Two Irreducible Components of the Moduli Space $\mathcal{M}_{1,3}^{\mathrm{can}}$

DISSERTATION
zur Erlangung
des DOKTORGRADES(DR. RER. NAT.)
der FAKULTÄT FÜR MATHEMATIK, PHYSIK UND INFORMATIK
der UNIVERSITÄT BAYREUTH

vorgelegt von
YIFAN CHEN
aus P. R. China

BAYREUTH
Tag der Einreichung: 13. Januar 2012
Tag des Kolloquiums: 23. Februar 2012

Angefertigt mit der Genehmigung der Fakultät für Mathematik, Physik und Informatik der Universität Bayreuth.

1. Gutachter: Prof. Dott. Fabrizio Catanese

2. Gutachter: Prof. Dr. Miles Reid, University of Warwick

3. Gutachter: Prof. Dr. Jin-Xing Cai, Peking University

Erklärung

Ich versichere eidesstattlich, dass ich die Arbeit selbständig verfasst und keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt habe.

Ich bestätige, dass ich keine frühere Promotionsversuche gemacht habe.

Unterschrift des Autors

Contents

Zusammenfassung

Das Ziel dieser Dissertation ist es zwei Familien von Flächen von allgemeinem Typ mit $p_g = 0$ und $K^2 = 3$ zu studieren. Genauer gesagt handelt es sich um die erweiterten Burniat Flächen mit $K^2 = 3$ und die Keum-Naie-Mendes Lopes-Pardini Flächen. Wir konzentrieren uns auf die lokalen Deformationen dieser Flächen und auf die Modulräume, die diesen Flächen entsprechen.

Die erweiterten Burniat Flächen mit $K^2 = 3$ wurden zuerst von Bauer und Catanese in [BC10-b] konstruiert, wo sie auch Burniat Flächen mit $K^2 = 3$ studierten (vgl. auch [Bu66] und [Pet77]). Sie haben gezeigt, dass der entsprechende Modulraum in dem Modulraum von Flächen von allgemeinem Typ irreduzibel, offen und von der Dimension 4 ist, und, dass der Abschluss dieses Modulraums eine irreduzible Komponente des Modulrams von Flächen von allgemeinem Typ ist.

Das erste Ziel dieser Arbeit ist, alle Degenerationen der erweiterten Burniat Flächen mit $K^2 = 3$ zu beschreiben. Dazu zeigen wir zuerst, dass die einparametrige Degeneration der kanonischen Modelle dieser Flächen eine endliche, flache $(\mathbb{Z}/2\mathbb{Z})^2$-Überlagerung von normalen singulären kubischen Flächen ist. Danach zeigen wir mittels der Klassifikationstheorie der kubischen Flächen und durch die Untersuchung des Verzweigungsorts dieser Überlagerungen, dass genau zwei Familien von Degenerationen existieren, die in [BC10-b] beschrieben wurden. Somit beweisen wir, dass die Vereinigung der Räume, beschrieben in [BC10-b], tatsächlich die ganze irreduzible Komponente des Modulrams ist.

Darüber hinaus studieren wir die lokalen Deformationen der Degenerationen der erweiterten Burniat Flächen mit $K^2 = 3$. Unter Zuhilfenahme des Struktursatzes der $(\mathbb{Z}/2\mathbb{Z})^2$-Überlagerungen sind wir in der Lage, die Dimensionen der Eigenräume der Kohomologiegruppen der Tangentialgarbe zu bestimmen. Wir zeigen, dass der Basisraum der Kuranishi Familie einer Fläche in einer der zwei Familien der Degenerationen glatt ist.

Im zweiten Teil der Dissertation untersuchen wir Keum-Naie Flächen mit $K^2 = 3$ ([Ke88] und [Na94]) und deren Deformationen, die von Mendes Lopes und Pardini konstruiert wurden. Wir nennen wir diese Flächen Keum-Naie-

Mendes Lopes-Pardini Flächen. In [MP04] wurde gezeigt, dass der Abschluss der entsprechenden Teilmenge dieser Flächen im Modulraum irreduzibel, *uniruled* und der Dimension 6 ist.

Wir konstruieren eine Unterfamilie dieser Flächen. Die Flächen in unserer Familie sind endliche flache $(\mathbb{Z}/2\mathbb{Z})^2$-Überlagerungen einer kubischen Fläche mit vier Knoten. Sie haben einen amplen kanonischen Divisor. Die bikanonische Abbildung dieser Fläche ist die Komposition der Überlagerung mit der antikanonischen Einbettung der kubischen Fläche. Daraus folgt, dass die bikanonische Abbildung dieser Fläche eine Komposition mit einer Involution aus der Galoisgruppe der Überlagerung ist, so dass die Quotientenfläche dieser Involution eine Enriques Fläche mit A_1-Singularitäten ist. Diese Eigenschaft charakterisiert alle Mendes Lopes-Pardini Flächen [MP04].

Unter Zuhilfenahme des Struktursatzes der $(\mathbb{Z}/2\mathbb{Z})^2$-Überlagerungen sind wir in der Lage eine obere Schranke für die Dimension der Kohomologiegruppen der Tangentialgarbe dieser Flächen zu geben. Durch Kombination unserer Ergebnisse und den Ergebnissen aus [MP04] zeigen wir, dass für eine generische Fläche S in unserer Unterfamilie $h^1(S, \Theta_S) = 6, h^2(S, \Theta_S) = 2$ gilt, und der Basisraum der Kuranishi Familie glatt ist. Somit zeigen wir, dass der Abschluss der Teilmenge des Modulraums, die den Keum-Naie-Mendes Lopes-Pardini Flächen entspricht, eine irreduzible Komponente ist.

Abstract

This thesis is devoted to the study of two families of surfaces of general type with $p_g = 0$ and $K^2 = 3$: extended Burniat surfaces with $K^2 = 3$ and Keum-Naie-Mendes Lopes-Pardini surfaces. We focus on the local deformations of these surfaces and the corresponding subsets in the Gieseker moduli space.

Extended Burniat surfaces with $K^2 = 3$ were constructed by Bauer and Catanese [BC10-b] in the course of studying Burniat surfaces with $K^2 = 3$ (cf. [Bu66] and [Pet77]). They showed that the corresponding subset in the moduli space is an irreducible open subset of dimension 4, and its closure is an irreducible component of the moduli space.

The first goal of this thesis is to describe all the degenerations of the extended Burniat surfaces with $K^2 = 3$. For this, we first show that the one parameter limits of the canonical models of these surfaces are finite flat $(\mathbb{Z}/2\mathbb{Z})^2$-covers of normal singular cubic surfaces. Then by applying the classification theory of cubic surfaces and by investigating the branch loci of such covers, we show that there are exactly two families of degenerations, which had been described in [BC10-b]. Thus we prove that the union of the loci described in [BC10-b] is indeed the full irreducible component in the moduli space.

We also study the local deformations of the degenerations of extended Burniat surfaces with $K^2 = 3$. Using the structure theorem for $(\mathbb{Z}/2\mathbb{Z})^2$-covers, we are able to calculate the dimensions of the eigenspaces of the cohomology groups of the tangent sheaves. We show that the base of the Kuranishi family of a surface in one of the two families of degenerations is smooth.

Another topic of this thesis is to study the Keum-Naie surfaces with $K^2 = 3$ (cf. [Ke88] and [Na94]) and their deformations constructed by Mendes Lopes and Pardini [MP04]. We call all these surfaces Keum-Naie-Mendes Lopes-Pardini surfaces. It is showed in [MP04] that the closure of the corresponding subset of such surfaces in the moduli space is irreducible, uniruled and of dimension 6.

We construct a subfamily of such surfaces. The surfaces in our family are finite flat $(\mathbb{Z}/2\mathbb{Z})^2$-covers of a 4-nodal cubic surface. They have ample canonical divisors. Moreover, the bicanonical maps of these surfaces are the composition of the covering morphisms and the anticanonical embedding of the 4-nodal cubic surface. It follows that the bicanonical map of such a surface is composed with an involution in the Galois group ($\cong (\mathbb{Z}/2\mathbb{Z})^2$) of the cover, such that the quotient of the surface by the involution is a nodal Enriques surface. This is a property characterizing all the Mendes Lopes-Pardini surfaces [MP04].

Again using the structure theorem for $(\mathbb{Z}/2\mathbb{Z})^2$-covers, we give upper bounds for the dimensions of the cohomology groups of the tangent sheaves of these surfaces. Combining the results in [MP04], we show that for a general surface S in our subfamily, $h^1(S, \Theta_S) = 6, h^2(S, \Theta_S) = 2$ and the base of the Kuranishi family of S is smooth. We thus show that the closure of the corresponding subset of the Keum-Naie-Mendes Lopes-Pardini surfaces is an irreducible component of the moduli space.

Acknowledgements

It is a pleasure to express my deep gratitude to my supervisor Prof. Fabrizio Catanese, for suggesting these research problems, for many guidance and ideas, and for his encouragement and support during my studies at Bayreuth.

Special thanks go to Prof. Ingrid Bauer, who gave me many stimulating advices, Prof. De-Qi Zhang, who pointed out the incompleteness in the proof of an important theorem, to Prof. Jin-Xing Cai, who taught me the basic theories of algebraic geometry, and to Matteo Penegini, who gave me many suggestions for writing the thesis.

I also thank Fabio Perroni, Masaaki Murakami and Stephen Coughlan for useful conversations and helping me to overcome some difficulties on mathematics. I am very grateful to Frau Rostock and Frau Nicodemus for helping me to solve many non mathematical problems.

I would like to thank Christian Gleißner, Michael Lönne, Mario Chan, Sascha Weigl, Wenfei Liu, Francesco Polizzi, and Zhiyi Tang for helpful and interesting discussions.

I am very grateful to the support of the Project of the Deutsche Forschungsgemeinschaft "Classification of algebraic surfaces and compact complex manifolds", and to the Mathematics Department for a stimulating and friendly environment.

Bayreuth, Yifan Chen
2012

Introduction

Complex algebraic surfaces of general type with geometric genus zero have a long history. Since the first example of minimal surfaces of general type with $p_g = 0$ was constructed by Campedelli [Cam32] in 1930's, these surfaces have received much attention and have been studied by many mathematicians.

Though much progress has been made in the theory of algebraic surfaces, the study of these special surfaces continues to be hard. A minimal smooth surface of general type with $p_g = 0$ satisfies $1 \leq K^2 \leq 9$. Examples for all possible values for K^2 are known in the literature (cf. [BHPV, Page 304, Table 14]). However, a classification is still missing. We refer to a recent survey [BCP09].

New surfaces have been constructed and the subsets in the moduli spaces corresponding to old and new examples need to be investigated. For fixed invariants χ and K^2, denote the Gieseker moduli space (cf. [Gies77]) for canonical models X having $\chi(\mathcal{O}_X) = \chi$ and $K_X^2 = K^2$ by $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$. Once a family of surfaces is constructed, important questions concerning information on the corresponding subset $\mathcal{M}$ in $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ would be:

- Determine the dimension of $\mathcal{M}$.
- Determine whether $\mathcal{M}$ is closed or not. If not, describe the surfaces in the closure $\overline{\mathcal{M}}$ of $\mathcal{M}$ in $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$.
- Determine whether $\mathcal{M}$ is open or not; if $\mathcal{M}$ is the image of an irreducible family, it amounts to the question whether $\overline{\mathcal{M}}$ is an irreducible component of $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ or not.
- Determine whether $\overline{\mathcal{M}}$ is a connected component of $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ or not.

This thesis is devoted to the study of two families of surfaces of general type with $K^2 = 3$ and $p_g = 0$: extended Burniat surfaces with $K^2 = 3$ and Keum-Naie-Mendes Lopes-Pardini surfaces.

P. Burniat constructed a series of surfaces of general type with $p_g = 0$ and $K^2 = 6, 5, 4, 3, 2$ in [Bu66]. These surfaces are singular $(\mathbb{Z}/2\mathbb{Z})^2$-covers of the projective plane branched on 9 lines forming different configurations.

Then Peters explained the construction of the minimal resolutions of Burniat surfaces in [Pet77] in the modern language of double covers. Another construction of these surfaces was given by Inoue in [In94]. See [BC11] for an introduction to Burniat surfaces. Following the terminology in [BC11], a Burniat surface is called primary if $K^2 = 6$, secondary if $K^2 = 5, 4$, tertiary if $K^2 = 3$ and quaternary if $K^2 = 2$. In particular, as stated in [BC11] and in [Ku04], there are two families of Burniat surfaces with $K^2 = 4$: the nodal type and the non-nodal type.

Burniat surfaces have been studied for a long time. Mendes Lopes and Pardini proved that primary Burniat surfaces form an irreducible connected component in the moduli space in [MP01]. In [Ku04], Kulikov corrected the errors of [Pet77] and [In94] on the torsion group of the quaternary Burniat surface and proved that the quaternary Burniat surface is one of the classical Campedelli surfaces, which had been completely described (cf. [Miy77] and [Reid79]). Recently, in [BC11] and [BC10-a], Bauer and Catanese gave another proof of Mendes Lopes-Pardini's result; they also showed moreover that the secondary Burniat surfaces with $K^2 = 5$ and $K^2 = 4$ of non nodal type form irreducible connected components in the moduli spaces (cf. [BC10-a, Theorem 0.2]). In [BC10-b], Bauer and Catanese introduced the extended Burniat surfaces with $K^2 = 4$, and realized Burniat surfaces with $K^2 = 4$ of nodal type as degenerations of these new surfaces. The whole irreducible connected component containing Burniat surfaces with $K^2 = 4$ of nodal type is thus completely described. After these results, the study of the moduli spaces of primary, secondary and quaternary Burniat surfaces can be considered complete.

For the study of the tertiary Burniat surfaces, Bauer and Catanese also introduced the extended Burniat surfaces with $K^2 = 3$ in [BC10-b]. They showed that the extended Burniat surfaces with $K^2 = 3$ and the tertiary Burniat surfaces, form an irreducible open subset $\mathcal{NEB}_3$ of dimension 4 in the moduli space $\mathcal{M}^{\mathrm{can}}_{1,3}$, normal and unirational (cf. [BC10-b, Theorem 0.1]). And by constructing two families of degenerations of extended Burniat surfaces, they showed that the closure $\overline{\mathcal{NEB}_3}$ is strictly larger than $\mathcal{NEB}_3$. Their results imply of course that $\overline{\mathcal{NEB}_3}$ is an irreducible component of $\mathcal{M}^{\mathrm{can}}_{1,3}$. In [NP11],

J. Neves and R. Pignatelli constructed a 4-dimensional family of canonical models of surfaces of general type with $p_g = 0$ and $K^2 = 3$ by using the method of unprojection (cf. [PR04]). Their family forms an open subset of the same irreducible component $\overline{\mathcal{NEB}_3}$.

To complete the investigation of tertiary Burniat surfaces, what remains to be done is to describe all the surfaces in $\overline{\mathcal{NEB}_3}$ and determine whether $\overline{\mathcal{NEB}_3}$ is a connected component or not.

The first result of this thesis is the following theorem.

Theorem 0.1. *$\overline{\mathcal{NEB}_3}$ is the union of $\mathcal{NEB}_3$, D_4-$\mathcal{GB}$ and $4A_1$-$\mathcal{GB}$.*

D_4-$\mathcal{GB}$ (respectively, $4A_1$-$\mathcal{GB}$) refers to the subset corresponding to the D_4-generalized Burniat surfaces (respectively, the $4A_1$-generalized Burniat surfaces), which are $(\mathbb{Z}/2\mathbb{Z})^2$-covers of a normal cubic surface with one D_4-singularity (respectively, 4 nodes) (cf. Section 7 and Section 8). These surfaces are the degenerations mentioned above, already described in [BC10-b, Section 7]. Theorem 0.1 shows that there are no more degenerations and describes $\overline{\mathcal{NEB}_3}$ completely.

Next we study the deformations of the generalized Burniat surfaces. For the D_4-generalized Burniat surfaces, we have a nice result.

Theorem 0.2. *Let S be a D_4-generalized Burniat surface and X be its canonical model. Then $h^1(S, \Theta_S) = 4, h^2(S, \Theta_S) = 0$ and the base of the Kuranishi family of S is smooth. Moreover, $\overline{\mathcal{NEB}_3}$ is the only irreducible component in $\mathcal{M}^{\mathrm{can}}_{1,3}$ containing $[X]$.*

For the $4A_1$-generalized Burniat surfaces, though we can calculate the dimensions of the cohomology groups of the tangent sheaves as follows, the study of the deformations of these surfaces still remains a problem.

Theorem 0.3. *Let S be a $4A_1$-generalized Burniat surface and X be its canonical model. Then the dimensions of the eigenspaces of the cohomology groups of the tangent sheaves Θ_S and Θ_X (for the $(\mathbb{Z}/2\mathbb{Z})^2$-action) are as follows.*

$$h^1(S, \Theta_S)^{inv} = 4, \quad h^1(S, \Theta_S)^{\chi_i} = 1, \quad h^2(S, \Theta_S)^{inv} = 0, \quad h^2(S, \Theta_S)^{\chi_i} = 1;$$
$$h^1(X, \Theta_X)^{inv} = 3, \quad h^1(X, \Theta_X)^{\chi_i} = 0, \quad h^2(X, \Theta_X)^{inv} = 0, \quad h^2(X, \Theta_X)^{\chi_i} = 1,$$

for $i = 1, 2, 3$.

In the course of describing $\overline{\mathcal{NEB}_3}$, we found another family of surfaces which are also $(\mathbb{Z}/2\mathbb{Z})^2$-covers of a 4 nodal cubic surface. The branch loci of the covers are very similar to the ones for the $4A_1$-generalized Burniat surfaces. However, we soon realized that this family lies in the family constructed by Mendes Lopes and Pardini in [MP04]. We give a short introduction to these surfaces.

J. H. Keum and later D. Naie ([Ke88], [Na94]) constructed several families of surfaces of general type with $p_g(S) = 0$ as double covers of nodal Enriques surfaces with 8 nodes (cf. [Na94, Théorème 2.10]). Later in [MP04], Mendes Lopes and Rita Pardini constructed a new family of surfaces with $K^2 = 3$ and $p_g = 0$. These new surfaces have the property that their bicanonical map is composed with an involution such that the quotient surface is a nodal Enriques surface with 7 nodes. Denote by $\mathcal{E}$ the corresponding subset of this new family in the moduli space $\mathcal{M}^{\text{can}}_{1,3}$. It turns out that the closure $\overline{\mathcal{E}}$ contains the Keum-Naie surfaces with $K^2 = 3$ (cf. [MP04, Example 3.5]). Moreover, they proved that $\overline{\mathcal{E}}$ is irreducible and uniruled of dimension 6. However, they pointed out that whether $\overline{\mathcal{E}}$ is an irreducible component or not remains a question (cf. [MP04, Remark 7.3]).

We will reconstruct a subset $\mathcal{E}'$ in $\mathcal{E}$. Our construction here has the advantage that the structure theorem for $(\mathbb{Z}/2\mathbb{Z})^2$-covers can be applied to calculate the cohomology groups of the tangent sheaves. We prove the following theorem.

Theorem 0.4. *(1) For a surface S in $\mathcal{E}'$, S is a smooth minimal surface and K_S is ample. Moreover, for a general surface S in $\mathcal{E}'$, $h^1(S, \Theta_S) = 6$, $h^2(S, \Theta_S) = 2$ and the base of the Kuranishi family of S is smooth.*

(2) $\overline{\mathcal{E}}$ is an irreducible component of the moduli space $\mathcal{M}^{\text{can}}_{1,3}$.

The thesis is organized as follows. Part I consists of the following preliminaries: Section 1 gives a brief introduction to the theory of bidouble covers. In Section 2 we quote results about involutions on rational double points. Section 3 is devoted to the classification and the geometry of normal cubic surfaces. It includes several subsections about resolutions of normal cubic surfaces, which will be used frequently in the later parts.

Part II is dedicated to studying limits of extended Burniat surfaces with $K^2 = 3$ in the moduli space. We first give an introduction to extended (nodal) Burniat surfaces with $K^2 = 3$ in Section 4. In Section 5 we show that limits of extended Burniat surfaces with $K^2 = 3$ are still bidouble covers of normal cubic surfaces and analyze the branch loci. In Section 6 we restrict the allowable classes of normal cubic surfaces to a small number of types according to the classification in section 3. In the following sections $7, 8$ and 9, for each type of normal cubic surfaces, we find out all the possible configurations of the branch loci of the bidouble covers. Finally we manage to prove Theorem 0.1.

We study the deformations of generalized Burniat surfaces in Part III. Section 10 puts together several tools to calculate the cohomology groups of the tangent sheaves of surfaces, including the idea from bidouble cover theory to decompose the cohomology groups into several character spaces, and methods for comparing the dimensions of the cohomology groups when contracting (-1)-curves or (-2)-curves. Section 11 is devoted to the calculation of $H^1(S, \Theta_S)$ and $H^2(S, \Theta_S)$ for the D_4-generalized Burniat surfaces and in the end we succeed to prove Theorem 0.2. In Section 12 we do the same thing for the $4A_1$-generalized Burniat surfaces and prove Theorem 0.3.

We investigate another family of surfaces in Part IV. Section 13 is a short introduction to Keum-Naie-Mendes Lopes-Pardini surfaces. In Section 14 we construct a subfamily of such surfaces as bidouble covers of a 4 nodal cubic surface. We study their local deformations and prove Theorem 0.4 in Section 15.

Notation and conventions

- A surface will mean a projective, irreducible and reduced surface defined over the complex number field $\mathbb{C}$ unless otherwise specified.
- A canonical surface will mean the canonical model of a minimal smooth surface of general type.
- We will only treat (extended) Burniat surfaces with $K^2 = 3$, so sometimes we call them briefly (extended) Burniat surfaces. The same convention will be used for Keum-Naie surfaces.
- For a smooth surface S and a sheaf $\mathcal{F}$ on S, we will denote by $h^k(S, \mathcal{F})$ the dimension of the cohomology group $H^k(S, \mathcal{F})$.
- For a surface S, we will denote by Θ_S the sheaf associated to the tangent bundle, Ω_S^p the sheaf of holomorphic p-forms on S, $p_g(S) := h^0(S, \Omega_S^2)$ the geometric genus, $q(S) := h^0(S, \Omega_S^1)$ the irregularity of S, $\chi(S) := 1 + p_g(S) - q(S)$ the holomorphic Euler-Poincaré characteristic and by K_S^2 the self-intersection number of the canonical divisor.
- Denote by $\equiv$ the linear equivalence for divisors and by $\stackrel{\text{num}}{\equiv}$ the numerical equivalence for divisors.
- An A_n-singularity of a surface is a singularity analytically isomorphic to $x^2 + y^2 + z^{n+1} = 0$. An A_1-singularity is also called a node.
- A $-m$-curve on a smooth surface is an irreducible smooth rational curve with self-intersection number $-m$, where m is a non-negative integer.
- The indices $i \in \{1, 2, 3\}$ should be understood as residue classes modulo 3 through the whole thesis.
- Denote by $G = \{0, g_1, g_2, g_3\}$ a group, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. And let $G^* = \{1, \chi_1, \chi_2, \chi_3\}$ be the group of characters of G, where $\chi_i(g_i) = 1$ and $\chi_i(g_{i+1}) = \chi_i(g_{i+2}) = -1$.

Figures

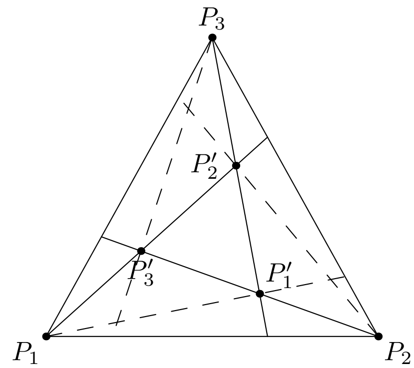


Figure 1: A plane model for a general $3A_1$-type cubic surface.

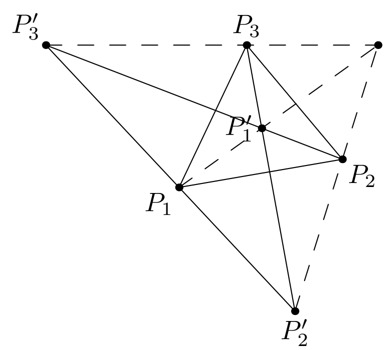


Figure 2: A plane model for a special $3A_1$-type cubic surface.

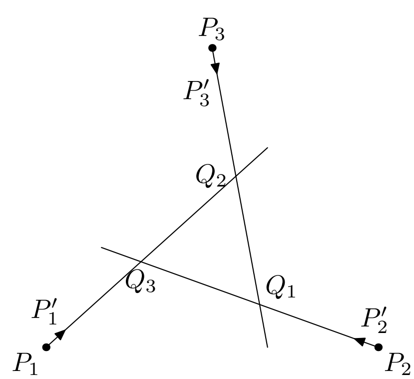


Figure 3: Another plane model for a general $3A_1$-type cubic surface.

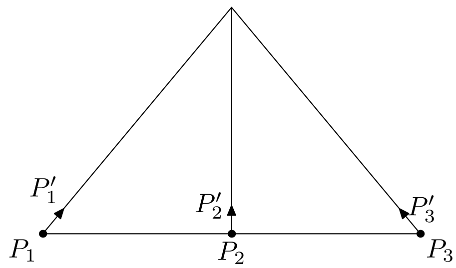


Figure 4: A plane model for the $D_4(1)$-type cubic surface.

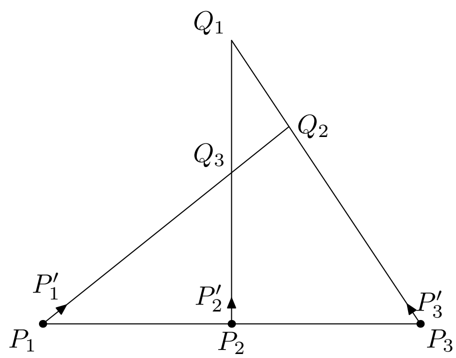


Figure 5: A plane model for the $D_4(2)$-type cubic surface.

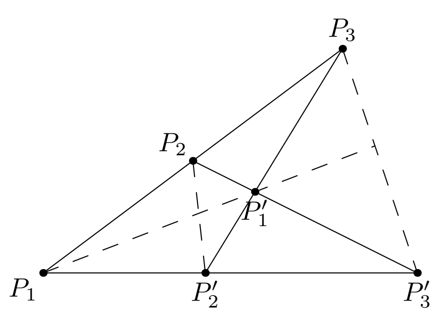


Figure 6: A plane model for the $4A_1$-type cubic surface.

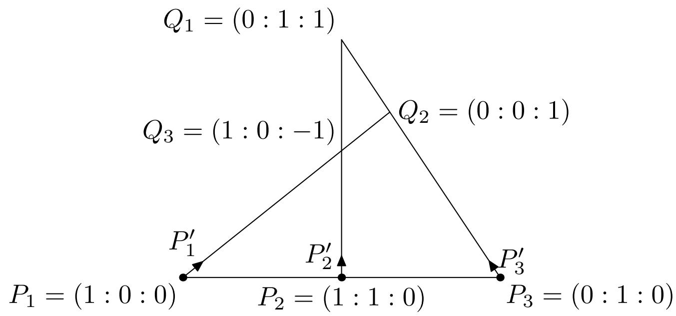


Figure 7: Coordinates for the proof of Proposition 11.4 and the calculation of $h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Gamma_3)(2L - 2E_1' - E_2' - E_3'))$.

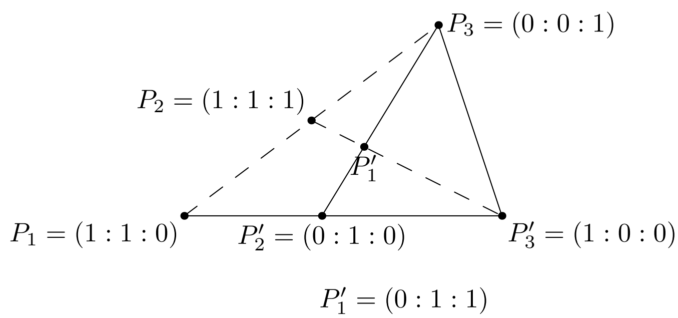


Figure 8: Coordinates for the proof of Proposition 12.3 and the calculation of $h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log N_1, \log N_3, \log \Gamma_3)(2L - E_1 - E_2 - E_1' - E_3'))$.

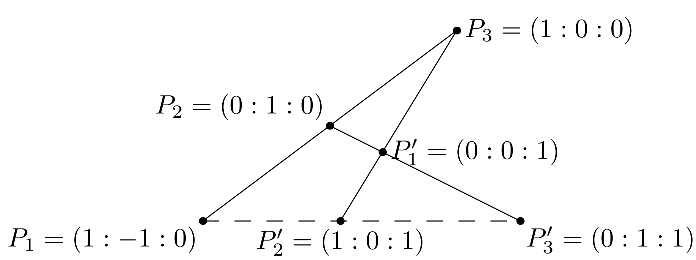


Figure 9: Coordinates for the proof of Proposition 15.2 and the calculation of $h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log N_2, \log N_3, \log Z)(2L - E_2 - E_2' - E_3'))$.

Part I

Preliminaries

1 Bidouble Covers of Surfaces

This section gives a brief introduction to the theory of bidouble covers. For simplicity, we restrict ourselves to the case of algebraic surfaces. We quote the results in [Cat84], [Par91] and [Cat99] without proof.

Definition 1.1 ([Cat84], [Par91, Definition 1.1])**.** Let $\widetilde{Y}$ be a normal surface. A bidouble cover of $\widetilde{Y}$ is a finite morphism $\pi\colon \widetilde{S} \to \widetilde{Y}$, together with a faithful G-action on $\widetilde{S}$ such that π exhibits $\widetilde{Y}$ as the quotient of $\widetilde{S}$ by G.

Definition 1.2. (1) Assume that $\pi\colon \widetilde{S} \to \widetilde{Y}$ is a bidouble cover between normal surfaces. We define the ramification locus R of π, to be the locus of points of $\widetilde{S}$ which have nontrivial stabilizers. The branch locus B of π is the image of R on $\widetilde{Y}$.

(2) For $i = 1, 2, 3$, define a branch divisor B_i corresponding to g_i, to be the image of all the 1**-dimensional** irreducible components of R, whose inertia groups are the subgroup $\{0, g_i\}$.

Here for a 1-dimensional irreducible component D of R, the inertia group H of D is defined as follows: $H = \{g \in G | gx = x \text{ for any } x \in D\}$ (cf. [Par91, Definition 1.2]).

Assume that $\widetilde{Y}$ is smooth and $\widetilde{S}$ is normal. Then by [Ber, Section 3], π is flat, and the ramification locus of π is of pure codimension 1 (cf. [Zar58]). It follows that the branch locus is also of pure codimension 1. The next theorem describes the structure of a bidouble cover under this assumption.

Theorem 1.1 ([Cat84, Section 1], [Par91, Theorem 2.1], [Cat99, Theorem 2])**.** *Let $\pi\colon \widetilde{S} \to \widetilde{Y}$ be a bidouble cover of surfaces. Assume that $\widetilde{Y}$ is smooth.*

(1) Assume that $\widetilde{S}$ is normal. Then

$$\pi_*(\mathcal{O}_{\widetilde{S}}) \cong \mathcal{O}_{\widetilde{Y}} \oplus \mathcal{O}_{\widetilde{Y}}(-\mathcal{L}_1) \oplus \mathcal{O}_{\widetilde{Y}}(-\mathcal{L}_2) \oplus \mathcal{O}_{\widetilde{Y}}(-\mathcal{L}_3),$$

where $\mathcal{L}_i$'s are divisors on $\widetilde{Y}$, and G acts on $\mathcal{O}_{\widetilde{Y}}(-\mathcal{L}_i)$ via the character χ_i. Moreover, there are three effective divisors $\Delta_1, \Delta_2, \Delta_3$ on $\widetilde{Y}$ such that

$$2\mathcal{L}_i \equiv \Delta_{i+1} + \Delta_{i+2}, \tag{1.1}$$

$$\mathcal{L}_i + \Delta_i \equiv \mathcal{L}_{i+1} + \mathcal{L}_{i+2}, \tag{1.2}$$

for $i = 1, 2, 3$, and Δ_i is the branch divisor corresponding to g_i.

(2) *Conversely, given three divisors $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and three effective divisors $\Delta_1, \Delta_2, \Delta_3$ on $\widetilde{Y}$, satisfying (1.1) and (1.2), we can associate a bidouble cover $\pi\colon \widetilde{S} \to \widetilde{Y}$ as follows (cf.* [BC11, Section 2]*):*
for each $i = 1, 2, 3$, locally let $\Delta_i = \operatorname{div}(\delta_i)$ and let u_i be a fibre coordinate of the geometric line bundle $\mathbb{L}_i$, whose sheaf of holomorphic sections is $\mathcal{O}_{\widetilde{Y}}(\mathcal{L}_i)$. Then $\widetilde{S} \subset \mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \mathbb{L}_3$ is given by the equations:

$$\begin{aligned} u_1u_2 &= \delta_3u_3, \quad u_3^2 = \delta_1\delta_2,\\ u_2u_3 &= \delta_1u_1, \quad u_1^2 = \delta_2\delta_3,\\ u_3u_1 &= \delta_2u_2, \quad u_2^2 = \delta_3\delta_1. \end{aligned} \tag{1.3}$$

According to this theorem, to construct a bidouble cover over a smooth surface $\widetilde{Y}$, it suffices to find divisors $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and effective divisors Δ_1, Δ_2, Δ_3 satisfying equations (1.1) and (1.2).

Remark 1.1. (1) If we sum up the left hand side and the right hand side of (1.2) for all $i = 1, 2, 3$, we obtain $\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 \equiv \Delta_1 + \Delta_2 + \Delta_3$.

(2) In the following sections, $\widetilde{Y}$ will be a rational surface, and thus $Pic(\widetilde{Y})$ has no torsion. Hence the equations (1.1) and (1.2) are equivalent. We usually just refer to equations (1.1), or just refer to $\Delta_1, \Delta_2, \Delta_3$, such that the sum of any two is even in $Pic(\widetilde{Y})$, without mentioning the $\mathcal{L}_i$'s.

Concerning the construction of a bidouble cover in Theorem 1.1 (2), the following proposition gives a criterion for the normality (respectively, smoothness) for $\widetilde{S}$.

Proposition 1.2 ([Par91, Proposition 3.1], [Cat99, Theorem 2])**.** *Let $\widetilde{Y}$ be a smooth surface, and let $\pi\colon \widetilde{S} \to \widetilde{Y}$ be the bidouble cover corresponding to the data $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and $\Delta_1, \Delta_2, \Delta_3$, satisfying (1.1) and (1.2). Then*

(1) $\widetilde{S}$ *is normal if and only if the total branch divisor* $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ *is reduced.*

(2) $\widetilde{S}$ *is smooth if and only if each* Δ_i *is smooth for* $i = 1, 2, 3$, *and the total branch divisor* Δ *has only normal crossing singularities.*

In Proposition 1.2 (2), if we do not require the condition "Δ has only normal crossing singularities", then $\widetilde{S}$ might have singularities.

Example 1.1. Assume that Δ_i intersects Δ_{i+1} transversely at a common point P for $i = 1, 2, 3$. Then the local equations (1.3) of $\widetilde{S}$ shows that $\pi^{-1}(P)$ consists of one point Q, which is a $\frac{1}{4}(1,1)$-singularity on $\widetilde{S}$. See [BC11, Section 2] for details.

The following theorem shows how to calculate the invariants of $\widetilde{S}$ from the covering data Δ_i's and $\mathcal{L}_i$'s.

Theorem 1.3 ([Cat84, Lemma 2.15], [Cat99, Section 2])**.** *Let* $\widetilde{Y}$ *be a smooth surface, and let* $\pi\colon \widetilde{S} \to \widetilde{Y}$ *be the bidouble cover associated to the data* $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ *and* $\Delta_1, \Delta_2, \Delta_3$, *satisfying (1.1) and (1.2). Assume that* $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ *is reduced and has only normal crossing singularities. Then*

(1) $\pi_*(\mathcal{O}_{\widetilde{S}}(K_{\widetilde{S}})) \cong \mathcal{O}_{\widetilde{Y}}(K_{\widetilde{Y}}) \oplus \left(\oplus_{i=1}^3 \mathcal{O}_{\widetilde{Y}}(K_{\widetilde{Y}} + \mathcal{L}_i)\right).$

(2) $2K_{\widetilde{S}} \equiv \pi^*(2K_{\widetilde{Y}} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3) \equiv \pi^*(2K_{\widetilde{Y}} + \Delta_1 + \Delta_2 + \Delta_3)$,

$$\pi_*(\mathcal{O}_{\widetilde{S}}(2K_{\widetilde{S}})) \cong \mathcal{O}_{\widetilde{Y}}(2K_{\widetilde{Y}} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3) \oplus \left(\oplus_{i=1}^3 \mathcal{O}_{\widetilde{Y}}(2K_{\widetilde{Y}} + \mathcal{L}_i + \mathcal{L}_{i+1})\right).$$

Corollary 1.4. *In the situation of Theorem 1.3,*

$$K_{\widetilde{S}}^2 = (2K_{\widetilde{Y}} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3)^2,$$

$$\chi(\mathcal{O}_{\widetilde{S}}) = 4\chi(\mathcal{O}_{\widetilde{Y}}) + \frac{1}{2}\sum_{i=1}^{3} \mathcal{L}_i(\mathcal{L}_i + K_{\widetilde{Y}}),$$

$$p_g(\widetilde{S}) = p_g(\widetilde{Y}) + \sum_{i=1}^{3} h^0(\widetilde{Y}, K_{\widetilde{Y}} + \mathcal{L}_i),$$

$$P_2(\widetilde{S}) = h^0(\widetilde{Y}, 2K_{\widetilde{Y}} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3) + \sum_{i=1}^{3} h^0(\widetilde{Y}, 2K_{\widetilde{Y}} + \mathcal{L}_i + \mathcal{L}_{i+1}).$$

2 Involutions on Rational Double Points

The previous section considered a bidouble cover $\pi\colon \widetilde{S} \to \widetilde{Y}$ when $\widetilde{Y}$ is a smooth surface. In our applications, both $\widetilde{S}$ and $\widetilde{Y}$ might have singularities. We would like to know when the quotient of a rational double point by a $\mathbb{Z}/2\mathbb{Z}$-action or a $(\mathbb{Z}/2\mathbb{Z})^2$-action remains a rational double point. This problem has been studied and solved in [Cat87]. We quote the main results and follow the notation in [Cat87] for convenience.

Let us first give a list of rational double points.

Table 1:

Singularities (X_0, x_0)	Equation
E_8	$z^2 + x^3 + y^5 = 0$
E_7	$z^2 + x(y^3 + x^2) = 0$
E_6	$z^2 + x^3 + y^4 = 0$
$D_n (n \geq 4)$	$z^2 + x(y^2 + x^{n-2}) = 0$
A_n	$z^2 + x^2 + y^{n+1} = 0$, or $uv + y^{n+1} = 0$

Definition 2.1 ([Cat87, Definition 1.3])**.** The involution τ of a rational double point (X_0, x_0) such that $\tau^*(z) = -z$, $\tau^*(x) = x$, $\tau^*(y) = y$ is called the trivial involution. Any involution σ conjugate to τ is also said to be trivial, and has the property that $X_0/\sigma \cong (\mathbb{C}^2, 0)$.

The next theorem classifies all the involutions on rational double points.

Theorem 2.1 ([Cat87, Theorem 2.1])**.** *The only involution acting on E_7, E_8 is the trivial one. The other rational double points admit the following nontrivial conjugacy classes of involutions:*

(a) $(x, y, z) \mapsto (x, -y, z)$ (E_6, D_n, A_{2k+1}),
(b) $(x, y, z) \mapsto (x, -y, -z)$ (E_6, D_n, A_{2k+1}),
(c) $(u, v, y) \mapsto (-u, v, -y)$ (A_{2n}),
(d) $(x, y, z) \mapsto (-x, y, -z)$ (A_n),
(e) $(u, v, y) \mapsto (-u, -v, -y)$ (A_{2k+1}).

The following theorems classify the quotients of rational double points by involutions. We also calculate the ramification loci of the quotient maps.

Theorem 2.2 ([Cat87, Theorem 2.2]). *The quotient of a rational double point by a nontrivial involution not of type (c),(e), is again a rational double point according to Table 2.*

Table 2:

Singularities (X_0, x_0)	Involutions	Quotients (Y_0, y_0)	Ramification locus
$E_6 : z^2 + x^3 + y^4 = 0$	$(x, y, z) \mapsto (x, -y, z)$	A_2	$z^2 + x^3 = 0$
$E_6 : z^2 + x^3 + y^4 = 0$	$(x, y, z) \mapsto (x, -y, -z)$	E_7	$(0, 0, 0)$
$D_n : z^2 + x(y^2 + x^{n-2}) = 0$	$(x, y, z) \mapsto (x, -y, z)$	A_1	$z^2 + x^{n-1} = 0$
$D_n : z^2 + x(y^2 + x^{n-2}) = 0$	$(x, y, z) \mapsto (x, -y, -z)$	D_{2n-2}	$(0, 0, 0)$
$A_{2k+1} : z^2 + x^2 + y^{2k+2} = 0$	$(x, y, z) \mapsto (x, -y, z)$	A_k	$z^2 + x^2 = 0$
$A_{2k+1} : z^2 + x^2 + y^{2k+2} = 0$	$(x, y, z) \mapsto (x, -y, -z)$	D_{k+3}	$(0, 0, 0)$
$A_n : z^2 + x^2 + y^{n+1} = 0$	$(x, y, z) \mapsto (-x, y, -z)$	A_{2n+1}	$(0, 0, 0)$

Theorem 2.3 ([Cat87, Theorem 2.4]). *The quotient B_k of the singularity A_{2k} by an involution of type (c) is defined in $\mathbb{C}^4$, with coordinates (u, w, t, η) by the ideal $I_k = (\eta w - t^2, uw + t\eta^k, ut + \eta^{k+1})$.*

The (reduced) exceptional divisor D of its minimal resolution T has normal crossings, consists of k smooth rational curves, and its Dynkin diagram is

$$\circ \!-\!\!-\!\!-\! \circ \qquad \cdots \qquad \circ \!-\!\!-\!\!-\! \circ \!-\!\!-\!\!-\! \underset{-3}{\circ}$$

Theorem 2.4 ([Cat87, Theorem 2.5]). *Let Z be the affine cone over the Veronese surface, i.e., the set of symmetric matrices*

$$\begin{pmatrix} x_1 & x_2 & x_6 \\ x_2 & x_3 & x_4 \\ x_6 & x_4 & x_5 \end{pmatrix}$$

of rank ≤ 1.

Then the quotient Y_{k+1} of the singularity A_{2k+1} by the involution (e) is the intersection of Z with the hypersurface $\phi = x_6 - x_3^{k+1} = 0$. In particular,

Y_{k+1} can also be defined as the singularity in $\mathbb{C}^5$ defined by the ideal

$$J_k = (x_1x_3 - x_2^2, x_2x_4 - x_3^{k+2}, x_3x_5 - x_4^2, x_1x_4 - x_2x_3^{k+1}, \\ x_2x_5 - x_3^{k+1}x_4, x_1x_5 - x_3^{2k+2}).$$

The exceptional divisor D in the minimal resolution T of Y_{k+1} has normal crossings, consists of $(k+1)$ smooth rational curves, and the associated Dynkin diagram is

$$\underset{-4}{\circ} \qquad \textit{for } k = 0$$

$$\underset{-3}{\circ} \text{———} \circ \text{———} \cdots \qquad \circ \text{———} \underset{-3}{\circ} \qquad \textit{for } k \geq 1$$

Remark 2.1. The Y_1-singularity (respectively, B_1-singularity) is the $\frac{1}{4}(1,1)$-singularity (respectively, the $\frac{1}{3}(1,1)$-singularity), i.e., the cone over the rational normal curve of degree 4 in $\mathbb{P}^4$ (respectively, of degree 3 in $\mathbb{P}^3$).

Consider the involution of type (e) on an A_1-singularity:

$$\sigma\colon (X_0, x_0) : uv + y^2 = 0 \to (X_0, x_0) : uv + y^2 = 0, \\ (u, v, y) \mapsto (-u, -v, -y).$$

Then by Theorem 2.4, the quotient $Y_0 := X_0/\sigma$ has a Y_1-singularity y_0. Let $\rho\colon X' \to X_0$ be the minimal resolution of x_0 and denote by N the (-2)-curve. Since N can be viewed as the projectivization of the tangent cone of X_0 to x_0, σ can be lifted to X' and it has N as fixed locus. We see that the image of N on the quotient X'/σ is a (-4)-curve. Hence X'/σ is the minimal resolution of (Y_0, y_0).

Theorem 2.5 ([Cat87, Theorem 2.7])**.** *Let (X_0, x_0) be a rational double point and let H be a subgroup of $Aut(X_0, x_0)$, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Then H is conjugate to a subgroup listed in Table 3.*

Remark 2.2 ([Cat87, Remark 2.8])**.** From Theorem 2.5 Table 3, we conclude that the quotient of a rational double point (X_0, x_0) by a faithful $(\mathbb{Z}/2\mathbb{Z})^2$-action is again a rational double point or a smooth point. This statement also holds for the case $(X_0, x_0) \cong (\mathbb{C}^2, 0)$. This remark will be very important in the proof of Theorem 5.2, Section 5.

Table 3:

Singularities (X_0, x_0)	Involutions	Quotients (Y_0, y_0)	Ramification locus
$E_6,\ D_n,\ A_{2k+1}$	$(x,y,z)\mapsto(x,-y,z)$ $(x,y,z)\mapsto(x,y,-z)$ $(x,y,z)\mapsto(x,-y,-z)$	smooth	
$A_n : z^2+x^2+y^{n+1}=0$	$(x,y,z)\mapsto(-x,y,z)$ $(x,y,z)\mapsto(x,y,-z)$ $(x,y,z)\mapsto(-x,y,-z)$	smooth	
$A_{2k+1} : z^2+x^2+y^{2k+2}=0$	$(x,y,z)\mapsto(x,-y,z)$ $(x,y,z)\mapsto(-x,y,-z)$ $(x,y,z)\mapsto(-x,-y,-z)$	A_{2k+1}	$z^2+x^2=0$ $(0,0,0)$ $(0,0,0)$
$A_{2k+1} : z^2+x^2+y^{2k+2}=0$	$(x,y,z)\mapsto(x,y,-z)$ $(x,y,z)\mapsto(-x,-y,-z)$ $(x,y,z)\mapsto(-x,-y,z)$	A_1	$x^2+y^{2k+2}=0$ $(0,0,0)$ $(0,0,0)$
$A_{2k+1} : z^2+x^2+y^{2k+2}=0$	$(x,y,z)\mapsto(x,-y,-z)$ $(x,y,z)\mapsto(-x,y,-z)$ $(x,y,z)\mapsto(-x,-y,z)$	D_{2k+4}	$(0,0,0)$ $(0,0,0)$ $(0,0,0)$
$A_{2k} : uv+y^{2k+1}=0$	$(u,v,z)\mapsto(-u,-v,y)$ $(u,v,z)\mapsto(-u,v,-y)$ $(u,v,z)\mapsto(u,-v,-y)$	A_{2k}	$(0,0,0)$ $u=0$ $v=0$

Remark 2.3. We make the following observation from the tables above: let $H \cong \mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$ be a subgroup of $Aut(X_0, x_0)$, consisting of involutions as in Table 2 or Table 3. Assume that the quotient (Y_0, y_0) is **singular.** Then either y_0 is an isolated branch locus, or there are at most two 1-dimensional irreducible components in the branch locus containing the singularity y_0. This remark will play an important role in the proof of Lemma 6.4, Section 6.

We also see that when singularities appear, the branch locus is not necessarily of codimension 1, unlike the case in Theorem 1.1 (1). To relate two cases, we can take a minimal resolution $\mu\colon \widetilde{Y}_0 \to Y_0$ of Y_0, and let $\widetilde{S}_0$ be the normalization of the fibre product of $\widetilde{Y}_0$ and X_0. Then compute the branch divisors $\Delta_1, \Delta_2, \Delta_3$ of the induced bidouble cover $\widetilde{S}_0 \to \widetilde{Y}_0$ explicitly (cf. Definition 1.2).

Example 2.1. Take an example from Table 3. Consider the cover of a D_4-singularity by an A_1-singularity,

$$(X_0, x_0) : z^2 + x^2 + y^2 = 0 \to (Y_0, y_0) : w^2 + uv(u + v) = 0,$$
$$(x, y, z) \mapsto (u, v, w) = (x^2, y^2, xyz),$$

with the G-action on (X_0, x_0) given by $g_1\colon (x, y, z) \mapsto (x, -y, -z)$, $g_2\colon (x, y, z) \mapsto (-x, y, -z)$, $g_3\colon (x, y, z) \mapsto (-x, -y, z)$.

Let N_1, N_2, N_3, Z be the exceptional curves of the minimal resolution μ, with $N_i.N_{i+1} = 0$ and $N_i.Z = 1$ for $i = 1, 2, 3$.

Then $\Delta_1 = N_1,\ \Delta_2 = N_2,\ \Delta_3 = N_3$, up to a permutation of $i = 1, 2, 3$.

Proof of the statement of Example 2.1. We follow the idea in [Cat87, Remark 2.3]. Note that for $i = 1, 2, 3$, $\mathrm{Fix}(g_i) = \{x_0\}$. Let $r\colon \widetilde{X} \to X_0$ be a minimal resolution of X_0 and let N be the (-2)-curve.

The group action lifts to $\widetilde{X}$. It acts freely outside N and N is invariant. We claim that each g_i has exactly two isolated fix points Q_i, Q_i' on $\widetilde{X}$, each of which has stabilizer $\{0, g_i\}$, and all these six points are on N. Actually, r is the blowup of X_0 at x_0. Thus N can be viewed as the projectivization of the tangent cone of X_0 to x_0. Hence we can view N as a projective curve $z^2 + x^2 + y^2 = 0$ in the plane with homogeneous coordinates z, x, y. The group action on N using the homogeneous coordinates is given by the the same formulae as the one acts on X_0. Hence the claim follows.

Blow up these points $\sigma\colon S_0 \to \widetilde{X}$ and denote by F_i (respectively F_i') the (-1)-curve corresponding to Q_i (respectively Q_i') and by Z' the strict transform of N, which is a (-8)-curve. The group action lifts to S_0. For each i, g_i has F_i and F_i' as fixed locus, while g_{i+1} and g_{i+2} permute F_i and F_i'. Thus S_0/G is smooth. The image of $F_i \cup F_i'$ is a (-2)-curve N_i, and the image of Z' is a (-2)-curve Z. Since $F_i.Z' = F_i'.Z' = 1$, it follows that $N_i.Z = 1$. Hence N_1, N_2, N_3, Z form the Dynkin diagram for the resolution of a D_4-singularity.

One sees that S_0/G is a minimal resolution of Y_0. It follows that $S_0/G \cong \widetilde{Y}_0$ and $S_0 \cong \widetilde{S}_0$. The discussion above shows that the statement about the branch divisors of $\widetilde{S}_0 \to \widetilde{Y}_0$ holds. □

3 Normal Cubic Surfaces

The classification of singular cubic surfaces was investigated by Schläfli [Sc64] and Cayley [Cay69] in the nineteenth century. This section introduces the classification as described in a more recent article [Sak10], which will be very important in this thesis. See also [BW79].

Assume that Y is a normal singular cubic surface, and P is a singular point of Y. Take a projective transformation sending P to $(0:0:0:1)$; then Y is defined by a homogeneous polynomial of degree 3,

$$F(x_0, x_1, x_2, x_3) = x_3 f_2(x_0, x_1, x_2) - f_3(x_0, x_1, x_2), \tag{3.1}$$

where $f_k(x_0, x_1, x_2)$ denotes a homogeneous polynomial of degree k.

Theorem 3.1 (cf. [Sak10]). *Let Y be a normal singular cubic surface in $\mathbb{P}^3$, defined by $F(x_0, x_1, x_2, x_3) = x_3 f_2(x_0, x_1, x_2) - f_3(x_0, x_1, x_2)$. Let C_2, C_3 be the two plane curves defined by f_2, f_3 respectively.*

Assume that rank $f_2 > 0$. Then,

(1) $Sing(C_2) \cap Sing(C_3) = \emptyset$.

(2) The rational map $\Phi\colon \mathbb{P}^2 \to Y$ defined by $(y_0 : y_1 : y_2) \mapsto$

$$(y_0 f_2(y_0, y_1, y_2) : y_1 f_2(y_0, y_1, y_2) : y_2 f_2(y_0, y_1, y_2) : f_3(y_0, y_1, y_2)),$$

is birational. Φ is the inverse map of the projection of Y with center P.

(3) Blowing up the intersection points of C_2 *with* C_3*, possibly including infinitely near points,* $\sigma\colon \widetilde{Y} \to \mathbb{P}^2$*, eliminates the fundamental points of* Φ*, and the induced morphism* $\mu\colon \widetilde{Y} \to Y$ *gives the minimal resolution of singularities of* Y *:*

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\mu} & Y \\ {\scriptstyle\sigma}\downarrow & \overset{\Phi}{\nearrow} & \\ \mathbb{P}^2 & & \end{array}$$

(4) Y *has only rational double points.*

The proof of Theorem 3.1 can be found in the first section in [Sak10]. We also quote the following main theorems in the same article concerning the classification of normal singular cubic surfaces. The following theorem is also stated in [BW79].

Theorem 3.2 ([Sak10, Theorem 1])**.** *Any normal singular cubic surface in* $\mathbb{P}^3$ *has either rational double points or a simple elliptic singularity* $\tilde{E}_6$ *as in Table 4. Moreover, the number of parameters and the number of lines on the surface, according to the types of singularities, are also listed in Table 4.*

Table 4:

Singularities	No. of parameters	No. of lines	Singularities	No. of parameters	No. of lines
A_1	3	21	A_2	2	15
$2A_1$	2	16	$2A_2$	1	7
$A_1 + A_2$	1	11	$3A_2$	0	3
$3A_1$	1	12	A_3	1	10
$A_1 + A_3$	0	7	A_4	0	6
$2A_1 + A_2$	0	8	A_5	0	3
$4A_1$	0	9	D_4	0	6
$A_1 + A_4$	0	4	D_5	0	3
$2A_1 + A_3$	0	5	E_6	0	1
$A_1 + 2A_2$	0	5	$\tilde{E}_6$	1	∞
$A_1 + A_5$	0	2			

Theorem 3.3 ([Sak10, Theorem 2])**.** *Let Y be a normal singular cubic surface in $\mathbb{P}^3$. Then Y is isomorphic to the projective surface in $\mathbb{P}^3$ defined by*

$$F(x_0, x_1, x_2) = x_3 f_2(x_0, x_1, x_2) - f_3(x_0, x_1, x_2),$$

where f_2, f_3 are given in Table 5, according to the types of singularities on Y. In Table 5, a, b, c are three distinct elements of $\mathbb{C}\backslash\{0,1\}$, d, e are elements of $\mathbb{C}\backslash\{0,-1\}$, and u is an element of $\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$.

Table 5:

Singularities	$f_2(x_0, x_1, x_2)$	$f_3(x_0, x_1, x_2)$
A_1	$x_0x_2 - x_1^2$	$(x_0 - ax_1)(-x_0 + (b+1)x_1 - bx_2)(x_1 - cx_2)$
$2A_1$	$x_0x_2 - x_1^2$	$(x_0 - 2x_1 + x_2)(x_0 - ax_1)(x_1 - bx_2)$
$A_1 + A_2$	$x_0x_2 - x_1^2$	$(x_0 - x_1)(-x_1 + x_2)(x_0 - (a+1)x_1 + ax_2)$
$3A_1$	$x_0x_2 - x_1^2$	$x_0x_2(x_0 - (a+1)x_1 + ax_2)$
$A_1 + A_3$	$x_0x_2 - x_1^2$	$(x_0 - x_1)(-x_1 + x_2)(x_0 - 2x_1 + x_2)$
$2A_1 + A_2$	$x_0x_2 - x_1^2$	$x_1^2(x_0 - x_1)$
$4A_1$	$x_0x_2 - x_1^2$	$(x_0 - x_1)(x_1 - x_2)x_1$
$A_1 + A_4$	$x_0x_2 - x_1^2$	$x_0^2x_1$
$2A_1 + A_3$	$x_0x_2 - x_1^2$	$x_0x_1^2$
$A_1 + 2A_2$	$x_0x_2 - x_1^2$	x_1^3
$A_1 + A_5$	$x_0x_2 - x_1^2$	x_0^3
A_2	x_0x_1	$x_2(x_0 + x_1 + x_2)(dx_0 + ex_1 - dex_2)$
$2A_2$	x_0x_1	$x_2(x_1 + x_2)(-x_1 + dx_2)$
$3A_2$	x_0x_1	x_2^3
A_3	x_0x_1	$x_2(x_0 + x_1 + x_2)(x_0 - ux_1)$
A_4	x_0x_1	$x_0^2x_2 + x_1^3 - x_1x_2^2$
A_5	x_0x_1	$x_0^3 + x_1^3 - x_1x_2^2$
$D_4(1)$	x_0^2	$x_1^3 + x_2^3$
$D_4(2)$	x_0^2	$x_1^3 + x_2^3 + x_0x_1x_2$
D_5	x_0^2	$x_0x_2^2 + x_1^2x_2$
E_6	x_0^2	$x_0x_2^2 + x_1^3$
$\tilde{E}_6$	0	$x_1^2x_2 - x_0(x_0 - x_2)(x_0 - ax_2)$

Remark 3.1. If two normal cubic surfaces Y_1 and Y_2 are isomorphic, then they are projectively equivalent. This follows from $\mathcal{O}_{Y_k}(-K_{Y_k}) = \mathcal{O}_{Y_k}(1)$.

Theorem 3.1 (3) gives an effective way, by blowing up points on the projective plane, to construct the minimal resolution of a normal cubic surface from its equation in Table 5. Now we study the resolutions of some cubic surfaces, which we will frequently refer to.

Conventions. In the following subsections, Y denotes a cubic surface, and $\widetilde{Y}$ denotes the minimal resolution of Y. $\widetilde{Y}$ is a blowup $\sigma\colon \widetilde{Y} \to \mathbb{P}^2$ at six points, possibly including infinitely near points. We will denote these six points by P_1, P_2, P_3, P'_1, P'_2, P'_3. Denote by E_i (respectively E'_i) the **total transform** of the point P_i (respectively P'_i) for $i = 1, 2, 3$, and by L the pullback of a general line by σ. As before, the indices $i \in \{1, 2, 3\}$ should be understood as residue classes modulo 3. See Figure 1 – Figure 6.

3.1 $3A_1$-type Cubic Surfaces

Resolution of a $3A_1$-type cubic surface. Assume that Y is a $3A_1$-type cubic surface. Then $\widetilde{Y}$ can be obtained as the blowup $\sigma\colon \widetilde{Y} \to \mathbb{P}^2$ of six points with the following configuration (see Figure 1):
P_1, P_2, P_3 are not collinear, and P_i, P'_{i+1}, P'_{i+2} are collinear for all $i = 1, 2, 3$.

Rational curves on $\widetilde{Y}$. $\widetilde{Y}$ has three disjoint (-2)-curves,

$$N_i = L - E_i - E'_{i+1} - E'_{i+2}, \text{ for } i = 1, 2, 3,$$

and twelve (-1)-curves,

$$E_i,\ E'_i,\ \Gamma_i := L - E_i - E'_i,\ G_i = L - E_{i+1} - E_{i+2}, \text{ for } i = 1, 2, 3.$$

$\Gamma_1, \Gamma_2, \Gamma_3$ are the only (-1)-curves which do not intersect any (-2)-curve. For each $i = 1, 2, 3$, $\widetilde{Y}$ has a pencil of rational curves C_i in the linear system

$$|2L - E_{i+1} - E_{i+2} - E'_{i+1} - E'_{i+2}|,$$

so that $C_i + \Gamma_i \equiv -K_{\widetilde{Y}}$. The singular elements in the pencil are

$$G_i + E_i + N_i,\ \Gamma_{i+1} + \Gamma_{i+2},\ N_{i+1} + N_{i+2} + 2E'_i.$$

Proof. $C_2 : y_0y_2 - y_1^2 = 0$ and $C_3 : y_0y_2(y_0 - (a+1)y_1 + ay_2) = 0$ intersect at six points

$$Q_0 = (1:1:1),\ Q_1 = (1:0:0),\ Q_2 = (0:0:1),\ Q_3 = (a^2:a:1),$$

and the infinitely near points Q'_k $(k = 1, 2)$ corresponding to the tangent line of C_2 to Q_k $(k = 1, 2)$: $Q_1Q'_1 : y_2 = 0$ and $Q_2Q'_2 : y_0 = 0$.

$\widetilde{Y}$ can be obtained by blowing up these six points. Apply the quadratic transformation centered at Q_0, Q_1, Q_2, namely, first blow up $\sigma_1 \colon Y' \to \mathbb{P}^2$ at Q_0, Q_1, Q_2, then blow down the strict transforms of Q_0Q_1, Q_0Q_2 and Q_1Q_2 to three points P_2, P_1 and P'_3 respectively. Denote the images of Q'_1, Q'_2, Q_3 by P'_2, P'_1, P_3 respectively. Then $P_1, \ldots, P'_3$ satisfy the configuration above. □

Remark 3.2. From the equation of a $3A_1$-type cubic surface, we see that it has one parameter a. If $a = -1$, there are three lines $x_0 = x_3 = 0,\ x_2 = x_3 = 0,\ x_0 - x_2 = x_3 = 0$ containing a smooth point $(0, 1, 0, 0)$ of Y. Correspondingly, three lines $P_iP'_i$'s pass through a common point in the configuration of $P_1, \ldots, P'_3$, and three (-1)-curves $\Gamma_1, \Gamma_2, \Gamma_3$ pass through a common point of $\widetilde{Y}$. See Figure 2.

3.2 $D_4(1)$-type and $D_4(2)$-type Cubic Surfaces

Resolution of the $D_4(1)$-type and the $D_4(2)$-type cubic surfaces. Assume that Y is a $D_4(1)$-type or a $D_4(2)$-type cubic surface. Then $\widetilde{Y}$ can be obtained as the blowup $\sigma \colon \widetilde{Y} \to \mathbb{P}^2$ of six points with the following configuration (see Figure 4 and Figure 5):

P_1, P_2, P_3 are three distinct collinear points on $\mathbb{P}^2$, and P'_i is an infinitely near point lying over P_i for all $i = 1, 2, 3$.

If Y is of $D_4(1)$-type, we require the three lines $P_iP'_i$'s to pass through a common point. If Y is of $D_4(2)$-type, we require the three lines $P_iP'_i$'s to form a triangle, with vertices Q_1, Q_2, Q_3, where Q_i is the intersection point of the lines $P_{i+1}P'_{i+1}$ and $P_{i+2}P'_{i+2}$.

Rational curves on $\widetilde{Y}$. In both cases, $\widetilde{Y}$ has four (-2)-curves,

$$N_i = E_i - E'_i,\ Z = L - E_1 - E_2 - E_3,\ \text{with } N_i.Z = 1 \text{ and } N_i.N_{i+1} = 0,$$

and six (-1)-curves

$$E_i',\ \Gamma_i := L - E_i - E_i',\ \text{for } i = 1, 2, 3.$$

For each $i = 1, 2, 3$, $\widetilde{Y}$ has a pencil of rational curves C_i in the linear system

$$|2L - E_{i+1} - E_{i+2} - E_{i+1}' - E_{i+2}'|,$$

so that $C_i + \Gamma_i \equiv -K_{\widetilde{Y}}$. The only singular element in the pencil is:

$$\Gamma_{i+1} + \Gamma_{i+2}.$$

Proof. (1) Assume that Y is of $D_4(1)$-type. $C_2 : y_0^2 = 0$ and $C_3 : y_1^3 + y_2^3 = 0$ intersect at six points:

$$P_1 = (0 : 1 : -1),\ P_2 = (0 : 1 : -\zeta),\ P_3 = (0 : 1 : -\zeta^2),$$

where ζ is a primitive cubic root of 1, and the infinitely near points P_i''s corresponding to the lines

$$P_1P_1' : y_1 + y_2 = 0,\ P_2P_2' : y_1 + \zeta^2 y_2 = 0,\ P_3P_3' : y_1 + \zeta y_2 = 0.$$

Note that these three lines intersect at a common point $(1 : 0 : 0)$.

(2) Assume that Y is of $D_4(2)$-type. $C_2 : y_0^2 = 0$ and $C_3 : y_1^3 + y_2^3 + y_0 y_1 y_2 = 0$ intersect at six points:

$$P_1 = (0 : 1 : -1),\ P_2 = (0 : 1 : -\zeta),\ P_3 = (0 : 1 : -\zeta^2),$$

and the infinitely near points P_i''s corresponding to the tangent lines of C_3 to P_i's:

$$P_1P_1' : -y_0 + 3y_1 + 3y_2 = 0,$$
$$P_2P_2' : -\zeta y_0 + 3y_1 + 3\zeta^2 y_2 = 0,$$
$$P_3P_3' : -\zeta^2 y_0 + 3y_1 + 3\zeta y_2 = 0.$$

Note that these three lines form a triangle with vertices

$$Q_1 = (-3 : 1 : 1),\ Q_2 = (-3 : \zeta : \zeta^2),\ Q_3 = (-3 : \zeta^2 : \zeta).$$

Then the conclusion follows from Theorem 3.1 (3). □

3.3 $4A_1$-type Cubic Surface

Resolution of the $4A_1$-type cubic surface. Assume that Y is a $4A_1$-type cubic surface. Then $\widetilde{Y}$ can be obtained as the blowup $\sigma\colon \widetilde{Y} \to \mathbb{P}^2$ of six points with the following configuration (see Figure 6):

P_1, P_2, P_3 are collinear, and P_i, P'_{i+1}, P'_{i+2} are collinear for all $i = 1, 2, 3$, i.e., $P_1, \ldots, P'_3$ are vertices of a complete quadrilateral.

Rational curves on $\widetilde{Y}$. $\widetilde{Y}$ has four disjoint (-2)-curves,

$$N_i = L - E_i - E'_{i+1} - E'_{i+2},\ Z = L - E_1 - E_2 - E_3,$$

and nine (-1)-curves,

$$E_i,\ E'_i,\ \Gamma_i := L - E_i - E'_i,\ \text{for } i = 1, 2, 3.$$

For each $i = 1, 2, 3$, $\widetilde{Y}$ has a pencil of rational curves C_i in the linear system

$$|2L - E_{i+1} - E_{i+2} - E'_{i+1} - E'_{i+2}|,$$

so that $C_i + \Gamma_i \equiv -K_{\widetilde{Y}}$. The singular elements in the pencil are:

$$\Gamma_{i+1} + \Gamma_{i+2},\ N_{i+1} + N_{i+2} + 2E'_i,\ Z + N_i + 2E_i.$$

Proof. $C_2 : y_0y_2 - y_1^2 = 0$ and $C_3 : (y_0 - y_1)(y_1 - y_2)y_1 = 0$ intersect at six points

$$Q_1 = (1:0:0),\ Q_2 = (0:0:1),\ Q_3 = (1:1:1),$$

and infinitely near points Q'_i corresponding to the tangent line of C_2 to Q_i :

$$Q_1Q'_1 : y_2 = 0,\ Q_2Q'_2 : y_0 = 0,\ Q_3Q'_3 : y_0 - 2y_1 + y_2 = 0.$$

$\widetilde{Y}$ can be obtained by blowing up these six points. Apply the quadratic transformation centered at Q_1, Q_2, Q_3, namely, first blow up $\sigma_1\colon Y' \to \mathbb{P}^2$ at Q_1, Q_2, Q_3, then blow down the strict transforms of Q_1Q_2, Q_2Q_3 and Q_3Q_1 to three points P'_3, P'_1 and P'_2 respectively. Denote the images of Q'_1, Q'_2, Q'_3 by P_1, P_2, P_3 respectively. Then $P_1, \ldots, P'_3$ satisfy the configuration above. □

The geometry of the $4A_1$-type cubic surface. We explain more about the geometry of the $4A_1$-type cubic surface Y. See the following figure.

Y has 4 nodes Q_0, Q_1, Q_2, Q_3, which do not lie in a plane. By Bézout's theorem, any line connecting two nodes is contained in Y. We can view Q_0, Q_1, Q_2, Q_3 as the vertices of a tetrahedron. The edges of the tetrahedron correspond to six lines of Y. The (-1)-curves E_i and E_i' on $\widetilde{Y}$ correspond to a pair of opposite edges of the tetrahedron, for $i = 1, 2, 3$.

There are three more lines l_1, l_2, l_3 of Y which do not pass any nodes. They lie in a plane and form a triangle. Each one of them intersects exactly one of the three pairs of opposite edges. The three (-1)-curves $\Gamma_1, \Gamma_2, \Gamma_3$ on $\widetilde{Y}$ correspond to these three lines. From this we see that the pencil of curves C_i on $\widetilde{Y}$ correspond to the residual conics cut by planes containing one of the l_i's.

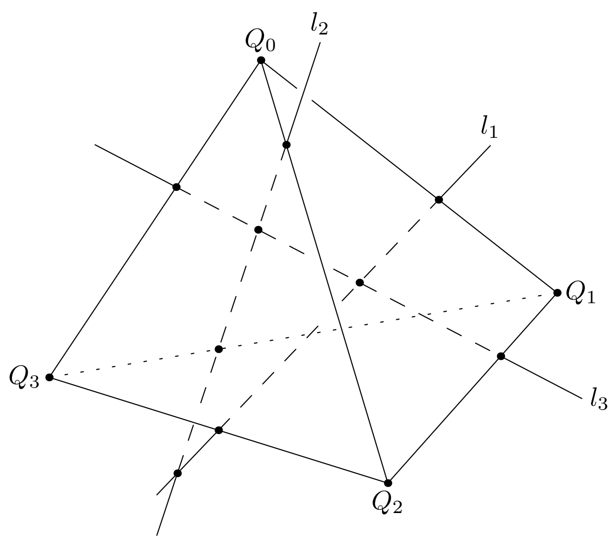


Figure 10: Singularities and lines of the $4A_1$-type cubic surface.

Part II

The Irreducible Component Containing the Extended Burniat Surfaces

4 Burniat Surfaces and Extended Burniat Surfaces

This section gives an introduction to the construction of the (extended) Burniat surfaces with $K^2 = 3$ and the main results on their moduli spaces obtained in [BC10-b].

Assume that Y is a $3A_1$-type cubic surface and $\widetilde{Y}$ is its minimal resolution. **Recall the notation introduced in Subsection 3.1. Assume that the lines P_iP_i''s do not pass through a common point. See Figure 1.**

Definition 4.1 ([Pet77], [BC10-b, Definition 1.1 and Definition 1.3])**.**

(1) Define strictly extended Burniat divisors on $\widetilde{Y}$ as follows:

$$\Delta_1 = \Gamma_1 + N_2 + C_3,\ \Delta_2 = \Gamma_2 + N_3 + C_1,\ \Delta_3 = \Gamma_3 + N_1 + C_2, \quad (4.1)$$

where all C_i's are **irreducible smooth** curves.

(2) If one or two of the three C_i's become **reducible** in the way

$$C_i = N_i + E_i + |L - E_{i+1} - E_{i+2}|,$$

then we define three new divisors by subtracting from Δ_{i+1} the divisor N_i, and subtracting from Δ_{i-1} the divisor N_i, and adding it to Δ_i.

These new divisors and the strictly extended Burniat divisors are all called extended Burniat divisors.

(3) If all three C_i's become reducible in the way above, then we get three new divisors, called nodal Burniat divisors:

$$\begin{aligned} D_1 &= |L - E_1 - E_2| + N_1 + \Gamma_1 + E_3, \\ D_2 &= |L - E_2 - E_3| + N_2 + \Gamma_2 + E_1, \\ D_3 &= |L - E_3 - E_1| + N_3 + \Gamma_3 + E_2. \end{aligned} \tag{4.2}$$

Definition 4.2 ([BC10-b, Definition 1.4])**.** A (strictly) extended Burniat surface with $K^2 = 3$ is the minimal model S of a bidouble cover $\pi\colon \widetilde{S} \to \widetilde{Y}$ associated to a (strictly) extended Burniat divisor.

A nodal Burniat surface with $K^2 = 3$ is the minimal model S of a bidouble cover $\pi\colon \widetilde{S} \to \widetilde{Y}$ associated to a nodal Burniat divisor.

Remark 4.1. (1) By Proposition 1.2, $\widetilde{S}$ in the definition is a smooth surface. However, it is not necessarily minimal. Whenever N_i is a connected component in Δ, $\pi^{-1}N_i$ is a disjoint union of two (-1)-curves.
(2) In particular, for a strictly extended Burniat divisor Δ, all N_i's are connected components in Δ. This implies that K_S is ample for a strictly extended Burniat surface S.
(3) Note that in Definition 4.1 (2), the procedure applied to the branch divisors is actually related to the procedure of normalization in the theory of bidouble covers (cf. [Cat99, Section 2, Remark 3]).

Theorem 4.1. *Let S be the minimal model of $\widetilde{S}$ in Definition 4.2. Then S is a surface of general type with $K_S^2 = 3$, $p_g(S) = q(S) = 0$.*

Moreover, $\pi_1^{\mathrm{top}}(S) \cong \mathcal{H}_8 \times \mathbb{Z}/2\mathbb{Z}$, where $\mathcal{H}_8$ is the quaternion group of order 8.

For the first statement see [BC10-b], or apply Corollary 1.4. For the second statement see [BC11, Theorem 3.2]. See also [In94].

Corollary 4.2 ([BC10-b, Remark 1.5])**.** *If X is the canonical model of an extended Burniat surface or a nodal Burniat surface S with $K_S^2 = 3$, then the bicanonical map of X realizes X as a finite bidouble cover of a $3A_1$-type cubic surface Y.*

In [BC10-b], Bauer and Catanese proved, among other things, the following theorem about the subset in the moduli space corresponding to extended Burniat surfaces and nodal Burniat surfaces with $K^2 = 3$.

Theorem 4.3 ([BC10-b, Proposition 5.7, Theorem 0.1 and Theorem 0.2])**.**

(1) The subset $\mathcal{NEB}_3$ of the moduli space of canonical surfaces of general type $\mathcal{M}^{\mathrm{can}}_{1,3}$ corresponding to extended Burniat surfaces and nodal Burniat surfaces with $K^2 = 3$ is an irreducible open set, normal, unirational of dimension 4.

(2) Let S be an extended Burniat surface or a nodal Burniat surface with $K_S^2 = 3$. Then $h^1(S, \Theta_S) = 4, h^2(S, \Theta_S) = 0$ and the base of the Kuranishi family of such a minimal model S is smooth.

(3) If X is the canonical model of an extended Burniat surface or a nodal Burniat surface S with $K_S^2 = 3$, then $Def(X, (\mathbb{Z}/2\mathbb{Z})^2) = Def(X)$.

Remark 4.2. (1) Here we give a geometric explanation of the dimension of $\mathcal{NEB}_3$: a $3A_1$-type cubic surface has one parameter (cf. Section 3), and each C_i moves in a pencil of curves. This gives the 4 dimensions.
(2) Theorem 4.3 is obtained by a more careful study of deformations of the extended Burniat surfaces (cf. [BC10-b, Proposition 5.7]), using bidouble cover theory. We will follow this method in Part III.
(3) Denote by $\mathcal{SEB}$ the subset of $\mathcal{NEB}_3$ corresponding to the strictly extended Burniat surfaces. Then $\mathcal{SEB}$ is a proper open subset of $\mathcal{NEB}_3$ of dimension 4.

Theorem 4.3 (1) and (2) imply that $\overline{\mathcal{NEB}_3}$ is an irreducible component in $\mathcal{M}^{\mathrm{can}}_{1,3}$. Here comes a natural **question**: is $\mathcal{NEB}_3$ is closed in $\mathcal{M}^{\mathrm{can}}_{1,3}$? Bauer and Catanese already showed that the answer is **No** (cf. [BC10-b, Section 7]). The aim of Part II is to complete the following task.

Task : Determine the irreducible component $\overline{\mathcal{NEB}_3}$ in $\mathcal{M}^{\mathrm{can}}_{1,3}$, i.e., describe all the surfaces corresponding to $\overline{\mathcal{NEB}_3} \setminus \mathcal{NEB}_3$.

5 One Parameter Limits of the Extended Burniat Surfaces

This section is the first step to study limits of extended Burniat surfaces with $K^2 = 3$ in the moduli space. We need the following proposition concerning normal Del Pezzo surfaces.

Let Y be a normal $\mathbb{Q}$-Gorenstein surface. Denote the dualizing sheaf of Y by ω_Y, and denote the associated Weil divisor by K_Y. Then there is a minimal positive integer m such that $\omega_Y^{\otimes m}$ is an invertible sheaf. So it makes sense to define K_Y to be ample or anti-ample. If K_Y is anti-ample, we call Y a *Del Pezzo surface*. Also note that Y is Gorenstein if and only if $m = 1$.

Proposition 5.1 ([HW81, Theorem 4.4 (ii)])**.** *Let Y be a normal Gorenstein Del Pezzo surface with $K_Y^2 = 3$. Then Y is a cubic surface in $\mathbb{P}^3$.*

The main result of this section is the following Theorem.

Theorem 5.2. *Let T be a smooth affine curve and $o \in T$, and let $F\colon \mathcal{X} \to T$ be a flat family of canonical surfaces. Suppose that $\mathcal{X}_t$ is the canonical model of an extended Burniat surface or a nodal Burniat surface with $K^2_{\mathcal{X}_t} = 3$ for $t \neq o \in T$. Then (after possibly shrinking T) there is a group action of $G := (\mathbb{Z}/2\mathbb{Z})^2$ on $\mathcal{X}$ and the quotient map $\Pi\colon \mathcal{X} \to \mathcal{Y} := \mathcal{X}/G$ yields a one parameter family of finite $(\mathbb{Z}/2\mathbb{Z})^2$-covers,*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ \Pi\ } & \mathcal{Y} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle F'} & \\ & T & \end{array}$$

(i.e., $\Pi_t\colon \mathcal{X}_t \to \mathcal{Y}_t$ is a finite $(\mathbb{Z}/2\mathbb{Z})^2$-cover), such that for each $t \neq o$, $\mathcal{Y}_t$ is a $3A_1$-type cubic surface, and $\mathcal{Y}_o$ is a normal cubic surface.

Remark 5.1. To study the limits of the extended Burniat surfaces with $K^2 = 3$, it suffices to require that $\mathcal{X}_t$ is a strictly extended Burniat surface for $t \neq o$ in Theorem 5.2. In fact, Remark 4.2 (3) implies that $\overline{\mathcal{SEB}} = \overline{\mathcal{NEB}_3}$.

Proof. Note that $\mathcal{X}$ is Gorenstein, since the base T is smooth and the fibres have only rational double points.

Since $\mathcal{X}\backslash F^{-1}(o) \to T\backslash\{o\}$ is a family of canonical models of extended Burniat surfaces or nodal Burniat surfaces with $K^2 = 3$, we have a $(\mathbb{Z}/2\mathbb{Z})^2$-action on $\mathcal{X}\backslash F^{-1}(o)$. This is the Galois group action inducing the bicanonical morphism (the key point is that we work on the canonical models, cf. [BC10-b, Theorem 0.2]).

Hence, by [Cat83, Theorem 1.8], the $(\mathbb{Z}/2\mathbb{Z})^2$-action extends to $\mathcal{X}$.

Let $\mathcal{Y}$ be the quotient of $\mathcal{X}$ by the group action, and let $\Pi\colon \mathcal{X} \to \mathcal{Y}$ be the quotient map. Set $\mathcal{X}_t := F^{-1}(t)$ and $\mathcal{Y}_t := F'^{-1}(t)$ for all $t \in T$. Then we have for all $t \in T : K_{\mathcal{Y}_t} = K_{\mathcal{Y}}|_{\mathcal{Y}_t},\ K_{\mathcal{X}_t} = K_{\mathcal{X}}|_{\mathcal{X}_t}$.

Moreover, $2K_{\mathcal{X}} = \Pi^*(2K_{\mathcal{Y}} + \mathcal{B})$, where $\mathcal{B}$ is the branch divisor of $\Pi\colon \mathcal{X} \to \mathcal{Y}$ (cf. Theorem 1.3). Since for $t \neq o$, we have $2K_{\mathcal{X}_t} = \Pi_t^*(-K_{\mathcal{Y}_t})$ (cf. Corollary 4.2), it follows that $2K_{\mathcal{X}} + \Pi^*K_{\mathcal{Y}} \equiv 0$ on $\mathcal{X}\backslash\mathcal{X}_o$.

Since $\mathcal{X}_o$ is irreducible, we obtain (after possibly shrinking T) that
$2K_{\mathcal{X}} + \Pi^*K_{\mathcal{Y}} \equiv 0$ on $\mathcal{X}$. In particular,

$$2K_{\mathcal{X}_t} = \Pi_t^*(-K_{\mathcal{Y}_t}) \text{ for all } t \in T, \tag{5.1}$$

which implies that $-K_{\mathcal{Y}_t}$ is ample and $K^2_{\mathcal{Y}_t} = K^2_{\mathcal{X}_t} = 3$ for all $t \in T$.

By construction, as the bicanonical image of $\mathcal{X}_t$ (cf. Corollary 4.2), $\mathcal{Y}_t$ is a cubic surface with three A_1-singularities for $t \neq o$, and $\mathcal{Y}_o$ is a normal $\mathbb{Q}$-Gorenstein surface.

We claim that $\mathcal{Y}_o$ is Gorenstein. Then $\mathcal{Y}_o$ is a normal cubic surface by Proposition 5.1.

We shall prove the claim by contradiction. Assume that $\mathcal{Y}_o$ is non-Gorenstein. Recall that

$$2K_{\mathcal{X}_o} \equiv \Pi_o^*(-K_{\mathcal{Y}_o}),\ K^2_{\mathcal{Y}_o} = 3, \tag{5.2}$$

and $-K_{\mathcal{Y}_o}$ is ample.

Step 1: All the possibilities of the non-Gorenstein locus of $\mathcal{Y}_o$ are (cf. Theorem 2.3, Theorem 2.4 and Remark 2.1)

(a) one B_1-singularity.

(b) one B_1-singularity and one Y_1-singularity.

(c) one Y_2-singularity.

(d) one Y_1-singularity.

(e) two Y_1-singularity.

In fact, $\mathcal{X}_o$ has at most rational double points. Hence by Remark 2.2, for a non-Gorenstein point q on $\mathcal{Y}_o$, $\Pi_o^{-1}(q)$ consists of two points p_1, p_2, and the stabilizers of p_1 and p_2 in G are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. By Theorem 2.3 and Theorem 2.4, either q is a B_k-singularity and both p_1 and p_2 are A_{2k}-singularities of $\mathcal{X}_o$, or q is a Y_{k+1}-singularity and both p_1 and p_2 are A_{2k+1}-singularities of $\mathcal{X}_o$ for some $k \geq 0$.

Hence an upper bound for the number of singularities of $\mathcal{X}_o$ would bound the number of non-Gorenstein singularities of $\mathcal{Y}_o$. Since the minimal resolution S_o of $\mathcal{X}_o$ has Picard number 7, S_o has at most six (-2)-curves (cf. [BHPV, Page 272, Proposition 2.5]). An easy calculation shows that the list of the non-Gorenstein singularities of $\mathcal{Y}_o$ stated above is complete.

Step 2: Let $\tilde{\mathcal{Y}}_o$ be the minimal resolution of $\mathcal{Y}_o$. Then $K^2_{\tilde{\mathcal{Y}}_o}$ is an integer. The resolution of a rational double point does not change K^2, while the resolution of a B_1-singularity (respectively, a Y_1-singularity) contributes $-\frac{1}{3}$ (respectively, -1) to K^2 (for example, cf. [Barlow99, Section 6]). Since $K^2_{\mathcal{Y}_o} = 3$ is an integer, case (a) and case (b) cannot occur.

Step 3: Assume that $\mathcal{Y}_o$ has exactly one Y_2-singularity q. The discussion in Step 1 shows that $\Pi_o^{-1}(q)$ consists of two A_3-singularities p_1, p_2 of $\mathcal{X}_o$ and p_1, p_2 are the only singularities of $\mathcal{X}_o$. Moreover, there is an involution $g \in G$ permuting p_1 and p_2.

Lift g to the minimal resolution S_o of $\mathcal{X}_o$, and denote it by $\hat{g}$. Denote by R the divisorial part of the fix locus of $\hat{g}$ and by t the trace of $\hat{g}^*\colon H^2(S_o,\mathbb{C}) \to H^2(S_o,\mathbb{C})$. Denote by N_1, N_2, N_3 (respectively, Z_1, Z_2, Z_3) the (-2)-curves of S_o lying over p_1 (respectively, p_2).

Note that $c_1(K_{S_o})$ and $c_1(N_1), \ldots, c_1(Z_3)$ are a basis of $H^2(S_o,\mathbb{C})$. Since g permutes p_1 and p_2 on $\mathcal{X}_o$, $N_1, \ldots, Z_3$ are disjoint from the fix locus of $\hat{g}$ on S_o. Moreover, $\hat{g}$ maps a (-2)-curve to a (-2)-curve, and $\hat{g}^*(c_1(K_{S_o})) = c_1(K_{S_o})$. Hence we conclude that $t = 1$.

Since $t = 2 - R^2$ (cf. [DMP02, Lemma 4.2]), it follows that $R^2 = 1$. In particular, R is non-empty. Since R is disjoint from the six (-2)-curves, $R \overset{num}{\equiv} rK_{S_o}$ for some rational number r. But $R^2 = 1$ and $K_{S_o}^2 = 3$ show that $r^2 = 3$. This is a contradiction and thus case (c) cannot occur.

Step 4: Assume that $\mathcal{Y}_o$ has one Y_1-singularity q. The discussion in Step 1 shows that $\Pi_o^{-1}(q)$ consists of two A_1-singularities p_1, p_2 of $\mathcal{X}_o$ and locally $(\mathcal{X}_o, p_j) \to (\mathcal{Y}_o, q)$ is described as in Theorem 2.4.

Take the minimal resolution $\mu\colon Y' \to \mathcal{Y}_o$ of q, and denote by E the exceptional curve. Then Y' is Gorenstein. Moreover,

$$K_{Y'}^2 = 2,\ K_{Y'}.E = 2,\ E^2 = -4.$$

Take the minimal resolution $\rho\colon X' \to \mathcal{X}_o$ of p_1 and p_2, and denote by N_1 and N_2 the corresponding (-2)-curves of X'.

By Remark 2.1, we have a morphism $\Pi'\colon X' \to Y'$ such that the following diagram commutes:

$$\begin{array}{ccc} X' & \xrightarrow{\ \rho\ } & \mathcal{X}_o \\ {\scriptstyle \Pi'}\downarrow & & \downarrow{\scriptstyle \Pi_o} \\ Y' & \xrightarrow[\ \mu\ ]{} & \mathcal{Y}_o \end{array}$$

We have the following equalities,

$$K_{Y'} \equiv \mu^* K_{\mathcal{Y}_o} - \frac{1}{2}E, \tag{5.3}$$
$$\Pi'^* E = 2N_1 + 2N_2,$$
$$2K_{X'} \equiv \rho^*(2K_{\mathcal{X}_o}).$$

These equalities together with (5.2) show that

$$2K_{X'} \equiv \Pi'^*(-K_{Y'} - E) + N_1 + N_2. \tag{5.4}$$

By the Riemann-Roch theorem for singular normal surfaces (cf. [Bla95, Section 1.2]),

$$\begin{aligned}\chi(\mathcal{O}_{Y'}(-K_{Y'}-E)) &= \frac{1}{2}(-K_{Y'}-E)(-2K_{Y'}-E)+\chi(\mathcal{O}_{Y'}) \\ &\quad + \sum_{y\in Sing(Y')} R_y(-K_{Y'}-E) \\ &= 4+\sum_{y\in Sing(Y')} R_y(-K_{Y'}-E). \end{aligned} \tag{5.5}$$

$R_y(-K_{Y'}-E)$ depends only on the local analytic isomorphism type of the singularity y and the local analytic divisor class of $-K_{Y'}-E$ at y. Since Y' has at most rational double points and E is disjoint from the singularities of Y', $R_y(-K_{Y'}-E)=0$ for all $y\in Sing(Y')$. Thus $\chi(\mathcal{O}_{Y'}(-K_{Y'}-E))=4$.

By Serre Duality, $H^2(Y',\mathcal{O}_{Y'}(-K_{Y'}-E))\cong H^0(Y',\mathcal{O}_{Y'}(2K_{Y'}+E))$. Note that $\mathcal{Y}_o$ is 2-Gorenstein, $\mathcal{O}_{\mathcal{Y}_o}(2K_{\mathcal{Y}_o})$ is invertible and $h^0(\mathcal{Y}_o,-2K_{\mathcal{Y}_o})=7$ (cf. [HP10, Proposition 2.6]). Since $2K_{Y'}+E=\mu^*(2K_{\mathcal{Y}_o})$ by (5.3), it follows that $h^0(Y',\mathcal{O}_{Y'}(2K_{Y'}+E))=0$. Hence by (5.5) $h^0(Y',\mathcal{O}_{Y'}(-K_{Y'}-E))\geq 4$.

Then $h^0(X',\mathcal{O}_{X'}(2K_{X'}))=4$ and (5.4) show that $h^0(Y',\mathcal{O}_{Y'}(-K_{Y'}-E))=4$, and N_1,N_2 are contained in the fixed part of $|2K_{X'}|$. This gives a contradiction to [Weng95, Theorem]. Thus case (d) is excluded.

Step 5: Assume that $\mathcal{Y}_o$ has two Y_1-singularity q_1 and q_2. The discussion in Step 1 shows that $\Pi_o^{-1}(q_1)$ (respectively $\Pi_o^{-1}(q_2)$) consists of two A_1-singularities p_1,p_2 (respectively p_3,p_4) of $\mathcal{X}_o$.

Take the minimal resolution $\mu\colon Y'\to\mathcal{Y}_o$ of q_1 and q_2, and denote by E_1 and E_2 the exceptional curves respectively. Then Y' is Gorenstein, and $K_{Y'}^2=1,\ K_{Y'}.E_1=K_{Y'}.E_2=2,\ E_1^2=E_2^2=-4$.
Take the minimal resolution $\rho\colon X'\to\mathcal{X}_o$ of $p_1,\dots,p_4$, and denote by $N_1,\dots,N_4$ the corresponding (-2)-curves of X' respectively.

By Remark 2.1, we have a morphism $\Pi'\colon X'\to Y'$ and the commutative diagram in Step 4. We have the following equalities,

$$\begin{aligned} K_{Y'} &\equiv \mu^* K_{\mathcal{Y}_o} - \frac{1}{2}E_1 - \frac{1}{2}E_2, \qquad (5.6)\\ \Pi'^* E_1 &= 2N_1 + 2N_2,\ \Pi'^* E_2 = 2N_3 + 2N_4 \\ 2K_{X'} &\equiv \rho^*(2K_{\mathcal{X}_o}), \end{aligned}$$

These equalities together with (5.2) show that

$$2K_{X'} \equiv \Pi'^*(-K_{Y'} - E_1 - E_2) + N_1 + N_2 + N_3 + N_4. \qquad (5.7)$$

Apply Riemann-Roch theorem for singular normal surfaces as in Step 4. Since Y' has at most rational double points and E_1, E_2 are disjoint from the singularities of Y', $R_y(-K_{Y'} - E_1 - E_2) = 0$ for all $y \in Sing(Y')$. Hence

$$\begin{aligned} \chi(\mathcal{O}_{Y'}(-K_{Y'} - E_1 - E_2)) &= \frac{1}{2}(-K_{Y'} - E_1 - E_2)(-2K_{Y'} - E_1 - E_2) \\ &\quad + \chi(\mathcal{O}_{Y'}) + \sum_{y \in Sing(Y')} R_y(-K_{Y'} - E_1 - E_2) \\ &= 4. \qquad (5.8) \end{aligned}$$

By Serre Duality,

$$H^2(Y', \mathcal{O}_{Y'}(-K_{Y'} - E_1 - E_2)) \cong H^0(Y', \mathcal{O}_{Y'}(2K_{Y'} + E_1 + E_2)).$$

Since $h^0(\mathcal{Y}_o, -2K_{\mathcal{Y}_o}) = 7$ and $2K_{Y'} + E_1 + E_2 = \mu^*(2K_{\mathcal{Y}_o})$ by (5.6), it follows that $h^0(Y', \mathcal{O}_{Y'}(2K_{Y'} + E_1 + E_2)) = 0$.
Hence by (5.8), $h^0(Y', \mathcal{O}_{Y'}(-K_{Y'} - E_1 - E_2)) \geq 4$.

Then $h^0(X', \mathcal{O}_{X'}(2K_{X'})) = 4$ and (5.7) show that
$h^0(Y', \mathcal{O}_{Y'}(-K_{Y'} - E_1 - E_2)) = 4$, and $N_1, \ldots, N_4$ are contained in the fixed part of $|2K_{X'}|$. This gives a contradiction to [Weng95, Theorem]. Thus case (e) is excluded.

Hence we conclude that $\mathcal{Y}_o$ is Gorenstein and thus complete the proof of Theorem 5.2. □

Corollary 5.3. *With the same situation as in Theorem 5.2 and denote by $\mathcal{B}_t$ the branch divisor of $\Pi_t \colon \mathcal{X}_t \to \mathcal{Y}_t$ for all $t \in T$. Then $\{\mathcal{B}_t\}_{t\in T}$ is a flat family of curves over T, and $\mathcal{B}_o$ is the limit of $\mathcal{B}_t$ ($t \neq o$). Moreover, $\mathcal{B}_o$ is reduced.*

Proof. By the closedness of branch locus of $\Pi \colon \mathcal{X} \to \mathcal{Y}$, $\mathcal{B}_o$ contains the limit of $\mathcal{B}_t$. (A priori, $\mathcal{B}_o$ might have other components which are not contained in the limit of $\mathcal{B}_t$. We shall show this cannot happen.)

We have $2K_{\mathcal{X}_t} \equiv \Pi_t^*(2K_{\mathcal{Y}_t} + \mathcal{B}_t)$ (cf. Theorem 1.3). Combining with (5.1) in the proof of Theorem 5.2, it follows that for all $t \in T$,

$$\mathcal{B}_t \equiv -3K_{\mathcal{Y}_t}.$$

By Theorem 5.2, for any $t \in T$, $\mathcal{O}_{\mathcal{Y}_t}(K_{\mathcal{Y}_t})$ is invertible and $K_{\mathcal{Y}_t} = K_{\mathcal{Y}}|_{\mathcal{Y}_t}$. Hence we see that $\mathcal{B}_o$ is the limit of $\mathcal{B}_t$ and $\{\mathcal{B}_t\}_{t\in T}$ is a flat family.

Since $\mathcal{X}_o$ is normal, by Proposition 1.2, $\mathcal{B}_o$ is reduced. □

From Definition 4.1, we see that $\mathcal{B}_t$ consists of lines and conics for $t \neq o$. Note that a smooth conic in the branch divisor $\mathcal{B}_t$ might degenerate into two lines of $\mathcal{Y}_o$, but a line in $\mathcal{B}_t$ remains as a line of $\mathcal{Y}_o$. Moreover, two coplanar curves in $\mathcal{B}_t$ remain coplanar on $\mathcal{Y}_o$.

Corollary 5.4. *With the same situation as in Theorem 5.2, the branch divisor $\mathcal{B}_o$ of $\Pi_o \colon \mathcal{X}_o \to \mathcal{Y}_o$ has the following properties:*

(1) (Lines and Conics) Let B_i be the branch divisor of Π_o corresponding to g_i, for $i = 1, 2, 3$. Then each B_i consists of a line l_i and a reduced (possibly reducible) conic c_{i+2}.

(2) (Three coplanar lines) l_1, l_2, l_3 are coplanar.

(3) (Hyperplane sections) For each $i = 1, 2, 3$, l_i and c_i form a hyperplane section of $\mathcal{Y}_o$.

(4) $\mathcal{B}_o \equiv -3K_{\mathcal{Y}_o}$.

Proof. It suffices to prove that in Theorem 5.2, for $t \neq o$, the branch divisor $\mathcal{B}_t$ of $\Pi_t \colon \mathcal{X}_t \to \mathcal{Y}_t$ satisfies all these properties. Then by Corollary 5.3, such properties will be preserved in the limit.

In Definition 4.1, note that Γ_i (respectively C_i or $E_i + (L - E_{i+1} - E_{i+2})$) corresponds to a line (respectively a conic) on $\mathcal{Y}_t$. Also note that $C_i + \Gamma_i \in |-K_{\widetilde{Y}}|$, which corresponds a hyperplane section on $\mathcal{Y}_t$. From the formulae for Δ_i in Definition 4.1, $\mathcal{B}_t$ satisfies all the properties for $t \neq 0$. □

6 Exclusion of Certain Cubic Surfaces

In this section, we will first determine a priori the possible types of the cubic surface $\mathcal{Y}_o$.

Theorem 6.1. *Let $\Pi_o \colon \mathcal{X}_o \to \mathcal{Y}_o$ be the bidouble cover as in Theorem 5.2. Then $\mathcal{Y}_o$ can only be one of the following types: $D_4(1)$, $D_4(2)$, $4A_1$ and $3A_1$.*

This theorem will be proved by the following propositions. We will refer to the number of lines on cubic surfaces (cf. Table 4) and the equations of cubic surfaces (cf. Table 5) in Section 3.

Proposition 6.2. *$\mathcal{Y}_o$ cannot be one of the following types:*
$\tilde{E}_6, A_1 + A_5, E_6, D_5$.

Proof. Since $\mathcal{X}_o$ has only rational double points, thus as a quotient of $\mathcal{X}_o$, $\mathcal{Y}_o$ has only rational singularities. Hence $\mathcal{Y}_o$ cannot be of $\tilde{E}_6$-type.

By Corollary 5.4 (2), $\mathcal{Y}_o$ has three coplanar lines. So types $A_1 + A_5$ and E_6 are excluded by the number of lines.

For type $D_5 : x_3x_0^2 - (x_0x_2^2 + x_1^2x_2) = 0$, the cubic surface has exactly three lines, $l_1 : x_0 = x_1 = 0$, $l_2 : x_0 = x_2 = 0$ and $l_3 : x_2 = x_3 = 0$. But these lines are not coplanar, thus type D_5 is excluded. □

A property of the local deformations of A_n-singularities helps to exclude more types.

Proposition 6.3. *$\mathcal{Y}_o$ cannot be one of the following types:*
$A_1, 2A_1, A_2, A_1 + A_2, 2A_2, A_3, A_4$.

Proof. The semiuniversal family of the local deformations of the A_n-singularity $xy - z^{n+1} = 0$ is

$$xy - z^{n+1} - a_{n-1}z^{n-1} - \ldots - a_1z - a_0 = 0,\ a_0, \ldots, a_{n-1} \in \mathbb{C}. \tag{6.1}$$

For fixed $a_0, \ldots, a_{n-1}$, the corresponding fiber has at most $[\frac{n+1}{2}]$ singularities. Since if $(0,0,z_0)$ is a singularity of (6.1) for fixed $a_0, \ldots, a_{n-1}$, then

$$z^{n+1} + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = (z - z_0)^k h(z),$$

for some polynomial $h(z)$ such that $h(z_0) \neq 0$ and $k \geq 2$.

Fix the type of the cubic surface $\mathcal{Y}_o$, the discussion above gives a upper bound for the number of singularities of $\mathcal{Y}_t$. By Theorem 5.2, $\mathcal{Y}_o$ can be deformed to cubic surfaces $\mathcal{Y}_t$ with 3 nodes. By counting the number of the singularities, the conclusion follows. □

There are still many cubic surfaces remaining as possible candidates for $\mathcal{Y}_o$. Note that, for $t \neq o$, three coplanar lines in the branch locus on $\mathcal{Y}_t$ (cf. the proof of Corollary 5.4), are disjoint from the singular locus of $\mathcal{Y}_t$, i.e., in the Definition 4.1, Γ_i is disjoint from any (-2)-curve for $i = 1, 2, 3$. We will prove that this property is preserved in the limit and this fact will exclude many types of cubic surfaces.

Lemma 6.4. *Let $\Pi_o \colon \mathcal{X}_o \to \mathcal{Y}_o$ be the bidouble cover as in Theorem 5.2. Assume that l_1, l_2, l_3 are the three coplanar lines as in Corollary 5.4 (2). Then these three lines are disjoint from the singular locus of $\mathcal{Y}_o$.*

Proof. Assume that l_1 contains a singularity P of $\mathcal{Y}_o$. By Corollary 5.4 (2), l_1, l_2, l_3 form a hyperplane section H_1 of $\mathcal{Y}_o$. It follows that P is a singularity of H_1. Hence at least one of l_2 and l_3 contains P. Assume that l_2 contains P.

By Corollary 5.4 (3), there is a conic c_1 in the branch locus such that l_1, c_1 form a hyperplane section H_2 of $\mathcal{Y}_o$. Thus P is a singularity of H_2 and c_1 contains P.

Hence there are at least three irreducible components, l_1, l_2 and some irreducible component of c_1 in the branch locus, containing the singularity P. This contradicts Remark 2.3. □

Proposition 6.5. *$\mathcal{Y}_o$ cannot be one of the following types:*
$A_5, 3A_2, A_1 + A_4, 2A_1 + A_3, A_1 + 2A_2, A_1 + A_3, 2A_1 + A_2$.

Proof. By Lemma 6.4, $\mathcal{Y}_o$ contains three coplanar lines, which are disjoint from the singularities. We shall use Table 4 and Table 5 to show that the cubic surfaces of the types listed above do not satisfy this property.

(1) An A_5-type cubic surface $x_3x_0x_1-(x_0^3+x_1^3-x_1x_2^2)=0$ has three lines, $l_1: x_0=x_1=0,\ l_2: x_0=0, x_1-x_2=0,\ l_3: x_0=0,\ x_1+x_2=0$, which all pass through the A_5-singularity $P=(0:0:0:1)$.

(2) A $3A_2$-type cubic surface $x_3x_0x_1-x_2^3=0$ has three lines, $l_1: x_0=x_2=0$, $l_2: x_1=x_2=0$, $l_3: x_2=x_3=0$, which form a triangle with the three A_2-singularities $P_1=(0:0:0:1)$, $P_2=(1:0:0:0)$, $P_3=(0:1:0:0)$ as the vertices.

(3) An (A_1+A_4)-type cubic surface $x_3(x_0x_2-x_1^2)-x_0^2x_1=0$ has four lines, $l_1: x_0=x_1=0$, $l_2: x_0=x_3=0$, $l_3: x_1=x_2=0$, $l_4: x_1=x_3=0$. l_1, l_3 contain the A_1-singularity $P_1=(0:0:0:1)$ and l_2, l_4 contain the A_4-singularity $P_2=(0:0:1:0)$.

(4) A $(2A_1+A_3)$-type cubic surface $x_3(x_0x_2-x_1^2)-x_0x_1^2=0$ has five lines, $l_1: x_0=x_1=0$, $l_2: x_1=x_2=0$, $l_3: x_1=x_3=0$, $l_4: x_0=x_3=0$, $l_5: x_2=x_0+x_3=0$, and three singularities $P_1=(0:0:1:0)(A_3)$, $P_2=(1:0:0:0)(A_1)$, $P_3=(0:0:0:1)(A_1)$. l_1, l_2, l_3 form a triangle with vertices P_1, P_2, P_3, and l_4 contains P_1. There is only one line l_5 which does not contain any singularity.

(5) An (A_1+2A_2)-type cubic surface $x_3(x_0x_2-x_1^2)-x_1^3=0$ has five lines, $l_1: x_0=x_1=0$, $l_2: x_1=x_2=0$, $l_3: x_1=x_3=0$, $l_4: x_0=x_1+x_3=0$, $l_5: x_2=x_1+x_3=0$, and it has three singularities $P_1=(0:0:0:1)(A_1)$, $P_2=(0:0:1:0)(A_2)$, $P_3=(1:0:0:0)(A_2)$. l_1, l_2, l_3 form a triangle with vertices P_1, P_2, P_3, l_4 contains P_2 and l_5 contains P_3.

(6) An (A_1+A_3)-type cubic surface

$$x_3(x_0x_2-x_1^2)-(x_0-x_1)(-x_1+x_2)(x_0-2x_1+x_2)=0$$

has two singularities $P=(0:0:0:1)(A_1)$, $Q=(1:1:1:0)(A_3)$. It has seven lines, $l_1: x_3=x_0-x_1=0$, $l_2: x_3=-x_1+x_2=0$, $l_3: x_3=x_0-2x_1+x_2=0$, $l_4: x_0=x_1=0$, $l_5: x_0=x_1=x_2$, $l_6: x_1=x_2=0$, $l_7: x_1=x_0+x_2-x_3=0$.

Note that l_1, l_2, l_3, l_5 meet at Q, and l_4, l_6 meet at P. There is only one line l_7 which does not contain any singularity.

(7) A $(2A_1+A_2)$-type normal cubic surface $x_3(x_0x_2-x_1^2)-x_1^2(x_0-x_1)=0$ has two A_1-singularities $P_1=(0:0:0:1)$, $P_2=(1:0:0:0)$, and one A_2-singularity $Q=(0:0:1:0)$. It has eight lines, $l_1: x_0=x_1=0$, $l_2: x_1=x_2=0$, $l_3: x_1=x_3=0$, $l_4: x_0-x_1=x_3=0$, $l_5: x_0=x_1=x_2$, $l_6: x_0=x_1-x_3=0$, $l_7: x_1=x_2=x_3$, $l_8: -x_0+x_1-x_3=x_2=0$.

Note that l_1,l_3,l_4,l_6 meet at Q, l_1,l_2,l_5 meet at P_1, l_2,l_3,l_7 meet at P_2. There is only one line l_8 which does not contain any singularity. □

Combining these three propositions with the classification of cubic surfaces (cf. Theorem 3.3), Theorem 6.1 follows.

7 D_4-generalized Burniat Surfaces

By Theorem 6.1, $\mathcal{Y}_o$ can be only one of the following types: $3A_1$, $D_4(1)$, $D_4(2)$ and $4A_1$. For each case we will either exclude it or find all the possible branch loci such that the associated bidouble cover $\mathcal{X}_o$ can be deformed to extended Burniat surfaces with $K^2=3$.

In order to apply the theory of Section 1 to smooth surfaces, we make the following conventions for the remaining sections of Part II.

Conventions Let $\Pi_o\colon \mathcal{X}_o \to \mathcal{Y}_o$ be the bidouble cover as in Theorem 5.2. Let $\mu\colon \widetilde{Y} \to \mathcal{Y}_o$ be the minimal resolution of $\mathcal{Y}_o$. Denote by $\widetilde{S}$ the normalization of the fiber product of $\mathcal{X}_o$ and $\widetilde{Y}$ over $\mathcal{Y}_o$, and $\pi\colon \widetilde{S}\to\widetilde{Y}$ the induced bidouble cover. Moreover, let Δ be the branch locus of the bidouble cover $\pi\colon \widetilde{S}\to\widetilde{Y}$. Write Δ as $\Delta=\Delta_1+\Delta_2+\Delta_3$ according to the group action (cf. Theorem 1.1, Section 1).

In view of Corollary 5.4, Δ has the following properties.

Proposition 7.1. *(1) Every irreducible component of Δ is a (-1)-curve, or a (-2)-curve or a 0-curve.*

(2) $-K_{\widetilde{Y}}.\Delta_i=3$ for $i=1,2,3$.

(3) $\mu_*(\Delta)\equiv -3K_{\mathcal{Y}_o}$.

Proof. By adjunction, for a smooth rational curve D, $-K_{\widetilde{Y}}.D = D^2 + 2$. Hence a (-1)-curve on $\widetilde{Y}$ corresponds to a line on $\mathcal{Y}_o$, and a 0-curve corresponds to a smooth conic. Thus (1) follows from Corollary 5.4. Effective divisors in the linear system $|-K_{\widetilde{Y}}|$ correspond to hyperplane sections of $\mathcal{Y}_o$. Note that $\mathcal{O}_{\mathcal{Y}_o}(K_{\mathcal{Y}_o})$ is invertible, $\mu_*(K_{\widetilde{Y}}) = K_{\mathcal{Y}_o}$ and $\mu^*(K_{\mathcal{Y}_o}) = K_{\widetilde{Y}}$. Since $\mu_*(\Delta_i) = B_i$ (cf. Corollary 5.4), (2) follows from the projection formula and Corollary 5.4 (1), and (3) follows from Corollary 5.4 (4). □

Remark 7.1. By Proposition 1.2, $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ is reduced. By Theorem 1.1, Δ_i's are divisors such that for any $i = 1, 2, 3$, $\Delta_i + \Delta_{i+1}$ is even in $Pic(\widetilde{Y})$. See also Remark 1.1. We will use this remark frequently in the following sections.

In this section we first deal with the case when $\mathcal{Y}_o$ has a D_4-singularity.

7.1 Configuration of Branch Divisors

Assume that $\mathcal{Y}_o$ is of $D_4(1)$-type or of $D_4(2)$-type. Let y_o be the D_4-singularity and $\widetilde{Y}$ be its minimal resolution. **Recall the notation introduced in Subsection 3.2. See Figure 4 and Figure 5.**

Lemma 7.2. $\Pi_o^{-1}(y_o)$ *consists of one point* x_o *and* x_o *is an* A_1*-singularity of* $\mathcal{X}_o$*. Moreover, locally,* $\Pi_o\colon (\mathcal{X}_o, x_o) \to (\mathcal{Y}_o, y_o)$ *is isomorphic to*

$$(X_0, x_0) : z^2 + x^2 + y^2 = 0 \to (Y_0, y_0) : w^2 + uv(u + v) = 0,$$
$$(x, y, z) \mapsto (u, v, w) = (x^2, y^2, xyz),$$

with the G*-action on* (X_0, x_0) *given by* $g_1\colon (x, y, z) \mapsto (x, -y, -z)$, $g_2\colon (x, y, z) \mapsto (-x, y, -z)$, $g_3\colon (x, y, z) \mapsto (-x, -y, z)$.

Proof. Consider the family of bidouble covers $\Pi\colon \mathcal{X} \to \mathcal{Y}$ in Theorem 5.2. For $t \neq o$, $\mathcal{Y}_t$ has three nodes $n_1(t), n_2(t), n_3(t)$. Their limits in $\mathcal{Y}_o$ must be the singularity y_o. Thus their inverse images under Π_t must have limit points in $\Pi_o^{-1}(y_o)$. By the construction (cf. Definition 4.1), for each $n_i(t)$, every point of $\Pi_t^{-1}(n_i(t))$ is fixed by g_i. Note that for any i, g_i and g_{i+1} generates G. Since $\Pi_o^{-1}(y_o)$ forms an orbit under the group action, the cardinality of $\Pi_o^{-1}(y_o)$ can only be $4, 2$, or 1. The argument above shows that $\Pi_o^{-1}(y_o)$ consists of one point x_o. By looking at Theorem 2.5 Table 3 where the quotient (Y_0, y_0) is a D_4-singularity, the conclusion follows.

It is easy to see that $u = x^2, v = y^2, w = xyz$ generate the ring of invariants for the action, and satisfy the equation $w^2 + uv(u+v) = 0$. □

Theorem 7.3. *Assume that $\mathcal{Y}_o$ has a D_4-singularity. Then*

(1) $\mathcal{Y}_o$ must be of $D_4(2)$-type.

(2) $\pi\colon \widetilde{S} \to \widetilde{Y}$ is isomorphic to the bidouble cover associated to the following branch divisors:

$$\Delta_1 = \Gamma_1 + N_2 + C_3,\ \Delta_2 = \Gamma_2 + N_3 + C_1,\ \Delta_3 = \Gamma_3 + N_1 + C_2,$$

where all C_i's are irreducible smooth curves.

Proof. First we consider the (-2)-curves. Lemma 7.2 and Example 2.1 show that one may assume that $\Delta_1 \geq N_2,\ \Delta_2 \geq N_3,\ \Delta_3 \geq N_1,\ \Delta \not\geq Z$, that N_1, N_2, N_3 are connected components of Δ, and that any irreducible component in $\Delta - N_1 - N_2 - N_3$ does not intersect any of the four (-2)-curves N_1, N_2, N_3, Z.

This shows that $(\Delta - N_i).N_i = 0, i = 1,2,3$ and $(\Delta - N_1 - N_2 - N_3).Z = 0$. It follows that $\Delta \equiv -3K_{\widetilde{Y}} + N_1 + N_2 + N_3$. In fact, by Proposition 7.1 (3) we may assume that

$$\Delta \equiv -3K_{\widetilde{Y}} + x_1N_1 + x_2N_2 + x_3N_3 + yZ, \text{ where } x_1, x_2, x_3, y \text{ are integers.}$$

The conditions above show that $x_1 = x_2 = x_3 = 1, y = 0$.

Second, we consider the (-1)-curves. Recall that $\widetilde{Y}$ contains exactly six (-1)-curves: $E_1', E_2', E_3', \Gamma_1, \Gamma_2, \Gamma_3$. Since $E_i'.N_i = 1$, the discussion above shows that $\Delta \not\geq E_i$ for $i = 1,2,3$. But Δ contains at least three (-1)-curves, thus $\Delta \geq \Gamma_1 + \Gamma_2 + \Gamma_3$.

Let $\Delta' := \Delta - N_1 - N_2 - N_3 - \Gamma_1 - \Gamma_2 - \Gamma_3 \equiv -2K_{\widetilde{Y}}$. Since we have considered all the (-2)-curves and all the (-1)-curves, Δ' consists of 0-curves. Note that Δ' is effective, reduced and is disjoint from all (-2)-curves. An easy argument using the following Lemma 7.4 shows that $\Delta' = C_1 + C_2 + C_3$.

Lemma 7.4. *Assume that C is a smooth rational curve on $\widetilde{Y}$ with $C^2 = 0$. If $C.N_1 = C.N_2 = C.N_3 = C.Z = 0$, then C belongs to one of the following linear systems: $|2L - E_{i+1} - E_{i+2} - E'_{i+1} - E'_{i+2}|$ for $i = 1, 2, 3$.*

Proof. We may assume that $C \equiv \lambda L - \sum_{i=1}^{3}(x_i E_i + y_i E'_i)$ in $Pic(\widetilde{Y})$, λ and x_i, y_i are integers. $C.N_1 = C.N_2 = C.N_3 = C.Z = 0$ show that $x_i = y_i$ for $i = 1, 2, 3$ and $\lambda - x_1 - x_2 - x_3 = 0$. Thus $C \equiv (x_1+x_2+x_3)L - \sum_{i=1}^{3} x_i(E_i + E'_i)$.

Then $C^2 = 0$ and $-K_{\widetilde{Y}}.C = 2$ imply $x_1 + x_2 + x_3 = 2, x_1^2 + x_2^2 + x_3^2 = 2$. Since C is effective and irreducible, the conclusion follows. □

We have seen

$$\begin{aligned}\Delta &= N_1 + N_2 + N_3 + \Gamma_1 + \Gamma_2 + \Gamma_3 + C_1 + C_2 + C_3 \\ &\equiv 9L - 2E_1 - 4E'_1 - 2E_2 - 4E'_2 - 2E_3 - 4E'_2.\end{aligned}$$

By Corollary 5.4 (1) and Proposition 7.1 (2), we have

$$\Delta_1 = N_2 + C_j + \Gamma_\alpha,\ \Delta_2 = N_3 + C_k + \Gamma_\beta,\ \Delta_3 = N_1 + C_l + \Gamma_\gamma,$$

where $\{j, k, l\} = \{\alpha, \beta, \gamma\} = \{1, 2, 3\}$. By Remark 7.1, each Δ_i has even coefficients in $E_1, E'_1, E_2, E'_2, E_3, E'_3$. So there are only two possibilities:

(a) $\Delta_i = \Gamma_i + N_{i+1} + C_{i+2}$, (b) $\Delta_i = \Gamma_{i+2} + N_{i+1} + C_i$, for each $i = 1, 2, 3$.

If $\mathcal{Y}_o$ is of $D_4(1)$-type, $\Gamma_1, \Gamma_2, \Gamma_3$ meet at a point P on $\widetilde{Y}$. Note that any other irreducible component of Δ does not pass through P. Then Example 1.1 shows that $\widetilde{S}$ has a $\frac{1}{4}(1, 1)$-singularity P', which is not a rational double point. Since the Γ_i's are disjoint from any (-2)-curves, $\widetilde{S} \to \mathcal{X}_o$ is locally isomorphic at P'. This contradicts that $\mathcal{X}_o$ is a canonical surface.

Thus $\mathcal{Y}_o$ must be of $D_4(2)$-type.

Note that there is an involution $\tau\colon \mathbb{P}^2 \to \mathbb{P}^2$ such that $\tau(P_1) = P_1, \tau(P'_1) = P'_1, \tau(P_2) = P_3,\ \tau(P'_2) = P'_3, \tau(P_3) = P_2, \tau(P'_2) = P'_3$ (for example, in the notation of Subsection 3.2, τ is defined by $(y_0 : y_1 : y_2) \mapsto (y_0 : y_2 : y_1)$). τ induces an involution on $\widetilde{Y}$. It maps the divisor classes of $\Delta_1, \Delta_2, \Delta_3$ in case (a) to the ones of $\Delta_2, \Delta_1, \Delta_3$ in case (b) respectively. Hence the bidouble covers associated to the two kinds of branch loci are essentially the same. □

Remark 7.2. If $\mathcal{Y}_o$ is of $D_4(1)$-type, then we already see that $\widetilde{S}$ has a $\frac{1}{4}(1,1)$-singularity. If we resolve this singularity and blow down the (-1)-curves $\pi^{-1}N_i$, we get a family of minimal smooth surfaces of general type with $K^2 = 2,\ p_g = q = 0$. We remark that the fundamental group of such a surface is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.

7.2 D_4-generalized Burniat Surfaces

Assume that $\mathcal{Y}_o$ is the $D_4(2)$-type cubic surface, and $\widetilde{Y}$ is its minimal resolution. **Recall the notation introduced in Subsection 3.2 and Figure 5.**

We define three effective divisors on $\widetilde{Y}$,

$$\Delta_i = \Gamma_i + N_{i+1} + C_{i+2} \equiv 3L - 2E_i - 2E_i' - 2E_{i+1}',\ i = 1,2,3, \tag{7.1}$$

where all C_i's are irreducible smooth curves. And define three divisors

$$\mathcal{L}_i = -K_{\widetilde{Y}} + E_i - E_{i+2}',\ i = 1,2,3. \tag{7.2}$$

Theorem 7.5 ([BC10-b, Section 7])**.** *Let $\pi\colon \widetilde{S} \to \widetilde{Y}$ be the bidouble cover associated to the above data $\Delta_1, \Delta_2, \Delta_3, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$. Then $\widetilde{S}$ is a smooth surface with $K_{\widetilde{S}}^2 = -3,\ p_g(\widetilde{S}) = q(\widetilde{S}) = 0$.*

Moreover, $|2K_{\widetilde{S}}| = \pi^*|-K_{\widetilde{Y}}| + \pi^*(N_1 + N_2 + N_3)$ and $P_2(\widetilde{S}) = 4$.

Proof. First note that Δ_i's and $\mathcal{L}_i$'s satisfy the equations (1.1) and (1.2). Since the total branch divisor Δ is normal crossing and each Δ_i is smooth, $\widetilde{S}$ is smooth by Proposition 1.2 (2).

Note that $\mathcal{L}_i^2 = 1,\ K_{\widetilde{Y}}.\mathcal{L}_i = -3$. By Corollary 1.4, $K_{\widetilde{S}}^2 = -3$ and $\chi(\mathcal{O}_{\widetilde{S}}) = 1$. From (7.2), one sees that $K_{\widetilde{Y}} + \mathcal{L}_i$ is not effective for all $i = 1,2,3$. Hence by Corollary 1.4, $p_g(\widetilde{S}) = p_g(\widetilde{Y}) = 0$. It follows that $q(\widetilde{S}) = 0$.

From (7.2), one sees that $2K_{\widetilde{S}} + \mathcal{L}_i + \mathcal{L}_{i+1}$ is not effective for all i and $\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 \equiv -3K_{\widetilde{Y}} + N_1 + N_2 + N_3$. By Theorem 1.3 (2) and Corollary 1.4,

$$\begin{aligned} 2K_{\widetilde{S}} &\equiv \pi^*(-K_{\widetilde{Y}} + N_1 + N_2 + N_3), \\ P_2(\widetilde{S}) &= h^0(\widetilde{Y}, -K_{\widetilde{Y}} + N_1 + N_2 + N_3) = h^0(\widetilde{Y}, -K_{\widetilde{Y}}) = 4. \end{aligned}$$

It follows that $|2K_{\widetilde{S}}| = \pi^*|-K_{\widetilde{Y}} + N_1 + N_2 + N_3| = \pi^*|-K_{\widetilde{Y}}| + \pi^*(N_1 + N_2 + N_3)$, since $N_1 + N_2 + N_3$ is the fixed part of $|-K_{\widetilde{Y}} + N_1 + N_2 + N_3|$. ∎

Definition 7.1. The minimal model of $\widetilde{S}$ in the Theorem 7.5 is called a **D_4-generalized Burniat surface**.

Corollary 7.6 ([BC10-b, Section 7])**.** *Let $f\colon \widetilde{S} \to S$ be the blow down of the six (-1)-curves $\pi^{-1}N_i$ for $i = 1, 2, 3$. Then S is a smooth minimal surface of general type with $K_S^2 = 3$, $p_g(S) = q(S) = 0$ and $P_2(S) = 4$. S has exactly one (-2)-curve Z'. Moreover, $f^*|2K_S| = \pi^*|-K_{\widetilde{Y}}|$ and the bicanonical linear system of S is base-point-free.*

Proof. Since each N_i, $i = 1, 2, 3$, forms a connected component of the branch locus, each $\pi^{-1}N_i$ is a disjoint union of two (-1)-curves. Note that Z is not in the branch locus, and $Z.N_i = 1$, $i = 1, 2, 3$. Then Hurwitz's Theorem shows that π^*Z is a smooth rational curve with self-intersection number -8. Let $f\colon \widetilde{S} \to S$ be the blow down of the six (-1)-curves. Then $K_S^2 = 3$ and the image of π^*Z is a (-2)-curve Z'.

Since p_g, q, P_2 are birational invariants, $p_g(S) = 0$ and $P_2(S) = 4$. Moreover, since $|2K_{\widetilde{S}}| = f^*|2K_S| + \pi^*(N_1 + N_2 + N_3)$ by the Theorem 7.5, we have $f^*|2K_S| = \pi^*|-K_{\widetilde{Y}}|$. $|-K_{\widetilde{Y}}|$ is base-point-free, thus $|2K_S|$ is base-point-free. Moreover, $-K_{\widetilde{Y}}$ is nef and big, so is K_S. Thus S is minimal and of general type. □

Corollary 7.7 ([BC10-b, Section 7])**.** *Let $\varphi\colon S \to X$ be the contraction of the (-2)-curve Z', i.e., X is the canonical model of S. Then X is a bidouble cover of the $D_4(2)$-type cubic surface $\mathcal{Y}_o$ by the bicanonical morphism.*

Moreover, X has an A_1-singularity, lying over the D_4-singularity of $\mathcal{Y}_o$, where the bicanonical morphism is totally ramified.

Proof. It follows from Corollary 7.6 and Lemma 7.2. □

8 $4A_1$-generalized Burniat Surfaces

8.1 Configuration of Branch Divisors

Assume that $\mathcal{Y}_o$ is the $4A_1$-type cubic surface. Let $\mu\colon \widetilde{Y} \to \mathcal{Y}_o$ be its minimal resolution. **Recall the notation introduced in Subsection 3.3 and Figure 6.**

Theorem 8.1. *$\pi\colon \widetilde{S} \to \widetilde{Y}$ is isomorphic to the bidouble cover associated to the following branch divisors:*

$$\Delta_1 = \Gamma_1 + N_2 + C_3,\ \ \Delta_2 = \Gamma_2 + N_3 + C_1,\ \ \Delta_3 = \Gamma_3 + N_1 + C_2,$$

where all C_i's are irreducible smooth curves.

Before giving the proof, we make the following remark.

Remark 8.1. (1) By Theorem 3.3, up to an isomorphism, there is exactly one $4A_1$-type cubic surface.
(2) It well known (cf. [Sak10, Theorem 3]), the automorphism group of a $4A_1$-type cubic surface is isomorphic to the symmetry group of four letters, which permutes the 4 nodes of the surface.

Proof. First we consider the (-1)-curves. All the (-1)-curves except Γ_1, Γ_2, Γ_3 intersect at least one (-2)-curve, which correspond to the lines in $\mathcal{Y}_o$ passing through singularities. By Corollary 5.4 (2) and Lemma 6.4, $\Delta \geq \Gamma_1 + \Gamma_2 + \Gamma_3$.

Next we consider 0-curves.

Lemma 8.2. *Fix $k \in \{1, 2, 3\}$. Assume that C is a reduced curve of $\widetilde{Y}$ such that $C \not\geq \Gamma_i, N_i$, for $i = 1, 2, 3$. If $\mu(C + \Gamma_k)$ is a hyperplane section of $\mathcal{Y}_o$, then C is a smooth irreducible curve in the linear system $|2L - E_{k+1} - E_{k+2} - E'_{k+1} - E'_{k+2}|$. It follows that C is disjoint from all N_i's and Z.*

Proof. Without loss of generality, assume that $k = 1$. Note that elements in $|-K_{\widetilde{Y}}|$ correspond to hyperplane sections of $\mathcal{Y}_o$ and $\Gamma_1 + (2L - E_2 - E_3 - E'_2 - E'_3) \equiv -K_{\widetilde{Y}}$. If C is a singular element in $|2L - E_2 - E_3 - E'_2 - E'_3|$, then $C = N_1 + Z + 2E_1$, or $C = N_2 + N_3 + 2E'_1$ or $C = \Gamma_2 + \Gamma_3$. Thus the first conclusion follows. The second conclusion follows from the calculation of intersection numbers. ☐

By Corollary 5.4 (3) and Lemma 8.2, one sees that Δ must contain a smooth curve C_i in the linear system $|2L - E_{i+1} - E_{i+2} - E'_{i+1} - E'_{i+2}|$ for each i.

We have shown that $\Delta \geq \Gamma_1 + \Gamma_2 + \Gamma_3 + C_1 + C_2 + C_3$. By Corollary 5.4 (1), (2) and (3), up to a permutation of $1, 2, 3$, one of the following two holds:

$$\begin{array}{lll} (a)\ \Delta_1 \geq \Gamma_1 + C_3, & \Delta_2 \geq \Gamma_2 + C_1, & \Delta_3 \geq \Gamma_3 + C_2, \\ (b)\ \Delta_1 \geq \Gamma_3 + C_1, & \Delta_2 \geq \Gamma_1 + C_2, & \Delta_3 \geq \Gamma_2 + C_3. \end{array}$$

Since the 3 nodes on $\mathcal{Y}_t$ are in the branch locus of π_t, thus at least 3 nodes of $\mathcal{Y}_o$ are in the branch locus of π_o. Equivalently, Δ contains at least three (-2)-curves. We distinguish two cases.

Case I: One of the four (-2)-curves is not in Δ.
Without loss of generality (cf. Remark 8.1 (2)), assume that $\Delta \not\geq Z$. Then

$$\Delta = \sum_{i=1}^{3}(\Gamma_i + C_i + N_i) \equiv 12L - \sum_{i=1}^{3}(4E_i + 5E_i').$$

Thus Δ_i has even coefficients in E_1, E_2, E_3 and odd coefficients in E_1', E_2', E_3' by Remark 7.1. It follows that if (a) holds then $\Delta_i = \Gamma_i + N_{i+1} + C_{i+2}$, and if (b) holds then $\Delta_i = \Gamma_{i+2} + N_{i+1} + C_i$.

Take an involution τ of $\mathbb{P}^2$ such that $\tau(P_1) = P_1$, $\tau(P_1') = P_1'$, $\tau(P_2) = P_3$, $\tau(P_2') = P_3'$. Then it follows that $\tau(P_3) = P_2$, $\tau(P_3') = P_2'$. It induces an involution on $\widetilde{Y}$ which maps the divisor classes of $\Delta_1, \Delta_2, \Delta_3$ in case (a) to the ones of $\Delta_2, \Delta_1, \Delta_3$ in case (b). Hence the bidouble covers associated to the two kinds of branch loci are essentially the same.

Case II: All the 4 nodes are contained in the branch locus, i.e,
$\Delta \geq N_1 + N_2 + N_3 + Z$. We intend to exclude this case.

Assume that (a) holds. Then we may assume that

$$\Delta_1 = \Gamma_1 + C_3 + \sum_{i=1}^{3} a_i N_i + a_4 Z,$$

$$\Delta_2 = \Gamma_2 + C_1 + \sum_{i=1}^{3} b_i N_i + b_4 Z,$$

$$\Delta_3 = \Gamma_3 + C_2 + \sum_{i=1}^{3} c_i N_i + c_4 Z,$$

for each $k = 1, 2, 3, 4$, exactly one of the a_k, b_k, c_k is 1 and the other two is 0, since Δ is effective and reduced. The following table gives the coefficients (up to sign) of E_1, E_2, E_3 in the branch divisors.

	Δ_1	Δ_2	Δ_3
E_1	$a_1 + a_4 + 2$	$b_1 + b_4$	$c_1 + c_4 + 1$
E_2	$a_2 + a_4 + 1$	$b_2 + b_4 + 2$	$c_2 + c_4$
E_3	$a_3 + a_4$	$b_3 + b_4 + 1$	$c_3 + c_4 + 2$

By Remark 7.1, $a_1 + a_4 + 2, b_1 + b_4, c_1 + c_4 + 1$ must be of the same parity. Since their sum is 5, they must be all odd integers. Thus either $(a_1, b_1, c_1) = (1, 0, 0)$ and $(a_4, b_4, c_4) = (0, 1, 0)$, or $(a_1, b_1, c_1) = (0, 1, 0)$ and $(a_4, b_4, c_4) = (1, 0, 0)$.

If the former holds, then the coefficients $a_3, b_3 + 2, c_3 + 2$ of E_3 cannot have the same parity. If the latter holds, then the coefficients $a_2 + 2, b_2 + 2, c_2$ of E_2 cannot have the same parity. So this case is excluded.

If (b) holds, a similar argument shows that Case II can be excluded. □

Remark 8.2. In the course of excluding Case II, we find another family of surfaces of general type which are also bidouble covers of the $4A_1$-type cubic surface, but branched on all the nodes. First construct the bidouble cover $\pi\colon \widetilde{S} \to \widetilde{Y}$ associated to the following data,

$$\Delta_1 = C_1 + \Gamma_2 + N_1 + N_2,\ \Delta_2 = C_2 + \Gamma_1 + N_3 + Z,\ \Delta_3 = C_3 + \Gamma_3.$$

Then blow down the eight (-1)-curves $\pi^{-1}N_i$ and $\pi^{-1}Z$, $f\colon \widetilde{S} \to S$. S is of general type with $K_S^2 = 3$ and $p_g(S) = 0$. S has 4 nodes coming from the nodes of the curve Δ_3. However, note that $\Delta_3 \equiv -K_{\widetilde{Y}}$, we can deform S to smooth surfaces by deforming Δ_3 to smooth curves. For details, see Section 14 in Part IV.

8.2 $4A_1$-generalized Burniat Surfaces

Assume that $\mathcal{Y}_o$ is the $4A_1$-type cubic surface, and $\widetilde{Y}$ is its minimal resolution. **Recall the notation introduced in Subsection 3.3 and Figure 6.**

We define three divisors on $\widetilde{Y}$,

$$\Delta_i = \Gamma_i + N_{i+1} + C_{i+2} \equiv 4L - 2E_i - 2E_{i+1} - 3E'_i - E'_{i+1} - E'_{i+2},\ i = 1, 2, 3, \quad (8.1)$$

where all C_i's are irreducible smooth curves. And define three divisors

$$\mathcal{L}_i = -K_{\widetilde{Y}} + L - E_{i+2} - E'_{i+1} - E'_{i+2} \equiv -K_{\widetilde{Y}} + N_i + E_i - E_{i+2},\ i = 1, 2, 3. \quad (8.2)$$

Theorem 8.3 ([BC10-b, Section 7]). *Let $\pi\colon \widetilde{S} \to \widetilde{Y}$ be the bidouble cover associated to the above data $\Delta_1, \Delta_2, \Delta_3, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$. Then $\widetilde{S}$ is a smooth surface with $K^2_{\widetilde{S}} = -3,\ p_g(\widetilde{S}) = q(\widetilde{S}) = 0$.*

Moreover, $|2K_{\widetilde{S}}| \equiv \pi^*| - K_{\widetilde{Y}}| + \pi^*(N_1 + N_2 + N_3)$ and $P_2(\widetilde{S}) = 4$.

Proof. Exactly the same proof as Theorem 7.5, verbatim. □

Definition 8.1. The minimal model of $\widetilde{S}$ in the Theorem 8.3 is called a $4A_1$**-generalized Burniat surface**.

Corollary 8.4 ([BC10-b, Section 7]). *Let $f\colon \widetilde{S} \to S$ be the blow down of six (-1)-curves $\pi^{-1}N_i$ for $i = 1, 2, 3$. Then S is a smooth minimal surface of general type with $K^2_S = 3$, $p_g(S) = q(S) = 0$ and $P_2(S) = 4$. S has exactly four (-2)-curves Z_1, Z_2, Z_3, Z_4. Moreover, $f^*|2K_S| = \pi^*| - K_{\widetilde{Y}}|$ and the bicanonical linear system of S is base-point-free.*

Proof. Since each $N_i,\ i = 1, 2, 3$, forms a connected component of the branch locus, each $\pi^{-1}N_i$ is a disjoint union of two (-1)-curves. Note that Z is disjoint from the branch locus, π^*Z is a disjoint union of four (-2)-curves. Let $f\colon \widetilde{S} \to S$ be the blow down of the six (-1)-curves coming from N_i, $i = 1, 2, 3$. Then $K^2_S = 3$, and the image of π^*Z is a disjoint union of four (-2)-curve Z_1, Z_2, Z_3, Z_4.

Since p_g, q, P_2 are birational invariants, $p_g(S) = 0$ and $P_2(S) = 4$. Moreover, since $|2K_{\widetilde{S}}| = f^*|2K_S| + \pi^*(N_1 + N_2 + N_3)$ by the Theorem 8.3, we have $f^*|2K_S| = \pi^*| - K_{\widetilde{Y}}|$. $| - K_{\widetilde{Y}}|$ is base-point-free, thus $|2K_S|$ is base-point-free. Moreover, $-K_{\widetilde{Y}}$ is nef and big, so is K_S. Thus S is minimal and of general type. □

Corollary 8.5 ([BC10-b, Section 7]). *Let $\varphi\colon S \to X$ be the contraction of the (-2)-curves*

Z_1, Z_2, Z_3, Z_4, i.e., X is the canonical model of S. Then X is a bidouble cover of the $4A_1$-type cubic surface $\mathcal{Y}_o$ by the bicanonical morphism. Moreover, X has 4 nodes lying over one node of $\mathcal{Y}_o$, where the bicanonical morphism is unramified.

9 Irreducible Component

9.1 Configuration of Branch Divisors on $3A_1$-type Cubic Surfaces

Assume that $\mathcal{Y}_o$ is a $3A_1$-type cubic surface. Let $\mu\colon \widetilde{Y} \to \mathcal{Y}_o$ be its minimal resolution. **Recall the notation introduced in Subsection 3.1. See Figure 1 and Figure 2.**

Theorem 9.1. *(1) Three (-1)-curves Γ_i do not pass through a common point, i.e., in the configuration of $P_1, \ldots, P_3'$, three lines P_1P_1', P_2P_2', P_3P_3' do not pass through a common point (cf. Remark 3.2).*

(2) $\mathcal{X}_o$ is the canonical model of an extended Burniat surface or a nodal Burniat surface S with $K_S^2 = 3$.

Proof. First we consider the (-2)-curves. By Theorem 5.2, three nodes of $\mathcal{Y}_o$ must be contained in the branch locus. Thus $\Delta \geq N_1 + N_2 + N_3$.

Next we consider the (-1)-curves. Note that $\Gamma_1, \Gamma_2, \Gamma_3$ are the only (-1)-curves which do not intersect any (-2)-curve. Thus by Corollary 5.4 (2) and by Lemma 6.4, $\Delta \geq \Gamma_1 + \Gamma_2 + \Gamma_3$.

Lemma 9.2. *Fix $k \in \{1, 2, 3\}$. Assume that C is a reduced curve of $\widetilde{Y}$ such that $C \not\geq \Gamma_i, N_i$ for any $i = 1, 2, 3$. If $\mu(C + \Gamma_k)$ is a hyperplane section of $\mathcal{Y}_o$, then there are two possibilities:*

(1) C is an irreducible smooth curve in the linear system $|2L - E_{k+1} - E_{k+2} - E'_{k+1} - E'_{k+2}|$ and $C.N_k = 0$.

(2) $C = G_k + E_k$ and $C.N_k = 2$, where $G_k = L - E_{k+1} - E_{k+2}$.

Proof. Note that $\Gamma_k + (2L - E_{k+1} - E_{k+2} - E'_{k+1} - E'_{k+2}) \equiv -K_{\widetilde{Y}}$. If $C \in |2L - E_{k+1} - E_{k+2} - E'_{k+1} - E'_{k+2}|$ is a reducible curve, then $C = G_k + N_k + E_k$, or $C = \Gamma_{k+1} + \Gamma_{k+2}$ or $C = N_{k+1} + N_{k+2} + 2E'_k$. The conclusion follows. □

By Corollary 5.4 (1), $\Delta = N_1 + N_2 + N_3 + \Gamma_1 + \Gamma_2 + \Gamma_3 + \hat{C}_1 + \hat{C}_2 + \hat{C}_3$, where $\hat{C}_i$ is a curve such that $\mu(\hat{C}_i + \Gamma_i)$ is a hyperplane section on $\mathcal{Y}_o$. By Corollary 5.4 (1), (2) and (3), up to a permutation of $1, 2, 3$, one of the following two holds:

$$(a)\ \Delta_1 \geq \Gamma_1 + \hat{C}_3, \quad \Delta_2 \geq \Gamma_2 + \hat{C}_1, \quad \Delta_3 \geq \Gamma_3 + \hat{C}_2,$$
$$(b)\ \Delta_1 \geq \Gamma_3 + \hat{C}_1, \quad \Delta_2 \geq \Gamma_1 + \hat{C}_2, \quad \Delta_3 \geq \Gamma_2 + \hat{C}_3.$$

So it suffices to determine $\Delta_1, \Delta_2, \Delta_3$. First assume that all the $\hat{C}_1, \hat{C}_2, \hat{C}_3$ are irreducible, denoted by C_1, C_2, C_3 as before. Then $\Delta \equiv 12L - 4E_1 - 4E_2 - 4E_3 - 5E'_1 - 5E'_2 - 5E'_3$. Remark 7.1 implies Δ_i has even coefficients on E_k and odd coefficients on E'_k. It follows that $\Delta_i = \Gamma_i + N_{i+1} + C_{i+2}$ or $\Delta_i = \Gamma_{i+2} + N_{i+1} + C_i$.

Take an automorphism τ of $\mathbb{P}^2$ such that $\tau(P_1) = P_1$, $\tau(P'_1) = P'_1$, $\tau(P_2) = P_3$, $\tau(P'_2) = P'_3$. Then it follows that $\tau(P_3) = P_2$, $\tau(P'_3) = P'_2$. It induces an involution on $\widetilde{Y}$ which maps the divisor classes of $\Delta_1, \Delta_2, \Delta_3$ in case (a) to the divisor classes of $\Delta_2, \Delta_1, \Delta_3$ in case (b) respectively. Hence the bidouble covers associated to the two kinds of branch loci above are essentially the same.

In either case, if $P_1P'_1, P_2P'_2, P_3P'_3$ meet at one point, then $\Gamma_1, \Gamma_2, \Gamma_3$ meet at a smooth point P. Then Example 1.1 shows that $\widetilde{S}$ and $\mathcal{X}_o$ would have a $\frac{1}{4}(1,1)$-singularity, which contradicts $\mathcal{X}_o$ has only rational double points. Hence $P_1P'_1, P_2P'_2, P_3P'_3$ do not meet at a common point. We see that these divisors are the strictly extended Burniat divisors (cf. Definition 4.1 (4.1)).

A similar discussion shows that (1) and (2) still hold, if one or more of the $\hat{C}_i$'s are reducible. For example, assume $\hat{C}_1 = G_1 + E_1$, $\hat{C}_2, \hat{C}_3$ are irreducible and (a) holds. Then $\Delta \equiv 11L - 3E_1 - 4E_2 - 4E_3 - 5E'_1 - 4E'_2 - 4E'_3$ and Δ_i has odd coefficients on E_1, E'_1 and even coefficients on E_2, E_3, E'_2, E'_3. Thus there is only one possibility, $\Delta_1 = \Gamma_1 + N_1 + N_2 + C_3, \Delta_2 = \Gamma_2 + G_1 + E_1, \Delta_3 = \Gamma_3 + C_2$. These are divisors in the Definition 4.1 (2). □

Remark 9.1. If $\Gamma_1, \Gamma_2, \Gamma_3$ pass through a common point, then we already see $\widetilde{S}$ has a $\frac{1}{4}(1,1)$-singularity. If we resolve this singularity and blow down the (-1)-curves $\pi^{-1}N_i$, we get a family of minimal smooth surfaces of general type with $K^2 = 2,\ p_g = q = 0$. We remark that the fundamental group of such a surface is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.

9.2 Closure of the Open Subset $\mathcal{NEB}_3$

Theorem 9.3. *Let $\overline{\mathcal{NEB}_3}$ be the closure of the open set $\mathcal{NEB}_3$ corresponding to nodal and extended Burniat surfaces with $K^2 = 3$. Then*

(1) $\overline{\mathcal{NEB}_3}$ contains a 3-dimensional family $4A_1$-$\mathcal{GB}$ of canonical models of the $4A_1$-generalized Burniat surfaces.

(2) $\overline{\mathcal{NEB}_3}$ contains a 3-dimensional family D_4-$\mathcal{GB}$ of canonical models of the D_4-generalized Burniat surfaces.

(3) $\overline{\mathcal{NEB}_3}$ is the union of $\mathcal{NEB}_3$, D_4-$\mathcal{GB}$ and $4A_1$-$\mathcal{GB}$.

Remark 9.2. (1) and (2) are already described in [BC10-b, proposition 7.1 and proposition 7.2]. Here (3) described $\overline{\mathcal{NEB}_3}$ completely.

Proof. (1) In the resolution of a $3A_1$-cubic surface in Subsection 3.1, if we let P_1, P_2, P_3 become collinear, more precisely, the point P_2 moves in the line joining P_1' and P_3' till it reaches the line joining P_1 and P_3, we get a one parameter family $F' \colon \mathcal{Y} \to T$ such that $\mathcal{Y}_t$ is of $3A_1$-type for $t \neq o$, and $\mathcal{Y}_o$ is of $4A_1$-type.

If we compare the strictly extended Burniat divisors (cf. Definition 4.1 (4.1), Section 2) and the $4A_1$-generalized Burniat divisors (cf. Subsection 8.2 (8.1)), we see that every $4A_1$-generalized Burniat surface can be deformed to strictly extended Burniat surfaces, and (1) follows.

(2) On the minimal resolution $\widetilde{Y}$ in Subsection 3.1, if we first blow down three (-1)-curves $L - E_{i+1} - E_{i+2}$ $(i = 1,2,3)$, and then the strict transforms of three (-2) curves N_i $(i = 1,2,3)$, we obtain another copy of the projective plane $\hat{\mathbb{P}}^2$, where one has blown up three points

$\hat{P}_i$ and again blown up three points $\hat{P}'_i$ $(i = 1, 2, 3)$, where $\hat{P}'_i$ is infinitely near to $\hat{P}_i$. See Figure 3.

If we denote by $\hat{\sigma}\colon \widetilde{Y} \to \hat{\mathbb{P}}^2$ the new blowup, by $\hat{E}_i$ (respectively $\hat{E}'_i$) the total transform of $\hat{P}_i$ (respectively $\hat{P}'_i$), and by $\hat{L}$ the pullback of a general line by $\hat{\sigma}$. Then we have

$$\begin{aligned} \hat{E}'_i &\equiv L - E_{i+1} - E_{i+2}, \\ \hat{E}_i &\equiv \hat{E}'_i + N_i \equiv 2L - E_i - E_{i+1} - E_{i+2} - E'_{i+1} - E'_{i+2}, \\ \hat{L} &\equiv 4L - 2\sum\nolimits_{i=1}^{3} E_i - \sum\nolimits_{i=1}^{3} E'_i. \end{aligned}$$

Use these formulae, it follows that

$$\Gamma_i = \hat{L} - \hat{E}_i - \hat{E}'_i,\ C_i \in |2\hat{L} - \hat{E}_{i+1} - \hat{E}_{i+2} - \hat{E}'_{i+1} - \hat{E}'_{i+2}|, N_i = \hat{E}_i - \hat{E}'_i. \tag{9.1}$$

Now if we let $\hat{P}_1, \hat{P}_2, \hat{P}_3$ become collinear (compare Figure 3 and Figure 5), then we get a one parameter family $F'\colon \mathcal{Y} \to T$ such that $\mathcal{Y}_t$ is of $3A_1$-type for $t \neq o$, and $\mathcal{Y}_o$ is of $D_4(2)$-type. If we compare the strictly extended Burniat divisors (cf. Definition 4.1 (4.1), Section 2) using (9.1) and the D_4-generalized Burniat divisors (cf. Theorem 7.3 (3)), we see that every D_4-generalized Burniat surface can be deformed to strictly extended Burniat surfaces.

(3) In Theorem 5.2, we show that a one parameter limit $\mathcal{X}_o$ of extended Burnait surface with $K^2 = 3$ is a bidouble cover of a cubic surface $\mathcal{Y}_o$. Then Theorem 6.1 and Theorem 7.3 (1) show that the cubic surface $\mathcal{Y}_o$ can only be one of the following types: $D_4(2), 4A_1, 3A_1$.

Corollary 9.1 shows that if $\mathcal{Y}_o$ is of $3A_1$-type, then $\mathcal{X}_o$ is the canonical model of an extended Burniat surface or a nodal Burniat surface with $K^2 = 3$. Theorem 7.3 (2) and Theorem 8.1 (2) confirm that the configurations of branch divisors of the bidouble covers are exactly those of the D_4-generalized Burniat surfaces and the $4A_1$-type generalized Burniat surfaces.

Thus the conclusion follows from (1) and (2). □

Part III

Deformations of Generalized Burniat Surfaces

From part II, we have describe $\overline{\mathcal{NEB}_3}$ completely. Now we start to study whether $\overline{\mathcal{NEB}_3}$ is a connected component in the moduli space or not. The first step is to study the local deformations of generalized Burniat surfaces.

10 Key Tools to Calculate the Cohomology Groups of the Tangent Sheaves

In order to study the deformations of a smooth surface $\widetilde{S}$, we sum up some results to calculate the dimensions of the cohomology groups of the tangent sheaf $\Theta_{\widetilde{S}}$. Some results are quoted without proof.

By Serre Duality, $H^k(\widetilde{S}, \Theta_{\widetilde{S}}) \cong H^{2-k}(\widetilde{S}, \Omega^1_{\widetilde{S}} \otimes \Omega^2_{\widetilde{S}})$ for $k = 0, 1, 2$. The first result is to calculate the cohomology groups of $\Omega^1_{\widetilde{S}} \otimes \Omega^2_{\widetilde{S}}$ using bidouble cover structure.

Theorem 10.1 ([Cat84, Theorem 2.16]). *Let $\widetilde{Y}$ be a smooth surface and let $\pi \colon \widetilde{S} \to \widetilde{Y}$ be a bidouble cover associated to the data $\Delta_1, \Delta_2, \Delta_3$, $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ satisfying equations (1.1) and (1.2). Assume that $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ is reduced and has only normal crossing singularities. Then*

$$\pi_*(\Omega^1_{\widetilde{S}} \otimes \Omega^2_{\widetilde{S}}) = \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}} \oplus \left(\oplus_{i=1}^3 \Omega^1_{\widetilde{Y}}(\log \Delta_i)(K_{\widetilde{Y}} + \mathcal{L}_i)\right),$$

the first summand is the G-invariant one, the others correspond to three nontrivial characters χ_i of G.

We need the following lemmas to calculate the cohomology groups of the direct summands.

Lemma 10.2 ([BC10-b, Lemma 5.1]). *Assume that C is a connected component of a smooth divisor $\Delta \subseteq \widetilde{Y}$, where $\widetilde{Y}$ is a smooth projective surface.*

Moreover, let M be a divisor on $\widetilde{Y}$ such that $(K_{\widetilde{Y}} + 2C + M).C < 0$. Then

$$H^0(\Omega^1_{\widetilde{Y}}(\log(\Delta - C))(C + M)) \cong H^0(\Omega^1_{\widetilde{Y}}(\log \Delta)(M)).$$

Proof. Since C is a connected component of a smooth divisor Δ, we have the following exact sequence,

$$0 \to \Omega^1_{\widetilde{Y}}(\log \Delta) \to \Omega^1_{\widetilde{Y}}(\log(\Delta - C))(C) \to \Omega^1_C(C) \to 0$$

Tensor it with the invertible sheaf $\mathcal{O}_{\widetilde{Y}}(M)$ and use the adjunction formula $\Omega^1_C = \mathcal{O}_C(K_{\widetilde{Y}} + C)$, to get the exact sequence,

$$0 \to \Omega^1_{\widetilde{Y}}(\log \Delta)(M) \to \Omega^1_{\widetilde{Y}}(\log(\Delta - C))(C + M) \to \mathcal{O}_C(K_{\widetilde{Y}} + 2C + M) \to 0$$

Since $(K_{\widetilde{Y}} + 2C + M).C < 0, H^0(C, \mathcal{O}_C(K_{\widetilde{Y}} + 2C + M)) = 0$, the associated exact sequence of cohomology groups shows that $H^0(\Omega^1_{\widetilde{Y}}(\log \Delta)(M)) \cong H^0(\Omega^1_{\widetilde{Y}}(\log(\Delta - C))(C + M))$. $\square$

Lemma 10.3 ([Cat84, Lemma 3.7], [CHKS06, Lemma 3, page 675])**.** *Let $\Delta = \cup_i \Delta_i$ be a union of smooth divisors $\Delta_1, \ldots, \Delta_k$ on a smooth surface $\widetilde{Y}$, such that Δ has only normal crossing singularities. Then*

(1) there is an exact sequence

$$0 \to \Omega^1_{\widetilde{Y}} \to \Omega^1_{\widetilde{Y}}(\log \Delta_1, \ldots, \log \Delta_k) \to \oplus_{i=1}^k \mathcal{O}_{\Delta_i} \to 0.$$

(2) In the cohomology exact sequence associated to the above exact sequence $\partial\colon \oplus_{i=1}^k H^0(\mathcal{O}_{\Delta_i}) \to H^1(\Omega^1_{\widetilde{Y}})$, if 1_{Δ_i} is the function which is $\equiv 1$ on Δ_i and 0 elsewhere, then $\partial(1_{\Delta_i}) = c_1(\Delta_i)$.

The next theorem studies how the dimensions of the cohomology groups of tangent sheaf change when blowing down a (-1)-curve.

Theorem 10.4 (cf. [Cat88, Lemma 9.22])**.** *Let S be a smooth surface, and $f\colon \widetilde{S} \to S$ be the blowup of S at a point p. Then $R^1 f_* \Theta_{\widetilde{S}} = 0$.*

Moreover, if S is of general type, then $h^0(\widetilde{S}, \Theta_{\widetilde{S}}) = h^0(S, \Theta_S) = 0$, $h^1(\widetilde{S}, \Theta_{\widetilde{S}}) = h^1(S, \Theta_S) + 2$ and $h^2(\widetilde{S}, \Theta_{\widetilde{S}}) = h^2(S, \Theta_S)$.

Proof. Let E be the exceptional curve of f. The sheaf $R^1 f_*\Theta_{\widetilde{S}}$ is supported on the point p, by formal function theorem (cf. [Har77, Theorem 11.1]), it suffices to show $H^1(E_n, \Theta_{\widetilde{S}} \otimes \mathcal{O}_{E_n}) = 0$, where E_n is the closed subscheme of $\widetilde{S}$ defined by $\mathcal{I}^n$, where $\mathcal{I}$ is the ideal sheaf of E.

There is an exact sequence

$$0 \to \frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} \to \mathcal{O}_{E_{n+1}} \to \mathcal{O}_{E_n} \to 0.$$

for all $n \geq 0$. Tensor the exact sequence by $\Theta_{\widetilde{S}}$, it remains exact since $\Theta_{\widetilde{S}}$ is a locally free sheaf. Note that $E_1 = E$ and $\frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} \cong \mathcal{O}_E(n)$, it suffices to show $H^1(E, \Theta_{\widetilde{S}} \otimes \mathcal{O}_E(n)) = 0$ for all $n \geq 0$.

We have a normal exact sequence

$$0 \to \Theta_E \to \Theta_{\widetilde{S}} \otimes \mathcal{O}_E \to \mathcal{O}_E(E) \to 0.$$

Tensor it with $\mathcal{O}_E(n)$, we get

$$0 \to \mathcal{O}_E(n+2) \to \Theta_{\widetilde{S}} \otimes \mathcal{O}_E(n) \to \mathcal{O}_E(n-1) \to 0.$$

Since $H^1(E, \mathcal{O}_E(n-1)) = 0$ for $n \geq 0$, one sees that $H^1(E, \Theta_{\widetilde{S}} \otimes \mathcal{O}_E(n)) = 0$. Hence we have shown that $R^1 f_*(\Theta_{\widetilde{S}}) = 0$. (See [Cat88, Lemma 9.22] for another proof).

There is an exact sequence (cf. [Ser06, page 73]),

$$0 \to \Theta_{\widetilde{S}} \to f^*\Theta_S \to \mathcal{O}_E(-E) \to 0.$$

By [Har77, Proposition 3.4, Chapter V] $R^k f_*\mathcal{O}_{\widetilde{S}} = 0$ for $k \geq 1$, then the projection formula shows that $R^k f_*(f^*\Theta_S) = R^k f_*\mathcal{O}_{\widetilde{S}} \otimes \Theta_S = 0$. Thus we have an exact sequence

$$0 \to f_*\Theta_{\widetilde{S}} \to \Theta_S \to f_*\mathcal{O}_E(-E) \to 0.$$

If S is of general type, $h^0(\widetilde{S}, \Theta_{\widetilde{S}}) = h^0(S, \Theta_S) = 0$ (cf. [Mats63]). Note that $f_*\mathcal{O}_E(-E)$ is supported on p, thus $H^k(S, f_*\mathcal{O}_E(-E)) = 0$ for $k \geq 1$. By the long exact sequence of cohomology associated to the last exact sequence above, and $\mathcal{O}_E(-E) \cong \mathcal{O}_E(1)$, we have $h^1(S, f_*\Theta_{\widetilde{S}}) = h^1(S, \Theta_S) + 2$ and $h^2(S, f_*\Theta_{\widetilde{S}}) = h^2(S, \Theta_S)$.

Finally since $R^k f_*\Theta_{\widetilde{S}} = 0$ for $k \geq 1$, Leray spectral sequence shows that $h^1(\widetilde{S}, \Theta_{\widetilde{S}}) = h^1(S, f_*\Theta_{\widetilde{S}})$ and $h^2(\widetilde{S}, \Theta_{\widetilde{S}}) = h^2(S, f_*\Theta_{\widetilde{S}})$. Hence the conclusion follows. □

The last theorem describes how the dimensions of the cohomology groups of the tangent sheaf change when contracting a (-2)-curve to a node.

Theorem 10.5 ([BW74, Proposition 1.10, Theorem 2.14]). *Let S be a minimal surface of general type, and let $\varphi\colon S \to X$ be a morphism contracting a (-2)-curve N of S to an A_1-singularity on X. Then*

$$\varphi_*\Theta_S = \Theta_X,\ H^1(S,\Theta_S) \cong H^1(X,\Theta_X) \oplus H^1_N(\Theta_S),\ H^2(S,\Theta_S) \cong H^2(X,\Theta_X).$$

Moreover, $\dim H^1_N(\Theta_S) = 1$.

11 Deformations of the D_4-generalized Burniat Surfaces

Throughout this section, we use the notation introduced in Subsection 3.2 and Subsection 7.2. See Figure 5.

We start to study the local deformations of the D_4-generalized Burniat surfaces. Let X be the canonical model of a D_4-generalized Burniat surface S. We intend to calculate the dimension of the tangent space to the base of the Kuranishi family of X, i.e., $\dim Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$. For this we first calculate $h^i(\widetilde{S}, \Theta_{\widetilde{S}})$ (cf. Theorem 7.5), using the bidouble cover structure as described in Theorem 10.1. Then we pass from $\widetilde{S}$ to the minimal model S, calculate $h^i(S,\Theta_S)$ by Theorem 10.4. Finally, we pass from S to the canonical model X by Theorem 10.5, and use the spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) \Rightarrow Ext^{p+q}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X).$$

By Serre Duality and Theorem 10.1,

$$H^k(\widetilde{S}, \Theta_{\widetilde{S}})^{\mathrm{inv}} = H^{2-k}(\widetilde{Y}, \Omega_{\widetilde{Y}}(\log\Delta_1, \log\Delta_2, \log\Delta_3) \otimes \Omega^2_{\widetilde{Y}}), \tag{11.1}$$

$$H^k(\widetilde{S}, \Theta_{\widetilde{S}})^{\chi_i} = H^{2-k}(\widetilde{Y}, \Omega_{\widetilde{Y}}(\log\Delta_i)(K_{\widetilde{Y}} + \mathcal{L}_i)), \tag{11.2}$$

for $k = 0,1,2$ and $i = 1,2,3$.

Since $\widetilde{S}$ is a surface of general type, $H^0(\widetilde{S}, \Theta_{\widetilde{S}}) = 0$. Therefore the right-hand sides of the equations equal 0 when $k = 0$.

Proposition 11.1. *$h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log\Delta_1, \log\Delta_2, \log\Delta_3) \otimes \Omega^2_{\widetilde{Y}}) = 0$ and $h^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log\Delta_1, \log\Delta_2, \log\Delta_3) \otimes \Omega^2_{\widetilde{Y}}) = 4$.*

Proof. By Lemma 10.3 (1), we have an exact sequence

$$0 \to \Omega^1_{\widetilde{Y}}(K_{\widetilde{Y}}) \to \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_{\widetilde{Y}}) \to \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_{\widetilde{Y}}) \to 0 \tag{11.3}$$

Note that $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}) = 0$ and $-K_{\widetilde{Y}}$ is effective, thus $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(K_{\widetilde{Y}})) = 0$. To prove the first equality, it suffices to show the boundary map

$$\delta \colon H^0(\widetilde{Y}, \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_{\widetilde{Y}})) \to H^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(K_{\widetilde{Y}}))$$

is injective.

Since Δ_i is a disjoint union of three smooth rational curves $\Gamma_i, N_{i+1}, C_{i+2}$,

$$H^0(\widetilde{Y}, \mathcal{O}_{\Delta_i}(K_{\widetilde{Y}})) \cong H^0(\widetilde{Y}, \mathcal{O}_{N_{i+1}}) \cong \mathbb{C}.$$

$|-K_{\widetilde{Y}}|$ is base-point-free, therefore there is a morphism $\mathcal{O}_{\widetilde{Y}}(K_{\widetilde{Y}}) \to \mathcal{O}_{\widetilde{Y}}$, which is not identically zero on any component of Δ_i's, in particular on N_i's. Now consider the commutative diagram coming from the above morphism $\mathcal{O}_{\widetilde{Y}}(K_{\widetilde{Y}}) \to \mathcal{O}_{\widetilde{Y}}$,

$$\begin{array}{ccccccccc} 0 & \to & \Omega^1_{\widetilde{Y}}(K_{\widetilde{Y}}) & \to & \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_{\widetilde{Y}}) & \to & \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_{\widetilde{Y}}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Omega^1_{\widetilde{Y}} & \to & \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) & \to & \oplus_{i=1}^3 \mathcal{O}_{\Delta_i} & \to & 0. \end{array}$$

It gives a commutative diagram of cohomology groups,

$$\begin{array}{ccc} \mathbb{C}^3 \cong H^0(\widetilde{Y}, \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_{\widetilde{Y}})) & \xrightarrow{\delta} & H^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(K_{\widetilde{Y}})) \\ \downarrow{\scriptstyle \psi_2} & \searrow^{\psi} & \downarrow \\ H^0(\widetilde{Y}, \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}) & \xrightarrow{\psi_1} & H^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}). \end{array}$$

By Lemma 10.3 (2), the image of the function identically equal to 1 on N_i maps under ψ_1 to the first Chern class of N_i. Because N_i's are disjoint (-2)-curves, their Chern classes are linearly independent in $H^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}})$. Thus the composite map ψ is injective.

Hence δ is also injective and $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}}) = 0$.

Since $H^2(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}}) = 0$, to calculate the dimension of $H^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}})$ is the same as to calculate $\chi(\Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}})$. By the exact sequence (11.3),

$$\chi(\Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}}) = \chi(\Omega^1_{\widetilde{Y}}(K_{\widetilde{Y}})) + \sum_{i=1}^{3} \chi(\mathcal{O}_{\Delta_i}(K_{\widetilde{Y}})).$$

Serre's Duality and Riemann-Roch theorem show that,

$$\chi(\Omega^1_{\widetilde{Y}}(K_{\widetilde{Y}})) = \chi(\Theta_{\widetilde{Y}}) = \frac{1}{2}c_1(\widetilde{Y})(c_1(\widetilde{Y}) - K_{\widetilde{Y}}) - c_2(\widetilde{Y}) + 2\chi(\mathcal{O}_{\widetilde{Y}}) = -4.$$

Note that Δ_i is a disjoint union of three smooth rational curves $\Gamma_i, N_{i+1}, C_{i+2}$. It follows that $\chi(\mathcal{O}_{\Delta_i}(K_{\widetilde{Y}})) = 0$ for $i = 1, 2, 3$.

Hence $\chi(\Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}}) = -4$ and it follows that $h^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}}) = 4$. □

In order to calculate $h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_i)(K_{\widetilde{Y}} + \mathcal{L}_i))$ for $i = 1, 2, 3$, we need the following lemmas.

Lemma 11.2. *Let $p_1 \colon W \to \mathbb{C}^2$ be the blowup of $\mathbb{C}^2$ at $(0,0)$, and let $p_2 \colon \Sigma \to W$ be the blowup of W at the intersection point O' of the strict transform of the line $l : y = 0$ with the exceptional curve E of p_1.*

Denoted by E' the exceptional curve of p_2 and by Γ the strict transform of the line l under the morphism $p = p_2 \circ p_1 \colon \Sigma \to W \to \mathbb{C}^2$. Then

(1) $p_*\Omega^1_\Sigma(-E') \subseteq \Omega^1_{\mathbb{C}^2}$ is the subsheaf of forms

$$\{\omega \in \Omega^1_{\mathbb{C}^2} | \omega = \alpha(x,y)dx + \beta(x,y)dy,\ \alpha(0,0) = 0\}.$$

(2) $p_*\Omega^1_\Sigma(-2E') \subseteq \Omega^1_{\mathbb{C}^2}$ is the subsheaf of forms

$$\{\omega \in \Omega^1_{\mathbb{C}^2} | \omega = \alpha(x,y)dx + \beta(x,y)dy,\ \alpha(0,0) = 0,\ \frac{\partial \alpha}{\partial x}(0,0) = 0,\ \beta(0,0) = 0\}.$$

(3) $p_*\Omega^1_\Sigma(\log \Gamma)(-E') \subseteq \Omega^1_{\mathbb{C}^2}(\log l)$ is the subsheaf of forms

$$\{\omega \in \Omega^1_{\mathbb{C}^2}(\log l) | \omega = \alpha(x,y)dx + \beta(x,y)\frac{dy}{y},\ \beta(0,0) = 0,\ \alpha(0,0) + 2\frac{\partial \beta}{\partial x}(0,0) = 0\}.$$

Proof. W can be covered by two affine coordinate charts $V_1 \cong \mathbb{C}^2(x,t)$ and $V_2 \cong \mathbb{C}^2(s,y)$, such that p_1 is given by

$$V_1 \to \mathbb{C}^2,\ (x,t) \mapsto (x, tx),$$
$$V_2 \to \mathbb{C}^2,\ (s,y) \mapsto (sy, y).$$

$p_2^{-1}(V_1)$ can be covered by two affine coordinate charts $U_{11} \cong \mathbb{C}^2(x,u)$ and $U_{12} \cong \mathbb{C}^2(v,t)$ such that the morphism $p\colon \Sigma \to \mathbb{C}^2$ is given by

$$U_{11} \to V_1 \to \mathbb{C}^2,\ (x,u) \mapsto (x,ux) \mapsto (x,x^2u),$$
$$U_{12} \to V_1 \to \mathbb{C}^2,\ (v,t) \mapsto (vt,t) \mapsto (vt,vt^2).$$

And similarly for $p_2^{-1}(V_2) = U_{21} \cup U_{22}$. Note that both E' and Γ are contained in $U_{11} \cup U_{12}$.

First use the coordinate chart U_{11}. Locally E' is defined by $x = 0$ and Γ is defined by $u = 0$.

(1) By Riemann's extension theorem, $p_*\Omega^1_\Sigma(-mE') \subseteq \Omega^1_{\mathbb{C}^2}$ for all $m \geq 0$. Assume that $\omega = \alpha(x,y)dx + \beta(x,y)dy$ for some holomorphic function $\alpha(x,y)$ and $\beta(x,y)$. Then

$$\begin{aligned} p^*\omega &= \alpha(x,x^2u)dx + \beta(x,x^2u)(x^2du + 2xudx) \\ &= (\alpha(x,x^2u) + 2xu\beta(x,x^2u))dx + \beta(x,x^2u)x^2du, \end{aligned}$$

Hence locally $p^*\omega$ belongs to the $\mathcal{O}_\Sigma$-module generated by xdx, xdu if and only if $\alpha(0,0) = 0$.

(2) By the calculation above, locally $p^*\omega$ belongs to the $\mathcal{O}_\Sigma$-module generated by x^2dx, x^2du if and only if $\alpha(x,x^2u) + 2xu\beta(x,x^2u)$ is divisible by x^2. Assume that

$$\alpha(x,y) = a + bx + cy + \textit{higher degree terms}, \tag{11.4}$$
$$\beta(x,y) = A + Bx + Cy + \textit{higher degree terms}, \tag{11.5}$$

$$\begin{aligned} a &= \alpha(0,0), & b &= \frac{\partial\alpha}{\partial x}(0,0), & c &= \frac{\partial\alpha}{\partial y}(0,0), \\ A &= \beta(0,0), & B &= \frac{\partial\beta}{\partial x}(0,0), & C &= \frac{\partial\beta}{\partial y}(0,0), \end{aligned}$$

then

$$\alpha(x,x^2u) + 2xu\beta(x,x^2u) = a + bx + 2Axu + x^2h(x,u),$$

for some holomorphic function $h(x,u)$. Thus $p^*\omega$ belongs to the $\mathcal{O}_\Sigma$-module generated by x^2dx, x^2du, if and only if $a = b = A = 0$.

(3) Observe that $p_*\Omega^1_\Sigma(\log \Gamma)(-E')$ consists of rational differential 1-forms ω which, when restricted to $\mathbb{C}^2 \setminus \{(0,0)\}$, yield sections of $\Omega^1_{\mathbb{C}^2}(\log l)$. In particular, $y\omega$ is a regular 1-form on $\mathbb{C}^2\setminus\{(0,0)\}$, which can be extended to a regular 1-form on $\mathbb{C}^2$. Assume that $\omega = \alpha_1(x,y)\frac{dx}{y} + \beta(x,y)\frac{dy}{y}$ for some holomorphic function $\alpha_1(x,y)$ and $\beta(x,y)$, then

$$\begin{aligned} p^*\omega &= \frac{\alpha_1(x,x^2u)}{x^2u}dx + 2\beta(x,x^2u)\frac{dx}{x} + \beta(x,x^2u)\frac{du}{u} \\ &= (\frac{\alpha_1(x,x^2u)}{x^3u} + \frac{2\beta(x,x^2u)}{x^2})xdx + \frac{\beta(x,x^2u)}{x}x\frac{du}{u}. \end{aligned}$$

Thus $p^*\omega$ belongs to the $\mathcal{O}_\Sigma$-module generated by $xdx, x\frac{du}{u}$ if and only if $\alpha_1(x,x^2u)+2xu\beta(x,x^2u)$ is divisible by x^3u and $\beta(x,x^2u)$ is divisible by x.

If $\alpha_1(x,x^2u) + 2xu\beta(x,x^2u)$ is divisible by x^3u, then $\alpha_1(x,x^2u)$ is divisible by u. This implies $\alpha_1(x,y) = y\alpha(x,y)$ for some holomorphic function $\alpha(x,y)$. Then

$$\begin{aligned} \omega &= \alpha(x,y)dx + \beta(x,y)\frac{dy}{y}, \\ p^*\omega &= \alpha(x,x^2u)dx + 2\beta(x,x^2u)\frac{dx}{x} + \beta(x,x^2u)\frac{du}{u}. \end{aligned}$$

If we write $\alpha(x,y), \beta(x,y)$ as (11.4) and (11.5), then one sees that $p^*\omega$ belongs to the $\mathcal{O}_\Sigma$-module generated by xdx, $x\frac{du}{u}$ if and only if $A = 0,\ a + 2B = 0$.

Hence we see that (1),(2),(3) hold locally. Similar calculation with other coordinate charts show the same results. □

Lemma 11.3. *Let l denote the line on the projective plane $\mathbb{P}^2$ defined by $x_1 = 0$. Then any $\omega \in H^0(\Omega^1_{\mathbb{P}^2}(\log l)(2))$ is of the form*

$$\begin{aligned} \omega = &(-Ax_1x_2 - Cx_1x_3 + Dx_2^2 + Ex_3^2 + Fx_2x_3)\frac{dx_1}{x_1} \\ &+ (Ax_1 - Dx_2 - Bx_3)dx_2 + (Cx_1 + Bx_2 - Fx_2 - Ex_3)dx_3, \end{aligned} \tag{11.6}$$

where $A, B, C, D, E, F \in \mathbb{C}$.

Proof. By [BC10-b, Lemma 5.2 (1)], the vector space $H^0(\Omega^1_{\mathbb{P}^2}(2))$ is 3-dimensional with a basis: $-x_2dx_1+x_1dx_2$, $-x_3dx_2+x_2dx_3$, $-x_3dx_1+x_1dx_3$.

By the exact sequence $0 \to \Omega^1_{\mathbb{P}^2}(2) \to \Omega^1_{\mathbb{P}^2}(\log l)(2) \to \mathcal{O}_l(2) \to 0$ and since $h^1(\Omega^1_{\mathbb{P}^2}(2)) = 0$ and $h^0(\mathcal{O}_l(2)) = 3$, we see that $h^0(\Omega^1_{\mathbb{P}^2}(\log l)(2)) = 6$. Moreover, it is easy to show that the following forms

$$x_2^2\frac{dx_1}{x_1} - x_2dx_2,\ x_3^2\frac{dx_1}{x_1} - x_3dx_3,\ x_2x_3\frac{dx_1}{x_1} - x_2dx_3$$

in the vector space $H^0(\Omega^1_{\mathbb{P}^2}(\log l)(2))$, are mapped to a basis of $H^0(\mathcal{O}_l(2))$. Hence these forms and the above basis of $H^0(\Omega^1_{\mathbb{P}^2}(2))$ are linearly independent in $H^0(\Omega^1_{\mathbb{P}^2}(\log l)(2))$. Then their linear combination

$$\begin{aligned} &A(-x_2dx_1 + x_1dx_2) + B(-x_3dx_2 + x_2dx_3) + C(-x_3dx_1 + x_1dx_3) \\ &+D(x_2^2\frac{dx_1}{x_1} - x_2dx_2) + E(x_3^2\frac{dx_1}{x_1} - x_3dx_3) + F(x_2x_3\frac{dx_1}{x_1} - x_2dx_3) \end{aligned}$$

is of the form (11.6). □

Proposition 11.4. $h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_i)(K_{\widetilde{Y}} + \mathcal{L}_i)) = 0$ *and* $h^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_i)(K_{\widetilde{Y}} + \mathcal{L}_i)) = 4$, *for* $i = 1, 2, 3$.

Proof. To prove the first equality for $i = 3$, note that by (7.2),

$$H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3)) = H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log N_1,\ \log C_2,\ \log \Gamma_3)(E_3 - E_2')).$$

Apply Lemma 10.2 to the curve C_2 and then to N_1,

$$H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3)) = H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Gamma_3)(2L - 2E_1' - E_2' - E_3')).$$

Without loss of generality, we may assume that

$$\mathbf{P_1 = (1:0:0), P_2 = (1:1:0), Q_1 = (0:1:1), Q_2 = (0:0:1).}$$

It follows that $\mathbf{P_3 = (0:1:0)}$ and $\mathbf{Q_3 = (1:0:-1)}$. See Figure 7.

Note that $\sigma_*(\Omega^1_{\widetilde{Y}}(\log \Gamma_3)(2L - 2E_1' - E_2' - E_3'))$ is a subsheaf of $\Omega^1_{\mathbb{P}^2}(\log l)(2)$, thus we can apply Lemma 11.3.

Any $\omega \in H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3))$, considered as an element of $H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log l)(2))$, is of the form (11.6).

Locally around the point $\mathbf{P_1} = (\mathbf{1}:\mathbf{0}:\mathbf{0})$, $x_1 = 1$ and the line P_1P_1' is defined by $x_2 = 0$. So locally we may write

$$\omega = \alpha(x_2, x_3)dx_3 + \beta(x_2, x_3)dx_2,$$
$$\alpha(x_2, x_3) = C + Bx_2 - Fx_2 - Ex_3,\ \beta(x_2, x_3) = A - Dx_2 - Bx_3.$$

Thus by Lemma 11.2 (2),

$$\alpha(0,0) = C = 0,\ \frac{\partial \alpha}{\partial x_3}(0,0) = -E = 0,\ \beta(0,0) = A = 0,$$

and then

$$\omega = (Dx_2^2 + Fx_2x_3)\frac{dx_1}{x_1} + (-Dx_2 - Bx_3)dx_2 + (B - F)x_2dx_3.$$

Locally around the point $\mathbf{P_3} = (\mathbf{0}:\mathbf{1}:\mathbf{0})$, $x_2 = 1$ and the line P_3P_3' is defined by $x_1 = 0$. So locally we may write

$$\omega = (D + Fx_3)\frac{dx_1}{x_1} + (B - F)dx_3.$$

Then by Lemma 11.2 (3), $D = 0,\ B + F = 0$, and then

$$\omega = F(x_2x_3\frac{dx_1}{x_1} + x_3dx_2 - 2x_2dx_3).$$

Locally around the point $\mathbf{P_2} = (\mathbf{1}:\mathbf{1}:\mathbf{0})$, $x_1 = 1$. P_2 is the intersection point of the line $x_3 = 0$, and the line $P_2P_2' : 1 - x_2 + x_3 = 0$. Let $x := x_3,\ y := 1 - x_2 + x_3$. Then locally

$$\omega = F(-2 - x + 2y)dx + F(-x)dy.$$

Thus by Lemma 11.2 (1), $F = 0,\ \omega = 0$.

Hence $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3)) = 0$.

Note that $H^2(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3)) = 0$, so to calculate the dimension of $H^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3))$ is equivalent to calculate $\chi(\Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3))$. Twist the following exact sequence with the invertible sheaf associated to the divisor $\mathcal{F} := K_{\widetilde{Y}} + \mathcal{L}_3$,

$$0 \to \Omega^1_{\widetilde{Y}} \to \Omega^1_{\widetilde{Y}}(\log \Delta_3) \to \mathcal{O}_{\Delta_3} \to 0,$$

we get

$$\chi(\Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3)) = \chi(\Omega^1_{\widetilde{Y}}(\mathcal{F})) + \chi(\mathcal{O}_{\Delta_3}(\mathcal{F})).$$

For the second summand, since Δ_3 is the disjoint union of rational curves N_1, C_2, Γ_3, and $\mathcal{F}.N_1 = 0,\ \mathcal{F}.C_2 = 1,\ \mathcal{F}.\Gamma_3 = 1$, we have

$$\chi(\mathcal{O}_{\Delta_3}(\mathcal{F})) = \chi(\mathcal{O}_{N_1}) + \chi(\mathcal{O}_{C_2}(1)) + \chi(\mathcal{O}_{\Gamma_3}(1)) = 5.$$

For the first summand, using the splitting principle, formally write

$$\Omega^1_{\widetilde{Y}} = \mathcal{O}_{\widetilde{Y}}(A_1) \oplus \mathcal{O}_{\widetilde{Y}}(A_2), \text{ and } A_1 + A_2 = K_{\widetilde{Y}},\ A_1.A_2 = c_2(Y) = 9.$$

Note that $\mathcal{F}^2 = -2$ and $\mathcal{F}.K_{\widetilde{Y}} = 0$, Riemann-Roch Theorem gives

$$\begin{aligned} \chi(\Omega^1_{\widetilde{Y}}(\mathcal{F})) &= \chi(\mathcal{O}_{\widetilde{Y}}(A_1 + \mathcal{F})) + \chi(\mathcal{O}_{\widetilde{Y}}(A_2 + \mathcal{F})) \\ &= \sum_{i=1}^{2} \frac{1}{2}(A_i + \mathcal{F})(A_i + \mathcal{F} - K_{\widetilde{Y}}) + 2\chi(\mathcal{O}_{\widetilde{Y}}) \\ &= -9. \end{aligned}$$

Hence $\chi(\Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3)) = -4$ and $h^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3)) = 4$.

Similarly, the statement also holds for $i = 1, 2$. ☐

Theorem 11.5. *Let $\pi\colon \widetilde{S} \to \widetilde{Y}$ be the bidouble cover as in Subsection 7.2. Let S be the minimal model of $\widetilde{S}$ and X the canonical model of $\widetilde{S}$ (cf. Subsection 8.1). The respective dimensions of the cohomology groups of the tangent sheaves $\Theta_{\widetilde{S}}, \Theta_S, \Theta_X$ are as follows.*

$$\begin{array}{lll} h^1(\widetilde{S}, \Theta_{\widetilde{S}}) = 16, & h^1(S, \Theta_S) = 4, & h^1(X, \Theta_X) = 3, \\ h^2(\widetilde{S}, \Theta_{\widetilde{S}}) = 0, & h^2(S, \Theta_S) = 0, & h^2(X, \Theta_X) = 0. \end{array}$$

Proof. By (11.1), (11.2), Proposition 11.1 and Proposition 11.4, $h^1(\widetilde{S}, \Theta_{\widetilde{S}}) = 16$ and $h^2(\widetilde{S}, \Theta_{\widetilde{S}}) = 0$.

Since S is obtained by blowing down six (-1)-curves (cf. Corollary 7.6) on $\widetilde{S}$, then by Theorem 10.4, $h^1(S, \Theta_S) = 4$ and $h^2(S, \Theta_S) = 0$.

X is obtained by contracting the (-2)-curve Z' on S (cf. Corollary 7.7), then by Theorem 10.5, $h^1(X, \Theta_X) = 3$ and $h^2(X, \Theta_X) = 0$. ☐

Corollary 11.6. *The base of the Kuranishi family of S is smooth.*

Corollary 11.7. $\dim Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = 4$ *and* $Ext^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = 0$.

Proof. Since X has a node as singular locus, the sheaf $\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ is supported on the singularity and has length 1. Then the exact sequence

$$\begin{aligned} 0 \to H^1(X, \Theta_X) &\to Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) &&\to H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) \\ \to H^2(X, \Theta_X) &\to Ext^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) &&\to 0, \end{aligned}$$

associated to the spectral sequence,

$$E_2^{pq} = H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) \Rightarrow Ext^{p+q}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$$

and Theorem 11.5 give the conclusion. □

Theorem 11.8. *Let $\mathcal{NEB}_3$ be the open subset of the moduli space of canonical surfaces of general type $\mathcal{M}^{\mathrm{can}}_{1,3}$ corresponding to the extended Burniat surfaces and nodal Burniat surfaces with $K^2 = 3$, and let $\overline{\mathcal{NEB}_3}$ be its closure. Let X be a canonical model of a D_4-generalized Burniat surface.*

Then $\overline{\mathcal{NEB}_3}$ is the only irreducible component in $\mathcal{M}^{\mathrm{can}}_{1,3}$ containing $[X]$. Moreover, the base of the Kuranishi family of deformations of X is smooth.

Proof. Recall that locally the germ of the complex space $(\mathcal{M}^{\mathrm{can}}_{1,3}, [X])$ is analytically isomorphic to the quotient of the base of the Kuranishi family by the finite group $Aut(X)$ and $Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ is the tangent space to the base of the Kuranishi family of X. We have the following inequalities,

$$\begin{aligned} \dim Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) &\geq \text{the dimension of the base of the Kuranishi family of } X \\ &= \text{the dimension of } \mathcal{M}^{\mathrm{can}}_{1,3} \text{ at the point } [X]. \end{aligned}$$

There is a subvariety $\overline{\mathcal{NEB}_3}$ of dimension 4 of $\mathcal{M}^{\mathrm{can}}_{1,3}$ passing the point $[X]$. Since $Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ also has dimension 4, in the above inequality, the equality holds.

Thus the base of the Kuranishi family of deformations of X is smooth, and $\overline{\mathcal{NEB}_3}$ is the only irreducible component in $\mathcal{M}^{\mathrm{can}}_{1,3}$ containing $[X]$. □

12 Deformations of the $4A_1$-generalized Burniat Surfaces

Throughout this section, we use the notation introduced in Subsection 3.3 and Subsection 8.2. See also Figure 6.

We start to study the local deformations of the $4A_1$-generalized Burniat surfaces by using the same method as in the case of the D_4-generalized Burniat surfaces.

By Serre Duality and Theorem 10.1,

$$H^k(\widetilde{S}, \Theta_{\widetilde{S}})^{\text{inv}} = H^{2-k}(\widetilde{Y}, \Omega_{\widetilde{Y}}(\log \Delta_1,\ \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}}), \tag{12.1}$$

$$H^k(\widetilde{S}, \Theta_{\widetilde{S}})^{\chi_i} = H^{2-k}(\widetilde{Y}, \Omega_{\widetilde{Y}}(\log \Delta_i)(K_{\widetilde{Y}} + \mathcal{L}_i)), \tag{12.2}$$

for $k = 0, 1, 2$ and $i = 1, 2, 3$.

Since $\widetilde{S}$ is a surface of general type, $H^0(\widetilde{S}, \Theta_{\widetilde{S}}) = 0$. Therefore the right-hand sides of the equations equal to 0 when $k = 0$.

Proposition 12.1. *$h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}}) = 0$ and $h^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}}) = 4$.*

Proof. Exactly the same proof of Proposition 11.1, verbatim. □

The complicate part is to calculate $h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_i)(K_{\widetilde{Y}} + \mathcal{L}_i))$. We need the following lemmas.

Lemma 12.2 ([BC10-b, Lemma 4.1])**.** *Let $p\colon \Sigma \to \mathbb{C}^2$ be the blowup of $\mathbb{C}^2$ at $(0,0)$. Denote by E the exceptional curve of p, and by D_1 the strict transform of the line $l_1 : y = 0$, and by D_2 the strict transform of the line $l_2 : x = 0$.*

(1) $p_*\Omega^1_\Sigma(-E) = m_o\Omega^1_{\mathbb{C}^2}$, where m_o is the ideal of $(0,0)$.

(2) $p_*\Omega^1_\Sigma(\log D_1, \log D_2) \subseteq \Omega^1_{\mathbb{C}^2}(\log l_1, \log l_2)$ is the subsheaf of forms

$$\{\omega \in \Omega^1_{\mathbb{C}^2}(\log l_1, \log l_2) | \omega = \alpha(x,y)\frac{dx}{x} + \beta(x,y)\frac{dy}{y},\ \alpha(0,0) + \beta(0,0) = 0\}.$$

(3) $p_*\Omega^1_\Sigma(\log D_1)(-E) \subseteq \Omega^1_{\mathbb{C}^2}(\log l_1)$ is the subsheaf of forms

$$\{\omega \in \Omega^1_{\mathbb{C}^2}(\log l_1) | \omega = \alpha(x,y)dx + \beta(x,y)\frac{dy}{y},\ \beta(0,0) = 0,$$
$$\frac{\partial \beta}{\partial y}(0,0) = 0,\ \alpha(0,0) + \frac{\partial \beta}{\partial x}(0,0) = 0\}.$$

(4) $p_*\Omega^1_\Sigma(\log D_1, \log D_2)(-E) \subseteq \Omega^1_{\mathbb{C}^2}(\log l_1, \log l_2)$ *is the subsheaf of forms*

$$\{\omega \in \Omega^1_{\mathbb{C}^2}(\log l_1, \log l_2) | \omega = \alpha(x,y)\frac{dx}{x} + \beta(x,y)\frac{dy}{y},\ \alpha(0,0) = 0, \\ \beta(0,0) = 0,\ \frac{\partial(\alpha+\beta)}{\partial x}(0,0) = 0,\ \frac{\partial(\alpha+\beta)}{\partial y}(0,0) = 0\}.$$

Proof. See [BC10-b, Lemma 4.1] for the proof and for a more general result. One can also prove it by a similar calculation to the one given in the proof of Lemma 11.2. □

Proposition 12.3. $h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_i)(K_{\widetilde{Y}} + \mathcal{L}_i)) = 1$ *and* $h^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_i)(K_{\widetilde{Y}} + \mathcal{L}_i)) = 5$, *for* $i = 1, 2, 3$.

Proof. To prove the first equality for $i = 3$, note that by (8.2),

$$\Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3) = \Omega^1_{\widetilde{Y}}(\log N_1, \log C_2, \log \Gamma_3)(N_3 + E_3 - E_2).$$

Apply Lemma 10.2 to N_3 and again to C_2,

$$H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3)) = \\ H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log N_1, \log N_3, \log \Gamma_3)(2L - E_1 - E_2 - E_1' - E_3')).$$

Without loss of generality, assume that

$$\mathbf{P_2 = (1:1:1),\ P_3 = (0:0:1),\ P_2' = (0:1:0),\ P_3' = (1:0:0).}$$

It follows that $\mathbf{P_1 = (1:1:0)}$ and $\mathbf{P_1' = (0:1:1)}$. Note that $\mathbf{N_3}$, $\mathbf{\Gamma_3}$, $\mathbf{N_1}$ are the strict transforms of $\mathbf{l_1 : x_1 = 0}$, $\mathbf{l_2 : x_2 = 0}$, $\mathbf{l_3 : x_3 = 0}$ respectively. See Figure 8.

One can view $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log N_1, \log N_3, \log \Gamma_3)(2L - E_1 - E_2 - E_1' - E_3'))$ as a subspace of $H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log l_1, \log l_2, \log l_3)(2))$.

By [BC10-b, Corollary 5.4], any $\omega \in H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log l_1, \log l_2, \log l_3)(2))$ can be written as:

$$\begin{aligned}
\omega &= \frac{dx_1}{x_1}(-a_{12}x_1x_2 - a_{13}x_3x_1 + a_{212}x_2^2 + a_{313}x_3^2 \\
&\quad - a_{121}x_1^2 - a_{131}x_1^2 - a_{231}x_2x_1 + a_{312}x_3x_2) \\
&+ \frac{dx_2}{x_2}(a_{12}x_1x_2 - a_{23}x_3x_2 - a_{212}x_2^2 + a_{121}x_1^2 \\
&\quad + a_{323}x_3^2 + a_{123}x_1x_3 - a_{232}x_2^2 - a_{312}x_3x_2) \\
&+ \frac{dx_3}{x_3}(a_{13}x_1x_3 + a_{23}x_2x_3 - a_{313}x_3^2 - a_{323}x_3^2 \\
&\quad + a_{131}x_1^2 + a_{232}x_2^2 - a_{123}x_1x_3 + a_{231}x_1x_2),
\end{aligned} \tag{12.3}$$

where $a_{12}, \ldots, a_{312}$ are complex numbers. Now we intend to use Lemma 12.2 to impose conditions on ω as in Proposition 11.4.

Around the point $\mathbf{P_3} = (\mathbf{0} : \mathbf{0} : \mathbf{1})$, $x_3 = 1$ and P_3 is the intersection point of $l_1 : x_1 = 0$ and $l_2 : x_2 = 0$. By Lemma 12.2 (2),

$$a_{313} + a_{323} = 0. \tag{12.4}$$

Around the point $\mathbf{P'_2} = (\mathbf{0} : \mathbf{1} : \mathbf{0}), x_2 = 1$ and P'_2 is the intersection point of $l_1 : x_1 = 0$ and $l_3 : x_3 = 0$. By Lemma 12.2 (2),

$$a_{212} + a_{232} = 0. \tag{12.5}$$

Around the point $\mathbf{P'_3} = (\mathbf{1} : \mathbf{0} : \mathbf{0}), x_1 = 1$ and P'_3 is the intersection point of $l_2 : x_2 = 0$ and $l_3 : x_3 = 0$. Use (12.4) and (12.5), locally we may write

$$\begin{aligned}
\omega &= \alpha(x_2, x_3)\frac{dx_2}{x_2} + \beta(x_2, x_3)\frac{dx_3}{x_3}, \\
\alpha(x_2, x_3) &= a_{12}x_2 - a_{23}x_3x_2 + a_{121} + a_{323}x_3^2 + a_{123}x_3 - a_{312}x_3x_2, \\
\beta(x_2, x_3) &= a_{13}x_3 + a_{23}x_2x_3 + a_{131} + a_{232}x_2^2 - a_{123}x_3 + a_{231}x_2.
\end{aligned}$$

By Lemma 12.2 (4),

$$\alpha(0,0) = a_{121} = 0, \tag{12.6}$$

$$\beta(0,0) = a_{131} = 0, \tag{12.7}$$

$$\frac{\partial(\alpha+\beta)}{\partial x_2}(0,0) = a_{12} + a_{231} = 0, \tag{12.8}$$

$$\frac{\partial(\alpha+\beta)}{\partial x_3}(0,0) = a_{13} = 0. \tag{12.9}$$

Up to this point, use (12.4), (12.5),(12.6),(12.7) (12.8) and (12.9) one can reduce (12.3) to

$$\begin{aligned}\omega =& \frac{dx_1}{x_1}(a_{212}x_2^2 + a_{313}x_3^2 + a_{312}x_3x_2)\\ &+ \frac{dx_2}{x_2}(a_{12}x_1x_2 - a_{23}x_3x_2 - a_{313}x_3^2 + a_{123}x_1x_3 - a_{312}x_3x_2)\\ &+ \frac{dx_3}{x_3}(a_{23}x_2x_3 - a_{212}x_2^2 - a_{123}x_1x_3 - a_{12}x_1x_2). \end{aligned} \tag{12.10}$$

Around the point $\mathbf{P_1'} = (\mathbf{0} : \mathbf{1} : \mathbf{1}) \in l_1 : x_1 = 0, x_3 = 1$, by (12.10) locally

$$\begin{aligned}\omega &= \alpha(x_1, x_2)dx_2 + \beta(x_1, x_2)\frac{dx_1}{x_1},\\ \alpha(x_1, x_2) &= \frac{a_{12}x_1x_2 - a_{23}x_2 - a_{313} + a_{123}x_1 - a_{312}x_2}{x_2},\\ \beta(x_1, x_2) &= a_{212}x_2^2 + a_{313} + a_{312}x_2.\end{aligned}$$

By Lemma 12.2 (3),

$$\beta(0, 1) = a_{212} + a_{313} + a_{312} = 0, \tag{12.11}$$

$$\frac{\partial \beta}{\partial x_1}(0, 1) = 0,$$

$$\alpha(0, 1) + \frac{\partial \beta}{\partial x_2}(0, 1) = -a_{23} - a_{313} + 2a_{212} = 0. \tag{12.12}$$

Around the point $\mathbf{P_1} = (\mathbf{1} : \mathbf{1} : \mathbf{0}) \in l_3 : x_3 = 0, x_1 = 1$, by (12.10) locally

$$\begin{aligned}\omega &= \alpha(x_2, x_3)dx_2 + \beta(x_2, x_3)\frac{dx_3}{x_3},\\ \alpha(x_2, x_3) &= \frac{a_{12}x_2 - a_{23}x_3x_2 - a_{313}x_3^2 + a_{123}x_3 - a_{312}x_3x_2}{x_2},\\ \beta(x_2, x_3) &= a_{23}x_2x_3 - a_{212}x_2^2 - a_{123}x_3 - a_{12}x_2.\end{aligned}$$

By Lemma 12.2 (3),

$$\beta(1, 0) = -a_{212} - a_{12} = 0, \tag{12.13}$$

$$\frac{\partial \beta}{\partial x_3}(1, 0) = a_{23} - a_{123} = 0, \tag{12.14}$$

$$\alpha(1, 0) + \frac{\partial \beta}{\partial x_2}(1, 0) = -2a_{212} = 0. \tag{12.15}$$

By (12.11), (12.12), (12.13), (12.14) and (12.15), we have

$$a_{12} = a_{212} = 0, a_{23} = a_{123} = -a_{313} = a_{312}.$$

Thus by (12.10) ω is some multiple of the following form

$$\omega_0 = \frac{dx_1}{x_1}(-x_3^2 + x_3x_2) + \frac{dx_2}{x_2}(-2x_3x_2 + x_3^2 + x_1x_3) + \frac{dx_3}{x_3}(x_2x_3 - x_1x_3).$$

Note that by Lemma 12.2 (1), the point $P_2 = (1:1:1)$ imposes no new condition on such a form. Conversely, by the above discussion and Lemma 12.2, ω_0 gives a non-zero element of $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3))$.

Hence $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3))$ has dimension 1.

Note that $H^2(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3)) = 0$, so calculating the dimension of $H^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3))$ is equivalent to calculating $\chi(\Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3))$. The same calculation (actually verbatim) in the proof of Proposition 11.4 shows that $\chi(\Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3)) = -4$. Thus $H^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3))$ has dimension 5.

Similarly, the statement also holds for $i = 1, 2$. ∎

Theorem 12.4. *Let $\pi\colon \widetilde{S} \to \widetilde{Y}$ be the bidouble cover as in Subsection 8.2. Let S be the minimal model of $\widetilde{S}$ and X the canonical model of $\widetilde{S}$ (cf. Section 8). The respective dimensions of the eigenspaces of the cohomology groups of the tangent sheaves $\Theta_{\widetilde{S}}, \Theta_S, \Theta_X$ according to the group action are as follows.*

$$\begin{aligned}
h^1(\widetilde{S}, \Theta_{\widetilde{S}})^{\mathrm{inv}} &= 4, & h^1(\widetilde{S}, \Theta_{\widetilde{S}})^{\chi_i} &= 5, & h^2(\widetilde{S}, \Theta_{\widetilde{S}})^{\mathrm{inv}} &= 0, & h^2(\widetilde{S}, \Theta_{\widetilde{S}})^{\chi_i} &= 1;\\
h^1(S, \Theta_S)^{\mathrm{inv}} &= 4, & h^1(S, \Theta_S)^{\chi_i} &= 1, & h^2(S, \Theta_S)^{\mathrm{inv}} &= 0, & h^2(S, \Theta_S)^{\chi_i} &= 1;\\
h^1(X, \Theta_X)^{\mathrm{inv}} &= 3, & h^1(X, \Theta_X)^{\chi_i} &= 0, & h^2(X, \Theta_X)^{\mathrm{inv}} &= 0, & h^2(X, \Theta_X)^{\chi_i} &= 1,
\end{aligned}$$

for $i = 1, 2, 3$.

Proof. The conclusion about $\widetilde{S}$ follows by (12.1), (12.2), Proposition 12.1 and Proposition 12.3.

Note that S is obtained by blowing down six (-1)-curves $\pi^{-1}N_i$ (cf. Corollary 8.4). Since $\pi^{-1}N_i$ consists two disjoint (-1)-curves whose stabilizers are $\{0, g_i\}$, the conclusion about S follows by Theorem 10.4.

X is obtained by contracting four (-2)-curves on S (cf. Corollary 8.5). The group G acts on the set of the four (-2)-curves transitively. Thus the conclusion about X follows by Theorem 10.5. □

Lemma 12.5. *The sheaf* $\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ *has support on the* 4 *nodes of* X*, such that every stalk over a node has length* 1*. Moreover, we have a decomposition of the global section group of* $\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$*, according to the group action,*

$$H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) = H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X))^{\mathrm{inv}} \oplus \\ \oplus_{i=1}^3 H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X))^{\chi_i},$$

and each direct summand has dimension 1.

Proof. Since the group acts transitively on four A_1-singularities, it induces the regular representation on $H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X))$. Hence the conclusion follows. □

Corollary 12.6. $\dim Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)^{\mathrm{inv}} = 4$ *and* $\dim Ext^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)^{\mathrm{inv}} = 0$.

Proof. We have an exact sequence

$$\begin{aligned} 0 \to H^1(X, \Theta_X) \ &\to Ext^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \ \to H^0(X, \mathcal{E}xt^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) \\ \to H^2(X, \Theta_X) \ &\to Ext^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \ \to 0, \end{aligned}$$

associated to the spectral sequence,

$$E_2^{pq} = H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) \Rightarrow Ext^{p+q}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X).$$

The exact sequence is a G-equivariant sequence of $\mathbb{C}$-vector spaces, since all sheaves have a natural G-linearization. Then the conclusion follows by Theorem 12.4 and Lemma 12.5. □

Unlike the case of the D_4-generalized surfaces, we cannot determine the deformations of the $4A_1$-generalized surfaces completely by using the bidouble cover structure to the $4A_1$-type cubic surface.

Part IV

The Irreducible Component containing the Keum-Naie-Mendes Lopes-Pardini Surfaces

13 Keum-Naie-Mendes Lopes-Pardini Surfaces

J. H. Keum and later D. Naie ([Ke88], [Na94]) constructed a family of surfaces of general type with $K^2 = 3$ and $p_g(S) = 0$. These surfaces are double covers of nodal Enriques surfaces with 8 nodes (cf. [Na94, Théorème 2.10]). Also these surfaces are different from the (extended) Burniat surfaces with $K^2 = 3$, since they have different fundamental groups.

Theorem 13.1 ([Na94, Théorème 3.1])**.** *If S is a Keum-Naie surface with $K^2 = 3$, then $\pi_1^{\text{top}}(S) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}$.*

Another property of Keum-Naie surfaces is that their bicanonical map factors through the covering map to the nodal Enriques surface and is of degree 4. Later, in the article [MP04], Mendes Lopes and Pardini gave an explicit construction of surfaces of general type whose bicanonical map is a morphism of degree 2. They proved the following theorem about the corresponding subset in the moduli space.

Theorem 13.2 ([MP04, Theorem 2.1, Theorem 7.1])**.** *Let $\mathcal{M}_{1,3}^{\text{can}}$ be the moduli space of canonical models of surfaces of general type with $\chi = 1$ and $K^2 = 3$. Let $\mathcal{E}$ be the subset of $\mathcal{M}_{1,3}^{\text{can}}$ consisting of the canonical surfaces with $p_g = 0$ whose bicanonical map is composed with an involution such that the quotient surface is birational to an Enriques surface.*

(1) If X belongs to $\mathcal{E}$ and τ is the involution satisfying the property above, then X/τ is a nodal Enriques surface with 7 nodes.

(2) The set $\mathcal{E}$ is constructible.

(3) The closure $\overline{\mathcal{E}}$ in $\mathcal{M}^{\mathrm{can}}_{1,3}$ is irreducible and uniruled of dimension 6.

(4) $\overline{\mathcal{E}}$ contains the Keum-Naie surfaces with $K^2 = 3$.

As pointed out in [MP04, Remark 7.2], there is a **question** left open: whether $\overline{\mathcal{E}}$ is an irreducible component of $\mathcal{M}^{\mathrm{can}}_{1,3}$ or not.

We will reconstruct a subset $\mathcal{E}'$ in $\mathcal{E}$ through bidouble covers of a $4A_1$-type cubic surface. Then by studying the deformations of the surfaces in $\mathcal{E}'$, we give an affirmative answer to this question.

14 A Subfamily of KNMP Surfaces

In this section we will construct the family of surfaces of general type already mentioned in Remark 8.2. The construction here is similar to (but different from) the one in [MP04, Example 3.6].

Assume that Y is a $4A_1$-type cubic surface, and $\widetilde{Y}$ is its minimal resolution. **Recall the notation introduced in Subsection 3.3 and Figure 6.** Especially recall that $\widetilde{Y}$ has a pencil of rational curves C_i in the linear system $|2L - E_{i+1} - E_{i+2} - E'_{i+1} - E'_{i+2}|$ for $i = 1, 2, 3$.

We define three effective divisors on $\widetilde{Y}$,

$$\begin{aligned}\Delta_1 &= C_1 + \Gamma_2 + N_1 + N_2 \equiv -K_{\widetilde{Y}} + 2L - 2E_2 - 2E'_2 - 2E'_3,\\ \Delta_2 &= C_2 + \Gamma_1 + N_3 + Z \equiv -K_{\widetilde{Y}} + 2L - 2E_1 - 2E_3 - 2E'_1,\\ \Delta_3 &= H \equiv -K_{\widetilde{Y}},\end{aligned} \tag{14.1}$$

where C_1, C_2, H are irreducible smooth curves. And define three divisors

$$\begin{aligned}\mathcal{L}_1 &= -K_{\widetilde{Y}} + L - E_1 - E_3 - E'_1 \equiv -K_{\widetilde{Y}} + \Gamma_1 - E_3,\\ \mathcal{L}_2 &= -K_{\widetilde{Y}} + L - E_2 - E'_2 - E'_3 \equiv -K_{\widetilde{Y}} + \Gamma_2 - E'_3,\\ \mathcal{L}_3 &= -K_{\widetilde{Y}} + 2L - E_1 - E_2 - E_3 - E'_1 - E'_2 - E'_3 \equiv -2K_{\widetilde{Y}} - L.\end{aligned} \tag{14.2}$$

Thoughout the following sections, we will assume that the divisor $\Delta := \Delta_1 + \Delta_2 + \Delta_3$ has only normal crossing singularities.

Theorem 14.1. *Let* $\pi\colon \widetilde{S} \to \widetilde{Y}$ *be the bidouble cover associated to the above data* $\Delta_1, \Delta_2, \Delta_3, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$. *Then* $\widetilde{S}$ *is a smooth surface with* $K^2_{\widetilde{S}} = -5$ *and* $p_g(\widetilde{S}) = q(\widetilde{S}) = 0$.

Moreover, $|2K_{\widetilde{Y}}| \equiv \pi^*|-K_{\widetilde{Y}}| + \pi^*(N_1 + N_2 + N_3 + Z)$ *and* $P_2(\widetilde{S}) = 4$.

Proof. Note that Δ_i's and $\mathcal{L}_i$'s satisfy the equations (1.1) and (1.2). Since the total branch divisor Δ has normal crossings and each Δ_i is smooth, $\widetilde{S}$ is smooth by Proposition 1.2 (2).

Note that $\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3 \equiv -3K_{\widetilde{Y}}+N_1+N_2+N_3+Z$, $\mathcal{L}_i^2 = 1$, $K_{\widetilde{Y}}.\mathcal{L}_i = -3$. By Corollary 1.4, $K^2_{\widetilde{S}} = -5$ and $\chi(\mathcal{O}_{\widetilde{S}}) = 1$. From (14.2) one sees that $K_{\widetilde{Y}}+\mathcal{L}_i$ is not effective, hence by Corollary 1.4, $p_g(\widetilde{S}) = p_g(\widetilde{Y}) = 0$. It follows that $q(\widetilde{S}) = 0$.

By Theorem 1.3, $2K_{\widetilde{S}} \equiv \pi^*(-K_{\widetilde{Y}} + N_1 + N_2 + N_3 + Z)$. Moreover, from (14.2) $2K_{\widetilde{Y}}+\mathcal{L}_i+\mathcal{L}_{i+1}$ is not effective for all i. Take $i = 2$ for example, assume that $|2K_{\widetilde{Y}} + \mathcal{L}_2 + \mathcal{L}_3|$ contains an effective divisor D. Then $D.N_1 = -2$, $(D - N_1).N_2 = -2$ and $(D - N_1 - N_2).Z = -1$ show that $D \geq N_1 + N_2 + Z$. But $D - N_1 - N_2 - Z \equiv E_1 - E_2'$, which is not effective. This gives a contradiction. Hence $2K_{\widetilde{Y}} + \mathcal{L}_2 + \mathcal{L}_3$ is not effective.

It follows that

$$\begin{aligned} P_2(\widetilde{S}) &= h^0(\widetilde{Y}, -K_{\widetilde{Y}} + N_1 + N_2 + N_3 + Z) = h^0(\widetilde{Y}, -K_{\widetilde{Y}}) = 4, \text{ and} \\ |2K_{\widetilde{S}}| &= \pi^*|-K_{\widetilde{Y}} + N_1 + N_2 + N_3 + Z| \\ &= \pi^*|-K_{\widetilde{Y}}| + \pi^*(N_1 + N_2 + N_3 + Z), \end{aligned}$$

since $N_1 + N_2 + N_3 + Z$ is the fixed part of $|-K_{\widetilde{Y}} + N_1 + N_2 + N_3 + Z|$. □

Corollary 14.2. *Let* $f\colon \widetilde{S} \to S$ *be the blow down of the eight* (-1)*-curves* $\pi^{-1}N_k$ $(k = 1, 2, 3)$ *and* $\pi^{-1}Z$. *Then* S *is a smooth minimal surface of general type with* $K^2_S = 3$, $p_g(S) = 0$ *and* $P_2(S) = 4$.

Moreover, K_S *is ample and* $|2K_S|$ *is base-point-free.* S *is a bidouble cover of the* $4A_1$*-type cubic surface* Y *through the bicanonical morphism.*

Proof. Since each N_k, $k = 1, 2, 3$, or Z forms a connected component of the branch locus, each $\pi^{-1}N_k$ or $\pi^{-1}Z$ is a disjoint union of two (-1)-curves. Let $f\colon \widetilde{S} \to S$ be the blow down of these eight (-1)-curves. Then $K^2_S = 3$.

Since p_g, q, P_2 are birational invariants, $p_g(S) = 0$ and $P_2(S) = 4$. Moreover, since $|2K_{\widetilde{S}}| = f^*|2K_S| + \pi^*(N_1+N_2+N_3+Z)$, by Theorem 14.1, we have $f^*|2K_S| = \pi^*|-K_{\widetilde{Y}}|$. Since $|-K_{\widetilde{Y}}|$ is base-point-free, $|2K_S|$ is base-point-free.

The minimal resolution $\mu\colon \widetilde{Y} \to Y$ contracts exactly the (-2)-curves N_1, N_2, N_3, Z. From the construction, we have a finite bidouble cover $p\colon S \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{f} & S \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ \widetilde{Y} & \xrightarrow{\mu} & Y \end{array}$$

and $2K_S \equiv p^*(-K_Y)$. Since $-K_Y$ is ample and p is finite, K_S is ample and thus S is minimal. It also shows that the bicanonical morphism of S is the composition of p and the anticanonical embedding of Y into $\mathbb{P}^3$. □

We denote by $\mathcal{E}'$ the corresponding subset of smooth surfaces constructed above in the moduli space $\mathcal{M}^{\text{can}}_{1,3}$.

Proposition 14.3. *$\mathcal{E}'$ is contained in $\mathcal{E}$.*

Proof. Given a surface S in $\mathcal{E}'$, consider the intermediate double cover

$$\hat{\pi}\colon \hat{S} \to \widetilde{Y}$$

associated to the data $2\mathcal{L}_3 \equiv \Delta_1 + \Delta_2$.

Standard formulae for double covers (for example, see [BHPV, Page 236-237]) show that

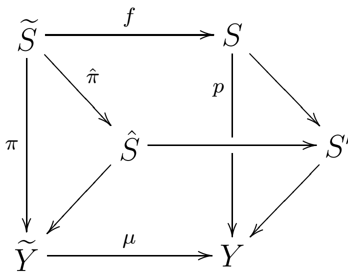
$$K_{\hat{S}} \equiv \hat{\pi}^*(K_Y + \mathcal{L}_3) \equiv \hat{\pi}^*(2L - E_1 - E_2 - E_3 - E_1' - E_2' - E_3'),$$

$$2K_{\hat{S}} \equiv \hat{\pi}^*(4L - 2E_1 - 2E_2 - 2E_3 - 2E_1' - 2E_2' - 2E_3') \equiv 2\hat{E}_1 + 2\hat{E}_2 + 2\hat{E}_3 + 2\hat{E}_4,$$

$$K^2_{\hat{S}} = -4,\ p_g(\hat{S}) = 0,$$

where $\hat{E}_k := \hat{\pi}^{-1}N_k$ and $\hat{E}_4 := \hat{\pi}^{-1}Z$ are (-1)-curves. Moreover, $\hat{S}$ has 7 nodes lying over the nodes of the curve $C_1 + C_2 + \Gamma_1 + \Gamma_2$.

Let $\hat{f}\colon \hat{S} \to S'$ be the blow down of the four (-1)-curves. We obtain a nodal Enriques surface S' with 7 nodes. The following diagram commutes:



Thus the bicanonical morphism $S \to Y \hookrightarrow \mathbb{P}^3$ of S factors through S'. By the definition of $\mathcal{E}$ (cf. Theorem 13.2), S belongs to $\mathcal{E}$. □

15 Local Deformations and Irreducible Component

In this section we will prove the following theorem.

Theorem 15.1. *(1) For a general surface S in $\mathcal{E}'$, $h^1(S, \Theta_S) = 6$, $h^2(S, \Theta_S) = 2$ and the base of the Kuranishi family of S is smooth.*

(2) $\overline{\mathcal{E}}$ is an irreducible component of the moduli space $\mathcal{M}^{\mathrm{can}}_{1,3}$.

The key point is to prove the following proposition.

Proposition 15.2. *For a general surface S in $\mathcal{E}'$, $h^2(S, \Theta_S) \leq 2$.*

Proof of Theorem 15.1 assuming Proposition 15.2.
Since $-h^1(S, \Theta_S) + h^2(S, \Theta_S) = 2K_S^2 - 10\chi(S) = -4$, by Proposition 15.2 $h^1(S, \Theta_S) \leq 6$. Since S is smooth and K_S is ample (cf. Corollary 14.2), the minimal model and the canonical model of S coincide. We have the following inequalities,

$$\begin{aligned} 6 \geq h^1(S, \Theta_S) &\geq \text{the dimension of the base of the Kuranishi family of } S \\ &= \text{the dimension of } \mathcal{M}^{\mathrm{can}}_{1,3} \text{ at the point } [S] \\ &\geq \text{the dimension of } \overline{\mathcal{E}}. \end{aligned}$$

Since the dimension of $\overline{\mathcal{E}}$ is 6 by Theorem 13.2, we see that all the equalities hold. The second equality shows that the base of the Kuranishi family of S is smooth. Since locally the germ of the complex space $(\mathcal{M}^{\mathrm{can}}_{1,3}, [S])$ is analytically isomorphic to the quotient of the base of the Kuranishi family by the finite group $Aut(S)$, it follows that $(\mathcal{M}^{\mathrm{can}}_{1,3}, [S])$ is irreducible. Since $\overline{\mathcal{E}}$ is irreducible by Mendes Lopes and Pardini's Theorem 13.2, the last equality shows that $\overline{\mathcal{E}}$ coincides with $\mathcal{M}^{\mathrm{can}}_{1,3}$ locally at $[S]$. It follows that $\overline{\mathcal{E}}$ is an irreducible component of $\mathcal{M}^{\mathrm{can}}_{1,3}$. $\square$

By Theorem 10.4, to prove Proposition 15.2, it suffices to show $h^2(\widetilde{S}, \Theta_{\widetilde{S}}) \leq 2$. By Serre Duality and Theorem 10.1,

$$\begin{aligned} H^2(\widetilde{S}, \Theta_{\widetilde{S}}) = {} & H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}}) \\ & \oplus \left(\oplus_{i=1}^3 H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_i)(K_{\widetilde{Y}} + \mathcal{L}_i)) \right). \end{aligned}$$

Thus it suffices to calculate the dimension of each summand.

Lemma 15.3. $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}}) = 0.$

Proof. By Lemma 10.3, we have an exact sequence

$$0 \to \Omega^1_{\widetilde{Y}}(K_{\widetilde{Y}}) \to \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_{\widetilde{Y}}) \to \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_{\widetilde{Y}}) \to 0 \tag{15.1}$$

Note that $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}) = 0$ and $-K_{\widetilde{Y}}$ is effective, thus $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(K_{\widetilde{Y}})) = 0$. To prove the claimed equality, it suffices to show the boundary map

$$\delta\colon H^0(\widetilde{Y}, \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_{\widetilde{Y}})) \to H^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(K_{\widetilde{Y}}))$$

is injective.

By (14.1),

$$\begin{aligned} & H^0(\widetilde{Y}, \mathcal{O}_{\Delta_1}(K_{\widetilde{Y}})) \cong H^0(\widetilde{Y}, \mathcal{O}_{N_1} \oplus \mathcal{O}_{N_2}) \cong \mathbb{C}^2, \\ & H^0(\widetilde{Y}, \mathcal{O}_{\Delta_2}(K_{\widetilde{Y}})) \cong H^0(\widetilde{Y}, \mathcal{O}_{N_3} \oplus \mathcal{O}_{Z}) \cong \mathbb{C}^2, \\ & H^0(\widetilde{Y}, \mathcal{O}_{\Delta_3}(K_{\widetilde{Y}})) = 0. \end{aligned}$$

Since $|-K_{\widetilde{Y}}|$ is base-point-free, there is a morphism $\mathcal{O}_{\widetilde{Y}}(K_{\widetilde{Y}}) \to \mathcal{O}_{\widetilde{Y}}$, which is not identically zero on any component of Δ_i's. Now consider the commutative diagram coming from the morphism $\mathcal{O}_{\widetilde{Y}}(K_{\widetilde{Y}}) \to \mathcal{O}_{\widetilde{Y}}$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1_{\widetilde{Y}}(K_{\widetilde{Y}}) & \longrightarrow & \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_{\widetilde{Y}}) & \longrightarrow & \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_{\widetilde{Y}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^1_{\widetilde{Y}} & \longrightarrow & \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) & \longrightarrow & \oplus_{i=1}^3 \mathcal{O}_{\Delta_i} & \longrightarrow & 0. \end{array}$$

It gives a commutative diagram of cohomology groups,

$$\begin{array}{ccc} \mathbb{C}^4 \cong H^0(\widetilde{Y}, \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_{\widetilde{Y}})) & \xrightarrow{\delta} & H^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(K_{\widetilde{Y}})) \\ \downarrow{\scriptstyle \psi_2} & \searrow^{\psi} & \downarrow \\ H^0(\widetilde{Y}, \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}) & \xrightarrow{\psi_1} & H^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}}). \end{array}$$

By Lemma 10.3, the image of the function identically equal to 1 on N_k $(k = 1, 2, 3)$, respectively on Z maps under ψ_1 to the first Chern class of N_k, respectively of Z. Because the N_k's and Z are 4 disjoint (-2)-curves, their Chern classes are independent in $H^1(\widetilde{Y}, \Omega^1_{\widetilde{Y}})$.

Thus the composite map ψ is injective. It follows that δ is also injective and $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1, \log \Delta_2, \log \Delta_3) \otimes \Omega^2_{\widetilde{Y}}) = 0$. □

To calculate other summands, we fix the coordinates of P_i and P'_i. **Without loss of generality, assume that**

$$\begin{aligned} \mathbf{P_1} = (\mathbf{1} : -\mathbf{1} : \mathbf{0}),\ \mathbf{P_2} = (\mathbf{0} : \mathbf{1} : \mathbf{0}),\ \mathbf{P_3} = (\mathbf{1} : \mathbf{0} : \mathbf{0}), \\ \mathbf{P'_1} = (\mathbf{0} : \mathbf{0} : \mathbf{1}),\ \mathbf{P'_2} = (\mathbf{1} : \mathbf{0} : \mathbf{1}),\ \mathbf{P'_3} = (\mathbf{0} : \mathbf{1} : \mathbf{1}). \end{aligned} \tag{15.2}$$

See Figure 9.

Lemma 15.4. $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3)) = 0$ *for a general* $H \in |-K_{\widetilde{Y}}|$.

Proof. Let $M := K_{\widetilde{Y}} + \mathcal{L}_3 = 2L - E_1 - E_2 - E_3 - E'_1 - E'_2 - E'_3$ (cf. (14.2)). Recall that $\Delta_3 = H \in |-K_{\widetilde{Y}}|$. Then $K_{\widetilde{Y}}.M = \Delta_3.M = 0$. For a general H, H is a smooth elliptic curve and $\mathcal{O}_{\Delta_3}(M)$ is a 2-torsion element, thus $H^0(\mathcal{O}_{\Delta_3}(M)) = 0$.

In fact, note that $2M \equiv N_1 + N_2 + N_3 + Z$. Take the double cover $\tilde{q} \colon \tilde{\Sigma} \to \widetilde{Y}$ associated to the data $2M \equiv N_1 + N_2 + N_3 + Z$, and blow down the (-1)-curves $\tilde{q}^{-1}N_i, i = 1, 2, 3$ and $\tilde{q}^{-1}Z$, $\eta \colon \tilde{\Sigma} \to \Sigma$. We have a morphism $q \colon \Sigma \to Y$

and the following commutative diagram

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{\eta} & \Sigma \\ {\scriptstyle \tilde{q}}\downarrow & & \downarrow{\scriptstyle q} \\ \widetilde{Y} & \xrightarrow[\mu]{} & Y. \end{array}$$

q only ramifies over the 4 nodes of Y and $q^*(-K_Y) \equiv -K_\Sigma$. Σ is a smooth Del Pezzo surface of degree 6, i.e., $K_\Sigma^2 = 6$ and $-K_\Sigma$ is very ample. By Bertini's theorem, a general curve C of $|-K_Y|$ is smooth and irreducible and $q^{-1}C$ is an irreducible smooth curve in $|-K_\Sigma|$. Since $|-K_{\widetilde{Y}}| = \mu^*|-K_Y|$ and a general element $H \in |-K_{\widetilde{Y}}|$ is disjoint from the (-2)-curves, the commutative diagram shows that $\tilde{q}^{-1}H$ is an irreducible smooth curve. Hence $\mathcal{O}_H(M)$ is a 2-torsion element.

Tensor the following exact sequence with $\mathcal{O}_{\widetilde{Y}}(M)$,

$$0 \to \Omega^1_{\widetilde{Y}} \to \Omega^1_{\widetilde{Y}}(\log \Delta_3) \to \mathcal{O}_{\Delta_3} \to 0,$$

we see that $h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(M)) = h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(M))$.

Since $\sigma_*\Omega^1_{\widetilde{Y}}(M)$ is a subsheaf of $\Omega^1_{\mathbb{P}^2}(2)$, one can view $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(M))$ as a subspace of $H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(2))$. By [BC10-b, Lemma 5.2], any form of $H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(2))$ can be written as

$$\omega = A(x_1dx_2 - x_2dx_1) + B(x_2dx_3 - x_3dx_2) + C(x_1dx_3 - x_3dx_1).$$

Evaluating at $P_2 = (0:1:0)$, by Lemma 12.2 (1), we get $A = B = 0$. Then evaluate at $P_3 = (1:0:0)$ and get $C = 0$.

Thus we see that $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_3)(K_{\widetilde{Y}} + \mathcal{L}_3)) = 0$. □

Proposition 15.5. $h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1)(K_{\widetilde{Y}} + \mathcal{L}_1)) \leq 1$ *and* $h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_2)(K_{\widetilde{Y}} + \mathcal{L}_2)) \leq 1$.

First we prove the following lemma.

Lemma 15.6.

$$h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1)(K_{\widetilde{Y}} + \mathcal{L}_1)) = h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log N_1, \log N_2, \log N_3, \log Z)(2L - E_2 - E_2' - E_3')),$$

$$h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_2)(K_{\widetilde{Y}} + \mathcal{L}_2)) = h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log N_1, \log N_2, \log N_3, \log Z)(2L - E_1 - E_3 - E_1')).$$

Proof. $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1)(K_{\widetilde{Y}}+\mathcal{L}_1)) = H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1)(\Gamma_1 - E_3))$ by (14.2). Note that Δ_1 is the disjoint union of C_1, Γ_2, N_1 and N_2. Since

$$(K_{\widetilde{Y}} + 2C_1 + \Gamma_1 - E_3).C_1 = -1 < 0,$$
$$(K_{\widetilde{Y}} + 2\Gamma_2 + \Gamma_1 - E_3 + C_1).\Gamma_2 = -2 < 0,$$

apply Lemma 10.2 to C_1 and then to Γ_2,

$$H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1)(K_{\widetilde{Y}} + \mathcal{L}_1)) \cong \\ H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log N_1, \log N_2)(\Gamma_1 - E_3 + C_1 + \Gamma_2)).$$

Since

$$\begin{aligned} \Gamma_1 - E_3 + C_1 + \Gamma_2 &\equiv 4L - E_1 - 2E_2 - 2E_3 - E_1' - 2E_2' - E_3' \\ &\equiv N_3 + Z + (2L - E_2 - E_2' - E_3'), \\ &(K_{\widetilde{Y}} + 2N_3 + (2L - E_2 - E_2' - E_3') + Z).N_3 = -3 < 0, \\ &(K_{\widetilde{Y}} + 2Z + 2L - E_2 - E_2' - E_3').Z = -3 < 0, \end{aligned}$$

apply Lemma 10.2 to N_3 and then to Z,

$$H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log N_1, \log N_2)(4L - E_1 - 2E_2 - 2E_3 - E_1' - 2E_2' - E_3')) \cong \\ H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log N_1, \log N_2, \log N_3, \log Z)(2L - E_2 - E_2' - E_3')).$$

Thus the first equality holds. A similar argument shows that the second equality also holds. □

Proof of Proposition 15.5. There is an automorphism
$\tau\colon \mathbb{P}^2 \to \mathbb{P}^2$ such that $\tau(P_1) = P_2', \tau(P_2) = P_1, \tau(P_1') = P_2, \tau(P_2') = P_1'$.
It follows that $\tau(P_3) = P_3', \tau(P_3') = P_3$. This automorphism induces an automorphism of $\widetilde{Y}$ and shows that

$$\Omega^1_{\widetilde{Y}}(\log N_1, \log N_2, \log N_3, \log Z)(2L - E_2 - E_2' - E_3') \cong \\ \Omega^1_{\widetilde{Y}}(\log N_1, \log N_2, \log N_3, \log Z)(2L - E_1 - E_3 - E_1').$$

Together with Lemma 15.6, it suffices to show

$$H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log N_1, \log N_2, \log N_3, \log Z)(2L - E_2 - E_2' - E_3')) \leq 1.$$

Tensor the following exact sequence with $\mathcal{O}_{\widetilde{Y}}(2L - E_2 - E_2' - E_3')$,

$$0 \to \Omega^1_{\widetilde{Y}}(\log N_2, \log N_3, \log Z) \to \Omega^1_{\widetilde{Y}}(\log N_1, \log N_2, \log N_3, \log Z) \to \mathcal{O}_{N_1} \to 0.$$

Since $(2L - E_2 - E_2' - E_3').N_1 = 0$, by the cohomology exact sequence associated to the above sequence, it suffices to show that

$$\mathbf{H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log N_2, \log N_3, \log Z)(2L - E_2 - E_2' - E_3')) = 0}. \tag{15.3}$$

View it as a subspace of $H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log l_1, \log l_2, \log l_3)(2))$. **Recall the coordinates** (15.2)**, note that $\mathbf{N_2, N_3, Z}$ are the strict transforms of $\mathbf{l_1 : x_1 = 0, l_2 : x_2 = 0, l_3 : x_3 = 0}$ respectively. See Figure 9.**

By [BC10-b, Corollary 5.4], any $\omega \in H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log l_1, \log l_2, \log l_3)(2))$ can be written as:

$$\begin{aligned} \omega \ = \ & \frac{dx_1}{x_1}(-a_{12}x_1x_2 - a_{13}x_3x_1 + a_{212}x_2^2 + a_{313}x_3^2 \\ & - a_{121}x_1^2 - a_{131}x_1^2 - a_{231}x_2x_1 + a_{312}x_3x_2) \\ + \ & \frac{dx_2}{x_2}(a_{12}x_1x_2 - a_{23}x_3x_2 - a_{212}x_2^2 + a_{121}x_1^2 \\ & + a_{323}x_3^2 + a_{123}x_1x_3 - a_{232}x_2^2 - a_{312}x_3x_2) \\ + \ & \frac{dx_3}{x_3}(a_{13}x_1x_3 + a_{23}x_2x_3 - a_{313}x_3^2 - a_{323}x_3^2 \\ & + a_{131}x_1^2 + a_{232}x_2^2 - a_{123}x_1x_3 + a_{231}x_1x_2), \end{aligned} \tag{15.4}$$

where $a_{12}, \ldots, a_{312}$ are complex numbers.

Now we intend to use Lemma 12.2 to impose conditions on ω.

At $\mathbf{P_2 = (0 : 1 : 0)}$, work on the coordinate chart $x_2 = 1$, and note that P_2 is the intersection point of $l_1 : x_1 = 0$ and $l_3 : x_3 = 0$. Locally,

$$\begin{aligned} \omega &= \alpha(x_1, x_3)\frac{dx_1}{x_1} + \beta(x_1, x_3)\frac{dx_3}{x_3}, \\ \alpha(x_1, x_3) &= (-a_{12}x_1 - a_{13}x_3x_1 + a_{212} + a_{313}x_3^2 \\ & \quad - a_{121}x_1^2 - a_{131}x_1^2 - a_{231}x_1 + a_{312}x_3), \\ \beta(x_1, x_3) &= (a_{13}x_1x_3 + a_{23}x_3 - a_{313}x_3^2 - a_{323}x_3^2 \\ & \quad + a_{131}x_1^2 + a_{232} - a_{123}x_1x_3 + a_{231}x_1) \end{aligned}$$

By Lemma 12.2 (4),

$$\alpha(0,0) = a_{212} = 0, \tag{15.5}$$
$$\beta(0,0) = a_{232} = 0, \tag{15.6}$$
$$\frac{\partial(\alpha+\beta)}{\partial x_1}(0,0) = -a_{12} = 0, \tag{15.7}$$
$$\frac{\partial(\alpha+\beta)}{\partial x_3}(0,0) = a_{312} + a_{23} = 0. \tag{15.8}$$

At $\mathbf{P_3} = (\mathbf{1}:\mathbf{0}:\mathbf{0})$, work on the coordinate chart $x_1 = 1$, and note that P_3 is the intersection point of $l_2 : x_2 = 0$ and $l_3 : x_3 = 0$. By Lemma 12.2 (2),

$$a_{121} + a_{131} = 0. \tag{15.9}$$

At $\mathbf{P'_1} = (\mathbf{0}:\mathbf{0}:\mathbf{1})$, work on the coordinate chart $x_3 = 1$, and note that P'_1 is the intersection point of $l_1 : x_1 = 0$ and $l_2 : x_2 = 0$. By Lemma 12.2 (2),

$$a_{313} + a_{323} = 0. \tag{15.10}$$

By (15.5), (15.6), (15.7), (15.8), (15.9) and (15.10),

$$\begin{aligned}\omega = {} & \frac{dx_1}{x_1}(-a_{13}x_3x_1 - a_{323}x_3^2 - a_{231}x_2x_1 - a_{23}x_3x_2) \\ & + \frac{dx_2}{x_2}(-a_{131}x_1^2 + a_{323}x_3^2 + a_{123}x_1x_3) \\ & + \frac{dx_3}{x_3}(a_{13}x_1x_3 + a_{23}x_2x_3 + a_{131}x_1^2 - a_{123}x_1x_3 + a_{231}x_1x_2),\end{aligned} \tag{15.11}$$

At $\mathbf{P'_2} = (\mathbf{1}:\mathbf{0}:\mathbf{1})$, work on the coordinate chart $x_1 = 1$, and note that P'_2 is on the line $l_2 : x_2 = 0$. Locally,

$$\omega = \alpha(x_2,x_3)dx_3 + \beta(x_2,x_3)\frac{dx_2}{x_2},$$
$$\alpha(x_2,x_3) = \frac{a_{13}x_3 + a_{23}x_2x_3 + a_{131} - a_{123}x_3 + a_{231}x_2}{x_3},$$
$$\beta(x_2,x_3) = -a_{131} + a_{323}x_3^2 + a_{123}x_3,$$

by Lemma 12.2 (3),

$$\beta(0,1) = -a_{131} + a_{323} + a_{123} = 0, \tag{15.12}$$
$$\frac{\partial\beta}{\partial x_2}(0,1) = 0,$$

$$\alpha(0,1)+\frac{\partial\beta}{\partial x_3}(0,1)=a_{13}+a_{131}+2a_{323}=0. \tag{15.13}$$

At $\mathbf{P_3'}=(\mathbf{0}:\mathbf{1}:\mathbf{1})$, work on the coordinate chart $x_2=1$, and note that P_3' is on the line $l_1: x_1=0$. Locally,

$$\omega=\alpha(x_1,x_3)dx_3+\beta(x_1,x_3)\frac{dx_1}{x_1},$$
$$\alpha(x_1,x_3)=\frac{a_{13}x_1x_3+a_{23}x_3+a_{131}x_1^2-a_{123}x_1x_3+a_{231}x_1}{x_3},$$
$$\beta(x_1,x_3)=-a_{13}x_3x_1-a_{323}x_3^2-a_{231}x_1-a_{23}x_3,$$

by Lemma 12.2 (3),

$$\beta(0,1)=-a_{323}-a_{23}=0, \tag{15.14}$$
$$\frac{\partial\beta}{\partial x_1}(0,1)=-a_{13}-a_{231}=0, \tag{15.15}$$
$$\alpha(0,1)+\frac{\partial\beta}{\partial x_3}(0,1)=-2a_{323}=0. \tag{15.16}$$

By (15.12), (15.13), (15.14), (15.15) and (15.16), $a_{23}=a_{323}=0, a_{231}=a_{131}=a_{123}=-a_{13}$. Thus by (15.11),

$$\omega=a_{13}[\frac{dx_1}{x_1}(-x_3x_1+x_2x_1)+\frac{dx_2}{x_2}(x_1^2-x_1x_3)+\frac{dx_3}{x_3}(2x_1x_3-x_1^2-x_1x_2)],$$

At $\mathbf{P_1}=(\mathbf{1}:-\mathbf{1}:\mathbf{0})$, work on the coordinate chart $x_1=1$, and note that P_1 is on the line $l_3: x_3=0$. Locally,

$$\omega=\alpha(x_2,x_3)dx_2+\beta(x_2,x_3)\frac{dx_3}{x_3},$$
$$\alpha(x_2,x_3)=\frac{a_{13}-a_{13}x_3}{x_2},$$
$$\beta(x_2,x_3)=2a_{13}x_3-a_{13}-a_{13}x_2,$$

by Lemma 12.2 (3),

$$\beta(-1,0)=0,$$
$$\frac{\partial\beta}{\partial x_3}(-1,0)=2a_{13}=0, \tag{15.17}$$
$$\alpha(-1,0)+\frac{\partial\beta}{\partial x_2}(-1,0)=-2a_{13}=0. \tag{15.18}$$

thus $a_{13} = 0$ and $\omega = 0$.

Hence $H^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log N_2, \log N_3, \log Z)(2L - E_2 - E_2' - E_3')) = 0$. It follows that $h^0(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log \Delta_1)(K_{\widetilde{Y}} + \mathcal{L}_1)) \leq 1$. $\square$

Now by Lemma 15.3, Lemma 15.4 and Proposition 15.5 we see that Proposition 15.2 holds and thus complete the proof of Theorem 15.1.

References

[BHPV] W. Barth, K. Hulek, C. Peters, and A. Van de Ven, *Compact complex surfaces, 2nd edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 4*, Springer 2004.

[Barlow99] R. N. Barlow, *Zero-cycles on Mumford's surface*, Math. Proc. Camb. Phil. Soc. (1999), 126, 5, 505-510.

[Be78] A. Beauville, *Complex algebraic surfaces*, Cambridge Univ. Press (1983) (translation of Surfaces algébriques complexes, Astérisque 54 Soc. Math. France, Paris (1978)).

[Ber] T. Berry, *Infinitesimal deformations of cyclic covers*, Acta Cient. Venezolana 35 (1984), 177-183.

[Bla95] B. Blache, *Riemann-Roch theorem for normal surfaces and applications*, Abh. Math. Sem. Univ.Hamburg 65 (1995), 307-340.

[Bu66] P. Burniat, *Sur les surfaces de genre* $P_{12} > 1$, Ann. Mat. Pura Appl. (4) 71 1966, 1-24.

[BC11] I. Bauer and F. Catanese, *Burniat surfaces I: fundamental groups and moduli of primary Burniat surfaces*, Classification of algebraic varieties-Schiermonnikoog 2009, C. Faber, G. van dr Geer, E. Looijenga editors, EMS Congress Reports (2011), 1-31.

[BC10-a] ______, *Burniat surfaces II: secondary Burniat surfaces form three connected components of the moduli space*, Invent. Math. 180 (2010), no. 3, 559-588.

[BC10-b] ______, *Burniat surfaces III: deformations of automorphisms and extended Burniat surfaces*, arXiv: 1012.3770.

[BCP09] I. Bauer, F. Catanese, and R. Pignatelli, *Surfaces of general type with geometric genus zero: a survey*, Proceedings of the conference Complex and Differential Geometry held in Hannover, September 14 - September 18, 2009.

[BW74] D. M. Burns Jr. and J. Wahl, *Local contributions to global deformations of surfaces*, Invent.Math. 26 (1974), 67-88.

[BW79] J. W. Bruce and C. T. C. Wall, *On the classification of cubic surfaces*, J. London Math. Soc. (2) 19 (1979), no. 2, 245-256.

[Cam32] L. Campedelli, *Sopra alcuni piani doppi notevoli con curva di diramazioni del decimo ordine*, Atti Acad. Naz. Lincei 15 (1932), 536-542.

[Cat83] F. Catanese, *Moduli of surfaces of general type*, Algebraic geometry—open problems (Ravello, 1982), Lecture Notes in Math., 997, Springer, Berlin-New York, (1983), 90-112.

[Cat84] ______, *On the moduli spaces of surfaces of general type*, J. Diff. Geom.19 (1984), no. 2, 483-515.

[Cat87] ______, *Automorphisms of rational double points and moduli spaces of surfaces of general type*, Compositio Math. 61 (1987), no.1, 81-102.

[Cat88] ______, *Moduli of algebraic surfaces*, Theory of moduli (Montecatini Terme, 1985), Lecture Notes in Math. 1337, 1-83.

[Cat99] ______, *Singular bidouble covers and the construction of interesting algebraic surfaces*, Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 97-120, Contemp. Math., 241, Amer. Math. Soc., Providence, RI, 1999.

[CHKS06] F. Catanese, S. Hosten, A. Khetan, and B. Sturmfels, *The maximum likelihood degree*, Amer.J.Math.128, no 3,(2006), 671-697.

[Cay69] A. Cayley, *A memoir on cubic surfaces*, Phil. Trans. Roy. Soc., 159 (1869), 231-326.

[Zhang01] D. -Q. Zhang, *Automorphisms of Finite Order on Rational Surfaces: With an appendix by I. Dolgachev*, Journal of Algebra, Volume 238, Issue 2, 15 April 2001, 560-589.

[DMP02] I. Dolgachev, M. Mendes Lopes, and R. Pardini, *Rational surfaces with many nodes*, Compositio Math. 132 (2002), 349-363.

[Gies77] D Gieseker, *Global moduli for surfaces of general type*, Invent. Math. 43 (1977), no. 3, 233-282.

[HP10] P. Hacking and Y. Prokhorov, *Smoothable del Pezzo surfaces with quotient singularities*, Compositio Math. 146 (2010), 169-192.

[Har77] R. Hartshorne, *Algebraic geometry. Graduate Texts in Mathematics, No. 52.*, Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.

[HW81] Fumio Hidaka and Kei-ichi Watanabe, *Normal Gorenstein surfaces with ample anti-canonical divisor*, Tokyo J. Math. Vol. 4, No. 2, 1981, 320-330.

[In94] M. Inoue, *Some new surfaces of general type*, Tokyo J. Math. 17 (1994), no. 2, 295-319.

[Ke88] J. H. Keum, *Some new surfaces of general type with* $p_g = 0$, *(preprint 1988).*

[Ku04] V. S. Kulikov, *Old examples and a new example of surfaces of general type with* $p_g = 0$, (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 68 (2004), no. 5, 123170; translation in Izv. Math. 68 no. 5, (2004), 9651008.

[Mats63] H. Matsumura, *On algebraic groups of birational transformations*, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur. (8) 34 (1963), 151-155.

[MP01] M. Mendes Lopes and R. Pardini, *A connected component of the moduli space of surfaces with $p_g = 0$*, Topology 40 (5) (2001), 977-991.

[MP04] ———, *A new family of surfaces with $p_g = 0$ and $K^2 = 3$*, Ann. Scient. Ec. Norm. Sup. 4^e série, 37 (2004), 507-531.

[Miy77] Y. Miyaoka, *On numerical Campedelli surfaces*, Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo, 1977, 113-118.

[Na94] D. Naie, *Surfaces d'Enriques et une construction de surfaces de type général avec $p_g = 0$*,, Math. Z. 215 (2) (1994), 269-280.

[NP11] J. Neves and R. Pignatelli, *Unprojection and deformations of tertiary Burniat surfaces.*

[PR04] S. Papadakis and M. Reid, *Kustin-Miller unprojection without complexes*, J. Algebraic Geometry 13 (2004), 563-577.

[Par91] R. Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. 417 (1991), 191-213.

[Pet77] C. A. M. Peters, *On certain examples of surfaces with $p_g = 0$ due to Burniat*, Nagoya Math. J. 66 (1977), 109-119.

[Reid79] M. Reid, *Surfaces with $p_g = 0, K^2 = 2$*, preprint (1979).

[Reid85] ———, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, (1987), 345-414.

[Sc64] L. Schläfli, *On the distribution of surfaces of the third order into species*, 193-247.

[Ser06] E. Sernesi, *Deformations of Algebraic Schemes*, Springer, Grundlehren 334, 2006.

[Weng95] L. Weng, *A result on bicanonical maps of surfaces of general type*, Osaka J. Math., 32 (1995), no. 2, 467-473.

[Sak10] Yoshiyuki Sakamaki, *Automorphism groups on normal singular cubic surfaces with no parameters*, Transactions of the American Mathematical Society Volume 362, Number 5, May 2010, 2641-2666.

[Zar58] O. Zariski, *On the purity of the branch locus of algebraic functions*, Proc. Nat. Acad. Sci. U. S. A. 44 (1958), 791-796.