Computation of Lyapunov functions and stability of interconnected systems

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Introduction

Lyapunov's Second Method, also commonly known as Lyapunov's Direct Method [75] (see also [45, 67, 85, 108]) has proven to be one the most useful tools for demonstrating stability properties. This is largely due to the fact that if one has a Lyapunov function at hand there is no need to explicitly generate system solutions in order to determine stability. Moreover, an estimate of the domain of attraction may be obtained via a Lyapunov function. Inspired by these properties of Lyapunov functions, researchers have been investigating the problem of computation of Lyapunov functions. As a consequence, various methods to compute Lyapunov functions have been proposed such as computation of Lyapunov functions by solving a partial differential equation with collocation [61, 26], graph theoretic methods for computation of complete Lyapunov functions [6, 62], and semidefinite optimization for sum-of-squares polynomials (known as the SOS method) for systems described by polynomial ordinary differential equations [82, 83]. Aside from these there are two methods we are particularly interested in:

Zubov's method:

In Section 34 of [45], Hahn reports that Zubov ([109, 111, 110]) finds that we can exactly determine the boundary of the domain of attraction with the help of a Lyapunov function. Such a Lyapunov function can be obtained by solving a partial differential equation (Hamilton-Jacobi-Bellman equation) (See [45, Theorem 34.1]), i.e., the domain of attraction of an asymptotically stable fixed point x^* of

$$\dot{x} = f(x) \tag{1}$$

could be characterized by the solution V of the partial differential equation

$$\langle \nabla V(x), f(x) \rangle = -h(x)(1 - V(x))\sqrt{1 + \|f(x)\|_2^2}. \tag{2}$$

Assuming $h(x)$ satisfies suitable conditions, the set $V^{-1}([0, 1))$ is equal to the domain of attraction. Since the solution of such a partial differential equation can be attained explicitly or be approximated by a numerical solution, Theorem 34.1 from [45] is very applicable in practice. Based on Zubov's method, numerical approaches of the approximation of the domain of attraction were developed in [46, 105, 66]. Zubov's method has been extended to compute robust Lyapunov functions and robust domain of attraction for dynamic systems with perturbations (see [9]). A robust Lyapunov function can be characterized as a unique viscosity solution of a partial differential equation. This result straightforwardly generalizes the classical Zubov equation. Zubov's method has been further extended to the computation of control Lyapunov functions for systems which are uniformly locally asymptotically null-controllable in [32]. Moreover, in [33] this method is used to compute Lyapunov functions for a finite nonlinear controlled systems subject to perturbation and state constraints. From these results, it is evident that Zubov's method plays an important role in the construction of Lyapunov functions and estimate of the domain of attraction.

Continuous and piecewise affine (CPA) method:

The CPA approach to construct Lyapunov functions for dynamic systems involves partitioning the state-space into a suitable triangulation (See Definition 1.4.4), defining values for the vertices of every simplex, and for every simplex taking the convex interpolation of those values at simplex vertices. This yields a continuous and piecewise affine (CPA) function. If the values at the vertices satisfy a system-dependent set of linear inequalities, then the resulting CPA function is a Lyapunov function.

Since the interpolation errors are incorporated in these linear inequalities, an important property of the CPA method is that it can deliver a true Lyapunov function, instead of a numerical approximation.

Based on the results of these two methods, we will study the following problems:

1. computation of CPA Lyapunov functions by the CPA method and construction from a converse Lyapunov theorem,

2. computation of iISS Lyapunov functions via the generalized Zubov's method and auxiliary systems,

3. computation of CPA ISS Lyapunov functions with the CPA method by solving linear optimization problems,

4. computation of iISS or (CPA) ISS Lyapunov functions for each subsystem of interconnected systems by the above two approaches for the investigation of stability of interconnected systems and estimate of the domain of attraction.

The results of this thesis are described in more details in the following.

Computation of CPA Lyapunov functions using Yoshizawa construction

An approach using linear programming to compute feasible values for the CPA function at the simplex vertices is proposed in [76] with refinements in [42, 5, 29]. This approach was then extended to discrete time systems in [28]. In each case, a Lyapunov function is obtained by solving a linear optimization problem. In these linear optimization problems, the values at the vertices of each simplex are introduced as variables, and the corresponding system-dependent set of linear inequalities are considered as constraints.

As the size of the linear programming increase, the cost of computation becomes more expensive. We investigate whether values at the vertices can be fixed by a less expensive method, with a subsequent fast test of the validity of the linear inequalities.

Classical converse Lyapunov theorems such as those developed by Massera [77] and Kurzweil [67] rely on integrating solutions (summing solution sequences) from the initial time to infinity. Yoshizawa [108] provided an alternative construction that involves taking the supremum over time of the norm of the solution. Initially this appears to provide no improvement towards a constructive approach, but it can be shown that this supremum is actually a maximum over a finite-time horizon. Furthermore, in many cases, this horizon may not be overly long. Thus, in Chapter 2, we construct a continuous and piecewise affine Lyapunov function based on a construction from the converse Lyapunov theorem first proposed by Yoshizawa.

We propose a method for constructing CPA Lyapunov functions for both continuous time and discrete time dynamic systems using the Yoshizawa construction for the values at the simplex vertices and subsequently verifying that the obtained CPA Lyapunov function is a

Lyapunov function by checking the validity of the linear inequalities from Theorems 2.1.4 and 2.2.4. Theorems 2.1.7 and 2.2.8 demonstrate that this method will always succeed if the CPA function has enough structure, i.e., if the triangulation has a sufficient number of vertices, and if the Yoshizawa construction meets certain conditions.

Stability of two interconnected systems and estimate of the domain of attraction

For higher dimensional systems, the direct computation of Lyapunov functions by the above methods becomes very expensive. In order to avoid expensive computation, we consider the whole system as a set of interconnected subsystems. Each subsystem is considered as a dynamic system with perturbations by treating other states' influence as perturbation. We then study stability of the whole system in terms of the stability of the subsystems and their interconnection.

In [78], it is summarized that one can construct a scalar or vector Lyapunov function for the whole system by imposing certain conditions on Lyapunov functions for each free subsystem (i.e., systems without inputs). In [107], small-gain-type theorems with linear gains are proposed to study general interconnected systems. If the spectral radius of the gain matrix is less than one, then the whole system is asymptotically stable.

In this work, we will analyse stability of interconnected systems by iISS or ISS small gain theorems. The concept of input to state stability (ISS) was first introduced by Sontag [88] in the late 1980s and has soon turned out to be one of the most influential concepts for characterizing stability of nonlinear systems with perturbations. Various types of ISS small gain theorem were then proposed such as [17, 18, 19, 57, 59, 57] where stability analysis of interconnected systems is presented.

Another notion playing an important part in investigating stability of interconnected systems is integral input to state stability (iISS). The concept was first proposed in [92]. The properties of iISS are described in [4]. iISS small gain theorems used to analyse the stability of interconnected systems were established e.g., in [51, 53].

In [13], ISS Lyapunov functions in implication formulation for dynamic systems with perturbations were obtained by the introduction of a suitable auxiliary system and Zubov's method for perturbed systems proposed in [9]. Stability of interconnected systems is then investigated by an ISS small gain theorem. Inspired by this idea, we propose a new technique for computing ISS Lyapunov functions in dissipative form as introduced in [73]. Based on this result, we consider how to construct iISS and ISS Lyapunov functions by Zubov's method for perturbed systems in Chapter 3.

In [3], the stability of two interconnected one dimensional systems is investigated. This result lays a foundation for the stability analysis of two iISS interconnected systems. Therefore, we restrict our attention to stability analysis of two interconnected systems in Chapter 3. We assume each subsystem is iISS. By introducing an auxiliary system for each subsystem which is uniformly asymptotically stable, we construct a robust Lyapunov function for the auxiliary system by Zubov's method for perturbed system. We then in Proposition 3.3.8 prove that such a robust Lyapunov function for the auxiliary system is a local iISS Lyapunov function for a fixed subsystem. Based on iISS Lyapunov functions for subsystems obtained by our proposed approach, we study stability of the whole system by a small gain theorem in comparison form, cf. Theorem 3.4.3. Moreover, an estimate of the domain of attraction of interconnected systems can be obtained.

By choosing appropriate comparison functions, we find that such an iISS Lyapunov function is also a local ISS Lyapunov function. Under certain conditions, we conclude in Theorem 3.5.2 that the interconnected system is asymptotically stable according to the small gain theorem in dissipative form, cf. Theorem 3.5.1. Furthermore, the domain of attraction can be estimated.

Computation of ISS Lyapunov functions, stability of interconenected systems, and estimate of the domain of attraction

In Chapter 3, we propose an approach to compute ISS Lyapunov functions which are solutions to partial differential equations (Hamilton-Jacobi-Bellman equations). In general, the solution to the Hamilton-Jacobi-Bellman equation is obtained numerically. Thus the computed ISS Lyapunov function is a numerical approximation of an ISS Lyapunov function but not a true ISS Lyapunov function. For discrete time systems, following the same auxiliary system approach, true ISS Lyapunov functions can be computed by a set oriented approach (see [36]). This numerical approach, however, does not carry over to the continuous time setting. Moreover, the detour via the auxiliary system introduces conservatism, since the resulting Lyapunov function and ISS gains strongly depend on the way the auxiliary system is constructed.

We thus propose a linear programming based algorithm for computing true ISS Lyapunov functions without introducing auxiliary systems. The approach for the computation of continuous and piecewise affine (CPA) Lyapunov functions by solving a linear optimization problem is first presented in [76]. In [40], it is proved that for exponentially stable equilibria the corresponding linear optimization problem always has a feasible solution. This result was extended to asymptotically stable systems [41], to asymptotically stable, arbitrarily switched, non-autonomous systems [42], and to asymptotically stable differential inclusions [5]. In these papers, true Lyapunov functions are constructed on compact subsets of the state space except possibly an arbitrarily small neighbourhood of the asymptotically stable equilibrium. Mainly inspired by [5], we propose an analogous linear programming based algorithm for computing CPA ISS Lyapunov functions for dynamic systems with perturbations.

We formulate a linear programming based algorithm for computing CPA ISS Lyapunov functions for continuous time dynamic systems with perturbations in Section 4.2. The algorithm relies on a linear optimization problem. We prove that the solution delivered by the linear optimization problem is a CPA ISS Lyapunov function for the considered continuous time dynamic system with perturbation on a compact set of state space excluding a small neighbourhood of the equilibrium in Theorem 4.2.6. Furthermore, we prove in Theorem 4.2.9 that if system has a C^2 ISS Lyapunov function, then the algorithm for the linear optimization problem always terminates successfully.

In Section 4.3, we extend the linear programming based algorithm for computing a CPA ISS Lyapunov function to discrete time systems with perturbations. Since the solution of the difference equation is a sequence of points rather than an absolutely continuous function, the constraints in the discrete time linear optimization problem (4.69) turn out to be stricter than the constraints in the continuous time linear optimization problem (4.36). Under appropriate conditions, we prove in Theorem 4.3.7 that the solution delivered by the algorithm is a CPA ISS Lyapunov function for the considered discrete time system with perturbation on a compact set of state space excluding a small neighbourhood of the origin. Moreover, the algorithm has a feasible solution, if discrete time system has a C^1 ISS Lyapunov function with bounded gradient.

These algorithms yields CPA ISS Lyapunov functions with linear gains, cf. (4.8) and (4.9). Based on these, in Section 4.4, we investigate the stability of the interconnected system by the small gain theorem in linear form, cf. Theorem 1.6.1. Furthermore, we estimate the domain of attraction of a small neighbourhood of the equilibrium.

Some of results presented this work have been published or submitted for publication in preliminary form; the construction of iISS Lyapunov functions by Zubov's method (see [73]), the construction of Lyapunov functions using Yoshizawa function (see [43, 72, 44]), and the computation of ISS Lyapunov function for continuous time dynamical systems with perturbations by linear programming (see [71, 70]).

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1 Preliminaries

Lyapunov's Second Method, also commonly known as Lyapunov's Direct Method is a very useful tool in stability analysis of dynamical systems. If there is a Lyapunov function at hand for a dynamic system, then the stability of the system is easily analysed. In this thesis, we are concerned with the computation of Lyapunov functions and stability analysis of interconnected systems.

In this chapter, we present basic notations and preliminary results from stability theory which serves as the foundation for this research. We list notations and recall concepts of comparison functions which are widely used in stability analysis in Section 1.1. We describe dynamic systems in Section 1.2. Definitions of stability and corresponding Lyapunov functions are presented in Section 1.3. In Section 1.4, we introduce the definition of nonsmooth Lyapunov functions based on the concept of Clarke's subdifferential, definitions of continuous and piecewise affine (CPA) function and CPA Lyapunov functions. In Section 1.5, we present a particular Lyapunov construction, i.e., Yoshizawa construction from a converse Lyapunov theorem. Finally, in Section 1.6, we describe three versions of small gain theorems which will be used to investigate stability of interconnected systems in Chapter 3 and Chapter 4.

1.1 Notations and comparison functions

Let $\mathbb{R}_{>0}$ and $\mathbb{R}_+$ denote the intervals $(0, +\infty)$ and $[0, +\infty)$, respectively. We denote positive integers and nonnegative integers by $\mathbb{Z}_{>0}$ and $\mathbb{Z}_+$, respectively. Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq 0\}$. For $x \in \mathbb{R}^n$, its transpose is denoted by $x^\top$. We define the *norms* $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$ and $\|x\|_\infty := \max_{i\in\{1,\dots,n\}} |x_i|$. The *induced matrix norm* is defined by $\|A\|_p := \max_{\|x\|_p=1} \|Ax\|_p$. By $\|u\|_{\infty,p} := \operatorname{ess\,sup}_{t\geq 0} \|u(t)\|_p$ we denote the *essential supremum norm* of a measurable function u. Let $\mathcal{B}_p(z, r) := \{x \in \mathbb{R}^n \,|\, \|x - z\|_p < r\}$ denote the set of points with distance less than r from z in the norm $\|\cdot\|_p$. The *inner product of* $x \in \mathbb{R}^n$ *and* $y \in \mathbb{R}^n$ is denoted as $\langle x, y\rangle$. For functions α_1, $\alpha_2 : \mathbb{R}_+ \to \mathbb{R}_+$, we use $\circ$ to denote *the composition of functions* α_1 *and* α_2, i.e., $\alpha_1 \circ \alpha_2(s)$ for all $s \geq 0$. For vectors $x, y \in \mathbb{R}^n$, the relationship $x \geq y$ is defined by $x_i \geq y_i$, $i \in \{1, \cdots, n\}$. The relations $\leq, <, >, =$ are defined in the same way. Given vectors $x, y \in \mathbb{R}^n$, $x \ngeq y$ means there exists at least one $j \in \{1, \cdots, n\}$ such that $x_j < y_j$. Similarly, the relationship $x \nleq y$ indicates there exists at least one $j \in \{1, \cdots, n\}$ such that $x_j > y_j$. A map $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is defined as a *monotone map*, if $\Gamma(x) \leq \Gamma(y)$ for $x \leq y$, $x, y \in \mathbb{R}^n_+$. The *identity function* is denoted by Id. For a set $\Omega \subset \mathbb{R}^n$, we denote the interior of Ω by Ω°, the closure of Ω by $\overline{\Omega}$, the boundary of Ω by $\partial\Omega$, and the complement of Ω by Ω^C. For $x, y \in \mathbb{R}^n$ with norm $\|\cdot\|_p$, we define the distance between x and y by $\operatorname{dist}(x, y) = \|x - y\|_p$. For $x \in \mathbb{R}^n$, and a compact and connected set $\mathcal{D} \in \mathbb{R}^n$ with norm $\|\cdot\|_p$, we let $\operatorname{dist}(x, \mathcal{D}) := \inf\{\operatorname{dist}(x, y) | y \in \mathcal{D}\}$ denote the distance between the point x and the set $\mathcal{D}$.

The following comparison function concepts play an important role in stability analysis of dynamic systems.

A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *positive definite* if it satisfies $\alpha(0) = 0$

and $\alpha(s) > 0$ for all $s > 0$. Let $\mathcal{P}$ denote the set of all positive definite functions.

A positive definite function α is of *class* $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is strictly increasing and of *class* $\mathcal{K}_\infty(\alpha \in \mathcal{K}_\infty)$ if it is of class $\mathcal{K}$ and unbounded.

A continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of *class* $\mathcal{L}$ ($\gamma \in \mathcal{L}$) if $\gamma(r)$ is strictly decreasing to 0 as $r \to +\infty$.

We call a continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ of *class* $\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if it is of class $\mathcal{K}_\infty$ in the first argument and of class $\mathcal{L}$ in the second argument.

It is obvious from the properties of $\mathcal{K}_\infty$-functions that for $\alpha \in \mathcal{K}_\infty$ its inverse $\alpha^{-1}(\cdot)$ exists and is of class $\mathcal{K}_\infty$. Note that for ρ_1, $\rho_2 \in \mathcal{K}$, $\rho_1 \circ \rho_2 \in \mathcal{K}$.

For more details of properties of comparison functions we refer to [2, 14, 35, 45, 92, 63].

1.2 Dynamical systems

Definition 1.2.1. A system $\Sigma = (T, X, \mathcal{U}_{ad}, \phi)$ including the following elements

- a time set T, either $T = \mathbb{Z}_+$ or $T = \mathbb{R}_+$
- a state space $(X, \|\cdot\|_X)$, and input value space $(U, \|\cdot\|_U)$
- the admissible input functions $\mathcal{U}_{ad} \subset \{u : T \to U\}$ with norm $\|\cdot\|_{\mathcal{U}_{ad}}$.
- a transition map $\phi : \mathcal{D}_\phi \to X$, where $\mathcal{D}_\phi$ is a subset of

$$\{(\sigma, \tau, x, u) |\, \sigma, \tau \in T,\ \tau \le \sigma,\ x \in X,\ u \in \mathcal{U}_{ad}\}$$

is called a dynamical system with perturbation, if the following properties hold:

1. *Existence*: for each initial state value $x \in X$, each input value function $u \in \mathcal{U}_{ad}$, initial time $\tau \in T$, there exists $\overline{T}_\tau(x) > \tau$ such that $\phi(\sigma, \tau, x, u) \in \mathcal{D}_\phi$ for all $\tau \le \sigma \le \overline{T}_\tau(x)$.
2. *Identity*: $\phi(\tau, \tau, x, u) = x$ hold for each $u \in \mathcal{U}_{ad}$, each state $x \in X$, and $\tau \in T$.
3. *Causality*: for each $(\sigma, \tau, x, u) \in \mathcal{D}_\phi$ with $\sigma \in (\tau, \overline{T}_\tau(x)]$, for each $\tilde{u} \in \mathcal{U}_{ad}$ such that $u(s) = \tilde{u}(s)$, $s \in [\tau, \sigma]$ it holds that $(\sigma, \tau, x, \tilde{u}) \in \mathcal{D}_\phi$ and $\phi(\sigma, \tau, x, u) = \phi(\sigma, \tau, x, \tilde{u})$.
4. *Semigroup property*: for each initial state value $x \in \mathbb{R}^n$, each function $u \in \mathcal{U}_{ad}$, and an initial time τ, if $\phi(\sigma, \tau, x, u) \in \mathcal{D}_\phi$, then $\phi(s, r, x, u) \in \mathcal{D}_\phi$ and $\phi(s, \tau, x, u) = \phi(s - r, \phi(r, \tau, x, u), u)$ hold for $\tau \le r \le s \le \sigma$.
5. *Continuity*: the map $(\sigma, \tau, x, u) \mapsto \phi(\sigma, \tau, x, u)$ is continuous in the sense: if $(\sigma_k, \tau^k, x_k, u_k) \in T \times T \times X \times \mathcal{U}_{ad}$ converges to (σ, τ, x, u) where $\sigma \in [\tau, \overline{T}_\tau(x)]$, then $\sigma_k \in [\tau_k, \overline{T}_{\tau^k}(x_k)]$ for k sufficiently large and $\lim\limits_{k \to +\infty} \phi(\sigma_k, \tau^k, x_k, u_k) = \phi(\sigma, \tau, x, u)$.

Here, $\phi(\sigma, \tau, x, u)$ denotes the state of a system at the time $\sigma \in T$ for initial state value $x \in X$ at initial time τ and admissible function $u \in \mathcal{U}_{ad}$. Let $T_{\max} \in (\tau, +\infty]$ denote the maximal time of the existence of solution of a system such that $\phi(\sigma, \tau, x, u) \in \mathcal{D}_\phi$ for $\sigma \in [\tau, T_{\max})$.

If $\mathcal{U}_{ad} = \{0\}$, then the system Σ is called a dynamical system without perturbation or simply, a dynamical system.

In this thesis we are particularly interested in time invariant dynamical systems and dynamical systems with perturbations.

Definition 1.2.2. A system with perturbation Σ is time invariant if for all $x \in X$, $u \in \mathcal{U}_{ad}$, $\sigma \geq \tau$ (σ, $\tau \in T$), and $s \geq -\tau$,

$$\phi(\sigma, \tau, x, u) = \phi(\sigma + s, \tau + s, x, u(\cdot + s)) \tag{1.1}$$

holds.

From Definition 1.2.2, it is known that if a dynamical system with perturbation is time invariant, then the trajectory of the system only depends on the initial state value and the input value function. Based on this fact, the trajectory of the system from initial time τ_0 can be obtained from another initial time τ_1 by transition in time. Thus, we let the initial time equal zero and $\phi(\sigma, x, u) := \phi(\sigma, 0, x, u)$. If $\mathcal{U}_{ad} = \{0\}$, we denote $\phi(\sigma, x, u)$ or simply, $\phi(\sigma, x)$.

In this thesis, we study the following time invariant, continuous and discrete time dynamical systems described by ordinary differential equations

$$\dot{x}(t) = f(x(t), u(t)), \quad \text{and difference equations} \tag{1.2}$$

$$x^+ = f(x, u), \tag{1.3}$$

respectively with vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, state $x \in \mathbb{R}^n$, input perturbation $u \in \mathbb{R}^m$, $t \in \mathbb{R}_+$. The set of *admissible input values* is denoted by $U_R := \operatorname{cl} \mathcal{B}_q(0, R) \subset \mathbb{R}^m$ for a constant $R > 0$ and the set of *admissible input functions* is defined by $\mathcal{U}_{ad} = \mathcal{U}_R := \{u : \mathbb{R} \to \mathbb{R}^m \text{ measurable} \mid \|u\|_{\infty,q} \leq R\}$ ($q \in \mathbb{R}_{>0}$ and $q \geq 1$). We assume that f is Lipschitz continuous and $f(0, 0) = 0$. From Theorem 2.2 of [102] and Chapter 2 of [93], solution to (1.2) with an initial condition exists and is unique.

If $u(t) = 0$ for all $t \geq 0$, time invariant, continuous time and discrete time dynamical systems are described by the following equations

$$\dot{x}(t) = f(x(t)), \tag{1.4}$$
$$x^+ = f(x). \tag{1.5}$$

In order to study stability of time invariant dynamical systems, we now introduce certain stability concepts.

1.3 Stability concepts

Let us start with the definition of equilibrium. Consider $X = \mathbb{R}^n$ endowed with norm $\|\cdot\|_p$, and $U = \mathbb{R}^m$ endowed with norm $\|\cdot\|_q$.

Definition 1.3.1. A point $x^* \in X$ is called an equilibrium or a fixed point of a time invariant dynamical system (with perturbation) Σ if $\phi(\sigma, x^*) = x^*$ ($\phi(\sigma, x^*, u) = x^*$) for all $\sigma \geq 0$ (and $u \in U_R$).

We assume solutions of time invariant dynamical system (with perturbation) Σ considered in the following and system (1.2) $-$ system (1.5) with initial state value are defined on T, respectively.

Definition 1.3.2. The equilibrium x^* of a time invariant dynamical system Σ is stable if for each $\epsilon > 0$ there exists a constant $\delta > 0$ such that

$$\|\phi(\sigma, x) - x^*\|_p \leq \epsilon \tag{1.6}$$

holds for all $\sigma \geq 0$ and $\|x - x^*\|_p \leq \delta$.

Definition 1.3.3. The equilibrium x^* of a time invariant dynamical system Σ is locally attractive if there exists a constant $\delta > 0$ such that

$$\lim_{\sigma \to +\infty} \|\phi(\sigma, x) - x^*\|_p = 0 \tag{1.7}$$

for $\|x - x^*\|_p \leq \delta$. If $\delta = +\infty$, then the equilibrium is globally attractive.

From now on, we assume $x^* = 0$ is an equilibrium of time invariant system Σ in Definitions 1.3.4 and 1.3.12, and systems (1.4) and (1.5), respectively.

Now we present the definition of asymptotic stability of the equilibrium $x^* = 0$.

Definition 1.3.4. The equilibrium of a time invariant dynamical system Σ is called locally (globally) asymptotically stable if it is both stable and locally (globally) attractive.

Based on Definition 1.3.4, in order to check if a system is asymptotically stable, we have to examine (1.6) and (1.7). However, in general the explicit solution of the system is not easy to compute. We introduce the concept of Lyapunov function which is widely used to verify if a system is asymptotically stable.

Definition 1.3.5. Let $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$.
(i) A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a local Lyapunov function for system (1.4) if there exist functions α_1, $\alpha_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{P}$ such that

$$\alpha_1(\|x\|_p) \leq V(x) \leq \alpha_2(\|x\|_p), \, \forall x \in \mathbb{R}^n, \tag{1.8}$$

$$\langle \nabla V(x), f(x) \rangle \leq -\alpha(\|x\|_p) \tag{1.9}$$

for all $x \in \mathcal{D}$. If $\mathcal{D} = \mathbb{R}^n$ then $V(x)$ is a global Lyapunov function for system (1.4).

(ii) A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a local or global Lyapunov function for system (1.5) if there exist functions α_1, $\alpha_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{P}$ such that (1.8) and

$$V(f(x)) - V(x) \leq -\alpha(\|x\|_p) \tag{1.10}$$

hold for all $x \in \mathcal{D}$ or $x \in \mathbb{R}^n$.

The continuous time converse Lyapunov theorem is presented in references e.g. [67, 77, 108] and [45, Theorem 49.1], and the discrete time converse Lyapunov theorem in [1, Theorem 5.12.5], [99, Theorem 1.7.6] and [30, 64]. We unify the converse Lyapunov theorem for two cases in the following theorem.

Theorem 1.3.6. *Consider system* (1.4) *or system* (1.5)*. The equilibrium of the system is locally (globally) asymptotically stable if and only if there exists a local (global) Lyapunov function.*

In this thesis, we also investigate how to estimate the domain of attraction of interconnected systems. Therefore, we recall the concept of the domain of attraction of time invariant dynamical system Σ at the asymptotically stable equilibrium 0.

Definition 1.3.7. Assume the time invariant dynamical system Σ is asymptotically stable at the equilibrium 0. The domain of attraction of system Σ at the origin is defined as

$$\mathcal{D}_d = \left\{ x \in \mathbb{R}^n \; : \; \lim_{\sigma \to +\infty,\, \sigma \in T} \phi(\sigma, x) = 0 \right\} .$$

Definition 1.3.8. We say a compact and connected set $D \subset \mathbb{R}^n$ is locally asymptotically stable for a time invariant dynamical system Σ if for each $\varepsilon > 0$ there exists a constant $\delta > 0$ such that

$$\operatorname{dist}(\phi(\sigma, x), D) \leq \varepsilon, \tag{1.11}$$

$$\lim_{\sigma \to +\infty} \operatorname{dist}(\phi(\sigma, x), D) = 0 \tag{1.12}$$

hold for all $\sigma \geq 0$ and $\operatorname{dist}(x, D) \leq \delta$. If $\delta = +\infty$, then D is called globally asymptotically stable.

Definition 1.3.9. Suppose a compact and connected set $D \subset \mathbb{R}^n$ is locally asymptotically stable for a time invariant dynamical system Σ. The domain of attraction of D of system Σ is defined as

$$\mathcal{D}_d(D) = \left\{ x \in \mathbb{R}^n \; : \; \lim_{\sigma \to +\infty,\, \sigma \in T} \operatorname{dist}(\phi(\sigma, x), D) = 0 \right\} .$$

If $D = \{0\}$ and the origin is an equilibrium for system Σ, then Definitions 1.3.8 and 1.3.9 are equivalent to Definitions 1.3.4 and 1.3.7, respectively.

Definition 1.3.10. A set $D \subset \mathbb{R}^n$ is called a positively invariant set for system (1.4) or (1.5) if for any initial condition $x \in D$, $\phi(\sigma, x) \in D$ for all $\sigma \in T$.

Remark 1.3.11. From the definition of the domain of attraction $\mathcal{D}_d$ ($\mathcal{D}_d(D)$), cf. Definition 1.3.7 (Definition 1.3.9), it is obvious that $\mathcal{D}_d$ ($\mathcal{D}_d(D)$) is a positively invariant set.

Definition 1.3.12. Let an open set $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$ be a positively invariant set for a time invariant dynamical system Σ. System Σ is $\mathcal{KL}$-stable at the origin on the open set $\mathcal{D}$ if there exists a function $\beta \in \mathcal{KL}$ such that

$$\|\phi(\sigma, x)\|_p \leq \beta(\|x\|_p, \sigma) \tag{1.13}$$

holds for all $x \in \mathcal{D}$ and all $\sigma \in T$.

Remark 1.3.13. Let $\mathcal{D}_d$ be the domain of attraction of system (1.4) or (1.5) at the origin. It is proved in [101, Proposition 1], [65, Proposition 2.2] that the concept of $\mathcal{KL}$-stability is equivalent to the concept of asymptotic stability of the origin for system (1.4) or (1.5), given $\overline{\mathcal{D}} \subset \mathcal{D}_d^\circ$. If $\mathcal{D} = \mathbb{R}^n$, then $\mathcal{KL}$-stability is equivalent to global asymptotic stability of the origin for system (1.4) or (1.5). The function $\beta \in \mathcal{KL}$ of (1.13) is called a stability estimate.

In the above, stability concepts and some results about dynamical systems without perturbation were presented. In the following, we recall stability concepts for dynamical systems with perturbations.

Robust stability is an important concept in the study of stability of systems with perturbation. We give the definition of robust stability in the following.

Definition 1.3.14. Let $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$. Consider system (1.2) or (1.3). Let $f(0,u) = 0$ for all $u \in U_R$. The origin is locally asymptotically stable for the system uniformly in $u \in \mathcal{U}_R$ if there exists a function $\beta \in \mathcal{KL}$ such that

$$\|\phi(\sigma, x, u)\|_p \leq \beta(\|x\|_p, \sigma) \tag{1.14}$$

holds for all $x \in \mathcal{D}$, $u \in \mathcal{U}_R$ and all $\sigma \in T$. If $\mathcal{D} = \mathbb{R}^n$, then the origin is globally asymptotically stable uniformly in $u \in \mathcal{U}_R$.

A corresponding concept of robust Lyapunov function is presented in the following definition.

Definition 1.3.15. Let $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$. Consider system (1.2) or (1.3). Let $f(0,u) = 0$ for all $u \in U_R$.

(i) A continuous differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a local robust Lyapunov function for system (1.2) if there exist functions α_1, $\alpha_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{P}$ such that

$$\alpha_1(\|x\|_p) \leq V(x) \leq \alpha_2(\|x\|_p),\ \forall x \in \mathbb{R}^n, \tag{1.15}$$

$$\langle \nabla V(x), f(x,u) \rangle \leq -\alpha(\|x\|_p) \tag{1.16}$$

hold for all $x \in \mathcal{D}$, and all $u \in U_R$. If $\mathcal{D} = \mathbb{R}^n$, then V is called a global robust Lyapunov function for system (1.2).

(ii) A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a local or global robust Lyapunov function for system (1.3) if there exist functions α_1, $\alpha_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{P}$ such that (1.15) and

$$V(f(x,u)) - V(x) \leq -\alpha(\|x\|_p) \tag{1.17}$$

are satisfied for all $x \in \mathcal{D}$ and all $u \in U_R$ or for all $x \in \mathbb{R}^n$ and all $u \in U_R$.

Definition 1.3.16. Consider system (1.2) or (1.3) with $f(0,u) = 0$ for all $u \in U_R$. Suppose the system is asymptotically stable at the origin uniformly in $u \in \mathcal{U}_R$. The robust domain of attraction of the system at the origin is defined by

$$\mathcal{D}_{rd} = \left\{ x \in \mathbb{R}^n : \begin{array}{l} \text{there exists } \beta \in \mathcal{L} \text{ such that } \|\phi(\sigma, x, u)\|_p \leq \beta(\sigma) \\ \text{for all } \sigma \in T,\ u \in \mathcal{U}_R. \end{array} \right\}.$$

Theorem 1.3.17. *[74, Theorem 2.9] [60] System* (1.2) *or* (1.3) *with* $f(0,u) = 0$ *for all* $u \in U_R$ *is globally uniformly asymptotically stable at the origin if and only if there exists a smooth robust Lyapunov function.*

According to Theorem 1.3.17, we can analyse robust stability of a system with perturbation using robust Lyapunov function instead of examining (1.14). (1.14) is not easy to check, since the explicit solution of the system may not easy to compute.

Definition 1.3.18. A set $D \subset \mathbb{R}^n$ is called a positively invariant set for system (1.2) or (1.3) if for any initial condition $x \in D$ and all $u \in \mathcal{U}_R$, $\phi(\sigma, x, u) \in D$ for all $\sigma \in T$.

In order to describe a characterization of $\mathcal{D}_{rd}$ for system (1.2), we introduce the following definition.

Definition 1.3.19. Consider system (1.2) with $f(0,u)=0$ for all $u \in U_R$. Assume system (1.2) is locally uniformly asymptotically stable at the origin, i.e., there exist a constant $\rho > 0$ and a function $\beta \in \mathcal{KL}$ such that

$$\|\phi(\sigma,x,u)\|_p \le \beta(\|x\|_p,\sigma) \tag{1.18}$$

holds for all $x \in \mathcal{B}_p(0,\rho)$, $u \in \mathcal{U}_R$ and $\sigma \in T$.

The first time of the trajectory touches the ball $\mathcal{B}_p(0,\rho)$ is defined by

$$t(x,u) = \inf\left\{\sigma > 0 : \phi(\sigma,x,u) \in \mathcal{B}_p(0,\rho)\right\}. \tag{1.19}$$

The following properties of $\mathcal{D}_{rd}$ for system (1.2) are shown in [10, Proposition 2.3].

Proposition 1.3.20. *Consider system* (1.2) *with* $f(0,u)=0$ *for all* $u \in U_R$ *and assume it is asymptotically stable at the origin uniformly in* $u \in \mathcal{U}_R$*, then*

1. $\mathcal{D}_{rd}$ *is an open, connected, positively invariant set with* $\overline{\mathcal{B}_p(0,\rho)} \subset \mathcal{D}_{rd}$*, where* ρ *from Definition 1.3.19.*
2. $\sup_{u\in\mathcal{U}_R}\{t(x,u)\} \to +\infty$ *for* $x \to x^0 \in \partial\mathcal{D}_{rd}$ *or* $\|x\|_p \to \infty$.
3. $\overline{\mathcal{D}_{rd}}$ *is a positively invariant set which is contractible to* 0 *(see [47]).*
4. *If for some* $\|u_0\| \le R$, $f(\cdot,u_0)$ *is of class* C^1*, then* $\mathcal{D}_{rd}$ *is* C^1*-diffeomorphic to* $\mathbb{R}^n$.

Definition 1.3.21. We say a compact and connected set $D \subset \mathbb{R}^n$ is locally (globally) uniformly asymptotically stable for system (1.2) or (1.3) if there exist a constant $\rho > 0$ and a function $\beta \in \mathcal{KL}$ such that

$$\operatorname{dist}(\phi(\sigma,x,u),D) \le \beta(\operatorname{dist}(x,D),\sigma) \tag{1.20}$$

holds for all x satisfying $\operatorname{dist}(x,D) \le \rho$ ($x \in \mathbb{R}^n$), $u \in \mathcal{U}_R$ and $\sigma \in T$.

If $D=\{0\}$ and for system (1.2) or (1.3), $f(0,u)=0$ for all $u \in U_R$, then Definition 1.3.21 is equivalent to Definition 1.3.14.

For a dynamical system with input perturbations which is not uniformly asymptotically stable, the concept of input to state stability (ISS) describes one type of stability. The concept plays an important part in stability analysis of systems with perturbations. Besides, when exploring stability of interconnected systems, we will assume all subsystems are input to state stable (ISS). Hence in the following, we introduce definitions of ISS and ISS Lyapunov function.

Definition 1.3.22. Let $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$. Consider system (1.2) or (1.3). The system is locally input to state stable (ISS) if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, for all $x \in \mathcal{D}$, $u \in \mathcal{U}_R$ and $\sigma \in T$, $\phi(\sigma,x,u)$ satisfies

$$\|\phi(\sigma,x,u)\|_p \le \beta(\|x\|_p,\sigma) + \gamma(\|u\|_{\infty,q}) \tag{1.21}$$

If $\mathcal{D}=\mathbb{R}^n$, then the system is globally ISS.

Remark 1.3.23. From Definition 1.3.22, we can conclude that for a given bounded $u \in \mathcal{U}_R$, the state will be bounded. The term $\gamma(\|u\|_{\infty,q})$ in (1.21) is called the ISS gain being referred to in small gain theorems. Furthermore, if the system is input to state stable, then the system with $u = 0$ is asymptotically stable at the origin, and the system with $\gamma = 0$ is robustly asymptotically stable at the origin.

It is not easy to verify that the system is ISS by checking the conditions from Definition 1.3.22, since in general the explicit solution of the system may not be easy to obtain. We introduce the concept of input to state stability Lyapunov function (ISS Lyapunov function) which plays a central role in examining whether system is ISS.

Definition 1.3.24. Let $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$.
(i) A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be a local ISS Lyapunov function in implication formulation for system (1.2) if there exist functions α_1, α_2, α, $\gamma \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x\|_p) \leq V(x) \leq \alpha_2(\|x\|_p),\ \forall x \in \mathbb{R}^n, \tag{1.22}$$

$$\|x\|_p \geq \gamma(\|u\|_q) \Rightarrow \langle \nabla V(x), f(x,u) \rangle \leq -\alpha(\|x\|_p) \tag{1.23}$$

hold for all $x \in \mathcal{D}$, $u \in U_R$. If $\mathcal{D} = \mathbb{R}^n$, then V is called a global ISS Lyapunov function in implication formulation.

(ii) A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be a local or global ISS Lyapunov function in implication formulation for system (1.3) if there exist functions α_1, α_2, α, $\gamma \in \mathcal{K}_\infty$ such that (1.22) and

$$\|x\|_p \geq \gamma(\|u\|_q) \Rightarrow V(f(x,u)) - V(x) \leq -\alpha(\|x\|_p) \tag{1.24}$$

are fulfilled for all $x \in \mathcal{D}$ and $u \in U_R$ or for all $x \in \mathbb{R}^n$ and $u \in U_R$.

For system (1.2) or (1.3), Remark 2.4 in [97] and Remark 3.3 in [59] state that the concept of ISS Lyapunov function in implication formulation is equivalent to the concept of ISS Lyapunov function in dissipative formulation which is described in the following.

Definition 1.3.25. Let $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$.
(i) A continuous differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be a local ISS Lyapunov function in dissipative formulation for system (1.2) if there exist functions α_1, α_2, α, $\beta \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x\|_p) \leq V(x) \leq \alpha_2(\|x\|_p),\ \forall x \in \mathbb{R}^n, \tag{1.25}$$

$$\langle \nabla V(x), f(x,u) \rangle \leq -\alpha(\|x\|_p) + \beta(\|u\|_q) \tag{1.26}$$

for all $x \in \mathcal{D}$ and $u \in U_R$. If $\mathcal{D} = \mathbb{R}^n$, then V is called a global ISS Lyapunov function in dissipative formulation.

(ii) A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a local or global ISS Lyapunov function in dissipative formulation for system (1.3) if there exist functions α_1, α_2, α, $\beta \in \mathcal{K}_\infty$ such that (1.25) and

$$V(f(x,u)) - V(x) \leq -\alpha(\|x\|_p) + \beta(\|u\|_q) \tag{1.27}$$

hold for all $x \in \mathcal{D}$ and $u \in U_R$ or for $x \in \mathbb{R}^n$ and $u \in U_R$.

In Lyapunov-type small gain theorems, cf. Theorem 1.6.1-Theorem 1.6.3, we refer to the term $\beta(\|u\|_q)$ in (1.26) and (1.27) as Lyapunov ISS gain or gain. The gain is called linear gain if $\beta(\|u\|_q)$ is a linear function of $\|u\|_q$. It is clear that if $u = 0$, then V is a Lyapunov function for system (1.2) or (1.3), and if $\beta = 0$, then V is a robust Lyapunov function.

From converse Lyapunov theorem (Theorem 1.3.6), it is known that system (1.4) or system (1.5) is asymptotically stable if and only if there exists a Lyapunov function. The next theorem describes such relationship between ISS and ISS Lyapunov function proposed in [97], and Theorem 1 in [59].

Theorem 1.3.26. *Consider system* (1.2) *or* (1.3)*. The system is ISS if and only if there exists a smooth ISS Lyapunov function in dissipative formulation.*

If f is continuous, then the system is ISS if and only if there exists a smooth ISS Lyapunov function in implication formulation.

Remark 1.3.27. If f is not continuous, the existence of an ISS Lyapunov function in implication form for system (1.3) does not imply it is ISS, which is demonstrated by Example 3.3 in [39]. In that paper, for f being not continuous, it is proved that system (1.3) is ISS if and only if there exists a strong ISS Lyapunov function in implication formulation defined by [39, Definition 4.1].

The concept of ISS is confined to the case bounded input-bounded output. However, in practice bounded inputs may yield unbounded output such as in the robotic example discussed in [4]. The notion of integral input to state stability (iISS) introduced by Sontag in [92] addresses this case.

Definition 1.3.28. Let $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$. System (1.2) is locally integral input to state stable (iISS) if there exist functions $\beta \in \mathcal{KL}$, $\gamma_1, \gamma_2 \in \mathcal{K}$ such that, for all $x \in \mathcal{D}$, $u \in \mathcal{U}_R$ and $\sigma \geq 0$, $\phi(\sigma, x, u)$ satisfies

$$\|\phi(\sigma, x, u)\|_p \leq \beta(\|x\|_p, \sigma) + \gamma_1(\int_0^\sigma \gamma_2(\|u\|_q)) \tag{1.28}$$

If $\mathcal{D} = \mathbb{R}^n$, then system (1.2) is globally iISS.

Remark 1.3.29. If $u = 0$ and system (1.2) is iISS, then system (1.2) is asymptotically stable at the equilibrium 0.

Definition 1.3.30. Let $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$. A continuously differentiable function V : $\mathbb{R}^n \to \mathbb{R}_+$ is said to be a local iISS Lyapunov function for system (1.2) if there exist functions α_1, $\alpha_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{K}$ such that

$$\alpha_1(\|x\|_p) \leq V(x) \leq \alpha_2(\|x\|_p),\ \forall x \in \mathbb{R}^n, \tag{1.29}$$

$$\langle \nabla V(x), f(x,u) \rangle \leq -\alpha(\|x\|_p) + \beta(\|u\|_q) \tag{1.30}$$

for all $x \in \mathcal{D}$ and $u \in U_R$. If $\mathcal{D} = \mathbb{R}^n$ then $V(x)$ is called a global iISS Lyapunov function.

The following theorem describes the relationship between iISS and iISS Lyapunov function. A proof is presented in [4].

Theorem 1.3.31. *System* (1.2) *is iISS if and only if there exists an iISS Lyapunov function.*

It is easier to check if a system is iISS by using iISS Lyapunov function than estimating the trajectory, since the explicit solution of the system may be hard to obtain and the condition (1.28) is not easy to examine.

Remark 1.3.32. Based on Definitions 1.3.22 and 1.3.28, if system (1.2) is ISS, then it is iISS. However, the converse is not always true. It depends on the possibility of finding a $\mathcal{K}_\infty$ function which bounds a positive definite function according to Definition 1.3.30.

In this section, we have listed important definitions which will be used in this thesis. In these definitions of stability and Lyapunov functions, we use $\|\cdot\|_p$, $\|\cdot\|_q$ norms. Because of the equivalence of norms, these definitions are equivalent to corresponding definitions in stability theory.

In this thesis, we will compute continuous and piecewise affine (CPA) Lyapunov functions, CPA ISS Lyapunov functions and iISS Lyapunov functions which may be not differentiable at some points (not smooth). Therefore, in the following Section 1.4, we introduce definitions of nonsmooth Lyapunov functions, CPA function and CPA Lyapunov functions.

1.4 Continuous and piecewise affine functions

In the sequel, in order to introduce the definition of continuous and piecewise affine (CPA) function on suitable triangulations of a compact set $\mathcal{D}$, we introduce the definition of a suitable triangulation. We first state basic concepts needed in the definition of a suitable triangulation (see [81, Section 1.1]).

Definition 1.4.1. The *convex hull* of vectors $x_0, x_1, \dots, x_m \in \mathbb{R}^n$ is given by

$$\text{co}\{x_0, \dots, x_m\} := \left\{ \sum_{i=0}^{m} \lambda_i x_i : 0 \le \lambda_i \le 1, \sum_{i=0}^{m} \lambda_i = 1 \right\}.$$

Definition 1.4.2. A set of vectors $x_0, x_1, \dots, x_m \in \mathbb{R}^n$ is called *affine independent* if $\sum_{i=1}^{m} \lambda_i (x_i - x_0) = 0$ implies $\lambda_i = 0$ for all $i = 1, \dots, m$.

This definition is independent of the numbering of the x_i, i.e., of the choice of the reference point x_0.

Definition 1.4.3. Let the vectors $x_0, x_1, \dots, x_m \in \mathbb{R}^n$ be affine independent. An m-simplex is defined by

$$\text{co}\{x_0, \dots, x_m\} := \left\{ \sum_{i=0}^{m} \lambda_i x_i : 0 \le \lambda_i \le 1, \sum_{i=0}^{m} \lambda_i = 1 \right\}.$$

The point defined by the vector x_i is called a vertex. The face of the m-simplex is defined as the convex hull of any nonempty subset of the $m + 1$ vertices.

Definition 1.4.4. We call a finite collection $\mathcal{T} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_N\}$ of n-simplices in $\mathbb{R}^n$ a *suitable triangulation* of $\mathcal{D}$ if

i) $\mathcal{S}_\nu, \mathcal{S}_\mu \in \mathcal{T}$, $\nu \neq \mu$, intersect in a common face or not at all.

ii) For $\mathcal{D}_\mathcal{T} := \cup_\nu \mathcal{S}_\nu$, $\mathcal{D}_\mathcal{T}^\circ$ is a connected neighbourhood of the origin.

iii) If $0 \in \mathcal{S}_\nu$, then 0 is a vertex of $\mathcal{S}_\nu$.

For each $x \in \mathcal{D}_\mathcal{T}$ we define the *active index set* $\mathcal{I}_\mathcal{T}(x) := \{\nu \in \{1, \dots, N\} | x \in \mathcal{S}_\nu\}$. We denote the *set of vertices of all simplices in* $\mathcal{T}$ by $\mathcal{V}_\mathcal{T}$. The *diameter* of a simplex $\mathcal{S}_\nu$ is defined as $\operatorname{diam}(\mathcal{S}_\nu) := \max_{x,y \in \mathcal{S}_\nu} \|x - y\|_2$.

Remark 1.4.5. Property i), often called shape regularity in the theory of finite element methods, is needed in order to parameterize every continuous function, linearly affine on every simplex, by specifying its values at the vertices, cf. Remark 1.4.10. Property ii) ensures that $\mathcal{D}_\mathcal{T}$ is a natural domain for a Lyapunov function and, without Property iii), a positive definite function linearly affine on each of the simplices could not have a local minimum at the origin.

Remark 1.4.6. If there is no suitable triangulation $\mathcal{T}$ such that $\mathcal{D}_\mathcal{T} = \mathcal{D}$, then we consider the suitable triangulation $\mathcal{D}_\mathcal{T} \subset \mathcal{D}$ instead of $\mathcal{D}$.

Remark 1.4.7. For an n-simplex $\mathcal{S}_\nu := \operatorname{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T}$ define its *shape-matrix* X_ν by writing the vectors $x_1 - x_0$, $x_2 - x_0, \dots, x_n - x_0$ in its rows subsequently, i.e.,

$$X_\nu = [(x_1 - x_0), (x_2 - x_0), \cdots, (x_n - x_0)]^T . \tag{1.31}$$

In this thesis, we define simplices by fixing an ordered set of vertices and considering the closed convex hull of those vertices. While simplices are usually defined by an unordered set of vertices, by insisting on an ordered set we obtain uniqueness of the shape matrix defined in (1.31).

Remark 1.4.8. In Theorems 2.1.7, 2.2.8, 4.2.9 and 4.3.8, we additionally require that the simplices in the suitable triangulation $\mathcal{T}$ have a certain regularity i.e. that they are not too close to being degenerate. To this end, let $\lambda_\nu := \|X_\nu^{-1}\|_2$. Then, $\lambda_\nu = \lambda_{\min}^{-1}$ holds, where $\lambda_{\min}$ is the smallest singular value of X_ν.

The regularity property now demands that we need to avoid grids with arbitrarily flat simplices. Formally, this means that there exists a positive constant $R_1 > 0$ such that all simplices $\mathcal{S}_\nu \in \mathcal{T}$ in the considered grids satisfy the inequality

$$\lambda_\nu \operatorname{diam}(\mathcal{S}_\nu) \le R_1. \tag{1.32}$$

Definition 1.4.9. For a suitable triangulation $\mathcal{T}$, and with $\mathcal{D}_\mathcal{T} := \cup_{\mathcal{S} \in \mathcal{T}} \mathcal{S}$, we define CPA[$\mathcal{T}$] as the set of continuous functions $g : \mathcal{D}_\mathcal{T} \to \mathbb{R}$ which are linearly affine on each simplex $\mathcal{S}_\nu$, i.e.,

$$g(x) = \langle w_\nu, x \rangle + a_\nu, \quad x \in \mathcal{S}_\nu, \tag{1.33}$$

where $w_\nu \in \mathbb{R}^n$ and $a_\nu \in \mathbb{R}$.

In the interior of any simplex, a function $g \in \mathrm{CPA}[\mathcal{T}]$ is differentiable and has a constant gradient, and we denote the gradient of a function $g \in \mathrm{CPA}[\mathcal{T}]$ in the interior of simplex $\mathcal{S}_\nu$ by ∇g_ν. In other words, with (1.33), for each $x \in \mathcal{S}_\nu^\circ$ we have

$$\nabla g_\nu := w_\nu = \nabla g(x). \tag{1.34}$$

Remark 1.4.10. A function $g \in \mathrm{CPA}[\mathcal{T}]$ is uniquely determined by its values at the vertices of the simplices of $\mathcal{T}$ as follows: let $\mathcal{S}_\nu = \mathrm{co}\{x_0, x_1, \ldots, x_n\} \in \mathcal{T}$. Every point $x \in \mathcal{S}_\nu$ can be written uniquely as a convex combination of its vertices, $x = \sum_{i=0}^n \lambda_i^x x_i$, $\lambda_i^x \geq 0$ for all $i = 0, 1, \ldots, n$, and $\sum_{i=0}^n \lambda_i^x = 1$. The value of g at x is given by $g(x) = \sum_{i=0}^n \lambda_i^x g(x_i)$. It is obvious that a CPA function is Lipschitz continuous.

We now address the question of how we construct a CPA function based on a given continuous function. The definition of a CPA approximation to a continuous function describes how.

Definition 1.4.11. Let $\mathcal{D} \subset \mathbb{R}^n$, $W : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, and $\mathcal{T}$ be a suitable triangulation of $\mathcal{D}$. The $\mathrm{CPA}[\mathcal{T}]$ *approximation* g *to* W *on* $\mathcal{D}_\mathcal{T}$ is the function $g \in \mathrm{CPA}[\mathcal{T}]$ defined by $g(x) = W(x)$ for all vertices x of all simplices in $\mathcal{T}$.

1.4.1 Continuous and piecewise affine Lyapunov functions

In order to introduce definitions of CPA Lyapunov functions, we first need the definition of Clarke's subdifferential for Lipschitz continuous functions, cf. [15, Theorem 2.5.1].

Definition 1.4.12. For a Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$, Clarke's subdifferential is given by

$$\partial_{Cl} V(x) := \mathrm{co}\left\{ \lim_{i\to\infty} \nabla V(x_i) \, : \, x_i \to x, \nabla V(x_i) \text{ and } \lim_{i\to\infty} \nabla V(x_i) \text{ exist} \right\}. \tag{1.35}$$

Before we introduce definitions of CPA Lyapunov functions, we state definitions of nonsmooth Lyapunov functions.

Definition 1.4.13. Let $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$. Consider system (1.4) or (1.5) with $f(0) = 0$.
(i) A Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a local *nonsmooth Lyapunov function* for the continuous time system (1.4) if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{P}$ such that

$$\alpha_1(\|x\|_p) \leq V(x) \leq \alpha_2(\|x\|_p), \, \forall x \in \mathbb{R}^n, \tag{1.36}$$

$$\langle \xi, f(x) \rangle \leq -\alpha(\|x\|_p), \; \forall \xi \in \partial_{Cl} V(x) \tag{1.37}$$

hold for all $x \in \mathcal{D}$. If $\mathcal{D} = \mathbb{R}^n$, then V is called a global *nonsmooth Lyapunov function.*

(ii) A Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a local *nonsmooth Lyapunov function* for the discrete time system (1.5) if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{P}$ such that (1.36) and

$$V(f(x)) - V(x) \leq -\alpha(\|x\|_p) \tag{1.38}$$

hold for all $x \in \mathcal{D}$. If $\mathcal{D} = \mathbb{R}^n$, then V is called a global *nonsmooth Lyapunov function.*

Definition 1.4.14. Let $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$.
(i) A Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be a local *nonsmooth ISS Lyapunov function in dissipative formulation* for the continuous time system (1.2) if there exist functions α_1, α_2, α, $\beta \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x\|_p) \leq V(x) \leq \alpha_2(\|x\|_p), \, \forall x \in \mathbb{R}^n, \tag{1.39}$$

$$\langle \xi, f(x,u) \rangle \leq -\alpha(\|x\|_p) + \beta(\|u\|_q), \; \forall \xi \in \partial_{Cl} V(x) \tag{1.40}$$

hold for all $x \in \mathcal{D}$, $u \in U_R$. If $\mathcal{D} = \mathbb{R}^n$ then V is called a global *nonsmooth ISS Lyapunov function in dissipative formulation.*

(ii) A Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be a local *nonsmooth ISS Lyapunov function in dissipative formulation* for the discrete time system (1.3) if there exist functions α_1, α_2, α, $\beta \in \mathcal{K}_\infty$ such that (1.39) and

$$V(f(x,u)) - V(x) \leq \alpha(\|x\|_p) + \beta(\|u\|_q) \tag{1.41}$$

hold for all $x \in \mathcal{D}$, $u \in U_R$. If $\mathcal{D} = \mathbb{R}^n$ then V is called a global *nonsmooth ISS Lyapunov function in dissipative formulation.*

If $\beta = 0$, $\alpha \in \mathcal{P}$ in (1.40) or (1.41), then V from Definition 1.4.14 is called a *nonsmooth robust Lyapunov function* for system (1.2) or system (1.3) with $f(0,u) = 0$ for all $u \in U_R$. If $\alpha \in \mathcal{P}$, $\beta \in \mathcal{K}$, then V from Definition 1.4.14 is called a *nonsmooth iISS Lyapunov function.*

Remark 1.4.15. Given a suitable triangulation $\mathcal{T}$ of a set $\mathcal{D}$ with $0 \in \mathcal{D}^\circ$ and $V \in \text{CPA}[\mathcal{T}]$

$$V(x) = \langle w_\nu, x\rangle + a_\nu, \quad x \in \mathcal{S}_\nu, \tag{1.42}$$

with $w_\nu \in \mathbb{R}^n$ and $a_\nu \in \mathbb{R}$, based on Definitions 1.4.4 and 1.4.9 and Remark 1.4.10, the identity $\partial_{Cl} V(x) = \text{co}\{w_\nu | \nu \in \mathcal{I}_\mathcal{T}(x)\}$ holds for $x \in \mathcal{D}_\mathcal{T}$.

Now we state definitions of CPA Lyapunov functions.

Definition 1.4.16. Consider system (1.4) or (1.5). Let $V \in \text{CPA}[\mathcal{T}]$, α_1, $\alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{P}$.

(i) If V satisfies

$$\alpha_1(\|x\|_p) \leq V(x) \leq \alpha_2(\|x\|_p), \tag{1.43}$$

$$\langle \nabla V_\nu, f(x)\rangle \leq -\alpha_3(\|x\|_p), \qquad \forall \nu \in \mathcal{I}_\mathcal{T}(x) \tag{1.44}$$

for $x \in \mathcal{D}_\mathcal{T}$, then V is called a CPA Lyapunov function for system (1.4).

(ii) If V satisfies (1.43) and

$$V(f(x)) - V(x) \leq -\alpha_3(\|x\|_p) \tag{1.45}$$

for $x \in \mathcal{D}_\mathcal{T}$, then V is called a CPA Lyapunov function for system (1.5).

Remark 1.4.17. Based on the linearity of the scalar product $\langle \xi, f(x)\rangle$ in the first argument, from the inequality in (1.44) we have for $x \in \mathcal{D}_\mathcal{T}^\circ$

$$\langle \xi, f(x)\rangle \leq -\alpha_3(\|x\|_p),\ \forall \xi \in \partial_{Cl} V(x). \tag{1.46}$$

V can be extended to be a positive definite function by choosing Lipschitz continuous function $V(x) > 0$ for $x \in \mathbb{R}^n \setminus \mathcal{D}_\mathcal{T}$. Thus, the CPA Lyapunov function V is a nonsmooth Lyapunov function.

In the following we introduce the definition of CPA ISS Lyapunov function in dissipative formulation.

Definition 1.4.18. (i) $V \in \text{CPA}[\mathcal{T}]$ is said to be a CPA ISS Lyapunov function in dissipative formulation for the continuous time system (1.2) if there exist functions α_1, α_2, α, $\beta \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x\|_p) \le V(x) \le \alpha_2(\|x\|_p), \tag{1.47}$$

$$\langle \nabla V_\nu, f(x,u) \rangle \le -\alpha(\|x\|_p) + \beta(\|u\|_q),\ \forall \nu \in \mathcal{I}_\mathcal{T}(x) \tag{1.48}$$

for all $x \in \mathcal{D}_\mathcal{T}$, $u \in U_R$.

(ii) $V \in \text{CPA}[\mathcal{T}]$ is said to be a CPA ISS Lyapunov function in dissipative formulation for the discrete time system (1.3) if functions α_1, α_2, α, $\beta \in \mathcal{K}_\infty$ such that (1.47) and

$$V(f(x,u)) - V(x) \le -\alpha(\|x\|_p) + \beta(\|u\|_q) \tag{1.49}$$

hold for all $x \in \mathcal{S}_\nu$, $u \in U_R$.

If $\beta = 0$ and $\alpha \in \mathcal{P}$ in (1.48) or (1.49), then V is a CPA robust Lyapunov function for system (1.2) or system (1.3) with $f(0,u) = 0$ for all $u \in U_R$. If $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{K}$, then V from Definition 1.4.18 is called a CPA iISS Lyapunov function.

Remark 1.4.19. From Remarks 1.4.10 and 1.4.17, and Definitons 1.4.16 and 1.4.18, a CPA (ISS) Lyapunov function is a nonsmooth (ISS) Lyapunov function.

Remark 1.4.20. The relationship between CPA (ISS) Lyapunov function and (ISS) Lyapunov function is discussed in Theorems 2.1.4, 2.2.4, 4.2.9 and 4.3.8.

1.5 Yoshizawa constructions

Based on Theorem 1.3.6, if system (1.4) or (1.5) is asymptotically stable, then there exists a Lyapunov function. In this section, we present the Yoshizawa constructions in Definitions 1.5.2 and 1.5.6 for continuous and discrete time dynamical systems, which were originally proposed by Yoshizawa in [108].

1.5.1 Continuous time case

Consider system (1.4) with $f(0) = 0$. Let $\mathcal{D} \subset \mathbb{R}^n$ be an open set containing the origin and positively invariant for system (1.4). We assume system (1.4) is $\mathcal{KL}$-stable at the origin on the set $\mathcal{D}$, i.e, there exists a stability estimate $\beta \in \mathcal{KL}$ such that solution $\phi(t,x)$ of system (1.4) fulfills

$$\|\phi(t,x)\|_2 \le \beta(\|x\|_2, t) \tag{1.50}$$

for all $x \in \mathcal{D}$ and $t \in \mathbb{R}_+$.

In what follows we will make use of Sontag's lemma on $\mathcal{KL}$-estimates [92, Proposition 7] ([63, Lemma 7]):

Lemma 1.5.1. *Given $\beta \in \mathcal{KL}$ and $\lambda > 0$, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that, for all $s, t \in \mathbb{R}_+$*

$$\alpha_1(\beta(s,t)) \le \alpha_2(s) e^{-\lambda t}. \tag{1.51}$$

Definition 1.5.2. Assume system (1.4) is $\mathcal{KL}$-stable with a stability estimate $\beta \in \mathcal{KL}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfy (1.51) with $\lambda = 2$. We call the function $V : \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$V(x) := \sup_{t\geq 0} \alpha_1(\|\phi(t,x)\|_2)e^t \tag{1.52}$$

a *Yoshizawa function.*

The following theorem emphasises what, in the sequel, are the important elements relating to the Yoshizawa function from [101, Section 5.1.2].

Theorem 1.5.3. *Suppose* (1.4) *is* $\mathcal{KL}$*-stable with stability estimate* $\beta \in \mathcal{KL}$*. Then the Yoshizawa function* (1.52) *is locally Lipschitz continuous on* $\mathcal{D}\backslash\{0\}$ *and satisfies*

$$\alpha_1(\|x\|_2) \leq V(x) \leq \alpha_2(\|x\|_2) \tag{1.53}$$

and the decrease condition

$$\langle \xi, f(x)\rangle \leq -\alpha_1(\|x\|_2), \quad \xi \in \partial_{Cl}V(x) \tag{1.54}$$

Thus V *is a nonsmooth Lyapunov function for system* (1.4)*.*

Furthermore, with $T_1 : \mathcal{D}\backslash\{0\} \to \mathbb{R}_+$ *defined by*

$$T_1(x) := \ln\left(\frac{\alpha_2(\|x\|_2)}{\alpha_1(\|x\|_2)}\right) + 1 \tag{1.55}$$

for all $x \in \mathcal{D}\backslash\{0\}$*, we have*

$$V(x) = \sup_{t\geq 0} \alpha_1(\|\phi(t,x)\|_2)e^t = \max_{t\in[0,T_1(x)]} \alpha_1(\|\phi(t,x)\|_2)e^t. \tag{1.56}$$

Proof. The boundedness property (1.53) and decrease property

$$V(\phi(t,x)) \leq V(x)e^{-t} \tag{1.57}$$

are demonstrated directly in [101, Section 5.1.2]. (1.54) can be derived by (1.53) and (1.57). It is also proved that the Yoshizawa function is locally Lipschitz continuous on $\mathcal{D} \setminus \{0\}$ in [101, Section 5.1.2]. Based on boundedness, decrease properties and Lipschitz continuity, it is obvious that V is a nonsmooth Lyapunov function for system (1.4).

In [101, Claim 2] it is shown for $\widehat{T}(x) : \mathcal{D} \setminus \{0\} \to \mathbb{R}_+$ given by

$$\widehat{T}(x) = -\ln\left(\frac{V(x)}{\alpha_2(\|x\|_2)}\right) + 1 \tag{1.58}$$

that the Yoshizawa function satisfies

$$V(x) = \sup_{t\geq 0} \alpha_1(\|\phi(t,x)\|_2)e^t = \max_{t\in[0,\widehat{T}(x)]} \alpha_1(\|\phi(t,x)\|_2)e^t. \tag{1.59}$$

Using the upper and lower bounds (1.53) we see that

$$0 \leq \widehat{T}(x) \leq -\ln\left(\frac{\alpha_1(\|x\|_2)}{\alpha_2(\|x\|_2)}\right) + 1 = \ln\left(\frac{\alpha_2(\|x\|_2)}{\alpha_1(\|x\|_2)}\right) + 1 = T_1(x) \tag{1.60}$$

giving the result of Theorem 1.5.3. □

Remark 1.5.4. From equations (1.56) it is obvious that for any $x \in \mathcal{D}\backslash\{0\}$ taking the maximum over any interval $[0, T_2]$ where $T_2 \geq T_1(x)$ will not change the value of the Yoshizawa function.

1.5.2 Discrete time case

The Yoshizawa construction (1.52) is extended to the discrete time case in [64]. Here we describe Yoshizawa construction for discrete time nonlinear system in detail. Let an open set $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$ be positively invariant for system (1.5) with $f(0) = 0$. In this section, we assume discrete time system (1.5) is $\mathcal{KL}$-stable on the open set $\mathcal{D} \subset \mathbb{R}^n$, i.e, there exists a function $\beta \in \mathcal{KL}$ such that $\phi(k, x)$ satisfies

$$\|\phi(k, x)\|_2 \le \beta(\|x\|_2, k) \tag{1.61}$$

for all $x \in \mathcal{D}$ and $k \in \mathbb{Z}_+$.

In order to define our candidate Lyapunov function, we use a version of Sontag's lemma on $\mathcal{KL}$-estimates Lemma 1.5.1.

Lemma 1.5.5. *For every $\mu \in (0, 1)$, $\beta \in \mathcal{KL}$ there exist α_1, $\alpha_2 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(\beta(s, k)) \le \alpha_2(s)\mu^{2k}, \ \forall s \in \mathbb{R}_+, \ \forall \ k \in \mathbb{Z}_+. \tag{1.62}$$

Proof. Sontag's lemma on $\mathcal{KL}$-estimates Lemma 1.5.1 states that, for any $\lambda > 0$ and $\beta \in \mathcal{KL}$ there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that

$$\alpha_1(\beta(s, k)) \le \alpha_2(s)e^{-\lambda k}, \ \forall s \in \mathbb{R}_+, \ \forall \ k \in \mathbb{Z}_+.$$

Given $\mu \in (0, 1)$, let $\lambda = -2 \ln \mu$ which satisfies $\lambda > 0$. Applying Lemma 1.5.1 with this $\lambda > 0$ then yields (1.62). □

We now define the Yoshizawa function in discrete time.

Definition 1.5.6. Consider system (1.5). We assume that system (1.5) is $\mathcal{KL}$-stable with a stability estimate $\beta \in \mathcal{KL}$, and α_1, $\alpha_2 \in \mathcal{K}_\infty$ satisfy (1.62). Then the function $V : \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$V(x) := \sup_{k \in \mathbb{Z}_+} \alpha_1(\|\phi(k, x)\|_2)\mu^{-k} \tag{1.63}$$

is called a Yoshizawa function.

Based on the results of [65, Section 6], we summarize some properties of the Yoshizawa function defined by (1.63) as the next theorem.

Theorem 1.5.7. *If system (1.5) is $\mathcal{KL}$-stable with a stability estimate $\beta \in \mathcal{KL}$, and α_1 is locally Lipschitz continuous, then the Yoshizawa function (1.63) is Lipschitz continuous on $\mathcal{D} \setminus \{0\}$ and satisfies the bounds*

$$\alpha_1(\|x\|_2) \le V(x) \le \alpha_2(\|x\|_2) \tag{1.64}$$

and the decrease condition

$$V(f(x)) - V(x) \le -(1 - \mu)\alpha_1(\|x\|_2). \tag{1.65}$$

Moreover, V is a nonsmooth Lyapunov function for system (1.5).

Further, for each $x \in \mathcal{D}$ there exists a positive integer $\overline{K(x)}$ such that

$$V(x) = \sup_{k \in \mathbb{Z}_+} \alpha_1(\|\phi(k, x)\|_2)\mu^{-k} = \max_{k \in \{0, \cdots, \overline{K(x)}\}} \alpha_1(\|\phi(k, x)\|_2)\mu^{-k}. \tag{1.66}$$

Proof. The properties that V is continuous, bounded and satisfies the decrease condition

$$V(\phi(1,x)) \leq V(x)\mu \tag{1.67}$$

are proved in [65, section 6]. Based on (1.64) and (1.67), we have (1.65).

Let $\lambda = \mu^{-1}$, the integer $K(x)$ is calculated explicitly in [65, Claim 7] as

$$K(x) = \left\lceil -\log_\lambda \left(\frac{V(x)}{\alpha_2(\|x\|_2)} \right) \right\rceil + 1, \quad x \neq 0 \tag{1.68}$$

and with the upper and lower bounds on V we get that

$$0 \leq K(x) \leq \left\lceil -\log_\lambda \left(\frac{\alpha_1(\|x\|_2)}{\alpha_2(\|x\|_2)} \right) \right\rceil + 1 = \left\lceil \log_\lambda \left(\frac{\alpha_2(\|x\|_2)}{\alpha_1(\|x\|_2)} \right) \right\rceil + 1 =: \overline{K(x)}. \tag{1.69}$$

Thus

$$\begin{aligned} V(x) = \sup_{k \in \mathbb{Z}_+} \alpha_1(\|\phi(k,x)\|_2)\mu^{-k} &= \max_{k \in \{0,\cdots,K(x)\}} \alpha_1(\|\phi(k,x)\|_2)\mu^{-k} \\ &= \max_{k \in \{0,\cdots,\overline{K(x)}\}} \alpha_1(\|\phi(k,x)\|_2)\mu^{-k}. \end{aligned} \tag{1.70}$$

Since α_1 and f are Lipschitz continuous, based on [65, Lemma 5.1] we get for all $x \in \mathcal{D}$ and $k \in \{0, \cdots, K(x)\}$ with $K(x)$ defined in (1.68) there exist $\delta > 0$, $L_x > 0$ such that

$$\|\alpha_1(\phi(k,x)) - \alpha_1(\phi(k,x+v))\|_2 \leq L_x \|v\|_2 \tag{1.71}$$

holds for $\|v\|_2 \leq \delta$.

Therefore, we obtain

$$\begin{aligned} V(x) = \max_{k \in \{0,\cdots,K(x)\}} \alpha_1(\|\phi(k,x)\|_2)\mu^{-k} &\leq \sup_{k \in \mathbb{Z}_+} \alpha_1(\|\phi(k,x+v)\|_2)\mu^{-k} \\ &+ \max_{k \in \{0,\cdots,K(x)\}} |\alpha_1(\|\phi(k,x)\|_2) - \alpha_1(\|\phi(k,x+v)\|_2)|\mu^{-k} \\ &\leq V(x+v) + L_x\|v\|_2\mu^{-K(x)}. \end{aligned} \tag{1.72}$$

Similarly, we have

$$V(x+v) \leq V(x) + L_{x+v}\|v\|_2\mu^{-K(x+v)}. \tag{1.73}$$

Then

$$-L_{x+v}\mu^{-K(x+v)}\|v\|_2 \leq V(x) - V(x+v). \tag{1.74}$$

According to inequalities (1.72) and (1.74), we conclude that V is locally Lipschitz continuous. It follows from these properties that V is a nonsmooth Lyapunov function for system (1.5). □

1.6 Small gain theorems

In this thesis, we consider large scale systems as interconnected low dimensional systems. In this section, we recall three versions of small gain theorems which provide tools to study stability of interconnected systems described by the following equations

$$\begin{cases} S_1 : & \dot{x}_1(t) = f_1(x_1(t), x_2(t), \cdots, x_M(t)) \\ & \vdots \\ S_M : & \dot{x}_M(t) = f_M(x_1(t), x_2(t), \cdots, x_M(t)) \end{cases} \tag{1.75}$$

with $x_i \in \mathbb{R}^{n_i}$, $n_i, M \in \mathbb{Z}_{>0}$, $i = 1, 2, \cdots, M$.

Let $x = (x_i^\top, x_2^\top, \cdots, x_M^\top)^\top$, $f(x) = (f_1^\top(x), f_2^\top(x), \cdots, f_M^\top(x))^\top$, $\sum_{i=1}^{M} n_i = n$, $f(0) = 0$. We assume that f is Lipschitz continuous.

In order to analyse the stability of system (1.75), we first analyse stability of the subsystems. For the interconnected systems (1.75), we consider each subsystem S_i as a dynamical system with perturbation by regarding the effect of other states x_j as perturbations as illustrated in Figure 1.1. In the following, we present three types of small gain theorems. Each

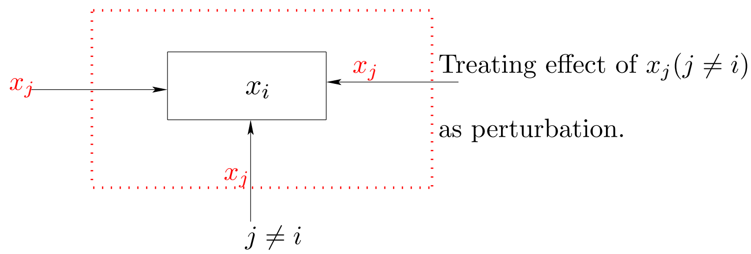

Figure 1.1: Subsystem

small gain theorem has its own advantage and disadvantage. Stability analysis of system (1.75) obtained by these three small gain theorems are then described.

1.6.1 Small gain theorem in linear form

Consider system (1.75). We assume that each subsystem is ISS and $A1$ holds for each subsystem.

A1: There exists an ISS Lyapunov function in dissipative formulation V_i satisfying the following linear inequality for each subsystem S_i of the interconnected systems (1.75), $i \in \{1, \cdots, M\}$

$$\langle \nabla V_i(x_i), \dot{x}_i \rangle \leq -a_{ii} \|x_i\|_p + \sum_{j=1, j\neq i}^{M} e_{ij} a_{ij} \|x_j\|_p, \tag{1.76}$$

with $e_{ij} = 0$ if the state x_j does not influence x_i and $e_{ij} = 1$ otherwise.

Let

$$A = \begin{bmatrix} -a_{11} & e_{12}a_{12} & \cdots & e_{1M}a_{1M} \\ e_{21}a_{21} & -a_{22} & \cdots & e_{2M}a_{2M} \\ & \cdots & & \\ e_{M1}a_{M1} & e_{M2}a_{M2} & \cdots & -a_{MM} \end{bmatrix}.$$

Now we state the small gain theorem in linear form [34, Theorem 1].

Theorem 1.6.1. *If the assumption A1 holds and the stated condition here is satisfied, then the interconnected systems (1.75) is asymptotically stable at the origin.*

$$(-1)^r \begin{vmatrix} -a_{11} & e_{12}a_{12} & \cdots & e_{1r}a_{1r} \\ e_{21}a_{21} & -a_{22} & \cdots & e_{2r}a_{2r} \\ & \cdots & & \\ e_{r1}a_{r1} & e_{r2}a_{r2} & \cdots & -a_{rr} \end{vmatrix} > 0, \quad r = 1, 2, \ldots, M.$$

Proof. According to the assumption and results of [22, Theorem 3.4], there exist an M-vector $b > 0$ for any M-vector $c > 0$ such that

$$c^\top = -b^\top A. \tag{1.77}$$

Let us introduce a positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$

$$V(x) = \langle b, V_{vec} \rangle, \tag{1.78}$$

where $V_{vec} = (V_1(x_i), \cdots, V_M(x_M))^\top$.

It follows that

$$\langle \nabla V(x), \dot{x} \rangle \le b^\top A W = -c^\top W < 0, \;\; x \neq 0, \tag{1.79}$$

where $W = (\|x_1\|_p, \|x_2\|_p, \cdots, \|x_M\|_p)^\top$.

Thus V is a Lyapunov function for system (1.75), and system (1.75) is then asymptotically stable at the origin. □

1.6.2 Small gain theorem in dissipative form

Consider system (1.75) and assume $A2$ is fulfilled for each subsystem.

A2: There exists an ISS Lyapunov function in dissipative formulation V_i satisfying the following nonlinear inequality for each subsystem S_i, $i \in \{1, \cdots, M\}$

$$\langle \nabla V_i(x_i), \dot{x}_i \rangle \le -\alpha_{ii}(V_i(x_i)) + \sum_{j=1, j\neq i}^{M} e_{ij} \beta_{ij}(V_j(x_j)), \tag{1.80}$$

where e_{ij} is defined as (**??**), α_{ii}, $\beta_{ij} \in \mathcal{K}_\infty$, $i \neq j$.

Let $\gamma_{ii} = 0$, $\gamma_{ij}(s) = e_{ij}\beta_{ij}(s)$, $i, j = 1, \cdots, M$, $i \neq j$. Define the matrix

$$A := \operatorname{diag}(\alpha_{11}, \cdots, \alpha_{MM}), \;\; \Gamma := ((\gamma_{ij})_{i,j=1,\cdots,M})_{M\times M}. \tag{1.81}$$

Let $s \in \mathbb{R}_+^M, s = (s_1^\top, \cdots, s_M^\top)^\top$. Furthermore, we define a monotone map $\Gamma : \mathbb{R}_+^M \to \mathbb{R}_+^M$

$$\Gamma(s) := \left(\sum_{j=1}^{M} \gamma_{1j}(s_j), \cdots, \sum_{j=1}^{M} \gamma_{Mj}(s_j) \right)^\top, \tag{1.82}$$

a diagonal operator $A : \mathbb{R}_+^M \mapsto \mathbb{R}_+^M$

$$A(s) := (\alpha_{11}(s_1), \cdots, \alpha_{MM}(s_M))^\top, \tag{1.83}$$

and a diagonal operator $E : \mathbb{R}_+^M \mapsto \mathbb{R}_+^M$

$$E := \operatorname{diag}\big((\operatorname{Id} + \varphi_1)(s_1), \cdots, (\operatorname{Id} + \varphi_M)(s_M)\big)^\top \tag{1.84}$$

with functions $\varphi_1, \cdots, \varphi_M \in \mathcal{K}_\infty$.

With the aid of these notations, we state the following small gain theorem in dissipative form [16, Theorem 4.5]

Theorem 1.6.2. *Consider the interconnected systems* (1.75) *and assume* $A2$ *holds and* Γ *is irreducible. If there exists a diagonal operator* E *of the form* (1.84) *such that the small gain condition*

$$E \circ \Gamma \circ A^{-1}(s) \not\geq s \,, \forall s \in \mathbb{R}_+^M \setminus \{0\} \,, \tag{1.85}$$

is satisfied, then there exists a continuously differentiable path $\theta : [0,\infty) \mapsto \mathbb{R}^M$*, such that* $\theta(0) = 0$ *and* θ' *is positive so that*

$$E \circ \Gamma \circ A^{-1}(\theta(s)) < \theta(s) \,, \quad \forall s \in (0,\infty) \,. \tag{1.86}$$

Assume further that there exist two constants c*,* C *such that*

$$0 < c < \frac{d}{ds}\theta_i^{-1} \circ \alpha_i(s) < C, \ \forall s \in (0,\infty) \,. \tag{1.87}$$

Then the interconnected systems (1.75) *is asymptotically stable at the origin. A Lyapunov function for the coupled system* (1.75) *is then given by*

$$v(x) := \max_{i \in \{1,2,\ldots,M\}} \theta_i^{-1} \circ \alpha_i(v_i(x_i)). \tag{1.88}$$

Proof. See the proof of [16, Theorem 4.5] □

1.6.3 Small gain theorem in comparison form

We consider the interconnected systems (1.75) with $M = 2$ and the initial condition $x^0 = (x_1^0, x_2^0)^\top$. In this section, we assume the following condition $A3$ is satisfied.

A3: There exists an iISS Lyapunov function V_i satisfying the following inequality for each subsystem S_i, $i \in \{1,2\}$

$$\langle \nabla V_i(x_i), \dot{x}_i \rangle \leq -\alpha_i(V_i(x_i)) + \beta_i(V_j(x_j)), \tag{1.89}$$

where $i, j = 1, 2$, $j \neq i$. $\alpha_i \in \mathcal{P}$, $\beta_i \in \mathcal{K}$ are Lipschitz continuous.

Let $v = (v_1, v_2)^\top$. We consider the following comparison system

$$\begin{cases} \dot{v}_1(t) = -\alpha_1(v_1(t)) + \beta_1(v_2(t)) \equiv F_1(v_1, v_2), \\ \dot{v}_2(t) = -\alpha_2(v_2(t)) + \beta_2(v_1(t)) \equiv F_2(v_1, v_2) \end{cases} \tag{1.90}$$

which evolves in $\mathbb{R}_+^2$. The solution to (1.90) with the initial condition $v(0)$ is denoted by $v(\cdot, v(0))$. α_i and β_i are from $A3$.

Define three subsets of state space

$$\begin{aligned} \Omega^{+-} &:= \{(v_1, v_2)^\top \in \mathbb{R}_+^2 : F_1(v_1, v_2) \geq 0 \text{ and } F_2(v_1, v_2) \leq 0)\}, \\ \Omega^{--} &:= \{(v_1, v_2)^\top \in \mathbb{R}_+^2 : F_1(v_1, v_2) \leq 0 \text{ and } F_2(v_1, v_2) \leq 0)\}, \\ \Omega^{-+} &:= \{(v_1, v_2)^\top \in \mathbb{R}_+^2 : F_1(v_1, v_2) \leq 0 \text{ and } F_2(v_1, v_2) \geq 0)\}. \end{aligned} \tag{1.91}$$

The typical shape of Ω is shown by Figure 1.2 which is Figure 1 of [3].

We present the next small gain theorem in comparison form [3, Theorem 1].

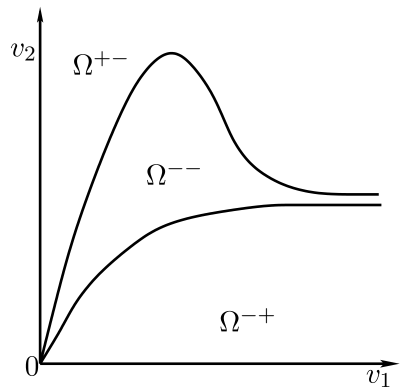

Figure 1.2: Typical shapes of the Ω regions

Theorem 1.6.3. *Consider system* (1.90)*. If*

$$\Omega^{-+} \cup \Omega^{--} \cup \Omega^{+-} = \mathbb{R}^2_+ \quad \textit{and} \quad \Omega^{+-} \cap \Omega^{--} \cap \Omega^{+-} = \{0\}. \tag{1.92}$$

and one of the following conditions are satisfied,

1) there exist constants $0 < L^- < L^+ \leq +\infty$ *such that*

$$\lim_{v \to +\infty} \sup \beta_1^{-1}(\alpha_1(v)) = L^+, \quad \lim_{v \to L^-} \sup \beta_2^{-1}(\alpha_2(v)) = +\infty. \tag{1.93}$$

2) there exist constants $0 < L^- < L^+ \leq +\infty$ *such that*

$$\lim_{v \to +\infty} \sup \beta_2^{-1}(\alpha_2(v)) = L^+, \quad \lim_{v \to L^-} \sup \beta_1^{-1}(\alpha_1(v)) = +\infty. \tag{1.94}$$

3)

$$\lim_{v \to +\infty} \sup \beta_1^{-1}(\alpha_1(v)) = +\infty, \quad \lim_{v \to +\infty} \sup \beta_2^{-1}(\alpha_2(v)) = +\infty. \tag{1.95}$$

then system (1.90) *is asymptotically stable at the origin.*

Proof. See the proof of Theorem 1 from [3]. □

Lemma 1.6.4. *Consider system* (1.90)*. If the initial conditions* $v^1(0), v^2(0)$ *satisfy* $v^1(0) > v^2(0)$*, then we have* $v(t, v^1(0)) > v(t, v^2(0))$ *for* $t \in [0, T_{\max})$*.*

Proof. See [3, Lemma 2.1]. □

Lemma 1.6.5. *Consider the interconnected systems* (1.75) *with* $M = 2$*. Assume A3 holds. For the corresponding comparison system* (1.90) *with the initial condition* $v(0) = (v_1(x_1^0), v_2(x_2^0))^\top$*, we further suppose that the conditions from Theorem 1.6.3 are fulfilled. Then the system* (1.75) *with* $M = 2$ *is asymptotically stable at the origin.*

Proof. Let $V = (V_1, V_2)^\top$ with V_1, V_2 satisfying (1.89) and $v(t)$ from (1.90). Using Lemma 1.6.4, we have $V(t) \leq v(t)$ for $t \in [0, T_{\max})$. By Theorem 1.6.3, we get system (1.90) is asymptotically stable at the origin. Then $V(x)$ is asymptotically stable at the origin. From the radially boundedness property of Lyapunov function (see condition (1.8)), we conclude that system (1.75) with $M = 2$ is asymptotically stable at the origin. □

In the above, we stated three versions of small gain theorems having their own pros and cons. The linear small gain theorem deals well with the case that the ISS Lyapunov function V_i for each subsystem satisfies a linear inequality such as (1.76). The small gain theorem in linear form is a special case of the small gain theorem in dissipative form. If V_i satisfies a nonlinear inequality like (1.80), then we may resort to a small gain theorem in dissipative form. Given that V_i is an iISS Lyapunov function fulfilling an inequality such as (1.89) for two interconnected systems, we may analyse stability of two interconnected systems by the small gain theorem in comparison form. However, the small gain theorem in comparison form could not be utilized to analyse stability of more than two interconnected subsystems. In Chapters 3 and 4, stability of interconnected system will be investigated by local versions of the above stated small gain theorems.

1.7 Notes and references

In Section 1.1, we recall notions of comparison functions which are widely used in stability theorems. These concepts could be found in any books on Lyapunov stability. One of the nice references is [45]. Besides, recently Kellett summarised properties of comparison functions and presented some new results in [63].

In Section 1.2, the definition of dynamical system with perturbation is introduced based on the definition of continuous dynamical system with control [80, Section 1.2], the definition of system in [93, Chapter 2] and the definition of local flow in Section 3.1.1 in [49].

In Section 1.3, we discuss stability of dynamical systems. The idea of stability can be traced back to the seventeenth century, which Leine points out in [69]. Then main stability ideas between the seventeenth century and the twentieth are described in [69]. The modern stability theory began with the concept of Lyapunov stability which was proposed by Lyapunov [75] in 1892. Many results about stability theory have been published e.g. in [45, 74, 75]. Particularly, Lyapunov's second method is a powerful and universal tool in investigating stability of dynamical systems. By this method, it is possible to verify if a dynamical system is asymptotically stable at the equilibrium without computing explicit solutions. The continuous and piecewise affine function notation and CPA Lyapunov function concept were used in computing Lyapunov functions by linear programming in [76, 40, 41, 42, 5]. The definition of continuous and piecewise affine function is described in [43, 72]. In addition, definitions of CPA Lyapunov function are presented in [72, 43].

From [101], it is known that Lyapunov originally posed the converse Lyapunov problem [75, Sect.20, Th.II]: if a dynamical system is asymptotically stable at the equilibrium, under what conditions does there exist a Lyapunov function? To answer this question, several classical converse Lyapunov theorems haven been proposed (see for instance [77, 67, 108, 45, 23, 30, 64], [1, Theorem 5.12.5] and [99, Theorem 1.7.6]). Based on the converse Lyapunov theorem proposed by Yoshizawa in [108], Yoshizawa constructions are established in [101, 64]. Additionally, properties of Yoshizawa constructions are described in detail in [101, 64]. Massera in [77] and Kurzweil in [67] develop Lyapunov constructions which rely on integrating solutions from the initial time to infinity, which can be also used to construct continuous and piecewise affine Lyapunov functions.

As perturbations are incorporated in dynamical systems, robust stability becomes important in control theory with application in robust nonlinear stabilization [74]. The definition of robust stability is presented in [95, Section 3.2], [60]. The converse robust Lyapunov function theorem for continuous time systems was proved in [74]. In [60], the converse robust Lyapunov function theorem is extended to discrete time systems. However, if under certain perturbations the dynamical systems is unstable, then the robust stability concept cannot deliver any information about the trajectories of dynamical systems. In [88], Sontag proposed the concept of input to state stability (ISS) which provides an estimate of the trajectory of a dynamical system with perturbation. If there is no perturbation, ISS is equal to asymptotic stability. Basic results of ISS are established in [88, 89, 90]. In [97], the converse ISS Lyapunov theorem is developed, and the equivalence between ISS and the existence of ISS Lyapunov functions is established. More details on ISS and ISS Lyapunov function can be found in [98, 38, 96, 94]. ISS is extended to discrete time dynamical systems in [59] where parallel results are presented. Furthermore, the equivalence of L^2 stability, L^∞ stability and ISS for linear dynamical systems is stated in [92]. However, for nonlinear dynamical systems L^2 stability does not equal ISS. But L^∞ stability equals to ISS, which still holds for nonlinear dynamical systems. Furthermore, properties of iISS and iISS Lyapunov function, and a converse iISS Lyapunov theorem are described in detail in [4]. These stability concepts and their properties play key roles in different areas such as stability analysis, observer design, feedback control design and small gain theorems, see for instance [84, 104, 103, 19, 100, 57].

Various versions of small gain theorems have been proposed by authors who study stability of control systems by the input/output approach in [91, 48]. Using properties of ISS discussed in [88, 89, 90, 97], a small gain theorem was proposed in [58]. Furthermore, the small gain theorem is extended to large scale systems in [19]. After the concept of ISS Lyapunov function is proposed, a Lyapunov type small gain theorem for two interconnected systems was first established in [57]. Via ISS Lyapunov functions in implication formulation for subsystems, a small gain theorem is developed for large scale systems and stability of the overall system is investigated in [19]. In [20], the Lyapunov function for interconnected systems constructed by a small gain theorem is only Lipschitz continuous rather than smooth. Assuming each subsystem has an ISS Lyapunov function in dissipative formulation, an alternative small gain theorem is established in [16] where a smooth iISS Lyapunov function is obtained for large scale systems. A smooth construction of an ISS Lyapunov function for two ISS interconnected systems is demonstrated in [50]. Furthermore, small gain theorems for two iISS interconnected systems are proposed in [51, 52]. Smooth constructions of ISS or iISS Lyapunov function for large scale systems are then presented in [54, 53, 55]. Moreover, small gain theorems for discrete time interconnected systems are also well investigated such as in [59, 68, 56, 37, 24, 25].

For large scale system without perturbations, the small gain theorems discussed in the last paragraph can be used to analyse stability of large scale system. In this thesis, we use three types of small gain theorems which are discussed in Section 1.6 to investigate stability of interconnected systems. Via computed iISS and ISS Lyapunov functions for subsystems, stability of the whole system can be also investigated by other small gain theorems mentioned in the above.

2 Computation of Lyapunov functions using the Yoshizawa constructions

In this chapter, we investigate the problem of computing CPA Lyapunov functions for dynamical systems by the CPA method. The procedure of the CPA method is described in the following.

1. Partition a compact subset of the state space into a suitable triangulation $\mathcal{T}$.

2. Define $V(x_i)$ at each vertex x_i of every simplex $\mathcal{S} \subset \mathcal{T}$.

3. Take the convex interpolation of those values $V(x_i)$ and obtain a function $V \in \mathrm{CPA}[\mathcal{T}]$.

4. Verify if the obtained function V satisfies a system-dependent set of linear inequalities which are conditions making sure the CPA function V is a Lyapunov function.

From the above procedure of the CPA method it is evident that the important step is the computation of the vertex values. The vertex values can be obtained by solving a linear optimization problem with a system-dependent set of linear inequality constraints in [76, 40, 41, 42, 5, 28]. As the size of the linear optimization problem increases, the cost of computation becomes more expensive. In this chapter, we will design a more efficient method to construct Lyapunov functions. To this end, we construct a CPA function based on vertex values obtained by the Yoshizawa functions. We then check if such a CPA function is a true Lyapunov function by examining linear inequalities which will be introduced in Theorems 2.1.4 and 2.2.4. Moreover, we demonstrate that this construction will always be feasible if the CPA function has enough structure, i.e. if the triangulation has a sufficient number of vertices, and if the Yoshizawa function for the system under consideration satisfies certain conditions: twice continuously differentiable for continuous time systems; differentiable with bounded gradient for discrete time systems.

In Section 2.1, we look at the problem of computing CPA Lyapunov functions for continuous time dynamical systems on a compact subset of the state space. We describe the construction of a CPA Lyapunov function on a given suitable triangulation and linear inequalities for vertices which are used to verify if a given CPA function is a Lyapunov function in Section 2.1.1. In Section 2.1.2, we recall the Yoshizawa function for the continuous time case. In Section 2.1.3, we illustrate the effectiveness of our proposed method by three numerical examples.

Motivated by the inspiring results of these examples, we further apply this new technique for computing Lyapunov functions for discrete time dynamical systems. Some parallel results are discussed in Section 2.2. The definition of CPA Lyapunov functions for discrete time dynamical systems is introduced in Section 2.2.1. We then present a finite number of inequalities which are used to check if a given CPA function is a true Lyapunov function for discrete time dynamical systems. The Yoshizawa function for the discrete time case is described in Section 2.2.2. In Section 2.2.3, several numerical examples are presented to show how our proposed method is applied.

2.1 Continuous time case

In this section, we consider a continuous time system described by ordinary differential equations

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is twice continuously differentiable, and $f(0) = 0$.

In this section, we investigate system (2.1) on a set $\mathcal{D}$ satisfying the following condition.

- $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$ is a compact set and positively invariant for system (2.1).

Remark 2.1.1. If $\mathcal{D}$ is not positively invariant, then it is possible that the solution $\phi(t, x)$ to (2.1) with the initial condition x may not be in $\mathcal{D}$.

Let $\mathcal{T} = \{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$ be a suitable triangulation of $\mathcal{D}$ with $\mathcal{D}_\mathcal{T} := \cup_{\mathcal{S}_\nu \in \mathcal{T}} \mathcal{S}$.

2.1.1 Continuous and piecewise affine Lyapunov function

Our subsequent results will be valid on the set $\mathcal{D} \subset \mathbb{R}^n$ excluding a fixed arbitrarily small neighbourhood of the origin. We define a CPA Lyapunov function that accounts for this.

Definition 2.1.2. Assume there exist functions α_1, $\alpha_2 \in \mathcal{K}_\infty$ and $V \in \mathrm{CPA}[\mathcal{T}]$ such that

$$\alpha_1(\|x\|_2) \leq V(x) \leq \alpha_2(\|x\|_2). \tag{2.2}$$

Let $\varepsilon > 0$ be such that

$$\max_{\|x\|_2 \leq \epsilon} V(x) < \min_{x \in \partial \mathcal{D}_\mathcal{T}} V(x). \tag{2.3}$$

If there is a constant $\alpha_3^* > 0$ such that

$$\langle \xi, f(x) \rangle \leq -\alpha_3^* \|x\|_2, \quad \xi \in \partial_{Cl} V(x) \tag{2.4}$$

for all $x \in \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0, \varepsilon)$, we call V a CPA *Lyapunov function* for system (2.1) on $\mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0, \varepsilon)$.

The implication of the existence of a CPA Lyapunov function for (2.1) on $\mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0, \varepsilon)$ is slightly weaker than asymptotic stability. For a set $\Omega \subset \mathbb{R}^n$, we denote the reachable set of system (2.1) from Ω at time $t \in \mathbb{R}_+$ by $\phi(t, \Omega) := \cup_{x \in \Omega} \phi(t, x)$.

Theorem 2.1.3. *Given $\varepsilon > 0$, assume that $V : \mathcal{D} \to \mathbb{R}_+$ is a* CPA *Lyapunov function for system* (2.1) *on $\mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0, \varepsilon)$. For every $c \geq 0$ define the sublevel set $L_{V,c} := \{x \in \mathcal{D}_\mathcal{T} : V(x) \leq c\}$ and let $m := \max_{\|x\|_2 \leq \varepsilon} V(x)$ and $M := \min_{x \in \partial \mathcal{D}_\mathcal{T}} V(x)$. Then, for every $c \in [m, M)$ we have $\mathcal{B}_2(0, \varepsilon) \subset L_{V,c} \subset \mathcal{D}_\mathcal{T}^\circ$ and, furthermore, there exists a $T_c \geq 0$ such that $\phi(t, L_{V,c}) \subset L_{V,m}$ for all $t \geq T_c$.*

In other words, a CPA Lyapunov function implies asymptotic stability of the set $L_{V,m}$. The proof is similar to [42, Theorem 6.16].

Proof. According to (2.3), we have $m < M$. The definition of m implies that $\mathcal{B}_2(0,\varepsilon) \subset L_{V,c}$ for $c \in [m, M)$. Furthermore, since $M := \min_{x\in\partial\mathcal{D}_\mathcal{T}} V(x)$, the inequality (2.3) implies $L_{V,c} \subset \mathcal{D}^\circ_\mathcal{T}$ for every $c \in [m, M)$.

We claim that if $x \in L_{V,m}$, then $\phi(t, x) \in L_{V,m}$ for all $t \geq 0$. Assume that this is not true. Then there exist some $t > 0$ and some $\delta > 0$ such that $V(\phi(t, x)) > m + \delta$. Let $\tau = \inf\{t \geq 0 : V(\phi(t, x)) \geq m + \delta\}$. According to the definition of m, $\phi(\tau, x) \in \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)$ and on this region (2.4) implies that $t \mapsto V(\phi(t, x))$ is a strictly decreasing function. Since $x \in L_{V,m}$ there exists a $\tau_1 \in [0, \tau)$ such that $V(\phi(\tau_1, x)) \geq m + \delta$, which is contrary to the definition of τ. Thus $\phi(t, x) \in L_{V,m}$ for all $t \geq 0$.

Let $x \in L_{V,c} \setminus L_{V,m}$. By the condition (2.4), there exists a $\hat{T}(x) > 0$ such that $V(\phi(\hat{T}(x), x)) = m$. Since f is twice continuously differentiable and V is Lipschitz continuous, $\hat{T} = \max\limits_{x\in L_{V,c}} \{\hat{T}(x)\}$ exists. Then, the above analysis yields that $V(\phi(t, x)) \leq m$ for $t > \hat{T}$. Thus $T_c = \hat{T}$.

Therefore, we conclude for every $c \in [m, M)$ there exists a $T_c \geq 0$ such that $\phi(t, L_{V,c}) \subset L_{V,m}$ for all $t \geq T_c$. The set $L_{V,m}$ is then attractive and forward invariant. □

The following theorem and corollary provide a set of linear inequalities so that, if a given CPA function satisfies the linear inequalities then it is a CPA Lyapunov function. The proofs of Theorem 2.1.4 and Corollary 2.1.5 are similar to [29, Theorem 1].

Theorem 2.1.4. *Let $\mathcal{T}$ be a suitable triangulation of $\mathcal{D}$, $V \in \mathrm{CPA}[\mathcal{T}]$ and $\varepsilon > 0$ be a small constant. We assume f from (2.1) is twice continuously differentiable. Let $\mathcal{S}_\nu = \mathrm{co}\{x^\nu_0, x^\nu_1, \ldots, x^\nu_n\} \in \mathcal{T}$ and $\mu_\nu \in \mathbb{R}_+$ satisfy*

$$\max_{\substack{i,j,k=1,2,\ldots,n\\ x\in\mathcal{S}_\nu}} \left| \frac{\partial^2 f_k}{\partial x_i \partial x_j}(x) \right| \leq \mu_\nu, \tag{2.5}$$

where f_k is the k-th element of f.

For each $\mathcal{S}_\nu$, for $i = 0, 1, \ldots, n$ define the constants

$$E_{i,\nu} := \frac{n\mu_\nu}{2} \|x^\nu_i - x^\nu_0\|_2 \left(\|x^\nu_i - x^\nu_0\|_2 + \mathrm{diam}(\mathcal{S}_\nu)\right). \tag{2.6}$$

Then, for every $\mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)$ such that the inequalities

$$\langle \nabla V_\nu, f(x^\nu_i)\rangle + \|\nabla V_\nu\|_1 E_{i,\nu} < 0 \tag{2.7}$$

hold for all vertices $x^\nu_i \in \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)$, $i = 0, 1, \ldots, n$, we have

$$\langle \nabla V_\nu, f(x)\rangle < 0$$

for all $x \in \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)$.

Proof. Let $H_{f_k}(x)$ denote the Hessian of f_k at $x \in \mathbb{R}^n$. For $x \in \mathcal{S}_\nu$, we see that, with $s \in \mathbb{R}^n$,

$$
\begin{aligned}
\max_{x\in\mathcal{S}_\nu} \|H_{f_k}(x)\|_2 &= \max_{x\in\mathcal{S}_\nu, \|s\|_2=1} \|H_{f_k}(x)s\|_2 = \max_{x\in\mathcal{S}_\nu, \|s\|_2=1} \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n h_{ij}(x)s_j\right)^2} \\
&\le \max_{\|s\|_2=1} \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n \mu_\nu |s_j|\right)^2} \le \max_{\|s\|_2=1} \sqrt{\sum_{i=1}^n n\mu_\nu^2 \sum_{j=1}^n |s_j|^2} \\
&\le \sqrt{n^2\mu_\nu^2} = n\mu_\nu, \qquad (2.8)
\end{aligned}
$$

where $h_{ij}(x)$ denotes $\frac{\partial^2 f_k}{\partial x_i \partial x_j}(x)$.

For $x \in \mathcal{S}_\nu$, x can be written as a convex combination of vertices, i.e., $x = \sum_{i=0}^n \lambda_i x_i^\nu$, where $\lambda_i \in [0,1]$, and $\sum_{i=0}^n \lambda_i = 1$. By Taylor's theorem

$$
\begin{aligned}
f_k(x) &= f_k(x_0^\nu) + \langle \nabla f_k(x_0^\nu), \sum_{i=0}^n \lambda_i (x_i^\nu - x_0^\nu)\rangle + \frac{1}{2}\sum_{i=0}^n \lambda_i (x_i^\nu - x_0^\nu)^\top H_{f_k}(z) \sum_{j=0}^n \lambda_j (x_j^\nu - x_0^\nu) \\
&= \sum_{i=0}^n \lambda_i \Big(f_k(x_0^\nu) + \langle \nabla f_k(x_0^\nu), (x_i^\nu - x_0^\nu)\rangle + \frac{1}{2}(x_i^\nu - x_0^\nu)^\top H_{f_k}(z) \sum_{j=0}^n \lambda_j (x_j^\nu - x_0^\nu)\Big) \qquad (2.9)
\end{aligned}
$$

for some z on the line segment between x_0^ν and $\sum_{i=0}^n \lambda_i x_i^\nu$.

Furthermore, by Taylor's theorem, we have for every $i = 0, 1, \dots, n$ that

$$
f_k(x_i^\nu) = f_k(x_0^\nu) + \langle \nabla f_k(x_0^\nu), (x_i^\nu - x_0^\nu)\rangle + \frac{1}{2}(x_i^\nu - x_0^\nu)^\top H_{f_k}(z_i)(x_i^\nu - x_0^\nu) \qquad (2.10)
$$

for some z_i on the line segment between x_0^ν and x_i^ν.

Combining (2.9) and (2.10) we obtain

$$
\begin{aligned}
&\left\| f_k(x) - \sum_{i=0}^n \lambda_i f_k(x_i^\nu) \right\|_2 \\
&\quad = \frac{1}{2}\left\| \sum_{i=0}^n \lambda_i (x_i^\nu - x_0^\nu)^T \left(H_{f_k}(z) \sum_{j=0}^n \lambda_i (x_j^\nu - x_0^\nu) - H_{f_k}(z_i)(x_i^\nu - x_0^\nu)\right)\right\|_2 \\
&\quad \le \frac{1}{2}\sum_{i=0}^n \lambda_i \|x_i^\nu - x_0^\nu\|_2 \left(\|H_{f_k}(z)\|_2 \left\| \sum_{j=0}^n \lambda_i (x_j^\nu - x_0^\nu)\right\|_2 + \|H_{f_k}(z_i)\|_2 \|x_i^\nu - x_0^\nu\|_2 \right). \qquad (2.11)
\end{aligned}
$$

Based on (2.8)

$$
\begin{aligned}
\left\| f_k\left(\sum_{i=0}^n \lambda_i x_i^\nu\right) - \sum_{i=0}^n \lambda_i f_k(x_i^\nu)\right\|_2 &\le \frac{1}{2}\sum_{i=0}^n \lambda_i n\mu_\nu \|x_i^\nu - x_0^\nu\|_2 \left(\max_{z\in\mathcal{S}_\nu} \|z - x_0^\nu\|_2 + \|x_i^\nu - x_0^\nu\|_2\right) \\
&\le \frac{1}{2}\sum_{i=0}^n \lambda_i n\mu_\nu \|x_i^\nu - x_0^\nu\|_2 (\|x_i^\nu - x_0^\nu\|_2 + \operatorname{diam}(\mathcal{S}_\nu)).
\end{aligned}
$$

Hence, for $x \in \mathcal{S}_\nu$

$$\begin{aligned}
\langle \nabla V_\nu, f(x)\rangle &= \sum_{i=0}^{n} \lambda_i \langle \nabla V_\nu, f(x_i^\nu)\rangle + \langle \nabla V_\nu, f(x)\rangle - \sum_{i=0}^{n} \lambda_i \langle \nabla V_\nu, f(x_i^\nu)\rangle \\
&\le \sum_{i=0}^{n} \lambda_i \langle \nabla V_\nu, f(x_i^\nu)\rangle + \|\nabla V_\nu\|_1 \left\| f(x) - \sum_{i=0}^{n} \lambda_i f(x_i^\nu) \right\|_\infty \\
&= \sum_{i=0}^{n} \lambda_i \langle \nabla V_\nu, f(x_i^\nu\rangle + \|\nabla V_\nu\|_1 \|f(x) - f(x_i^\nu)\|_\infty \\
&\le \sum_{i=0}^{n} \lambda_i (\langle \nabla V_\nu, f(x_i^\nu)\rangle + \|\nabla V_\nu\|_1 E_{i,\nu}).
\end{aligned}$$

Since $\langle \nabla V_\nu,\ f(x_i^\nu)\rangle + \|\nabla V_\nu\|_1 E_{i,\nu} < 0$ for every vertex $x_i^\nu \in \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)$, we conclude that

$$\langle \nabla V_\nu, f(x)\rangle < 0 \tag{2.12}$$

holds for $x \in \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)$. □

Corollary 2.1.5. *Assume that $V \in$ CPA$[\mathcal{T}]$ from Theorem 2.1.4 is positive definite and that the constant $\varepsilon > 0$ satisfies (2.3). If the inequalities (2.7) are satisfied for all vertices of $\mathcal{S}_\nu \in \mathcal{T}$ with $\mathcal{S}_\nu \cap \mathcal{B}_2(0,\varepsilon)^C \neq \emptyset$, then V is a* CPA *Lyapunov function for system* (2.1) *on $\mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)$.*

Proof. According to the proof of Theorem 2.1.4 and the assumptions, there exists $\delta_1 > 0$ such that

$$\langle \nabla V_\nu, f(x)\rangle < -\delta_1 \tag{2.13}$$

holds for $x \in \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)$. Let $\alpha_3^* = \frac{\delta_1}{\|x\|_2}$. It is easy to check that V satisfies (2.4) on $\mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)$. Since V is positive definite, there exist functions $\alpha_1,\ \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x\|_2) \le V(x) \le \alpha_2(\|x\|_2) \tag{2.14}$$

holds for $x \in \mathcal{D}_\mathcal{T}$.

Therefore V is a CPA Lyapunov function for system (2.1) on $\mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)$. □

Remark 2.1.6. The usefulness of Theorem 2.1.4 is that it reduces the verification that a function $V \in$ CPA$[\mathcal{T}]$ is a Lyapunov function for system (2.1) to the verification of a finite number of inequalities (2.7). Finding a candidate CPA Lyapunov function can be done as in [5, 29, 42, 76] and Chapter 4, via linear programming. In these papers, the vertex values are introduced as optimization variables, the inequalities (2.7) are considered as constraints, and the objective of the linear optimization problem is to minimize $\max_{\nu \in \{1,2,\ldots,N\}} \|\nabla V_\nu\|_1$. If the linear optimization problem has a feasible solution, then the CPA function V is a CPA Lyapunov function. It is proved that if system is asymptotically stable, then the linear optimization problem has a feasible solution. Alternatively, as in this chapter, one can define $V \in$ CPA$[\mathcal{T}]$ by computing suitable values at the vertices of the simplices of $\mathcal{T}$ and then verify the linear inequalities (2.7). The benifit of the method proposed here is that the cost of computation is lower cheap, see Section 2.3.

In this chapter, the vertex values will be computed using the Yoshizawa function. According to Theorem 1.5.3, the Yoshizawa function is a nonsmooth Lyapunov function. In order to see under what conditions the CPA function constructed based on vertex values assigned by the given Lyapunov functions is a Lyapunov function, in the next theorem we consider CPA approximations to functions. To this end, the simplices in our suitable triangulation are needed to have a certain regularity (see Remark 1.4.8).

Theorem 2.1.7. *Let $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$ be simply connected compact neighbourhoods of the origin such that $\overline{\mathcal{C}^\circ} = \mathcal{C}$, $\overline{\mathcal{D}^\circ} = \mathcal{D}$, and $\mathcal{C} \subset \mathcal{D}^\circ$. Assume that $W : \mathbb{R}^n \to \mathbb{R}_+$ is a Lyapunov function for system* (2.1) *and twice continuously differentiable on $\mathcal{D}$. Let $\varepsilon > 0$ satisfy*

$$\max_{\|x\|_2 \leq \varepsilon} W(x) < \min_{x \in \mathcal{D} \setminus \mathcal{C}^\circ} W(x). \tag{2.15}$$

Then for every $R_1 > 0$ there exists a $\delta_{R_1} > 0$ so that, for any suitable triangulation $\mathcal{T}$ satisfying

$$\mathcal{C} \subset \mathcal{D}_\mathcal{T} \subset \mathcal{D}, \tag{2.16}$$

$$\max_{\mathcal{S}_\nu \in \mathcal{T}} \operatorname{diam}(\mathcal{S}_\nu) \leq \delta_{R_1}, \qquad \textit{and} \tag{2.17}$$

$$\max_{\mathcal{S}_\nu \in \mathcal{T}} \operatorname{diam}(\mathcal{S}_\nu) \|X_\nu^{-1}\|_2 \leq R_1, \qquad \textit{with } X_\nu \textit{ defined in } (1.31) \tag{2.18}$$

the CPA$[\mathcal{T}]$ *approximation V to W on $\mathcal{D}_\mathcal{T}$ is a CPA Lyapunov function for system* (2.1) *on $\mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0, \varepsilon)$.*

Proof. Given $R_1 > 0$ is sufficiently large, there is no problem to have suitable triangulations satisfying (2.16), (2.17) and (2.18). In fact, one can take any δ_{R_1} between zero and ε that is smaller than $\inf\{\|x - y\| : x \in \mathcal{C}, y \in \mathcal{D}^C\}$ and the triangulation $\mathcal{T}^{\mathcal{C}}_{K,b}$ defined in Definition 5.1.2 ([29, Definition 13]) with $K = 0$ and $b = \delta_{R_1}/\sqrt{n}$. For the rest of proof assume that we have such a suitable triangulation $\mathcal{T}$.

Since $W \in C^2(\mathcal{D})$ is a Lyapunov function for system (2.1), there exists a positive definite function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\langle \nabla W(x), f(x) \rangle \leq -\alpha(|x|), \text{ for } x \in \mathcal{D}\ . \tag{2.19}$$

For an arbitrary but fixed $\mathcal{S}_\nu = \operatorname{co}\{x_0, x_1, \cdots, x_n\} \in \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0, \varepsilon)$ define

$$W_\nu := \begin{pmatrix} W(x_1) - W(x_0) \\ W(x_2) - W(x_0) \\ \vdots \\ W(x_n) - W(x_0) \end{pmatrix}. \tag{2.20}$$

As in part (iii) of the proof of [5, Theorem 4.6], we obtain

$$W_\nu - \langle X_\nu^\top, \nabla W(x_0) \rangle = \frac{1}{2} \begin{pmatrix} (x_1 - x_0)^T H_W(z_1)(x_1 - x_0) \\ (x_2 - x_0)^T H_W(z_2)(x_2 - x_0) \\ \vdots \\ (x_n - x_0)^T H_W(z_n)(x_n - x_0) \end{pmatrix},$$

where $H_W(z_i)$ is the Hessian of W at $z_i = x_0 + \xi_i(x_i - x_0)$ for some $\xi_i \in [0,1]$.

Following the same reasoning used in deriving (2.8), we obtain

$$\left\|(x_i - x_0)^T H_W(z_i)(x_i - x_0)\right\|_2 \leq \frac{1}{2}\delta_{R_1}^2 \|H_W(z_i)\|_2. \tag{2.21}$$

Then

$$\|W_\nu - X_\nu \nabla W(x_0)\|_2 \leq \frac{1}{2} n^{\frac{3}{2}} A \delta_{R_1}^2, \tag{2.22}$$

where $A := \max_{\substack{z\in\mathcal{D},\\ i,j=1,2,\ldots,n}} \left|\frac{\partial^2 W}{\partial x_i \partial x_j}(z)\right|$.

It follows from (2.8) that

$$\|\nabla W(x_i) - \nabla W(x_0)\|_2 \leq nA\delta_{R_1}. \tag{2.23}$$

From this we get the inequality

$$\|X_\nu^{-1} W_\nu - \nabla W(x_i)\|_2 \leq nA\delta_{R_1}\left(\frac{1}{2}n^{\frac{1}{2}}R_1 + 1\right). \tag{2.24}$$

Define

$$C := \sup_{x\in\mathcal{D}} \|f(x)\|_2 \tag{2.25}$$

and observe that since $f(x)$ is twice continuously differentiable in $x \in \mathcal{D}$, $C < +\infty$.

For each vertex $x_i \in \mathcal{S}_\nu \in \mathcal{D}_\mathcal{T}$, let $V(x_i) = W(x_i)$. It is obvious that $V(x_i)$ is positive definite for $x_i \in \mathcal{S}_\nu \in \mathcal{D}_\mathcal{T}$.

Choose one simplex $\mathcal{S}_\nu = \mathrm{co}\{x_0, x_1, \cdots, x_n\} \in \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)$ and let $x \in \mathcal{S}_\nu$. Since $V \in \mathrm{CPA}[\mathcal{T}]$, we see that $V(x) = V(x_0) + \langle \nabla V_\nu, (x - x_0)\rangle$. Then taking $x = x_i \in \mathcal{S}_\nu$ for all $i \in \{1, \cdots, n\}$, using the fact that $V(x_i) = W(x_i)$, and the definitions W_ν, (2.20), and X_ν, (1.31), we get

$$\nabla V_\nu = X_\nu^{-1} W_\nu, \tag{2.26}$$

Hence

$$V(x) = V(x_0) + \langle X_\nu^{-1} W_\nu, (x - x_0)\rangle. \tag{2.27}$$

Let $h := \max_{\mathcal{S}_\nu \in \mathcal{T}} \mathrm{diam}(\mathcal{S}_\nu)$. Since $\nabla W(x)$ is bounded for $x \in \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)$, there exists a positive constant G such that

$$\begin{aligned}\|\nabla V_\nu\|_2 = \|X_\nu^{-1} W_\nu\|_2 &\leq \|X_\nu^{-1}\|_2 \max_{z\in\mathcal{D}_\mathcal{T}\setminus\mathcal{B}_2(0,\varepsilon)} \|\nabla W(z)\|_2 h \\ &\leq R_1 \max_{z\in\mathcal{D}_\mathcal{T}\setminus\mathcal{B}_2(0,\varepsilon)} \|\nabla W(z)\|_2 =: G \end{aligned} \tag{2.28}$$

holds uniformly in ν. Let $\nabla V_{\nu,i}(x)$ denote i-th component of $\nabla V_\nu(x)$. We then see that $|\nabla V_{\nu,i}(x)| \leq G$ for $x \in \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)$.

Using (2.19), (2.24), (2.25) and (2.26), we then obtain that

$$\begin{aligned}\langle \nabla V_\nu, f(x_i)\rangle &= \langle (\nabla W(x_i) + \nabla V_\nu - \nabla W(x_i)), f(x_i)\rangle \\ &\leq -\alpha(\|x_i\|_2) + \|X_\nu^{-1} W_\nu - \nabla W(x_i)\|_2 \|f(x_i)\|_2 \\ &\leq -\alpha(\|x_i\|_2) + nA\delta_{R_1}(\frac{1}{2}n^{\frac{1}{2}}R_1 + 1)C. \end{aligned} \tag{2.29}$$

Let $B = \max_{\nu=1,2,\cdots,N} \mu_\nu$. If there exists $\delta_{R_1} \in (0, \varepsilon)$ such that

$$-\alpha(\|x_i\|_2) + nA\delta_{R_1}(\frac{1}{2}n^{\frac{1}{2}}R_1 + 1)C + n^2 B\delta_{R_1}^2 G < 0 \tag{2.30}$$

holds for all $x_i \in \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0, \varepsilon)$, then the linear constraints

$$\langle \nabla V_\nu, f(x_i)\rangle + \|\nabla V_\nu\|_1 E_{i,\nu} < 0 \tag{2.31}$$

are satisfied for all $x_i \in \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0, \varepsilon)$. It is obvious by inspection that such a δ_{R_1} exists.

Furthermore, since for $x \in \mathcal{S}_\nu$ $V(x)$ is defined as interpolated values of $W(x)$ at vertices of $\mathcal{S}_\nu$, we obtain by (2.15)

$$\max_{\|x\|_2 \le \varepsilon} V(x) \le \max_{\|x\|_2 \le \varepsilon} W(x) < \min_{x \in \mathcal{D}\setminus\mathcal{C}^\circ} W(x) \le \min_{x \in \mathcal{D}\setminus\mathcal{C}^\circ} V(x). \tag{2.32}$$

Because $W(x)$ is positive definite, so is $V(x)$. Consequently, Corollary 2.1.5 proves the theorem. □

Remark 2.1.8. Theorem 2.1.7 is more constructive than it might seem at first glance since a given suitable triangulation $\mathcal{T}$ can be manipulated to deliver a new suitable triangulation $\mathcal{T}^*$ with smaller simplices without increasing their degeneracy. From (2.30), we obtain that for $R_1 < \sqrt{\frac{\alpha(\epsilon)}{n^{\frac{3}{2}}A}}$, there exists $\delta_{R_1} > 0$ such that (2.30) holds. As a consequence, it is always possible to find a suitable triangulation that admits a CPA Lyapunov function approximating a twice continuously differentiable Lyapunov function.

2.1.2 Yoshizawa construction of Lyapunov functions

In order to construct a CPA Lyapunov function, we now turn to the question of how to define the vertex values of each simplex . We propose using a numerical approximation of the Yoshizawa function defined in Definition 1.5.2.

We suppose system (2.1) is $\mathcal{KL}$ stable on $\mathcal{D}$, i.e., there exists $\beta \in \mathcal{KL}$ such that

$$\|\phi(t, x)\|_2 \le \beta(\|x\|_2, t),\ \forall x \in \mathcal{D},\ t \in \mathbb{R}_+. \tag{2.33}$$

From Lemma 1.5.1, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\beta(s, t)) \le \alpha_2(s)e^{-2t}, \quad s, t \in \mathbb{R}_+. \tag{2.34}$$

According to Definition 1.5.2 and Theorem 1.5.3, we have the Yoshizawa function

$$V(x) = \sup_{t \ge 0} \alpha_1(\|\phi(t, x)\|_2)e^t = \max_{t \in [0, T_1(x)]} \alpha_1(\|\phi(t, x)\|_2)e^t. \tag{2.35}$$

is a nonsmooth Lyapunov function for system (2.1), where

$$T_1(x) = \ln\left(\frac{\alpha_2(\|x\|_2)}{\alpha_1(\|x\|_2)}\right) + 1 \tag{2.36}$$

Our intention is to calculate $V(x)$ for each x that is a simplex vertex. In order to do this, we clearly need a solution to (2.1) from each such x. As a closed form solution is generally not available, we will resort to numerical integration in order to calculate $V(x)$ given by (2.35). For this approach to be numerically tractable, we require that the time horizon $T_1(x)$ given by (2.36) not be too large. We present two examples of stability estimates and derive $T_1(x)$ in each case.

Example 1 - Exponential Stability Estimates

Suppose the stability estimate is given by

$$\|\phi(t,x)\|_2 \leq \alpha(\|x\|_2)e^{-\mu t}, \quad \mu \in \mathbb{R}_{>0},$$

where $\alpha(s) \geq s$, $\alpha \in \mathcal{K}_\infty$. Then we can define

$$\alpha_1(s) = s^{2/\mu}, \quad \text{and} \quad \alpha_2(s) = (\alpha(s))^{2/\mu}$$

so that

$$\alpha_1(\alpha(\|x\|_2)e^{-\mu t}) \leq (\alpha(\|x\|_2))^{2/\mu}e^{-2t} = \alpha_2(\|x\|_2)e^{-2t}.$$

We therefore see that, in this case, an upper bound for the time horizon to optimize over is given by

$$T_1(x) \leq \frac{2}{\mu}\ln\left(\frac{\alpha(\|x\|_2)}{\|x\|_2}\right) + 1 \tag{2.37}$$

where the assumption that $\alpha(s) \geq s$ for all $s \in \mathbb{R}_+$ guarantees that $T_1(x) \geq 1$.

If $\alpha(s) = Ms$ for some $M > 1$, then an upper bound for the time horizon necessary to optimize over is independent of the point x and is given by

$$T_1(x) = -\ln M^{-2/\mu} + 1 = \tfrac{2}{\mu}\ln M + 1.$$

Example 2

With the functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ given by

$$\alpha_1^{-1}(s) = e^s - 1, \quad \alpha_2(s) = Ms$$

we choose functions $\beta \in \mathcal{KL}$ satisfying

$$\beta(s,t) \leq \exp(Mse^{-2t}) - 1$$

and the optimization horizon bound is given by

$$T_1(x) \leq \ln\left(\frac{M\|x\|_2}{\ln(1 + \|x\|_2)}\right) + 1.$$

The horizon length grows with increasing $\|x\|_2$ but not too quickly. For example, with $M = 10$: $\|x\|_2 = 1$ yields $T_1(x) = 3.67$ and $\|x\|_2 = 100$ yields $T_1(x) = 6.38$.

Remark 2.1.9. There are two difficulties we encounter in calculating the stability estimate (2.35). The first difficulty lies in finding a stability estimate $\beta \in \mathcal{KL}$. There seems to be little that can be done to circumvent this problem.

The second difficulty is that Sontag's lemma on $\mathcal{KL}$-estimates is not constructive and, to the best of our knowledge, given an arbitrary $\beta \in \mathcal{KL}$, there are currently no constructive techniques for finding $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

2.1.3 Examples

In this section, three numerical examples are presented to demonstrate the construction of CPA Lyapunov functions using the Yoshizawa function. In each case, we first define a suitable triangulation on a region of the state space that includes the origin in its interior.

While the Yoshizawa function (2.35) provides the exact value for a Lyapunov function at each vertex, under the assumption that we have an exact solution of the differential equation. As that is not feasible in practice, we will use numerical integration to obtain an approximation solution. For each example we calculate a stability estimate $\beta \in \mathcal{KL}$ and, with the triangulation defined, at each simplex vertex, x, we calculate an approximate solution to (2.1) on the interval $[0, T_1(x)]$ with the vertex as the initial condition. We can then calculate the value of the Yoshizawa function using (2.35). Finally, we check the inequality (2.7) to verify that the function defined by taking the convex interpolation on each simplex of the Yoshizawa function values is, in fact, a CPA Lyapunov function on a region $\mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_2(0, \varepsilon)$.

In what follows, we use a simple Euler scheme to perform the numerical integration. More complicated integration schemes were also investigated, but for the examples below these provided no benefit; i.e., the region on which the inequalities (2.7) are satisfied was essentially the same.

Based on the above analysis, the procedure of computation CPA Lyapunov functions for system (2.1) is summarized as the following.

1. Calculate a stability estimate $\beta \in \mathcal{KL}$ such that $\|\phi(t,x)\|_2 \leq \beta(\|x\|_2, t)$.

2. Choose α_1, $\alpha_2 \in \mathcal{K}_\infty$ satisfying $\alpha_1(\beta(\|x\|_2, t) \leq \alpha_2(\|x\|_2)e^{-2t}$.

3. Define the Yoshizawa function by $V(x) = \max_{t \in [0, T_1(x)]} \alpha_1(\|\phi(t,x)\|_2)e^t$.

4. Partition the subset $\mathcal{D}$ into a suitable triangulation $\mathcal{T}$.

5. Compute a numerical solution to (2.1) on the interval $[0, T_1(x_i)]$ with the vertex x_i as the initial condition by Euler integration scheme.

6. Define values $V(x_i)$ at vertex x_i of every simplex $\mathcal{S}_\nu \in \mathcal{T}$ based on the Yoshizawa function (2.35) with the computed numerical solution to (2.1) from the last step.

7. Take the convex interpolation of these values $V(x_i)$ at vertex x_i of each simplex $\mathcal{S}_\nu \in \mathcal{T}$, i.e., for $x \in \mathcal{S}_\nu$ $x = \sum_{i=0}^n \lambda_i x_i$ $(0 \leq \lambda_i \leq 1)$, then

$$V(x) = \sum_{i=0}^{n} \lambda_i V(x_i) \tag{2.38}$$

   and then obtain a CPA function $V : \mathcal{D}_{\mathcal{T}} \to \mathbb{R}_+$.

8. Calculate μ_ν such that (2.5) holds.

9. Verify if V satisfies the system-dependent set of linear inequalities (2.7) for all vertices in the set $\mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_2(0, \varepsilon)$.

Triangulation

We now describe the construction of the suitable triangulations used in the subsequent examples. We present the process used for the two dimensional examples.

Fix positive integers $K \in \mathbb{Z}_{>0}$, $k \in \mathbb{Z}_+$ and define the preliminary vertex set given by

$$\overline{\mathcal{V}} := \{(i,j)^\top \in \mathbb{Z}^2_{>0} : (i,j) \in [-K,K]^2 \setminus (-k,k)^2\}. \tag{2.39}$$

Simplex edges are then defined in two regions. For each $(i,j)^\top \in (-k,k)^2 \cap \overline{\mathcal{V}}$, place an edge between $(i,j)^\top$ and the origin. This gives the central, fan-like region in Figure 2.1. For each $(i,j)^\top \in \overline{\mathcal{V}} \setminus (-k,k)^2$, edges are placed between vertices satisfying $\{(i,j)^\top, (i+1,j)^\top\}$, $\{(i,j)^\top, (i,j+1)^\top\}$, and if $sign(i) = sign(j)$ between $\{(i,j)^\top, (i+1,j+1)^\top\}$ and if $sign(i) = -sign(j)$ between $\{(i,j)^\top, (i+1,j-1)^\top\}$. This yields the triangulation shown in Figure 2.1.

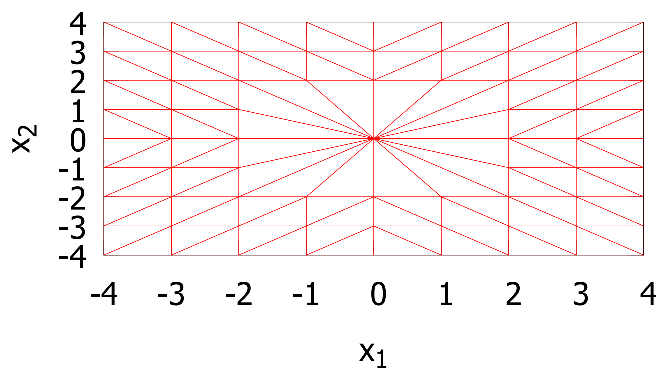

Figure 2.1: The triangulation before scaling by $F(\cdot)$.

To obtain the triangulation used for the examples, we transfer the vertices from these triangulations into new vertices by the mapping $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ defined by

$$F(x) = \begin{cases} 1.2x\frac{10^{-4}\|x\|^2_\infty}{\|x\|_2}, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{2.40}$$

Based on the new vertices, we get the new triangulation used in computation. For the case where $K = 4$ and $k = 2$, this yields the triangulation shown in Figure 2.2.

Note that in defining the initial vertex set we fixed a large square, characterized by K, and exclude a smaller square, characterized by k, from interior. However, for systems with complicated dynamics or for equilibria with complicated regions of attraction, particularly in higher dimensions, it can be useful to define more complicated regions. Using the two dimensional case as an example, one straightforward modification to achieve this is to allow different constants defining the initial vertex region; i.e., choose $K_i, k_i \in \mathbb{Z}_{>0}$, $i = 1,2,3,4$, and then define the preliminary vertex set using the regions $[-K_1, K_2] \times [-K_3, K_4]$ and $[-k_1, k_2] \times [-k_3, k_4]$.

Let

$$\mathcal{T}^\epsilon := \{\mathcal{S}_\nu \,|\, \mathcal{S}_\nu \cap \mathcal{B}^C_2(0,\epsilon) \neq \emptyset\} \subset \mathcal{T} \quad \text{and} \quad \mathcal{D}^\epsilon_{\mathcal{T}} := \bigcup_{\mathcal{S}_\nu \in \mathcal{T}^\epsilon} \mathcal{S}_\nu. \tag{2.41}$$

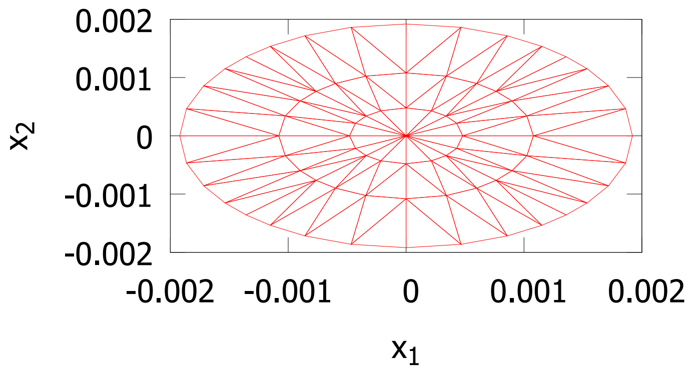

Figure 2.2: Triangulation after scaling by $F(\cdot)$.

Given a suitable triangulation, for $V \in \mathrm{CPA}[\mathcal{T}]$ we have

$$\max_{y \in \partial(\mathcal{D}_{\mathcal{T}} \setminus \mathcal{D}^{\epsilon}_{\mathcal{T}})} V(y) \geq V(x), \text{ vertex } y,\, x \in \mathcal{D}_{\mathcal{T}} \setminus \mathcal{D}^{\epsilon}_{\mathcal{T}}. \tag{2.42}$$

Example 3 - Linear System

Consider the linear system

$$\dot{x} = Ax = \begin{bmatrix} 1 & 1 \\ -5 & -3 \end{bmatrix} x. \tag{2.43}$$

We observe that the origin is globally exponentially stable as the eigenvalues are at $-1 \pm i$ and, by solving the Lyapunov equation $A^{\top}P + PA = -\mathrm{Id}$, a Lyapunov function is given by

$$V(x) = x^{\top}Px = x^{\top} \begin{bmatrix} 4.5 & 1 \\ 1 & 0.5 \end{bmatrix} x. \tag{2.44}$$

By explicitly calculating the solutions of (2.43) we see that the system satisfies the stability estimate

$$\|\phi(t,x)\|_2 \leq 7\|x\|_2 e^{-t}, \quad \forall x \in \mathbb{R}^2,\ t \in \mathbb{R}_+. \tag{2.45}$$

From Section 2.1.2, with $\alpha(s) = 7s$ and $\mu = 1$ we see that $\alpha_1(s) = s^2$, $\alpha_1(s) = 49s^2$, and $T_1(x) = 4.892$.

According to the above proposed procedure, we define a suitable triangulation as described in the above with $K = 90$ and $k = 20$. The values at the simplex vertices are given by approximating the solution of (2.43) by numerical integration over the time interval $[0, 4.892]$ and computing the value of the Yoshizawa function (2.35). This then defines a CPA function V_1 as shown in Figure 2.3. It is straightforward to numerically verify that the inequalities (2.7) are satisfied for all simplex vertices where $\mathcal{S}_{\nu} \cap \mathcal{B}_2^C(0, 0.048) \neq \emptyset$. Hence, V_1 is a CPA Lyapunov function on $\mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_2(0, 0.048) = \mathcal{B}_2(0, 0.972) \setminus \mathcal{B}_2(0, 0.048)$.

The function V defined by (2.44) which is shown by Figure 2.4 has a similar though slightly different shape. The CPA Lyapunov function V_1 does not allow for an explicit formulation as the quadratic Lyapunov function V. In order to compare the quadratic Lyapunov function V and CPA Lyapunov function V_1, level curves of V_1 and V are shown in Figures 2.5 and 2.6, respectively.

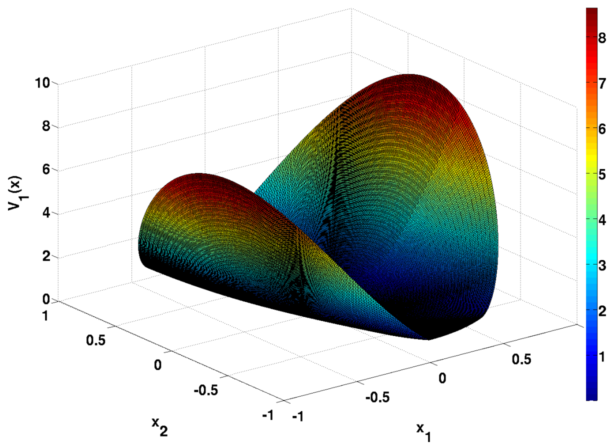

Figure 2.3: CPA Lyapunov function V_1 for system (2.43).

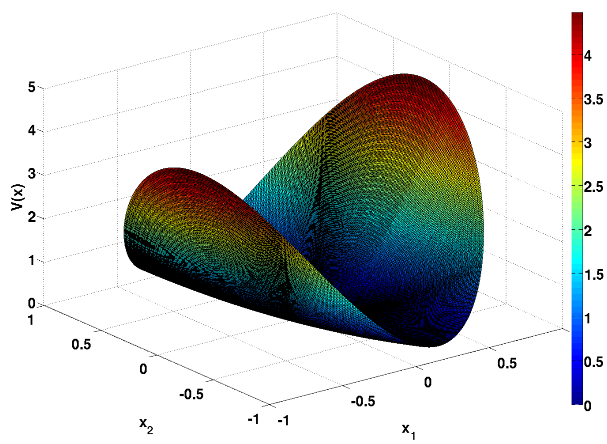

Figure 2.4: Lyapunov function V for system (2.43).

Example 4 - Simple Nonlinear System

Consider

$$\dot{x} = -x^3 \tag{2.46}$$

which has the solutions

$$\phi(t,x) = \frac{x}{\sqrt{1+2x^2t}}, \quad \forall x \in \mathbb{R}, \ t \in \mathbb{R}_+. \tag{2.47}$$

We observe that the norm of the solution is in fact a $\mathcal{KL}$ function and we can verify that the functions

$$\alpha_1(s) = \alpha_2(s) = \begin{cases} 0, & s = 0, \\ s \exp\left(-\frac{1}{s^2}\right), & s > 0 \end{cases} \tag{2.48}$$

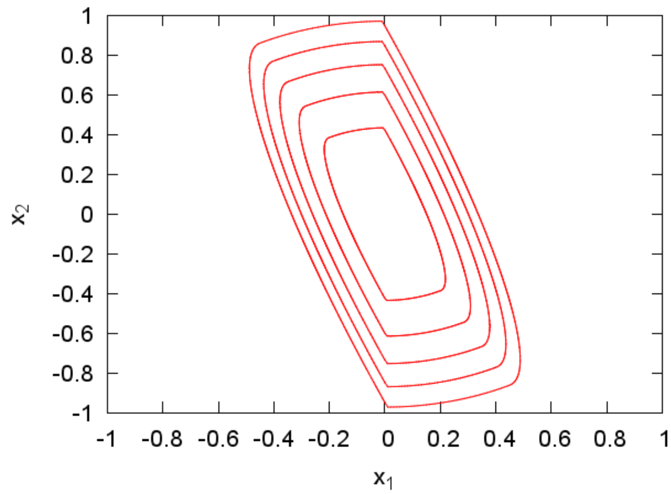

Figure 2.5: Level curves of V_1 for values 0.189, 0.378, 0.567, 0.756 and 0.945.

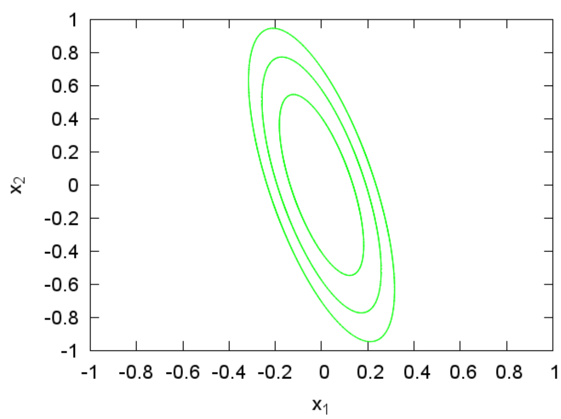

Figure 2.6: Level curves of V for values 0.083, 0.166 and 0.249.

are such that

$$\alpha_1\left(\frac{s}{\sqrt{1+2s^2t}}\right) \leq \alpha_2(s)e^{-2t}. \tag{2.49}$$

We define a suitable triangulation with $K = 200$ and $k = 19$, where modifying the procedure of triangulation for a one dimensional system is straightforward. We calculate the values at the simplex vertices by approximating (2.35) and a convex interpolation of these values on each simplex then yields a CPA function V_2. We verify that the inequalities (2.7) are satisfied for all simplex vertices where $\mathcal{S}_\nu \cap [0, 0.043]^C \neq \emptyset$ and hence V_2 is a CPA Lyapunov function on $[-4.8, 4.8] \setminus (-0.043, 0.043)$, where the outer limits of $\mathcal{D}_\mathcal{T}$ come from $F(K) = F(200) = 4.8$.

We note that, for any $p \in \mathbb{Z}_{>0} \geq 1$ and $c > 0$, a Lyapunov function is given by

$$V(x) = cx^{2p}, \quad \forall x \in \mathbb{R}. \tag{2.50}$$

Figure 2.7 shows the CPA Lyapunov function V_2 for system (2.46) for $4.8 \geq \|x\|_2 \geq$

0.04332. In order to nicely compare the known Lyapunov function V and the CPA Lyapunov function V_2 in Figure 2.7, we choose $p = 2$, $c = 0.01$.

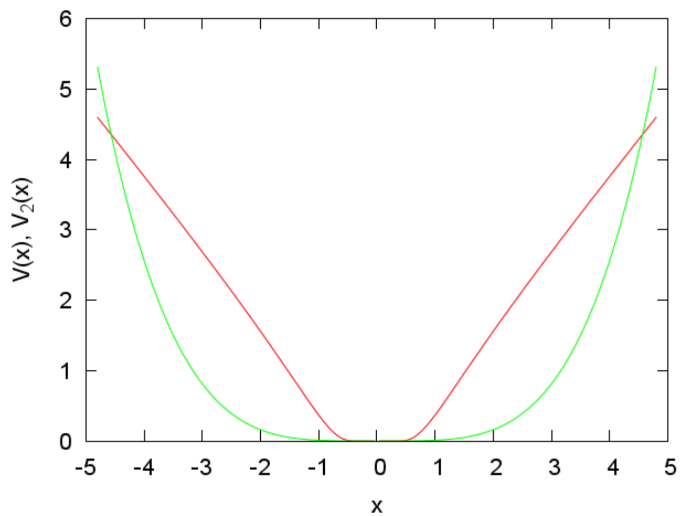

Figure 2.7: Lyapunov functions $V(x) = 0.01x^4$ (green curve) and V_2 (red curve) for system (2.46).

Example 5 - Nonlinear System

Consider the two-dimensional nonlinear system given by

$$\begin{cases} \dot{x}_1 = -x_2 - (1 - x_1^2 - x_2^2)x_1, \\ \dot{x}_2 = x_1 - (1 - x_1^2 - x_2^2)x_2. \end{cases} \tag{2.51}$$

This system has the unit circle as a periodic orbit and the origin as a locally asymptotically stable equilibrium. On any compact subset of the unit ball, the simple quadratic

$$V(x) = x_1^2 + x_2^2 \tag{2.52}$$

is a known Lyapunov function.

Fix $R \in (0, 1)$. Then, for any initial condition satisfying

$$x_1^2 + x_2^2 \leq R$$

we have the stability estimate

$$\|\phi(t, x)\|_2 \leq \|x\|_2 e^{-(1-R)t} \tag{2.53}$$

and, from Example 1 in Section 2.1.2, we can calculate

$$\alpha_1(s) = \alpha_2(s) = s^{2/(1-R)}$$

and $T_1(x) = 1$.

For this example, with $R = 0.94478$, the CPA function V_3 is shown in Figure 2.8. The triangulation is defined with $K = 90$ and $k = 10$, yielding a region that is coincident with the region which the stability estimate is valid; i.e., on $\mathcal{B}_2(0, \sqrt{R})$. The inequalities (2.7) hold for all simplex vertices where $\mathcal{S}_\nu \cap \mathcal{B}_2^C(0, 0.012) \neq \emptyset$. Thus, V_3 is a CPA Lyapunov function on $\mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0, 0.012) = \mathcal{B}_2(0, 0.972) \setminus \mathcal{B}_2(0, 0.012)$. For comparison, Figure 2.8 also shows the known Lyapunov function (2.52). Besides, level curves of V_3 and V are shown in Figures 2.10 and 2.9, respectively.

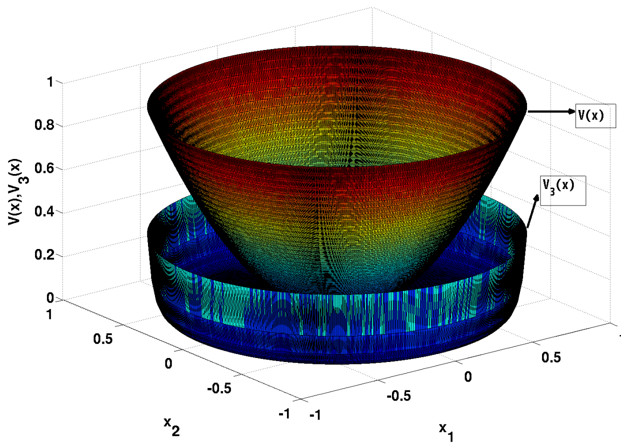

Figure 2.8: Lyapunov functions V and V_3 for system (2.51).

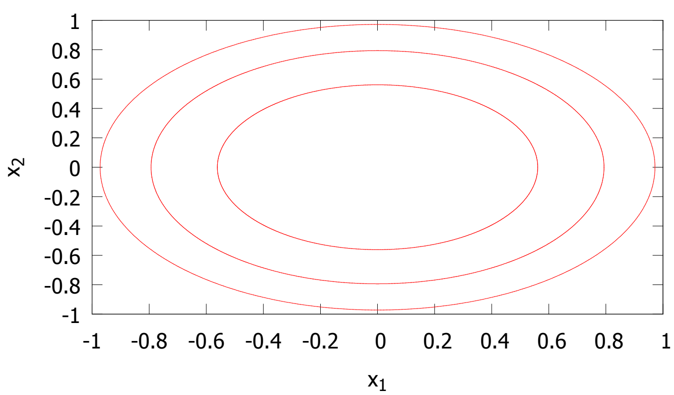

Figure 2.9: Level curves of V for values 0.1192, 0.2383 and 0.3575.

2.1.4 Conclusion

In this section, we proposed a new method to compute a CPA Lyapunov function given the system is $\mathcal{KL}$-stable. From the results of these three examples it is obvious that the method is feasible.

As Theorem 1.5.3 mentioned, it was demonstrated that the Yoshizawa-Lyapunov function (2.35) is locally Lipschitz continuous except possibly at the origin. However, in order to make

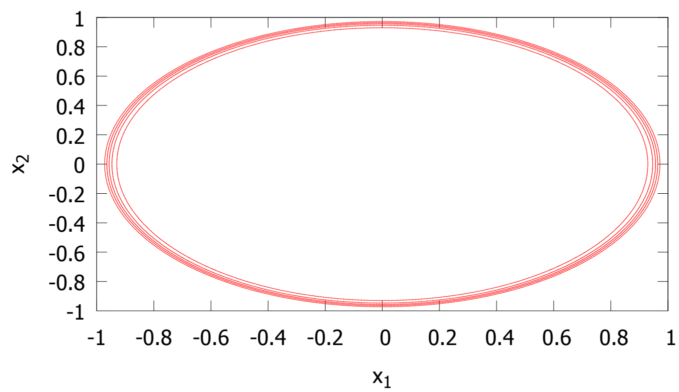

Figure 2.10: Level curves of V_3 for values 0.1890, 0.3779, 0.5669, 0.7558 and 0.9448.

use of Theorem 2.1.7 we would clearly prefer that the Yoshizawa-Lyapunov function be twice continuously differentiable. While it is known that, in general, there exists a Lyapunov function that inherits the regularity property of the vector field defining (2.1) without requiring the use of smoothing techniques (see [7]), the numerical example of Section 2.1.3 indicates that this is not true for the Yoshizawa function (see Figure 2.5). It is seen from Figure 2.5 that the Yoshizawa function is not differentiable.

2.2 Discrete time case

In this section, we apply the above approach of computing a CPA Lyapunov function to the discrete time nonlinear system described by

$$x^+ = f(x), \tag{2.54}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous, and $f(0) = 0$.

We observe that deriving such discrete time results from their continuous time counterparts is nontrivial due to the fact that solutions in the discrete time setting are sequences of points rather than absolutely continuous functions as in the continuous time setting.

We consider system (2.54) on a compact set $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$. In this section, we further assume the following conditions hold.

- The set $\mathcal{D}$ is positively invariant for system (2.54).
- $\mathcal{T} = \{\mathcal{S}_1, \dots, \mathcal{S}_N\}$ is a suitable triangulation of $\mathcal{D}$ with $\mathcal{D}_\mathcal{T} = \cup_{\mathcal{S}_\nu \in \mathcal{T}} \mathcal{S}_\nu$.

2.2.1 Continuous and piecewise affine Lyapunov function

In the following, we present the definition of a CPA Lyapunov function for system (2.54) on a closed, connected set $\mathcal{O} \subset \mathcal{D}_\mathcal{T} \subset \mathbb{R}^n$ excluding a fixed arbitrary small neighbourhood of the origin.

Definition 2.2.1. Let $V \in \mathrm{CPA}[\mathcal{T}]$, α_1, $\alpha_2 \in \mathcal{K}_\infty$ and $\mathcal{O} \subset \mathcal{D}_\mathcal{T}$ be a closed, connected set such that

(i) $0 \in \mathcal{O}^\circ$;

(ii) there exists no $\mathcal{S}_\nu$ with $x, y \in \mathcal{S}_\nu$ satisfying $x \in \mathcal{O}, y \in \mathcal{D}_\mathcal{T} \setminus \mathcal{O}$;

(iii) $f(x) \in \mathcal{D}_\mathcal{T}$, for $x \in \mathcal{O}$; and

(iv) $\alpha_1(\|x\|_2) \le V(x) \le \alpha_2(\|x\|_2)$, for $x \in \mathcal{D}_\mathcal{T}$.

Define the constant

$$q^* = \inf\{q \in \mathbb{R}_+ : \|f(x)\|_2 \le q\|x\|_2,\ x \in \mathcal{O}\}. \tag{2.55}$$

Since f is locally Lipschitz continuous on $\mathcal{O}$, q^* exists.

Let $\varepsilon > 0$ satisfy

$$\begin{cases} \max\limits_{\|x\|_2 \le q^*\varepsilon} V(x) < \min\limits_{x \in \partial\mathcal{O}} V(x), \\ & \text{for } q^* \ge 1, \\ \mathcal{B}_2(0, q^*\varepsilon) \subset \mathcal{O}. \end{cases} \tag{2.56}$$

or

$$\max_{\|x\|_2 \le \varepsilon} V(x) < \min_{x \in \partial\mathcal{O}} V(x), \qquad \text{for } q^* < 1. \tag{2.57}$$

If there is a constant $\alpha_3^* > 0$ such that

$$V(f(x)) - V(x) \le -\alpha_3^*\|x\|_2 \tag{2.58}$$

holds for all $x \in \mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$, then V is called a CPA *Lyapunov function* for (2.54) on $\mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$.

We denote the set of solutions of (2.54) at time $k \in \mathbb{Z}_+$ from a compact set $\mathcal{C} \subset \mathbb{R}^n$ by $\phi(k, \mathcal{C}) := \bigcup_{x \in \mathcal{C}} \phi(k, x)$. Define the sublevel sets of V by

$$L_{V,c} := \{x \in \mathcal{O} : V(x) \le c\}, \quad c \ge 0. \tag{2.59}$$

Theorem 2.2.2. *Let the function $V : \mathcal{D}_\mathcal{T} \to \mathbb{R}_+$ be a* CPA *Lyapunov function for system (2.54) on $\mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$ with appropriate $q^*, \varepsilon \in \mathbb{R}_{>0}$ as in Definition 2.2.1 and where $\mathcal{O} \subset \mathbb{R}^n$ satisfies conditions (i)-(iii) from Definition 2.2.1. Define*

$$m := \begin{cases} \max_{\|x\|_2 \le q^*\varepsilon} V(x), & \text{if} \quad q^* \ge 1, \\ \max_{\|x\|_2 \le \varepsilon} V(x), & \text{if} \quad q^* < 1, \end{cases}$$

and $M := \min_{x \in \partial\mathcal{O}} V(x)$. Then, for certain $c \in (m, M)$, $\mathcal{B}_2(0,\varepsilon) \subset L_{V,c} \subset \mathcal{O}^\circ$ and there exists a $K_c \in \mathbb{Z}_+$ such that $\phi(k, L_{V,c}) \subset L_{V,m}$ for all $k \ge K_c$.

Proof. Using (2.56), (2.57) and the definitions of m, M, we have $m < M$. It follows directly by the definitions of m and M and the continuity of V that $\mathcal{B}_2(0,\varepsilon) \subset L_{V,c} \subset \mathcal{O}^\circ$.

For $x \in L_{V,c} \setminus \mathcal{B}_2(0,\varepsilon)$ we get by (2.58) that $f(x) \in L_{V,c}$. For $x \in \mathcal{B}_2(0,\varepsilon)$ we get by (2.55) that $\|f(x)\|_2 < q^*\varepsilon$ for $q^* \ge 1$, and $\|f(x)\|_2 < \varepsilon$ for $q^* < 1$. Hence, by the definition of m we get $f(x) \in L_{V,m} \subset L_{V,c}$. Thus $L_{V,c}$ is positively invariant. The last assertion of the theorem now follows from (2.58) with $K_c \ge (c - m)/(\alpha_3\varepsilon)$. $\square$

Remark 2.2.3. The conditions of Theorem 2.2.2 are more restrictive than those in Theorem 2.1.3. These more restrictive conditions are required because the solution of (2.54) is a sequence of points rather than an absolutely continuous function. Similar to the results for continuous time case in Theorem 2.1.3, Theorem 2.2.2 provides an estimate of the domain of attraction for the positively invariant set $L_{V,m}$.

We state the criteria for verifying that a CPA function is a CPA Lyapunov function in Theorem 2.2.4 and Corollary 2.2.6.

Theorem 2.2.4. *Let $\mathcal{T}$ be a suitable triangulation, a closed, connected set $\mathcal{O} \subset \mathbb{R}^n$ satisfy conditions (i)-(iii) from Definition 2.2.1, and $\varepsilon > 0$ be a small constant. Let $V \in \mathrm{CPA}[\mathcal{T}]$ and C, $L_\nu \geq 0$ satisfy*

$$\|f(x) - f(y)\|_2 \leq L_\nu \|x - y\|_2, \text{ for } x, y \in \mathcal{S}_\nu, \tag{2.60}$$

$$\max_{\nu=1,\dots,N} \|\nabla V_\nu\|_2 \leq C. \tag{2.61}$$

If for each $\mathcal{S}_\nu \subset \mathcal{O}$, $\mathcal{S}_\nu \cap \mathcal{B}_2(0,\varepsilon)^C \neq \emptyset$ the inequalities

$$V(f(x_i^\nu)) - V(x_i^\nu) + CL_\nu \operatorname{diam}(\mathcal{S}_\nu) < 0 \tag{2.62}$$

hold for all $i = 0, 1, \dots, n$, then

$$V(f(x)) - V(x) < 0 \tag{2.63}$$

for all $x \in \mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$.

Proof. Let $x \in \mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$ be arbitrary. Then there exists a $\mathcal{S}_\nu \subset \mathcal{O}$ such that $x \in \mathcal{S}_\nu$; i.e., $x = \sum_{i=0}^n \lambda_i x_i^\nu$ where $\sum_{i=0}^n \lambda_i = 1$. Let $C_\nu = \|\nabla V_\nu\|_2$. By Remark 1.4.10 and inequalities (2.60) and (2.61), we have

$$\begin{aligned} V(f(x)) - V(x) &= V(f(x)) - \sum_{i=0}^n \lambda_i V(f(x_i^\nu)) + \sum_{i=0}^n \lambda_i V(f(x_i^\nu)) - \sum_{i=0}^n \lambda_i V(x_i^\nu) \\ &= \sum_{i=0}^n \lambda_i [V(f(x)) - V(f(x_i^\nu)) + V(f(x_i^\nu)) - V(x_i^\nu)] \\ &\leq \sum_{i=0}^n \lambda_i [V(f(x_i^\nu)) - V(x_i^\nu) + C_\nu L_v \|x - x_i^\nu\|_2] \\ &\leq \sum_{i=0}^n \lambda_i [V(f(x_i^\nu)) - V(x_i^\nu) + CL_\nu \operatorname{diam}(\mathcal{S}_\nu)]. \end{aligned} \tag{2.64}$$

Based on (2.62), we conclude that $V(f(x)) - V(x) < 0$ for all $x \in \mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$. □

Lemma 2.2.5. *If $f(x)$ is a C^2 function, then another set of inequalities, i.e. (2.5) and*

$$V(f(x_i^\nu)) - V(x_i^\nu) + \|\nabla V_\nu\|_1 E_{i,\nu} < 0, \quad E_{i,\nu} \text{ defined in (2.6)} \tag{2.65}$$

could be given to make sure the inequality (2.63) holds for $x \in \mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$. A better estimate of the interpolation error can be given by (2.6).

Proof. See the proofs of Theorems 2.1.4 and 2.2.4. □

Corollary 2.2.6. *Let $V \in \mathrm{CPA}[\mathcal{T}]$ from Theorem 2.2.4 be positive definite and the constant $\varepsilon > 0$ satisfy (2.56) or (2.57) as appropriate. If the inequalities (2.62) are satisfied for all $\mathcal{S}_\nu \subset \mathcal{O}$ with $\mathcal{S}_\nu \cap \mathcal{B}_2(0,\varepsilon)^C \neq \emptyset$, then V is a CPA Lyapunov function for system (2.54) on $\mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$.*

Proof. According to (2.62) and the inequality (2.64), there exists $\delta_i^\nu > 0$ such that

$$V(f(x)) - V(x) \leq -\sum_i^n \lambda_i \delta_i^\nu$$

for $x \in \mathcal{S}_\nu \subset \mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$. Since the number of vertices is finite, there exists a $\delta > 0$ satisfying $\delta \leq \delta_i^\nu$ for $i = 1, \ldots, n$, $\nu = 1, \ldots, N$. Therefore,

$$V(f(x)) - V(x) \leq -\delta < 0$$

holds for $x \in \mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$. Let $r_1 > 0$ such that $\mathcal{O} \subset \mathcal{B}_2(0, r_1)$. Therefore there exists a constant $\alpha_3^* = \frac{\delta}{r_1} > 0$ such that $V \in \mathrm{CPA}[\mathcal{T}]$ satisfies (2.58) on $\mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$. Hence $V \in \mathrm{CPA}[\mathcal{T}]$ is a CPA Lyapunov function for system (2.54) on $\mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$. □

Remark 2.2.7 explains why we compute a CPA Lyapunov function using Yoshizawa function for discrete time system (2.54). The reason is similar to Remark 2.1.6.

Remark 2.2.7. From Theorem 2.2.4 and Corollary 2.2.6, for a candidate Lyapunov function $V \in \mathrm{CPA}[\mathcal{T}]$, the verification that V is a Lyapunov function for system (2.54) is done by checking that V is positive definite and that the inequality (2.62) holds for each vertex. The problem then is to find a candidate Lyapunov function. In order to obtain a CPA candidate Lyapunov function, a CPA function was obtained by solving a linear programming problem in [28]. In that paper, the vertex values are introduced as optimization variables, the inequalities (2.64) are considered as optimization constraints. The objective is to minimize $\max\limits_{\nu \in \{1,\ldots,N\}} \|V_\nu\|_1$. If the linear optimization problem has a feasible solution, then the CPA function V is a Lyapunov function. In this section, we compute the value at each vertex by using the Yoshizawa function (see Definition 1.5.6), and then verify the inequality (2.62) for each vertex, since the cost of the similar approach used in Section 2.1 is much cheaper than solving a linear optimization problem.

We now state conditions under which the CPA approximation to a Lyapunov function is also a Lyapunov function. It lays the foundation of constructing Lyapunov function using the Yoshizawa function. For the following theorem, the simplices in our suitable triangulation are supposed to have a certain regularity (see Remark 1.4.8).

Theorem 2.2.8. *Let $\mathcal{O}, \mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$ be simply connected compact neighbourhoods of the origin such that $\overline{\mathcal{O}^\circ} = \mathcal{O}$, $\overline{\mathcal{C}^\circ} = \mathcal{C}$, $\overline{\mathcal{D}^\circ} = \mathcal{D}$, $\mathcal{C} \subset \mathcal{O}^\circ$, $\mathcal{O} \subset \mathcal{D}^\circ$ and $f(x) \in \mathcal{D}^\circ$ for $x \in \mathcal{O}$. Further, assume that $W \in C^1(\mathcal{D})$ is a Lyapunov function for system (2.54) and there exists a constant*

$L > 0$ such that $\|\nabla W(x)\|_2 \le L$ for $x \in \mathcal{D}$. Let $\varepsilon > 0$ be such that

$$\begin{cases} \max\limits_{\|x\|_2 \le q^*\varepsilon} W(x) < \min\limits_{x\in\mathcal{O}\setminus\mathcal{C}^\circ} W(x), & \text{for } q^* \ge 1, \\ \\ \mathcal{B}_2(0, q^*\varepsilon) \subset \mathcal{O}, & \text{for } q^* \ge 1, \\ \\ \max\limits_{\|x\|_2 \le \varepsilon} W(x) < \min\limits_{x\in\mathcal{O}\setminus\mathcal{C}^\circ} W(x), & \text{for } q^* < 1. \end{cases} \tag{2.66}$$

Then for every $R_1 > 0$ there exists a constant $\delta_{R_1} > 0$ such that, for any suitable triangulation $\mathcal{T}$ satisfying (i)-(iii) from Definition 2.2.1 and

$$\max_{\mathcal{S}_\nu\in\mathcal{T}} \operatorname{diam}(\mathcal{S}_\nu) \le \delta_{R_1}, \text{ and} \tag{2.67}$$

$$\max_{\mathcal{S}_\nu\in\mathcal{T}} \operatorname{diam}(\mathcal{S}_\nu)\|X_\nu^{-1}\|_2 \le R_1, \text{ with } X_\nu \text{ defined in } (1.31), \tag{2.68}$$

the CPA$[\mathcal{T}]$ *approximation V to W on $\mathcal{D}_\mathcal{T}$ is a CPA Lyapunov function for system* (2.54) *on $\mathcal{O} \setminus \mathcal{B}_2(0, \varepsilon)$.*

Proof. The proof is similar to the proof of Theorem 2.1.7.

If $R_1 > 0$ is sufficiently large, then there is no problem to have suitable triangulations satisfying (i)-(iii) from Definition 2.2.1, (2.67) and (2.68). Actually, δ_{R_1} can be chosen between zero and ε that is smaller than $\inf\{\|x - y\| : x \in \mathcal{C}, y \in \mathcal{O}^C\}$ and the triangulation $\mathcal{T}^{\mathcal{C}}_{K,b}$ introduced in Definition 5.1.2 ([29, Definition 13]) with $K = 0$ and $b = \delta_{R_1}/\sqrt{n}$. For the rest of proof assume that we have such a suitable triangulation $\mathcal{T}$.

Since $W(x)$ is a Lyapunov function for (2.54) on $\mathcal{D}$, we get that there exists a positive definite function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$W(f(x)) - W(x) \le -\alpha(\|x\|_2), \text{ for } x \in \mathcal{D} . \tag{2.69}$$

For an arbitrary but fixed $\mathcal{S}_\nu = \operatorname{co}\{x_0, x_1, \cdots, x_n\} \subset \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0, \varepsilon)$ define

$$W_\nu := \begin{pmatrix} W(x_1) - W(x_0) \\ W(x_2) - W(x_0) \\ \vdots \\ W(x_n) - W(x_0) \end{pmatrix}. \tag{2.70}$$

For each vertex $x_i \in \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}$, let $V(x_i) = W(x_i)$. It is obvious that $V(x_i)$ is positive definite for $x_i \in \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}$.

Choose one $\mathcal{S}_\nu = \operatorname{co}\{x_0, x_1, \cdots, x_n\} \subset \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0, \varepsilon)$ and let $x \in \mathcal{S}_\nu$. Since $V \in$ CPA$[\mathcal{T}]$, $V(x) = V(x_0) + \langle \nabla V_\nu, (x - x_0)\rangle$. Then taking $x = x_i \in \mathcal{S}_\nu$ for all $i \in \{1, \cdots, n\}$, using the fact that $V(x_i) = W(x_i)$, and the definitions W_ν, (2.70), and X_ν, (1.31), we get

$$\nabla V_\nu = X_\nu^{-1} W_\nu. \tag{2.71}$$

Hence

$$V(x) = V(x_0) + \langle X_\nu^{-1} W_\nu, (x - x_0)\rangle. \tag{2.72}$$

There exists a constant $C = R_1 L$ such that

$$\begin{aligned}\|\nabla V_\nu\|_2 = \|X_\nu^{-1} W_\nu\|_2 &\le \|X_\nu^{-1}\|_2 \max_{z \in \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)} \|\nabla W(z)\|_2 \delta_{R_1} \\ &\le R_1 \max_{z \in \mathcal{D}_\mathcal{T} \setminus \mathcal{B}_2(0,\varepsilon)} \|\nabla W(z)\|_2 = C \end{aligned} \tag{2.73}$$

holds uniformly in ν.

Let x_i be an arbitrary vertex of an arbitrary simplex $\mathcal{S}_\nu \subset \mathcal{O}$. Since $f(x_i) \in \mathcal{D}_\mathcal{T}$, there exists an $\mathcal{S}_\mu := \mathrm{co}\{y_0^\mu, y_1^\mu, \dots, y_n^\mu\} \in \mathcal{T}$ such that $f(x_i) = \sum_{j=0}^{n} \mu_j y_j^\mu \in \mathcal{S}_\mu$ with $\sum_{j=0}^{n} \mu_j = 1$. We have assigned $V(x) = W(x)$ for all vertices x of all simplices $\mathcal{S}_\nu$. Hence

$$\begin{aligned} V(f(x_i)) - V(x_i) = \sum_{j=0}^{n} \mu_j W(y_j^\mu) - W(x_i) &= \sum_{j=0}^{n} \mu_j W(y_j^\mu) - W\left(\sum_{j=0}^{n} \mu_j y_j^\mu\right) \\ &+ W\left(\sum_{j=0}^{n} \mu_j y_j^\mu\right) - W(x_i). \end{aligned} \tag{2.74}$$

It follows that

$$V(f(x_i)) - V(x_i) \le L\delta_{R_1} - \alpha(\|x_i\|_2), \tag{2.75}$$

It is obvious that for every $R_1 > 0$ there exists a suitable $\delta_{R_1} \in (0, \varepsilon)$ such that

$$L\delta_{R_1} - \alpha(\|x_i\|_2) + CL_\nu \delta_{R_1} < 0 \tag{2.76}$$

holds for all $x_i \in \mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$. Therefore the linear constraints (2.62) are fulfilled for all $x_i \in \mathcal{O} \setminus \mathcal{B}_2(0,\varepsilon)$.

Based on (2.66), and for $x \in \mathcal{S}_\nu$ $V(x)$ is defined as the interpolated value of $W(x)$ at the vertices of $\mathcal{S}_\nu$, we get that

$$\begin{cases} \max\limits_{\|x\|_2 \le q^*\varepsilon} V(x) \le \max\limits_{\|x\|_2 \le q^*\varepsilon} W(x) < \min\limits_{x \in \mathcal{O} \setminus \mathcal{C}^\circ} W(x) \le \min\limits_{x \in \mathcal{O} \setminus \mathcal{C}^\circ} V(x), & \text{for } q^* \ge 1, \\ \mathcal{B}_2(0, q^*\varepsilon) \subset \mathcal{O}, & \text{for } q^* \ge 1, \\ \max\limits_{\|x\|_2 \le \varepsilon} V(x) \le \max\limits_{\|x\|_2 \le \varepsilon} W(x) < \min\limits_{x \in \mathcal{O} \setminus \mathcal{C}^\circ} W(x) \le \min\limits_{x \in \mathcal{O} \setminus \mathcal{C}^\circ} V(x), & \text{for } q^* < 1. \end{cases} \tag{2.77}$$

Since $W(x)$ is positive definite, so is $V(x)$. The theorem is then proved by Corollary 2.2.6. □

The essence of the following Remark 2.2.9 is the same as in Remark 2.1.8.

Remark 2.2.9. Since a given suitable triangulation $\mathcal{T}$ can be manipulated to deliver a new suitable triangulation $\mathcal{T}^*$ with smaller simplices without increasing their degeneracy, it is always possible to find a suitable triangulation that admits a CPA Lyapunov function approximating a differentiable Lyapunov function with bounded derivative.

2.2.2 Yoshizawa construction of Lyapunov functions

We now address the problem of how to calculate the vertex values of each simplex. We assume system (2.54) is $\mathcal{KL}$-stable on $\mathcal{D}$, i.e., there exists a function $\beta \in \mathcal{KL}$ such that

$$\|\phi(k,x)\|_2 \leq \beta(\|x\|_2, k), \quad \forall x \in \mathcal{D},\ k \in \mathbb{Z}_+. \tag{2.78}$$

Based on Lemma 1.5.5, the definition of Yoshizawa function (see Definition 1.5.6) and Theorem 1.5.7, we have the Yoshizawa function

$$V(x) = \sup_{k\in\mathbb{Z}_+} \alpha_1(\|\phi(k,x)\|_2)\mu^{-k} = \max_{k\in\{0,\cdots,\overline{K(x)}\}} \alpha_1(\|\phi(k,x)\|_2)\mu^{-k} \tag{2.79}$$

is a nonsmooth Lyapunov function for system (2.54), where $\overline{K(x)} = \left\lceil \log_\lambda\left(\frac{\alpha_2(\|x\|_2)}{\alpha_1(\|x\|_2)}\right)\right\rceil + 1$, $\lambda = \frac{1}{\mu}$, $\mu \in (0,1)$ and functions α_1, $\alpha_2 \in \mathcal{K}_\infty$ satisfying (1.62).

We calculate V for each simplex vertex x, based on equation (2.79). In order to make this method numerically tractable, here we assume that the time horizon $\overline{K(x)}$ not to be too large. We consider discrete time versions of Example 1 and Example 2 from Section 2.1.2. Given stability estimate, $\overline{K(x)}$ for each example can be derived.

For system (2.54), our proposed approach of constructing a CPA Lyapunov function is the following:

1: Obtain a stability estimate $\beta \in \mathcal{KL}$ so that (2.78) holds.

2: Find $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying inequality (1.62).

3: Define a suitable triangulation $\mathcal{T} = \{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$ on a subset $\mathcal{D}$ of the domain of attraction of the state space with the equilibrium in its interior.

4: Calculate the values $V(x_i)$ at vertex x_i of each simplex $\mathcal{S}_\nu$ via the Yoshizawa function defined by (2.79).

5: Construct a CPA function V via convex interpolation of the vertex values $V(x_i)$ of each simplex $\mathcal{S}_\nu$, i.e., for $x \in \mathcal{S}_\nu$ $x = \sum_{i=0}^{n} \lambda_i x_i$ $(0 \leq \lambda_i \leq 1)$, then

$$V(x) = \sum_{i=0}^{n} \lambda_i V(x_i). \tag{2.80}$$

6: Calculate L_ν such that (2.60) is satisfied.

7: Check the inequality (2.62) for each vertex.

Remark 2.2.10. Given a stability estimate such as the discrete time versions of Example 1 and Example 2 from Section 2.1.2 , we can construct $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying inequality (1.62). However, for general systems, there is no explicit method to compute stability estimates and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

Remark 2.2.11. Let us consider (2.62) again. If the vertex x_i^ν is very close to the origin, the interpolation term $CL_\nu \operatorname{diam}(\mathcal{S}_\nu)$ may be predominant on the left hand side of the inequality (2.62). In order to make sure the inequality (2.62) holds for all vertices, we have to exclude a small neighbourhood of the equilibrium. If system (2.54) is exponentially stable, CPA Lyapunov functions can be computed on the whole set $\mathcal{T}$ with the fan shape triangulation of the small neighbourhood of the equilibrium such as Figure 2.1 in Section 2.1.3, which is demonstrated by the CPA function $V_{11}(x)$ for the following Example 1 (2.81). The corresponding theoretical results are discussed in [28].

From Theorem 2.2.4 and Corollary 2.2.6, if (2.62) holds for each vertex, then such a CPA function is a CPA Lyapunov function. If the Yoshizawa function (2.79) is a differentiable function and the derivative is bounded, based on Theorem 2.2.8 our method always succeeds on a subset of the domain of attraction. However, from Theorem 1.5.7 the Yoshizawa function (2.79) is only Lipschitz continuous. Thus a subject of future work is to investigate under what conditions the Yoshizawa function (2.79) is differentiable.

2.2.3 Examples

In this section, we present three numerical examples to demonstrate the effectiveness of the proposed method. In each case, we define the suitable triangulation by the same method described in Section 2.1.3. Then, for each example a CPA Lyapunov function is computed by our proposed approach.

Example 1 - Linear System

Consider the system

$$x^+ = Ax = \begin{bmatrix} 0.25 & 0.25 \\ -0.125 & -0.25 \end{bmatrix} x. \tag{2.81}$$

Let $x = (x_1, x_2)^\top$. We observe that the origin is globally exponentially stable as the eigenvalues of A are at $\pm\frac{\sqrt{2}}{8}$. We solve the so-called discrete Lyapunov equation,

$$A^\top PA = P - 0.25\text{Id} \tag{2.82}$$

where P is a symmetric positive definite matrix and thus obtain that

$$V(x) = x^\top Px = x^\top \begin{bmatrix} 0.2815 & -0.0235 \\ -0.0235 & 0.2698 \end{bmatrix} x \tag{2.83}$$

is a Lyapunov function shown in Figure 2.11 for system (2.81).

We observe that

$$\|\phi(k,x)\|_2 \le \left(\frac{\sqrt{2}}{8}\right)^k \|x\|_2 \le e^{-k}\|x\|_2 \tag{2.84}$$

and so (2.81) has a stability estimate $\beta \in \mathcal{KL}$ given by

$$\beta(s,k) = se^{-k}.$$

With $\alpha_1(s) = s^2 = \alpha_2(s)$, then $\overline{K(x)} = 1$. By Theorem 1.5.7,

$$V_y(x) = \max_{k\in\{0,1\}} \alpha_1(\|\phi(k,x)\|_2)e^k, \tag{2.85}$$

is the Yoshizawa function for system (2.81).

We defined a suitable triangulation as described in Section 2.1.3 with $K = 80$, $k = 5$ and the map $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$, $F(x) = 0.01x$. Let $\mathcal{O} = \mathcal{D}_{\mathcal{T}} = [-0.8, 0.8]^2$. The value at each vertex of the simplex are given by (2.85). This defines a CPA function V_1 shown in Figure 2.12. It is straightforward to numerically verify that the inequalities (2.62) are satisfied for all simplex vertices where $\mathcal{S}_\nu \cap (\mathcal{O} \setminus (-0.05, 0.05)^2) \neq \emptyset$. Therefore, V_1 is a CPA Lyapunov function on $\mathcal{O} \setminus (-0.05, 0.05)^2 = [-0.8, 0.8]^2 \setminus (-0.05, 0.05)^2$. For comparison, the level curves of V and V_1 are shown in Figures 2.13 and 2.14, respectively.

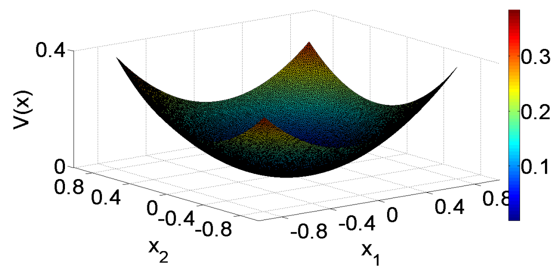

Figure 2.11: Lyapunov function V for system (2.81) on $[-0.8, 0.8]^2 \setminus (-0.05, 0.05)^2$.

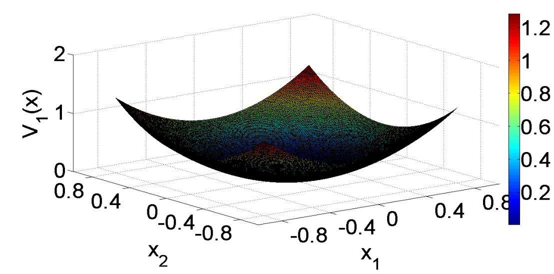

Figure 2.12: CPA Lyapunov function V_1 for system (2.81) on $[-0.8, 0.8]^2 \setminus (-0.05, 0.05)^2$.

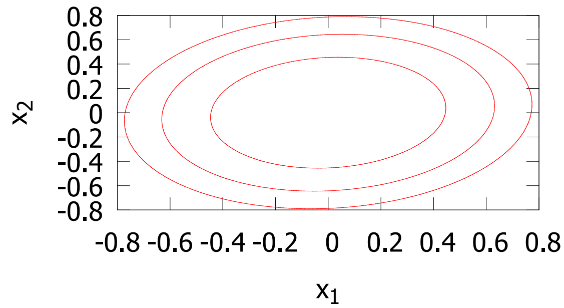

Figure 2.13: Level curves of V for values 0.0571, 0.1142 and 0.1713.

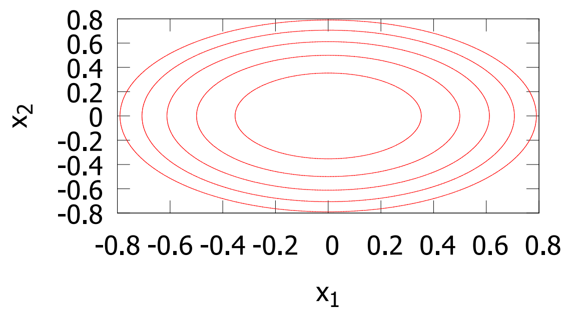

Figure 2.14: Level curves of V_1 for values 0.128, 0.256, 0.384, 0.512 and 0.64.

Since system (2.81) is exponentially stable, a CPA Lyapunov function V_{11} shown in Figure 2.15 is obtained by our proposed method under the same triangulation with the fan shape triangulation of the small neighbourhood of the origin as Figure 2.1 shows.

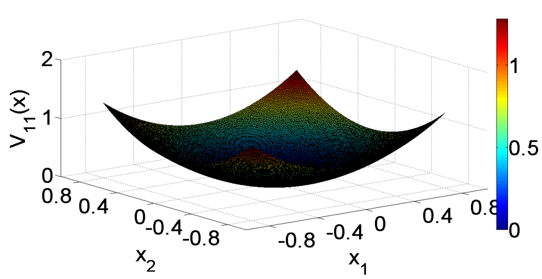

Figure 2.15: CPA Lyapunov function V_{11} for system (2.81) on $[-0.8, 0.8]^2$.

Example 2 - Simple Nonlinear System

Consider the one-dimensional system

$$x^+ = \begin{cases} \frac{1}{2}x^2, & \text{if} \quad |x| \le 1, \\ \frac{1}{2}\sqrt{|x|}, & \text{if} \quad |x| > 1. \end{cases} \tag{2.86}$$

It is obvious that $|x^+| \le \frac{1}{2}|x|$. Let $\mu = \frac{\sqrt{2}}{2}$, $\alpha_1(s) = \alpha_2(s) = s$. Then $\overline{K(x)} = 1$ and the stability estimate satisfies

$$|\phi(k,x)| \le \left(\frac{1}{2}\right)^k |x| \le |x|\mu^{2k}. \tag{2.87}$$

It follows that

$$V_y(x) = \max_{k\in\{0,1\}} \alpha_1(|\phi(k,x)|)\mu^{-k} \tag{2.88}$$

is the Yoshizawa function for system (2.86).

We define a suitable triangulation as described in Section 2.1.3 with $K = 200$, $k = 5$ and the map $F : \mathbb{R} \mapsto \mathbb{R}$, $F(x) = 0.025x$. Let $\mathcal{O} = [-4.975, 4.975] \subset \mathcal{D}_{\mathcal{T}} = [-5, 5]$. We calculate the values of (2.88) at the simplex vertices and a convex interpolation of these values on each simplex vertex, which then delivers a CPA function V_2. We then numerically verify that the inequalities (2.62) are satisfied for all simplex vertices where $\mathcal{S}_\nu \cap (\mathcal{O} \setminus (-0.125, 0.125)) \ne \emptyset$. Therefore, V_2 shown in Figure 2.16 is a CPA Lyapunov function on $[-4.975, 4.975] \setminus (-0.125, 0.125)$.

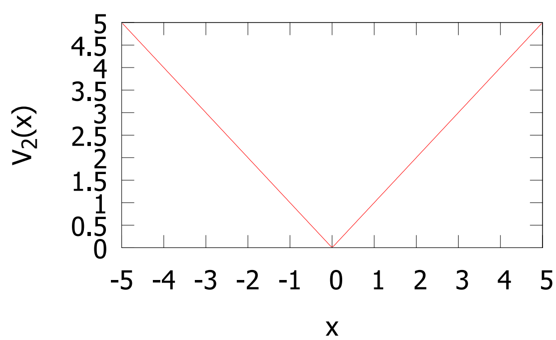

Figure 2.16: CPA Lyapunov function V_2 for system (2.86).

Example 3 - Nonlinear System

Consider the two-dimensional nonlinear system described by

$$\begin{cases} x^+ = -0.125y - 0.125(1 - x^2 - y^2)x, \\ y^+ = 0.125x - 0.125(1 - x^2 - y^2)y. \end{cases} \tag{2.89}$$

Let $z := (x, y)^\top$. For $\|z\|_2 < 1$, it is easy to get that $\|z^+\|_2 \le \frac{\sqrt{2}}{8}\|z\|_2$. Like Example 1, the stability estimate is given by

$$\|\phi(k,z)\|_2 \le \|z\|_2 e^{-k}. \tag{2.90}$$

Let $\alpha_1(s) = s^2 = \alpha_2(s)$, then $\overline{K(z)} = 1$. Thus

$$V_y(z) = \max_{k\in\{0,1\}} \alpha_1(\|\phi(k,z)\|_2)e^k. \tag{2.91}$$

is the Yoshizawa function for system (2.89).

The suitable triangulation is defined by the way stated in Section 2.1.3 with $K = 80$, $k = 5$ and the map $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$, $F(x) = 0.01x$. Let $\mathcal{O} = [-0.8, 0.8]^2 = \mathcal{D}_{\mathcal{T}}$. A CPA Lyapunov function V_3 is computed and shown in Figure 2.17 on $\mathcal{O} \setminus (-0.05, 0.05)^2 = [-0.8, 0.8]^2 \setminus (-0.05, 0.05)^2$. The level curves of V_3 are demonstrated in Figure 2.18. In order to compare with V_3, we present another Lyapunov function.

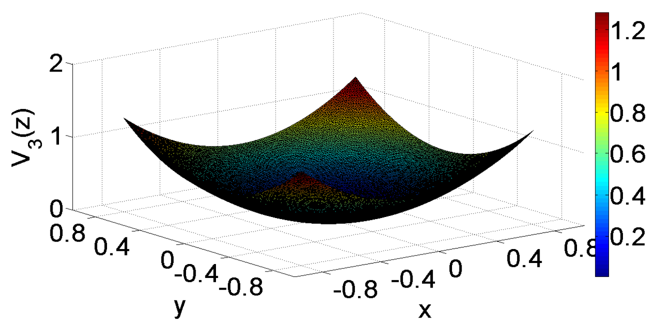

Figure 2.17: CPA Lyapunov function V_3 for system (2.89).

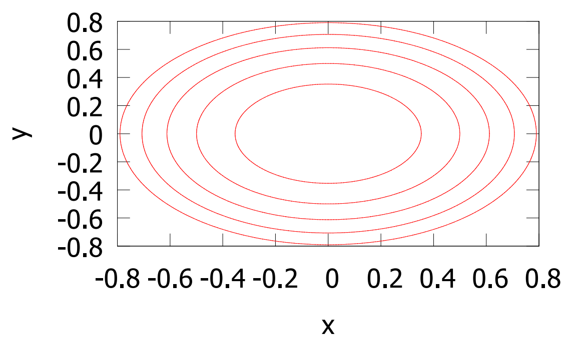

Figure 2.18: Level curves of V_3 for values 0.128, 0.256, 0.384, 0.512 and 0.64.

On any compact subset of the unit ball, the simple quadratic function

$$V(z) = x^2 + y^2 \tag{2.92}$$

is a known Lyapunov function which is shown in Figure (2.19). Its level curves are shown in Figure 2.20.

Remark 2.2.12. From Figures 2.18 and 2.19, we obtain that $V_3(z)$ is similar to $V(z)$. The reason for this property is that $\alpha_1(s) = s^2$ determines the property of Yoshizawa function in some degree. That is also explains why $V_1(x)$ is similar to $V(x)$ for Example 1. For Example 2, $\alpha_1 = \alpha_2 = \|x\|_2$ and $\frac{1}{\sqrt{2}} < 1$ lead to $V_2(x) = \|x\|_2$.

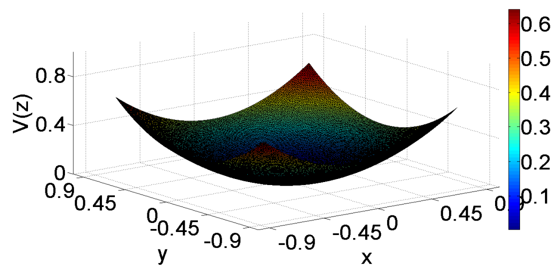

Figure 2.19: Lyapunov function V for system (2.89).

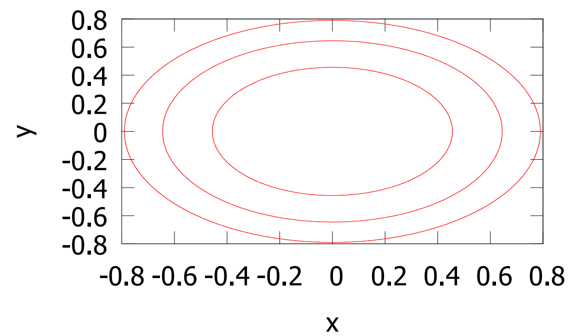

Figure 2.20: Level curves of V for values 0.213, 0.426 and 0.64.

2.3 Concluding remarks and open questions

In this chapter, a novel technique for the computation of CPA Lyapunov functions has been proposed given the system is $\mathcal{KL}$-stable. For a suitable triangulation of a compact subset of state space, the Yoshizawa functions values at vertices of each simplex are computed. Then we construct a CPA function based on these vertex values. Furthermore, we check if such a CPA function is a Lyapunov function by verifying the inequalities (2.7) and (2.62). If the CPA function is not a Lyapunov function, then we could re-calculate the Yoshizawa functions values at vertices and construct a CPA function after refining the suitable triangulation. The obtained CPA Lyapunov function is a true Lyapunov function rather a numerical approximation of a Lyapunov function, since the interpolation errors are incorporated in the linear inequalities (2.7) and (2.62).

For continuous time dynamical systems, we assume that the Yoshizawa function (2.35) is C^2. Then there exists a suitable triangulation such that the constructed CPA function is a Lyapunov function (see Theorem 2.1.7). However, the Yoshizawa function is only Lipschitz continuous. Thus, there may not exist a suitable triangulation such that the computed CPA function is a Lyapunov function.

For discrete time dynamical systems, if the Yoshizawa function (2.79) is differentiable with bounded derivative, then there exists a suitable triangulation such that the constructed CPA function is a Lyapunov function (see Theorem 2.2.8). However, it is only proved that the Yoshizawa function is Lipschitz continuous. If we cannot prove that the Yoshizawa function is differentiable, then there exists a possibility that our approach for computation of Lyapunov function cannot succeed. However, it is worth to point out that the cost of computation of CPA Lyapunov functions by our proposed method is less expensive than by solving a linear optimization problem which also delivers a true Lyapunov function. For Example 1 from Section 2.2.3, it took 12 hours to get the solution by our proposed method using using a Laptop computer with a 32 bit, AMD Athlom (tm) II 360 Dual-core Processor 2.30GHZ. However, it took more than 36 hours to solving the corresponding linear optimization problem for the same suitable triangulation.

The triangulation of the state space described in Section 2.1.3 was proposed in [27, Section 2]. If system is exponentially stable, then we could construct a CPA Lyapunov function on a subset of state space without excluding the small neighbourhood of the equilibrium with a suitable triangulation, which is discussed in [28, 40]. This is why we use this type of suitable triangulation.

In Section 2.1.2 and Section 2.2.2, we present two types of stability estimate. For each type of stability, the formulation of α_1, α_2 are given. However, for general cases, there are some difficult problems of finding a stability estimate $\beta \in \mathcal{KL}$, and constructing functions α_1, $\alpha_2 \in \mathcal{K}_\infty$ such that (1.51) and (1.62) hold. Besides, the cost of computation becomes more expensive as the dimension of considered systems increases. Because of these difficulties, the proposed methods can not be widely applied in computing Lyapunov functions for general dynamical systems.

Based on the above summary, for large scale systems, it is not easy to construct a Lyapunov function by the proposed method. In order to analyse stability of a large scale system, we consider it as interconnected systems. We will investigate stability of interconnected systems in Chapter 3. We will first study stability of each subsystem by Zubov's method for perturbed systems, since Zubov's method delivers a maximal Lyapunov function. Then stability of interconnected systems will be analysed by small gain theorems which play a central part in stability analysis of interconnected systems.

3 Stability of two interconnected systems and estimate of the domain of attraction

After the concepts of ISS and iISS were introduced, many small gain theorems have been proposed. The stability of interconnected systems has been investigated using small gain theorems and ISS, iISS Lyapunov functions for the subsystems. We are interested in the small gain theorem in comparison form (Theorem 1.6.3) since the small gain theorem can be used to study stability of two interconnected iISS systems. In this chapter, our aim is to investigate stability of two interconnected systems which are locally iISS. To this end, we first study how to construct local iISS Lyapunov functions for dynamical systems with perturbations. We then restrict our attention to analysing stability of the coupling of two nonlinear systems in a feedback interconnection by small gain theorems.

In order to construct local iISS Lyapunov functions for perturbed systems, we introduce auxiliary systems which are locally uniformly asymptotically stable at the equilibrium in Section 3.2. This idea is inspired by the result of [13] where an ISS Lyapunov function in implication formulation is computed via an auxiliary system. In [13], it is proved that if there exists an ISS Lyapunov function in implication formulation for the considered system, then there exists an auxiliary system such that the ISS Lyapunov function in implication formulation is a robust Lyapunov function for the auxiliary system. A robust Lyapunov function for the auxiliary system is then computed by Zubov's method. Furthermore, in [13] the robust Lyapunov function is proved to be an ISS Lyapunov function in implication formulation for the original system on a subset of the robust domain of attraction of the auxiliary system. In this chapter, under certain conditions, we prove that a robust Lyapunov function for the introduced auxiliary system is an iISS Lyapunov function for the original system on the robust domain of attraction of the auxiliary system. Therefore, the problem of constructing an iISS Lyapunov function for the original system is reduced to constructing a robust Lyapunov function for the auxiliary system.

We recall Zubov's method developed in [9, 10, 11] for computing robust Lyapunov functions for perturbed systems which are uniformly asymptotically stable at the equilibrium in Section 3.3. A first order partial differential equation (Hamilton-Jacobi-Bellman equation) is formulated which has a unique viscosity solution vanishing at the fixed point. This viscosity solution is a maximal Lyapunov function on the robust domain of attraction for perturbed systems. By Zubov's method for perturbed systems we obtain a robust Lyapunov function for the introduced auxiliary system and then show that it is an iISS Lyapunov function for the original system.

From Section 3.3.1 on, we put our emphasis on investigating stability of two interconnected nonlinear systems which are locally iISS. A local iISS Lyapunov function for each subsystem can be obtained by the introduction of an auxiliary system and Zubov's method for a perturbed system. In Section 3.4, a local version of the small gain theorem in comparison form (Theorem 1.6.3) is introduced. We prove that if each subsystem of the two interconnected systems are locally iISS and the conditions of the local version of small gain theorem in comparison form hold, then the interconnected system is asymptotically stable at the equilibrium. Moreover, an estimate of the domain of attraction of the two interconnected systems at the equilibrium is attained.

Based on the obtained iISS Lyapunov functions, we further state that the iISS Lyapunov function is a local ISS Lyapunov function in dissipative formulation for each subsystem on a compact subset of the domain of attraction of the corresponding auxiliary system in Section 3.5. A local version of the small gain theorem in dissipative form (Theorem 1.6.2) is presented. In order to compare with the results obtained by the local version small gain theorem in comparison form, stability of two interconnected ISS systems will be investigated by the local version of small gain theorem in dissipative form. If each subsystem is locally ISS and conditions from the local version of the small gain theorem in dissipative form hold, then the interconnected system is asymptotically stable at the equilibrium. The procedure of estimating the domain of attraction of the coupled system at the equilibrium is described based on the local version of the small gain theorem in dissipative form.

Additionally, we illustrate the main results of this chapter by an academic example in Section 3.6. For this example, we compare estimates of the domain of attraction obtained by these two small gain theorems and subsystems' iISS and ISS Lyapunov functions.

3.1 Problem statement

We consider a system described by

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \tag{3.1}$$

where $x_i \in \mathbb{R}^{n_i}$, $n_i \in \mathbb{Z}_{>0}$, $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$.

Let $n = n_1 + n_2$, $x = (x_1, x_2)^\top$, $f = (f_1, f_2)^\top$, and x^0 denote the initial condition.

Then we may equivalently write the system as

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x^0 \end{cases} \tag{3.2}$$

We assume

- f is Lipschitz continuous,
- system (3.1) has a fixed point in $x = 0$ which is locally asymptotically stable.

Each subsystem of system (3.1) is treated as a dynamical system with perturbation by considering the effect of the other state as perturbation. A dynamical system with perturbations is of the type

$$\dot{x}(t) = f(x(t), u(t)) \tag{3.3}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the perturbation input and $t \in \mathbb{R}_+$. The set of admissible input values is denoted by $U_R := \mathcal{B}_2(0, R) \subset \mathbb{R}^m$, the set of admissible input functions are defined by $\mathcal{U}_R := \{u : \mathbb{R} \to \mathbb{R}^m \text{ measurable} \,|\, \|u\|_\infty \leq R\}$, $R > 0$, and $f(0, 0) = 0$.

For each subsystem of system (3.1), x_i, x_j $(j \neq i)$ and f_i are considered as state x, the perturbation input u and the function f in (3.3), respectively.

We further assume

- each subsystem of system (3.1) is locally iISS.

In order to investigate stability of interconnected systems (3.1) by local versions of the small gain theorem in dissipative form (Theorem 1.6.2) and the small gain theorem in comparison form (Theorem 1.6.3), we propose a new technique for computing iISS Lyapunov functions for system (3.3).

3.2 Auxiliary system

In order to construct an iISS Lyapunov function for system (3.3), we introduce an auxiliary system

$$\dot{x} = f_\eta(x,u) := f(x, \overline{\eta}(x)u) - \eta(\|u\|_2)x\,, \tag{3.4}$$

where $x \in \mathbb{R}^n$, $u \in U_R$, $\eta \in \mathcal{K}_\infty$ and a Lipschitz continuous function $\overline{\eta} : \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions.

1. If $f(0,u) = 0$ for all $u \in U_R$, then $\overline{\eta} := 1$ is a constant.

2. If $f(0,u) \neq 0$ for certain $u \in U_R$, then

$$\begin{cases} |\overline{\eta}(x)| \leq 1, \text{ for all } x \in \mathbb{R}^n, \\ \\ \overline{\eta}(0) = 0, \\ \\ \overline{\eta}(x) = 1, \text{ for } \|x\|_2 \geq \delta,\ \delta > 0 \text{ is small enough.} \end{cases} \tag{3.5}$$

Based on the conditions on $\overline{\eta}$, we have $f_\eta(0,u) = 0$ for all $u \in U_{\mathbb{R}}$.

In the whole chapter, we assume

- f_η is bounded in $\mathbb{R}^n \times U_R$.

Let $\overline{L}_x$, $\overline{L}_u \in \mathbb{R}_{>0}$ denote the Lipschitz constants for f with respect to x, u respectively. We denote the solutions of system (3.4) with initial condition x^0 by $x_\eta(\cdot, x^0, u)$. Let $\mathcal{D}_\eta$ denote the robust domain of attraction of system (3.4), see Definition 1.3.16.

We explain the reason for the introduction of an auxiliary system by the next lemma.

Lemma 3.2.1. *If there exist a robust Lyapunov function V for system* (3.4)*, and constants K_1, $K_2 \in \mathbb{R}_{>0}$ such that for $x \in \mathcal{D}_\eta$*

$$|\langle \nabla V(x), x\rangle| \leq K_1, \tag{3.6}$$

$$\|\nabla V(x)\|_2 \leq K_2, \tag{3.7}$$

then V is an iISS Lyapunov function for system (3.3) *on $\mathcal{D}_\eta$.*

Proof. According to the definition of robust Lyapunov function (Definition 1.3.15), V is radially unbounded, moreover there exists a positive definite function α such that

$$\langle \nabla V(x), f_\eta(x,u) \rangle \leq -\alpha(\|x\|_2), \tag{3.8}$$

for $x \in \mathcal{D}_\eta$.

Based on the definition of f_η, we have

$$\begin{aligned} \langle \nabla V(x), f(x,u) \rangle &= \langle \nabla V(x), (f(x,u) - f_\eta(x,u) + f_\eta(x,u)) \rangle \\ &= \langle \nabla V(x), f_\eta(x,u) \rangle + \langle \nabla V(x), (f(x,u) - f(x, \overline{\eta}(x)u)) \rangle \\ &\quad + \langle \nabla V(x), x \rangle \eta(\|u\|_2) \\ &\leq -\alpha(\|x\|_2) + \langle \nabla V(x), x \rangle \eta(\|u\|_2) + \|\nabla V(x)\|_2 \overline{L}_u |1 - \overline{\eta}(x)| \|u\|_2. \end{aligned} \tag{3.9}$$

Using (3.6) and (3.7), we obtain that V is an iISS Lyapunov function for system (3.3) on $\mathcal{D}_\eta$. □

Remark 3.2.2. From the above discussion, the problem of computing an iISS Lyapunov function for system (3.3) is transformed into the problem of constructing a robust Lyapunov function for system (3.4).

We assume that $x^* = 0$ is locally uniformly asymptotically stable for system (3.4) cf. Definition 1.3.14. We restate it as following.

(H1) there exists a constant $r > 0$ and a function β of class $\mathcal{KL}$ such that $\|x_\eta(t, x^0, u)\|_2 \leq \beta(\|x^0\|_2, t)$ for any $x^0 \in \mathcal{B}_2(0, r)$, any $u \in \mathcal{U}_R$, and all $t \geq 0$.

By Sontag's lemma on $\mathcal{KL}$-estimates cf. Lemma 1.5.1 for any $\beta \in \mathcal{KL}$ there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\beta(r,t) \leq \alpha_2(\alpha_1(r)e^{-t}). \tag{3.10}$$

In the sequel we will work primarily with the functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

3.3 Zubov's method for dynamical systems with perturbation

In this section, we recall from [10] how to formulate a Zubov type equation that allows for the computation of a maximal robust Lyapunov function on the robust domain of attraction for system (3.4). Further we prove that under certain conditions the robust Lyapunov function computed by Zubov's method for perturbed systems satisfies the conditions of Lemma 3.2.1. To this end an optimal control problem is defined using a running cost g, which is chosen in such a manner that the function $g : \mathbb{R}^n \times U_R \to \mathbb{R}$ is continuous and satisfies

(H2)

- (i) Using α_2^{-1} from (3.10), there exists a $d > 0$ such that $g(x,u) \leq d\alpha_2^{-1}(\|x\|_2)$ for all $x \in \mathbb{R}^n$, $u \in U_R$. Furthermore, $g(x,u) > 0$ if $x \neq 0$.
- (ii) There exists a constant $g_0 > 0$ such that $\inf\{g(x,u) \mid x \notin \mathcal{B}_2(0,r),\ u \in U_R\} \geq g_0$.
- (iii) For each $P > 0$ there exists $L_P > 0$ such that $|g(x,u) - g(y,u)| \leq L_P \|x - y\|_2$ for all $\|x\|_2, \|y\|_2 \leq P$, and all $\|u\|_2 \leq R$.

We now introduce the value function of a suitable optimal control problem related to system (3.4). Consider the functional $J_\eta : \mathbb{R}^n \times \mathcal{U}_R \to \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$J_\eta(x,u) := \int_0^{+\infty} g(x_\eta(t), u(t))dt\,,$$

and the optimal value function

$$v_\eta(x) := \sup_{u\in\mathcal{U}_R} 1 - e^{-J_\eta(x,u)}\,. \tag{3.11}$$

Since g is nonnegative we immediately obtain that $v_\eta(x) \in [0,1]$ for all $x \in \mathbb{R}^n$. Furthermore, standard techniques from optimal control imply that v_η satisfies a dynamic programming principle, i.e. for each $t > 0$ we have

$$v_\eta(x) = \sup_{u\in\mathcal{U}_R} \Big\{(1 - G(x,t,u)) + G(x,t,u)v_\eta(x_\eta(t,x,u))\Big\}, \tag{3.12}$$

with

$$G(x,t,u) := \exp\left(-\int_0^t g(x_\eta(\tau,x,u),u(\tau))d\tau\right). \tag{3.13}$$

An application of the chain rule shows $1 - G(x,t,u) = \int_0^t G(x,\tau,u)g(x_\eta(\tau,x,u),u(\tau))d\tau$ implying

$$v_\eta(x) = \sup_{u\in\mathcal{U}_R} \Big\{ \int_0^t G(x,\tau,u)g(x_\eta(\tau,x,u),u(\tau))d\tau + G(x,t,u)v_\eta(x_\eta(t,x,u))\Big\}.$$

The next proposition [10, Proposition 3.1] shows the relationship between $\mathcal{D}_\eta$ and v_η.

Proposition 3.3.1. *If $(H1) - (H2)$ hold, then*

(i) $v_\eta(x) < 1$ if and only if $x \in \mathcal{D}_\eta$. (ii) $v_\eta(x) = 0$ if and only if $x = 0$.

(iii) v_η is continuous on $\mathbb{R}^n$. (iv) $v_\eta(x) \to 1$ for $x \to x^0 \in \partial\mathcal{D}_\eta$ or $\|x\|_2 \to \infty$.

From Proposition 3.3.1, v_η is bounded on $\mathbb{R}^n$. By the dynamic programming principle (3.12), v_η can be characterized as the unique viscosity solution (see Definition 5.2.1) of the extended Zubov equation

$$\sup_{\|u\|_2\le R} \{\langle\nabla v_\eta(x), f_\eta(x,u)\rangle + (1 - v_\eta(x))g(x,u)\} = 0 \tag{3.14}$$

with $v_\eta(0) = 0$.

The following Theorem 3.3.2 and Proposition 3.3.4 recall Theorems 3.8 and 4.1 of [10]. Theorem 3.3.5 states the content of Propositions 4.2 and 4.3 of [10].

Theorem 3.3.2. *If $(H1)$ holds and there exists a function $g : \mathbb{R}^n \times U_R \longrightarrow \mathbb{R}$ such that $(H2)$ is satisfied, then (3.14) has a unique bounded and continuous viscosity solution v_η on $\mathbb{R}^n$ satisfying $v_\eta(x) = 0$ for $x = 0$. Furthermore, this function coincides with v_η from (3.11). In particular the characterization*

$$\mathcal{D}_\eta = \{x \in \mathbb{R}^n \,|\, v_\eta(x) < 1\} \tag{3.15}$$

holds.

Remark 3.3.3. If f_η is unbounded in $\mathbb{R}^n \times U_R$, let $\tilde{f}_\eta(x,u) = \frac{f_\eta(x,u)}{1+\|f_\eta(x,u)\|_2}$, $\tilde{g}(x,u) = \frac{g(x,u)}{1+\|f_\eta(x,u)\|_2}$, and then we consider the following system

$$\dot{x} = \tilde{f}_\eta(x,u). \tag{3.16}$$

Assume $\tilde{g}$ satisfies $(H2)$ and $\tilde{g}(x,u) \to \infty$ as $\|u\|_2 \to \infty$ for each $x \in \mathbb{R}^n$. Based on Remark 4.2 and Lemma 4.5 of [12], Theorem 3.3.2 can be proved.

Proposition 3.3.4. *Assume $(H1)$ and $(H2)$ hold and consider the unique viscosity solution v_η of* (3.14) *with $v_\eta(0) = 0$. Then the function v_η is a robust Lyapunov function for the system* (3.4) *on $\mathcal{D}_\eta$. More precisely we have*

$$v_\eta(x(t,x^0,u)) - v_\eta(x^0) \le \left[1 - e^{-\int_0^t g(x(\tau),u(\tau))d\tau}\right](v_\eta(x(t,x^0,u)) - 1) < 0$$

for all $x^0 \in \mathcal{D}_\eta \setminus \{0\}$ and all functions $u \in \mathcal{U}_R$.

In order to prove that v_η is Lipschitz continuous, we introduce the following conditions.

(H3) $f_\eta(\cdot,u)$ and $g(\cdot,u)$ are globally Lipschitz continuous in $\mathcal{D}_\eta$, with constants L_f, $L_g > 0$ uniformly in $u \in U_R$.

(H4) There exist an open neighbourhood $\mathcal{W}$ of 0 and constants $K > 0$, $s > L_f$ such that for all x, $y \in \mathcal{W}$ and $u \in U_R$ the inequality

$$|g(x,u) - g(y,u)| \le K\alpha_2^{-1}(\max\{\|x\|_2, \|y\|_2\})^s \|x-y\|_2$$

holds, again with α_2 from (3.10).

Theorem 3.3.5. *If the conditions $(H1) - (H4)$ hold and $g_0 > L_f$, then v_η is Lipschitz continuous in $\mathbb{R}^n$.*

Proof. We recall parts of the proofs of Propositions 4.2 and 4.3 in [10].

According to (H_1), there exists a finite time $T_1 > 0$ such that $x_\eta(t,x,u) \in \mathcal{W} \cap \mathcal{B}_2(0,r)$ for all $t > T_1$, $x \in \mathcal{B}_2(0,r)$, $u \in \mathcal{U}_R$.

Let $V(x) = \sup\limits_{u\in\mathcal{U}_R} \int_0^{+\infty} g(x_\eta(t,x,u),u)dt$. It follows that $v_\eta(x) = 1 - e^{-V(x)}$. Now fix $x, y \in \mathcal{B}_2(0,r)$, then we have

$$\begin{aligned}
|V(x) - V(y)| &\le \sup_{u\in\mathcal{U}_R} \int_0^{+\infty} |g((x_\eta(t,x,u),u) - g(x_\eta(t,y,u),u)|dt \\
&\le \sup_{u\in\mathcal{U}_R} \int_0^{T_1} |g(x_\eta(t,x,u),u) - g(x_\eta(t,y,u),u)|dt \\
&\quad + \sup_{u\in\mathcal{U}_R} \int_{T_1}^{+\infty} |g(x_\eta(t,x,u),u) - g(x_\eta(t,y,u),u)|dt \\
&\le \int_0^{T_1} L_g e^{L_f t}\|x-y\|_2 dt + \int_{T_1}^{+\infty} K\alpha_1(r)^s e^{-s(t-T_1)} e^{L_f t}\|x-y\|_2 dt \\
&\le \left(L_g \frac{e^{L_f T_1} - 1}{L_f} + K\alpha_1(r)^s e^{sT_1} \frac{e^{(L_f - s)T_1}}{s - L_f}\right)\|x-y\|_2 =: L_0\|x-y\|_2.
\end{aligned} \tag{3.17}$$

Thus, we conclude that V is Lipschitz continuous on $\mathcal{B}_2(0,r)$ with Lipschitz constant L_0.

For $x \in \mathcal{D}_\eta$, note that $T_x = \sup\{t(x,u) : u \in U_R\}$ (see Definition 1.3.19 of $t(x,u)$ with $\mathcal{B}_p(0,\rho^x) = \mathcal{B}_2(0,r)$, $\phi(t,x,u) = x_\eta(t,x,u)$). We have $V(x) \geq g_0 T_x$, where g_0 is given by (H_2). Let $x, y \in \mathcal{D}_\eta$, and assume without loss of generality that $T_x \geq T_y$. There exists a control $u \in \mathcal{U}_R$ for any $\varepsilon > 0$ such that

$$\begin{aligned} |V(x) - V(y)| &\leq \int_0^{T_x} |g(x_\eta(t,x,u),u) - g(x_\eta(t,y,u),u)|dt \\ &\quad + |V(x_\eta(T_x,x,u)) - V(x_\eta(T_x,y,u))| + \varepsilon \\ &\leq \int_0^{T_x} L_g e^{L_f t}\|x-y\|_2 dt + L_0 e^{L_f T_x}\|x-y\|_2 + \varepsilon \\ &\leq \left(L_0 + \frac{L_g}{L_f}\right)\exp\left(\frac{L_f V(x)}{g_0}\right)\|x-y\|_2. \end{aligned} \tag{3.18}$$

Therefore, V is locally Lipschitz continuous in $\mathcal{D}_\eta$ with a constant of the form $L\exp\left(\frac{L_f V(x)}{g_0}\right)$, where $L = L_0 + \frac{L_g}{L_f}$. Then we have

$$|v_\eta(x) - v_\eta(y)| \leq L\exp\left(-1 + \frac{L_f}{g_0}\right)\|x-y\|_2. \tag{3.19}$$

Thus, v_η is Lipschitz continuous in $\mathbb{R}^n$ with Lipschitz constant L. □

Remark 3.3.6. If the conditions of Theorem 3.3.5 hold, then the viscosity solution v_η of (3.14) is Lipschitz continuous. By Rademacher's theorem [21, Theorem 5.8.6] v_η is differentiable almost everywhere. We then have that

$$\sup_{\|u\|_2 \leq R}\{\langle \nabla v_\eta, f_\eta(x,u)\rangle + (1 - v_\eta(x))g(x,u)\} = 0 \tag{3.20}$$

holds almost everywhere on $\mathcal{D}_\eta$.

Using the definition of f_η and Clarke's subdifferential (see Definition 1.4.12), we get that $v_\eta(x)$ satisfies

$$\langle \xi, f(x,u)\rangle \leq -(1 - v_\eta(x))g(x,u) + \langle \xi, x\rangle\ \eta(\|u\|_2) + L\overline{L}_u|1 - \overline{\eta}(x)|\|u\|_2\,, \tag{3.21}$$

for all $x \in \mathcal{D}_\eta$, $\xi \in \partial_{Cl} v_\eta(x)$ with the constraint $\|u\|_2 \leq R$.

Based on Clarke's subdifferential, it is reasonable to consider v_η on the set where v_η is differentiable in the following. In order to get an iISS Lyapunov function, an estimate of the term $\langle \nabla v_\eta(x), x\rangle$ is required. The following Proposition 3.3.7 is our new results showing that the term $\langle \nabla v_\eta(x), x\rangle$ is bounded .

Proposition 3.3.7. *If the assumptions of Theorem 3.3.5 hold, then there exists a constant* $K_1 > 0$ *such that*

$$\|\nabla v_\eta(x)\|_2 \|x\|_2 \leq K_1. \tag{3.22}$$

Proof. Using (3.18), we have

$$|V(x) - V(y)| \leq \left(L_0 + \frac{L_g}{L_f}\right) \exp\left(\frac{L_f V(x)}{g_0}\right) \|x - y\|_2. \tag{3.23}$$

It follows that

$$\|\nabla v_\eta(x)\|_2 \leq L \exp\left(\left(\frac{L_f}{g_0} - 1\right) V(x)\right). \tag{3.24}$$

According to (iv) of Proposition 3.3.1,

$$\lim_{x \to \partial \mathcal{D}_\eta} V(x) = +\infty. \tag{3.25}$$

Then we obtain that

$$\lim_{x \to \partial \mathcal{D}_\eta} \|\nabla v_\eta(x)\|_2 \|x\|_2 \leq \lim_{x \to \in \partial \mathcal{D}_\eta} L \exp\left(\left(\frac{L_f}{g_0} - 1\right) V(x)\right) \|x\|_2 = 0.$$

Therefore, there exists a constant $K_1 > 0$ such that

$$\|\nabla v_\eta(x)\|_2 \|x\|_2 \leq K_1. \tag{3.26}$$

□

Proposition 3.3.8. *If the assumptions of Theorem 3.3.5 hold, then v_η is an iISS Lyapunov function for system* (3.3)*.*

Proof. Under the assumptions of Theorem 3.3.5, we get that v_η is a robust Lyapunov function for the auxiliary system (3.4) by Theorem 3.3.2 and Proposition 3.3.4.

According to Proposition 3.3.7, we have

$$\begin{aligned} \langle \nabla v_\eta(x), f(x,u) \rangle &\leq -(1 - v_\eta(x)) g(x,u) + \langle \nabla v_\eta(x), x \rangle \eta(\|u\|_2) + L\overline{L}_u |1 - \overline{\eta}(x)| \|u\|_2 \\ &\leq -(1 - v_\eta(x)) g(x,u) + K_1 \eta(\|u\|_2) + L\overline{L}_u |1 - \overline{\eta}(x)| \|u\|_2 \end{aligned} \tag{3.27}$$

for $x \in \mathcal{D}_\eta$ and $u \in U_R$.

Since $(1 - v_\eta(x)) g(x,u)$ is nonnegative on $\mathcal{D}_\eta$, there exists a positive definite function ρ such that

$$\rho(v_\eta(x)) \leq (1 - v_\eta(x)) g(x,u) \tag{3.28}$$

holds for $x \in \mathcal{D}_\eta$, $u \in U_R$.

Let function $\overline{\beta} \in \mathcal{K}_\infty$ such that $\overline{\beta}(\|u\|_2) = K_1 \eta(\|u\|_2) + L\overline{L}_u |1 - \overline{\eta}(x)| \|u\|_2$. Then, we obtain that

$$\langle \nabla v_\eta(x), f(x,u) \rangle \leq -\rho(v_\eta(x)) + \overline{\beta}(\|u\|_2) \tag{3.29}$$

for $x \in \mathcal{D}_\eta$, $u \in U_R$.

Hence, v_η is an iISS Lyapunov function for system (3.3). □

3.3.1 Coupled systems

We now consider the interconnected system (3.1). For each subsystem, we assume there exists an auxiliary system such as (3.4) which is locally uniformly asymptotically stable at the origin. We impose the conditions of Theorem 3.3.5 on the corresponding auxiliary system of each subsystem. Then applying Zubov's method for perturbed systems to each auxiliary system, we obtain positive definite functions $v_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ which are iISS Lyapunov functions for the subsystems. Let $\mathcal{D}_{\eta_i}$ denote the robust domain of attraction of each auxiliary system. According to Proposition 3.3.8, there exist $\underline{\alpha}_i$, $\overline{\alpha}_i$, $\beta_i \in \mathcal{K}_\infty$, $\rho_i \in \mathcal{P}$ such that

$$\underline{\alpha}_i(\|x_i\|) \le v_i(x_i) \le \overline{\alpha}_i(\|x_i\|) \tag{3.30}$$

$$\langle \nabla v_i, f_i(x_1, x_2)\rangle \le -\rho_i(v_i(x_i)) + \beta_i(v_j(x_j)) \tag{3.31}$$

hold for $x_i \in \mathcal{D}_{\eta_i}$, $x_j \in \mathcal{D}_{\eta_j}$ and $i \neq j$, $i, j = 1, 2$.

3.4 A local small gain theorem in comparison form

For the subsystems of system (3.1), we have

$$\begin{cases} \langle \nabla v_1(x_1), f_1(x_1, x_2)\rangle \le -\rho_1(v_1(x_1)) + \beta_1(v_2(x_2)), \\ \langle \nabla v_2(x_2), f_2(x_1, x_2)\rangle \le -\rho_2(v_2(x_2)) + \beta_2(v_1(x_1)), \end{cases} \tag{3.32}$$

where $(x_1, x_2)^\top$ on $\mathcal{D}_{\eta_1} \times \mathcal{D}_{\eta_2}$, and $v = (v_1, v_2)^\top$.

In this section we assume furthermore that the functions ρ_i and β_i are locally Lipschitz continuous.

In order to analyse stability and estimate the domain of attraction for system (3.1), we first study the following comparison system

$$\begin{cases} \dot{V}_1 = -\rho_1(V_1) + \beta_1(V_2) \equiv F_1(V_1, V_2), \\ \dot{V}_2 = -\rho_2(V_2) + \beta_2(V_1) \equiv F_2(V_1, V_2) \end{cases} \tag{3.33}$$

which evolves in $\mathbb{R}^2_+$. The initial condition is $V^0 = V(x^0) = (V_1(x_1^0), V_2(x_2^0))^\top$.

Let $V = (V_1, V_2)^\top$, and $F(V) = (F_1(V), F_2(V))^\top$. We denote the solution of (3.33) with the initial condition V^0 and the domain of attraction of system (3.33) by $V(\cdot, V^0)$ and $\mathcal{D}_V$, respectively. Define $V^{-1}(\mathcal{D}_V) = \{x \in \mathbb{R}^n : V(x) \in \mathcal{D}_V\}$.

Let $\mathcal{D}$ denote the domain of attraction of system (3.1). The relationship between $\mathcal{D}_V$ and $\mathcal{D}$ is described in Proposition 3.4.1.

Proposition 3.4.1. *Consider system* (3.1) *with the initial condition* $x^0 = (x_1^0, x_2^0)^\top \in \mathbb{R}^n$. *If* $V^0(x^0) \in \mathcal{D}_V$, *then* $x^0 \in \mathcal{D}$. *In other words*

$$V^{-1}(\mathcal{D}_V) \subset \mathcal{D}.$$

Proof. Fix $x^0 \in \mathbb{R}^n$ and consider the function defined by $t \mapsto v(\phi(t, x^0))$. Then $v(\phi(t, x^0))$ satisfies (3.32) for all t in the interval of existence of the solution $\phi(t, x^0)$ to (3.1). Choosing the initial condition $V^0 := V(x^0)$ for (3.33), by assumption we obtain that the corresponding solution $V(\cdot, V^0)$ satisfies

$$v(\phi(t, x^0)) \le V(\phi(t, x^0)). \tag{3.34}$$

Since the assumption $V^0 \in \mathcal{D}_V$,

$$\lim_{t\to\infty} V(t) = 0. \tag{3.35}$$

Therefore, we have

$$\lim_{t\to\infty} v(t) = 0. \tag{3.36}$$

Thus, each Lyapunov function v_i for each subsystem of system (3.1) satisfies

$$\lim_{t\to\infty} v_i(t) = 0,\ i = 1, 2. \tag{3.37}$$

From this we obtain that

$$\lim_{t\to\infty} x_i(t) = 0,\ i = 1, 2. \tag{3.38}$$

Since the set $\mathcal{D}_V$ is positively invariant, so is the set $\{x^0 : V^0(x^0) \in \mathcal{D}_V\}$. Therefore $x^0 \in \mathcal{D}$. □

We assume $S = U \cap \mathbb{R}^2_+$, where U is an open neighbourhood of $(0,0)^\top$ and define

$$\begin{aligned} \Omega^{+-} &:= \{V \in S : F_1(V) \geq 0 \text{ and } F_2(V) \leq 0)\}, \\ \Omega^{--} &:= \{V \in S : F_1(V) \leq 0 \text{ and } F_2(V) \leq 0)\}, \\ \Omega^{-+} &:= \{V \in S : F_1(V) \leq 0 \text{ and } F_2(V) \geq 0)\}. \end{aligned} \tag{3.39}$$

Typical shape of the Ω regions constituting the set S is shown in Figure 3.1.

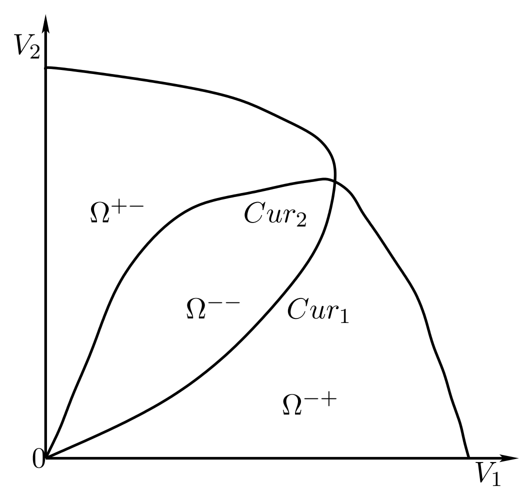

Figure 3.1: Typical shape of the Ω regions constituting the set S, $Cur_1 := \{V \in S : \rho_2(V_2) = \beta_2(V_1)\}$, $Cur_2 := \{V \in S : \rho_1(V_1) = \beta_1(V_2)\}$

In order to prove a set is positively invariant, we introduce the concept of the Bouligand cone.

Definition 3.4.2. Let S be a subset of $\mathbb{R}^n$. The Bouligand cone $T^B_S(x)$ to S at x is defined by

$$T^B_S(x) := \{\xi \in \mathbb{R}^n : \exists \xi_n \in S \ \text{ and } \ t_n \downarrow 0 \ \text{ satisfying } \ \xi_n \longrightarrow x \ \text{ and } \ \frac{\xi_n - x}{t_n} \longrightarrow \xi\}. \tag{3.40}$$

In the following, we use the notations

$$\lambda_0 = \min\{V_1 : V > 0,\, V \in \overline{\Omega^{--}} \cap \partial S\},$$
$$\lambda_1 = \inf\{V_1 : V > 0,\, V \in \overline{\Omega^{-+}} \cap \partial S\}, \tag{3.41}$$
$$\lambda_2 = \inf\{V_2 : V > 0,\, V \in \overline{\Omega^{+-}} \cap \partial S\}.$$

From the definitions of λ_i $(i = 0, 1, 2)$, it is obvious that λ_i $(i = 0, 1, 2)$ exist and are positive.

We define

$$W_0 := \{V^0 \in \Omega^{--} :\; V_1^0 < \lambda_0\},$$
$$W_1 := \{V^0 \in \Omega^{-+} :\; V_1^0 < \lambda_1\},$$
$$W_2 := \{V^0 \in \Omega^{+-} :\; V_2^0 < \lambda_2\},$$
$$S_1 := W_0 \cup W_1 \cup W_2.$$

Theorem 3.4.3. *If there exists a set $S \subset \mathbb{R}^2_+$ such that*

$$\Omega^{-+} \cup \Omega^{--} \cup \Omega^{+-} = S \;\; and \quad \Omega^{+-} \cap \Omega^{--} \cap \Omega^{+-} = \{0\}, \tag{3.42}$$

then system (3.33) *is locally asymptotically stable at the origin on S_1.*

In order to prove Theorem 3.4.3, we introduce the following lemma.

Lemma 3.4.4. *If the assumptions of Theorem 3.4.3 hold, then the set W_0 is positively invariant for system* (3.33). *Moreover, for all initial conditions V^0 in W_0 we have for the corresponding solution: $\lim\limits_{t \longrightarrow \infty} V(t, V^0) = 0$.*

Proof. Based on the conditions, we obtain that the origin is the only equilibrium of system (3.33) in S. Let $V \in W_0$.

If $\dot{V}_1(t) = 0$ and $\dot{V}_2(t) = 0$, then $V = 0$ is the equilibrium and invariance trivially holds. For the rest we use an analysis similar to the proof of Lemma 2.2 in [3].

In the case $\dot{V}_1 < 0$ and $\dot{V}_2 = 0$, in order to get the Bouligand cone $T^B_{W_0}(V)$ to W_0 at V, we take any sequence $V_n = (V_{1n}, V_{2n})^\top = (V_1 - h\varepsilon_n, V_2)^\top$ with $\varepsilon_n \downarrow 0$, $h > 0$. For sufficiently small ε_n one has $\dot{V}_{1n}(t) < 0$, and because the function β_2 is continuous and increasing,

$$\dot{V}_{2n}(t) = -\rho_2(V_2) + \beta_2(V_1 - h\varepsilon_n) < -\rho_2(V_2) + \beta_2(V_1) = 0.$$

Hence, $V_n \in W_0^\circ$ and

$$(-h, 0)^\top = \lim_{n \to +\infty} \frac{V_n - V}{\varepsilon_n} \in T^B_{W_0}(V). \tag{3.43}$$

Therefore, $F(V) \in T^B_{W_0}(V)$.

For the case $\dot{V}_1 < 0$ and $\dot{V}_2 = 0$, by a similar analysis we have that

$$(0, -h)^\top = \lim_{n \to +\infty} \frac{V_n - V}{\varepsilon_n} \in T^B_{W_0}(V). \tag{3.44}$$

Thus, $F(V) \in T^B_{W_0}(V)$.

If V satisfies $\dot{V}_1 < 0$ and $\dot{V}_2 < 0$, then it is obvious that $F(V) \in T^B_{W_0}(V)$.

In conclusion, we have $F(V) \in T^B_{W_0}(V)$ for all $V \in W_0$. According to Theorem 3.8 in [15] (see Theorem 5.3.1), W_0 is positively invariant.

For $V^0 \in W_0$, both components of V are monotonically non-increasing. Since the positive orthant is positively invariant and there is only one equilibrium in S, it follows that $\lim_{t\longrightarrow\infty} V(t, V^0) = 0$. □

Proof. [of Theorem 3.4.3] Let $V^0 \in S_1$. If $V^0 \in W_0$ there is nothing to prove because of Lemma 3.4.4.

For $V^0 \in W_1$, we claim that $V(t, V^0)$ enters W_0 in finite time.

To prove this, we assume that $V(t, V^0) \in W_1$ for all $t \in [0, T_{\max})$ for $V^0 \in W_1$. It is obvious that $V_1(t)$ is bounded and non-increasing. Therefore, $V_1(t, V_1^0) \longrightarrow \bar{V}_1 \geq 0$ as $t \longrightarrow T_{\max}$. Hence, $V(t)$ cannot enter the set $W_{11} := \{V \in W_1 : V_1(t) \geq \lambda_1\}$. We define $W_{12} := \Omega^{-+} \setminus W_{11}$. Since $\dot{V}_2(t) > 0$ and W_{12} is bounded, in finite time V will enter W_0 through one point of Cur_1, where $Cur_1 := \{V \in S : \rho_2(V_2) = \beta_2(V_1)\}$ (See Figure 3.1). According to Lemma 3.4.4, we get $T_{\max} = +\infty$ and $\lim_{t\longrightarrow+\infty} V(t, V^0) = 0$ for $V_0 \in W_1$.

For $v^0 \in W_2$, in a similar way we prove that v will enter W_0 in finite time through one point of Cur_2, where $Cur_2 := \{V \in S : \rho_1(V_1) = \beta_1(V_2)\}$ (See Figure 3.1). Based on Lemma 3.4.4, we have that $T_{\max} = +\infty$ and $\lim_{t\longrightarrow+\infty} V(t, V^0) = 0$ for $V_0 \in W_2$.

Using Lemmas 1.6.4 and 3.4.4, we obtain that system (3.33) is locally asymptotically stable at the origin. □

Proposition 3.4.5. *The lower estimate of the domain of attraction for system* (3.33) *is* S_1.

Proof. It follows from Theorem 3.4.3 that $S_1 \subset \mathcal{D}_V$. □

Remark 3.4.6. By Proposition 3.4.1 and 3.4.5, we get that $V^{-1}(S_1) \subset \mathcal{D}$.

In the following we describe the whole procedure of estimating the domain of attraction for system (3.1) at the origin.

1. Choose functions $\eta_i \in \mathcal{K}_\infty$, $\overline{\eta}_i$ $(i = 1, 2)$ such that the corresponding auxiliary system (3.4) is uniformly asymptotically stable at the origin and compute the maximal robust Lyapunov function v_i on $\mathcal{D}_{\eta i}$ by solving the corresponding extended Zubov equation (3.14).

2. Choose functions $\overline{\alpha}_i$, $\underline{\alpha}_i \in \mathcal{K}_\infty$ such that (3.30) is satisfied, and locally Lipschitz continuous functions ρ_i and β_i such that (3.31) hold.

3. Consider the comparison system (3.33) and calculate the sets Ω^{+-}, Ω^{--}, Ω^{-+} and λ_i $(i = 0, 1, 2)$.

4. Check the assumptions of Theorem 3.4.3.

5. Compute W_i $(i = 0, 1, 2)$ and get an estimate of the domain of attraction for system (3.1) at the origin by

$$V^{-1}(\mathcal{D}_V) \subset \mathcal{D}. \tag{3.45}$$

We claim that provided all steps in the construction can be completed successfully, the interconnected system (3.1) is locally asymptotically stable at the origin and an estimate of the domain of attraction for system (3.1) at the origin is given by $V^{-1}(\mathcal{D}_V)$. This is the gist of the following theorem.

Theorem 3.4.7. *Consider the coupled system* (3.1). *For each subsystem, we assume that the corresponding auxiliary system is locally uniformly asymptotically stable at the origin. If the assumptions of Theorem 3.3.5 hold, then for each subsystem there exists a local iISS Lyapunov function v_i satisfying* (3.31). *Consider the corresponding comparison system* (3.33). *If the assumptions of Theorem 3.4.3 hold, then system* (3.1) *is locally asymptotically stable at the origin and an estimate of the domain of attraction for system* (3.1) *at the origin is defined by*

$$V^{-1}(\mathcal{D}_V) \subset \mathcal{D}. \tag{3.46}$$

Proof. Under the assumptions, for each subsystem of system (3.1) there exists a local iISS Lyapunov function v_i which satisfies (3.31). Let $x^0 \in V^{-1}(\mathcal{D}_V)$. According to Theorem 3.4.3, the corresponding comparison system (3.33) with the initial condition $V^0 = (V_1(x_1^0), V_2(x_2^0))^\top$ is locally asymptotically stable at the origin. Using Lemma 1.6.4 and Proposition 3.4.1, we conclude that system (3.1) is locally asymptotically stable at the origin and $V^{-1}(\mathcal{D}_V) \subset \mathcal{D}$. □

3.5 A local small gain theorem in dissipative form

In this section, we introduce a small gain theorem which is applied to analysing stability of interconnected ISS systems. For this theorem, we need to obtain ISS Lyapunov functions in dissipative formulation for all subsystems. To this end, for (3.31) we choose $\gamma_i \in \mathcal{K}_\infty$ such that $\gamma_i(s) \le \rho_i(s)$ on subsets $[0, s_i^*]$, $i = 1, 2$, $s_i^* \in [0, 1)$. Let $\widetilde{\mathcal{D}}_{\eta_i} = \{x_i : v_i(x_i) \in [0, s_i^*]\} \subset \mathcal{D}_{\eta_i}$, $i = 1, 2$. By (3.31), we have

$$\dot{v}_i(x_i) \le -\gamma_i(v_i(x_i)) + \beta_i(v_j(x_j)), \tag{3.47}$$

for $x_i \in \widetilde{\mathcal{D}}_{\eta_i}$ and $x_j \in \widetilde{\mathcal{D}}_{\eta_j}$, $i, j = 1, 2$ and $i \neq j$. Thus, v_i is a local ISS Lyapunov function in dissipative formulation for the subsystem.

We define the matrix

$$\Gamma = \begin{pmatrix} 0 & \beta_1 \\ \beta_2 & 0 \end{pmatrix},$$

which defines a monotone map $\Gamma : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+^2$ by

$$\Gamma(s) := (\beta_1(s_2), \beta_2(s_1))^\top, \; s \in \mathbb{R}_+^2, \; s = (s_1, s_2)^\top. \tag{3.48}$$

Furthermore define the diagonal operator $A : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+^2$

$$A(s) := (\gamma_1(s_1), \gamma_2(s_2))^\top, \; s \in \mathbb{R}_+^2, \; s = (s_1, s_2)^\top. \tag{3.49}$$

Now we state the local version of the small gain theorem in dissipative form (Theorem 1.6.2).

Theorem 3.5.1. *Consider the coupled system* (3.1) *and assume that for each subsystem there exists a local ISS Lyapunov function in dissipative formulation* v_i *in the sense of* (3.47) *for* $x_i \in \widetilde{\mathcal{D}}_{\eta_i}$. *We further assume that* $\Gamma \circ A^{-1}$ *satisfies the local small gain conditions on* $[0, s^*]$, *i.e.,*

$$\Gamma \circ A^{-1}(s^*) < s^* \quad \text{and} \quad \Gamma \circ A^{-1}(s) \not\geq s, \quad \forall s \in (0, s^*), \tag{3.50}$$

where $s^* = (s_1^*, s_2^*)^\top$. *Then there exist strictly increasing functions* $\theta_i : [0,1] \longrightarrow [0, s_i^*]$ *such that*

$$\Gamma \circ A^{-1}(\theta(r)) < \theta(r), \ \text{for } r \in (0,1] \tag{3.51}$$

with $\theta(r) = (\theta_1(r), \theta_2(r))^\top$.

Furthermore, a local nonsmooth Lyapunov function for system (3.1) *is defined by*

$$v(x_1, x_2) := \max\left\{\theta_1^{-1} \circ \gamma_1(v_1(x_1)), \theta_2^{-1} \circ \gamma_2(v_2(x_2))\right\}. \tag{3.52}$$

Proof. Proposition 5.2 in [86] (see Proposition 5.4.1) states that the local small gain conditions imply that there exist strictly increasing functions $\theta_i : [0,1] \longrightarrow [0, s_i^*]$ such that (3.51) holds.

Note that $\theta_i^{-1} : [0, s_i^*] \longrightarrow [0,1]$ is well defined. Now we assume first that for a given x, $v(x)$ satisfies $v(x) = \theta_1^{-1} \circ \gamma_1(v_1(x_1)) > \theta_2^{-1} \circ \gamma_2(v_2(x_2))$.

We denote $z_i = \gamma_i(v_i(x_i))$. According to the assumptions, we attain that

$$\begin{aligned} \dot{v}_1(x_1) &\leq -\gamma_1(v_1(x_1)) + \beta_1(v_2(x_2)) \\ &= -\theta_1 \circ \theta_1^{-1}(z_1) + \beta_1 \circ \gamma_2^{-1}(z_2) \\ &\leq -\theta_1 \circ \theta_1^{-1}(z_1) + \beta_1 \circ \gamma_2^{-1} \circ \theta_2 \circ \theta_1^{-1}(z_1). \end{aligned} \tag{3.53}$$

For $\xi = \theta_1^{-1}(z_1)$, by (3.51) we have

$$\Gamma \circ A^{-1}(\theta(\xi)) < \theta(\xi), \ \ \xi \neq 0. \tag{3.54}$$

Then using (3.53) for $x_1 \neq 0$ we obtain that

$$\begin{aligned} \dot{v}_1(x_1) &\leq -\theta_1 \circ \theta_1^{-1}(z_1) + \beta_1 \circ \gamma_2^{-1} \circ \theta_2 \circ \theta_1^{-1}(z_1) \\ &< (\beta_1 \circ \gamma_2^{-1} \circ \theta_2 \circ \theta_1^{-1} - \mathrm{Id}) \circ \gamma_1(v_1(x_1)) < 0. \end{aligned} \tag{3.55}$$

Hence, under the assumption that $v(x) = \theta_1^{-1} \circ \gamma_1(v_1(x_1))$, for $x \neq 0$ we get that

$$\dot{v}(x) = \langle \nabla \theta_1^{-1} \circ \gamma_1(v_1(x_1)), \dot{v}_1(x_1) \rangle < 0 \tag{3.56}$$

holds.

The same argument applies vice versa if $v(x) = \theta_2^{-1} \circ \gamma_2(v_2(x_2)) > \theta_1^{-1} \circ \gamma_1(v_1(x_1))$.

As v is defined by the maximization of Lipschitz continuous functions v_i, for $x \in \widetilde{\mathcal{D}}_{\eta_1} \times \widetilde{\mathcal{D}}_{\eta_2}$ the Clarke's subdifferential of $v(x)$ at x (see Definition 1.4.12) is the set

$$\partial_{Cl} v(x) = \mathrm{co}\{\nabla(\theta_i^{-1} \circ \gamma_i(v_i(x_i))) | \theta_i^{-1} \circ \gamma_i(v_i(x_i)) = v(x_1, x_2)\}.$$

Based on the above analysis, we have

$$\langle \xi, f(x_1, x_2) \rangle < 0 \tag{3.57}$$

for $x \neq 0$, $\xi \in \partial_{Cl} v(x)$.

Using Lipschitz continuity and the decreasing property of v, we conclude that v is a nonsmooth Lyapunov function for system (3.1). □

The procedure of estimating the domain of attraction for system (3.1) is described as the following.

1. Choose functions $\eta_i \in \mathcal{K}_\infty$, $\overline{\eta}_i$ $(i = 1, 2)$ such that the corresponding auxiliary system (3.4) is uniformly asymptotically stable at the origin and compute the maximal robust Lyapunov function v_i on $\mathcal{D}_{\eta_i}$ by solving the corresponding extended Zubov equation (3.14).

2. Choose functions $\gamma_i, \beta_i \in \mathcal{K}_\infty$ such that (3.47) is satisfied, and calculate s_i^*.

3. Check if $\Gamma \circ A^{-1}(s)$ satisfies the local small gain conditions (3.50) on $(0, s^*]$.

4. Choose the path[1] θ along which $\Gamma \circ A^{-1}$ is decreasing, i.e., fulfills (3.51).

5. Define the Lyapunov function for system (3.1) by

$$v(x) := \max\left\{\theta_1^{-1} \circ \gamma_1(v_1(x_1)), \theta_2^{-1} \circ \gamma_2(v_2(x_2))\right\} , \tag{3.58}$$

   and let $\tau := \min\left\{\theta_1^{-1} \circ \gamma_1(s_1^*), \theta_2^{-1} \circ \gamma_2(s_2^*)\right\}$.

We claim that provided all steps in the construction can be completed successfully then with this choice of v we have that $v^{-1}([0, \tau])$ is a subset of the domain of attraction for system (3.1). This is the main content of the following theorem.

Theorem 3.5.2. *Consider system* (3.1)*. Assume that for each subsystem the corresponding auxiliary system is locally uniformly asymptotically stable at the origin. Furthermore, we suppose the assumptions of Theorem 3.3.5 and that for chosen γ_i, β_i $(i = 1, 2)$, $\Gamma \circ A^{-1}(s)$ satisfies the small gain conditions* (3.50) *for $s \in S$, where $S = [0, s_1^*] \times [0, s_2^*]$. Then there exists a path $\theta : [0, 1] \longrightarrow S$ such that $\theta(0) = 0$, $\theta(1) = s^*$ and the component functions of θ are continuous and strictly increasing. Moreover, the function v defined in* (3.52) *is a local Lyapunov function for system* (3.1)*. An estimate of the domain of attraction for system* (3.1) *at the origin is given by $\mathcal{D}_x := v^{-1}([0, \tau]) \subset \mathcal{D}$.*

Proof. Under the assumptions, each subsystem is local ISS and has a local ISS Lyapunov function in dissipative formulation v_i satisfying (3.47).

The choice of τ ensures that $v(x_1, x_2) \le \tau < 1$ implies that $x_1 \in \widetilde{\mathcal{D}}_{\eta_1}$ and $x_2 \in \widetilde{\mathcal{D}}_{\eta_2}$, because of Proposition 3.3.1 (i). Then by similar proof as the proof of Theorem 3.5.1, we arrive that $\dot{v}(x) < 0$, for all the points $x \in \widetilde{\mathcal{D}}_{\eta_1} \times \widetilde{\mathcal{D}}_{\eta_2}$, $x \neq 0$. Thus, $v(x)$ is a nonsmooth Lyapunov function for system (3.1) and $\mathcal{D}_x \subset \mathcal{D}$. □

Remark 3.5.3. The structure of our methods for estimating the domain of attraction for system (3.1) is shown in Figure 3.2.

[1] This can be done numerically, see [86].

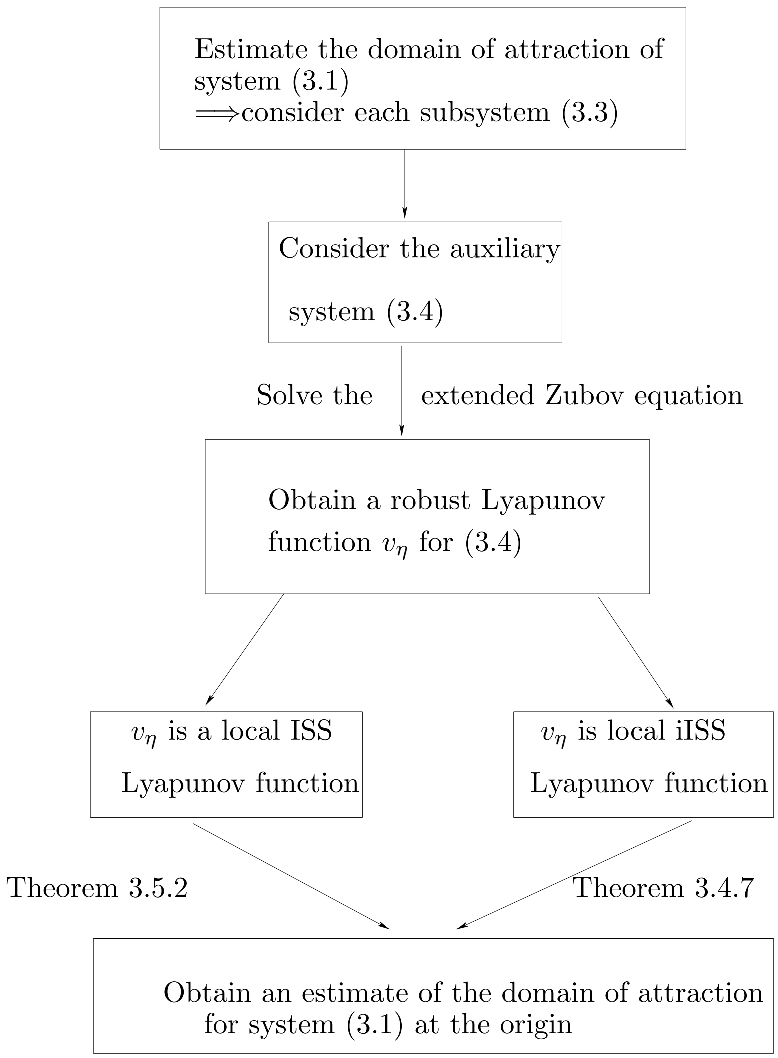

Figure 3.2: The structure of the proposed methods for estimating the domain of attraction for system (3.1).

3.6 Example

In this section, we present a academic example to illustrate the comparison of main results of Theorems 3.4.7 and 3.5.2.

Consider the system described by

$$\begin{cases} \dot{x}_1 = -x_1 + x_1^3 + x_1 x_2^2, \\ \dot{x}_2 = -x_2 + x_2^3 + x_2 x_1^2 \end{cases} \tag{3.59}$$

with $(x_1, x_2)^\top$ in $\mathbb{R}^2$ and the initial condition $x^0 = (x_1^0, x_2^0)^\top$.

3.6.1 Computation of iISS Lyapunov functions for subsystems

1. For each subsystem, we consider the auxiliary system described in the form

$$\dot{x} = f_\eta(x,u) := f(x,u) - \eta(\|u\|_2)x = -x + x^3 + xu^2 - xu^2, \tag{3.60}$$

where $x, u \in \mathbb{R}$ and $|u| \leq 1$.

Let $g(x,u) = 2|x|$. Define the function v by

$$v(x) = \begin{cases} 1 - e^{-V(x)} = \dfrac{2|x|}{1+|x|}, & x \in (-1,1), \\ 1, \quad x \notin (-1,1), \end{cases} \tag{3.61}$$

where $V : (-1,1) \longrightarrow \mathbb{R}_+$ is given by

$$V(x) = -\ln(1-|x|) + \ln(1+|x|) \tag{3.62}$$

solving

$$\inf_{|u|\leq 1} \{-\langle \nabla V(x), f_\eta(x,u)\rangle - g(x,u)\} = 0, \quad x \in (-1,1). \tag{3.63}$$

Using (3.61), we get that v is bounded and continuous on $(-1,1)$. By Theorem 3.3.2, we have v is a unique viscosity solution of

$$\sup_{|u|\leq 1} \{\langle \nabla v(x), f_\eta(x,u)\rangle + (1-v(x))g(x,u)\} = 0, \quad x \in (-1,1) \tag{3.64}$$

with $v(0) = 0$.

According to Theorem 3.3.2, $(-1,1)$ is a robust domain of attraction for system (3.60) at the origin.

Based on (3.60), we obtain that

$$\begin{aligned} \langle \nabla v(x), f(x)\rangle &\leq -(1-v(x))g(x_1,x_2) + \langle \nabla v(x), x\rangle u^2 \\ &\leq -\Big(1 - \frac{2|x|}{1+|x|}\Big)2|x| + \frac{2|x|}{(1+|x|)^2}u^2. \end{aligned} \tag{3.65}$$

2. Applying the above procedure to each subsystem of system (3.59), we have

$$\langle \nabla v_1(x_1), f_1(x_1,x_2)\rangle \leq -\Big(1 - \frac{2|x_1|}{1+|x_1|}\Big)2|x_1| + \frac{2|x_1|}{(1+|x_1|)^2}x_2^2, \tag{3.66}$$

$$\langle \nabla v_2(x_2), f_1(x_1,x_2)\rangle \leq -\Big(1 - \frac{2|x_2|}{1+|x_2|}\Big)2|x_2| + \frac{2|x_2|}{(1+|x_2|)^2}x_1^2 \tag{3.67}$$

for $|x_1| < 1$ and $|x_2| < 1$.

Using $v_i(x_i) = \dfrac{2|x_i|}{1+|x_i|}$ and $\dfrac{2|x_i|}{(1+|x_i|)^2} \leq \dfrac{1}{2}$, we get that

$$\langle \nabla v_1(x_1), f_1(x_1,x_2)\rangle \leq -(1-v_1)\frac{2v_1}{2-v_1} + \frac{v_2^2}{2(2-v_2)^2}, \tag{3.68}$$

$$\langle \nabla v_2(x_2), f_2(x_1,x_2)\rangle \leq -(1-v_2)\frac{2v_2}{2-v_2} + \frac{v_1^2}{2(2-v_1)^2}. \tag{3.69}$$

3. Let $\rho_i(v_i) = -(1-v_i)\dfrac{2v_i}{2-v_i}$, and $\beta_j(v_j) = \dfrac{v_j^2}{2(2-v_j)^2}$ $(i, j = 1, 2,\ i \neq j)$. It is obvious that β_j is an increasing function and $\beta_j(0) = 0$. Thus we treat β_j as a $\mathcal{K}_\infty$ function on $[0, 1)$. Since ρ_i is positive definite for $v_i \in [0, 1)$, v_i is a local iISS Lyapunov function for the subsystem.

3.6.2 Estimate of the domain of attraction by Theorem 3.4.7

In order to estimate the domain of attraction for system (3.59), we consider the comparison system described by

$$\begin{cases} \dot{v}_1 = -(1-v_1)\dfrac{2v_1}{2-v_1} + \dfrac{v_2^2}{2(2-v_2)^2} = F_1(v), \\ \dot{v}_2 = -(1-v_2)\dfrac{2v_2}{2-v_2} + \dfrac{v_1^2}{2(2-v_1)^2} = F_2(v), \end{cases} \tag{3.70}$$

where $v = (v_1,\, v_2)^\top \in [0, 1)^2$ and the initial condition $v^0 = v(x^0)$.

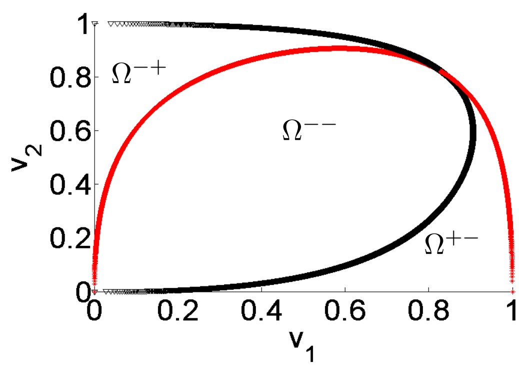

Figure 3.3: red curve: $F_1(v) = 0$, black curve: $F_2(v) = 0$

We define

$$\begin{aligned} \Omega^{+-} &:= \{v^\top \in \mathbb{R}_+^2 : F_1(v_1, v_2) \geq 0 \text{ and } F_2(v_1, v_2) \leq 0)\}, \\ \Omega^{--} &:= \{v^\top \in \mathbb{R}_+^2 : F_1(v_1, v_2) \leq 0 \text{ and } F_2(v_1, v_2) \leq 0)\}, \\ \Omega^{-+} &:= \{v^\top \in \mathbb{R}_+^2 : F_1(v_1, v_2) \leq 0 \text{ and } F_2(v_1, v_2) \geq 0)\}. \end{aligned} \tag{3.71}$$

Let $S = \Omega^{+-} \cup \Omega^{--} \cup \Omega^{-+}$. In Figure 3.3, the positive intersection point is $(0.8246, 0.8246)^\top$.

Remark 3.6.1. From Figure 3.3, it is known that S satisfies the condition of Theorem 3.4.3. Then by Theorem 3.4.7 the comparison system (3.70) is locally asymptotically stable at the origin. A lower estimate of the domain of attraction for system (3.70) at the origin is given by $S_1 = W_0 \cap W_1 \cap W_2$ with $\lambda_0 = \lambda_1 = \lambda_2 = 0.8246$, i.e., $S_1 = [0, 0.8246]^2$.

Remark 3.6.2. S_1 is a lower estimate of the domain of attraction for system (3.70) at the origin. Based on S_1, we can obtain a bigger estimate S_2 by checking if the solution to (3.70) with the initial condition $v^0 \in S \backslash S_1$ enters S_1 in sufficiently long time T_1. Let $T_1 = 10^7$. Figure 3.4 shows that S_2 (the yellow section) is a bigger estimate of the domain of attraction for system (3.70). The size of the set S_2 is dependent on the length of $[0, T_1]$.

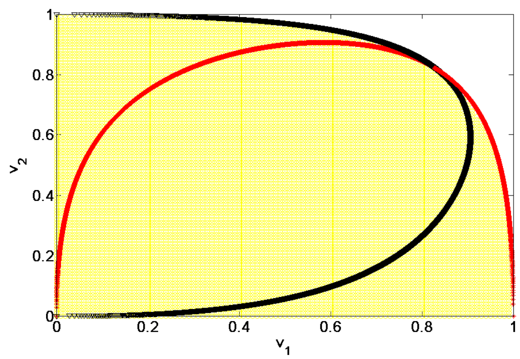

Figure 3.4: red curve: $F_1(v) = 0$, black curve : $F_2(v) = 0$, yellow section is an estimate of the domain of attraction for system (3.70) at the origin.

Remark 3.6.3. According to Theorem 3.4.7 and Remark 3.6.2, an estimate of the domain of attraction for system 3.59 at the origin is given by $v^{-1}(S_2) = \{(x_1, x_2)^\top \in \mathbb{R}^2 : (v_1(x_1), v_2(x_2))^\top \in S_2\}$ shown in Figure 3.5.

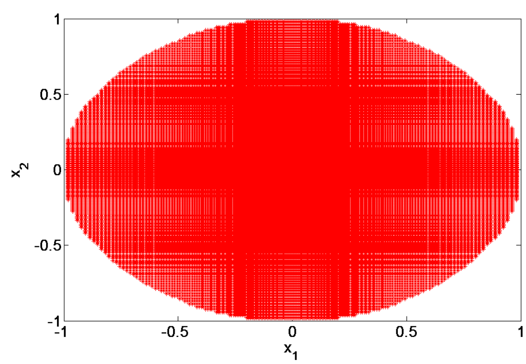

Figure 3.5: An estimate of the domain of attraction for system (3.59) at the origin.

3.6.3 Estimate of the domain of attraction by Theorem 3.5.2

In order to estimate the domain of attraction of system (3.59) by Theorem 3.5.2, we have to choose functions $\gamma_i \in \mathcal{K}_\infty$ $(i = 1, 2)$ such that $\rho_i(v_i) \geq \gamma_i(v_i)$ for $v_i \in [0, m_i]$ $(m_i < 1)$. In practice, m_i should be chosen so close to 1 that we could get a better estimate of the domain of attraction.

Here, we let $\gamma_i(v_i) = cv_i^k$ with $k = 1, 2$. Of course, other formulations of γ_i can be used. We choose these types of formulation because it is easily to obtain results. In the following, for each type of γ_i we study how to choose a c such that the conditions of Theorem 3.5.2 hold and the interval $[0, m_i]$ $(i = 1, 2)$ is as large as possible.

Remark 3.6.4. Let $\gamma_i(v_i) = cv_i$ $(c > 0)$. The largest estimate of the domain of attraction $\mathcal{D}_x = [-0.4085, 0.4085]^2$ for system (3.59) is obtained when $c = 0.58$.

Procedure for obtaining the results of Remark 3.6.4:

Let

$$cv_i = \frac{2v_i(1-v_i)}{2-v_i}, \quad \text{for } v_i \in [0,1)\ , \tag{3.72}$$

then $v_i = \dfrac{2c-2}{c-2}$. Based on Figure 3.6, it is reasonable to consider $c \in (0,1]$.

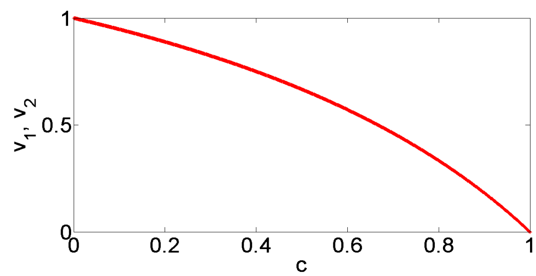

Figure 3.6: $v_i = \dfrac{2c-2}{c-2}$.

We define the operators needed in Theorem 3.5.2.

$$\Gamma(s) = \Big(\frac{s_2^2}{2(2-s_2)^2}, \frac{s_1^2}{2(2-s_1)^2}\Big)^\top, \quad \forall s = (s_1, s_2)^\top \in [0,1)^2. \tag{3.73}$$

The diagonal operator $A : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ is given by

$$A(s) = (cs_1, cs_2)^\top, \quad s = (s_1, s_2)^\top \in [0,1)^2. \tag{3.74}$$

Then

$$\Gamma \circ A^{-1}(s) = \Big(\frac{s_2^2}{2(2c-s_2)^2}, \frac{s_1^2}{2(2c-s_1)^2}\Big)^\top, \quad \forall s \in \Big[0, \frac{2c-2}{c-2}\Big]^2. \tag{3.75}$$

The main task now is to check which $c \in (0,1]$ satisfies the following conditions (A1), (A2) and insures the set $\Big[0, \dfrac{2c-2}{c-2}\Big]^2$ is the biggest one.

(A1)

$$cv_i \le \rho_i(v_i) = \frac{2v_i(1-v_i)}{2-v_i}, \quad \text{for } v_i \in \Big[0, \frac{2c-2}{c-2}\Big]. \tag{3.76}$$

(A2) Local small gain conditions are

$$\Gamma \circ A^{-1}(s) \not\ge s, \quad \forall s \in \Big(0, \frac{2c-2}{c-2}\Big]^2, \tag{3.77}$$

$$\Gamma \circ A^{-1}(s) < s, \quad \text{for } s = \Big(\frac{2c-2}{c-2}, \frac{2c-2}{c-2}\Big)^\top. \tag{3.78}$$

First, we choose $c \in (0,1]$ under which (3.76), (3.77) and (3.78) hold using Matlab with the step size $h = 0.01$ for the variable c and the step size $h_1 = 0.0001$ for variable v_i ($i = 1, 2$).

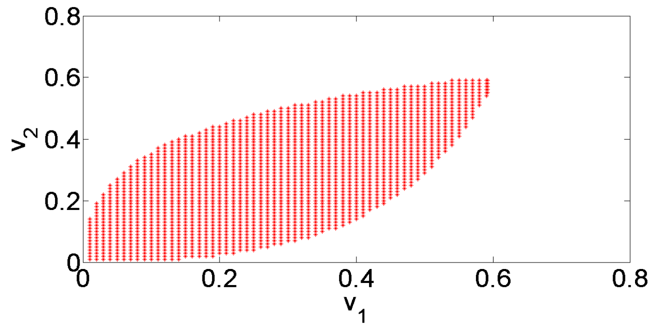

Figure 3.7: $c = 0.58$. Red stars represent points satisfying (3.79).

Second, for each c obtained from the first step, we check if there exists a set of points as Figure 3.7 shows satisfying

$$\Gamma \circ A^{-1}(s) < s, \quad s \in \Big(0, \frac{2c-2}{c-2}\Big]^2. \tag{3.79}$$

Third, for each c obtained from the second step, we calculate the maximum of $s_1 \cdot s_2$ for s_1, s_2 satisfying (3.79).

From the above procedure, we obtain that $c = 0.58$ is the candidate, and $s_1^* = 0.5915 = s_2^*$.

Based on Figure 3.7, we choose $\theta : \theta_1(r) = 0.5915r = \theta_2(r)$ such that $\Gamma \circ A^{-1}(\theta(r)) < \theta(r)$ for $r \in [0, 1]$. According to Theorem 3.5.1, a local nonsmooth Lyapunov function for system (3.59) is then defined by

$$v(x_1, x_2) := \max\left\{\frac{cv_1(x_1)}{0.5915}, \frac{cv_2(x_2)}{0.5915}\right\}, \ c = 0.58. \tag{3.80}$$

Utilizing $|x_i| = \dfrac{v_i}{2 - v_i}$, we get an estimate of the domain of attraction for (3.59) of the form $\mathcal{D}_x = [-0.4085, 0.4085]^2$.

Remark 3.6.5. Let $\gamma_i(v_i) = cv_i^2$ $(c > 0)$. When $c = 0.61$, the largest estimate of the domain of attraction $\mathcal{D}_x = [-0.4388, 0.4388]^2$ for system (3.59) is obtained.

Procedure for obtaining the results of Remark 3.6.5:

Let $cv_i^2 = \dfrac{2v_i(1 - v_i)}{2 - v_i}$ for $v_i \in [0, 1)$, then $v_i = 1 + \dfrac{1 - \sqrt{c^2 + 1}}{c}$. According to Figure 3.8, in the following we choose $c \in (0, 100]$.

Define the operators used in Theorem 3.5.1

$$\Gamma(s) = \Big(\frac{s_2^2}{2(2 - s_2)^2}, \frac{s_1^2}{2(2 - s_1)^2}\Big)^\top, \quad \forall s = (s_1, s_2)^\top \in [0, 1). \tag{3.81}$$

The diagonal operator $A : R_+^2 \to R_+^2$ is defined as

$$A(s) = (cs_1^2, cs_2^2)^\top, \quad s = (s_1, s_2)^\top \in [0, 1)^2. \tag{3.82}$$

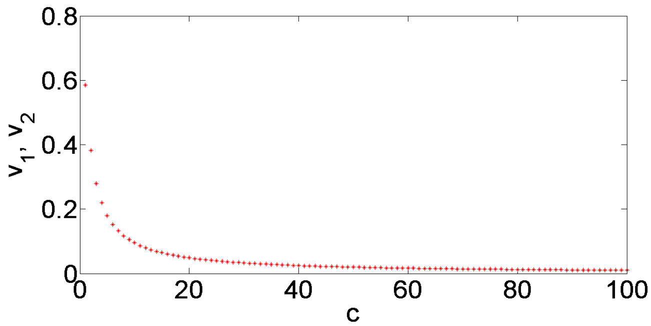

Figure 3.8: $v_i = 1 + \frac{1-\sqrt{c^2+1}}{c}$.

Then

$$\Gamma \circ A^{-1}(s) = \left(\frac{s_2}{2(2\sqrt{c}-\sqrt{s_2})^2}, \frac{s_1}{2(2\sqrt{c}-\sqrt{s_1})^2}\right)^\top, \quad \forall (s_1,s_2)^\top \in \left[0, 1+\frac{1-\sqrt{c^2+1}}{c}\right]^2. \tag{3.83}$$

As what we did for Remark 3.6.4, in the following we choose which $c \in (0,100]$ satisfies the conditions (A3), (A4) and ensures the set $\left[0, 1+\frac{1-\sqrt{c^2+1}}{c}\right]^2$ is the largest one.

(A3)

$$cv_i^2 \leq \frac{2v_i(1-v_i)}{2-v_i}, \text{ for } v_i \in \left[0, 1+\frac{1-\sqrt{c^2+1}}{c}\right]. \tag{3.84}$$

(A4) The local small gain conditions are

$$\Gamma \circ A^{-1}(s) \not\geq s, \quad \forall s \in \left(0, 1+\frac{1-\sqrt{c^2+1}}{c}\right]^2, \tag{3.85}$$

$$\Gamma \circ A^{-1}(s) < s, \quad \text{for } s = \left(1+\frac{1-\sqrt{c^2+1}}{c}, 1+\frac{1-\sqrt{c^2+1}}{c}\right)^\top. \tag{3.86}$$

By the same way as in obtaining results of Remark 3.6.4, we get that for $c = 0.61$ the maximum of $s_1 \cdot s_2$ is the biggest one with $s_1^* = 0.71 = s_2^*$, and the set of points $(s_1, s_2)^\top$ satisfying $\Gamma \circ A^{-1}(s) < s$ shown in Figure 3.9 exists, and the above conditions $(A3)$ and $(A4)$ are fulfilled.

According to what is shown in Figure 3.9, we choose $\theta(r) = (0.71r, 0.71r)^\top$ so that $\Gamma \circ A^{-1}(\theta(r)) < \theta(r)$, for $r \in [0,1]$. By Theorem 3.5.2, a local nonsmooth Lyapunov function for system (3.59) is then defined by

$$v(x_1,x_2) := \max\left\{\frac{cv_1^2(x_1)}{0.71}, \frac{cv_2^2(x_2)}{0.71}\right\}, \; c = 0.61. \tag{3.87}$$

Using $|x_i| = \frac{v_i}{2-v_i}$, an estimate of the domain of attraction for the original system (3.59) is given by $\mathcal{D}_x = [-0.4388, 0.4388]^2$.

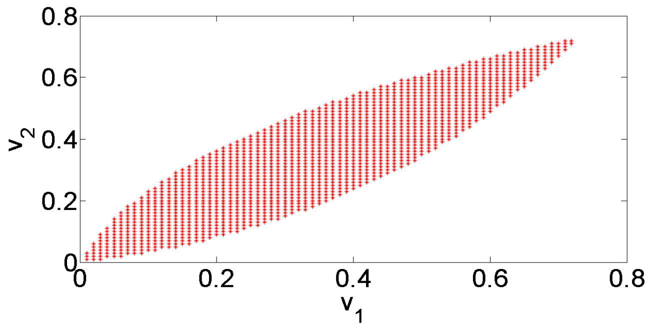

Figure 3.9: $c = 0.61$. Red stars represent points satisfying $\Gamma \circ A^{-1}(s) < s$.

Remark 3.6.6. The results for Example (3.59) demonstrate the effectiveness of our proposed methods. From the above results, it is obvious that for system (3.59) the way of estimating the domain of attraction by Theorem 3.4.3 is better than by Theorem 3.5.1. The reason is the following. When we analyse stability of (3.59) by the small gain theorem in dissipative form, it is necessary to find a $\mathcal{K}_\infty$ function γ_i such that $\gamma_i \leq \rho_i$. Thus, we have to exclude the points $x_i \in \mathcal{D}_{\eta_i}$ which do not satisfy $\gamma_i(v_i(x_i)) \leq \rho_i(v_i(x_i))$. This leads to the fact that the estimate of the domain of attraction obtained by Theorem 3.5.1 is smaller than by Theorem 3.4.3.

3.7 Concluding remarks and open questions

In this chapter, a new approach of computing iISS and ISS Lyapunov functions was proposed. An iISS or ISS Lyapunov function can be obtained with the help of an auxiliary system and Zubov's method for the considered system with perturbation. By Zubov's method, a maximal robust Lyapunov function can be attained for the auxiliary system. According to Proposition 3.3.7, we proved that such a maximal robust Lyapunov function is an iISS or ISS Lyapunov function for the original system. Using ISS or iISS Lyapunov function computed by our proposed technique for each subsystem of two interconnected systems, stability of the whole system is investigated by small gain theorems. Furthermore, estimates of the domain of attraction were obtained.

However, we cannot prove that if system (3.3) is locally iISS, then there exist functions $\eta \in \mathcal{K}_\infty$, $\overline{\eta}$ such that the auxiliary system (3.4) is uniformly asymptotically stable. On a compact subset of state space excluding a small neighbourhood of the equilibrium, we have the results summarized in Theorem 3.7.1.

Theorem 3.7.1. *Let $\mathcal{D} \subset \mathbb{R}^n$ be a compact set of state space. If system* (3.3) *is locally iISS on $\mathcal{D}$, then there exist a function $\eta \in \mathcal{K}_\infty$ and a constant $\epsilon > 0$ such that system described by*

$$\dot{x} = f(x,u) + \eta(\|u\|_2) f(x,0) \tag{3.88}$$

is uniformly asymptotically stable on $\mathcal{D} \setminus \mathcal{B}_2(0,\epsilon)$.

Proof. According to the assumption and Theorem 1.3.31, there exists a smooth iISS Lyapunov function V for system (3.3) satisfying

$$\langle \nabla V(x), f(x,u) \rangle \leq -\alpha(\|x\|_2) + \beta(\|u\|_2) \tag{3.89}$$

where $\alpha \in \mathcal{P}$, $\beta \in \mathcal{K}_\infty$.

Then we have

$$\langle \nabla V(x), f(x,u) + \eta(\|u\|_2) f(x,0)\rangle \leq -\alpha(\|x\|_2) + \beta(\|u\|_2) + \langle \nabla V(x), f(x,0)\rangle \eta(\|u\|_2). \quad (3.90)$$

Since V is a Lyapunov function for system (3.3) with $u = 0$, $\langle \nabla V(x), f(x,0)\rangle < 0$ for $x \neq 0$. Hence, there exists a constant $\delta > 0$ such that

$$\langle \nabla V(x), f(x,0)\rangle \leq -\delta \quad (3.91)$$

for $x \in \mathcal{D} \setminus \mathcal{B}_2(0,\epsilon)$.

Thus we choose $\eta(\|u\|_2) = \frac{1}{\delta}\beta(\|u\|_2)$. Following (3.90), we have system (3.4) is uniformly asymptotically stable on the set $\mathcal{D} \setminus \mathcal{B}_2(0,\epsilon)$. □

Remark 3.7.2. By Theorem 3.7.1 and Zubov's method for a perturbed system with a uniformly asymptotically stable set discussed in [31], we can obtain a local iISS Lyapunov function for system (3.3) on $\mathcal{D} \setminus \mathcal{B}_2(0,\epsilon)$.

If (3.4) is replaced with (3.88), then it is not necessary to prove Proposition 3.3.7 under the condition f is Lipschitz continuous.

Remark 3.7.3. Given system (3.3) is locally iISS, there are some problems we will investigate in the future.

(1) Give a formulation of $\eta(\|u\|_2)$ such that the auxiliary system (3.4) is uniformly asymptotically stable on a compact subset of state space.

(2) How to choose suitable $\rho_i \in \mathcal{P}$, $\beta_i \in \mathcal{K}_\infty$ satisfying (3.31), and $\gamma_i \in \mathcal{K}_\infty$ insuring (3.47) holds.

(3) Extend the small gain theorem in comparison form to more than 2 dimensional systems.

The results of Chapter 2 and this chapter inspire us to consider the following problem in the future.

- Extend the method for computing Lyapunov functions proposed in Chapter 2 to construction of robust Lyapunov functions for perturbed systems. If we could get CPA robust Lyapunov functions for auxiliary systems by this method, then such Lyapunov functions may be proved to be CPA iISS and ISS Lyapunov functions by similar argument of Lemma 3.2.1.

Remark 3.7.4. In order to make sure the viscosity solution to the extended Zubov's equation is an ISS Lyapunov function, we need to find $\gamma_i \in \mathcal{K}_\infty$ such that

$$\gamma_i(v_i(x)) \leq (1 - v_i(x_i)) g(x_i, u) \quad (3.92)$$

holds for all $u \in U_R$. Since the term $(1 - v_i(x_i))$ converges to 0 as x_i goes to the boundary of $\mathcal{D}_{\eta_i}$, the inequality (3.92) only holds on a subset of $\mathcal{D}_{\eta_i}$. Therefore, the domain where system (3.59) is ISS is smaller than the domain where system (3.59) is iISS. Thus the estimate of the domain of attraction by Theorem 3.5.1 is smaller than by Theorem 3.4.3. However, since Theorem 3.4.3 is only applied to analyse stability of two interconnected systems, there is no help in stability analysis of more than two interconnected systems. The solution to the

Hamilton-Jacobi-Bellman equation is generally computed by numerical methods. Therefore, we only get a numerical approximation of a Lyapunov function. Another drawback is that in Zubov's method the gain cannot be influenced directly via the optimal control criteria. These motivates us to consider the problem of how to compute an ISS Lyapunov function. From the results of Chapter 2, it is known that we can obtain a Lyapunov function rather than a numerical approximation by the CPA method. Hence, based on the idea of the CPA method we will design linear programming based algorithms for constructing ISS Lyapunov functions rather than numerical approximations on subsets of state space in the next chapter. Furthermore, we analyse stability of interconnected systems by the small gain theorem in linear form, since the computed ISS Lyapunov functions satisfy linear inequalities and the small gain theorem in linear form can be used to investigate stability for more than two interconnected systems.

4 Computation of ISS Lyapunov functions and stability of interconnected systems

From the results on estimates of the domain of attraction by the small gain theorem in dissipative form in Section 3.6, we observe that the framework of ISS is useful in stability analysis of two interconnected systems. More than that, the ISS notion plays an important part in the stability analysis of large scale systems. If subsystems are ISS, then stability of large scale systems can be analyzed by ISS small gain theorems discussed in Section 1.7. In Chapter 3, we proved that ISS Lyapunov functions in dissipative formulation can be computed using Zubov's method and auxiliary systems. The obtained ISS Lyapunov function in dissipative formulation is in fact a viscosity solution to a partial differential equation (Hamilton-Jacobi-Bellman equation). On the one hand, since the solution to the partial differential equation is usually computed by numerical methods, we just get a numerical approximation of an ISS Lyapunov function in dissipative formulation. On the other hand, the gain cannot be influenced via an optimization criterion. Motivated by these results, in this chapter we investigate how to compute true local ISS Lyapunov functions for low dimensional systems, as the knowledge of ISS Lyapunov functions leads to the knowledge of ISS gains which may be used in a small gain based stability analysis.

The linear programming based algorithm [5, 40, 41, 42, 76] for computing continuous and piecewise affine (CPA) Lyapunov functions yields true Lyapunov functions since it incorporates the interpolation errors in the linear constraints. Hence, it is interesting to design a linear programming based algorithm for computing CPA ISS Lyapunov functions in dissipative formulation for perturbed systems. We first investigate how to compute CPA ISS Lyapunov functions in dissipative formulation for continuous time dynamic systems with perturbations. We then apply the proposed method to the computation of CPA ISS Lyapunov functions in dissipative formulation for discrete time dynamic systems with perturbations. Based on CPA ISS Lyapunov functions in dissipative formulation obtained by solving linear optimization problems, we further study the stability of interconnected systems via the small gain theorem in linear form (Theorem 1.6.1).

In Section 4.1, we present preliminaries for Section 4.2 and Section 4.3.

In Section 4.2, we propose a linear programming based algorithm for computing CPA ISS Lyapunov functions in dissipative formulation for continuous time dynamic systems with perturbations. This algorithm relies on a linear optimization problem and delivers a CPA function. We will prove that, if the algorithm has a feasible solution, then it is a CPA ISS Lyapunov function in dissipative formulation for our considered system on a compact subset excluding a small neighbourhood of the origin. The CPA ISS Lyapunov function in dissipative formulation delivered by the algorithm is a viscosity subsolution of a partial differential equation associated with the perturbed system. Moreover, if there exists a C^2 ISS Lyapunov function for the perturbed system, then the algorithm with a suitable triangulation can terminate successfully. In the end, we illustrate the effectiveness of our proposed method by two numerical examples.

Inspired by the nice results of Section 4.2, in Section 4.3 we extend the method for constructing a CPA ISS Lyapunov function in dissipative formulation to discrete time dynamic systems with perturbations. We obtain some parallel results. In Section 4.3.1, we describe the linear programming based algorithm for computing a CPA ISS Lyapunov function in dissipative formulation on a compact subset of state space excluding a small neighbourhood of the origin. Furthermore, we prove that if the system has a local C^1 ISS Lyapunov function with a bounded gradient, then there exist suitable triangulations such that the algorithm has a feasible solution. In Section 4.3.2, we present two numerical examples to show how our proposed approach is applied.

The subject of Section 4.4 is to investigate stability of interconnected continuous time dynamic systems. We assume subsystems are locally ISS. Based on Theorem 4.2.6, a CPA ISS Lyapunov function in dissipative formulation for each subsystem can be constructed by solving a linear optimization problem (4.36). These CPA ISS Lyapunov functions in dissipative formulation satisfy linear inequalities. In Section 4.4.1, we demonstrate how to study stability of interconnected ISS systems using the small gain theorem in linear form (Theorem 1.6.1) and CPA ISS Lyapunov functions in dissipative formulation obtained from the proposed algorithm in Section 4.2.

4.1 Preliminaries

In this chapter, we specify the norms on the state space and the input value space as $\|\cdot\|_2$, $\|\cdot\|_1$ respectively. The reason for choosing these norms are explained in Remark 4.1.2.

In Section 4.2, we will study the problem of computing a CPA ISS Lyapunov function in dissipative formulation for a continuous time dynamic system with perturbation described by ordinary differential equations

$$\dot{x}(t) = f(x(t), u(t)) \tag{4.1}$$

with vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, state $x \in \mathbb{R}^n$ and perturbation input $u \in \mathbb{R}^m$, $t \in \mathbb{R}_+$. The admissible input values are given by $U_R := \operatorname{cl} B_1(0, R) \subset \mathbb{R}^m$ for a constant $R > 0$ and the admissible input functions by $u \in \mathcal{U}_R := \{u : \mathbb{R}_+ \to U_R \text{ measurable}\}$. Additionally, we assume $f(0, 0) = 0$.

In Section 4.3, we will investigate the problem of computing a CPA ISS Lyapunov function in dissipative formulation for a discrete time dynamic system with perturbation described by the following difference equation

$$x^+ = f(x, u), \tag{4.2}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, state $x \in \mathbb{R}^n$, and perturbation input $u \in \mathbb{R}^m$. The admissible input values are given by U_R and the admissible input functions by $\mathcal{U}_R$. We require $f(0, 0) = 0$.

For our algorithmic construction of Lyapunov functions for continuous and discrete time dynamic systems with perturbation, we need certain regularity properties of f which also determine certain inequalities imposed in the algorithm. To this end, we require one of the following two hypotheses.

(H1) The map $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is globally Lipschitz continuous.

(H2) The vector field f is twice continuously differentiable.

In regards of (H1) we fix the following notation: For each $u \in U_R$, $L_x(u)$ is the Lipschitz constant of the map $x \mapsto f(x,u)$, and for each $x \in \mathbb{R}^n$, $L_u(x)$ is the Lipschitz constant for the function $u \mapsto f(x,u)$. Moreover, by (H1) there exist constants $\overline{L}_x$ and $\overline{L}_u$ such that

$$\overline{L}_x \geq L_x(u) > 0,\ \overline{L}_u \geq L_u(x) > 0 \tag{4.3}$$

for all $x \in \mathbb{R}^n$, $u \in U_R$. Since we will only consider compact subsets of the state space $\mathbb{R}^n$ in the following, (H1) holds if f is locally Lipschitz in x and u and the constants $\overline{L}_x$, $L_x(u)$ etc. may be chosen with respect to the compact set of interest.

Consider system (4.1) or (4.2). Theorem 1.3.26 states that the ISS property of the system is equivalent to the existence of a smooth, i.e. C^∞, ISS Lyapunov function for the system. While this result guarantees the existence of smooth ISS Lyapunov functions our numerical techniques will not generate a smooth function. In the following we will numerically construct continuous and piecewise affine and thus nonsmooth ISS Lyapunov functions in dissipative formulation defined in Definition 1.4.14. For convenience, we restate the definition of nonsmooth ISS Lyapunov functions in dissipative formulation with the specified norms.

Definition 4.1.1. Let $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$.

(i) A Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be a local *nonsmooth ISS Lyapunov function in dissipative formulation* for system (4.1) if there exist functions α_1, α_2, α, $\beta \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x\|_2) \leq V(x) \leq \alpha_2(\|x\|_2), \quad \forall x \in \mathbb{R}^n, \tag{4.4}$$
$$\langle \xi, f(x,u) \rangle \leq -\alpha(\|x\|_2) + \beta(\|u\|_1) \quad \forall\, \xi \in \partial_{Cl} V(x) \tag{4.5}$$

hold for all $x \in \mathcal{D}$, $u \in U_R$ and $\xi \in \partial_{Cl} V(x)$. If $\mathcal{D} = \mathbb{R}^n$ then V is called a global *nonsmooth ISS Lyapunov function in dissipative formulation.*

(ii) A Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be a local *nonsmooth ISS Lyapunov function in dissipative formulation* for system (4.2) if if there exist functions α_1, α_2, α, $\beta \in \mathcal{K}_\infty$ such that (4.4) and

$$V(f(x,u)) - V(x) \leq -\alpha(\|x\|_2) + \beta(\|u\|_1) \tag{4.6}$$

hold for all $x \in \mathcal{D}$ and $u \in U_R$. If $\mathcal{D} = \mathbb{R}^n$ then V is called a global *nonsmooth ISS Lyapunov function in dissipative formulation.*

Remark 4.1.2. The particular norms chosen in the formulation of the ISS Lyapunov function in dissipative formulation in (4.5) and (4.6) do not play a role from the conceptual point of view: as all norms in $\mathbb{R}^n$ are equivalent, different norms will only lead to different numerical values of the gains. The particular formulations we have chosen will turn out to be useful in deriving easy estimates, see the end of proofs of Theorems 4.2.6 and 4.3.7.

In order to simplify the algorithm to be proposed in this chapter, we will restrict ourselves to ISS Lyapunov functions in dissipative formulation which satisfy (4.5) or (4.6) with linear functions $\alpha(s) = s$ and $\beta(s) = rs$ for some fixed $r > 0$. The following proposition shows that on compact subsets of the state space excluding a ball around the origin this can be done without loss of generality.

Proposition 4.1.3. *If there exists a nonsmooth ISS Lyapunov function in dissipative formulation W for system (4.1) or (4.2) on a compact set $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$, then for any $\epsilon > 0$ and $\sigma > 0$ there exist positive constants $C, r > 0$ such that $V(x) := CW(x)$ satisfies*

$$V(x) \geq \|x\|_2 \quad \forall x \in \mathcal{D} \setminus \mathcal{B}_2(0, \epsilon) \tag{4.7}$$

and $\forall x \in \mathcal{D} \setminus \mathcal{B}_2(0, \epsilon)$, $u \in U_R$ (U_R from Definition 4.1.1)

$$\langle \xi, f(x, u) \rangle \leq -\sigma \|x\|_2 + r\|u\|_1 \quad \forall\, \xi \in \partial_{Cl} V(x), \quad \textit{or} \tag{4.8}$$

$$V(f(x, u)) - V(x) \leq -\sigma \|x\|_2 + r\|u\|_1. \tag{4.9}$$

Proof. According to (4.5) or (4.6), for $x \in \mathcal{D} \setminus \mathcal{B}_2(0, \epsilon)$, $u \in U_R$, $W(x)$ satisfies

$$\langle \xi, f(x, u) \rangle \leq -\alpha(\|x\|_2) + \beta(\|u\|_1), \quad \xi \in \partial_{Cl} W(x), \quad \text{or,} \tag{4.10}$$

$$W(f(x, u)) - W(x) \leq -\alpha(\|x\|_2) + \beta(\|u\|_1). \tag{4.11}$$

In order to construct C and r we now distinguish two cases.

Case 1: $\limsup_{s \to 0} \beta(s)/s$ is bounded.
In this case we define

$$C := \min \Big\{ c > 0 \,:\, c\alpha_1(\|x\|_2) \geq \|x\|_2 \text{ and } c\alpha(\|x\|_2) \geq \sigma\|x\|_2,\ \forall x \in \mathcal{D} \setminus \mathcal{B}_2(0, \epsilon) \Big\}.$$

Then, there exists $r > 0$ satisfying

$$C\beta(\|u\|_1) \leq r\|u\|_1 \quad \text{for all } u \in U_R. \tag{4.12}$$

Case 2: $\limsup_{s \to 0} \beta(s)/s$ is unbounded.
In this case we choose C as

$$C := \min \Big\{ c > 0 \,:\, c\alpha_1(\|x\|_2) \geq \|x\|_2 \text{ and } c\alpha(\|x\|_2) \geq \sigma\|x\|_2 + \epsilon,\ \forall x \in \mathcal{D} \setminus \mathcal{B}_2(0, \epsilon) \Big\}.$$

Then we have

$$\langle \xi, f(x, u) \rangle \leq -\sigma\|x\|_2 - \epsilon + C\beta(\|u\|_1), \quad \xi \in \partial_{Cl} V(x), \quad \text{or} \tag{4.13}$$

$$V(f(x, u)) - V(x) \leq -\sigma\|x\|_2 - \epsilon + C\beta(\|u\|_1) \tag{4.14}$$

holds for $x \in \mathcal{D} \setminus \mathcal{B}_2(0, \epsilon)$, $u \in U_R$.
It is possible to find a constant $r > 0$ such that

$$\begin{cases} C\beta(\|u\|_1) \leq r\|u\|_1, & \text{if } C\beta(\|u\|_1) \geq \epsilon,\ u \in U_R, \\ C\beta(\|u\|_1) \leq r\|u\|_1 + \epsilon, & \text{if } C\beta(\|u\|_1) \leq \epsilon,\ u \in U_R. \end{cases} \tag{4.15}$$

In both cases, a straightforward calculation shows that $V(x) = CW(x)$ satisfies the desired inequalities. □

Remark 4.1.4. It may not always be possible to choose $\epsilon = 0$ in Proposition 4.1.3. However, in general the linear programming approach of computing Lyapunov functions only works outside a neighbourhood of the origin, anyway, cf. Remarks 2.2.11, 4.2.2 and 4.3.3, such that the need to remove $\mathcal{B}_2(0, \epsilon)$ does not introduce additional limitations into our approach.

We will propose algorithms for computing a local ISS Lyapunov function in dissipative formulation defined on a suitable triangulation of a compact set $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}^\circ$ and valid for perturbation inputs from a suitable triangulation of $U_R \subset \mathbb{R}^m$. The algorithms use linear programming and the representation of the function on a suitable triangulation in order to obtain a numerical representation as a continuous and piecewise affine function. By taking into account interpolation errors, the algorithms yields true ISS Lyapunov functions in dissipative formulation, not only approximative ones.

Let $\mathcal{T} = \{\mathcal{S}_1, \dots, \mathcal{S}_N\}$, $\mathcal{T}_u = \{\mathcal{S}^u_\kappa \mid \kappa = 1, \dots, N_u\}$ be suitable triangulations of $\mathcal{D}$, U_R, respectively. We assume $\mathcal{D}_\mathcal{T} = \cup_{\mathcal{S}\in\mathcal{T}}\mathcal{S}$ and $U_R^\mathcal{T} = \cup_{\mathcal{S}^u\in\mathcal{T}_u}\mathcal{S}^u$. We briefly write $h_{x,\nu} = \operatorname{diam}(\mathcal{S}_\nu)$, $h_{u,\kappa} = \operatorname{diam}(\mathcal{S}^u_\kappa)$ and $h_x = \max_{\nu=1,\dots,N} h_{x,\nu}$, $h_u = \max_{\kappa=1,\dots,N_u} h_{u,\kappa}$. For each $x \in \mathcal{D}_\mathcal{T}$ we recall the active index set $I_\mathcal{T}(x) := \{\nu \in \{1, \dots, N\} \,|\, x \in \mathcal{S}_\nu\}$ defined in Section 1.4. For the simplices $\mathcal{T}_u$, we additionally assume that

$$\text{for each simplex } \mathcal{S}^u_\kappa \in \mathcal{T}_u, \text{ the vertices of } \mathcal{S}^u_\kappa \text{ are in the same closed orthant.} \tag{4.16}$$

Observe that (4.16) implies that the map $u \mapsto \|u\|_1$ is contained in $\mathrm{CPA}[\mathcal{T}_u]$.

According to Definition 1.4.9, $V \in \mathrm{CPA}[\mathcal{T}]$, i.e., there exist constants $a_\nu \in \mathbb{R}$, $w_\nu \in \mathbb{R}^n$, $\nu = 1, \dots, N$, such that

$$V|_{\mathcal{S}_\nu}(x) = \langle w_\nu, x\rangle + a_\nu \qquad \forall\, x \in \mathcal{S}_\nu,\ \mathcal{S}_\nu \in \mathcal{T} \tag{4.17}$$

$$\nabla V_\nu := \nabla V|_{\mathcal{S}^\circ_\nu} = w_\nu \qquad \forall\, \mathcal{S}_\nu \in \mathcal{T}. \tag{4.18}$$

We denote the k-th component of the vector ∇V_ν by $\nabla V_{\nu,k}$ $(k = 1, 2, \dots, n)$.

Remark 4.1.5. The algorithms will construct an ISS Lyapunov function in dissipative formulation $V \in \mathrm{CPA}[\mathcal{T}]$. In particular, this means that the inequality (4.8) or (4.9) has to be satisfied. To this end, from Definition 1.4.18 we should make sure that

$$\langle \nabla V_\nu, f(x,u)\rangle \le -\sigma\|x\|_2 + r\|u\|_1 \quad \forall \nu \in I_\mathcal{T}(x), \quad \text{or} \tag{4.19}$$

$$V(f(x,u)) - V(x) \le -\sigma\|x\|_2 + r\|u\|_1 \tag{4.20}$$

holds for $x \in \mathcal{D}_\mathcal{T}$, $u \in U_R^\mathcal{T}$.

Therefore, an inequality of this type will be used for ensuring (4.8) or (4.9) in the algorithms.

As in [42, 5] and Chapter 2, the key idea for the numerical computation of a true Lyapunov function lies in incorporating estimates for the interpolation errors on $\mathcal{T}$ – and in this section also on $\mathcal{T}_u$ – into the constraints of a linear optimization problem. In order to derive an estimate for the error terms, we introduce the following Proposition 4.1.6 which extends the results used in deriving (2.11) to a function with two arguments. Here, for a function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ which is twice continuously differentiable with respect to their first arguments, we denote the Hessian of $g(x,u)$ with respect to x at z by

$$H_g(z,u) = \begin{bmatrix} \left.\frac{\partial^2 g(x,u)}{\partial x_1^2}\right|_{x=z} & \cdots & \left.\frac{\partial^2 g(x,u)}{\partial x_1 \partial x_n}\right|_{x=z} \\ & \cdots & \\ \left.\frac{\partial^2 g(x,u)}{\partial x_n \partial x_1}\right|_{x=z} & \cdots & \left.\frac{\partial^2 g(x,u)}{\partial x_n^2}\right|_{x=z} \end{bmatrix}.$$

For the first argument $x \in \mathcal{S}_\nu$, let

$$H_x(u) := \max_{z\in\mathcal{S}_\nu} \|H_g(z,u)\|_2, \tag{4.21}$$

and let $K_x : U_R^{\mathcal{T}} \to \mathbb{R}_+$, $\overline{K}_x \in \mathbb{R}_+$, respectively, denote a bounded function and a positive constant satisfying

$$\max_{\substack{z\in\mathcal{S}_\nu \\ r,s=1,2,\ldots,n}} \left| \frac{\partial^2 g(z,u)}{\partial x_r \partial x_s} \right| \leq K_x(u) \leq \overline{K}_x \quad (u \in U_R^{\mathcal{T}}). \tag{4.22}$$

For a function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ which is Lipschitz continuous in the first argument, we define $L_x(u)$ as Lipschitz constant of $g(x,u)$ with respect to x.

In the next proposition which is proved in a similar way to [5, Proposition 4.1, Lemma 4.2 and Corollary 4.3], we state properties of scalar functions $g : \mathcal{D}_{\mathcal{T}} \times U_R^{\mathcal{T}} \to \mathbb{R}$ or vector functions $g : \mathcal{D}_{\mathcal{T}} \times U_R^{\mathcal{T}} \to \mathbb{R}^n$ with respect to its first argument. Analogous properties hold with respect to the second argument.

Proposition 4.1.6. *Consider a convex combination $x = \sum_{i=0}^n \lambda_i x_i \in \mathcal{S}_\nu$, $\sum_{i=0}^n \lambda_i = 1$, $1 \geq \lambda_i \geq 0$, $\mathcal{S}_\nu = \mathrm{co}\{x_0, x_1, \ldots, x_n\}$, $u \in U_R^{\mathcal{T}}$ and a function $g : \mathcal{D}_{\mathcal{T}} \times U_R^{\mathcal{T}} \to \mathbb{R}^p$ with components $g(x,u) = (g_1(x,u), g_2(x,u), \ldots, g_p(x,u))$, $p \in \mathbb{Z}_{>0}$.*

(a) If $g(x,u)$ is Lipschitz continuous in x with the bounds $L_x(u)$, $\overline{L}_x$ from (4.3)*, then*

$$\left\| g\left(\sum_{i=0}^n \lambda_i x_i, u\right) - \sum_{i=0}^n \lambda_i g(x_i,u) \right\|_\infty \leq L_x(u) h_{x,\nu} \leq \overline{L}_x h_{x,\nu} \quad (u \in U_R^{\mathcal{T}}). \tag{4.23}$$

(b) If $g_j(x,u)$ is twice continuously differentiable with respect to x with the bound $H_x(u)$ from (4.21) *on its second derivative for some $j = 1, \ldots, p$, then*

$$\begin{aligned} \left| g_j\left(\sum_{i=0}^n \lambda_i x_i, u\right) - \sum_{i=0}^n \lambda_i g_j(x_i,u) \right| &\leq \frac{1}{2} \sum_{i=0}^n \lambda_i H_x(u) \|x_i - x_0\|_2 \left(\max_{z\in\Gamma_\nu} \|z - x_0\|_2 + \|x_i - x_0\|_2 \right) \\ &\leq H_x(u) h_{x,\nu}^2. \end{aligned} \tag{4.24}$$

Under the same differentiability assumption for all $j = 1, \ldots, p$, the estimate

$$\left\| g\left(\sum_{i=0}^n \lambda_i x_i, u\right) - \sum_{i=0}^n \lambda_i g(x_i,u) \right\|_\infty \leq n K_x(u) h_{x,\nu}^2 \leq n\overline{K}_x h_{x,\nu}^2 \quad (u \in U_R^{\mathcal{T}}) \tag{4.25}$$

holds by assuming the bounds from (4.22).

Proof. The estimate (4.25) is an immediate consequence of (4.24) and the estimate

$$H_x(u) = \max_{z\in\mathcal{S}_\nu} \|H_g(z,u)\|_2 \leq n K_x(u) \leq n\overline{K}_x. \tag{4.26}$$

The proof of (4.26) follows from the following observation. Let $M \in \mathbb{R}^{n\times n}$, $|M|$ the matrix obtained by taking the absolute value componentwise, $\overline{r}$ an upper bound for the absolute values of the entries in M and E the matrix with all entries equal to 1. Then we have $\|M\|_2 \leq \| \, |M| \, \|_2 \leq \overline{r}\|E\|_2 = n\overline{r}$. Using inequalities (2.9), (2.10) and (2.11), we conclude that (4.24) holds. □

4.2 Computation of ISS Lyapunov functions for continuous time dynamic systems with perturbations

In this section we are going to introduce the linear programming based algorithm for the computation of a CPA ISS Lyapunov function in dissipative formulation for system (4.1) on $\mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_2(0, \epsilon)$.

Remark 4.2.1. Inspired by the proposed method for computing iISS (ISS) Lyapunov function in Chapter 3, we introduce auxiliary functions $\eta_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $\eta_2 : \mathbb{R}^m \to \mathbb{R}_+$.

For every $x \in \mathcal{S}_\nu = \text{co}\{x_0, x_1, \cdots, x_n\}$, $x = \sum_{i=0}^{n} \lambda_i x_i$ $(1 \geq \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1)$, we define

$$\eta_1(x) = \sum_{i=0}^{n} \lambda_i \eta_1(x_i). \tag{4.27}$$

For every $u \in \mathcal{S}_\kappa^u = \text{co}\{u_0, u_1, \cdots, u_m\}$, $u = \sum_{j=0}^{m} \lambda_j u_j$ $(1 \geq \mu_j \geq 0, \sum_{j=0}^{m} \mu_j = 1$), we define

$$\eta_2(u) = r \sum_{j=0}^{m} \mu_j \|u_j\|_1 \quad r \geq 0. \tag{4.28}$$

With the help of auxiliary functions η_1 and η_2, we may introduce the auxiliary system for system (4.1)

$$\dot{x} = f_\eta(x, u) := f(x, u) - \eta_2(u)\eta_1(x). \tag{4.29}$$

Then, using arguments similar to Lemma 3.2.1, it can be shown that a CPA robust Lyapunov function for system (4.29) is a CPA ISS Lyapunov function in dissipative formulation for system (4.1). However, it turns out that for computation purposes this detour via the auxiliary system is not efficient, as it leads to an algorithm in which two linear optimization problems have to be solved and furthermore introduces conservatism into the estimates. To this end, we will not explicitly use the auxiliary system. The way of designing the algorithm for computing CPA ISS Lyapunov functions in dissipative formulation is, however, inspired by the structure of (4.29).

The basic idea of the algorithm is to impose conditions on $V \in \text{CPA}[\mathcal{T}]$ in the vertices x_i of the simplices $\mathcal{S}_\nu \in \mathcal{T}$ which ensure that the function V satisfies the inequalities (4.4) and (4.8) with $\sigma = 1$ on the whole set $\mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_2(0, \epsilon)$. Note that $V \in \text{CPA}[\mathcal{T}]$ is completely determined by its values in the vertices of the simplices in $\mathcal{T}$.

The properness condition (4.4) is satisfied if the condition

$$V(x_i) \geq \|x_i\|_2 \tag{4.30}$$

holds for every vertex $x_i \in \mathcal{S}_\nu$, $V(0) = 0$ and $V \in \text{CPA}[\mathcal{T}]$.

It follows that

$$V(x) = \sum_{i=0}^{n} \lambda_i V(x_i) \geq \sum_{i=0}^{n} \lambda_i \|x_i\|_2 \geq \|x\|_2 \, . \tag{4.31}$$

Note that this does indeed imply (4.4) for all $x \in \mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_2(0, \epsilon)$.

In order to make sure that $V \in \mathrm{CPA}[\mathcal{T}]$ satisfies (4.19) for all $x \in \mathcal{S}_\nu \in \mathcal{T}$, $u \in \mathcal{S}^u_\kappa \in \mathcal{T}_u$ via imposing inequalities in the node values $V(x_i)$, we need to incorporate an estimate of the interpolation error into the inequalities. To this end, we demand that

$$\langle \nabla V_\nu, f(x_i, u_j)\rangle - r\|u_j\|_1 + \|\nabla V_\nu\|_1 A_{\nu,\kappa} \leq -\|x_i\|_2, \tag{4.32}$$

for all $i = 0, 1, 2, \dots, n$, $j = 0, 1, \dots, m$. Here $A_{\nu,\kappa} \geq 0$ denotes a bound for the interpolation error of f in the points (x, u) with $x \in \mathcal{S}_\nu \in \mathcal{T}$, $u \in \mathcal{S}^u_\kappa \in \mathcal{T}_u$, $x \neq x_i$, $u \neq u_j$.

Remark 4.2.2. Close to the origin the positive term $\|\nabla V_\nu\|_1 A_{\nu,\kappa}$ may become predominant on the left hand side of (4.32), rendering (4.32) infeasible. Thus, we have to exclude a small neighbourhood of the origin $\mathcal{B}_2(0, \epsilon)$. An estimate of $A_{\nu,\kappa}$ can be obtained by Proposition 4.1.6.

4.2.1 The algorithm for the computation of ISS Lyapunov functions based on a linear programming problem

Now we have collected all the preliminaries to formulate the linear programming based algorithm for computing a local ISS Lyapunov function in dissipative formulation $V \in \mathrm{CPA}[\mathcal{T}]$ for (4.1). In this algorithm, the values $V(x_i)$ are considered as optimization variables. Since it is desirable to obtain an ISS Lyapunov function in which the influence of the perturbation is as small as possible, in this case the objective of the linear optimization problem is to minimize r in (4.8).

We define the subsets

$$\mathcal{T}^\epsilon := \{\mathcal{S}_\nu \,|\, \mathcal{S}_\nu \cap \mathcal{B}^C_2(0, \epsilon) \neq \emptyset\} \subset \mathcal{T} \quad \text{and} \quad \mathcal{D}^\epsilon_\mathcal{T} := \bigcup_{\mathcal{S}_\nu \in \mathcal{T}^\epsilon} \mathcal{S}_\nu. \tag{4.33}$$

In the following algorithm we will only impose the conditions (4.30) in the nodes $x_i \in \mathcal{D}_\mathcal{T}$ and (4.32) in nodes $x_i \in \mathcal{S}_\nu \in \mathcal{T}^\epsilon$.

Algorithm

We solve the following linear optimization problem.

$$\text{Inputs:} \begin{cases} \epsilon, \\ \text{all vertices } x_i \text{ of all simplices } \mathcal{S}_\nu \in \mathcal{T}, \\ \text{all vertices } u_j \text{ of all simplices } \mathcal{S}^u_\kappa \in \mathcal{T}_u, \\ h_{x,\nu} \text{ of each simplex } \mathcal{S}_\nu \in \mathcal{T}^\epsilon, \\ h_{u,\kappa} \text{ of each simplex } \mathcal{S}^u_\kappa \in \mathcal{T}_u, \\ \text{Choose } \overline{L}_x, \overline{L}_u \text{ from (4.3) if } f \text{ satisfies (H1)}, \\ \text{or choose } \overline{K}_x, \overline{K}_u \text{ from (4.25) with respect to } x, u, \text{ respectively}, \\ \quad \text{for } g(x,u) = f(x,u) \text{ from (4.1) if } f \text{ satisfies (H2)}. \end{cases} \tag{4.34}$$

$$\text{Optimization variables:} \begin{cases} V_{x_i} = V(x_i) \text{ for all vertices } x_i \text{ of each simplex } \mathcal{S}_\nu \in \mathcal{T}, \\ C_{\nu,k} \text{ for } k = 1, 2, \dots, n \text{ and every } \mathcal{S}_\nu \in \mathcal{T}^\epsilon, \\ r \in \mathbb{R}_+. \end{cases} \tag{4.35}$$

Optimization problem: (4.36)

minimize r

subject to

$(A1):$ $V_{x_i} \geq \|x_i\|_2$ for all vertices x_i of each simplex $\mathcal{S}_\nu \in \mathcal{T}$, and $V(0) = 0$.
$(A2):$ $|\nabla V_{\nu,k}| \leq C_{\nu,k}$ for each simplex $\mathcal{S}_\nu \in \mathcal{T}^\epsilon$, $k = 1, 2, \ldots, n$.
$(A3):$ $\max_{x_i \in \partial(\mathcal{D}_\mathcal{T} \setminus \mathcal{D}_\mathcal{T}^\epsilon)} V_{x_i} < \min_{x_j \in \partial \mathcal{D}_\mathcal{T}} V_{x_j}$.

For all vertices x_i of each simplex $\mathcal{S}_\nu \in \mathcal{T}^\epsilon$, all vertices u_j of each simplex $\mathcal{S}_\kappa^u \in \mathcal{T}_u$, one of the conditions $(A4)$ or $(A5)$ is required:

$(A4):$ $\langle \nabla V_\nu, f(x_i, u_j) \rangle - r\|u_j\|_1 + (\overline{L}_x h_{x,\nu} + \overline{L}_u h_{u,\kappa}) \sum_{k=1}^n C_{\nu,k} \leq -\|x_i\|_2,$
if f satisfies (H1),

$(A5):$ $\langle \nabla V_\nu, f(x_i, u_j) \rangle - r\|u_j\|_1 + (n\overline{K}_x h_{x,\nu}^2 + m\overline{K}_u h_{u,\kappa}^2) \sum_{k=1}^n C_{\nu,k} \leq -\|x_i\|_2,$
if f satisfies (H2).

Remark 4.2.3. (i) By (4.31), the condition $(A1)$ yields that $V(x) \geq \|x\|_2$ for $x \in \mathcal{D}_\mathcal{T}^\epsilon$ and $V(0) = 0$.

(ii) The condition $(A2)$ defines linear constraints on the optimization variables V_{x_i} and $C_{\nu,k}$.

(iii) The linear constraint $(A3)$ makes sure the level set $\{x \in \mathcal{D}_\mathcal{T} | V(x) \leq \max_{x \in \partial(\mathcal{D}_\mathcal{T} \setminus \mathcal{D}_\mathcal{T}^\epsilon)} V(x)\}$ includes the set $\mathcal{B}_2(0, \epsilon)$. If system (4.1) is locally ISS, the condition $(A3)$ is not necessary.

Remark 4.2.4. If the linear optimization problem (4.36) has a feasible solution, then the values $V(x_i) = V_{x_i}$ from this feasible solution at all vertices x_i of all simplices $\mathcal{S}_\nu \in \mathcal{T}$ and the condition $V \in \mathrm{CPA}[\mathcal{T}]$ uniquely define a continuous and piecewise affine function

$$V : \mathcal{D}_\mathcal{T} \to \mathbb{R}. \tag{4.37}$$

Remark 4.2.5. It follows from Proposition 4.1.6 that instead of the term $n\overline{K}_x h_{x,\nu}^2 + m\overline{K}_u h_{u,\kappa}^2$ in (A5) one may use the sharper estimate

$$\frac{n\overline{K}_x}{2}\left(\|x_i - x_0\|_2 \left(\max_{k=1,2,\ldots,n} \|x_k - x_0\|_2 + \|x_i - x_0\|_2\right)\right) + \frac{mK_u(x_i)}{2}\left(\|u_j - u_0\|_2 \left(\max_{k=1,2,\ldots,m} \|u_k - u_0\|_2 + \|u_i - u_0\|_2\right)\right)$$

with $K_u(x_i)$ satisfying (4.22) with respect to u. The latter is used in our numerical experiments.

In the following, we formulate and prove our two main results. We show that any feasible solution of our algorithm defines a CPA ISS Lyapunov function in dissipative formulation on $\mathcal{D}_\mathcal{T}^\epsilon$ and give conditions under which our algorithm will yield such a feasible solution. We start with the former.

Theorem 4.2.6. *If assumption (H1) or (H2) holds and the linear optimization problem* (4.36) *has a feasible solution, then the function V from* (4.37) *is a* CPA *ISS Lyapunov function in dissipative formulation on $\mathcal{D}_\mathcal{T}^\epsilon$, i.e., it satisfies* (4.4) *and* (4.8) *for all $x \in \mathcal{D}_\mathcal{T}^\epsilon$ and all $u \in U_R^\mathcal{T}$.*

Proof. Consider the convex combinations $x = \sum_{i=0}^{n} \lambda_i x_i \in \mathcal{S}_\nu$, $\mathcal{S}_\nu = \mathrm{co}\{x_0, x_1, \ldots, x_n\} \in \mathcal{T}$, $\sum_{i=0}^{n} \lambda_i = 1$, $1 \geq \lambda_i \geq 0$, and $u = \sum_{j=0}^{m} \mu_j u_j \in \mathcal{S}_\kappa^u$, $\mathcal{S}_\kappa^u = \mathrm{co}\{u_0, u_1, \ldots, u_m\} \in \mathcal{T}_u$, $\sum_{j=0}^{m} \mu_j = 1$, $1 \geq \mu_j \geq 0$.

First note that by (4.31) we have $V(x) \geq \|x\|_2$ for all $x \in \mathcal{D}_\mathcal{T}$. Thus in (4.4) we may choose α_1 to be the identity and the existence of α_2 follows by Lipschitz continuity.

In order to prove inequality (4.8) with $\sigma = 1$ for $x \in \mathcal{D}_\mathcal{T}^\epsilon$ we compute

$$\begin{aligned}
\langle \nabla V_\nu, f(x,u)\rangle &= \sum_{i=0}^{n} \lambda_i \langle \nabla V_\nu, f(x_i, \sum_{j=0}^{m} \mu_j u_j)\rangle + \langle \nabla V_\nu, f(\sum_{i=0}^{n} \lambda_i x_i, \sum_{j=0}^{m} \mu_j u_j)\rangle \\
&\quad - \sum_{i=0}^{n} \lambda_i \langle \nabla V_\nu, f(x_i, \sum_{j=0}^{m} \mu_j u_j)\rangle \\
&\leq \sum_{i=0}^{n} \lambda_i \langle \nabla V_\nu, f(x_i, \sum_{j=0}^{m} \mu_j u_j)\rangle + \|\nabla V_\nu\|_1 \left\| f(\sum_{i=0}^{n} \lambda_i x_i, u) - \sum_{i=0}^{n} \lambda_i f(x_i, u) \right\|_\infty \\
&\leq \sum_{i=0}^{n} \lambda_i \sum_{j=0}^{m} \mu_j \langle \nabla V_\nu, f(x_i, u_j)\rangle + \|\nabla V_\nu\|_1 \left\| f(\sum_{i=0}^{n} \lambda_i x_i, u) - \sum_{i=0}^{n} \lambda_i f(x_i, u) \right\|_\infty \\
&\quad + \sum_{i=0}^{n} \lambda_i \|\nabla V_\nu\|_1 \left\| f(x_i, u) - \sum_{j=0}^{m} \mu_j f(x_i, u_j) \right\|_\infty .
\end{aligned}$$

According to Proposition 4.1.6 the constraints from $(A4)$ or $(A5)$ ensure that V satisfies

$$\langle \nabla V_\nu, f(x,u)\rangle \leq -\sum_{i=0}^{n} \lambda_i \|x_i\|_2 + r \sum_{i=0}^{m} \mu_j \|u_j\|_1 \leq -\|x\|_2 + r\|u\|_1 .$$

Thus we have shown (4.8) with $\sigma = 1$ holds for all $x \in \mathcal{D}_\mathcal{T}^\epsilon$ and all $u \in U_R^\mathcal{T}$. □

In the last step we have used the equality $\sum_{i=0}^{m} \mu_j \|u_j\|_1 = \|u\|_1$, which is true by assumption (4.16). Indeed, this assumption ensures that the signs of the entries of u_j coincide in each simplex.

In the following Corollaries 4.2.7 and 4.2.8, we prove that V from Theorem 4.2.6 is a viscosity subsolution of partial differential equations (4.38) and (4.43) on $\mathcal{D}_\mathcal{T}^\epsilon$, respectively.

Corollary 4.2.7. *If assumption (H1) or (H2) holds, and the linear optimization problem* (4.36) *has a feasible solution, then the function V from* (4.37) *is a viscosity subsolution of the partial differential equation* (4.38) *on $\mathcal{D}_\mathcal{T}^\epsilon$*

$$H(x, V(x), DV(x)) = 0 \tag{4.38}$$

with the Hamiltonian

$$H(x, W, p) = \sup_{u \in U_R^\mathcal{T}} \{\langle p, f(x,u)\rangle + \|x\|_2 - r\|u\|_1\} \tag{4.39}$$

defined for $x, p \in \mathbb{R}^n$, $W \in \mathbb{R}$.

Proof. It follows from the proof of Theorem 4.2.6

$$\langle \nabla V_\nu, f(x,u)\rangle \le -\|x\|_2 + r\|u\|_1 \tag{4.40}$$

for all $x \in \mathcal{D}^\epsilon_\mathcal{T}$ and all $u \in U^\mathcal{T}_R$.

According to Remark 4.1.5, we obtain

$$\langle \xi, f(x,u)\rangle \le -\|x\|_2 + r\|u\|_1 \qquad \forall\, \xi \in \partial_{Cl} V(x) \tag{4.41}$$

for all $x \in \mathcal{D}^\epsilon_\mathcal{T}$ and all $u \in U^\mathcal{T}_R$.

Based on Remark 5.2.4, the following inequality holds for V and all $x \in \mathcal{D}^\epsilon_\mathcal{T}$

$$\sup_{u\in U^\mathcal{T}_R} \{\langle p, f(x,u)\rangle + \|x\|_2 - r\|u\|_1\} \le 0 \quad \forall\, p \in D^+V(x) \ \ \text{(see in Remark 5.2.2)}. \tag{4.42}$$

Therefore the function V from (4.37) is a viscosity subsolution of the partial differential equation (4.38) on $\mathcal{D}^\epsilon_\mathcal{T}$ □

The partial differential equation (4.38) can be transformed into a Hamilton-Jacobi-Bellman equation (4.43) which is studied e.g. in [35, Sec. 3.5].

Corollary 4.2.8. *Under the assumptions of Theorem 4.2.7, the function V from (4.37) is a viscosity subsolution of the partial differential equation (4.43) on $\mathcal{D}^\epsilon_\mathcal{T}$:*

$$\sup_{u\in U^\mathcal{T}_R:\ 2r\|u\|_1 \le \|x\|_2} \{\langle \nabla V(x), f(x,u)\rangle + \frac{1}{2}\|x\|_2\} = 0 \tag{4.43}$$

Proof. The proof is similar to Corollary 4.2.7, but we use

$$\langle \xi, f(x,u)\rangle \le -\frac{1}{2}\|x\|_2 \qquad \forall\, \xi \in \partial_{Cl} V(x) \tag{4.44}$$

for all $x \in \mathcal{D}^\epsilon_\mathcal{T}$ satisfying $\|x\|_2 \ge 2r\|u\|_1$, $u \in U^\mathcal{T}_R$. □

Now we turn to the second main objective of this section. We derive conditions under which the linear programming problem has a feasible solution. To this end, we need a certain regularity property of the simplices in our suitable triangulations (see Remark 1.4.8). In order to formalize these, we recall the following notation defined in Section 1.4.

For an n-simplex $\mathcal{S}_\nu := \text{co}\{x_0, x_1, \ldots, x_n\} \in \mathcal{T}$, its shape-matrix is

$$X_\nu = \left[(x_1 - x_0), (x_2 - x_0), \cdots, (x_n - x_0)\right]^T. \tag{4.45}$$

Let $\lambda_\nu := \|X_\nu^{-1}\|_2$. Moreover, $\lambda_\nu = \lambda_{\min}^{-1}$ holds, where $\lambda_{\min}$ is the smallest singular value of X_ν. We define

$$\lambda^* := \max_{\nu=1,2,\ldots,N} \lambda_\nu. \tag{4.46}$$

Theorem 4.2.9. *Consider system (4.1) which satisfies (H1) or (H2) and which has a C^2 ISS Lyapunov function in dissipative formulation $W : \mathbb{R}^n \to \mathbb{R}$ on $\mathcal{D}$. Let $\epsilon > 0$ and $R_1 > 0$.*

Then for every $R_1 > 0$ there exist $\delta_{R_1} > 0$, $\delta_u > 0$ such that, for all suitable triangulations $\mathcal{T}$, $\mathcal{T}_u$ satisfying

$$\max_{\mathcal{S}_\nu \in \mathcal{T}} \operatorname{diam}(\mathcal{S}_\nu) \le \delta_{R_1}, \tag{4.47}$$

$$\max_{\mathcal{S}^u_\kappa \in \mathcal{T}_u} \operatorname{diam}(\mathcal{S}^u_\kappa) \le \delta_u, \tag{4.48}$$

$$\lambda^* h_x \le R_1 \tag{4.49}$$

the linear programming problem from our algorithm has a feasible solution and delivers an ISS Lyapunov function in dissipative formulation $V \in \mathrm{CPA}[\mathcal{T}]$ on $\mathcal{D}^\epsilon_{\mathcal{T}}$.

Proof. Applying Proposition 4.1.3 we may without loss of generality assume that W satisfies (4.7) and (4.8) with $\sigma = 2$ and some $r > 0$.

For an arbitrary but fixed $\mathcal{S}_\nu = co\{x_0, x_1, \ldots, x_n\} \in \mathcal{T}^\epsilon$, we define

$$W_\nu := \begin{pmatrix} W(x_1) & - & W(x_0) \\ W(x_2) & - & W(x_0) \\ & \vdots & \\ W(x_n) & - & W(x_0) \end{pmatrix}.$$

Using proof of Theorem 2.1.7, we get

$$\|W_\nu - X_\nu \nabla W(x_0)\|_2 \le \frac{1}{2} n^{\frac{3}{2}} A h_x^2, \tag{4.50}$$

where $A := \max_{\substack{z \in \mathcal{D} \\ i,j=1,2,\ldots,n}} \left| \frac{\partial^2 W}{\partial x_i \partial x_j}(z) \right|$,

and

$$\|X_\nu^{-1} W_\nu - \nabla W(x_i)\|_2 \le n A h_x (\frac{1}{2} n^{\frac{1}{2}} R_1 + 1).$$

After these preliminary considerations, we now assign values to the variables $V_{x_i} = V_i(x_i)$ and $C_{\nu,k}$ of the linear programming problem from the algorithm and show that they fulfill the constraints.

For each vertex $x_i \in \mathcal{S}_\nu \in \mathcal{T}$, we let $V(x_i) = V_{x_i} = W(x_i)$. Since W satisfies (4.7), it is obvious that $V(x_i) = V_{x_i} \ge \|x_i\|_2$ for $x \in \mathcal{T}^\epsilon$. The constraint $(A3)$ is obviously satisfied. It thus remains to show $(A4)$ or $(A5)$ for some $r > 0$.

To this end, choosing one simplex $\mathcal{S}_\nu = co\{x_0, x_1, \ldots, x_n\} \in \mathcal{T}^\epsilon$ we get

$$\nabla V_\nu = X_\nu^{-1} W_\nu, \tag{4.51}$$

since V is linear affine on the simplex $\mathcal{S}_\nu$ and

$$V(x) = V(x_0) + \left\langle X_\nu^{-1} W_\nu, (x - x_0) \right\rangle = V(x_0) + W_\nu^\top (X_\nu^\top)^{-1} (x - x_0). \tag{4.52}$$

For the variables $C_{\nu,k}$, we set

$$C_{\nu,k} := \|\nabla V_\nu\|_2 = \|X_\nu^{-1} W_\nu\|_2, \quad k = 1, \ldots, n. \tag{4.53}$$

Thus $C_{\nu,k} \geq |\nabla V_{\nu,k}|$ for each $\mathcal{S}_\nu \in \mathcal{T}^\epsilon$. Since $\nabla W(x)$ is bounded on $\mathcal{D}$ and (4.49) holds, there exists a positive constant $\overline{C}$ such that

$$\begin{aligned} C_{\nu,k} = \|X_\nu^{-1} W_\nu\|_2 &\leq \|X_\nu^{-1}\|_2 \max_{z\in\mathcal{D}_\mathcal{T}^\epsilon} \|\nabla W(z)\|_2 h_x \\ &\leq R_1 \max_{z\in\mathcal{D}_\mathcal{T}^\epsilon} \|\nabla W(z)\|_2 =: \overline{C} \end{aligned} \tag{4.54}$$

holds for all ν and k. From this analysis and the fact that W satisfies (4.8) with $\sigma = 2$, we obtain that

$$\begin{aligned} \langle \nabla V_\nu, f(x_i, u_j)\rangle - r\|u_j\|_1 &= \langle \nabla W(x_i) + \nabla V_\nu - \nabla W(x_i), f(x_i, u_j)\rangle - r\|u_j\|_1 \\ &\leq -2\|x_i\|_2 + \|X_\nu^{-1} W_\nu - \nabla W(x_i)\|_2 \|f(x_i, u_j)\|_2 \\ &\leq -2\|x_i\|_2 + nAh(\frac{1}{2} n^{\frac{1}{2}} R_1 + 1)D, \end{aligned}$$

where $D := \sup_{x\in\mathcal{D}, u\in U_R} \|f(x,u)\|_2 < \infty$.

Now let $h = \max\{h_x, h_u\}$. If (H1) holds, i.e., in the Lipschitz case, the linear constraint from $(A4)$ is fulfilled whenever $h > 0$ is so small that for all vertices x_i of simplices in $\mathcal{T}^\varepsilon$

$$nAh(\frac{1}{2} n^{\frac{1}{2}} R_1 + 1)D + n\overline{C}(\overline{L}_x + \overline{L}_u)h \leq \|x_i\|_2. \tag{4.55}$$

In case (H2), i.e., f is C^2, the linear constraint from $(A5)$ is satisfied if $h > 0$ is so small that

$$nAh(\frac{1}{2} n^{\frac{1}{2}} R_1 + 1)D + n\overline{C}(n\overline{K}_x + m\overline{K}_u)h^2 \leq \|x_i\|_2, \tag{4.56}$$

where $\overline{K}_x, \overline{K}_u$ are the constants satisfying inequality (4.22) with respect to x, u, respectively. Thus, the theorem is proved. □

4.2.2 Examples

In this section we illustrate the algorithm by two examples. In order to highlight the fact that our algorithm minimizes the gain r in (4.8) for $\sigma = 1$, in our first example we compare the result of our algorithm with two CPA ISS Lyapunov functions in dissipative formulation, for which a closed-form expression is derived following the construction of the proof of Theorem 4.2.9. Our second example shows the result of our algorithm for an example for which no closed-form ISS Lyapunov function is known.

Example 1

We consider the following system which is adapted from [79]

$$\begin{aligned} \dot{x}_1 &= -x_1[1 - (x_1^2 + x_2^2)] + 0.1 x_2 u^2, \\ \dot{x}_2 &= -x_2[1 - (x_1^2 + x_2^2)], \end{aligned} \tag{4.57}$$

where $x \in \mathcal{D} = \mathcal{B}_2(0, 0.588) \subset \mathbb{R}^2$, $u \in U_R = \{u \in \mathbb{R} : |u| \leq 4.41\}$.

For this example, we obtain two CPA ISS Lyapunov functions in dissipative formulation V and v on $\mathcal{D}$ based on two different functions W_1, W_2 following the construction in the proof of Theorem 4.2.9. We compare them with the numerical CPA ISS Lyapunov function in dissipative formulation V_1 delivered by the algorithm.

(1) For constructing a theoretical CPA ISS Lyapunov function in dissipative formulation V we start with the quadratic function candidate $W_1(x) = x_1^2 + x_2^2$. It is obvious that W_1 is twice differentiable. For the dynamics from (4.57) we obtain

$$\begin{aligned}\langle \nabla W_1(x), f(x,u)\rangle &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &\leq -2\|x\|_2^2(1-\|x\|_2^2) + 0.05u^2.\end{aligned} \tag{4.58}$$

Here $\alpha(\|x\|_2) = 2\|x\|_2^2(1-\|x\|_2^2)$ is an increasing function whenever $\|x\|_2 \in [0, \frac{\sqrt{2}}{2}]$ and can thus be extended as a $\mathcal{K}_\infty$ function for $\|x\|_2 > \frac{\sqrt{2}}{2}$. Hence, for $\beta(|u|) = 0.05|u|^2 \in \mathcal{K}_\infty$, W_1 is an ISS Lyapunov function in dissipative formulation for system (4.57) on $\mathcal{D}$.

Now we follow the proof of Theorem 4.2.9 in order to construct a CPA ISS Lyapunov function in dissipative formulation V satisfying the constraints in our algorithm. To this end, let $\epsilon = 0.048$. Then the appropriate rescaling constant C in Proposition 4.1.3 is given by $C = C_1 = \frac{1}{\epsilon(1-\epsilon^2)}$. Indeed, by replacing $W_1(x)$ with $V(x) := C_1 W_1(x) = C_1(x_1^2 + x_2^2)$ we obtain

$$\begin{aligned}\langle \nabla V_1(x), \dot{x}\rangle &= 2C_1x_1\dot{x}_1 + 2C_1x_2\dot{x}_2 \\ &\leq -2C_1\|x\|_2^2(1-\|x\|_2^2) + C_1\frac{1}{20}u^2 \\ &\leq -2\|x\|_2 + C_1\frac{1}{20}u^2,\end{aligned} \tag{4.59}$$

for $x \in \mathcal{D} \setminus \mathcal{B}_2(0,\epsilon)$ and $|u| \leq 4.41$.

For u satisfying $|u| \leq 4.41$ we now need to find $r_1 > 0$ with

$$r_1|u| \geq \frac{1}{20}C_1u^2. \tag{4.60}$$

Since in the algorithm the objective is to minimize r, we select the minimal r satisfying this inequality which is given by $r_1 = \frac{4.41}{20\epsilon(1-\epsilon^2)} = 4.6044$.

Now, linear interpolation of this $W_1(x)$ on a sufficiently fine grid $\mathcal{T}$ yields the desired function $V(x) = C_1W_1(x)$, which is plotted in Figure 4.1.

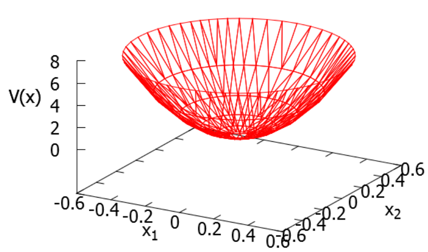

Figure 4.1: Theoretical CPA ISS Lyapunov function in dissipative formulation V based on the function W_1 for system (4.57), $\epsilon = 0.048$, $r_1 = 4.6044$.

(2) In the construction in the proof of Theorem 4.2.9 we rescale the function via Proposition 4.1.3 to satisfy $W(x) \geq \|x\|_2$. Thus it appears reasonable to start with $W_2(x) = \|x\|_2$ as an ISS Lyapunov function candidate. Following the same steps as in (1), we can show that W_2 is also an ISS Lyapunov function in dissipative formulation on $\mathcal{D} \setminus \mathcal{B}_2(0,\epsilon)$. A rescaling $v(x) := C_2 W_2(x)$ along Proposition 4.1.3 yields $C_2 = \frac{2}{1-\epsilon^2}$ and $r_2 = \frac{4.41}{10(1-\epsilon^2)} = 0.4410$. The resulting interpolated v is shown in Figure 4.2.

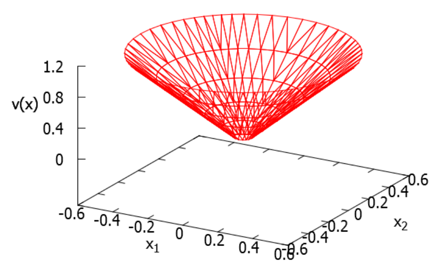

Figure 4.2: Theoretical CPA ISS Lyapunov function in dissipative formulation v based on the function W_2 for system (4.57), $\epsilon = 0.048$, $r_2 = 0.4410$.

(3) From the algorithm we get the numerical CPA ISS Lyapunov function in dissipative formulation V_1 shown in Figure 4.3 with $r = 0.420909$.

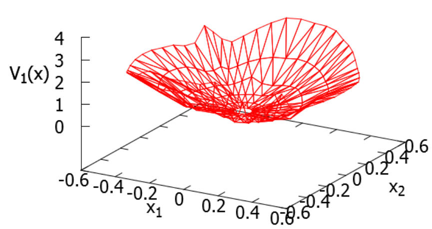

Figure 4.3: Numerical CPA ISS Lyapunov function in dissipative formulation V_1 delivered by the algorithm for system (4.57), $\epsilon = 0.048$, $r = 0.420909$.

The suitable triangulations of subsets of state space, and input value space are obtained by the way described in Section 2.1.3. For the triangulation of $\mathcal{D}$, $K = 7$, $k = 2$, and the map $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ from (4.61). For the triangulation of U_R, we use $K = 21$, $k = 0$ and the map $G : \mathbb{R}^m \mapsto \mathbb{R}^m$ from (4.62). The parameter $\rho > 0$ in the map F controls the size of the resulting vertices. For our computations we used $\rho = 0.012$. For G we used $\gamma = 0.01$.

$$F(x) = \begin{cases} \rho x \|x\|_\infty^2 / \|x\|_2 & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases} \tag{4.61}$$

$$G(u) = \gamma u|u|. \tag{4.62}$$

As expected, the optimization based algorithmic approach yields the smallest possible gain parameter r. For finer grids, even smaller values of r can be obtained and it appears that r converges to a lower bound $\underline{r} > 0.4$.

In Figures 4.4–4.5 we include a comparison of the calculated CPA ISS Lyapunov function in dissipative formulation V_1 with the CPA ISS Lyapunov functions in dissipative formulation V and v. Note that the CPA ISS Lyapunov function in dissipative formulation is not unique, but the calculated one is more similar to v.

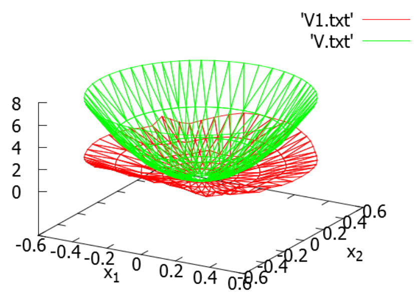

Figure 4.4: Difference between CPA ISS Lyapunov functions in dissipative formulation V and V_1 for system (4.57).

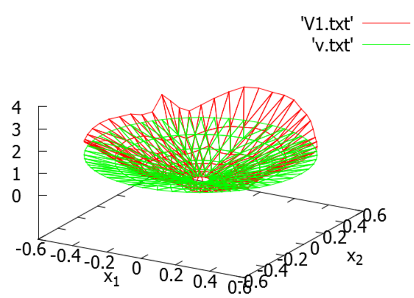

Figure 4.5: Difference between CPA ISS Lyapunov functions in dissipative formulation v and V_1 for system (4.57).

Example 2: Synchronous generator with varying damping

We consider the following model adapted from [46] which is described by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_2 - \sin(x_1 + u) + \sin(u), \end{aligned} \tag{4.63}$$

on $\mathcal{D} = \mathcal{B}_2(0, 2.352) \subset \mathbb{R}^2$, $U_R = [-0.3, 0.3] \subset \mathbb{R}$.

Compact sets $\mathcal{D}$ and U_R are partitioned into suitable triangulations in the same way as described in Section 2.1.3. For the suitable triangulation of $\mathcal{D}$, we let $K = 14$, $k = 1$, and utilize the map from (4.61) with $\rho = 0.012$. The suitable triangulation of U_R is obtained with $K = 5$, $k = 0$ and the map from (4.62) with $\gamma = 0.012$.

The algorithm yields a CPA ISS Lyapunov function in dissipative V_2 shown in Figure 4.6 for system (4.63). Note that for this example an analytical ISS Lyapunov function is not known and that our numerical analysis yields a numerical value for the (in our approach) linear ISS gain.

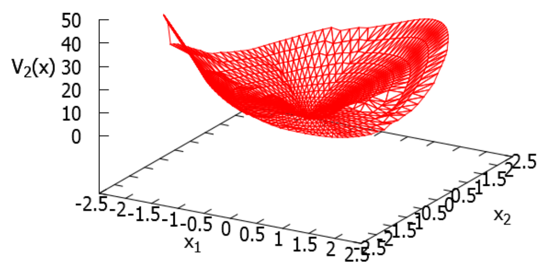

Figure 4.6: CPA ISS Lyapunov function in dissipative formulation V_2 given by the algorithm for system (4.63), $\epsilon = 0.012$, $r = 19.7621$.

Remark 4.2.10. In our algorithm we construct grids on U_R and $\mathcal{D}$, respectively. If $\mathcal{D}$ is a two-dimensional set and the number of vertices for gridding U_R increases by 1, then the number of constraints in the linear program increases at least by $(48 + 6(N_v + 2)(N_v - 4) + 12(N_v - 2))$, where $N_v = \min\{N_{v,1}, N_{v,2}\}$ and $N_{v,i}$ is the number of vertices intersecting the x_i-axis. Similarly, the number of constraints increases in higher space dimensions. Thus, the gridding of U_R renders the number of constraints much larger than the number of optimization variables. It is hence much faster to solve the corresponding dual optimization problem than to solve the primal problem. For numerical computations we used the GNU Linear Programming Kit (GLPK)[1] , Gurobi[2] and CPLEX[3], respectively. We experienced that Gurobi and CPLEX carry out a significantly better preprocessing of the constraints which eliminates much more redundant ones and thus both methods can solve the optimization problem much faster than GLPK.

4.2.3 Conclusion

In this section, we proved that on suitable triangulations of compact subsets of state space and input value space, the algorithm for solving the linear optimization problem (4.36) delivers a true CPA ISS Lyapunov function in dissipative formulation for system (4.1) on a

[1] http://www.gnu.org/software/glpk/
[2] http://www.gurobi.com/
[3] http://www.ibm.com/software/commerce/optimization/cplex-optimizer/

compact subset of state space excluding a small neighbourhood of the origin (Theorem 4.2.6). Furthermore, the linear gain is obtained by the algorithm which plays an important role in the stability analysis of interconnected ISS subsystems. Under a finer triangulation, the gain parameter r in (4.8) may be the smallest one. Moreover, if system (4.1) has a C^2 ISS Lyapunov function, then there exist suitable triangulations such that the algorithm has a feasible solution (Theorem 4.2.9). It is known that, if system (4.1) is ISS, there exists a smooth ISS Lyapunov function (see Theorem 1.3.26). Therefore, our proposed algorithm for computing CPA ISS Lyapunov functions in dissipative formulation always has a feasible solution.

These promising results motivate us to extend this method to discrete time dynamic systems with perturbations.

4.3 Computation of ISS Lyapunov functions for discrete time dynamic systems with perturbations

In this section, we consider discrete time dynamic systems with perturbation, i.e., system (4.2). The objective is to design a parallel linear programming based algorithm for constructing CPA ISS Lyapunov functions in dissipative formulation for system (4.2) on a suitable triangulation of a compact subset $\mathcal{D}$ of state space.

Let $\mathcal{O} \subset \mathcal{D}_{\mathcal{T}}$ be a closed and compact neighbourhood of the origin. We assume $\mathcal{O}$ satisfies the following conditions:

- $\mathcal{O} \subset \mathcal{D}_{\mathcal{T}}$, $0 \in \mathcal{O}^\circ$,
- $x \in \mathcal{O}$ implies $f(x,u) \in \mathcal{D}_{\mathcal{T}}$ for all $u \in \mathcal{T}_u$. Here f is from (4.2), and
- there exists no $\mathcal{S}_\nu$ with $x, y \in \mathcal{S}_\nu$ satisfying $x \in \mathcal{O}$, $y \in \mathcal{D}_{\mathcal{T}} \setminus \mathcal{O}$.

Remark 4.3.1. Since the solution to equation (4.2) is not continuous, the constraints on the set $\mathcal{O}$ are necessary, in order to make sure that the point $f(x,u)$ is in the set $\mathcal{D}_{\mathcal{T}}$ for $x \in \mathcal{O}$, $u \in \mathcal{T}_u$.

In order to make sure a function $V \in \mathrm{CPA}[\mathcal{T}]$ fulfills the inequalities (4.4) and (4.9) with $\sigma = 1$ on the whole set $\mathcal{O} \setminus \mathcal{B}_2(0,\epsilon)$, we have to impose some constraints on V at the vertices x_i of $\mathcal{S}_\nu$, since $V \in \mathrm{CPA}[\mathcal{T}]$ is determined by its values in the vertices of the simplices in $\mathcal{T}$.

To ensure the properness condition (4.4), we impose the condition

$$V(x_i) \geq \|x_i\|_2, \tag{4.64}$$

for every vertex $x_i \in \mathcal{S}_\nu$, $V(0) = 0$ and $V \in \mathrm{CPA}[\mathcal{T}]$, as we did in the Section 4.2.

In order to make sure that V satisfies (4.9) for all $x \in \mathcal{S}_\nu \subset \mathcal{O}$, $u \in \mathcal{S}^u_\kappa \subset U^{\mathcal{T}}_R$ via imposing inequalities in the node values $V(x_i)$, an estimate of the interpolation error should be incorporated into the inequalities. To this end, we demand that

$$V(f(x_i,u_j)) - V(x_i) - r\|u_j\|_1 + \|\nabla V_\nu\|_1 A_{\nu,\kappa} \leq -\|x_i\|_2, \tag{4.65}$$

for all $i = 0, 1, 2, \ldots, n$, $j = 0, 1, \ldots, m$. Here $A_{\nu,\kappa} \geq 0$ represents a bound for the interpolation error of f in the points (x,u) with $x \in \mathcal{S}_\nu \subset \mathcal{O}$, $u \in \mathcal{S}^u_\kappa \subset U^{\mathcal{T}}_R$, $x \neq x_i$, $u \neq u_j$.

Remark 4.3.2. The constraints imposed above are similar to the constraints required in the algorithm proposed in Section 4.2. But here we use the difference between $V(f(x_i,u_j))$ and $V(x_i)$ instead of the gradient ∇V_ν in the inequality (4.65).

The following remark similar to Remark 4.2.2 explains why we need to exclude a small neighbourhood of the origin $\mathcal{B}_2(0,\epsilon)$.

Remark 4.3.3. For $u_j = 0$, $V(f(x_i,u_j))$ converges to 0 as x_i goes to the origin. Meanwhile the interpolation error term may be predominant on the left hand side of the inequality (4.65) when x_i is very close to the origin. In order to make sure that (4.65) holds, we have to exclude a small neighbourhood of the origin $\mathcal{B}_2(0,\epsilon)$. Therefore, we will just consider the problem of constructing of a CPA ISS Lyapunov function in dissipative formulation on $\mathcal{O}\setminus\mathcal{B}_2(0,\epsilon)$. An estimate of $A_{\nu,\kappa}$ can be computed using Proposition 4.1.6.

4.3.1 The linear programming based algorithm for the computation of ISS Lyapunov functions

Based on the above preliminaries, we now can describe the linear programming based algorithm for computing a local ISS Lyapunov function in dissipative formulation $V \in \mathrm{CPA}[\mathcal{T}]$ for system (4.2). In this algorithm, $V(x_i)$ are introduced as optimization variables. We want to reduce the influence of perturbation in our considered system as much as possible. Therefore, for fixed $\sigma = 1$ the objective of the linear optimization problem is to minimize r in (4.9).

In the following algorithm, we will only impose the condition (4.64) in the nodes $x_i \in \mathcal{D}_\mathcal{T}$ and (4.65) in nodes $x_i \in \mathcal{S}_\nu \subset \mathcal{O}\setminus\mathcal{B}_2(0,\epsilon)$.

We define the subsets

$$\mathcal{T}^\epsilon := \{\mathcal{S}_\nu | \mathcal{S}_\nu \cap \mathcal{B}_2^C(0,\epsilon) \neq \emptyset\} \subset \mathcal{T},\ \mathcal{O}^\epsilon := \bigcup_{\mathcal{S}_\nu \in \mathcal{T}^\epsilon} (\mathcal{S}_\nu \cap \mathcal{O}). \tag{4.66}$$

Next we describe the linear programming based algorithm for the computation of CPA ISS Lyapunov functions in dissipative formulation for system (4.2) on $\mathcal{O}\setminus\mathcal{B}_2(0,\epsilon)$.

Algorithm

We solve the following linear optimization problem.

$$\text{Inputs:} \begin{cases} \epsilon, \\ \text{all vertices } x_i \text{ of all simplices } \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}, \\ \text{all vertices } u_j \text{ of all simplices } \mathcal{S}^u_\kappa \in \mathcal{T}_u, \\ h_{x,\nu} \text{ of each simplex } \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}, \\ h_{u,\kappa} \text{ of each simplex } \mathcal{S}^u_\kappa \in \mathcal{T}_u, \\ \text{Choose } \overline{L}_x, \overline{L}_u \text{ from (4.3) if } f \text{ satisfies (H1)}, \\ \text{or choose } \overline{K}_x, \overline{K}_u \text{ from (4.25) with respect to } x, u, \text{ respectively}, \\ \quad \text{for } g(x,u) = f(x,u) \text{ from (4.1) if } f \text{ satisfies (H2)}. \end{cases} \tag{4.67}$$

$$\text{Optimization variables:} \begin{cases} V_{x_i} = V(x_i) \text{ for all vertices } x_i \text{ of each simplex } \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}, \\ C_{\nu,k} \text{ for } k = 1,2,\ldots,n \text{ and every } \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}, \\ r \geq 0,\ \overline{C} \geq 0. \end{cases} \tag{4.68}$$

Optimization problem: (4.69)

minimize r

subject to

$(C1):$ $V_{x_i} \geq \|x_i\|_2$ for all vertices x_i of each simplex $\mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}$, and $V(0) = 0$,
$(C2):$ $|\nabla V_{\nu,k}| \leq C_{\nu,k}$ for each simplex $\mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}$, $k = 1, 2, \ldots, n$,
$(C3):$ $C_{\nu,k} \leq \overline{C}$ for each simplex $\mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}$, $k = 1, 2, \ldots, n$,
$(C4):$ $f(x_i, u_j) \in \mathcal{D}_\mathcal{T}$ for all vertices $x_i \in \mathcal{O}^\epsilon$, $u_j \in U_R^\mathcal{T}$,
$(C5):$ $\max\limits_{x_i \in \partial(\mathcal{O} \setminus \mathcal{O}^\epsilon)} V_{x_i} < \min\limits_{x_j \in \partial \mathcal{O}} V_{x_j}$,
For all vertices x_i of each simplex $\mathcal{S}_\nu \subset \mathcal{O}^\epsilon$, all vertices u_j of each simplex $\mathcal{S}_\kappa^u \in \mathcal{T}_u$, one of the conditions $(C6)$, $(C7)$ is required:
$(C6):$ $V(f(x_i, u_j)) - V(x_i) - r\|u_j\|_1 + n\overline{C}(\overline{L}_x h_{x,\nu} + \overline{L}_u h_{u,\kappa}) \leq -\|x_i\|_2$,
if f satisfies (H1),
$(C7):$ $V(f(x_i, u_j)) - V(x_i) - r\|u_j\|_1 + n\overline{C}(n\overline{K}_x h_{x,\nu}^2 + m\overline{K}_u h_{u,\kappa}^2) \leq -\|x_i\|_2$,
if f satisfies (H2).

The following Remark 4.3.4 addresses the same issues as in Remark 4.2.3.

Remark 4.3.4. (i) From Remark 4.2.3 it is obvious that $V(x) \geq \|x\|_2$ for all $x \in \mathcal{D}_\mathcal{T}$, and $V(0) = 0$.

(ii) The condition $(C2)$ defines linear constraints on the optimization variables V_{x_i}, $C_{\nu,k}$.

(iii) Constraint $(C3)$ is necessary since $f(x, u)$ and x may not be in the same simplex. The constant $\overline{C}$ plays an important role in the proof of Theorem 4.3.7.

(iv) The condition $(C5)$ ensures that the set $\mathcal{B}_2(0, \epsilon)$ is a subset of the level set $\{x \in \mathcal{O} | V(x) \leq \max\limits_{x_i \in \partial(\mathcal{O} \setminus \mathcal{O}^\epsilon)} V(x_i)\}$. If system (4.2) is locally ISS, condition $(C5)$ is not necessary.

Remark 4.3.5. If the linear optimization problem (4.69) has a feasible solution, then the values V_{x_i} from this feasible solution at all vertices x_i of all simplices $\mathcal{S}_\nu \in \mathcal{T}$ and the condition $V \in \mathrm{CPA}[\mathcal{T}]$ uniquely define a continuous and piecewise affine function

$$V : \mathcal{T} \to \mathbb{R}. \tag{4.70}$$

Remark 4.3.6. For the estimate of the interpolation errors in $(C7)$, we use the sharp estimate given in Remark 4.2.5 for our numerical experiments.

We show that the feasible solution delivered by the algorithm is a true CPA ISS Lyapunov function in dissipative formulation for system (4.2) on $\mathcal{O}^\epsilon$ by the next theorem.

Theorem 4.3.7. *If assumption (H1) or (H2) holds and the linear optimization problem* (4.69) *has a feasible solution, then the function V from* (4.70) *is a* CPA *ISS Lyapunov function in dissipative formulation for system* (4.2) *on $\mathcal{O}^\epsilon$, i.e., it satisfies* (4.4) *and* (4.9) *with $\sigma = 1$ for all $x \in \mathcal{O}^\epsilon$ and all $u \in U_R^\mathcal{T}$.*

Proof. Consider the convex combinations $x = \sum_{i=0}^n \lambda_i x_i \in \mathcal{S}_\nu$, $\mathcal{S}_\nu = \mathrm{co}\{x_0, x_1, \ldots, x_n\} \in \mathcal{T}$ and $\mathcal{S}_\nu \subset \mathcal{O}^\epsilon$, $\sum_{i=0}^n \lambda_i = 1$, $1 \geq \lambda_i \geq 0$, and $u = \sum_{j=0}^m \mu_j u_j \in \mathcal{S}_\kappa^u$, $\mathcal{S}_\kappa^u = \mathrm{co}\{u_0, u_1, \ldots, u_m\} \in \mathcal{T}_u$, $\sum_{j=0}^m \mu_j = 1$, $1 \geq \mu_j \geq 0$.

From the proof of Theorem 4.2.6, it is known that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that (4.4) holds on $\mathcal{O}^\epsilon$.

In the following we prove that inequality (4.9) holds for $\sigma = 1$. We calculate

$$\begin{aligned} V(f(x,u)) - V(x) &= V(f(x,u)) - \sum_{i=0}^{n} \lambda_i V(f(x_i,u)) + \sum_{i=0}^{n} \lambda_i V(f(x_i,u)) - \sum_{i=0}^{n} \lambda_i V(x_i) \\ &\le \sum_{i=0}^{n} \lambda_i n \overline{C} \| f(x,u) - f(x_i,u)\|_\infty - \sum_{i=0}^{n} \lambda_i \sum_{j=0}^{m} \mu_j V(f(x_i,u_j)) \\ &\quad + \sum_{i=0}^{n} \lambda_i \sum_{j=0}^{m} \mu_j V(f(x_i,u_j)) + \sum_{i=0}^{n} \lambda_i V(f(x_i,u)) - \sum_{i=0}^{n} \lambda_i V(x_i) \\ &\le \sum_{i=0}^{n} \lambda_i n \overline{C} \| f(x,u) - f(x_i,u)\|_\infty \\ &\quad + \sum_{i=0}^{n} \lambda_i \sum_{j=0}^{m} \mu_j n \overline{C} \| f(x_i,u) - f(x_i,u_j)\|_\infty \\ &\quad + \sum_{i=0}^{n} \lambda_i \sum_{j=0}^{m} \mu_j V(f(x_i,u_j)) - \sum_{i=0}^{n} \lambda_i V(x_i). \end{aligned}$$

According to Proposition 4.1.6, the constraints from $(C6)$ or $(C7)$ ensure that V satisfies

$$V(f(x,u)) - V(x) \le -\sum_{i=0}^{n} \lambda_i \|x_i\|_2 + r \sum_{i=0}^{m} \mu_j \|u_j\|_1 \le -\|x\|_2 + r\|u\|_1 .$$

Thus the inequality (4.9) with $\sigma = 1$ is satisfied for all $x \in \mathcal{O}^\epsilon$ and all $u \in U_R^{\mathcal{T}}$. □

Next we turn our attention to our second main result, i.e., the conditions under which our proposed algorithm yields a feasible solution. In order to derive these conditions, we have to assume that the simplices in our suitable triangulation satisfy a certain regularity (see Remark 1.4.8).

Theorem 4.3.8. *Consider system* (4.2) *which satisfies (H1) or (H2) and which has a C^1 ISS Lyapunov function in dissipative formulation $W : \mathbb{R}^n \to \mathbb{R}$ on $\mathcal{D}$. Furthermore, we assume that there exists $C \ge 0$ such that $\|\nabla W(x)\|_2 \le C$. Let $\epsilon > 0$ and $R_1 > 0$. Then for every $R_1 > 0$ there exist $\delta_{R_1} > 0$, $\delta_u > 0$ such that, for any triangulations $\mathcal{T}$, $\mathcal{T}_u$ satisfying*

$$\max_{\mathcal{S}_\nu \in \mathcal{T}} \operatorname{diam}(\mathcal{S}_\nu) \le \delta_{R_1}, \tag{4.71}$$

$$\max_{\mathcal{S}_\kappa^u \in \mathcal{T}_u} \operatorname{diam}(\mathcal{S}_\kappa^u) \le \delta_u, \tag{4.72}$$

$$\lambda^* h_x \le R_1, \quad \textit{with } \lambda^* \textit{ defined in } (4.46) \tag{4.73}$$

the linear programming problem from our algorithm has a feasible solution and delivers an ISS Lyapunov function $V \in \mathrm{CPA}[\mathcal{T}]$ on $\mathcal{O}^\epsilon$.

Proof. Applying Proposition 4.1.3 we may, without loss of generality, assume that W satisfies (4.7) and (4.9) with $\sigma = 2$ and some $r > 0$.

For each vertex $x_i \in \mathcal{S}_\nu \in \mathcal{T}$, we let $V_{x_i} = V(x_i) = W(x_i)$. Based on the proof of Theorem 4.2.9 we conclude that $V \in \text{CPA}[\mathcal{T}]$ satisfies (4.4) and there exists a constant $\overline{C} \geq 0$ such that $\max\limits_{\nu\in\{1,\ldots,N\}} \left\{ \max\limits_{k\in\{1,2,\ldots,n\}} |\nabla V_{\nu,k}| \right\} \leq \overline{C}$. It is obvious that the constraint $(C5)$ is fulfilled. In the following, we show that $(C6)$ and $(C7)$ satisfied.

Let x_i be an arbitrary vertex of an arbitrary simplex $\mathcal{S}_\nu \subset \mathcal{O}^\epsilon$ and u_j of $\mathcal{S}^u_\kappa \subset U^{\mathcal{T}}_R$. Since $f(x,u) \in \mathcal{D}_\mathcal{T}$ for all $x \in \mathcal{O}^\epsilon$ and all $u \in U^{\mathcal{T}}_R$, there exists a simplex $\mathcal{S}_\iota = \text{co}\{y_1, y_2, \ldots, y_n\} \in \mathcal{T}$ such that $f(x_i, u_j) = \sum\limits_{k=0}^{n} \lambda_k y_k \in \mathcal{S}_\iota$ with $\sum\limits_{k=0}^{n} \lambda_k = 1$. We have assigned $V(x) = W(x)$ for all vertices x of all simplices $\mathcal{S}_\nu$. Hence, for $x_i \in \mathcal{O}^\epsilon$ we have

$$\begin{aligned} V(f(x_i,u_j)) - V(x_i) &= \sum_{k=0}^{n} \lambda_k W(y_k) - W(x_i) \\ &= \sum_{k=0}^{n} \lambda_k W(y_k) - W(\sum_{k=0}^{n} \lambda_k y_k) + W(\sum_{k=0}^{n} \lambda_k y_k) - W(x_i) \\ &\leq C\delta_{R_1} - 2\|x_i\|_2 + r\|u_j\|_1. \end{aligned}$$

It is obvious that for $R_1 > 0$ there exist suitable $\delta_{R_1} > 0$, $\delta_u > 0$ such that

$$C\delta_{R_1} + n\overline{C}(\overline{L}_x\delta_{R_1} + \overline{L}_u\delta_u) \leq \|x_i\|_2, \text{ if } f \text{ satisfies (H1).} \tag{4.74}$$

$$C\delta_{R_1} + n\overline{C}(n\overline{K}_x\delta^2_{R_1} + m\overline{K}_u\delta^2_u) \leq \|x_i\|_2, \text{ if } f \text{ satisfies (H2).} \tag{4.75}$$

holds for all $x_i \in \mathcal{O}^\epsilon$ and all $u_j \in U^{\mathcal{T}}_R$. Thus the theorem is proved. □

Remark 4.3.9. Since inequality (4.50) is not needed in the proof of Theorem 4.3.8, we only require that $W(x)$ is a C^1 function with bounded gradient.

4.3.2 Examples

In this section, we present two numerical examples to demonstrate our proposed algorithm. Our first example describes a nonlinear dynamic system with one perturbation. Our second example illustrates that we can deal with the case with more than one perturbations.

Example 1

The system is described by the following difference equations

$$\begin{cases} x_1^+ = x_2, \\ x_2^+ = -0.2x_2 + 0.1\sin(x_1 + u) + sin(u), \end{cases} \tag{4.76}$$

where $x \in \mathcal{D} = [-0.225, 0.225]^2 \subset \mathbb{R}^2$, $U_R = [-0.12, 0.12] \subset \mathbb{R}$. We let $\mathcal{O} = [-0.195, 0.195]^2$.

A suitable triangulation of $\mathcal{D} = [-0.225, 0.225]^2$ is obtained as described in Section 2.1.3 with $K = 15$, $k = 1$ and the following map F

$$F : \mathbb{R}^2 \mapsto \mathbb{R}^2,\ F(s) = 0.015s. \tag{4.77}$$

A suitable triangulation of $U_R = [-0.12, 0.12]$ is obtained as in Section 2.1.3 with $K = 8$, $k = 1$ and the following map

$$G : \mathbb{R} \mapsto \mathbb{R},\ G(s) = 0.015s. \tag{4.78}$$

The algorithm for system (4.76) on $\mathcal{O} \setminus (-0.015, 0.015)^2$ yields a CPA ISS Lyapunov function in dissipative formulation V_1 as shown in Figure 4.7. The gain parameter is $r = 2.89335$.

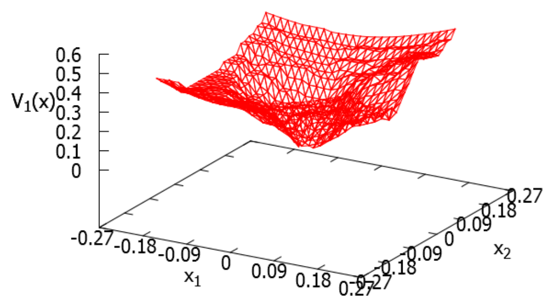

Figure 4.7: CPA ISS Lyapunov function in dissipative formulation V_1 delivered by the algorithm for system (4.76) with the gain parameter $r = 2.89335$.

Example 2

We consider the following system with two one dimensional perturbations

$$x^+ = 0.5x^2 + 0.1 \sin u_1 + 0.2 w_1, \tag{4.79}$$

where $x \in \mathcal{D} = [-0.45, 0.45] \subset \mathbb{R}$, $u_1, w_1 \in U_R = [-0.225, 0.225] \subset \mathbb{R}$. Let $\mathcal{O} = [-0.435, 0.435]$.

We partition the compact set $\mathcal{D}$ into a suitable triangulation as described in Section 2.1.3 with $K = 30$, $k = 1$ and the map from G (4.78). Similarly, the compact set U_R is partitioned into a suitable triangulation with $K = 15$, $k = 0$ and the map G from (4.78). For system (4.79), the objective of the linear optimization problem (4.69) is to minimize the sum of gain parameters r_1, r_2, since two perturbations are incorporated in this system.

A CPA ISS Lyapunov function in dissipative formulation V_2 is delivered by the algorithm for system (4.79) on $\mathcal{O} \setminus (-0.015, 0.015)$ which is shown in Figure 4.8. The gain parameters are $r_1 = 0.334841$, $r_2 = 0.669958$. Figure 4.8 shows that the CPA ISS Lyapunov function in dissipative formulation V_2 is smooth in $\mathcal{O} \setminus (-0.015, 0.015)$.

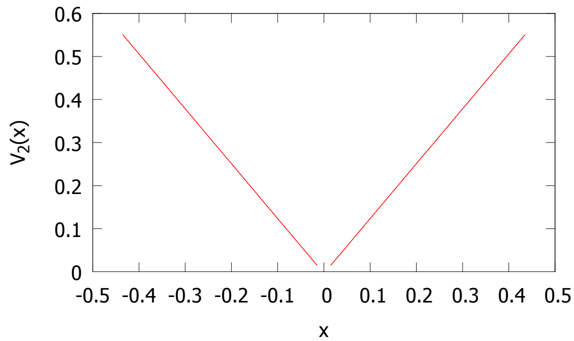

Figure 4.8: CPA ISS Lyapunov function in dissipative formulation V_2 delivered by the algorithm for system (4.79) with gain parameters $r_1 = 0.203388$, $r_2 = 0.40704$.

4.3.3 Conclusion

In this section, the method for computing CPA ISS Lyapunov functions in dissipative formulation is successfully extended to discrete time systems with perturbations. However, the constraints of the linear optimization problem (4.69) are more restrictive than for the continuous time case since the solution of (4.2) is a sequence of points which is not absolutely continuous. As Remark 4.3.9 describes, the conditions of Theorem 4.3.8 are a little more relaxed than that of Theorem 4.2.9. Utilizing the results presented in Section 4.3.2, the linear programming based algorithm for the computation of a CPA ISS Lyapunov function in dissipative formulation can be applied to systems with more than one type of input perturbations.

4.4 Stability of interconnected ISS systems and estimate of the domain of attraction

In this section, we turn our attention to the stability analysis of interconnected continuous time ISS systems. First, for each subsystem a CPA ISS Lyapunov function in dissipative formulation is computed by our proposed approach. Then stability of interconnected systems is looked into via the small gain theorem in linear form.

In the following we investigate the stability of interconnected continuous time systems which are described by the following equations

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \ldots, x_s), \\ \qquad \vdots \\ \dot{x}_s = f_s(x_1, x_2, \ldots, x_s), \\ x_i(0) = x_i^0, \quad i = 1, 2, \ldots, s, \end{cases} \tag{4.80}$$

where $x_i \in \mathbb{R}^{n_i}$, $\sum_{i=1}^{s} n_i = n$, $x = (x_1, x_2, \ldots, x_s)^\top$, $f(x) = (f_1(x), f_2(x), \ldots, f_s(x))^\top$, $f(0) = 0$.

We treat each subsystem as a dynamic system with perturbations by regarding the inputs of other states as perturbations.

We assume that f_i satisfies $(H1)$ or $(H2)$ and that each subsystem is locally ISS.

Define $e_{ij} = 0$ if the state x_j does not influence x_i and $e_{ij} = 1$ otherwise.

Theorem 4.4.1. *Given a suitable triangulation $\mathcal{T}_i$ of a subset $\mathcal{D}_i$ of state space x_i and a suitable triangulation $\mathcal{T}_j$ of a subset $\mathcal{D}_j$ of the input perturbation value space x_j. If the linear optimization problem* (4.36) *has a feasible solution for each subsystem, then the function V_i defined by* (4.37) *is a* CPA *ISS Lyapunov function in dissipative formulation, i.e., it satisfies* (4.4) *and*

$$\langle \nabla V_{\nu,i}(x_i), f_i(x_1, x_2, \ldots, x_s)\rangle \leq -\|x_i\|_2 + \sum_{j=1, j\neq i}^{s} \sqrt{n_j} e_{ij} r_{ij} \|x_j\|_2 \tag{4.81}$$

for all $x_i \in \mathcal{S}_\nu^i \subset \mathcal{D}_{i\mathcal{T}}^{\epsilon_i}$ and all $x_j \in \mathcal{D}_{j\mathcal{T}}$ $(i \neq j)$, $\mathcal{D}_{i\mathcal{T}}^{\epsilon_i} := \mathcal{D}_{i\mathcal{T}} \setminus \mathcal{B}_2(0, \epsilon_i)$, $\epsilon_i > 0$.

Proof. This result is directly obtained from Theorem 4.2.6. □

Theorem 4.4.2. *If the conditions of Theorem 4.4.1 and Theorem 1.6.1 are satisfied, then the interconnected system* (4.80) *is locally asymptotically stable on $\overline{\mathcal{D}}$ defined by* (4.82).

Proof. Let $V = (V_1, \ldots, V_s)^\top$. From the proof of Theorem 1.6.1 it is known that there exists an s-vector $b > 0$ such that $W = \langle b, V\rangle$ is a CPA Lyapunov function for the whole system. Define the set

$$\overline{\mathcal{D}} = \{x \in \mathcal{D}_{1\mathcal{T}} \times \ldots \times \mathcal{D}_{s\mathcal{T}} \setminus \Big(\mathcal{B}_2(0, \epsilon_1) \times \ldots \times \mathcal{B}_2(0, \epsilon_s)\Big) \,:\, W(x) < \min_{x \in \partial(\mathcal{D}_{1\mathcal{T}} \times \ldots \times \mathcal{D}_{s\mathcal{T}})} W(x)\}. \tag{4.82}$$

The interconnected system (4.80) is locally asymptotically stable on $\overline{\mathcal{D}}$, since W is a CPA Lyapunov function for system (4.80) on $\overline{\mathcal{D}}$. □

4.4.1 Examples

In this section we present three examples to demonstrate how to analyse the stability of interconnected systems by Theorem 4.4.2. In order to compare results of Chapter 3 and this chapter, the first example is the academic example studied in Chapter 3. The second and third examples show that the proposed approach for computing CPA ISS Lyapunov functions in dissipative formulation is very useful in stability analysis of complicated and practical systems for which it is difficult to construct a Lyapunov function.

Example 1

We consider the academic example studied in Chapter 3.

$$\begin{cases} \Delta_1 : \dot{x}_1 = -x_1 + x_1^3 + x_1 x_2^2, \\ \Delta_2 : \dot{x}_2 = -x_2 + x_2^3 + x_2 x_1^2, \end{cases} \tag{4.83}$$

where $x = (x_1, x_2)^\top$ in $\mathcal{D} = [-0.70225, 0.70225]^2 \subset \mathbb{R}^2$.

The suitable triangulation of $[-0.70225, 0.70225]$ is obtained in the same way as described in Section 2.1.3 with $K = 265$, $k = 1$ and the map $\rho : \mathbb{R} \mapsto \mathbb{R}$.

$$\rho(s) = 10^{-5}|s|s. \tag{4.84}$$

System (4.83) is considered as two interconnected one dimensional systems. For subsystem Δ_i, we obtain a CPA ISS Lyapunov function in dissipative formulation V_i (see Figure 4.9,

$V_1 = V_2$) by solving the corresponding linear optimization problem (4.36). The gain parameter is $r_i = 0$, which means that the perturbation has no influence in the stability of the state x_i. Furthermore, V_i satisfies

$$\begin{aligned} \langle \xi, \dot{x}_i \rangle &\leq -\|x_i\|_2 + r_i \|x_j\|_1 \\ &\leq -\|x_i\|_2 + r_i \|x_j\|_2, \;\; \forall \xi \in \partial_{Cl} V_i(x_i), \;\; i \neq j. \end{aligned} \tag{4.85}$$

According to inequalities (4.85), we define

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

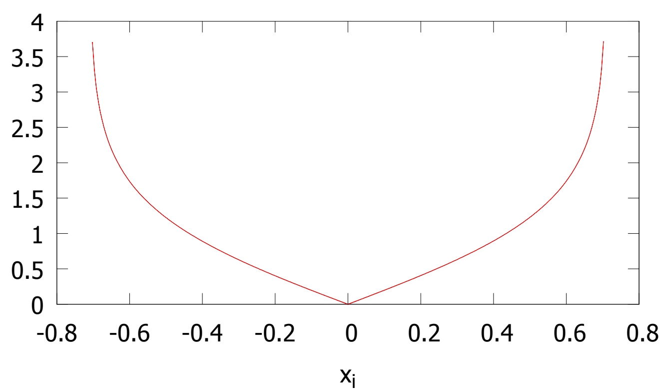

Figure 4.9: CPA ISS Lyapunov function in dissipative formulation V_i delivered by the algorithm for subsystem Δ_i.

It is obvious that the conditions of Theorem 1.6.1 hold. Thus system (4.83) is locally asymptotically stable in $\mathcal{D}_1 \setminus (-10^{-5}, 10^{-5})^2$ with $\mathcal{D}_1$ defined by (4.86).

Let $c = (1,1)^\top$. According to Theorem 1.6.1 we have $b = (1,1)^\top$ such that $c^\top = -b^\top A$. Let $V(x) = \langle b, (V_1, V_2)^\top \rangle$. Therefore, V is a CPA Lyapunov function for system (4.87). An estimate of the domain of attraction of $(-10^{-5}, 10^{-5})^2$ (see Figure 4.10) is given by

$$\mathcal{D}_1 = \{x \in [-0.70225, 0.70225]^2 \; : \; V(x) < \min_{x \in \partial \mathcal{D}} V(x) = 3.7166738\}. \tag{4.86}$$

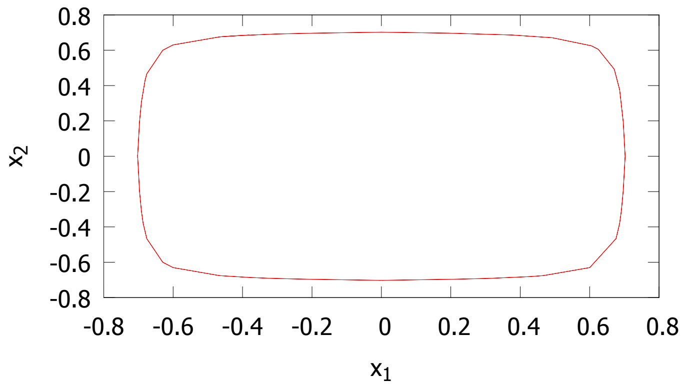

Figure 4.10: An estimate of the domain of attraction of system (4.83) is the inside of the red curve.

Remark 4.4.3. Compared with the estimate of the domain of attraction by the local version of the small gain theorem in comparison form in Section 3.4, the estimate obtained here is not bigger. In order to obtain a bigger $\mathcal{D}_1$, we construct suitable triangulations of a set bigger than $\mathcal{D}$ and then solve the linear optimization problem (4.36) again. But the gain parameters r_i yielded by the algorithm for system Δ_i on the bigger $\mathcal{D}_1$ do not satisfy the conditions of Theorem 1.6.1. However, the domain of attraction obtained here is bigger than the estimate of the domain of attraction obtained by the small gain theorem in dissipative form. Moreover, the proposed approach in this chapter can deal with more complicate cases, which is demonstrated by the following two examples.

Example 2

We study stability of the following system adapted from the practical model of [87]. Here the practical model of [87] is investigated with fixed control, i.e. (33), (34) of [87] and without perturbations.

$$\begin{cases} \dot{l}_B = \frac{10^{-4}}{0.07}\left\{0.8211 + 0.2896\sqrt{19.6(l_S+0.34)} - 0.6168\sqrt{19.6(l_B+0.33)}\right\} \\ \dot{v}_B = \frac{10^{-6}}{l_B+0.33}\Big\{0.8211(-100v_B - 0.55) + 0.3787(-12.05 - 100v_B) \\ \qquad + 0.2896\sqrt{19.6(l_s+0.34)}(100v_s - 100v_B + 5.5)\Big\} \\ \dot{l}_S = \frac{10^{-4}}{0.07}\left\{0.9901 + 0.947\sqrt{19.6(l_B+0.33)} - 0.6739\sqrt{19.6(l_S+0.34)}\right\} \\ \dot{v}_S = \frac{10^{-6}}{l_S+0.34}\Big\{0.9901(-100v_S - 6.05) \\ \qquad + 0.2947\sqrt{19.6(l_B+0.33)}(100v_B - 100v_S - 5.5) + 5.1123\Big\} \end{cases} \tag{4.87}$$

We consider this model as two interconnected subsystems, i.e., subsystem S_1 with states $x_1 = (l_B, v_B)^\top$, subsystem S_2 with states $x_2 = (l_S, v_B)^\top$.

Consider each subsystem of (4.87) on $\mathcal{D} = \mathcal{B}_2(0, 0.05) \subset \mathbb{R}^2$ with $U_R = \mathcal{D}$. A suitable triangulation of $\mathcal{D}$ is constructed as described in Section 2.1.3 with $K = 5$, $k = 4$ and the map $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$,

$$F(s) = \begin{cases} 0.2s\frac{10^{-2}\|s\|_\infty^2}{\|s\|_2}, & s \neq 0, \\ 0, & s = 0. \end{cases} \tag{4.88}$$

By our proposed approach we get a CPA ISS Lyapunov function in dissipative formulation for system S_1 on $\mathcal{D} \setminus \mathcal{B}_2(0, 0.032)$ as shown in Figure 4.11. The gain parameter is $r_1 = 0$ and V_1 satisfies

$$\begin{aligned} \langle \xi, \dot{x}_1 \rangle &\leq -\|x_1\|_2 + r_1\|x_2\|_1 \\ &\leq -\|x_1\|_2 + \sqrt{2}r_1\|x_2\|_2, \ \forall \xi \in \partial_{Cl}V_1(x_1). \end{aligned} \tag{4.89}$$

Similarly, a CPA ISS Lyapunov function in dissipative formulation for system S_2 on $\mathcal{D} \setminus \mathcal{B}_2(0, 0.032)$ is obtained by the same proposed approach. It is shown in Figure 4.12. The gain parameter is $r_2 = 3.78$ and V_2 satisfies

$$\begin{aligned} \langle \xi, \dot{x}_2 \rangle &\leq -\|x_2\|_2 + r_2\|x_1\|_1 \\ &\leq -\|x_2\|_2 + \sqrt{2}r_2\|x_1\|_2, \quad \forall \xi \in \partial_{Cl}V_2(x_2) \end{aligned} \tag{4.90}$$

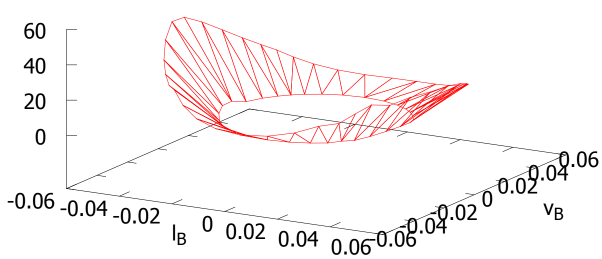

Figure 4.11: CPA ISS Lyapunov function in dissipative formulation V_1 delivered by the algorithm for subsystem S_1.

Based on inequalities (4.89) and (4.90) we define

$$A = \begin{bmatrix} -1 & 0 \\ 5.3457 & -1 \end{bmatrix}.$$

Through calculation, we have that the conditions of Theorem 1.6.1 are satisfied. Therefore, the whole system is locally asymptotically stable in $\mathcal{D}_d' \setminus \mathcal{B}_2(0, 0.032) \times \mathcal{B}_2(0, 0.032)$ with $\mathcal{D}_d'$ defined by (4.91).

Let $c = (0.6453, 1)^\top$. According to Theorem 1.6.1 we have $b = (6, 1)^\top$ such that $c^\top = -b^\top A$. Then a CPA Lyapunov function for system (4.87) is defined by $V = \langle b, (V_1, V_2)^\top \rangle$.

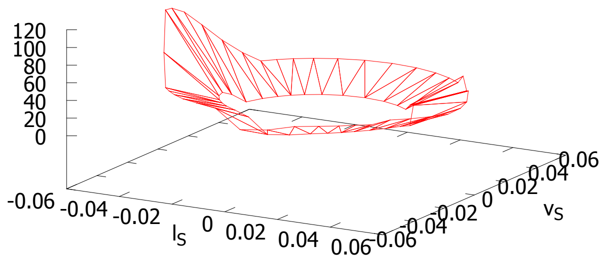

Figure 4.12: CPA ISS Lyapunov function in dissipative formulation V_2 delivered by the algorithm for subsystem S_2.

Based on this Lyapunov function an estimate of the domain of attraction of $\mathcal{D}_2 = \{(x_1, x_2)^\top \in \mathcal{D} \times \mathcal{D} : V(x) \leq \max_{x \in \partial(\mathcal{B}_2(0,0.032) \times \mathcal{B}_2(0,0.032))} V(x) = 46.7922\}$ is obtained, i.e.,

$$\mathcal{D}_d^{'} = \{(x_1, x_2)^\top \in \mathcal{D} \times \mathcal{D} : V(x_1, x_2) < \min_{x \in \partial(\mathcal{D} \times \mathcal{D})} V(x_1, x_2) = 46.8102\}. \tag{4.91}$$

Remark 4.4.4. Compared with the set $\mathcal{D}$, the excluded neighbourhood of the origin is quite big. When choosing a smaller neighbourhood of the origin, we cannot get that the whole system is locally asymptotically stable. This means the origin of the overall system is not locally asymptotically stable. When we consider system (4.87) on a bigger set than $\mathcal{D}$, the obtained gain parameters r_1, r_2 will not satisfy the conditions of Theorem 1.6.1.

Example 3

Consider the system

$$\begin{cases} \dot{z}_1 = -(2 + \sin z_3) z_1 + z_3, \\ \dot{z}_2 = (0.1 \sin z_5 - 1) z_2 + 0.1 z_2 e^{-z_3^2}, \\ \dot{z}_3 = -(\sin z_1 + 2) z_3 + 0.1 z_4, \\ \dot{z}_4 = \sin(0.1 z_3 + z_4) - z_4(2 + \sin(0.1 z_1)), \\ \dot{z}_5 = (\sin z_2 - 1) z_5 + 0.1 z_3 e^{-z_1^2}, \end{cases} \tag{4.92}$$

which is adapted from [106, Example 5.3]. We divide this model into three interconnected systems, system $\overline{S}_1$ with states $x_1 = (z_1, z_3)^\top$, system $\overline{S}_2$ with state $x_2 = z_4$ and system $\overline{S}_3$ with states $x_3 = (z_2, z_5)^\top$.

We consider each subsystem as a dynamic system with perturbations. We study systems $\overline{S}_1$, $\overline{S}_3$ on $\mathcal{D} = \mathcal{B}_2(0, 0.072) \subset \mathbb{R}^2$ and system $\overline{S}_2$ on $\mathcal{D}_1 = [-0.072, 0.072] \subset \mathbb{R}$.

A suitable triangulation of $\mathcal{D}$ is obtained as described in Section 2.1.3 with $K = 6$, $k = 1$ and the map F from (4.88). A suitable triangulation of $\mathcal{D}_1$ is attained with $K = 6$, $k = 1$ and map $G_1 : \mathbb{R} \mapsto \mathbb{R}$,

$$G_1(s) = 0.002 s|s|. \tag{4.93}$$

With the proposed method we compute a CPA ISS Lyapunov function in dissipative formulation V_1 for system $\overline{S}_1$ on $\mathcal{B}_2(0, 0.072)\backslash\mathcal{B}_2(0, 0.002)$, see Figure 4.13. The gain parameter is $r_1 = 0.0620467$. V_1 satisfies

$$\langle \xi, \dot{x}_1 \rangle \leq -\|x_1\|_2 + r_1\|x_2\|_2, \ \forall \xi \in \partial_{Cl} V_1(x_1). \tag{4.94}$$

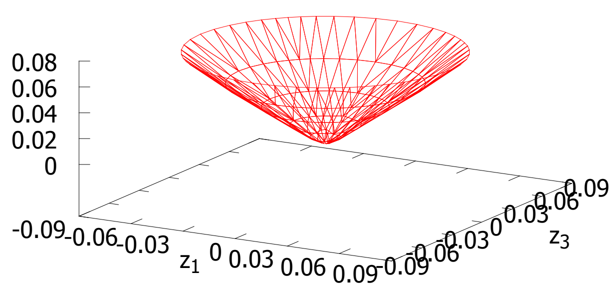

Figure 4.13: CPA ISS Lyapunov function in dissipative formulation V_1 computed by the algorithm for subsystem $\overline{S}_1$, $r_1 = 0.0620467$.

A CPA ISS Lyapunov function in dissipative formulation V_2 (see Figure 4.14) is computed by our proposed approach for system $\overline{S}_2$ on $[-0.072, 0.072] \setminus (-0.002, 0.002)$. The gain parameter is $r_2 = 0.114601$ and V_2 satisfies

$$\begin{aligned}\langle \xi, \dot{x}_2 \rangle &\leq -\|x_2\|_2 + r_2\|x_1\|_1 \\ &\leq -\|x_2\|_2 + \sqrt{2} r_2 \|x_1\|_2, \ \forall \xi \in \partial_{Cl} V_2(x_2).\end{aligned} \tag{4.95}$$

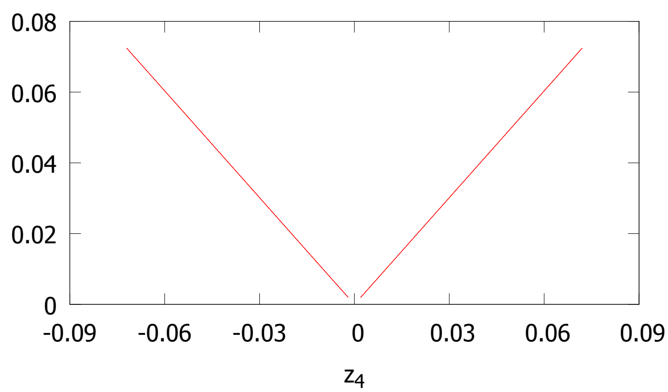

Figure 4.14: CPA ISS Lyapunov function in dissipative formulation V_2 computed by the algorithm for subsystem $\overline{S}_2$, $r_2 = 0.114601$.

Similarly, using our proposed approach we obtain a CPA ISS Lyapunov function in dissipative formulation V_3 (see Figure 4.15) for system $\overline{S}_3$ on $\mathcal{B}_2(0, 0.072) \setminus \mathcal{B}_2(0, 0.002)$. The gain parameter is $r_3 = 0.107921$ and V_3 satisfies

$$\begin{aligned}\langle \xi, \dot{x}_3 \rangle &\leq -\|x_3\|_2 + r_3\|x_1\|_1 \\ &\leq -\|x_3\|_2 + \sqrt{2} r_3 \|x_1\|_2, \ \forall \xi \in \partial_{Cl} V_3(x_3). \end{aligned} \tag{4.96}$$

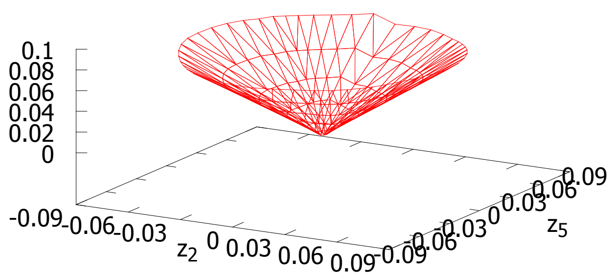

Figure 4.15: CPA ISS Lyapunov function in dissipative formulation V_3 computed by the algorithm for subsystem $\overline{S}_3$, $r_3 = 0.107921$.

Based on inequalities (4.94), (4.95) and (4.96), we define

$$A = \begin{bmatrix} -1 & 0.06240467 & 0 \\ 0.1621 & -1 & 0 \\ 0.15026 & 0 & -1 \end{bmatrix}.$$

Remark 4.4.5. Let $\tilde{\mathcal{D}} = \mathcal{D} \times \mathcal{D}_1 \times \mathcal{D}$. The conditions of Theorem 1.6.1 are fulfilled. Thus system (4.92) is asymptotically stable on $\mathcal{D}_d' \setminus \mathcal{B}_2(0, 0.002) \times (-0.002, 0.002) \times \mathcal{B}_2(0, 0.002)$ with $\mathcal{D}_d'$ defined by (4.97).

Let $c = (0.6853, 0.9389533, 1)^\top$, $x = (x_1^\top, x_2^\top, x_3^\top)^\top$. According to Theorem 1.6.1 there exists a vector $b = (1, 1, 1)^\top$ such that $c^\top = -b^\top A$. Then a CPA Lyapunov function is defined by $V(x) = \langle b, (V_1, V_2, V_3)^\top \rangle$ for system (4.92). An estimate of the domain of attraction of $\mathcal{B}_2(0, 0.002) \times (-0.002, 0.002) \times \mathcal{B}_2(0, 0.002)$ is given by

$$\mathcal{D}_d' = \{x \in \tilde{\mathcal{D}} : V(x) < \min_{x \in \partial \tilde{\mathcal{D}}} V(x) = 0.0786\}. \tag{4.97}$$

Remark 4.4.6. It is difficult to construct Lyapunov functions for systems (4.87) and (4.92) without considering them as interconnected systems. Even for the subsystems, there is no known analytic ISS Lyapunov function except for the one dimensional system. By our proposed method, CPA ISS Lyapunov functions in dissipative formulation are computed for subsystems, and then CPA Lyapunov functions are defined for systems (4.87) and (4.92) and estimates of domain of attraction are obtained.

4.4.2 Conclusion

In this section, the stability of interconnected continuous time ISS systems was investigated. We assume each subsystem is locally ISS. Then for each subsystem, a CPA ISS Lyapunov function in dissipative formulation is computed by the method described in Section 4.2. Using the small gain theorem in linear form (Theorem 1.6.1) we analysed the stability of the whole system.

For interconnected discrete time ISS systems, stability of the overall system can be studied by a similar method and parallel results can be obtained.

In the inequalities of the small gain theorem in linear form (Theorem 1.6.1), $\|\cdot\|_2$ is used. However, $\|u\|_1$ is utilized in the inequality obtained from the linear optimization problem. In order to analyse stability of the interconnected systems by the small gain theorem in linear form, a new inequality with $\|\cdot\|_2$ is deduced from the obtained inequality from the algorithm, see (4.85), (4.89), (4.90), (4.95) and (4.96). In order to avoid this step, $\|x\|_1$, $\|u\|_1$ could be used in the linear optimization problem and the linear inequalities of the small gain theorem in linear form.

4.5 Concluding remarks

In this chapter approaches of computing CPA ISS Lyapunov functions in dissipative formulation for dynamic systems with perturbations were proposed. The linear programming based algorithm for the computation of CPA ISS Lyapunov functions in dissipative formulation for continuous dynamic systems with perturbations was first described in Section 4.2. A CPA ISS Lyapunov function in dissipative formulation can be obtained by solving the linear optimization problem (4.36) and the gain parameter delivered by the algorithm can be the smallest one by refining the suitable triangulation. We then successfully adapted this algorithm to the computation of ISS Lyapunov functions for discrete time systems with perturbations. The solutions to the linear optimization problems are true CPA ISS Lyapunov functions in dissipative formulation which are not numerical approximations. Theorems 4.2.9 and 4.3.8 prove that the proposed algorithms for the computation of CPA ISS Lyapunov functions in dissipative formulation always have feasible solutions. Furthermore, we analysed the stability of interconnected continuous time ISS systems. For interconnected ISS systems, based on the CPA ISS Lyapunov functions in dissipative formulation obtained by our proposed approach for each subsystem, we can analyse the stability of the overall system via the small gain theorem in linear form. In Section 4.4, we presented examples to show how to investigate the stability of interconnected ISS systems. From results of the examples in Section 4.4.1, we conclude that the proposed method for computing a CPA ISS Lyapunov function in dissipative formulation can play an important role in the stability analysis of interconnected systems.

However, the proposed method has some disadvantages. The computed CPA ISS Lyapunov function in dissipative formulation is a CPA function which does not allow for an explicit formulation such as a quadratic Lyapunov function. For more than three dimensional system, the CPA Lyapunov function cannot easily be shown in a figure. As experienced in our experiments, the cost of computing ISS Lyapunov functions by solving a linear optimization problem becomes more expensive as the dimension of the considered system increases.

5 Appendix

5.1 Definition of the triangulation $\mathcal{T}^{\mathcal{C}}_{K,b}$

In order to state the definition of the triangulation $\mathcal{T}^{\mathcal{C}}_{K,b}$, we first recall the following definition.

Definition 5.1.1 ([29, Def. 1])**.** Denote by S the set of all subsets of $\mathcal{D} \in \mathbb{R}^n$ that fulfill:

i) $\mathcal{D}$ is compact.

ii) The interior $\mathcal{D}^\circ$ of $\mathcal{D}$ is a connected open neighbourhood of the origin.

iii) $\mathcal{D} = \overline{\mathcal{D}^\circ}$.

For the construction we use the set S_n of all permutations of the numbers $1, 2, \ldots, n$, the characteristic functions $\chi_{\mathcal{J}}(i)$ equal to one if $i \in \mathcal{J}$ and equal to zero if $i \notin \mathcal{J}$, and the standard orthonormal basis e_1, e_2 $\ldots$, e_n of $\mathbb{R}^n$. Further, we use the functions $R^{\mathcal{J}} : \mathbb{R}^n \to \mathbb{R}^n$, defined for every $\mathcal{J} \subset \{1, 2, \ldots, n\}$ by

$$R^{\mathcal{J}}(x) := \sum_{i=1}^{n} (-1)^{\chi_{\mathcal{J}}(i)} x_i e_i. \tag{5.1}$$

Definition 5.1.2 ([29, Def. 13])**.** Let $\mathcal{C} \in S$ be a given subset of $\mathbb{R}^n$. We will define a triangulation $\mathcal{T}^{\mathcal{C}}_{K,b}$ ($K \in \mathbb{Z}_+$) of a $\mathcal{D} \in S$, $\mathcal{C} \subset \mathcal{D}$, that approximates $\mathcal{C}$. To construct the triangulation $\mathcal{T}^{\mathcal{C}}_{K,b}$, we first define the triangulation $\mathcal{T}^{std}$, $\mathcal{T}^{std}_K$, and $\mathcal{T}^{std}_{K,b}$ as intermediate steps.

1. The standard triangulation $\mathcal{T}^{std}$ consists of the simplices

$$\mathcal{S}_{z,\mathcal{J},\sigma} := \operatorname{co}\left\{ R^{\mathcal{J}}\left(z + \sum_{i=1}^{j} e_{\sigma(i)} \right) : j = 0, 1, 2, \ldots, n \right\} \tag{5.2}$$

   for all $z \in \mathbb{Z}^n_+$ and $\mathcal{J} \subset \{1, 2, \ldots, n\}$, and all $\sigma \in S_n$.

2. Choose a $K \in \mathbb{Z}_+$ and consider the intersections of the n-simplices $\mathcal{S}_{z,\mathcal{J},\sigma}$ in $\mathcal{T}^{std}$ and the boundary $[-2^K, 2^K]^n$. We are only interested in those intersections that are $(n-1)$-simplices, i.e., we take every simplex with vertices $x_j := R^{\mathcal{J}}\left(z + \sum_{i=1}^{j} e_{\sigma_i}\right)$, $j \in \{0, 1, 2, \ldots, n\}$, where exactly one vertex satisfies $\|x_{j^*}\|_\infty \neq 2^K$ and the other n of $n+1$ vertices satisfy $\|x_j\|_\infty = 2^K$ for $j \in \{0, 1, 2, \ldots, n\} \setminus \{j^*\}$. Then we replace the vertex x_{j^*} by 0. Thus, we obtain a new triangulation of $[-2^K, 2^K]^n$, which is denoted by $\mathcal{T}^{std}_K$.

3. Now choose a constant $b > 0$ and scale down the triangulation $\mathcal{T}_K^{std}$ of the hypercube $[-2^K, 2^K]^n$ and the triangulation $\mathcal{T}^{std}$ outside the hypercube $[-2^K, 2^K]^n$ with the mapping $x \mapsto \rho x$, where $\rho := 2^{-K}b$. We denote by $\mathcal{T}_{K,b}^{std}$ by the resulting set of n-simplices, i.e.

$$\mathcal{T}_{K,b}^{std} = \rho\mathcal{T}_K^{std} \cup \rho\{\mathcal{S} \in \mathcal{T}^{std} : \mathcal{S} \cup [-2^K, 2^K]^n = \emptyset\}. \tag{5.3}$$

4. As a final step define

$$\mathcal{T}_{K,b}^{\mathcal{C}} := \{\mathcal{S}_\nu \in \mathcal{T}_{K,b}^{std} : \mathcal{S}_\nu \cap \mathcal{C}^\circ \neq \emptyset\}. \tag{5.4}$$

and set

$$\mathcal{D} := \bigcup_{\mathcal{S}_\nu \in \mathcal{T}_{K,b}^{\mathcal{C}}} \mathcal{S}_\nu. \tag{5.5}$$

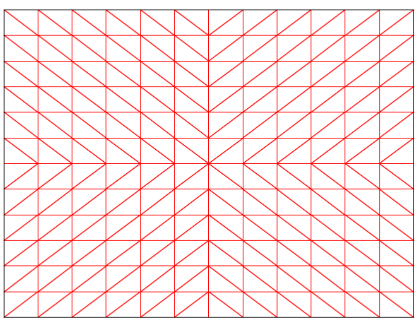

Figure 5.1: $\mathcal{T}^{std} = \mathcal{T}_{0,b}^{std}$.

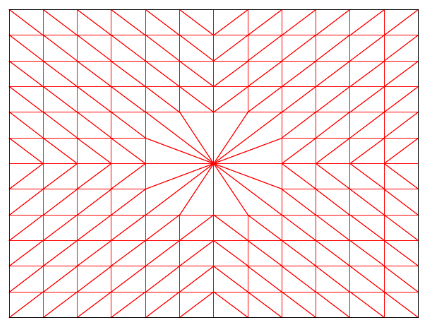

Figure 5.2: $\mathcal{T}_{1,b}^{std}$.

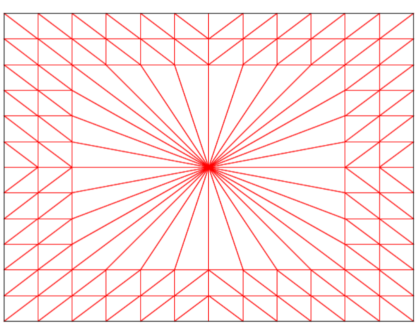

Figure 5.3: $\mathcal{T}_{2,b}^{std}$.

The two parameters b and K of the triangulation $\mathcal{T}_{K,b}^{std}$ refer to the size of the hypercube $[-b, b]^n$ covered by its simplicial fan at the origin and to the fineness of the triangulation, respectively. For schematic pictures of some of these triangulations in $2D$ see Figures 5.1, 5.2 and 5.3 (see Figure 1 in [28]). For similar pictures in $3D$ see Figure 1 in [27].

5.2 Viscosity solution

We now recall the definition of viscosity solutions. For more details of this theory we refer to [8, Sec. II.1 and III.2].

Here, C^1-test functions are used to avoid the gradient of the solution at points of non-differentiability in the domain.

Definition 5.2.1 ([8, Chap. II, Def. 1.1])**.**
Given an open subset Ω of $\mathbb{R}^n$ and a continuous function $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, we consider the partial differential equation

$$H(x, W, DW) = 0 \qquad \forall\, x \in \Omega \tag{5.6}$$

for a continuous function $W : \mathbb{R}^n \to \mathbb{R}$. We say that a continuous function $W : \mathbb{R}^n \to \mathbb{R}$ is a *viscosity subsolution*(resp. *supersolution*) of the equation if for all test functions $\phi \in C^1(\Omega)$ and $x \in \arg\max_\Omega(W - \phi)$ (resp. $x \in \arg\min_\Omega(W - \phi)$) we have

$$H(x, W(x), D\phi(x)) \leq 0 \tag{5.7}$$

$$\text{resp. } H(x, W(x), D\phi(x)) \geq 0. \tag{5.8}$$

A continuous function $W : \Omega \to \mathbb{R}$ is said to be a *viscosity solution* of (5.6) if W is a viscosity supersolution and a viscosity subsolution of (5.6).

Remark 5.2.2. Based on [8, Chap. II, Lemma 1.7], the set of derivatives $D\phi(x)$ for $x \in \arg\min_\Omega(W - \phi)$ coincides with the set

$$D^-W(x) := \{p \in \mathbb{R}^n \colon \liminf_{y \to x, y \in \Omega} \frac{W(y) - W(x) - \langle p, y - x\rangle}{\|x - y\|_2} \geq 0\} \tag{5.9}$$

and the set of derivatives $D\phi(x)$ for $x \in \arg\max_\Omega(W - \phi)$ equals the following set:

$$D^+W(x) := \{p \in \mathbb{R}^n \colon \limsup_{y \to x, y \in \Omega} \frac{W(y) - W(x) - \langle p, y - x\rangle}{\|x - y\|_2} \leq 0\} \tag{5.10}$$

Therefore, one can equivalently define viscosity solution by the sets $D^-W(x)$ and $D^+W(x)$ which are called *sub-* and *superdifferentials*, respectively, i.e.

$$H(x, W(x), p) \leq 0 \qquad \forall\, p \in D^+W(x), \tag{5.11}$$

$$\text{resp. } H(x, W(x), p) \geq 0 \qquad \forall\, p \in D^-W(x). \tag{5.12}$$

Remark 5.2.3. A Lipschitz continuous viscosity solution satisfies the partial differential equation (5.6) almost everywhere due to [8, Chap. II, Proposition 1.9].

Remark 5.2.4. For any locally Lipschitz continuous function $W : \Omega \to \mathbb{R}$ the sub- and superdifferentials satisfy (cf. [8, Chap. II, (4.6)])

$$D^-W(x) \cup D^+W(x) \subseteq \partial_{Cl} W(x) \qquad \forall\, x \in \Omega \tag{5.13}$$

with $\partial_{Cl} W(x)$ defined by (1.35).

5.3 Strong invariance

Consider

$$\dot{x} = F(x), \tag{5.14}$$

$x \in \mathbb{R}^n$, F is Lipschitz continuous.

Let $S \subset \mathbb{R}^n$. We say (S, F) is *strongly invariant* (positively invariant) if for each initial condition $x^0 \in S$ the corresponding trajectory $x(t) \in S$ for $t \in \mathbb{R}_+$.

We recall two properties of Theorem 3.8 in [15] which are used in the proof of Theorem 3.4.3.

Theorem 5.3.1. *(Two properties of [15, Theorem 3.8, p198]) Let F be Lipschitz continuous. Then the following are equivalent:*

(a): $F(x) \in \mathrm{T}_S^B(x)$, $\forall x \in S$, $\mathrm{T}_S^B(x)$ is defined by (3.40).

(b): (S, F) is strongly invariant.

5.4 Path

A continuous mapping $\Theta : \mathbb{R}_+^n \mapsto \mathbb{R}_+^n$ is called a *monotone operator* if $x \leq y$ $(x, y \in \mathbb{R}_+^n)$ implies $\Theta(x) \leq \Theta(y)$ and a *strictly monotone operator* if $x < y$ implies $\Theta(x) < \Theta(y)$.

We now recall [86, Proposition 5.2].

Proposition 5.4.1. *Assume the mapping $\Theta : \mathbb{R}_+^n \mapsto \mathbb{R}_+^n$ is a strictly monotone operator and Θ satisfies $\Theta(s) \ngeq s$, $s \in \mathbb{R}_+^n$. Let $\Omega(\Theta) := \{s \in \mathbb{R}_+^n : \Theta(s) < s\}$. Then for any $s \in \Omega(\Theta)$ there exists a continuous path $\sigma : [0, 1] \to \mathbb{R}_+^n$ such that $\Theta(\sigma(r)) < \sigma(r)$ for all $r \in (0, 1]$, each σ_i is strictly increasing, and $\sigma(0) = 0$, $\sigma(1) = s$. Moreover, σ can be chosen to be piecewise linear on $(0, 1]$.*

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