} \tag{7}$$

whereas I_t^{FT} denotes the total investment at time t (all particular investment amounts up to time t) of a feedback trader FT. Rising prices lead to an increasing gain of the linear feedback trader if $I_t^{FT} > 0$ and thus his investment is increasing, too. Analogously, falling prices lower the gain and the trader disinvests. In Baumann [2015a] it is shown that in case only one feedback trader is acting on the market with a price process described by (1), it holds that

$$I_t > 0 \ \forall t, \tag{8}$$

$$\Delta I_t > 0 \ \forall t, \text{ and} \tag{9}$$

$$\text{if } \exists t : \Delta I_t > \Delta I_{t-1} \tag{10}$$

$$\Rightarrow \Delta I_{t+1} > \Delta I_t. \tag{11}$$

This is important as it will be shown that, together with the results of Section 3, the price explosion effects of feedback traders, that would have occurred when acting without fundamentalists, can be compensated

[11] The relative price change $\frac{p_t - p_{t-1}}{p_{t-1}}$ is called return on investment (ROI).

by fundamentalists — at least up to a certain degree.

Two typical investment paths can be identified in the scenario only one feedback-based trader is acting on the market. The two paths are shown in Figure 4a and Figure 4b where the stock price p_t is indicated with a solid line and the feedback trader's investment with a dashed one. If I_0^{FT} lies below a specific threshold, I_t^{FT} converges (Figure 4a), if it is above this threshold the investment explodes (Figure 4b). In Baumann [2015a], a non-closed formula is given for determining the threshold. Specific values can only be derived through simulation like the one in Figure 4 and algorithmically localizing the threshold.

Control-based trading strategies like the one presented in Barmish and Primbs [2014] are interesting for further analysis since there exist several notable results in literature, for example the guarantee of non-negative profit for the so-called simultaneous-long-short (SLS) strategy that has initial investment zero and consists of two linear feedback strategies for continuously differentiable prices (Arbitrage!). For prices following a GBM, for price processes allowing for jumps (Merton's Jump Diffusion Model (MJDM)), and for all essentially linearly representable prices one can expect positive profit for the SLS strategy, i.e., $\mathbb{E}[g_t^{SLS}] > 0$ while $I_0^{SLS} = 0$.[12] However, all of these settings assume the price taker property, as the price process is defined independently of the traders' investments. In contrast, here we drop the assumption of the price taker property but consider an interactive market model, as introduced in Baumann [2015a], as we want to examine the price's behavior under heterogeneous agents.

By transforming equation (7), the feedback trader's investment rate, we see that linear feedback traders follow a strategy that can be written as

$$I_t^{FT} = I_{t-1}^{FT} + K \cdot I_{t-1}^{FT} \cdot \frac{p_i - p_{i-1}}{p_{i-1}} \tag{12}$$

$$= I_{t-1}^{FT} + K \cdot I_{t-1}^{FT} \cdot (e^{M^{-1}\Delta I_{t-1}} - 1) \tag{13}$$

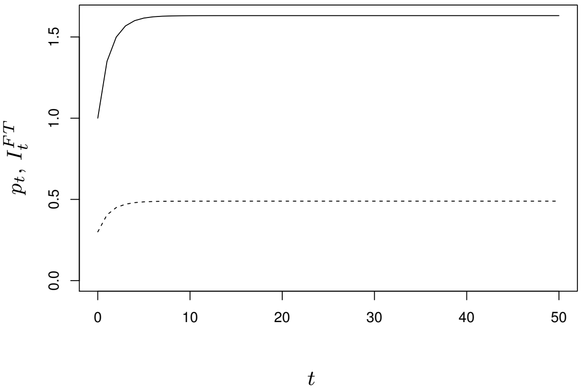
which leads to an investment of

$$I_t^{FT} = I_{t-1}^{FT} + K \cdot I_{t-1}^{FT} \cdot (e^{M^{-1}\Delta I_{t-1}^{FT}} - 1), \tag{14}$$

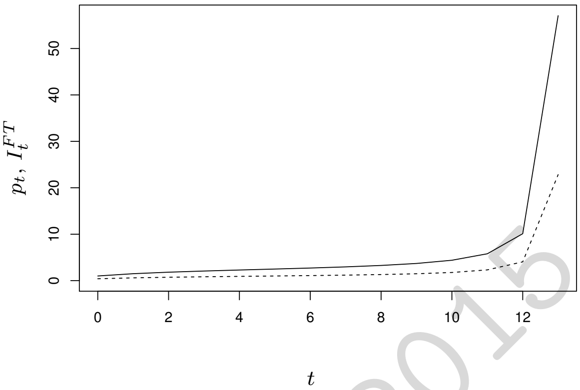
in case only one trader, the linear feedback trader, is acting on the market.

To sum up, the idea behind linear feedback traders' trading strategy is that money can be made by

[12] In Barmish and Primbs [2011] it is shown that a so-called SLS strategy which is the sum of two particular and opposed linear feedback strategies with initial investment zero always makes a positive profit under the assumption of continuously differentiable prices. Furthermore, if in a continuous time model prices follow a GBM the SLS strategy is expected to yield non-negative profit (as proven in Barmish and Primbs [2011]). In Baumann [2015b] it is shown that even for a discontinuous price process characterized through MJDM the result of positive expectation for SLS trading is still valid, independent of intensity, type, and height of jumps. All, continuously differentiable prices, GBM, and MJDM fulfill the so-called price taker property which is usually assumed in literature about feedback trading (see, e.g., Barmish [2011], Barmish and Primbs [2011]). Furthermore, Baumann and Grüne [2015] show that $I_0^{SLS} = 0$ and $\mathbb{E}[g_t^{SLS}] > 0$ under the price taker assumption even hold for all essentially linearly representable prices. This is a whole class of price models containing GBM and MJDM.



(a) I_0^{FT} below a specific threshold: I_t^{FT} (dashed line) converges. $\{p_0 = 1, M = 1, T = 50, FT(I_0^{FT} = 0.3, K = 1)\}$.



(b) I_0^{FT} above a specific threshold: I_t^{FT} (dashed line) diverges. $\{p_0 = 1, M = 1, T = 13, FT(I_0^{FT} = 0.4, K = 1)\}$.

Figure 4: Investment of feedback traders is indicated with a dashed line, development of the stock price p_t is indicated with a solid line; note the different scaling of the vertical axes

following the price trend.

2.4 Fundamentalists

As explained in Section 1, fundamentalists buy assets when the price is below the fundamental value $f_t > 0$ and sell assets if the price is above the fundamental value. Against this background, it is of particular interest how much fundamentalists do invest or disinvest in the respective cases. For deterministic fundamental values f_t, i.e., the fundamental value is a function in t, one possibility to determine the investment rates is

$$\Delta I_t^{FU} = M \cdot \ln \frac{f_{t+1}}{p_t}. \tag{15}$$

In this case fundamentalists do not need to estimate the fundamental value but it is fix and known for sure. Traders following investment rule (15) could be called *strong fundamentalists* because their investment strategy could push the price back to its fundamental value at any time. If the strong fundamentalist is the only trader buying/selling at time t, then for any $p_t > 0$ and f_{t+1} it follows:

$$p_{t+1} = p_t \cdot e^{\ln \frac{f_{t+1}}{p_t}} \tag{16}$$

$$= p_t \cdot \frac{f_{t+1}}{p_t} \tag{17}$$

$$= f_{t+1} \tag{18}$$

In Section 2.5, a fundamentalist trading on a distorted fundamental value is presented. But as it will turn out, this distortion does not affect the general behavior of the market model. To sum up the characteristics of fundamentalists, it can be said that if their view is correct, they can make money because the price will converge to the fundamental value (see Section 1).

2.5 Expectations and Noise

As for some types of traders, e.g., informed speculators [De Long et al., 1990b], it is important that they are subject to rational expectations we want to discuss whether feedback traders and fundamentalists have expectations and, if this is the case, whether these are rational.

Feedback traders and trend followers in general are not subject to rational expectation formation, as they only assume the existence of a trend. For example, with the knowledge of the current slope of the asset price development ($p_t - p_{t-1}$) they forecast the further direction of the trend. By contrast, fundamentalists are assumed to have rational expectations (see, e.g., Drescher and Herz [2012]). Generally, they pursue the strategy

$$\Delta I_t^{FU} = M \cdot \ln \frac{\mathbb{E}[f_{t+1}|f_t]}{p_t}. \tag{19}$$

As it can be observed in real markets, price fluctuations are seldom purely rational. There is always noise and uncertainty in the market, a factor considered essential by many economists (see, e.g., Black [1986], De Long et al. [1990a]). Some reasons for noise are, for instance, that traders are making mistakes, trade on noisy information, which is taken for real, or simply enjoy trading and do not follow a somehow rational strategy.

For the further analysis in the work at hand, we do not assume that traders are making mistakes, as this would lead to a completely unexpected, unsystematically behavior. Furthermore, both feedback traders and fundamentalists follow a specified strategy. The only way of how noise could enter the market is through noisy information. However, the traders' investments as well as the price, announced by the market maker (see Figure 1), are not distorted. The only information that could be noisy is that one about the fundamental value. In this case, the fundamentalist has to estimate f_{t+1} at time t and trade according to (19). Since it is unreasonable that $|f_{t+1} - \mathbb{E}[f_{t+1}]|$ becomes arbitrary large but exploding prices imply $|p_t - f_t| \to \infty$, the effects of noisy information do not play a decisive role.

That is why we a priori consider f_t being a deterministic fundamental value in the presented work.

3 Proof of Limitations of Fundamentalists' Stabilizing Effects

Now we show analytically and mathematically rigorously shown that fundamentalists are not always able to stabilize markets through their trading actions. As simulations indicate that this seems to be true, it is inductively proven that effects of linear feedback traders dominate those of fundamentalists and destabilize markets.

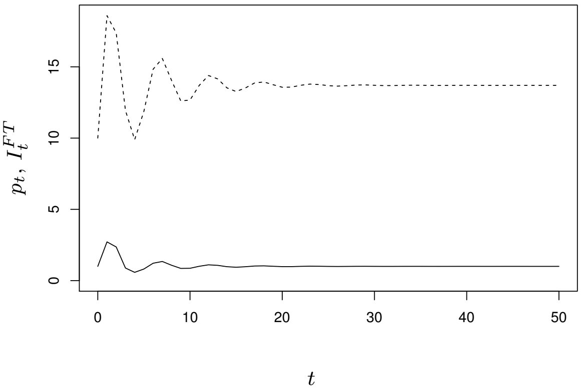
After having defined the pricing model (1) and the traders, the next task is to check whether fundamentalists defined according to formula (15) are able to stabilize the price when trading simultaneously on the market with linear feedback traders following (3) and (4). To simplify the notation, we set $f_t \equiv 1$. This is one special case, but if we can show the destabilizing effects of feedback traders' investment strategy for this case, the claim that fundamentalists do not always have market stabilizing effects, holds in general. The proof shall proceed without using technical trading restrictions as, for example, limiting the feedback traders' amount of investment.

These two trader types are suitable for analyzing the problem because when it turns out that prices explode for appropriately chosen parameters I_0^{FT} and K of linear feedback traders even when acting on a market together with fundamentalists, with an investment strategy that could bring prices close to the fundamental value in every point of time, then this shows that there exist chartists' rules, in this case the linear feedback strategy as a special trend follower rule, that are able to outperform the effects of strong fundamentalists. Why it is enough to consider only linear feedback traders and fundamentalists and no other types of traders, some of them presented in Baumann and Baumann [2015], becomes obvious with the following consideration: If the feedback traders' investment goes to infinity which means prices explode, then also the absolute value of fundamentalists' investment goes to infinity. Thus, the relatively small investment[13] of other possible traders, compared to the exploding investments of feedback traders and fundamentalists, may be neglected at least for our analysis.[14]

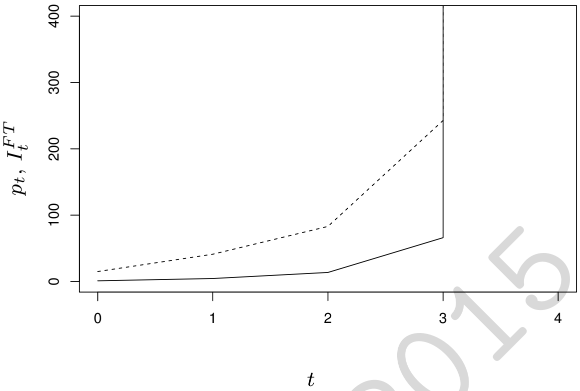
Simulations show that there exist two typical price developments, see Figures 5a and 5b. In Figure 5a, fundamentalists' effects predominate and the price stabilizes around the fundamental value. In Section 4, this converging investment effect is shown in case $K = 1$, where K is the feedback parameter from (3), and a specific limit value for I_0^{FT} is computed. In Figure 5b, the market development is, however, not that obvious. At a first glance, the figure might suggest that prices explode. But as the simulation software reaches its limits, it is not sure whether or not prices level out in these simulation scenarios. We are therefore in need of an analytical examination. In cases like those shown in the simulated Figure 5b, the proposition of Theorem 1 provides guarantee whether the supposed exploding investment of feedback traders is in fact exploding, or whether this impression only emerges due to simulational insufficiencies and the investment is eventually stabilized but with a greater amplitude as, for example, in Figure 5a. In order to simplify the expressions in the model, we assume that $f_t \equiv 1$, as already mentioned above,

[13] Trend followers invest a lot if prices rise strongly and fundamentalist disinvest a lot if price exceeds the fundamental value greatly, i.e., investment of trend followers goes against infinity and that one of fundamentalists goes against minus infinity. For traders which neither predicate their investment on the distance of fundamental value and price nor on the slope of the price it is quite unreasonable that their investment goes against (minus) infinity.

[14] For moving average traders (MA) and noise traders (NO), both presented in Baumann and Baumann [2015] and needed for a valid market model as also shown in Baumann and Baumann [2015], usually $|\Delta I_t^{MA}| \leq \Delta I^*$ and $P(|\Delta_t^{NO}| > B) \to 0$ holds for $B \to \infty$. Thus, only $|\Delta I_t^{FT}|$ and $|\Delta I_t^{FU}|$ can become arbitrarily large which is the only interesting contribution for bubble analysis.



(a) Price and feedback trader's investment converging, i.e., fundamentalists' effects predominate; parameters $\{p_0 = 1, M = 10, T = 50, FU(f_t \equiv 1), F(I_0^{FT} = 10, K = 0.5)\}$



(b) Price and feedback trader's investment diverging, i.e., feedback traders' effects predominate; parameters $\{p_0 = 1, M = 10, T = 3, FU(f_t \equiv 1), F(I_0^{FT} = 15, K = 0.5)\}$

Figure 5: Two typical situations in a market involving feedback traders and fundamentalists (notice: T differs in the two figures for better recognizability)

and also $p_0 = 1$ for all upcoming equations. This choice is just one possible scaling but does not change the model's dynamics in general. It holds:[15]

$$\begin{aligned}
\Delta I_t^{FU} &= M \cdot \ln \frac{f_{t+1}}{p_t} && (20)\\
&= -M \cdot \ln e^{M^{-1} I_{t-1}} && (21)\\
&= -I_{t-1} && (22)\\
&= -I_{t-1}^{FT} - I_{t-1}^{FU} && (23)\\
\Rightarrow\ I_t^{FU} &= -I_{t-1}^{FT} && (24)\\
\Rightarrow\ \Delta I_t^{FU} &= -\Delta I_{t-1}^{FT} && (25)
\end{aligned}$$

With this, we can specify equation (13) that describes the investment of the feedback traders:

$$\begin{aligned}
\Delta I_t^{FT} &= K \cdot I_{t-1}^{FT} (e^{M^{-1}(\Delta I_{t-1}^{FT} + \Delta I_{t-1}^{FU})} - 1) && (26)\\
&= K \cdot I_{t-1}^{FT} (e^{M^{-1}(\Delta I_{t-1}^{FT} - \Delta I_{t-2}^{FT})} - 1) && (27)\\
&= K \cdot I_{t-1}^{FT} (e^{M^{-1} \Delta^2 I_{t-1}^{FT}} - 1) && (28)
\end{aligned}$$

Theorem 1 tells us conditions for the feedback trader's investment I^{FT} and its derivatives for which prices

[15] We define a process α_t as $(\alpha_t)_{t \in \mathbb{Z}} \subset \mathbb{R}$ with $\alpha_t = 0\ \forall t < 0$. Furthermore, we define the Δ-operator as $\Delta^k \alpha_t := \Delta^{k-1}\alpha_t - \Delta^{k-1}\alpha_{t-1}$, $\Delta^1 \alpha_t := \Delta \alpha_t = \alpha_t - \alpha_{t-1}$, and $\Delta^0 \alpha_t := \alpha_t$. A price process p_t is strictly positive, i.e., $(p_t)_t > 0$ for all $t \geq 0$.

explode. It should be noted that the following implication holds:

$$\Delta^k I_{t-1}^{FT} > a \ \wedge \ \Delta^{k+1} I_t^{FT} > b \ \Rightarrow \ \Delta^k I_t^{FT} > a + b. \tag{29}$$

We obtain this directly from the definition of the delta operator which is equivalent to

$$\Delta^k I_t^{FT} = \Delta^{k+1} I_t^{FT} + \Delta^k I_{t-1}^{FT}. \tag{30}$$

Theorem 1. For the investment of the positive linear feedback trader (3) interacting with a strong fundamentalist (15) on the market model (1), under conditions

$$\Delta^3 I_t^{FT} > M \cdot \ln 2, \tag{31}$$

$$\Delta^2 I_t^{FT} > M \cdot \ln 2 \cdot \max\left\{1, K^{-1}\right\}, \tag{32}$$

$$\Delta I_{t-1}^{FT} > 0, \text{ and} \tag{33}$$

$$I_{t-2}^{FT} > 0 \tag{34}$$

it follows that

$$\Delta^k I_{t+1}^{FT} > M \cdot \ln 2 \ \forall k \in \{0, 1, 2, 3\} \tag{35}$$

and

$$\Delta^2 I_{t+1}^{FT} > M \cdot \ln 2 \cdot K^{-1}. \tag{36}$$

This means, the feedback trader's investment, the slope of investment, the curvature of investment, and the increase of the curvature of the investment are strictly greater than $M \cdot \ln 2$ for all $t \geq t^*$ for some t^*. All in all, this is a fast exploding investment, which leads to an equally fast exploding price, as equation (24) states that fundamentalists always respond one period later with the inversed investment of feedback traders. In other words, the feedback trader's investment increases, the rate of increase increases, and the rate of this growth increases. Furthermore, all of this growth rates are bounded from below. Since the investment of the fundamentalist is minus the investment of the feedback trader from one period before the ratio of the (dis-)invested amounts is strictly getting larger, i.e., the feedback trader's exploding effect predominates the fundamentalist's stabilizing one. Theorem 1 is proven by induction in the following.

Proof. It is enough to show (35) for $k = 3$ as all other inequalities can then be derived from (29) resp. (30).

$$\frac{1}{K}\Delta^3 I_{t+1}^{FT} = \frac{1}{K}(\Delta^2 I_{t+1}^{FT} - \Delta^2 I_t^{FT}) \tag{37}$$
$$= \frac{1}{K}(\Delta I_{t+1}^{FT} - 2\Delta I_t^{FT} + \Delta I_{t-1}^{FT}) \tag{38}$$
$$= I_t^{FT}(e^{M^{-1}\Delta^2 I_t^{FT}} - 1) \tag{39}$$
$$- 2I_{t-1}^{FT}(e^{M^{-1}\Delta^2 I_{t-1}^{FT}} - 1) \tag{40}$$
$$+ I_{t-2}^{FT}(e^{M^{-1}\Delta^2 I_{t-2}^{FT}} - 1) \tag{41}$$
$$= (I_{t-2}^{FT} + \Delta I_{t-1}^{FT} + \Delta I_t^{FT})(e^{M^{-1}\Delta^2 I_t^{FT}} - 1) \tag{42}$$
$$- 2(I_{t-2}^{FT} + \Delta I_{t-1}^{FT})(e^{M^{-1}\Delta^2 I_{t-1}^{FT}} - 1) \tag{43}$$
$$+ I_{t-2}^{FT}(e^{M^{-1}\Delta^2 I_{t-2}^{FT}} - 1) \tag{44}$$
$$= I_{t-2}^{FT}(e^{M^{-1}\Delta^2 I_t^{FT}} - 1) \tag{45}$$
$$+ \Delta I_{t-1}^{FT}(e^{M^{-1}\Delta^2 I_t^{FT}} - 1) \tag{46}$$
$$+ \Delta I_t^{FT}(e^{M^{-1}\Delta^2 I_t^{FT}} - 1) \tag{47}$$
$$- 2I_{t-2}^{FT}(e^{M^{-1}\Delta^2 I_{t-1}^{FT}} - 1) \tag{48}$$
$$- 2\Delta I_{t-1}^{FT}(e^{M^{-1}\Delta^2 I_{t-1}^{FT}} - 1) \tag{49}$$
$$+ I_{t-2}^{FT}(e^{M^{-1}\Delta^2 I_{t-2}^{FT}} - 1) \tag{50}$$
$$= I_{t-2}^{FT}(e^{M^{-1}\Delta^2 I_t^{FT}} - 2e^{M^{-1}\Delta^2 I_{t-1}^{FT}} + e^{M^{-1}\Delta^2 I_{t-2}^{FT}}) \tag{51}$$
$$+ 2\Delta I_{t-1}^{FT}(e^{M^{-1}\Delta^2 I_t^{FT}} - e^{M^{-1}\Delta^2 I_{t-1}^{FT}}) \tag{52}$$
$$+ \Delta I_t^{FT}(e^{M^{-1}\Delta^2 I_t^{FT}} - 1) \tag{53}$$

We evaluate these summands separately:

$$(52) = 2\Delta I_{t-1}^{FT}(e^{M^{-1}\Delta^2 I_{t-1}^{FT} + M^{-1}\Delta^3 I_t^{FT}} - e^{M^{-1}\Delta^2 I_{t-1}^{FT}}) \tag{54}$$
$$= 2\Delta I_{t-1}^{FT} e^{M^{-1}\Delta^2 I_{t-1}^{FT}}(e^{M^{-1}\Delta^3 I_t^{FT}} - 1) \tag{55}$$
$$> 2\Delta I_{t-1}^{FT} e^{M^{-1}\Delta^2 I_{t-1}^{FT}}(2 - 1) \tag{56}$$
$$> 0 \tag{57}$$

$$(53) = (\Delta I_{t-1}^{FT} + \Delta^2 I_t^{FT})(e^{M^{-1}\Delta^2 I_t^{FT}} - 1) \tag{58}$$
$$> 0 + M \cdot \max\{\ln 2, \frac{\ln 2}{K}\} \tag{59}$$

$$> M \cdot \frac{\ln 2}{K} \tag{60}$$

$$(51) = I_{t-2}^{FT}(e^{M^{-1}\Delta^2 I_{t-2}^{FT} + M^{-1}\Delta^3 I_{t-1}^{FT} + M^{-1}\Delta^3 I_t^{FT}} \tag{61}$$

$$- 2e^{M^{-1}\Delta^2 I_{t-2}^{FT} + M^{-1}\Delta^3 I_{t-1}^{FT}} + e^{M^{-1}\Delta^2 I_{t-2}^{FT}}) \tag{62}$$

$$= I_{t-2}^{FT} e^{M^{-1}\Delta^2 I_{t-2}^{FT}}(e^{M^{-1}\Delta^3 I_{t-1}^{FT}}(e^{M^{-1}\Delta^3 I_t^{FT}} - 2) + 1) \tag{63}$$

$$> I_{t-2}^{FT} e^{M^{-1}\Delta^2 I_{t-2}^{FT}}(e^{M^{-1}\Delta^3 I_{t-1}^{FT}}(2 - 2) + 1) \tag{64}$$

$$= I_{t-2}^{FT} e^{M^{-1}\Delta^2 I_{t-2}^{FT}} \tag{65}$$

$$> 0 \tag{66}$$

As a result, we get

$$\Delta^3 I_{t+1}^{FT} > M \cdot \ln 2. \tag{67}$$

□

That the conditions for the endogenous variables $I_{t-2}^{FT}, \Delta I_{t-1}^{FT}, \Delta^2 I_t^{FT}, \Delta^3 I_t^{FT}$ may be fulfilled for some t (and some parameter assignment) is shown in Table 1 in which the investment development of the feedback trader and its derivatives are listed for $I_0^{FT} = 15$, $K = 0.5$, and $M = 10$. That means exogenous variables exist so that prices may explode. This demonstrates that feedback traders' effects are able to outperform fundamentalists' effects.

On the other hand, Table 2 shows a situation where price would explode if only feedback traders would act on the market. Equations (8)–(10) hold for the feedback traders, so, according to Baumann [2015a], their investment causes a bubble when acting without any other traders. If fundamentalists enter the market, too, price explosion is, however, prevented, as the investment rates tend to 0 at time $t = 80$ in Table 2. Clearly, the conditions of Theorem 1 for feedback traders are not satisfied.

To conclude this section, we summarize that even a strong fundamentalist's investment rule, i.e., a fundamentalist's strategy without any restrictions including a possibly infinitely large investment amount,

	I_t^{FT}	ΔI_t^{FT}	$\Delta^2 I_t^{FT}$	$\Delta^3 I_t^{FT}$
$t=0$	15	15	15	15
$t=1$	41.112668	26.112668	11.112668	-3.88733197
$t=2$	83.0106859	41.8980179	15.7853499	4.67268186
$t=3$	242.716956	159.70627	117.808252	102.022902
$t=4$	15864296.3	15864053.3	15863893.8	15863776

Table 1: The boxed table entries fulfill the conditions of Theorem 1 for $t = 3$ for which prices explode; market parameters are as in Figure 5b

always following investment rule (15) where a large price may lead to an even larger negative investment rate, is not able to stabilize the market when a trader with a really simply constructed linear feedback strategy with an adequate initial investment is acting on the market, too. Market failures can happen, prices may explode, and the investment rule of a strong fundamentalist cannot prevent this.

4 Further Results for $K = 1$

After having shown that the pricing model as described in (1) together with a quite simple chartist rule is able to cause financial bubbles even when acting on the market together with strong fundamentalists, some further results and special features of the market model shall be given when the linear feedback trader is a reasonable all-in feedback trader, i.e., a linear feedback trader that invests all of his gain but nothing more, i.e., $K = 1$. For two specific cases, formula simplifications for the feedback trader's investment are given. The first case is a market with only a linear feedback trader, the second case is a market with a linear feedback trader and a fundamentalist. Furthermore, as already mentioned in Section 3, if existing, i.e., when no bubble occurs, the limit values of the investment is calculated for these cases. For simplicity we set $M = 1$.

4.1 Case 1: All-in Linear Feedback Trader

In the first case with only one linear feedback trader, equation (14) simplifies to:

$$I_t^{FT} = I_{t-1}^{FT} + I_{t-1}^{FT} \cdot (e^{\Delta I_{t-1}^{FT}} - 1) \tag{68}$$

$$= I_{t-1}^{FT} \cdot e^{\Delta I_{t-1}^{FT}} \tag{69}$$

$$= I_0^{FT} \cdot e^{\Delta I_0^{FT}} \cdots e^{\Delta I_{t-1}^{FT}} \tag{70}$$

$$= I_0^{FT} \cdot e^{I_{t-1}^{FT}} \tag{71}$$

	I_t^{FT}	ΔI_t^{FT}	$\Delta^2 I_t^{FT}$	$\Delta^3 I_t^{FT}$
$t = 0$	0.4	0.4	0.4	0.4
$t = 1$	0.59672988	0.19672988	-0.2032701	-0.6032701
...				
$t = 5$	0.39734101	0.02131807	0.05226206	-0.069616
...				
$t = 80$	≈0.4	≈0	≈0	≈0

Table 2: The table shows a situation where price would explode without fundamentalists but is stabilized by them. The investment parameters are the same as for Figure 4b where prices explode. The boxed cells fulfill (8)-(10).

In the case that I_t^{FT} converges to some value it follows

$$\lim_{t\to\infty} I_t^{FT} = -lw(-I_0^{FT}), \tag{72}$$

where lw denotes the Lambert-W-function.[16]

4.2 Case 2: All-in Linear Feedback Trader and Fundamentalist

In the second case, we need to reshape equations (13) and (28) where the fundamentalist's investment rate is considered, too, to get the simplified feedback trader's investment rate:

$$I_t^{FT} = I_{t-1}^{FT} + I_{t-1}^{FT}(e^{\Delta^2 I_{t-1}^{FT}} - 1) \tag{73}$$

$$= I_0^{FT} \cdot e^{\Delta^2 I_0^{FT}} \cdots e^{\Delta^2 I_{t-1}^{FT}} \tag{74}$$

$$= I_0^{FT} \cdot e^{\Delta I_{t-1}^{FT}} \tag{75}$$

In the case only linear feedback traders and fundamentalists acting on the market – which is also the setting of the main result in Section 3 with $K = 1$ – it is easy to calculate the limit of the feedback trader's investment under the assumption that a limit exists (see, e.g., Figure 6): If $I_t^{FT} \to c \in \mathbb{R} \Rightarrow \Delta I_t^{FT} \to 0$ and thus by using (75):

$$\lim_{t\to\infty} I_t^{FT} = I_0^{FT} \tag{76}$$

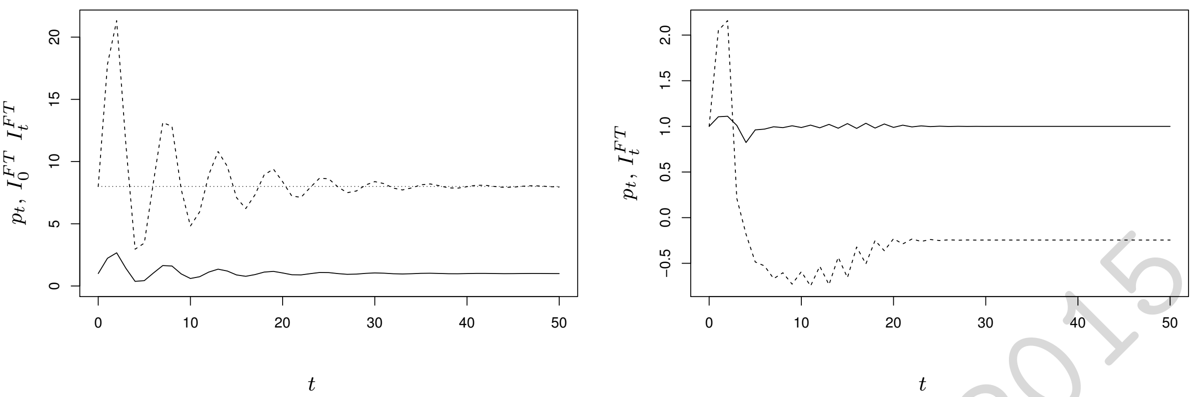
This fact is illustrated in Figure 6a. Furthermore, it should be noticed that due to (28) (market with feedback trader and fundamentalist) for $K \in (0, 1]$ we have $I_t^{FT} \geq 0$, i.e., the linear feedback trader is a long trader. In contrast, for $K > 1$, negative investments I_t^{FT} may occur, too, as shown in Figure 6b.

5 Conclusion

Our analysis indicates that the behavior of trend followers may lead to exploding price processes irrespective of fundamentalists' investment decisions. In particular, Theorem 1 and its proof analytically show that a fundamentalist's investment strategy, i.e., a strategy that pushes prices towards its fundamental value, can be insufficient to dominate linear feedback trading strategies. However, the boundaries for feedback traders' investments to cause a bubble appear to be higher (equations (31)–(34)) when fundamentalists are involved (cf. equations (8)–(10)). Although the results indicate that fundamentalists have a stabilizing effect, this effect is limited up to some threshold values (cf. Table 2).

Moreover, the analysis also shows that for identical investment decisions price movements are more

[16] The Lambert-W-function is the inverse function of $f(x) = x \cdot e^x$.



(a) Feedback trader's investment converging to the initial investment I_0^{FT}; parameters $\{p_0 = 1, M = 10, T = 50, FU(f_t \equiv 1), F(I_0^{FT} = 8, K = 1)\}$

(b) Negative investment of feedback trader; parameters $\{p_0 = 1, M = 10, T = 50, FU(f_t \equiv 1), F(I_0^{FT} = 1, K = 10)\}$

Figure 6: Two particular situations in a market model involving feedback traders and fundamentalists

volatile at higher price levels compared to lower price levels. This finding highlights that even this simply constructed market model is able to capture certain market phenomena appropriately.

Given our results and the fact that financial bubbles are associated with high economic costs an important question arises: as fundamentalists do not appear to be an adequate market stabilizing force, one could ask if another type of trader, perhaps the market maker, is able to stabilize prices in a market conform way and, if so, how such a trader could be constructed.

Generally, our analysis supports the view that intervention measures or at least some kind of incentive systems is necessary in order to stabilize asset markets, and thus to prevent the emergence of financial bubbles.

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