

Optimal control of the convergence time in the Hegselmann–Krause dynamics

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Abstract

We study the optimal control problem of minimizing the convergence time in the discrete Hegselmann–Krause model of opinion dynamics. The underlying model is extended with a set of strategic agents that can freely place their opinion at every time step. Indeed, if suitably coordinated, the strategic agents can significantly lower the convergence time of an instance of the Hegselmann–Krause model. We give several lower and upper worst-case bounds for the convergence time of a Hegselmann–Krause system with a given number of strategic agents, while still leaving some gaps for future research.

1 Introduction

The dynamics of opinion formation using agent-based models has been studied for more than half a century, see e.g. [12] for a (partial) overview of different models. Here we consider one specific model, which generated at lot of research papers in the social simulation community, the so called bounded confidence model also known as Hegselmann–Krause model. As originally defined in [14], see also [12], the Hegselmann–Krause model considers discrete time and a finite set of agents with opinions in \mathbb{R}^d (or \mathbb{R} as a special case). At each time step the opinion of every agent is updated by averaging certain other opinions. However, an agent is influenced only by those opinions that are *near* to his or her own opinion. We will give a precise specification of the model in Section 2.

One key issue of a model for opinion dynamics is the question of convergence, see e.g. [3, 13, 18, 22, 17]. Here by convergence we mean a final stable state, where the agent’s opinions remain the same in all subsequent time steps. Some authors also say that the system is *in equilibrium* or has *frozen*. If convergence is guaranteed, the next question is about the necessary number of time steps to reach this state. For the Hegselmann–Krause model the convergence time has been upper bounded by $n^{O(n)}$ in [5]. The author’s conjecture of a polynomial convergence time was, to the best of our knowledge, first proven for the special case of dimension $d = 1$ in [19]. Their upper bound $O(n^5)$ was subsequently improved to $O(n^4)$ in [23] and to $O(n^3)$ in [2, 21]. A polynomial upper bound for a general dimension d was presented in [2]. For dimension $d = 1$ a $\Omega(n)$ worst-case lower bound was given in [19]. This was improved recently to $\Omega(n^2)$ in [10]. For dimension $d = 2$ an example yielding a lower bound of $\Omega(n^2)$ was also given in [2]. Some first few exact values of the worst-case convergence time in dimension $d = 1$ were determined in [16] using integer linear programming techniques.

Recently researchers started to look at the problem of controlling or steering an instance of an opinion dynamics system towards a desired state, see [24, 1, 11, 4]. Exogenous interventions into the systems dynamics were e.g. studied in [20, 7, 6]. The *desired state* in [11] is a *conviction interval*, i.e., a

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subset of the opinion space. The practical problem that one can have in mind is that of a political or commercial campaign. Via media channels, speeches or personal communications the opinion dynamics can be influenced. Formally such a controllable exogenous influence was modeled by introducing *strategic* agents in [11]. Here we will use strategic agents to accelerate the convergence time in the Hegselmann–Krause model, i.e., we study the optimal control problem minimizing the convergence time.

The remaining part of the paper is organized as follows. In Section 2 we formally introduce the Hegselmann–Krause dynamics with strategic agents and collect some theoretical observations about the system dynamics. The optimal control problem is studied in Section 3, i.e., we give several lower and upper worst-case bounds for the convergence time of a Hegselmann–Krause system with a given number of strategic agents. We draw a conclusion and propose some future lines of research in Section 4.

2 The Hegselmann–Krause dynamics with strategic agents

As originally defined, the Hegselmann–Krause model considers a one-dimensional continuous opinion space \mathbb{R}^1 , a set $N = \{1, \dots, n\}$ of agents – these are the non-strategic agents – and discrete time. At each time $t \in \mathbb{N}$ each agent $i \in N$ has a certain opinion $x_i(t) \in \mathbb{R}$. The opinion of an agent $i \in N$ is influenced at time t only by those agents which have a similar opinion, more precisely, where the distance of the respective opinions is at most ε . In our rescaled version of an opinion space \mathbb{R} we can assume w.l.o.g. $\varepsilon = 1$. The new opinion $x_i(t+1)$ of agent i is then determined as the mean of all opinions² that influence agent i at time t .

Here we want to extend the model by strategic agents. To this end we denote the set of strategic agents by S and the set of all agents by $A = N \cup S$. The opinion $x_s(t)$ of a strategic agent $s \in S$ can be freely chosen at each time $t \in \mathbb{N}$ as any real number. However, determining the new opinion at time $t+1$, the non-strategic agents make no difference by whom they are influenced. To be more precise, we set

$$x_i(t+1) = \frac{\sum_{j \in A: \|x_i(t) - x_j(t)\| \leq 1} x_j(t)}{|\{j \in A : \|x_i(t) - x_j(t)\| \leq 1\}|} \quad (1)$$

for all $i \in N$. Here, $\|\cdot\|$ denotes the Euclidean norm (or the absolute value, as it makes no difference in dimension 1). As a model extension we may also use Equation (1) for strategic agents at time steps where they do not freely reset their opinion. However, in this paper we assume that each strategic agent freely chooses an opinion at every time step.

We say that two agents $i, j \in A$ are *neighbors* at time t if $\|x_i(t) - x_j(t)\| \leq 1$, which induces an *influence graph* \mathcal{G}_t . For brevity, we denote the *neighborhood* of an agent $i \in A$ at time t by $\mathcal{N}_i(t) = \{j \in A : \|x_i(t) - x_j(t)\| \leq 1\}$, i.e., the set of neighbors. With this, Equation (1) can be rewritten to $x_i(t+1) = \sum_{j \in \mathcal{N}_i(t)} x_j(t) / |\mathcal{N}_i(t)|$.

The Hegselmann–Krause dynamics, with or without strategic agents, is rather complicated and hard to treat analytically, i.e., besides convergence not too many theoretical results are known. For the special case of dimension 1 at the very least the ordering of the opinions of the agents is preserved, as observed in several papers, see e.g. [15, Lemma 2]:

Proposition 1 *If $x_i(0) \leq x_j(0)$ for $i, j \in N$, then $x_i(t) \leq x_j(t)$ for all $t \in \mathbb{N}$.*

This well known observation is clearly also true, if strategic agents are present. If i or j is a strategic agent, then we can not have such a result, since, by definition, strategic agents can choose their opinion freely. W.l.o.g. we assume $x_i(0) \leq x_j(0)$ for all $i \leq j$, $i, j \in N$ in the remaining part of the paper.

For dimension 1 and the absence of strategic agents there is another well known theoretical insight. If two agents i and j are, at a certain time step t , in two different connectivity components of \mathcal{G}_t , then this property will be preserved for all times $t' > t$. As a consequence, if $\|x_{i+1}(t) - x_i(t)\| > 1$, then $\|x_{i+1}(t') - x_i(t')\| > 1$ for all $t' > t$ (using the assumed ordering of the starting opinions). However, this needs not to be true if at least one strategic agent is present. It can indeed be easily shown, see e.g.

¹In some papers the unit interval $[0, 1]$ or other intervals of the real line are used. For a finite number of agents those variants are equivalent via scaling.

²Note that every agent influences itself.

[11], that one strategic agent suffices to bring any configuration of starting positions to a consensus in a finite number of time steps, i.e., the opinions of all non-strategic agents coincide after some rounds.

For the other direction we remark that it is almost always possible, given a suitably large number of strategic agents and time steps, to move the opinions of any two non-strategic agents as far apart from each other as desired. If albeit $x_i(t) = x_j(t)$, where $i, j \in N$, at a certain time t , then we have $x_i(t') = x_j(t')$ for all $t' > t$, independently of the precise opinions of the strategic agents.

Definition 2 For each $x \in \mathbb{R}$ and each $t \in \mathbb{N}$ we define the weight of x at t as $w_t(x) = |\{i \in N : x_i(t) = x\}|$. With this, the weight of agent $i \in N$ at t is defined as $w_t(x_i(t))$.

Proposition 3 We have $0 \leq w_t(x) \leq n$ and $w_t(x) \in \mathbb{N}$ for all $x \in \mathbb{R}$, $t \in \mathbb{N}$. For all $t \in \mathbb{N}$ and all $i \in N$ we have $w_t(x_i(t)) \leq w_{t+1}(x_i(t))$, i.e., the weight of a non-strategic agent is weakly increasing.

If we use the strategic agents to control the system dynamics, we can ensure that some properties will be preserved if we choose the opinions of the strategic agents accordingly. To this end we denote by $\mathcal{N}'_i(t) = \{j \in N : \|x_i(t) - x_j(t)\| \leq 1\}$ the set of non-strategic neighbors of a non-strategic agent i . The graph arising from \mathcal{G}_t by restricting the vertex set to non-strategic agents is denoted by \mathcal{G}'_t . With this we can state:

Definition 4 We call a non-strategic agent $i \in N$ frozen if $|\mathcal{N}'_i(t)| = w_t(x_i(t))$. For the vertex set \mathcal{C} of a connectivity component of \mathcal{G}'_t we denote by $l(\mathcal{C}, t)$ the agent with the smallest index in \mathcal{C} and by $r(\mathcal{C}, t)$ the agent with the largest index in \mathcal{C} . The width $w(\mathcal{C}, t)$ of \mathcal{C} at t is given by $x_{r(\mathcal{C}, t)}(t) - x_{l(\mathcal{C}, t)}(t) \in \mathbb{R}_{\geq 0}$.

Due to our assumed ordering of the starting opinions, we have $x_{l(\mathcal{C}, t)}(t) \leq x_i(t) \leq x_{r(\mathcal{C}, t)}(t)$ for all $i \in \mathcal{C}$, i.e., $l(\mathcal{C}, t)$ and $r(\mathcal{C}, t)$ mark the ends of the interval of connectivity component \mathcal{C} and $w(\mathcal{C}, t)$ denotes the corresponding length. We remark that $w(\mathcal{C}, t) = 0$ if and only if all non-strategic agents of \mathcal{C} are frozen at time t .

Proposition 5 Given two non-strategic agents $i, j \in N$ and a time $t \in \mathbb{N}$ we have $x_i(t+1) = x_j(t+1)$ if and only if $\mathcal{N}_i(t) = \mathcal{N}_j(t)$.

If the influence graph \mathcal{G}'_t at time t decomposes into the connectivity components $\mathcal{C}_1, \dots, \mathcal{C}_h$, then we denote the width of the entire configuration, i.e., the sum over the widths of the connectivity components, by $w(\mathcal{G}'_t, t) = \sum_{i=1}^h w(\mathcal{C}_i, t)$.

The width of a connectivity component can only increase if a strategic agents is placed behind the ends of the corresponding interval but still within the influence range.

Proposition 6 Let \mathcal{C} be a connectivity component of the influence graph \mathcal{G}'_t at a certain time $t \in \mathbb{N}$. If we have $x_a(t) \geq x_{l(\mathcal{C}, t)}(t)$ and $x_a(t) \leq x_{r(\mathcal{C}, t)}(t)$ for all $i \in \mathcal{C}$ and all $a \in \mathcal{N}_i(t)$, then we have $x_{l(\mathcal{C}, t)}(t) \leq x_j(t+1) \leq x_{r(\mathcal{C}, t)}(t)$ for all $j \in \mathcal{C}$.

So, one can place the opinions of the strategic agents such that the width of the entire configuration does not increase.

Since the range of influence is at most 1, each non-strategic agent can not move too far within one time step.

Proposition 7 For each $i \in N$ and each $t \in \mathbb{N}$ we have $\|x_i(t) - x_i(t+1)\| \leq 1$.

We say that the system has converged at time t if all non-strategic agents are frozen, i.e., if we have for every $i, j \in N$ either $x_i(t) = x_j(t)$ or $\|x_i(t) - x_j(t)\| > 1$. Without the influence of any strategic agents, we may simply assume that they all place their opinions far apart or at one of the positions $x_i(t)$, the opinions of the non-strategic agents will remain unchanged, i.e., stable for all times $t' > t$. However, the strategic agents may convert frozen agents to non-frozen ones. This can make sense, if one wants to end up with a consensus. Nevertheless, we define the convergence time of a Hegselmann–Krause system (HK system for short) as the smallest time t such that the system has converged at time t . Obviously this definition of convergence time depends on the starting opinions $x_i(0)$ of the non-strategic agents and all opinions $x_j(t)$, or rules to compute them, of the strategic agents. As already observed in the

introduction, the convergence time of a HK system with n non-strategic agents and no strategic agent is upper bounded by $O(n^3)$, while the slowest known sequence of examples reaches $\Omega(n^2)$. In the following section we will consider the optimization problem of lowering the convergence time using strategic agents, i.e., we consider an optimal control problem.

3 Lowering the convergence time using strategic agents

Given a HK system we ask how to optimally coordinate the opinions of the $m = |S|$ strategic agents so that the convergence time is minimized, i.e., we consider an optimal control problem. We call the corresponding time the *optimal convergence time*, given the starting positions of the non-strategic agents and the number m of strategic agents. Mostly we will be interested in general assertions than considerations of specific examples. So, by $f(n, m)$ we denote the supremum of the optimal convergence time over all HK instances with n non-strategic agents and m strategic agents. In this notation our current knowledge on the convergence time can be written as $f(n, 0) \in O(n^3)$ and $f(n, 0) \in \Omega(n^2)$. Obviously, we have $f(n, m) \leq f(n', m)$ and $f(n, m) \geq f(n, m')$ for all $n' \geq n$ and all $m' \geq m$.

We start by considering the case of $m = 1$ strategic agent and mimic the proof strategy for the best known upper bound of the convergence time from [2].

Lemma 8 *Given a HK system with $m = 1$ strategic agent, the opinion of this agent at time t can be chosen in such a way such that either*

- (1) *the weight of a non-strategic agent increases and the width $w(\mathcal{G}'_t, t)$ does not increase or*
- (2) *the width $w(\mathcal{G}'_t, t)$ decreases by at least $\frac{1}{n+1}$*

at time $t + 1$, whenever the HK system has not converged at time t .

PROOF. Let us denote the connectivity components of \mathcal{G}'_t by $\mathcal{C}_1, \dots, \mathcal{C}_h$. If there exists an index $1 \leq g \leq h$ with $0 < w(\mathcal{C}_g, t) \leq 1$, then we can place the opinion of the strategic agent far away from the other opinions so that no agent is influenced. Since $w(\mathcal{C}_g, t) \leq 1$ we have $\|x_i(t) - x_j(t)\| \leq 1$, so that $x_i(t+1) = x_j(t+1)$ for all $i, j \in \mathcal{C}_g$. As $w(\mathcal{C}_g, t) > 0$ the weights of the agents in \mathcal{C}_g increase by at least 1 each in this case. Obviously, the width of the influence graph does not increase.

Otherwise we choose an index $1 \leq g \leq h$ with $w(\mathcal{C}_g, t) > 1$ and set $x_s(t) = x_{l(\mathcal{C}_g, t)}(t) + 1$, where s denotes the strategic agent. With this we have

$$x_{l(\mathcal{C}_g, t)}(t+1) = \frac{1}{|\mathcal{C}_g| + 1} \cdot \left(x_s(t) + \sum_{i \in \mathcal{C}_g} x_i(t) \right) \geq x_{l(\mathcal{C}_g, t)}(t) + \frac{1}{n+1},$$

since $x_i(t) \geq x_{l(\mathcal{C}_g, t)}(t)$ for all $i \in \mathcal{C}_g$ and $|\mathcal{C}_g| \leq n$. Using Proposition 6 we conclude $w(\mathcal{C}_g, t+1) \leq w(\mathcal{C}_g, t) - \frac{1}{n+1}$. The widths of the other connectivity components do not increase. It may happen that a certain connectivity component, e.g. \mathcal{C}_g itself, decomposes into several components in the considered time step, i.e., \mathcal{C}_g may not be a connectivity component at time $t+1$. Nevertheless, the summed widths of the respective components is not larger than it was originally, while in one component of \mathcal{C}_g a contraction of at least $\frac{1}{n+1}$ occurs. Thus, we have $w(\mathcal{G}'_{t+1}, t+1) \leq w(\mathcal{G}'_t, t) - \frac{1}{n+1}$. \square

Corollary 9 $f(n, 1) \in O(n^2)$.

PROOF. Since the weight of every agent is a non-negative integer being at most equal to n and it is weakly increasing over time, Case (1) of Lemma 8 can occur at most n^2 times. (A more refined analysis would yield that we need to consider this case at most $n - 1$ times.) Since the maximum difference between two neighbored agents is 1, we have $w(\mathcal{G}'_0, 0) \leq n - 1$. Thus, Case (2) of Lemma 8 can occur at most $(n - 1) \cdot (n + 1) \leq n^2$ times. \square

Using one strategic agent the upper bound for the (optimal) convergence time could be improved from $O(n^3)$ to $O(n^2)$, but still we do not know whether the strategic agent was necessary, since $f(n, 0) \in O(n^2)$ may also be true. To see, that already one strategic agent can significantly decrease the (optimal) convergence time we consider a specific parametric example:

Lemma 10 For $n \in \mathbb{N}$ consider the HK system with starting opinions $x_i(0) = i - 1$ for all $1 \leq i \leq n$, i.e., the equidistant configuration. The convergence time of this example is in $\Omega(n)$ while its optimal convergence time for $m = 1$ strategic agent is in $O(n^{3/4})$.

PROOF. Via induction one can easily show that $x_i(t) = i - 1$ for all $t + 1 \leq i \leq n - t$, i.e., the movement starts from the two ends of the chain of neighbors and concerns two additional agents at each time step. Thus the convergence time is at least $\frac{n-2}{2}$. Actually, the convergence time of the equidistant configuration is given by $\frac{5n}{6} + O(1)$. An exact formula was stated in [16] without a proof and rigorously proven in [9].

For the other direction we assume w.l.o.g. $n \geq 3^4 = 81$ and set $k = \lfloor n^{1/4} \rfloor$. At the beginning all n non-strategic agents are in a single connectivity component and each neighbor of an agent is distance 1 apart. We inductively will cut off groups of k agents.

At step 1 there are n remaining agents with distances $1, \dots, 1$. Let x_1 be the position of the opinion of the k th non-strategic agent at time 0. Setting the opinion of the strategic agent s to $x_1 - 1$ at time 0, the first k non-strategic agents are separated from the last $n - k$ ones at time 1. For the remaining steps we ignore the first k agents.

At time 1 the distances between the remaining $n - k$ agents are given by $1, \dots, 1, \frac{1}{2}$. For step 2 we consider the opinion x_2 of the k th non-strategic agent counted from the right at time 1. Setting the opinion of the strategic agent to $x_2 + 1$ at time 1, the last k non-strategic agents are separated from the first $n - 2k$ ones at time 2. For the remaining steps we ignore the last k agents.

At time 2 we have to consider $n - 2k$ agents with distances $\frac{1}{2}, 1, \dots, 1$. So, up to symmetry we are in the same situation as in the previous step. We iterate the separation process using the strategic agent until we end up with at most $\lceil n^{3/4} \rceil + 2$ groups of cardinality at most k after at most $\lceil n^{3/4} \rceil + 2$ time steps.

The dynamics of each of these groups can be considered separately. After the separation process has finished we place the opinion of the strategic agent far away from all other opinions so that it does not play a role. Since any HK system with at most k non-strategic agents converges in $O(k^3)$ time, the overall convergence time is in $O(n^{3/4})$. \square

Next we consider the example from [10] yielding the quadratic lower bound $f(n, 0) \in \Omega(n^2)$:

Lemma 11 For an integer $k \geq 10$ we consider the so-called dumbbell configuration with $3k + 1$ non-strategic agents, where

- (1) k agents have starting opinion $-\frac{1}{k}$,
- (2) one item has starting opinion i for each $0 \leq i \leq k$, and
- (3) k agents have starting opinion $k + \frac{1}{k}$.

The convergence time of this example is in $\Omega(n^2)$ while its optimal convergence time for $m = 1$ strategic agent is in $O(n^{3/4})$.

PROOF. The convergence time of the dumbbell configuration has been already treated in [10], so that we only consider the optimal convergence time.

At time $t = 0$ we place the strategic agent at 2. Using the usual ordered numbering of the non-strategic agents, we obtain

- $x_i(1) = -\frac{1}{k+1}$ for all $1 \leq i \leq k$, $x_{k+1}(1) = 0$,
- $x_{k+2}(1) = \frac{5}{4}$, $x_{k+3}(1) = 2$, $x_{k+4}(1) = \frac{11}{4}$,
- $x_{k+1+i}(1) = i$ for all $4 \leq i \leq k$, and $x_j(1) = k + \frac{1}{k+1}$ for all $2k + 2 \leq j \leq 3k + 1$.

At time $t = 1$ we place the strategic agent at $k - 2$. With this, we obtain

- $x_i(2) = -\frac{k}{(k+1)^2}$ for all $1 \leq i \leq k + 1$,

- $x_{k+2}(2) = \frac{13}{8}$, $x_{k+3}(2) = 2$, $x_{k+4}(2) = \frac{19}{8}$,
- $x_{k+5}(2) = \frac{9}{2}$, $x_{k+1+i} = i$ for all $5 \leq i \leq k-4$,
- $x_{2k-2}(2) = k - \frac{11}{4}$, $x_{2k-1}(2) = k-2$, $x_{2k}(2) = k - \frac{5}{4}$,
- $x_{2k+1}(2) = k - \frac{1}{(k+1)(k+2)}$, and $x_{2k+1+i}(2) = k + \frac{k}{(k+1)^2}$ for all $1 \leq i \leq k$.

At time $t = 3$ the agents $k+2$, $k+3$, and $k+4$ will converge to the joint opinion 2. Similarly, at time $t = 4$ the agents $2k-2$, $2k-1$, and $2k$ will converge to the joint opinion $k-2$. The last $k+1$ agents will converge to a joint opinion at time $t = 3$. The remaining agents from $k+5$ to $2k-3$ form a chain of (almost) equal distances, i.e., with a single exception all distances are equal to 1, while the exceptional distance is equal to $\frac{1}{2}$ – a case that has already been considered in the proof of Lemma 10. So, reusing the corresponding reasoning, we conclude a convergence time of $O(n^{3/4})$. \square

So, we have seen that the control of a single strategic agent can accelerate the convergence time of the equidistant configuration by at least $\Omega(n^{1/4})$. Improving the upper bound for the convergence time $f(n, 0)$, i.e., in the absence of strategic agents, would increase this gap even more. Besides a tighter analysis, an improved strategy for the strategic agent is also conceivable. For the dumbbell configuration the demonstrated acceleration is at least of order $\Omega(n^{5/4})$. However, the *power* of a single strategic agent alone is limited, as we will see in the next two results.

Lemma 12 *For each integer $k \geq 21$ consider the HK system given by*

- k^2 non-strategic agents with starting opinion $-\frac{2}{3}$,
- k non-strategic agents with starting opinion 0,
- k^2 non-strategic agents with starting opinion $\frac{2}{3}$, and
- $m = 1$ strategic agent.

Let $x_1(t)$ denote the opinion of the agents in (a), $x_2(t)$ denote the opinion of the agents in (b), and $x_3(t)$ denote the opinion of the agents in (c) at time $t \in \mathbb{N}$. Then, for all $0 \leq t \leq \frac{k}{8}$ we have

- $-\frac{2}{3} - \frac{t}{k} \leq x_1(t) \leq -\frac{2}{3} + \frac{t}{k}$,
- $\frac{2}{3} - \frac{t}{k} \leq x_3(t) \leq \frac{2}{3} + \frac{t}{k}$,
- $-\left(\frac{t}{k} + \frac{t^2}{k^2}\right) \leq x_2(t) \leq \left(\frac{t}{k} + \frac{t^2}{k^2}\right)$,
- $x_2(t) - x_1(t) \leq 1 - \frac{1}{k}$,
- $x_3(t) - x_2(t) \leq 1 - \frac{1}{k}$, and
- $x_3(t) - x_1(t) > 1$

independently of the precise opinions of the strategic agent.

PROOF. We remark that inequalities (4)-(6) say that the agents of type (a) influence agents of types (a) and (b), agents of type (b) influence all non-strategic agents, and agents of type (c) influence agents of types (b) and (c). The strategic agent may influence agents of none, some or all types.

We prove by induction on t and observe that for $t = 0$ all statements are valid. Now let $t \in \mathbb{N}$, with $t \leq \frac{k}{8} - 1$, and we assume that all six inequalities are valid for t .

Since the strategic agent can pull with a force of at most 1, we have

$$x_1(t+1) \geq x_1(t) + \frac{-1 + k^2 \cdot 0 + k \cdot 0}{1 + k^2 + k} \geq x_1(t) - \frac{1}{k} \stackrel{(1)}{\geq} -\frac{2}{3} - \frac{t+1}{k}.$$

Using Inequality (4) at time t we conclude

$$x_1(t+1) \leq x_1(t) + \frac{1 + k^2 \cdot 0 + k \cdot \left(1 - \frac{1}{k}\right)}{1 + k^2 + k} \stackrel{(1)}{\leq} x_1(t) + \frac{1}{k} \stackrel{(1)}{\leq} -\frac{2}{3} + \frac{t+1}{k}.$$

So, Inequality (1) is also valid at time $t+1$. The validity of Inequality (2) is proven analogously.

In order to prove Inequality (3) we ask how left the opinions of agents of type (b) can be at time $t+1$. The extreme situation occurs if the strategic agent pulls with force 1 to the left and the agents of types (a), (b), and (c) are located as left as possible. So, we obtain

$$\begin{aligned} x_2(t+1) &\geq \frac{-\left(1 + \frac{t}{k} + \frac{t^2}{k^2}\right) + k^2 \cdot \left(-\frac{2}{3} - \frac{t}{k}\right) + k \cdot \left(0 - \frac{t}{k} - \frac{t^2}{k^2}\right) + k^2 \cdot \left(\frac{2}{3} - \frac{t}{k}\right)}{1 + k^2 + k + k^2} \\ &= -\frac{2tk + t + 1}{2k^2 + k + 1} - \frac{\frac{t}{k} + \frac{t^2}{k} + \frac{t^2}{k^2}}{2k^2 + k + 1} \\ &\geq -\frac{t+1}{k} - \frac{(t+1)^2}{k^2}. \end{aligned}$$

Analogously, we conclude $x_2(t+1) \leq \frac{t+1}{k} + \frac{(t+1)^2}{k^2}$ so that Inequality (3) is also valid at time $t+1$.

Setting $t' = t+1$ and using inequalities (1) and (3) at time t' , we have $x_2(t') - x_1(t') \leq \frac{2}{3} + \frac{2t'}{k} + \frac{t'^2}{k^2}$. For all $0 \leq t' \leq -k + \frac{1}{3} \cdot \sqrt{12k^2 - 9k}$ the right hand side is at most $1 - \frac{1}{k}$. Since $k \geq 15$ and $t' \leq \frac{k}{8}$ we have $t' \leq \frac{k}{8} \leq -k + \frac{1}{3} \cdot \sqrt{12k^2 - 9k}$, so that Inequality (4) is valid for all $t \leq \frac{k}{8} - 1$. Analogously, we conclude the validity of Inequality (5).

Setting $t' = t+1$ and using inequalities (1) and (2), we have $x_3(t') - x_1(t') \geq \frac{4}{3} - \frac{2t'}{k} \geq \frac{13}{12} > 1$, so that Inequality (6) is valid for all $t \leq \frac{k}{8}$. \square

Corollary 13 $f(n, 1) \in \Omega(\sqrt{n})$.

We remark that the lower bound from Corollary 13 can be increased to $f(n, 1) \in \Omega(n^{2/3})$, see Lemma 18. However, the result from Lemma 12 has the advantage, that it uses a connected starting configuration (instead of many separated connectivity components) and does not rely on the convergence analysis of the dumbbell configuration.

If the number of strategic agents is sufficiently increased, with respect to the number of non-strategic agents, then the optimal convergence time can be decreased down to a constant.

Theorem 14 For each $n \in \mathbb{N}$ we have $f(n, 9n) \leq 2$ and $f(n, 9n) \in \Theta(1)$.

PROOF. At first we determine a set of positions where we place the opinions of the strategic agents in the first round. To this end, let us denote the connectivity components of \mathcal{G}'_0 by $\mathcal{C}_1, \dots, \mathcal{C}_h$. For each index $1 \leq i \leq h$ we proceed as follows: We set $w_i = \lceil w(\mathcal{C}_i) \rceil$, i.e., w_i is a non-negative integer such that $w(\mathcal{C}_i) \in (w_i - 1, w_i]$. For each connectivity component \mathcal{C}_i we will use k_i out of the $9n$ strategic agents in the first round.³

If $w_i \leq 1$ we place strategic agents for \mathcal{C}_i at $k_i = 0$, i.e., no, positions. To ease the subsequent notation we set $p_0^i = \frac{1}{2} \cdot (x_{l(\mathcal{C}_i, 0)}(0) + x_{r(\mathcal{C}_i, 0)}(0))$, i.e., we choose the center of the corresponding interval of opinions.

Now assume $w_i \geq 2$. If w_i is even, we place strategic agents at the $k_i = w_i/2$ positions $p_j^i = x_{l(\mathcal{C}_i, 0)}(0) + 1 + (2 + \varepsilon_i) \cdot j$, where $0 \leq j < k_i$, $j \in \mathbb{N}$. Here we choose $\varepsilon_i > 0$ suitably small such that $p_{k_i-1}^i \leq x_{r(\mathcal{C}_i, 0)}(0)$ and the open intervals $(p_j^i + 1, p_{j+1}^i - 1)$ do not contain opinions of non-strategic agents at time 0 for $0 \leq j < k_i - 1$. If w_i is odd, we place strategic agents at the $k_i = (w_i + 1)/2$ positions $p_j^i = x_{l(\mathcal{C}_i, 0)}(0) + (2 + \varepsilon_i) \cdot j$, where $0 \leq j < k_i$. Here we again choose ε_i suitably small such that $p_{k_i-1}^i \leq x_{r(\mathcal{C}_i, 0)}(0)$ and the open intervals $(p_j^i + 1, p_{j+1}^i - 1)$ do not contain opinions of non-strategic agents at time 0 for $0 \leq j < k_i - 1$.

With this we have $x_{l(\mathcal{C}_i, 0)}(0) \leq p_j^i \leq x_{r(\mathcal{C}_i, 0)}(0)$ for all $0 \leq j < k_i$, i.e., strategic agents at these positions influence only agents from \mathcal{C}_i . Further more, if $k_i > 0$, each agent in \mathcal{C}_i is influenced by

³Note that an illustrative example will be given directly after Lemma 15.

strategic agents from exactly one position p_j^i . It remains to determine the number of strategic agents that should be placed at position p_j^i .

Let b be the number of non-strategic agents with a starting opinion which is at most 1 apart from p_j^i . By a we denote the number of non-strategic agents with starting opinion in $\left[p_j^i - 2, p_j^i - 1\right)$. Similarly, by c we denote the number of non-strategic agents with starting opinion in $\left(p_j^i + 1, p_j^i + 2\right]$. We place exactly $3 \cdot (a + b + c)$ non-strategic agents at position p_j^i at time 0. Let g be an arbitrary non-strategic agent with starting opinion in $\left[p_j^i - 1, p_j^i\right]$ and $\delta = p_j^i - x_g(0)$. With this we have

$$x_g(1) \geq \frac{a_1 \cdot (p_j^i - 1 - \delta) + b_1 \cdot (p_j^i - \delta) + 3 \cdot (a + b + c) \cdot p_j^i}{a_1 + b_1 + 3 \cdot (a + b + c)},$$

where $a_1 \leq a$ and $b_1 \leq b$. Since $\delta \leq 1$ and $a_1 + b_1 \leq a + b + c$, we have

$$x_g(1) \geq p_j^i - \frac{2 \cdot (a_1 + b_1)}{(a_1 + b_1) + 3(a + b + c)} \geq p_j^i - \frac{1}{2}.$$

For the other direction we have

$$x_g(1) \leq \frac{1 \cdot (p_j^i - \delta) + 3 \cdot (a + b + c) \cdot p_j^i + b_2 \cdot (p_j^i + 1 - \delta) + c_1 \cdot (p_j^i + 2 - \delta)}{1 + 3 \cdot (a + b + c) + b_2 + c_1},$$

where $b_2 \leq b - 1$ and $c_1 \leq c$. Since $\delta \geq 0$ and $b_2 + c_1 + 1 \leq a + b + c$, we have

$$x_g(1) \leq p_j^i + \frac{2 \cdot (b_2 + c_1 + 1)}{(b_2 + c_1 + 1) + 3 \cdot (a + b + c)} \leq p_j^i + \frac{1}{2}.$$

Similarly, we conclude $p_j^i - \frac{1}{2} \leq x_g(1) \leq p_j^i + \frac{1}{2}$ for every non-strategic agent g with starting opinion in $\left[p_j^i, p_j^i + 1\right]$.

If $k_i = 0$, i.e., when we place no strategic agents for \mathcal{C}_i , then we also have $p_0^i - \frac{1}{2} \leq x_g(1) \leq p_0^i + \frac{1}{2}$ for every non-strategic agent $g \in \mathcal{C}_i$.

When determining the parameters a , b , and c for the strategic agents at the positions p_j^i every non-strategic agent is counted at most thrice, so that the number of placed strategic agents at time 0 is at most $3 \cdot 3 \cdot n = 9n$. We place the remaining strategic agents, if there are any, far away from all other opinions, so that they do not influence non-strategic agents.

At time 1 the connectivity graph \mathcal{G}'_1 consists of unions of complete graphs, i.e., every non-strategic agent i and every $j \in \mathcal{N}'_i(1)$ we have $\mathcal{N}'_i(1) = \mathcal{N}_i(1)$, since the opinions of the non-strategic agents are clustered into intervals of length at most 1 at time 1.

No influence of strategic agents is needed to cause convergence at time 2, so that we place the $9n$ strategic agents far away from the opinions of the non-strategic agents. Thus, we have $f(n, 9n) \leq 2$. For $n \geq 2$ we can consider the example with starting positions given by i for all $0 \leq i \leq n - 1$, which has not converged at time 0, so that $f(n, 9n) \geq 1$ for $n \geq 2$ and $f(n, 9n) \in \Theta(1)$. \square

We remark that essentially we have used the capabilities of external control in the proof of Theorem 14 in a single round only. Theorem 14 states that $9n$ strategic agents are sufficient to control any HK system in such a way that it converges in a constant number of time steps. For $n \geq 5$ the convergence is, in general, as fast as possible, i.e. more strategic agents will not help to accelerate the convergence.

Lemma 15 *For $n \geq 5$ consider the example where the starting opinions of the non-strategic agents are given by $x_1(0) = 0$, $x_2(0) = \frac{1}{4}$, $x_3(0) = \frac{2}{3}$, $x_4(0) = \frac{5}{4}$, $x_5(0) = \frac{4}{3}$, and $x_i(0) = 6$ for $6 \leq i \leq n$. Independently from the number m of strategic agents and their precise positions, the optimal convergence time for this example is at least 2.*

PROOF. Since $x_i(0) \in \left[0, \frac{4}{3}\right]$ for all $i \in \{1, 2, 3, 4, 5\}$, we can use Proposition 7 to conclude $x_i(1) \in \left[-1, \frac{7}{3}\right]$ for all $i \leq 5$ and $x_i(1) \geq 5$ for all $i > 5$. Due to Proposition 5 all non-strategic agents $i \in \{1, 2, 3, 4, 5\}$ have different opinions at time 1. Thus, by the pigeonhole principle, there exist two non-strategic agents $i, j \in \{1, 2, 3, 4, 5\}$ with $0 < \|x_i(1) - x_j(1)\| \leq 1$, i.e., the system has not converged at time 1. \square

In order to illustrate the proof of Theorem 14, we consider an example: Assume starting positions given by the vector $x(0) = (0, 1.5, 2.4, 3.6, 4.5, 4.8, 5.1, 5.6, 5.9, 6.5, 6.7, 10, 10.7, 11.5, 11.9, 12.4, 13.3, 14.2)$, i.e., we have $n = 18$ non-strategic agents and $m = 162$ strategic agents. \mathcal{G}'_0 consists of four connectivity components with vertex sets $\{1\}$, $\{2, 3\}$, $\{4, \dots, 11\}$, and $\{12, \dots, 18\}$. The corresponding integers w_i are given by $w_1 = 0$, $w_2 = 1$, $w_3 = 4$, and $w_4 = 5$. According to the strategy proposed in the proof of Theorem 14, we set $p_0^1 = 0$, $k_1 = 0$, $p_0^2 = 1.95$, $k_2 = 0$, $k_3 = 2$, and $k_4 = 3$. Since $w_3 = 4$ is even we choose $p_0^3 = 4.6$ and $p_1^3 = 6.6 + \varepsilon_3$. Here, $\varepsilon_3 > 0$ has to be chosen such that the open interval $(p_0^3 + 1, p_1^3 - 1) = (5.6, 5.6 + \varepsilon_3)$ is free of non-strategic agents, i.e., we have to guarantee $\varepsilon_3 \leq 0.3$. The second condition $p_1^3 \leq x_{12}(0)$ yields $\varepsilon_3 \leq 0.1$, so that we choose $\varepsilon_3 = 0.1$. Since $w_4 = 5$ is odd, we choose $p_0^4 = 10$, $p_1^4 = 12 + \varepsilon_4$, and $p_2^4 = 14 + 2\varepsilon$. Here the open intervals $(11, 11 + \varepsilon_4)$ and $(13 + \varepsilon_4, 13 + 2\varepsilon_4)$ have to be free of non-strategic agents, i.e., $\varepsilon_4 \leq 0.15$ is sufficient. The condition $p_2^4 \leq x_{18}(0)$ on the other hand requires $\varepsilon_4 \leq 0.1$, so that we choose $\varepsilon_4 = 0.1$. Since $w_1, w_2 \leq 1$, we place no strategic agents at p_0^1 and p_0^2 . For p_0^3 , there are $a = 0$ non-strategic agents with an opinion at time 0 in $[p_0^3 - 2, p_0^3 - 1) = [2.6, 3.6)$, $b = 5$ agents in $[p_0^3 - 1, p_0^3 + 1] = [3.6, 5.6]$, and $c = 2$ agents in $(p_0^3 + 1, p_0^3 + 2] = (5.6, 6.6]$. Thus, we place $3 \cdot (a + b + c) = 21$ strategic agents at $p_0^3 = 4.6$. By the analogous consideration we determine that we have to place 18 strategic agents at $p_1^3 = 6.7$, 12 agents at $p_0^4 = 10$, 15 agents at $p_1^4 = 12.1$, and 9 agents at $p_2^4 = 14.2$. Having placed 75 out of the 162 strategic agents so far, we have to place the remaining 87 far away from all non-strategic agents, e.g., at position 20.7. With this, the opinions of the non-strategic agents at time 1 are given by $x(1) \approx (0, 1.95, 1.95, 4.5522, 4.584, 4.664, 4.7115, 4.7885, 6.5391, 6.6045, 6.6524, 10.05, 10.1467, 12, 12.0722, 12.1368, 13.975, 14.1182)$. At time 1 we have 7 connectivity components, each of these having a width of at most 1. Placing the strategic agents far away, e.g., at 20.7, the HK system converges to $x(2) \approx (0, 1.95, 1.95, 4.66, 4.66, 4.66, 4.66, 6.5987, 6.5987, 6.5987, 10.0983, 10.0983, 12.0697, 12.0697, 12.0697, 14.0466, 14.0466)$ at time 2.

Lemma 16 *For each $c_1, c_2 \in \mathbb{R}_{>0}$ and each $\alpha \in [0, 1)$ we have $f(n, c_1 \cdot n^\alpha) \geq c_2$ for all sufficiently large $n \in \mathbb{N}$.*

PROOF. We consider the HK system with $\lfloor \frac{n}{k} \rfloor$ equidistant configurations consisting of k non-strategic agents, where $k = 2c_2 + 2$, each. The remaining $n - \lfloor \frac{n}{k} \rfloor \cdot k \geq 0$ non-strategic agents are placed far away from the equidistant configurations. If one of connectivity components of \mathcal{G}'_0 , corresponding to an equidistant configurations, is never influenced by any strategic agent, then it takes at least c_2 time steps till convergence. Up to time $c_2 - 1$ at most $c_2 \cdot c_1 \cdot n^\alpha$ connectivity components of \mathcal{G}'_0 could be influenced by at least one strategic agent. Thus, for n sufficiently large, c_2 time steps are not enough so that all initial connectivity components can be affected at least once. \square

So far we have seen, that using a single strategic agent the optimal convergence time is upper bounded by $O(n^2)$ and using $9n$ strategic agents the optimal convergence time is upper bounded by the constant 2. Next we deal with the case in between these two extremes.

Theorem 17 *For each $\alpha \in [0, 1]$ we have $f(n, n^\alpha + 12) \in O(n^{2-2\alpha})$.*

PROOF. As usual, we assume that the non-strategic agents are ordered with respect to their starting opinions, i.e., $x_1(0) \leq \dots \leq x_n(0)$. In a first step we use our strategic agents to influence each non-strategic agent at most once in order to guarantee that after this step the connectivity components have either a *low width* or a *high density*, i.e., many members compared to the corresponding width. We will formulate our reasoning in the style of an algorithm, which we then analyze later on.

Step 1:

- (1.1) $t \leftarrow 1, h \leftarrow 1, u \leftarrow n^\alpha + 12$
- (1.2) let \mathcal{C} be the connectivity component in \mathcal{G}'_t with $h \in \mathcal{C}$
- (1.3) if $x_h(t) + 4 \geq x_{r(\mathcal{C}, t)}(t)$ then
 - $h \leftarrow 1 + \max\{i : i \in \mathcal{C}\}$
 - if $h > n$ then STOP else go to (1.2) end if
- end if

- (1.4) $k \leftarrow |\{i \in \mathcal{C} : x_h(t) \leq x_i(t) \leq x_h(t) + 4\}|$
- (1.5) if $k > (n^\alpha + 12)/12$ then
 if $x_{h+1}(t) + 4 \leq x_{r(\mathcal{C},t)}(t)$ then
 $h \leftarrow h + 1$, go to (1.4)
 else
 $h \leftarrow 1 + \max\{i : i \in \mathcal{C}\}$
 if $h > n$ then STOP else go to (1.2) end if
 end if
end if
- (1.6) choose $a, b \in \mathcal{C}$ with $x_h(t) + 1 \leq x_a(t), x_b(t) \leq x_h(t) + 3$ and $x_a(t) < x_b(t)$ such that $x_a(t) = x_{b-1}(t)$
- (1.7) if $u < 6k$ then
 $t \leftarrow t + 1$
 $u \leftarrow n^\alpha + 12$
 go to (1.3)
end if
- (1.8) place $3k$ strategic agents at $x_a(t) - 1$ and $x_b(t) + 1$ each
- (1.9) $u \leftarrow u - 6k$
- (1.10) if $x_b(t) + 4 \geq x_{r(\mathcal{C},t)}(t)$ then
 $h \leftarrow 1 + \max\{i : i \in \mathcal{C}\}$
 if $h > n$ then STOP else go to (1.2) end if
else
 $h \leftarrow \min\{i \in \mathcal{C} : x_i(t) > x_b(t) + 4\}$
 go to (1.3)
end if

By t we denote the current time, by h we denote the index of a non-strategic agent that we do not have considered before, and by u we denote the number of strategic agents that we can still use at time t . In (1.1) we initialize these three variables, i.e., we start at time $t = 1$ with the first non-strategic agent $h = 1$ and still can use all $m = n^\alpha + 12$ strategic agents.

By \mathcal{C} we denote the connectivity component of agent h at time t . Whenever we change h to a non-strategic agent outside of \mathcal{C} we go to (1.2) in the remaining steps of the algorithm.

In (1.3) we check whether the so far unconsidered part of \mathcal{C} is *rather short*. If this is the case, we increase h to a non-strategic agent of the *next* connectivity component. Here $h > n$ is used as a stopping criterion.

Reaching (1.4), we can assume $x_h(t) + 4 \leq x_{r(\mathcal{C},t)}(t)$ and denote the number of non-strategic agents with an opinion at time t from the interval $[x_h(t), x_h(t) + 4]$ by k .

In (1.5) we check whether there are *many*, compared to the number of strategic agents, non-strategic agents with an opinion in the mentioned sub interval of the opinion space of length 4. If this is the case, we increase the index h .

Reaching (1.6), we can assume $k \leq \overbrace{(n^\alpha + 12)/12}^{\geq 1}$ and choose two non-strategic agents $a, b \in \mathcal{C}$ satisfying certain technical constraints. The existence of $a, b \in \mathcal{C}$ can be concluded from the fact that the non-strategic agents of \mathcal{C} are connected, so that the interval $[x_h(t) + 1, x_h(t) + 3]$ of length 2 contains at least two different opinions of non-strategic agents at time t .

In (1.7) it is checked if we can still use $6k$ strategic agents at time t . If not, the time is increased to the next time step.

Reaching (1.8), we can assume that we can still use $6k$ strategic agents at time t . (1.8) describes the precise placement of the strategic agents and (1.9) performs the bookkeeping of the number of used strategic agents.

In (1.10) we ensure that any subsequent placement of strategic agents in (1.8) will not interfere with non-strategic agents treated so far.

After a finite number of time steps the algorithm of Step (1) stops with $h > n$. There exists exactly one place in the algorithm, where t is increased by one, i.e., sub step (1.7). Since here the condition $u < 6k$

is satisfied, at least $(t - 1) \cdot \frac{n^\alpha}{2}$ strategic agents have been placed in total. In (1.4) and (1.8) we ensure that each 6 placed strategic agents correspond to a non-strategic agents without double counting. Thus, the algorithm of Step (1) finishes after $O(n^{1-\alpha})$ time steps.

In (1.8) strategic agents are used to split connectivity components. To be precise, we have

$$x_a(t+1) \leq \frac{3k \cdot (x_a(t) - 1) + 1 \cdot x_a(t) + k_1 \cdot (x_a(t) + 1)}{3k + 1 + k_1} \leq x_a(t) - \frac{1}{2},$$

where $k_1 \leq k - 1$. Similarly, we deduce $x_b(t+1) \geq x_b(t) + \frac{1}{2}$, i.e., $x_b(t+1) - x_a(t+1) > 1$. We remark that any placement of strategic agents in the subsequent operations does not clash with our previous placement of strategic agents, even if they are performed at the same time t . To be more precise, if two strategic agents are placed at the same time within the range $[x_{l(\mathcal{C},t)}(t), x_{r(\mathcal{C},t)}(t)]$ of the same connectivity component, then their distance is either zero or at least 2. If two strategic agents are placed at the same time in two different connectivity components, then there do not exist non-strategic agents which are influenced by both strategic agents, since the opinions of the two strategic agents are within the ranges of the respective connectivity components.

There are exactly two reason why the algorithm of Step (1) does not further split connectivity components. Either they have a *low width* or a *high density*. If the condition in (1.5) is satisfied at time t , then at time $t + 2$ agent h is contained in a connectivity component with density, i.e., number of non-strategic agent in the connectivity component divided by its length, at least $\frac{n^\alpha}{144}$. Note that the first and the last sub interval of length 4 might not have a high density, while the remaining part, i.e., those agents where the condition in (1.5) applies, must indeed have a high density, since otherwise the connectivity component would have been partitioned into several connectivity components by the algorithm. It is easy to check that the remaining connectivity components have a width of at most 8 after Step (1) has been completed.

In Step (2) we will get rid of the *small* connectivity components with a *low density*. We treat each non-strategic agent at most once starting from $h = 1$. The time for Step (2) starts one time step after the completion of Step (1).

Step 2: Let \mathcal{C} be the connectivity component of our current agent h at the current time t . If $w(\mathcal{C}, t) = 0$ we do nothing for h , i.e., we consider the next non-strategic agent. If $0 < w(\mathcal{C}, t) \leq 1$ we again do nothing and have $w(\mathcal{C}, t + 1) = 0$ at time $t + 1$. If $1 < w(\mathcal{C}, t) \leq 8$ and $|\mathcal{C}| \leq \frac{n^\alpha}{2}$, we place $|\mathcal{C}|$ strategic agents at $x_{l(\mathcal{C},t)}(t) + 1$ until the width of the connectivity component is at most 1, which happens after at most 14 time steps, since the length of the interval is decreased by at least $\frac{1}{2}$ at every time step. We can clearly perform several such operations at the same time as long as we have a sufficient number of strategic agents available. Since running off available strategic agents means that we have used at least half of them, Step (2) can be done in $O(n^{1-\alpha})$ time steps.

Step 3: Let t be one time step after the completion of Step (1). At time t every connectivity component with a width larger than 8 has a density of at least $\frac{n^\alpha}{144}$. Thus $w(\mathcal{G}'_t, t) \in O(n^{1-\alpha})$. Also the number of those connectivity components is in $O(n^{1-\alpha})$. During the subsequent time steps some of these connectivity components may split further. However, we will consider at most $O(n^{1-\alpha})$ different connectivity components in the following sub steps, since each treated connectivity component has a width larger than 1.

At a given time t we choose a connectivity component \mathcal{C} with width larger than 1 and place all $n^\alpha + 12$ strategic agents at $x_{l(\mathcal{C},t)}(t) + 1$. Since the number of non-strategic agents at position $x_{l(\mathcal{C},t)}(t)$ is at most n , this decreases the width of the corresponding connectivity component by at least $\frac{1}{2n^{1-\alpha}}$. Thus, we have $w(\mathcal{G}'_{t+1}, t + 1) \leq w(\mathcal{G}'_t, t) - \frac{1}{2n^{1-\alpha}}$, so that it last at most $O(n^{2-2\alpha})$ time steps until all connectivity components have width zero, i.e., the system has converged after the completion of Step (3). Note again that connectivity components of width at most 1 collapse into connectivity components of width zero in one time step. \square

To illustrate the proof of Theorem 17 we consider our previous example again. Setting $\alpha = \frac{1}{4}$, we can use $m = 14$ strategic agents to control the HK system of the other $n = 18$ non-strategic agents. The agents 1, 2, and 3 of the first two connectivity components of \mathcal{G}'_0 are skipped at (1.1), so that we

continue with $h = 4$. The number k of non-strategic agents with an opinion at time 0 within the interval $[x_h(t), x_h(t) + 3] = [3.6, 6.6]$ is given by $k = 7$. This number is larger than $\frac{n^\alpha}{12} < 1$, so that we increase h in (1.2). Since $4.5 + 3 > 6.7$ we skip the entire connectivity component consisting of the agents $4, \dots, 11$ and continue with $h = 12$. Here the interval $h(t), x_h(t) + 3]$ again contains too many agents so that h is set to 13. Also, the corresponding interval $h(t), x_h(t) + 3]$ contains too many agents, so that finally also the fourth connectivity component is skipped. They last two connectivity intervals are skipped since their densities of $\frac{8}{3.1} \approx 2.58$ and $\frac{7}{4.2} \approx 1.67$ are too high, i.e., at least as large as $\frac{n^\alpha}{144} < 1$. Since also the density of connectivity component 2 is large enough, there is nothing to do in Step 2, so that we proceed with Step 3. Since we have performed no real action, we still are at time $t = 0$. Agent 1 is already frozen. Agents 1 and 2 are contained in a connectivity component with width at most 1, so that no external influence is needed. At the next time step $t = 1$ they get frozen at opinion 1.95. Agents $4, \dots, 11$ are contained in a connectivity component of high density and length 3.1. So, we place all $m = 14$ agents at time 0 at position 4.6. At time 1 the opinions of the non-strategic agents are given by $x(1) \approx (0, 1.95, 1.95, 4.53, 4.58, 4.69, 4.75, 4.86, 5.96, 6.18, 6.37, 10.35, 10.73, 11.63, 11.93, 12.28, 13.3, 13.75)$, i.e., agents $4, \dots, 8$ and agents $9, 10, 11$ both form a connectivity component of width at most 1. Since the width of the connectivity component of agents $12, \dots, 16$ is larger than 1, we place all $m = 14$ agents at time 1 at position $x_{12}(1) + 1 = 11.35$. At time 2 all connectivity components have a width of at most 1, so that the HK system has converged at time 3.

Lemma 18 *For each $\alpha \in [0, 1]$ we have $f(n, n^\alpha) \in \Omega\left(n^{\frac{2-2\alpha}{3}}\right)$.*

PROOF. We set $k = \lfloor n^\beta \rfloor$, where $\beta = \frac{1-\alpha}{3}$, and consider the HK system of $m = \lfloor n/k \rfloor$ dumbbell configurations consisting of $k-2$, $k-1$, or k non-strategic agents each (in the construction of a dumbbell configuration, we assume $k \equiv 1 \pmod{3}$). Of course, we also have to assume that k is sufficiently large. The possibly non-empty set of remaining non-strategic agents is placed far away from the other m connectivity components at time 0. If not affected by at least one strategic agent, the convergence time of an arbitrary dumbbell connectivity component is in $\Omega(n^{2\beta}) = \Omega\left(n^{\frac{2-2\alpha}{3}}\right)$. Within $n^{2\beta}$ time steps, at most $n^{2\beta+\alpha}$ of the m dumbbell connectivity components can be affected by at least a single strategic agent. Thus, we have $f(n, n^\alpha) \in \Omega\left(n^{\frac{2-2\alpha}{3}}\right)$. \square

We remark that for $\alpha = 0$, we have $m = 1$, so that Lemma 18 improves the lower bound from Corollary 13 to $\Omega(n^{2/3})$. If we replace the use of many dumbbell configurations by the same number of equidistant configurations an $\Omega\left(n^{\frac{1-\alpha}{2}}\right)$ lower bound can be concluded.

4 Conclusion

We have demonstrated that the convergence times of $\Omega(n)$ of the equidistant configuration and of $\Omega(n^2)$ of the dumbbell configuration can both be reduced to $O(n^{3/4})$ using a single strategic agent. It would be very interesting to see optimized strategies and/or an improved analysis further reducing the upper bound of the optimal convergence time for these two specific examples for $m = 1$. Given a concrete HK system and a number m of strategic agents, the optimal convergence time might be determined using an exact algorithm based on integer linear programming, similar to those ILP formulations presented in [11, 16]. However, we do not go into details here and propose the development of (practically) efficient exact algorithms for the optimal control problem of minimizing the convergence time in the Hegselmann–Krause model as a research challenge.

We have further analyzed an example, where the optimal convergence time using a single strategic agent is lower bounded by $\Omega(n^{2/3})$. So, either this bound should be improved or the optimal convergence time of the two previously mentioned examples is better than $O(n^{3/4})$. Since the upper bound for $f(n, 1)$ is still $O(n^2)$, it would be nice if the considerable large gap could be narrowed. If enough strategic agents are available, any given HK system could be controlled in such a way that it converges in 2 time steps. To be more precise, we have shown that $m = 9n$ agents are sufficient. In some sense, the number of strategic agents must be at least as large as a constant fraction of the number of non-strategic agents in order to guarantee convergence in a constant number of time steps. So, we ask for the following

tightening: Given a constant $c_2 \in \mathbb{R}_{>0}$, what is the minimum constant $c_1 \in \mathbb{R}_{>0}$, of course depending on c_2 , such that we have $f(n, c_1 n) \geq c_2$ for all sufficiently large n ?

For the cases in between the two extreme situations of a single and that of very many strategic agents, we have proven a smooth upper bound meeting the best known bounds for the extreme situations up to a constant. However, we do not think that this general construction is tight. Indeed, there is a considerable gap to the presented lower bound. So, we ask for improvements.

Instead of distinguishing the stable state as the desired state one can also ask for the minimal time needed to reach a consensus. However, the resulting problem is rather similar to the one studied in this paper.

Even without the presence of strategic agents there is a considerable gap between the best known upper $O(n^3)$ and lower bound $\Omega(n^2)$ for convergence in the 1-dimensional Hegselmann–Krause model. As demonstrated by the example analyzed in [16], the estimation in the proof of the upper bound in [2] can be tight up to a constant for $\Omega(n)$ time steps. So, it seems that additional ideas are needed.

All of our considerations were restricted to the one-dimensional case. The same questions can clearly be asked for higher dimensions and then for different norms.

There are variants of the HK model where it is still not known whether the system converges or not. One notable example is the heterogeneous version of a HK system, i.e., where the neighborhood radii of the agents are not necessarily the same. Here, one strategic agent is sufficient to guarantee convergence in a finite number of time steps. Another example is the Hegselmann–Krause dynamics on the one-dimensional boundary of a circle assuming asymmetric influence ranges. For convergence results on the circle in the absence of strategic agents we refer the reader to [8].

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