Numerical Computation of Control Lyapunov Functions in the Sense of Generalized Gradients*

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***Abstract*— The existence of a control Lyapunov function with the weak infinitesimal decrease via the Dini or the proximal subdifferential and the lower Hamiltonian characterizes asymptotic controllability of nonlinear control systems and differential inclusions. We study the class of nonlinear differential inclusions with a right-hand side formed by the convex hull of active $\mathbf{C}^2$ functions which are defined on subregions of the domain. For a simplicial triangulation we parametrize a control Lyapunov function (clf) for nonlinear control systems by a continuous, piecewise affine (CPA) function via its values at the nodes and demand a suitable negative upper bound in the weak decrease condition on all vertices of all simplices.**

Applying estimates of the proximal subdifferential via active gradients we can set up a mixed integer linear problem (MILP) with inequalities at the nodes of the triangulation which can be solved to obtain a CPA function. The computed function is a clf for the nonlinear control system.

We compare this novel approach with the one applied to compute Lyapunov functions for strongly asymptotically stable differential inclusions and give a first numerical example.

***Index Terms*— control Lyapunov functions, asymptotic controllability, nonlinear control systems, continuous, piecewise affine functions, mixed integer linear programming**

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I. Preliminaries

Let us introduce the *nonlinear control system*

$$\dot{x}(t) = f(x(t), u(t)) \quad (\text{a.e. } t \in I = [t_0, T]), \tag{1}$$

$$u(t) \in U \quad (\text{a.e. } t \in I) \tag{2}$$

with the Lipschitzian right-hand side $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and the compact, nonempty control set $U \subset \mathbb{R}^m$. Throughout the paper, we assume that the solutions exist on the interval $[0, \infty)$ and that the equilibrium of the system is the origin. We can associate the following *differential inclusion*

$$\dot{x}(t) \in F(x(t)) \quad (\text{a.e. } t \in I) \tag{3}$$

with the set-valued map $F(x) = \bigcup_{u \in U} \{f(x, u)\}$ which is Lipschitz w.r.t. the Hausdorff distance. The control system is *relaxed*, if $F(x)$ has convex images for all $x \in \mathbb{R}^n$.

We will calculate a Lyapunov function with *continuous, piecewise affine (CPA)* functions. Hence, we need the following generalized differentiation concepts for nonsmooth functions: the *generalized gradients* forming the *Clarke's subdifferential*

$$\partial_{\mathrm{Cl}} V(x) = \operatorname{co} \Big\{ \lim_{i \to \infty} \nabla V(x_i) \,|\, \lim_{i \to \infty} x_i = x, \\ \nabla V(x_i) \text{ as well as } \lim_{i \to \infty} \nabla V(x_i) \text{ exist} \Big\}$$

for Lipschitz functions (see [10, Subsec. 0.1]) and, for continuous functions, the *Dini subderivate* in direction $v \in \mathbb{R}^n$ as well as the *Dini-(Hadamard) subdifferential* defined as

$$DV(x; v) := \liminf_{\substack{w \to v \\ t \downarrow 0}} \frac{V(x + tw) - V(x)}{t},$$

$$\partial_{\mathrm{D}} V(x) := \{\zeta \in \mathbb{R}^n \,|\, \langle \zeta, v \rangle \le DV(x; v) \text{ for all } v \in \mathbb{R}^n\}$$

with the standard inner product $\langle \zeta, v \rangle := \sum_{i=1}^n \zeta_i v_i$ for $\zeta = (\zeta_1, \ldots, \zeta_n)$, $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ as well as the *proximal subdifferential* ([9], [10]) defined as

$$\partial_{\mathrm{P}} V(x) = \{\zeta \in \mathbb{R}^n \,|\, \exists \delta > 0 \ \exists \sigma > 0 \ \forall y \in B_\delta(x) : \\ V(x) + \langle \zeta, y - x \rangle - \sigma \|y - x\|_2^2 \le V(y)\},$$

where $\|\cdot\|_2$ denotes the Euclidean norm. For Lipschitz functions $V(\cdot)$ the Dini subderivate (sometimes called lower Dini-Hadamard derivative as in [11]) coincides with the lower Dini derivative ([10, Chap. 3, Exercise 4.1], [1, Sec. 6.1, Proposition 2]).

We will now follow the notations in [3]. We assume G is the union of the simplices of a *simplicial triangulation* $\mathcal{T} = \{T_\nu : \nu = 1, \ldots, N\}$ with $T_\nu \subset G$, $\nu = 1, \ldots, N$. Each simplex is the convex hull of $n+1$ affinely independent vectors x_i, $i = 0, \ldots, n$, and the intersection of two simplices is either empty or a common face of both simplices. Let us denote the active index set

$$I_{\mathcal{T}}(x) := \{\nu \in \{1, \ldots, N\} : x \in T_\nu\}.$$

From the values at the nodes x_i, $i = 0, \ldots, n$, for each simplex T_ν, the values of the continuous, piecewise affine function $V(\cdot)$ can be constructed via the convex combination

$$V(x) = \sum_{i=0}^n \lambda_i V(x_i) \quad (\text{for all } x \in T_\nu)$$

given by the barycentric coordinates of $x = \sum_{i=0}^n \lambda_i x_i$, i.e. $\sum_{i=0}^n \lambda_i = 1$ and $\lambda_i \ge 0$ for $i = 0, \ldots, n$. If a

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function $V : \mathbb{R}^n \to \mathbb{R}$ is continuous, piecewise affine on the triangulation $\mathcal{T}$, Clarke's subdifferential is given by

$$\partial_{\mathrm{Cl}} V(x) = \mathrm{co}\,\{\nabla V_\nu \,|\, \nu \in I_{\mathcal{T}}(x)\}\,. \tag{4}$$

We consider a compact, nonempty set $G \subset \mathbb{R}^n$ divided into M closed *subregions* $G_\mu \subset \mathbb{R}^n$ such that $\bigcup_{\mu=1,\dots,M} G_\mu = G$. The *active index set* is introduced as

$$I_G(x) := \{\mu \in \{1,\dots,M\} \,|\, x \in G_\mu\}\,.$$

Furthermore, Lipschitz continuous vector fields $f_\mu : G_\mu \to \mathbb{R}^n$, $\mu = 1,\dots,M$, are given and we consider the corresponding differential inclusion with right-hand side

$$F(x) = \mathrm{co}\,\{f_\mu(x) \,:\, \mu \in I_G(x)\}\,. \tag{5}$$

Important cases are *polytopic differential inclusions* for which all subregions G_μ coincide with the domain G and M is the maximal number of vertices of the right-hand side and *switched systems* for which the domain is partitioned into M subregions G_μ whose interiors are pairwise disjoint.

The control system is *strongly asymptotically stable*, if every solution $x(\cdot)$ of (3) is defined on $[0,\infty)$ and for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$x(0) \in B_\delta(0) \Rightarrow x(t) \in B_\varepsilon(0) \quad (\text{for all } t \geq 0) \tag{6}$$

and

$$\lim_{t\to\infty} x(t) = 0 \tag{7}$$

It is called *asymptotically controllable* resp. *weakly asymptotically stable*, if there exists a control $u(\cdot)$ with the corresponding solution $x(\cdot)$, which is defined on $[0,\infty)$ and satisfies (6) and (7).

With the help of the *upper* resp. *lower Hamiltonian* for F

$$H(x,p) := \max_{u\in U} \langle p, f(x,u)\rangle \quad (\text{for all } x,p \in \mathbb{R}^n)\,,$$
$$h(x,p) := \min_{u\in U} \langle p, f(x,u)\rangle \quad (\text{for all } x,p \in \mathbb{R}^n)$$

defined as in [10, Sec. 4.1], we can later characterize strongly and weakly asymptotic stability. Both Hamiltonians are also used in the study of strong and weak invariance (see [10, Sec. 4.2 and 4.3] and [12]). Let us first remark that the upper Hamiltonian is convex w.r.t. p, since it is a maximum of linear function, whereas the lower Hamiltonian fulfills $h(x,p) = -H(x,-p)$ and is concave w.r.t. p. Furthermore, both functions are Lipschitz, since the upper Hamiltonian equals the support function of the convex compact set $F(x)$ in direction p.

Definition 1.1: The function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *smooth (strong) Lyapunov function* for system (1)–(2), if the following conditions hold:
(i) $V(\cdot)$ is *positive definite*, i.e.

$$V(0) = 0, \quad V(x) > 0 \quad (\text{for all } x \neq 0)\,,$$

(ii) *proper*, i.e. the level sets

$$\{x \in \mathbb{R}^n \,|\, V(x) \leq \alpha\}$$

are compact for all $\alpha \in \mathbb{R}$,
(iii) $V(\cdot)$ in C^∞ and there exists a C^∞ function $W : \mathbb{R}^n \to \mathbb{R}$ which is positive definite and the *strong infinitesimal decrease* holds:

$$\langle \nabla V(x), f(x,u)\rangle \leq -W(x) \quad (\text{for all } x \neq 0,\ u \in U)\,,$$
$$\text{i.e. } H(x, \nabla V(x)) \leq -W(x) \quad (\text{for all } x \neq 0)$$

In this case, (V,W) is called *smooth (strong) Lyapunov pair*.

Definition 1.2: The function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *Lipschitz (strong) Lyapunov function in the sense of generalized gradients* for system (1)–(2), if the following conditions hold:
(i) and (ii) of Definition 1.1 hold,
(iii) $V(\cdot)$ is Lipschitz and there exists a Lipschitz function $W : \mathbb{R}^n \to \mathbb{R}$ which is positive definite and the *strong infinitesimal decrease in the sense of generalized gradients* hold for all $u \in U$:

$$\langle \zeta, f(x,u)\rangle \leq -W(x) \ (\text{for all } x \neq 0, \zeta \in \partial_{\mathrm{Cl}} V(x))$$
$$\text{i.e. } \sup_{\zeta\in\partial_{\mathrm{Cl}} V(x)} H(x,\zeta) \leq -W(x) \ (\text{for all } x \neq 0) \tag{8}$$

In this case, (V,W) is called *Lipschitz (strong) Lyapunov pair in the sense of generalized gradients*.
(V,W) is called *(strong) Lyapunov pair in the Dini sense*, if $V(\cdot)$ is continuous and (8) is replaced by the Dini subderivate

$$\sup_{u\in U} DV(x; f(x,u)) \leq -W(x) \quad (\text{for all } x \neq 0)\,. \tag{9}$$

The pair is called *(strong) Lyapunov pair in the proximal sense*, if $V(\cdot)$ is continuous and (8) is replaced with the help of the proximal subdifferential

$$\sup_{\zeta\in\partial_{\mathrm{P}} V(x)} H(x,\zeta) \leq -W(x) \quad (\text{for all } x \neq 0)\,. \tag{10}$$

By using only the definition parts with the upper Hamiltonian

$$H(x,p) = \max_{\xi\in F(x)} \langle p, \xi\rangle \quad (\text{for all } x,p \in \mathbb{R}^n)\,,$$

(strong) Lyapunov functions and pairs can be introduced for the differential inclusion (3) as well.

The existence of a smooth Lyapunov pair characterizes strongly asymptoptic stability.

Proposition 1.3 ([9, Theorem 1.2]): Let the set-valued map F have compact convex values and be upper-semicontinuous.
Then the control system is strongly asymptotically stable if and only if there exist a smooth Lyapunov pair (V,W).

Although the existence of a smooth Lyapunov function is a very strong result (see the longer discussion of the history of research in [20] on this topic and [14] for the consequences for discrete inclusions), we would like to present a nonsmooth version which will be very similar to the characterization of asymptotic controllability in the smooth case.

Proposition 1.4: The control system is strongly asymptotically stable if and only if there exists a Lipschitz strong Lyapunov pair (V,W) in the sense of generalized gradients.

The same holds for a Lipschitz strong Lyapunov pair in the Dini or proximal sense.

Proof: We will only sketch the proof: the necessity for (9) follows from [9, Proposition 4.2], the sufficiency of the condition (8) follows e.g. from [3, Theorem 3.3] (see references therein).

The necessity of the infinitesimal decrease (8) resp. (10) follows easily from the inclusions

$$\partial_{\mathrm{P}} V(x) \subset \partial_{\mathrm{D}} V(x) \subset \partial_{\mathrm{Cl}} V(x) \qquad (11)$$

in [10, Sec. 3.4] and from the convexity of the upper Hamiltonian $H(x,\cdot)$. The condition (8) also shows the sufficiency of the two other decrease conditions. ■

In [3] we used the condition (8) of infinitesimal decrease to set up a linear optimization problem for a continuous, piecewise affine function with the constraints

$$\sup_{\zeta \in \partial_{\mathrm{Cl}} V(x_i)} \widetilde{H}(x_i, \zeta) \leq -W(x_i) \quad (\text{for all } x_i \neq 0),$$

at all vertices x_i of the simplices of a triangulation $\mathcal{T}$, where $\widetilde{H}(x,p) = \max_{u \in U}(\langle p, f(x,u) + E(x,u,p)\rangle)$ is a perturbed Hamiltonian and $E(x,u,p) \geq 0$ represent interpolation errors. For continuous, piecewise affine Lyapunov functions $V(\cdot)$ and a switched system, together with suitable entities $E_{i,\mu,\nu}$ from $E(x_i,\cdot,\nabla V_\nu)$, this yields the condition

$$\max_{\nu \in I_{\mathcal{T}}(x_i)} \max_{\mu \in I_G(x_i)} [\langle \nabla V_\nu, f_\mu(x_i)\rangle + E_{i,\mu,\nu}] \leq -\|x_i\|_2 .$$

II. Control Lyapunov Functions

We start with the definition of smooth control Lyapunov functions (see e.g. [8, (5)]).

Definition 2.1: The function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *smooth control Lyapunov function (clf)* for system (1)–(2), if the following conditions hold:
(i) and (ii) of Definition 1.1 hold,
(iii) $V(\cdot)$ in C^∞ and there exists a C^∞ function $W : \mathbb{R}^n \to \mathbb{R}$ which is positive definite and there exists $u \in U$ such that the *weak infinitesimal decrease*

$$\begin{aligned} \langle \nabla V(x), f(x,u)\rangle &\leq -W(x) \quad (\text{for all } x \neq 0), \\ \text{i.e. } h(x, \nabla V(x)) &\leq -W(x) \quad (\text{for all } x \neq 0) \end{aligned}$$

holds. In this case, (V,W) is called *smooth control Lyapunov pair* resp. *smooth weak Lyapunov pair*.

Most common is the definition of a nonsmooth control Lyapunov functions, i.e. a Lipschitz function, with the help of Dini subderivates or proximal subgradients (see e.g. [8, (4) and (6)], [19]). Sometimes as in [17], the weak infinitesimal decrease is formulated with generalized gradients as well.

Definition 2.2: The function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *Lipschitz control Lyapunov function in the sense of generalized gradients* for system (1)–(2), if the following conditions hold:
(i) and (ii) of Definition 1.1 hold,
(iii) $V(\cdot)$ is Lipschitz and there exists a Lipschitz function $W : \mathbb{R}^n \to \mathbb{R}$ which is positive definite and the *weak infinitesimal decrease in the sense of generalized gradients* holds for some $u \in U$:

$$\langle \zeta, f(x,u)\rangle \leq -W(x) \ (\text{for all } x \neq 0, \zeta \in \partial_{\mathrm{Cl}} V(x)),$$

$$\text{i.e. } \sup_{\zeta \in \partial_{\mathrm{Cl}} V(x)} h(x,\zeta) \leq -W(x) \ (\text{for all } x \neq 0) \qquad (12)$$

In this case, (V,W) is called *Lipschitz control Lyapunov pair* resp. *Lipschitz weak Lyapunov pair in the sense of generalized gradients*.
(V,W) is called *control Lyapunov pair in the Dini sense*, if $V(\cdot)$ is continuous and (12) is replaced by the Dini subderivate

$$\inf_{u \in U} DV(x; f(x,u)) \leq -W(x) \quad (\text{for all } x \neq 0). \qquad (13)$$

The pair is called *control Lyapunov pair in the proximal sense*, if $V(\cdot)$ is continuous and (12) is replaced with the help of the proximal subdifferential

$$\sup_{\zeta \in \partial_{\mathrm{P}} V(x)} h(x,\zeta) \leq -W(x) \quad (\text{for all } x \neq 0). \qquad (14)$$

As before, control Lyapunov functions and pairs can be introduced for the differential inclusion (3) with the help of the lower Hamiltonian

$$h(x,p) = \min_{\xi \in F(x)} \langle p, \xi\rangle \quad (\text{for all } x, p \in \mathbb{R}^n).$$

The asymptotic controllability is equivalent to the existence of a control Lyapunov function in the Dini or proximal sense. There are numerous publications of this fact, see e.g. [18], [8], [16], [7, Theorem 4.3] and the extension to weakly uniformly globally asymptotically stable closed sets in [15]. Let us add that we cannot guarantee the equivalence of asymptotic controllability to the existence of a control Lyapunov function in the sense of generalized gradients as in the proof of Proposition 1.4, since this proof essentially uses the convexity of the upper Hamiltonian and the lower Hamiltonian is not convex in general but concave.

Proposition 2.3:
(i) The control system is asymptotically controllable if and only if there exists a control Lyapunov pair (V,W) in the Dini sense.
(ii) The same holds for a control Lyapunov pair in the proximal sense.

Proof: (i) The equivalence to the existence of a control Lyapunov function in the Dini sense is proved in the pioneering work [18, Theorem 2.5] (see also e.g. [8, Theorem 2]).
(ii) Due to [10, Chap. 4, 5.3 Proposition] both conditions (13) and (14) for infinitesimal decrease are equivalent. ■

The smoothness properties of control Lyapunov functions are an ongoing research, see e.g. [7, Sec. 5]. Important subcases are Lipschitz continuity, semiconcavity and continuous differentiability. The linear case

$$f(x,u) = Ax + Bu$$

with matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $U = \mathbb{R}^m$ as control set are an important example in which the controllability yields a smooth (even quadratic) control Lyapunov function ([7, Subsec. 1.3]).

The conditions for a control Lyapunov function to yield infinitesimal decrease are usually formulated in the Dini or proximal sense. Our numerical algorithm will use the CPA approximation so that the proximal or Dini subdifferential of such a continuous, piecewise affine function has to be calculated. As a first step, we currently use the corresponding decrease condition in the sense of generalized gradients as in [17], since the formula for Clarke's subdifferential of a continuous, piecewise affine function (stated in (4)) is simpler to formulate and calculate.

We can use the estimate $\partial_{\mathrm{P}} V(x) \subset \partial_{\mathrm{Cl}} V(x)$ (which is usually a rather rough one) and condition (12), i.e.

$$\max_{\zeta \in \partial_{\mathrm{Cl}} V(x)} \min_{u \in U} \langle \zeta, f(x,u) \rangle \leq -\|x\|_2 \quad (\text{for all } x \neq 0)\,.$$

The case of regularity of the Lyapunov function constitutes an important case in which the conditions for infinitesimal decrease in the Dini sense and in the sense of generalized gradients do not differ ([10, Sec. 1.2, 2.3, 2.6 and 3.4]). Important but rather special examples are C^1 or convex control Lyapunov functions. Note that the class of Lipschitz continuous, regular functions is also studied in [2] to establish invariance principles with a certain set-valued derivative in the context of Lyapunov functions. Let us add the important restriction that continuous, piecewise affine functions are not automatically regular functions in the sense of [10, Sec. 2.4].

We summarize this in the following remark providing special cases in which the replacement of the weak decrease condition (14) by (12) does not matter.

Remark 2.4: If the control Lyapunov function $V(\cdot)$ is regular ([10, Sec. 2.4]), i.e. the Dini subderivate $DV(x;p)$ equals the support function of the Clarke's subdifferential $\partial_{\mathrm{Cl}} V(x)$ for all directions $p \in \mathbb{R}^n$, both conditions (12) and (14) for weak infinitesimal decrease are equivalent due to (11).

The infinitesimal decrease (12) in the sense of generalized gradients is sufficient for asymptotic controllability.

Furthermore, the existence of a Lipschitz control Lyapunov pair in the sense of generalized gradients is even equivalent to the existence of a smooth control Lyapunov pair (see [17, Theorem 2.7]).

For switched systems and a continuous, piecewise affine function $V(\cdot)$ we get the estimates

$$\begin{aligned} & \max_{\nu \in I_{\mathcal{T}}(x)} \min_{\mu \in I_G(x)} \langle \nabla V_\nu, f_\mu(x) \rangle \\ = & \max_{\nu \in I_{\mathcal{T}}(x)} \min_{u \in U} \langle \nabla V_\nu, f(x,u) \rangle \leq -\|x\|_2 \quad (\text{for all } x \neq 0) \end{aligned}$$

With the lower Hamiltonian this condition can be reduced to

$$\max_{\nu \in I_{\mathcal{T}}(x)} h(x, \nabla V_\nu) \leq -\|x\|_2 \quad (\text{for all } x \neq 0)\,.$$

For each vertex x_i from the simplex T we introduce the constraints

$$\max_{\nu \in I_{\mathcal{T}}(x_i)} \min_{\mu \in I_G(x_i)} \left[\langle \nabla V_\nu, f_\mu(x_i) \rangle + E_{i,\mu,\nu} \right] \leq -\|x_i\|_2 \quad (15)$$

where the $E_{i,\mu,\nu}$ are adequate positive entities from Section III, for the calculation of a control Lyapunov function which is a continuous, piecewise affine function.

III. Approach with Mixed Integer Programming

First numerical calculations of control Lyapunov functions can be found in [4], [5], [6] based on Zubov's method which requires the solution of a Hamilton-Jacobi partial differential equation. Here, we would like to introduce a method which is based on linear optimization with mixed integer variables.

In [3] a linear programming problem to compute CPA (strong) Lyapunov functions in the sense of generalized gradients for strongly asymptotically stable differential inclusions was proposed. In this paper we adapt this algorithm to compute CPA control Lyapunov functions, also in the sense of generalized gradients, for asymptotically controllable systems. Our new approach uses the solution to a mixed integer linear programming (MILP) problem to achieve this.

In both cases the following proposition, that can be proved identically to [3, Theorem 4.5], is of essential importance:

Proposition 3.1: Let f_μ be a C^2 function on $T_\nu = \mathrm{co}\,\{x_0, x_1, \ldots, x_n\} \subset \mathbb{R}^n$ and let $V : T_\nu \to \mathbb{R}$ be an affine function. Denote by ∇V_ν the gradient of V on T_ν, i.e. $V(x) = \langle \nabla V_\nu, x \rangle + a_\nu$ for an $a_\nu \in \mathbb{R}$ and all $x \in T_\nu$. Then:

(i) If $V(x_i) \geq \|x_i\|_2$ for $i = 0, 1, \ldots, n$, then $V(x) \geq \|x\|_2$ for all $x \in T_\nu$.

(ii) Let $\mathrm{co}\,\{x_{i_0}, x_{i_1}, \ldots, x_{i_k}\}$, $0 \leq k \leq n$, be a face of T_ν. If for all $j = 0, \ldots, k$ we have

$$\langle \nabla V_\nu, f_\mu(x_{i_j}) \rangle + n B_\nu^\mu h_\nu^2 \|\nabla V_\nu\|_1 \leq -\|x_{i_j}\|_2, \quad (16)$$

where $h_\nu := \mathrm{diam}(T_\nu) = \max_{i,j=1,\ldots,n} \|x_i - x_j\|_2$, and B_ν^μ is an upper bound on the second order derivatives of the scalar components $f_{\mu,j}$ of f_μ on T_ν, i.e.

$$B_\nu^\mu \geq \max_{r,s,j=1,\ldots,n} \max_{x \in T_\nu} \left| \frac{\partial^2 f_{\mu,j}}{\partial x_r \partial x_s}(x) \right|,$$

then

$$\langle \nabla V_\nu, f_\mu(x) \rangle \leq -\|x\|_2$$

for all $x \in \mathrm{co}\,\{x_{i_0}, x_{i_1}, \ldots, x_{i_k}\}$.

□

For the MILP problem proposed below, it is important to note that because $V : T_\nu \to \mathbb{R}$ is affine, the vector

$$\nabla V_\nu = \left((x_1 - x_0, \ldots, x_n - x_0)^\top \right)^{-1} \begin{pmatrix} V(x_1) - V(x_0) \\ \ldots \\ V(x_n) - V(x_0) \end{pmatrix}$$

is linear in the values of V at the vertices of T_ν (see [3]) and that by introducing the auxiliary variables $C_{\nu,i}$, $i = 1, \ldots, n$, (16) can be implemented by the the linear constraints

$$-C_{\nu,i} \leq \nabla V_{\nu,i} \leq C_{\nu,i}, \quad i = 1, \ldots, n, \quad (17)$$

where $\nabla V_{\nu,i}$ is the i-th component of ∇V_ν, and

$$\langle \nabla V_\nu, f_\mu(x_{i_j}) \rangle + n B_\nu^\mu h_\nu^2 \sum_{i=1}^n C_{\nu,i} \leq -\|x_{i_j}\|_2, \; j = 0, \ldots, k.$$

We propose a MILP problem to compute a CPA control Lyapunov function for asymptotically controllable systems. Thus, we compute a CPA function $V : G \to \mathbb{R}$ such that

$$\max_{\nu \in I_{\mathcal{T}}(x)} \min_{\mu \in I_G(x)} \langle \nabla V_\nu, f_\mu(x) \rangle \leq -\|x\|_2. \quad (18)$$

As simplifying assumptions in this section assume that

– G_μ is a union of simplices in $\mathcal{T}$, (19)

– $G = \bigcup_{\mu=1,\dots,M} G_\mu = \bigcup_{\nu=1,\dots,N} T_\nu$, (20)

– the vector fields $f_\mu : G_\mu \to \mathbb{R}^n$ in (5) are in C^2 (21)

for all $\mu = 1, \dots, M$.

To ensure that $V(x) \geq \|x\|_2$ for all $x \in G$, we include the constraints $V(x_i) \geq \|x_i\|_2$ for all vertices x_i of all simplices T_ν of $\mathcal{T}$, in our MILP problem. This is identical to the approach in [3], where additionally the constraints

$$\langle \nabla V_\nu, f_\mu(x_i)\rangle + nB_\nu^\mu h_\nu^2 \|\nabla V_\nu\|_1 \leq -\|x_i\|_2$$

for every simplex $T_\nu = \mathrm{co}\{x_0, \dots, x_n\} \in \mathcal{T}$, every vertex x_i of T_ν, and every $\mu \in I_G(x_i)$, yields

$$\max_{\nu \in I_\mathcal{T}(x)} \max_{\mu \in I_G(x)} \langle \nabla V_\nu, f_\mu(x)\rangle \leq -\|x\|_2$$

for all $x \in G$.

To enforce (18) for all $x \in G$, it suffices by Proposition 3.1, if for every $T_\nu = \mathrm{co}\{x_0, \dots, x_n\} \in \mathcal{T}$ there is a μ^* with $T_\nu \subset G_{\mu^*}$, such that for every $i = 0, \dots, n$ and every $\tau \in I_\mathcal{T}(x_i)$ we have

$$\langle \nabla V_\tau, f_{\mu^*}(x_i)\rangle + nB_\tau^{\mu^*} h_\tau^2 \|\nabla V_\tau\|_1 \leq -\|x_i\|_2 \tag{22}$$

(cf. the proof of Proposition 3.3 below). Our control strategy is to use a control that is constant on every simplex of the triangulation.

To implement this let us first define the set

$$S_\nu := \{\mu \in \{1, \dots, M\} : T_\nu \subset G_\mu\} \tag{23}$$

for every simplex $T_\nu \in \mathcal{T}$. Thus $\mu \in S_\nu$ if and only if T_ν is contained in the domain of f_μ. Now for every $T_\nu \in \mathcal{T}$ introduce the binary variables $z_{\nu,\mu} \in \{0,1\}$ for all $\mu \in S_\nu$. Consider the constraints

$$z_{\nu,\mu} \cdot \big(\langle \nabla V_\tau, f_\mu(x_i)\rangle + nB_\tau^\mu h_\tau^2 \|\nabla V_\tau\|_1 + \|x_i\|_2\big) \leq 0 \tag{24}$$

for all $T_\nu = \mathrm{co}\{x_0, \dots, x_n\} \in \mathcal{T}$, all $\mu \in S_\nu$, all $i = 0, \dots, n$, and all $\tau \in I_\mathcal{T}(x_i)$, together with

$$\sum_{\mu \in S_\nu} z_{\nu,\mu} = 1 \tag{25}$$

for all $T_\nu \in \mathcal{T}$. These constraints imply (22) because for every $T_\nu \in \mathcal{T}$ there is a $\mu^* \in S_\nu$ such that $z_{\nu,\mu^*} = 1$.

The constraints (24), however, are not linear. We replace (24) by the equivalent *linear* constraints

$$\begin{aligned} &\langle \nabla V_\tau, f_\mu(x_i)\rangle \\ &+ nB_\tau^\mu h_\tau^2 \|\nabla V_\tau\|_1 + z_{\nu,\mu} \cdot (\|x_i\|_2 + K) \leq K, \end{aligned} \tag{26}$$

where $K > 0$ is a constant so large, that the case $z_{\nu,\mu} = 0$ is not limiting, i.e. the constraints

$$\langle \nabla V_\tau, f_\mu(x_i)\rangle + nB_\tau^\mu h_\tau^2 \|\nabla V_\tau\|_1 \leq K \tag{27}$$

can be satisfied for all $\mu \in S_\nu$ and all $\tau \in I_\mathcal{T}(x_i)$.

Note that (24) can always be replaced by (26) using a large enough K and by choosing a priori a common upper bound $\overline{C}$ in (17) and assuming solvability of (24), K is bounded in terms of $\overline{C}$.

Because of (25) there is for every $T_\nu \in \mathcal{T}$ a $\mu^* \in S_\nu$ such that $z_{\nu,\mu^*} = 1$, and then

$$\langle \nabla V_\tau, f_{\mu^*}(x_i)\rangle + nB_\tau^{\mu^*} h_\tau^2 \|\nabla V_\tau\|_1 + \|x_i\|_2 + K \leq K$$

is equivalent to (22). If $z_{\nu,\mu} = 0$, then (26) reduces to the condition (27) on K.

We now propose our algorithm and then prove that its result yields a parameterization of a continuous, piecewise affine function $V : G \to \mathbb{R}$ that is a control Lyapunov function for the differential inclusion used in the construction of the MILP problem of the algorithm.

Algorithm 3.2: Consider the system (3) with $F(x)$ as in (5) and assume that the conditions (19)–(20) hold.
Then, we solve the following MILP feasibility problem:
The variables of the MILP problem are:

- $V(x_i) \in \mathbb{R}$ for every vertex x_i of every simplex $T_\nu = \mathrm{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T}$,
- $z_{\nu,\mu} \in \{0,1\}$ for every $T_\nu \in \mathcal{T}$ and every $\mu \in S_\nu$

The linear constraints of the MILP feasibility problem are:

(C1) We set $V(0) = 0$. For every vertex x_i of every simplex $T_\nu = \mathrm{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T}$:

$$V(x_i) \geq \|x_i\|_2$$

(C2) For every $T_\nu \in \mathcal{T}$:

$$\sum_{\mu \in S_\nu} z_{\nu,\mu} = 1$$

(C3) For every vertex x_i of every simplex $T_\nu = \mathrm{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T}$, $x_i \neq 0$, and every $\tau \in I_\mathcal{T}(x_i)$ and every $\mu \in S_\nu$:

$$\begin{aligned} &\langle \nabla V_\tau, f_\mu(x_i)\rangle + nB_\tau^\mu h_\tau^2 \|\nabla V_\tau\|_1 \\ &\qquad + z_{\nu,\mu} \cdot (\|x_i\|_2 + K) \leq K \end{aligned} \tag{28}$$

Here, the following constants of the MILP feasibility problem are used: the values B_ν^μ are chosen as in Proposition 3.1 for every $T_\nu \in \mathcal{T}$ and every μ such that $T_\nu \subset G_\mu$ as well as a large positive constant K satisfying (27).

If the MILP problem possesses a feasible solution, then a value $V(x_i)$ has been computed for every vertex x_i of every simplex $T_\nu \in \mathcal{T}$. By abuse of notation we use these values to parameterize a continuous function $V : G \to \mathbb{R}$ that is affine on every simplex $T_\nu \in \mathcal{T}$. We do this by fixing the value of the function V at the vertex x_i to the value of the variable $V(x_i)$ from the feasible solution to the MILP problem.

□

As in [3] we need, in the general case, to exclude an arbitrary neighbourhood of the origin from the constraints (C3), i.e. we do not demand infinitesmal decrease of the clf close to the origin. Because of this the essential implication of the constraints (C1) on the clf V is that it must have a minimum at the origin. We refer to [3] for details.

Let us discuss the constraints (28) in (C3) with Figure 1. In this 2d example, the two subregions G_1 and G_2 coincide and

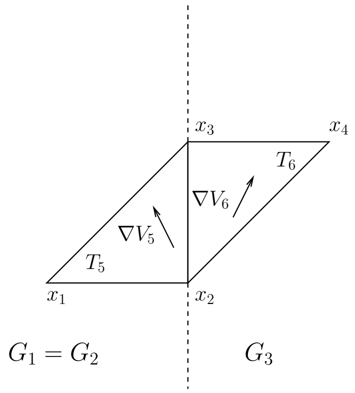

Fig. 1. Gradient conditions (C3) on two adjacent triangles.

have a common border with G_3. In $G_1 = G_2$ the dynamics is given by $f_1(\cdot)$ and $f_2(\cdot)$, while $f_3(\cdot)$ describes it on G_3. For simplification assume that $B_5^1 = B_5^2 = B_6^3 = 0$. The triangle $T_5 = \operatorname{co}\{x_1, x_2, x_3\}$ belongs to the triangulation of $G_1 = G_2$, while the triangle $T_6 = \operatorname{co}\{x_2, x_3, x_4\}$ belongs to the triangulation of G_3. As required, the intersection $T_5 \cap T_6$ is the common face $\operatorname{co}\{x_2, x_3\}$. We have $\{5\} \subset I_{\mathcal{T}}(x_1)$, $\{5, 6\} \subset I_{\mathcal{T}}(x_2)$, $\{5, 6\} \subset I_{\mathcal{T}}(x_3)$ and $S_5 = \{1, 2\}$, $S_6 = \{3\}$. We must set $z_{6,3} = 1$, which implies

$$\langle \nabla V_6, f_3(x_i) \rangle \leq -\|x_i\|_2, \quad i = 2, 3, 4$$

and, because of the maximum in (18), we also need

$$\langle \nabla V_5, f_3(x_i) \rangle \leq -\|x_i\|_2, \quad i = 2, 3.$$

We can either set $z_{5,1} = 1, z_{5,2} = 0$ or $z_{5,1} = 0, z_{5,2} = 1$. This corresponds to the minimum in (18). The first choice delivers the constraints

$$\langle \nabla V_5, f_1(x_i) \rangle \leq -\|x_i\|_2, \quad i = 1, 2, 3$$

and

$$\langle \nabla V_6, f_1(x_i) \rangle \leq -\|x_i\|_2, \quad i = 2, 3,$$

while the second choice delivers

$$\langle \nabla V_5, f_2(x_i) \rangle \leq -\|x_i\|_2, \quad i = 1, 2, 3$$

and

$$\langle \nabla V_6, f_2(x_i) \rangle \leq -\|x_i\|_2, \quad i = 2, 3.$$

We now state our main proposition.

Proposition 3.3: If the mixed integer problem in Algorithm 3.2 has a feasible solution, the continuous, piecewise affine function $V : G \to \mathbb{R}$ from Algorithm 3.2 fulfills both

$$V(x) \geq \|x\|_2, \tag{29}$$

$$\max_{\substack{\nu \in I_{\mathcal{T}}(x) \\ 0 \notin T_\nu}} \min_{\mu \in I_G(x)} \langle \nabla V_\nu, f_\mu(x) \rangle \leq -\|x\|_2 \tag{30}$$

for all $x \in G$.

Moreover, if we define the sets $G_\mu^* \subset G_\mu$ through $G_\mu^* := \bigcup_{z_{\nu,\mu}=1} T_\nu$ and the active index set

$$I_{G^*}(x) := \{\mu \in \{1, \ldots, M\} \,:\, x \in G_\mu^*\}, \tag{31}$$

then

$$\max_{\substack{\nu \in I_{\mathcal{T}}(x) \\ 0 \notin T_\nu}} \max_{\mu^* \in I_{G^*}(x)} \langle \nabla V_\nu, f_{\mu^*}(x) \rangle \leq -\|x\|_2. \tag{32}$$

Proof: That $V(x) \geq \|x\|_2$ for all $x \in G$ was shown in Proposition 3.1(i). To show (30) and (32) fix $T_\nu = \operatorname{co}\{x_0, x_1, \ldots, x_n\} \in \mathcal{T}$, $0 \notin T_\nu$, and let $\mu^* \in S_\nu$ be such that $z_{\nu,\mu^*} = 1$. Since $\nu \in I_{\mathcal{T}}(x_i)$ for all vertices x_i the constraints (28) and Proposition 3.1 deliver

$$\langle \nabla V_\nu, f_{\mu^*}(x) \rangle \leq -\|x\|_2$$

for all $x \in T_\nu$. Now let $T_\tau \in \mathcal{T}$ be such that

$$T_\tau \cap T_\nu = \operatorname{co}\{x_{i_0}, x_{i_1}, \ldots, x_{i_k}\},$$

$0 \leq k < n$, is a face of T_ν. Then $\tau \in I_{\mathcal{T}}(x_{i_j})$ for $j = 0, \ldots, k$ and the constraints (28) ensure

$$\langle \nabla V_\tau, f_{\mu^*}(x_{i_j}) \rangle + n B_\tau^{\mu^*} h_\tau^2 \|\nabla V_\tau\|_1 \leq -\|x_{i_j}\|_2$$

for all $j = 0, \ldots, k$. Thus, by Proposition 3.1 we also have

$$\langle \nabla V_\tau, f_{\mu^*}(x) \rangle \leq -\|x\|_2$$

for all $x \in \operatorname{co}\{x_{i_0}, x_{i_1}, \ldots, x_{i_k}\}$. Since this holds true for all $T_\nu \in \mathcal{T}$, $0 \notin T_\nu$, and all faces of T_ν that are intersections of T_ν with other simplices in the triangulation $\mathcal{T}$, (30) and (32) follow for all $x \in G$. ■

Since we restrict the infinitesimal decrease condition (32) to $L := \{T_\nu \,:\, 0 \notin T_\nu\}$ we must take care that sublevel sets of V are neighbourhoods of L. This can be done by verifying $\max_{x \in G \setminus L} V(x) < \min_{x \in \partial G} V(x)$ a posteriori or by including constraints in the MILP that enforce this inequality.

Note that if $G \subset \mathbb{R}^n$ is a neighborhood of the origin, Proposition 3.3 implies that if the MILP problem in Algorithm 3.2 has a feasible solution, then $(V, \|\cdot\|_2)$ is a weak Lyapunov pair for the system (5), i.e.

$$\dot{x}(t) \in \operatorname{co}\{f_\mu(x) \,:\, \mu \in I_G(x)\},$$

and a strong Lyapunov pair for a kind of a closed-loop system

$$\dot{x}(t) \in \operatorname{co}\{f_\mu(x) \,:\, \mu \in I_{G^*}(x)\}. \tag{33}$$

Thus, apart from computing a CPA control Lyapunov function V for the original system, the MILP problem computes a controlling strategy through the sets G_μ^*.

IV. Numerical Example

As an example of the application of Algorithm 3.2 we consider the control system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= u(t)|x_2(t)| - x_1(t), \\ |u(t)| &\leq 4 \end{aligned}$$

in [7, Example in Subsec. 8.1] and the associated differential inclusion

$$\dot{x}(t) \in \operatorname{co}\{f_\mu(x) \,:\, \mu \in I_G(x)\}, \tag{34}$$

where f_1 and f_2 are given by the formulas

$$f_1(x) := \begin{pmatrix} x_2 \\ 4|x_2| - x_1 \end{pmatrix} \text{ and } \tag{35}$$

$$f_2(x) := \begin{pmatrix} x_2 \\ -4|x_2| - x_1 \end{pmatrix} \text{ for all } x = (x_1, x_2) \in \mathbb{R}^2. \tag{36}$$

We consider G as polytope with 104 vertices that approxi-

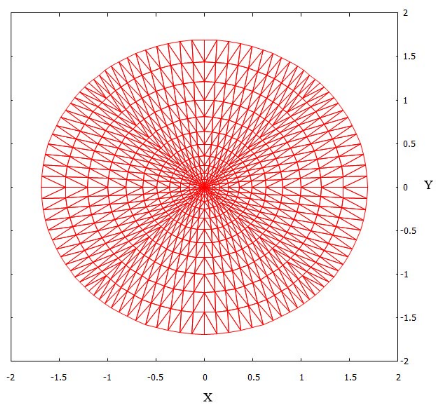

Fig. 2. The triangulation $\mathcal{T}$ used in our example.

mates a circular disc with the origin as center and the radius 1.7 and set $G_1 = G_2 = G$ for the subdomains. The chosen triangulation with

$$G = \bigcup_{T_\nu \in \mathcal{T}} T_\nu$$

is depicted in Figure 2. As shown in [7, Example in Subsec. 8.1], (34) is asymptotically controllable with

$$V(x) = 9x_1^2 + 4x_1x_2 + x_2^2$$

as a smooth control Lyapunov function.

First, we apply the MILP problem from Algorithm 3.2 to compute a control Lyapunov function for (34) together with the sets G_1^* and G_2^*. By Proposition 2.3(iv) and due to the weak decrease condition

$$\min_{u \in U} \langle \nabla V(x), f(x,u) \rangle$$
$$\leq \langle \nabla V(x), f(x, -4\,\mathrm{sgn}(x_2)) \rangle \leq -4\|x\|_2^2$$

there exists a clf in the sense of generalized gradients. As the problem is a feasibility problem its objective is free. We set it to minimize the following objective function

$$\max_{T_\nu \in \mathcal{T}} \|\nabla V_\nu\|_\infty\,. \tag{37}$$

The constants B_ν^μ can be set equal to zero for all μ and ν, since the interior of a triangle does not hit the x_1-axis and each f_μ is C^2 on all triangles, and we set $K := 1000$.

For solving the MILP problem, we used the Gurobi Optimizer, Version 5.6 from [13] (free for academic use). Although the triangulation is rather fine and the number of variables and constraints are considerably high, the Gurobi Optimizer needed only 31 sec. to solve the problem. The

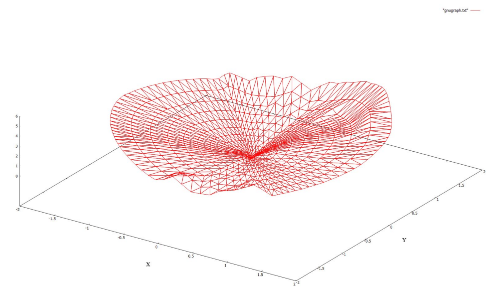

Fig. 3. The computed CPA control Lyapunov function for the system (34).

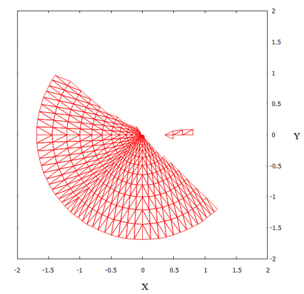

Fig. 4. The computed domain G_1^* for f_1 in the system (34).

objective value for the solution was $\max_{T_\nu \in \mathcal{T}} \|\nabla V_\nu\|_\infty = 4.414$.

The computed CPA control Lyapunov function $V_1(\cdot)$ is depicted in Figure 3. The sets G_1^* and G_2^* identifying a possible control strategy for $V_1(\cdot)$ are shown in Figures 4 and 5. Note that $V_1(\cdot)$ is also a strong Lyapunov function for the closed-loop system

$$\dot{x}(t) \in \mathrm{co}\,\{f_\mu(x) \,:\, \mu \in I_{G^*}(x)\}.$$

To control this result, we follow another strategy by using the feedback $k(x) = -4\,\mathrm{sgn}(x_2)$ for $x = (x_1, x_2)$ from [7, Example in Subsec. 8.1] to define subregions

$$G_1' := \{x = (x_1, x_2) \in G \,:\, x_2 \leq 0\},$$
$$G_2' := \{x = (x_1, x_2) \in G \,:\, x_2 \geq 0\}$$

of the polytope G in such a way that the active index set

$$I_{G'}(x) := \{\mu \in \{1, 2\} \,:\, x \in G_\mu'\}$$

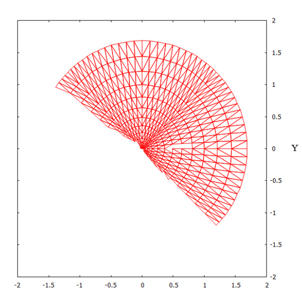

Fig. 5. The computed domain G_2^* for f_2 in the system (34).

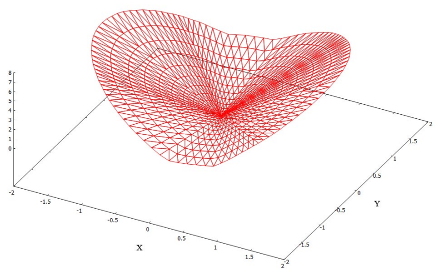

Fig. 6. Computed strong CPA Lyapunov function for (38). It is also a control Lyapunov function for (34).

defines a closed-loop system

$$\dot{x}(t) \in \mathrm{co}\,\{f_\mu(x) \,:\, \mu \in I_{G'}(x)\}. \tag{38}$$

Due to the partitioning in these subregions motivated by the feedback, $V(\cdot)$ is also a strong Lyapunov function for (38).

The linear programming problem from [3] is solved to compute a strong CPA Lyapunov function for (38). Again the objective of the problem can be chosen freely, so we use the same objective function (37) as before. The computation time was 0.23 sec. on a modern PC. The computed CPA (strong) Lyapunov function $V_2(\cdot)$ in Figure 6 differs from the CPA control Lyapunov function in Figure 3. Nevertheless, the objective function had exactly the same value 4.414 as in the computation of the CPA control Lyapunov function.

The sets G_1' and G_2' from the feedback law are however different to the sets G_1^* and G_2^*, cf. Figures 4 and 5 which originates from the different calculation of a CPA control resp. strong Lyapunov function in both approaches (cf. Figures 3 and 6).

V. Conclusions

Control Lyapunov functions are usually difficult to obtain analytically. Hence, our approach presented in Algorithm 3.2 is a first step towards the numerical calculation of a control Lyapunov function via the approximation with continuous, piecewise affine functions. Since the mixed integer problem is based on the weak infinitesimal decrease in the sense of generalized gradients, it has the important restriction that it can only have a feasible solution if the control system possesses a smooth control Lyapunov function. Our computed Lyapunov function is then also a clf in the Dini and proximal sense. The same holds if the clf is regular in the sense of Proposition 2.3(iii). The aim of future research will be the direct use of the weak infinitesimal decrease in the Dini or proximal sense which should enter as constraints in the mixed integer linear programming problem.

Further, two more issues must be addressed if the proposed method is to be generally applicable. The first issue is the generation of an appropriate triangulation for the problem at hand. In the case of strong asymptotic stability this can achieved be algorithmically generate a sequence of increasingly refined triangulations fulfilling certain properties [3]. Weak asymptotic stability is somewhat more involved, but essentially a triangulation with enough structure to support an appropriate feedback control as well as the structure of an associated clf should do the job. Note that on each of the simplices of our triangulation the control and the gradient of the clf are constant.

The second issue is the numerical complexity of the method. Generally MILPs are NP-hard and are usually solved by using advanced heuristic methods like the cutting-plane method combined with branch-and-bound. It remains a challenging task to study for which classes of control systems our method can compute an appropriate control in a reasonable time or if the method can be improved to reduce its computational complexity.

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