Stabilization with discounted optimal control*

Vladimir Gaitsgory
Department of Mathematics
Macquarie University
NSW 2109, Australia
vladimir.gaitsgory@mq.edu.au

Lars Grüne
Mathematisches Institut
Universität Bayreuth
95440 Bayreuth, Germany
lars.gruene@uni-bayreuth.de

Neil Thatcher
Weapons and Combat Systems Division
Defence Science and Technology Organisation
Edinburgh SA 5111, Australia
neil.thatcher@dsto.defence.gov.au

February 16, 2015

Abstract: We provide a condition under which infinite horizon discounted optimal control problems can be used in order to obtain stabilizing controls for nonlinear systems. The paper gives a mathematical analysis of the problem as well as an illustration by a numerical example.

Keywords: Stabilization, Discounted optimal control, Lyapunov function

1 Introduction

Stabilization is one of the central tasks in control engineering. Given a desired equilibrium point, the task consists of designing a control — often in form of a feedback law — which ensures that the desired equilibrium becomes asymptotically stable for the controlled system, i.e., that solutions starting close to the equlibrium remain close and that all solutions eventually converge to the equilibrium.

It is well known that optimal control methods can be used for the design of asymptotically stabilizing controls by choosing the objective in such a way that it penalizes states away from the desired equilibrium. This approach is appealing since optimal control problems often allow for analytical or numerical solution techniques and thus provide a constructive approach to the stabilization problem. For linear systems, the classical (infinite horizon) linear quadratic regulator, going back to [19] (see also, e.g., the textbooks [1, Chapter 3] or [26, Section 8.2]), is one example for this approach. Here, the optimal control can be obtained from the solution of the algebraic Riccati-equation.

*This paper was written while Lars Grüne was visiting the Department of Mathematics of Macquarie University, Sydney, Australia. The research was supported by the Australian Research Council (ARC) Discovery Grants DP130104432 and DP120100532 and by the European Union under the 7th Framework Programme FP7-PEOPLE-2010-ITN, Grant agreement number 264735-SADCO.

In this paper, we consider this problem for general nonlinear systems. The nonlinear generalization of the infinite horizon linear quadratic problem — i.e., the infinite horizon undiscounted optimal control problem — can still be used in order to theoretically characterize stabilizing controls and related control Lyapunov functions, see, e.g., [25, 5]. However, numerically this problem is very difficult to solve. Direct methods, which can efficiently solve finite horizon nonlinear optimal control problems [2] fail here since infinite horizon problems are still infinite dimensional after a discretization in time. Dynamic programming methods apply only after suitable regularizations [4]. One popular way out of these difficulties is receding horizon or model predictive control (MPC) [22, 15], in which the infinite horizon problem is circumvented by the iterative solution of finite horizon problems.

In the present paper we present another way to circumvent the use of infinite horizon undiscounted optimal control problems by using infinite horizon *discounted* optimal control problems, instead. These problems allow for various types of numerical approximation techniques, see, e.g., [8, 9, 10, 11, 12] and the references therein. The reason for them being easier to solve numerically lies in the fact that while formally defined on an infinite time horizon, due to the discounting everything that happens after a long time contributes only very little to the value of the optimization objective, hence the effective time horizon can be considered as finite. We show that a condition similar to what can be found in the MPC literature can be used in order to establish that the discounted optimal value function is a Lyapunov function, from which asymptotic stability can be concluded. As such, the approach in this paper has some similarities with [21], with the difference that here we consider continuous time systems and that our condition allows to conclude asymptotic stability as opposed to the merely practical asymptotic stability statement in [21]. The results presented in this paper are also related to asymptotic turnpike theorems establishing that, under certain conditions, optimal or near optimal solutions of optimal control problems considered on an infinite time horizon converge (as times goes to infinity) to optimal solutions of certain "steady state" optimization problems (see, e.g., [3, 6, 7, 24, 28] and the references therein). However, our results are obtained under a different set of assumptions and with the use of different techniques than in the aforementioned works.

The paper is organized as follows. After defining the problem and the necessary background in Section 2, the main stability result is formulated and proved in Section 3. To this end we utilize a condition involving a bound on the optimal value function. In Section 4 it is shown how different controllability properties can be used in order to establish this bound. The performance of the resulting controls are illustrated by a numerical example in Section 5 and brief conclusions can be found in Section 6.

2 Problem formulation

For discount rate $C > 0$ we consider the discounted optimal control problem

$$\text{minimize } J(y_0, u(\cdot)) = \int_0^\infty e^{-Ct} g(y(t), u(t))dt \tag{2.1}$$

with respect to the control functions $u(\cdot)$. Here $y(t)$ is given by the control system

$$\dot{y}(t) = f(y(t), u(t)) \tag{2.2}$$

and the minimization is subject to the initial condition $y(0) = y_0$ and the control and state constraints $u(t) \in U \subseteq \mathbb{R}^m$, $y(t) \in Y \subseteq \mathbb{R}^n$. The map $f : Y \times U \to \mathbb{R}^n$ is assumed to be continuous and Lipschitz in y. With $\mathcal{U}$ we denote the set of measurable and locally Lebesgue integrable functions with values in U. We assume that the set Y is viable, i.e., for any $y_0 \in Y$ there exists at least one $u(\cdot) \in \mathcal{U}$ with $y(t) \in Y$ for all $t \geq 0$. Control functions with this property will be called *admissible* and the fact that we impose the state constraints when solving (2.1) implies that the minimization in (2.1) is carried out over the set of admissible control functions, only.

We define the optimal value function of the problem as

$$V(y_0) := \inf_{u(\cdot)\in\mathcal{U} \text{ admissible}} J(y_0, u(\cdot)).$$

We remark that we will not need any regularity assumption on V like, e.g., continuity in this paper. For a given initial value, an admissible control $u^\star(\cdot) \in \mathcal{U}$ is called an *optimal control* if $J(y_0, u^\star(\cdot)) = V(y_0)$ holds. While we do not assume the existence of optimal controls in the remainder of this paper, we will point out where its existence simplifies or improves the results.

Our goal is to design the running cost g in (2.1) in such a way that a desired equilibrium $\bar{y}$ is asymptotically stable for (approximately) optimal trajectories. Loosely speaking, this means that trajectories starting close to $\bar{y}$ remain close and eventually converge to $\bar{y}$. Formally, “remaining close” means that for each $\varepsilon > 0$ there is $\delta > 0$ such that the implication

$$\|y_0 - \bar{y}\| \leq \delta \quad \Rightarrow \quad \|y(t) - \bar{y}\| \leq \varepsilon \text{ for all } t \geq 0$$

holds while convergence is formalized in the usual way as $\lim_{t\to\infty} y(t) = \bar{y}$. Here, we assume that $\bar{y} \in Y$ is an equilibrium, i.e., that there exists a control value $\bar{u} \in U$ with $f(\bar{y}, \bar{u}) = 0$.

Remark 2.1: In the literature, asymptotic stability for controlled systems is often only used for feedback controls $u(t) = F(y(t))$. Here we use it in a more general sense also for time dependent control functions which, of course, may be generated by a feedback law.

In order to achieve asymptotic stability, we impose the following structure of g.

Assumption 2.2: Given $\bar{y} \in Y$, $\bar{u} \in U$, the running cost $g : Y \times U \to \mathbb{R}$ satisfies

(i) $g(y, u) > 0$ for $y \neq \bar{y}$

(ii) $g(\bar{y}, \bar{u}) = 0$.

This assumption states that g penalizes deviations of the state y from the desired state $\bar{y}$ and the hope is that this forces the optimal solution which minimizes the integral over g to converge to $\bar{y}$. A typical simple choice of g satisfying this assumption is the quadratic penalization

$$g(y, u) = \|y - \bar{y}\|^2 + \lambda\|u - \bar{u}\|^2 \tag{2.3}$$

with $\lambda \geq 0$.

It is well known that undiscounted optimal control can be used in order to enforce asymptotic stability of the (approximately) optimally controlled system. Prominent approaches using this fact are the linear quadratic optimal controller or model predictive control

(MPC). In the latter, the infinite horizon (undiscounted) optimal control problem is replaced by a sequence of finite horizon optimal control problems. Unless stabilizing terminal constraints or costs are used, this approach is known to work whenever the optimization horizon of the finite horizon problems is sufficiently large, cf. e.g. [18, 13, 23] or [15, Chapter 6]. The idea of using discounted optimal control for stabilization bears some similarities with this finite horizon approach, as in discounted optimal control the far future only contributes very weakly to the value of the functional J in (2.1), i.e., the effective optimization horizon is also finite.

It thus comes at no surprise that also the conditions we are going to use in order to deduce stability are similar to conditions which can be found in the MPC literature. More precisely, we will use the following assumption on the optimal value function.

Assumption 2.3: There exists $K > C$ such that

$$KV(y) \leq g(y,u) \tag{2.4}$$

holds for all $y \in Y$, $u \in U$.

This assumption in fact involves two conditions: as a first condition, inequality (2.4) expresses that the optimal value function can be bounded from above by the running cost, at all. In the MPC literature, a similar condition is used, more precisely

$$\text{there exists } \gamma > 0 \text{ with } V_0(y) \leq \gamma g(y,u) \text{ for all } y \in Y, u \in U, \tag{2.5}$$

where V_0 denotes the undiscounted optimal value function, i.e., with $C = 0$. This condition is either used directly, e.g., in [27, 17] or implicitly as the consequence of a controllability condition as, e.g., in [14, 16, 23], see also Section 4 for the relation between controllability and (2.4). Under condition (2.5), it is known that MPC stabilizes the system if the finite optimization horizon is sufficiently long, where the horizon must be the longer the larger γ is. Using that $V \leq V_0$ for all $C > 0$ (due to non-negativity of g), it is easily seen that (2.5) implies (2.4) whenever $C < 1/\gamma$. Hence, the second condition involved in Assumption (2.3) is that C is sufficently small. Particularly, for large γ the discount rate C must be chosen very small, or equivalently, the effective horizon over which we optimize must be very long. This is in perfect accordance with the requirement that a large γ also requires long optimization horizons in MPC.

3 Stability results

In this section we are going to derive the stability results. Before we present the result in its full generality, we briefly explain our key argument under the simplifying assumption that for a given initial value $y_0 \in Y$ the optimal control $u^\star(\cdot)$ exists. Denoting the optimal trajectory by $y^\star(\cdot)$, the dynamic programming principle states that for all $t \geq 0$ we have

$$V(y_0) = \int_0^t e^{-Cs} g(y^\star(s), u^\star(s)) ds + e^{-Ct} V(y^\star(t))$$

implying

$$V(y^\star(t)) = e^{Ct} V(y_0) - e^{Ct} \int_0^t e^{-Cs} g(y^\star(s), u^\star(s)) ds.$$

Since the map $t \mapsto V(y^\star(t))$ is absolutely continuous (see the discussion after Lemma 3.1, below), we can differentiate it for almost all t [20, Chap. IX, § 2, Corollary to Theorem 1] and under Assumption 2.3 we obtain

$$\begin{aligned}\frac{d}{dt}V(y^\star(t)) &= Ce^{Ct}V(y_0) - Ce^{Ct}\int_0^t e^{-Cs}g(y^\star(s),u^\star(s))ds - g(y^\star(t),u^\star(t)) \\ &= CV(y^\star(t)) - g(y^\star(t),u^\star(t)) \;\leq\; -(K-C)V(y^\star(t)).\end{aligned}$$

Then the Gronwall-Bellman inequality implies

$$V(y^\star(t)) \leq e^{-(K-C)t}V(y_0),$$

which tends to 0 as $t \to \infty$ and thus — under suitable additional assumptions detailed in Assumption 3.7, below — implies asymptotic stability. In fact, the optimal value function V is a Lyapunov function of the system.

In general, however, for nonlinear problems one cannot expect that the true optimal control is computable. Often, however, numerical methods may be available which allow us to compute approximately optimal controls in open loop or feedback form. Theorem 3.4 shows that under suitable accuracy requirements, made precise in Assumption 3.3, one can still obtain exponential convergence of $V(y(t))$ to 0. Afterwards, we will introduce conditions under which the convergence $V(y(t)) \to 0$ together with suitable bounds implies asymptotic stability. We will also briefly discuss the case when approximately optimal controls cannot be computed with arbitrary accuracy.

For the proof of Theorem 3.4 we need two preparatory lemmas. For the proof of the first lemma, observe that the definition of J yields

$$J(y_0,u) = \int_0^t e^{-Cs}g(y(s),u(s))ds + e^{-Ct}J(y(t),u(t+\cdot)). \tag{3.1}$$

Lemma 3.1: Assume that $J(y_0,u) \leq V(y_0) + \varepsilon$ for some $\varepsilon > 0$. Then the inequality $J(y(t),u(t+\cdot)) \leq V(y(t)) + e^{Ct}\varepsilon$ holds.

Proof: Assume that $J(y(t),u(t+\cdot)) > V(y(t)) + e^{Ct}\varepsilon$ holds. Then from (3.1) we obtain

$$\begin{aligned}J(y_0,u) &= \int_0^t e^{-Cs}g(y(s),u(s))ds + e^{-Ct}J(y(t),u(t+\cdot)) \\ &> \int_0^t e^{-Cs}g(y(s),u(s))ds + e^{-Ct}V(y(t)) + \varepsilon \\ &\geq \inf_{u(\cdot)\in\mathcal{U}\text{ admissible}}\left\{\int_0^t e^{-Cs}g(y(s),u(s))ds + e^{-Ct}V(y(t))\right\} + \varepsilon \;=\; V(y_0)+\varepsilon\end{aligned}$$

where we used the optimality principle in the last step. This yields a contradiction to the assumption and shows the claim. □

For the proof of the next lemma, observe that (3.1) implies the identity

$$J(y(t),u(t+\cdot)) = e^{Ct}J(y_0) - e^{Ct}\int_0^t e^{-Cs}g(y(s),u(s))ds \tag{3.2}$$

which shows that $t \mapsto J(y(t),u(t+\cdot))$ is an absolutely continuous function. Note that, setting $y = y^\star$ and $u = u^\star$, this implies that $t \mapsto V(y^\star(t))$ is absolutely continuous, provided an optimal control $u^\star$ exists.

Lemma 3.2: Let Assumption 2.3 hold, let $\tau \geq 0$, $T > 0$ and let $u(\cdot)$ and the corresponding trajectory $y(\cdot)$ satisfy

$$J(y(\tau), u(\tau + \cdot)) \leq V(y(\tau)) + \varepsilon(\tau) \tag{3.3}$$

for a value $\varepsilon(\tau) > 0$. Then for all $t \in [0, T]$ the inequality

$$V(y(\tau + t)) \leq e^{-(K-C)t} V(y(\tau)) + 2e^{Ct}\varepsilon(\tau)$$

holds.

Proof: From Lemma 3.1 applied with $y_0 = y(\tau)$ and $u = u(\tau + \cdot)$ we obtain the inequality

$$J(y(\tau + t), u(\tau + t + \cdot)) \leq V(y(\tau)) + e^{Ct}\varepsilon(\tau) \tag{3.4}$$

for all $t \in [0, T]$. Now we abbreviate $J(t) = J(y(t), u(t + \cdot))$ and consider the absolutely continuous function $t \mapsto J(\tau + t)$. Since this function is absolutely continuous, it is differentiable for almost every $t \geq 0$ [20, Chap. IX, § 2, Corollary to Theorem 1] and using Assumption 2.3 and (3.3) we obtain

$$\begin{aligned}
\frac{d}{dt} J(\tau + t) &= -Ce^{Ct} \int_0^t e^{-Cs} g(y(\tau + s), u(\tau + s)) ds - e^{Ct} e^{-Ct} g(y(\tau + t), u(\tau + t)) \\
&\qquad + Ce^{Ct} J(\tau) \\
&= CJ(\tau + t) - g(y(\tau + t), u(\tau + t)) \\
&\leq (C - K) J(\tau + t) + Ke^{Ct}\varepsilon(\tau),
\end{aligned}$$

where in the last step we have used that $-g(y(\tau + t), u(\tau + t)) \leq -KV(y(\tau + t)) \leq -KJ(y(\tau+t), u(\tau+t+\cdot)) + Ke^{Ct}\varepsilon(\tau)$ due to (3.4). Then the Gronwall-Bellman inequality and (3.3) yield

$$\begin{aligned}
J(\tau + t) &\leq e^{-(K-C)t} J(\tau) + K \int_0^t e^{(K-C)(s-t)} e^{Cs} \varepsilon(\tau) ds \\
&= e^{-(K-C)t} J(\tau) + K\varepsilon(\tau) e^{-(K-C)t} \int_0^t e^{(K-C)s} e^{Cs} ds \\
&= e^{-(K-C)t} J(\tau) + K\varepsilon(\tau) e^{-(K-C)t} \frac{1}{K}(e^{Kt} - 1) \\
&= e^{-(K-C)t} J(\tau) + \varepsilon(\tau)(e^{Ct} - e^{-(K-C)t}) \;\leq\; e^{-(K-C)t} J(\tau) + \varepsilon(\tau) e^{Ct}.
\end{aligned}$$

Using (3.3) and $K > 0$, we obtain

$$\begin{aligned}
V(y(\tau + t)) &\leq J(\tau + t) \;\leq\; e^{-(K-C)t} J(\tau) + e^{Ct}\varepsilon(\tau) \\
&\leq e^{-(K-C)t}(V(\tau) + \varepsilon(\tau)) + e^{Ct}\varepsilon(\tau) \\
&= e^{-(K-C)t} V(\tau) + (e^{-Kt} + 1) e^{Ct}\varepsilon(\tau) \;\leq\; e^{-(K-C)t} V(\tau) + 2e^{Ct}\varepsilon(\tau)
\end{aligned}$$

which finishes the proof. $\square$

We note that the optimal control $u^\star(\cdot)$ and the corresponding optimal trajectory $y^\star(\cdot)$ (provided they exist) satisfy (3.3) for all $\tau \geq 0$ with $\varepsilon(\tau) = 0$.

The following assumption defines the "level of accuracy" needed for an approximately optimal control function in order to be asymptotically stabilizing. Examples for constructions of such control functions will be discussed after the subsequent theorem.

Assumption 3.3: Given an initial value $y_0 \in Y$, an admissible control function $u(\cdot) \in \mathcal{U}$, the corresponding trajectory $y(\cdot)$ with $y(0) = y_0$, a value $\sigma > 0$ and times $0 = \tau_0 < \tau_1 < \tau_2 \ldots$ with $0 < \Delta_{\min} \leq \tau_{i+1} - \tau_i \leq \Delta_{\max}$ for all $i \in \mathbb{N}$, we assume that for all $i \in \mathbb{N}$ and all $t \in [0, \tau_{i+1} - \tau_i)$ the function $u(\cdot)$ satisfies (3.3) with $\tau = \tau_i$ and $\varepsilon(\tau) = \sigma e^{-K\Delta_{\max}} V(y(\tau))/2$.

Theorem 3.4: Let Assumption 2.3 hold and let $\sigma > 0$, $\Delta_{\min} > 0$ be such that $\lambda = K - C - \ln(1+\sigma)/\Delta_{min} > 0$. Consider an initial value $y_0 \in Y$ and an admissible control function $u(\cdot) \in \mathcal{U}$ satisfying Assumption 3.3. Then the optimal value function along the corresponding solution $y(\cdot)$ satisfies the estimate

$$V(y(t)) \leq (1+\sigma)e^{-\lambda t} V(y_0)$$

for all $t \geq 0$. Particularly, $V(y(t))$ tends to 0 exponentially fast as $t \to \infty$.

Proof: By Assumption 3.3, $y(\cdot)$ and $u(\cdot)$ satisfy the assumption of Lemma 3.2 for all $\tau = \tau_0, \tau_1, \tau_2, \ldots$ with $\varepsilon(\tau) = \sigma e^{-K\Delta_{\max}} V(y(\tau))/2$ and $T = \tau_{i+1} - \tau_i$. Applying this lemma with $\tau = \tau_i$, for $t \in [0, \tau_{i+1} - \tau_i]$ (implying $t \leq \Delta_{\max}$) we obtain the inequality

$$\begin{aligned} V(y(\tau_i + t)) &\leq e^{-(K-C)t} V(y(\tau_i)) + 2e^{Ct}\sigma e^{-K\Delta_{\max}} V(y(\tau_i))/2 \\ &\leq e^{-(K-C)t}(1+\sigma)V(y(\tau_i)) \quad \leq \quad (1+\sigma)e^{-\lambda t} V(y(\tau_i)). \end{aligned}$$

For $t = \tau_{i+1} - \tau_i$, since $1 + \sigma = e^{(K-C-\lambda)\Delta_{\min}} \leq e^{(K-C-\lambda)t}$, we moreover obtain

$$V(y(\tau_i + t)) \leq e^{-(K-C)t}(1+\sigma)V(y(\tau_i)) \leq e^{-\lambda t} V(y(\tau_i)).$$

From the last inequality a straightforward induction yields

$$V(y(\tau_i)) \leq e^{-\lambda \tau_i} V(y_0)$$

for all $i \in \mathbb{N}$. For arbitrary $t \geq 0$ let $i \in \mathbb{N}$ be maximal with $\tau_i \leq t$ and set $s := t - \tau_i \in [\tau_i, \tau_{i+1})$. Then we obtain

$$V(y(t)) = V(y(\tau_i + s)) \leq (1+\sigma)e^{-\lambda s} V(y(\tau_i)) \leq (1+\sigma)e^{-\lambda s} e^{-\lambda \tau_i} V(y_0) = (1+\sigma)e^{-\lambda t} V(y_0),$$

i.e., the desired inequality. □

The following remark outlines three possibilities of how a control function meeting Assumption 3.3 can be constructed.

Remark 3.5: (i) optimal control If the optimal control $u^\star(\cdot)$ exists, then $u(\cdot) = u^\star(\cdot)$ will satisfy (3.3) with $\varepsilon(\tau) = 0$ for all $\tau \geq 0$. Thus, the optimal trajectory $y^\star(\cdot)$ satisfies the estimate from Theorem 3.4 for $\sigma = 0$.

(ii) moving horizon control Assume that we have a method (e.g., a numerical algorithm) in order to compute approximately optimal open loop control functions, i.e., control functions satisfying $J(y_\tau, u^\tau) \leq V(y_\tau) + \varepsilon$ for small $\varepsilon > 0$, can be computed for any given initial value. Then, given y_0 and $\sigma > 0$, the control $u(\cdot)$ can be constructed by the following algorithm:

For $\tau = 0, 1, 2, \ldots$:

(a) Compute $u^\tau(\cdot)$ such that $\quad J(y_\tau, u^\tau) \leq V(y_\tau) + \sigma e^{-K} V(y_\tau)/2.$

(b) Set $u(t+\tau) := u^\tau(t)$ for $t \in [0,1)$ and $y_{\tau+1} := y^\tau(1)$, where $y^\tau(\cdot)$ solves $\dot y^\tau(t) = f(y^\tau(t), u^\tau(t))$ with $y^\tau(0) = y_\tau$.

A straightforward induction then yields that (3.3) holds for all $\tau = 0, 1, 2, \ldots$ with $\varepsilon(\tau) = \sigma e^{-K} V(y_\tau)/2$. This implies Assumption 3.3 for $\tau_i = i$ and $\Delta_{\min} = \Delta_{\max} = 1$.

(iii) feedback control If an approximately optimal feedback law $F : Y \to U$ is known which generates $u(\cdot)$ via $u(t) = F(y(t))$ and whose accuracy ε can be successively reduced to 0 (see also Remark 3.6, below) as $y(t)$ approaches $\bar y$, then this feedback law can be used in order to construct control functions as in (ii) which satisfy Assumption 3.3. Particularly, this construction applies when an optimal feedback law $F^\star$ is known.

Remark 3.6: In practice it may not be possible to compute u_τ in Remark 3.5(ii) or to evaluate F in Remark 3.5(iii) with arbitrary accuracy, e.g., if numerical methods are used. If $\varepsilon_0 > 0$ denotes the smallest achievable error, then a straightforward modification of the proof of Theorem 3.4 shows that $V(y(t))$ does not converge to 0 but only to an interval around 0 with radius proportional to ε_0.

In order to derive asymptotic stability from Theorem 3.4 we impose the following additional assumption. For its formulation, we recall that a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, unbounded and satisfies $\alpha(0) = 0$.

Assumption 3.7: There are functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that the inequality

$$\alpha_1(\|y - \bar y\|) \leq V(y) \leq \alpha_2(\|y - \bar y\|)$$

holds for all $y \in Y$.

We note that the upper bound in Assumption 3.7 is immediate from Assumption 2.3 provided $y \mapsto \inf_{u \in U} g(y,u)$ satisfies a similar inequality. Typical choices of g like the quadratic penalization (2.3) will always satisfy such a bound. Regarding the lower bound, we show existence for g from (2.3) in the case when f is bounded on $Y \times U$: first observe that

$$J(y, u(\cdot)) \geq \int_0^\infty e^{-Ct} \|y(t) - \bar y\|^2 dt.$$

Since the solution satisfies $y(t) = y_0 + \int_0^t f(y(s), u(s))ds$, we obtain

$$\|y(t) - \bar y\| \geq \|y_0 - \bar y\| - \int_0^t \|f(y(s), u(s))\| ds \geq \|y_0 - \bar y\| - Mt$$

for $M := \sup_{(y,u) \in Y \times U} \|f(y,u)\|$. Choosing $\tau = \min\{\|y_0 - \bar y\|/(2M), 1\}$, for all $t \in [0, \tau]$ we obtain

$$\|y(t) - \bar y\| \geq \|y_0 - \bar y\| - \frac{1}{2}\|y_0 - \bar y\| \geq \frac{1}{2}\|y_0 - \bar y\|.$$

Together this yields

$$\begin{aligned} J(y, u(\cdot)) &\geq \int_0^\tau e^{-Ct} \|y(t) - \bar y\|^2 dt \;\geq\; e^{-C\tau} \int_0^\tau \frac{1}{4}\|y_0 - \bar y\|^2 dt \\ &\geq e^{-C} \frac{1}{4} \|y_0 - \bar y\|^2 \min\{\|y_0 - \bar y\|/(2M),\, 1\} \end{aligned}$$

i.e., the lower bound in Assumption 3.7 with $\alpha_1(r) = e^{-C}\min\{r^3/(8M), r/4\}$. We note that for any fixed $C_{\max} > 0$ this bound is uniform for all $C \in (0, C_{\max}]$.

The following theorem shows that under Assumption 3.7 the assertion of Theorem 3.4 implies asymptotic stability.

Theorem 3.8: Let Assumptions 2.3 and 3.7 hold and let $\sigma > 0$ be such that $\lambda = -(K - C) + \ln(1+\sigma) < 0$. Then the point $\bar{y}$ is asymptotically stable for the trajectories $y(\cdot)$ corresponding to the control functions $u(\cdot)$ satisfying Assumption 3.3.

Proof: Convergence $y(t) \to \bar{y}$ follows immediately from the fact that $V(y(t)) \to 0$ and $\|y(t) - \bar{y}\| \le \alpha_1^{-1}(V(y(t)))$, noting that the inverse function of a $\mathcal{K}_\infty$ function is again a $\mathcal{K}_\infty$ function.

In order to prove stability, let $\varepsilon > 0$. For all $t \ge 0$ we have

$$\|y(t) - \bar{y}\| \le \alpha_1^{-1}(V(y(t))) \le \alpha_1^{-1}((1+\sigma)V(y_0)) \le \alpha_1^{-1}((1+\sigma)\alpha_2(\|y_0 - \bar{y}\|)).$$

Thus, for $\|y_0 - \bar{y}\| \le \delta = \alpha_2^{-1}(\alpha_1(\varepsilon)/(1+\sigma))$ we obtain $\|y(t) - \bar{y}\| \le \varepsilon$ and thus the desired stability estimate. $\square$

Remark 3.9: In the situation of Remark 3.6, convergence $y(t) \to \bar{y}$ may no longer hold. Instead, $y(t)$ converges to a neighbourhood of $\bar{y}$ whose size depends on ε_0, a property known as *practical asymptotic stability.*

4 Controllability conditions

In this section we give sufficient controllability conditions under which Assumption 2.3 holds for sufficiently small discount rate $C > 0$. We will provide both finite time and exponential controllability conditions. For sake of conciseness, we restrict ourselves to the quadratic running cost (2.3) for which we will consider both the case $\lambda = 0$ and $\lambda > 0$. For further cost functions as well as alternative controllability properties we refer to the upcoming PhD thesis of the third co-author.

4.1 Finite time controllability

Assumption 4.1: Let $Y \times U$ be compact and assume there exists $\beta > 0$ such that for any initial condition $y(0) = y_0 \in Y$ there exists an admissible control $\hat{u}(\cdot) \in \mathcal{U}$ which will drive our system from y_0 to $\bar{y}$ in time $t(y_0) \le \beta\|y_0 - \bar{y}\|^2$.

Proposition 4.2: Under Assumption 4.1, the optimal value function for g from (2.3) with any $\lambda \ge 0$ satisfies Assumption 2.3 for all $0 < C < \frac{1}{(1+\lambda)M\beta}$.

Proof: Let $\hat{y}(\cdot)$ denote the solution corresponding to $\hat{u}(t)$ starting in y_0. Since Y and U are assumed to be compact, for any $(y, u) \in Y \times U$ there exists a constant M such that $\|\hat{y}(t) - \bar{y}\|^2 \le M$ and $\|\hat{u}(t) - \bar{u}\|^2 \le M$. Using, moreover, the inequality $1 - e^{-x} \le x$ we obtain

$$
\begin{aligned}
V(y) &\le \int_0^{t(y)} e^{-C\tau}\left(\|\hat{y}(\tau)-\bar{y}\|^2+\lambda\|\hat{u}(\tau)-\bar{u}\|^2\right)d\tau \;\le\; (1+\lambda)M\int_0^{t(y)} e^{-C\tau}d\tau \\
&= \frac{(1+\lambda)M}{C}[1-e^{-Ct(y)}] \;\le\; (1+\lambda)Mt(y) \\
&\le (1+\lambda)M\beta\|y-\bar{y}\|^2 \;\le\; (1+\lambda)M\beta g(y,u)
\end{aligned}
$$

implying (2.4) with $K=\frac{1}{(1+\lambda)M\beta}$. For Assumption 2.3 to be satisfied, we need $K>C$. Hence, the assumption holds whenever

$$C<\frac{1}{(1+\lambda)M\beta}.$$

□

4.2 Exponential controllability

Assumption 4.3: (i) For any initial condition $y(0)=y_0\in Y$ there exists an admissible control $\hat{u}(\cdot)\in\mathcal{U}$ which will exponentially drive our system from y_0 to $\bar{y}$, i.e., such that the corresponding solution $\hat{y}(\cdot)$ satisfies

$$\|y(t)-\bar{y}\|\le Me^{-\delta t}\|y_0-\bar{y}\|$$

where $\delta>0$ and $M\ge 1$.

(ii) The control functions from (i) satisfy

$$\|\hat{u}(t)-\bar{u}\|\le Me^{-\delta t}\|y_0-\bar{y}\|$$

with $\delta>0$ and $M\ge 1$ from (i).

Proposition 4.4: Under Assumption 4.3(i), the optimal value function for g from (2.3) with $\lambda=0$ satisfies Assumption 2.3 for all $0<C<\frac{2\delta}{(M^2-1)}$. If, in addition, Assumption 4.3(ii) holds, then Assumption 2.3 also holds for any $\lambda>0$ for all $0<C<\frac{2\delta}{((1+\lambda)M^2-1)}$.

Proof: For $\lambda=0$ we have

$$
\begin{aligned}
V(y) &\le \int_0^{\infty} e^{-C\tau}\|\hat{y}(\tau)-\bar{y}\|^2d\tau \;\le\; \int_0^{\infty} e^{-C\tau}M^2e^{-2\delta\tau}\|y_0-\bar{y}\|^2d\tau \\
&= \frac{M^2}{C+2\delta}\|y_0-\bar{y}\|^2 \;\le\; \frac{M^2}{C+2\delta}g(y,u).
\end{aligned}
$$

and in case Assumption 4.3(ii) holds, for any $\lambda>0$ we have

$$
\begin{aligned}
V(y) &\le \int_0^{\infty} e^{-C\tau}[\|\hat{y}(\tau)-\bar{y}\|^2+\lambda\|\hat{u}(\tau)-\bar{u}\|^2]d\tau \\
&\le \int_0^{\infty} e^{-C\tau}M^2e^{-2\delta\tau}(1+\lambda)\|y_0-\bar{y}\|^2d\tau \\
&= \frac{(1+\lambda)M^2}{C+2\delta}\|y_0-\bar{y}\|^2 \;\le\; \frac{(1+\lambda)M^2}{C+2\delta}g(y,u).
\end{aligned}
$$

Thus, in both cases we obtain (2.4) with

$$K = \frac{C + 2\delta}{(1+\lambda)M^2}$$

For Assumption 2.3 to be satisfied, we again need $K > C$, which holds if

$$\frac{C + 2\delta}{(1+\lambda)M^2} > C \quad \Leftrightarrow \quad C < \frac{2\delta}{((1+\lambda)M^2 - 1)}.$$

$\square$

5 Numerical example

Consider the following control system

$$\dot{y}_1(t) = -y_1(t) + y_1(t)y_2(t) - y_1(t)u(t), \tag{5.1}$$

$$\dot{y}_2(t) = +y_2(t) - y_1(t)y_2(t), \tag{5.2}$$

where

$$u \in U = [0,1] \subset \mathbb{R}^1, \tag{5.3}$$

$$y = (y_1, y_2) \in Y = \{(y_1, y_2) \colon y_1 \in [0.6, 1.6], y_2 \in [0.6, 1.6]\} \subset \mathbb{R}^2. \tag{5.4}$$

With $u(t) \equiv 0$, the system (5.1)-(5.2) becomes the Lotka-Volterra equations, the general solution of which has the form of closed curves described by the equality (see also Figure 5.1 below)

$$\ln y_2(t) - y_2(t) + \ln y_1(t) - y_1(t) = K. \tag{5.5}$$

It can be readily seen that the set $\mathcal{S}$ of steady state admissible pairs, that is the pairs $(\bar{y}, \bar{u}) \in Y \times U$ such that $f(\bar{y}, \bar{u}) = 0$ is defined by the equation

$$\mathcal{S} = \{(\bar{y}, \bar{u}) \colon \bar{y} = (1, \bar{u} + 1), \ \forall \bar{u} \in [0, \ 0.6]\}.$$

Consider the problem of stabilizing the system to the point $\bar{y} = (1, \ 1.26)$ from the initial condition $y_0 = (1.4, \ 1.4)$. In accordance with results described above, the stabilizing control can be found via solving the optimal control problem

$$\inf_{u(\cdot) \in \mathcal{U} \text{ admissible}} \int_0^\infty e^{-Ct}[(y_1(t) - 1)^2 + (y_2(t) - 1.26)^2 + (u(t) - 0.26)^2] dt. \tag{5.6}$$

It can be checked (using a local analysis via linearization and a global analysis via La Salle's invariance principle with (5.5) as a Lyapunov function) that the desired steady state is exponentially stable using the linear feedback $u = k(y_1 - \bar{y}_1) + 0.26$ for any $k > 0$. Hence, the system is exponentially controllable and thus for sufficiently small $C > 0$ Proposition 4.4 implies Assumption 2.3. From this, from the boundedness of f and from the quadratic form of g the bounds in Assumption 3.7 follow, cf. the discussion after this assumption. Note that this analysis does not provide us with an easily computable explicit bound on C. Moreover, this bound would be conservative, given that our assumptions are sufficient but

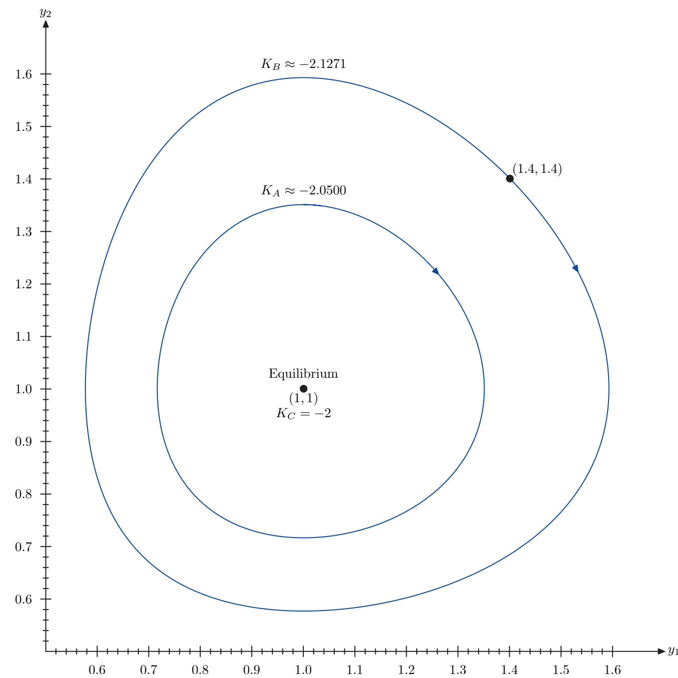

Figure 5.1: Two Lotka-Volterra closed curves characterised by the constants $K_A \approx -2.0500$ and $K_B \approx -2.1271$. The evolution of state is in a clockwise direction about the equilibrium point at $(1, 1)$ which is associated with the constant $K_C = -2$.

not necessary. To this end and also to investigate the effect of C on the optimally controlled dynamics, we next investigate the optimal dynamics numerically for varying $C > 0$.

From results of [10] and [11] it follows that a near optimal solution of the problem (5.6) can be constructed on the basis of the solution of the semi-infinite (SI) linear programming (LP) problem

$$\min_{\gamma \in W_K(y_0)} \int_{U\times Y} [(y_1 - 1)^2 + (y_2 - 1.26)^2 + (u - 0.26)^2] \gamma(dy_1, dy_2, du), \tag{5.7}$$

where

$$\begin{aligned} W_K(y_0) = \{\gamma \in \mathcal{P}(Y \times U) \ : \ \int_{U\times Y} [\frac{\partial(y_1^{l_1} y_2^{l_2})}{\partial y_1}(-y_1 + y_1 y_2 - y_1 u_) \\ + \frac{\partial(y_1^{l_1} y_2^{l_2})}{\partial y_2}(y_2 - y_1 y_2) + C(1.4^{l_1 + l_2} - (y_1^{l_1} y_2^{l_2}))] \gamma(dy_1, dy_2, du) \ = \ 0 \\ \forall \ \text{integers} \ \ l_1, l_2 \quad \text{such that} \quad 0 \le l_1 + l_2 \le K\}. \end{aligned} \tag{5.8}$$

Here, $\mathcal{P}(Y \times U)$ is the space of probability measures defined on Borel subsets of $Y \times U$ and $K \in \mathbb{N}$ determines the accuracy of the solution. The problem dual to the SILP problem (5.7) is of the form

$$\sup_{\mu,\lambda_{l_1,l_2}} \{\mu \ : \ \mu \ \leq \ (y_1-1)^2+(y_2-1.26)^2+(u-0.26)^2+\sum_{0\leq l_1+l_2\leq K} \lambda_{l_1,l_2}[\frac{\partial(y_1^{l_1}y_2^{l_2})}{\partial y_1}(-y_1+y_1y_2-y_1u)$$

$$+\frac{\partial(y_1^{l_1}y_2^{l_2})}{\partial y_2}(y_2-y_1y_2)+C(1.4^{l_1+l_2}-(y_1^{l_1}y_2^{l_2}))] \ \ \forall(y_1,y_2)\in Y, \quad \forall u\in U\}. \tag{5.9}$$

Denote by $\{ \ \mu^K, \lambda^K_{l_1,l_2}\}$ an optimal solution of the problem (5.9) and denote by $\psi^K(y)$ the function

$$\psi^K(y) = \sum_{0\leq l_1+l_2\leq K} \lambda^K_{l_1,l_2} y_1^{l_1} y_2^{l_2}.$$

Denote also

$$a^K(y_1,y_2) = \frac{1}{2}\Big(\frac{\partial\psi^K(y_1,y_2)}{\partial y_1}\Big)y_1 + 0.26. \tag{5.10}$$

In [11] it has been shown that the control

$$u^K(y) = \begin{cases} a^K(y_1,y_2), & \text{if} \quad 0\leq a^K(y_1,y_2) \ \ \leq 1, \\ 0, & \text{if} \qquad a^K(y_1,y_2) \ \ <0, \\ 1, & \text{if} \qquad a^K(y_1,y_2) \ \ >1. \end{cases} \tag{5.11}$$

that minimizes the expression

$$\min_{u\in U}\{(u-0.26)^2+\frac{\partial\psi^K(y)}{\partial y_1}(-y_1u)\}$$

can serve as an approximation for the optimal control for K large enough.

The SILP problem (5.7) and its dual problem (5.9) were solved (using a simplex method based technique similar to one used in [10] and [11]) for different values of the discount rates C . The resultant state trajectories are depicted in Figure 5.2.

As one can see in all the cases, the state trajectories converge to the selected steady state point $(1,\ 1.26)$. We note that the approximately optimal control is in feedback form here, hence our stability theorem applies due to Remark 3.5(ii). The deviation from $\bar{y}$ caused by the limited numerical accuracy as discussed in Remark 3.6 does not have a visible effect in this example.

6 Conclusion

In this paper we have given a condition under which discounted optimal control problems can be used for the stabilization of nonlinear systems with the optimal value function acting as a control Lyapunov function. The condition is similar to those found in the model predictive control literature for MPC schemes without terminal conditions. A numerical example has illustrated the performance of the resulting controls.

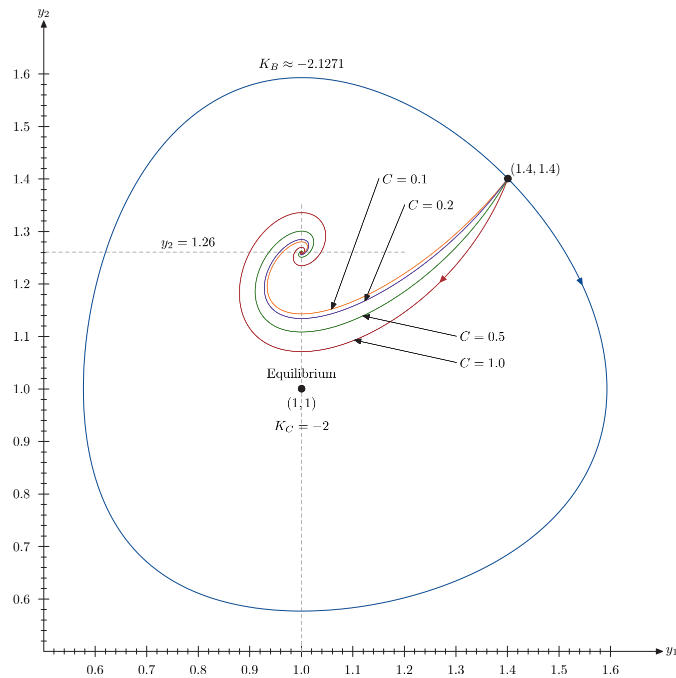

Figure 5.2: Near-optimal state trajectories for problem (5.7)

References

[1] B. D. O. Anderson and J. Moore, *Optimal control: linear quadratic methods*, Prentice-Hall, Engelwood Cliffs, 1990.

[2] J. T. Betts, *Practical methods for optimal control and estimation using nonlinear programming*, SIAM, Philadelphia, second ed., 2010.

[3] W. A. Brock and J. Sheinkman, *Global asymptotic stability of optimal control systems with applications to the theory of economic growth*, J. Econ. Theory, 12 (1976), pp. 164–190.

[4] F. Camilli, L. Grüne, and F. Wirth, *A regularization of Zubov's equation for robust domains of attraction*, in Nonlinear Control in the Year 2000, Volume 1, A. Isidori, F. Lamnabhi-Lagarrigue, and W. Respondek, eds., Lecture Notes in Control and Information Sciences 258, NCN, Springer-Verlag, London, 2000, pp. 277–290.

[5] F. Camilli, L. Grüne, and F. Wirth, *Control Lyapunov functions and Zubov's method*, SIAM J. Control Optim., 47 (2008), pp. 301–326.

[6] D. A. Carlson, A. B. Haurie, and A. Leizarowitz, *Infinite horizon optimal control — Deterministic and Stochastic Systems*, Springer-Verlag, Berlin, second ed., 1991.

[7] D. Cass and K. Shell, *The structure and stability of competitive dynamical system*, J. Econ. Theory, 12 (1976), pp. 30–70.

[8] M. Falcone, *Numerical solution of dynamic programming equations.* Appendix A in Bardi, M. and Capuzzo Dolcetta, I., Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations, Birkhäuser, Boston, 1997.

[9] M. Falcone and R. Ferretti, *Semi-Lagrangian approximation schemes for linear and Hamilton-Jacobi equations*, SIAM, Philadelphia, 2013.

[10] A. Gaitsgory and M. Quincampoix, *Linear programming approach to deterministic infinite horizon optimal control problems with discounting*, SIAM J. Control Optim., 48 (2009), pp. 2480–2512.

[11] V. Gaitsgory, S. Rossomakhine, and N. Thatcher, *Approximate solutions of the HJB inequality related to the infinite horizon optimal control problem with discounting*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms, 19 (2012), pp. 65–92.

[12] D. Grass, J. P. Caulkins, G. Feichtinger, G. Tragler, and D. A. Behrens, *Optimal control of nonlinear processes*, Springer-Verlag, Berlin, 2008.

[13] G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel, *Model predictive control: for want of a local control Lyapunov function, all is not lost*, IEEE Trans. Automat. Control, 50 (2005), pp. 546–558.

[14] L. Grüne, *Analysis and design of unconstrained nonlinear MPC schemes for finite and infinite dimensional systems*, SIAM J. Control Optim., 48 (2009), pp. 1206–1228.

[15] L. Grüne and J. Pannek, *Nonlinear Model Predictive Control. Theory and Algorithms*, Springer-Verlag, London, 2011.

[16] L. Grüne, J. Pannek, M. Seehafer, and K. Worthmann, *Analysis of unconstrained nonlinear MPC schemes with time varying control horizon*, SIAM J. Control Optim., 48 (2010), pp. 4938–4962.

[17] L. Grüne and A. Rantzer, *On the infinite horizon performance of receding horizon controllers*, IEEE Trans. Automat. Control, 53 (2008), pp. 2100–2111.

[18] A. Jadbabaie and J. Hauser, *On the stability of receding horizon control with a general terminal cost*, IEEE Trans. Automat. Control, 50 (2005), pp. 674–678.

[19] R. Kalman, *Contributions to the theory of optimal control*, Bol. Soc. Mat. Mex, 5 (1960), pp. 102–119.

[20] I. P. Natanson, *Theory of functions of a real variable*, Frederick Ungar Publishing Co., New York, 1955. Translated by Leo F. Boron with the collaboration of Edwin Hewitt.

[21] R. Postoyan, L. Buşoniu, D. Nešić, and J. Daafouz, *Stability of infinite-horizon optimal control with discounted cost*, in Proceedings of the 53rd IEEE Conference on Decision and Control, 2014, pp. 3903–3908.

[22] J. B. Rawlings and D. Q. Mayne, *Model Predictive Control: Theory and Design*, Nob Hill Publishing, Madison, Wisconsin, 2009.

[23] M. Reble and F. Allgöwer, *Unconstrained model predictive control and suboptimality estimates for nonlinear continuous-time systems*, Automatica, 48 (2011), pp. 1812–1817.

[24] R. T. Rockafellar, *Saddle points of Hamiltonian systems in convex Lagrange problems having nonzero discount rate*, J. Econ. Theory, 12 (1976), pp. 71–113.

[25] E. D. Sontag, *A Lyapunov-like characterization of asymptotic controllability*, SIAM J. Control Optim., 21 (1983), pp. 462–471.

[26] E. D. Sontag, *Mathematical Control Theory*, Springer Verlag, New York, 2nd ed., 1998.

[27] S. E. Tuna, M. J. Messina, and A. R. Teel, *Shorter horizons for model predictive control*, in Proceedings of the 2006 American Control Conference, Minneapolis, Minnesota, USA, 2006.

[28] A. Zaslavski, *Turnpike Phenomenon and Infinite Horizon Optimal Control*, Springer International Publishing, 2014.