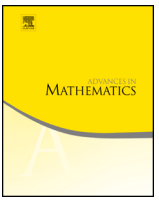
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Slope zero tensors, curvature type tensors, uniformizing variations of Hodge structure and quotients of bounded symmetric domains



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ABSTRACT

We prove an equivalence between two approaches to characterizing complex-projective varieties X with klt singularities and ample canonical divisor that are uniformized by bounded symmetric domains. In order to do so, we show how to construct a uniformizing variation of Hodge structure from a slope zero tensor and vice versa. As a consequence, we generalize various uniformization results of Catanese and Di Scala to the singular setting. For example, we prove that X is a quotient of a bounded symmetric domain of tube type by a group acting properly discontinuously and freely in codimension one if and only if X admits a slope zero tensor. As a key step in the proof, we establish the compactness of the holonomy group of the singular Kähler–Einstein metric on X_{reg}.

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1. Introduction

The problem of characterizing smooth projective varieties X uniformized by a bounded symmetric domain $\mathcal{D}$ is classical and has been thoroughly studied by Yau, Simpson and Catanese–Di Scala, among others [10,11,29,31]. It has also been solved more generally for projective varieties with klt singularities and for orbifolds [6,18,19,26].

There are two different approaches to the problem:

- Simpson's approach is of a Hodge-theoretic nature and provides a criterion for uniformization by $\mathcal{D} = {}^{G_0}/_{K_0}$ in terms of the existence of a *uniformizing variation of Hodge structure (uVHS)* for G_0 on X. The existence of such a uVHS is in turn equivalent to a reduction of structure group of the tangent bundle $\mathcal{T}_X$ to $K = (K_0)_{\mathbb{C}}$ together with a certain Chern class equality depending on $\mathcal{D}$.
- The approach of Catanese–Di Scala is more differential-geometric and provides a uniformization criterion in terms of the existence of a holomorphic tensor σ on X having certain properties. This yields several uniformization results, the most general of which (in terms of $\mathcal{D}$) is as follows: X is uniformized by a bounded symmetric domain $\mathcal{D}$ without higher-dimensional ball factors if and only if there is a tensor σ with the three properties listed in Theorem 1.3.

Now fix a bounded symmetric domain $\mathcal{D} = {}^{G_0}/_{K_0}$ without higher-dimensional unit balls as factors. It is then clear that on X, the existence of a uVHS for G_0 is equivalent to the existence of a tensor σ as above (since both are equivalent to the universal cover of X being isomorphic to $\mathcal{D}$). However, to the best of our knowledge there is yet no way to see this equivalence directly, i.e. without passing to the universal cover. One of the main goals of this article is to build a bridge between these two very different objects, in the more general setting of projective varieties with klt singularities. More precisely, we construct a uVHS given a tensor (in the sense of Catanese–Di Scala) and vice versa, without passing to the universal cover. As a consequence, we can generalize the uniformization results of Catanese–Di Scala to the singular setting.

Theorem 1.1 *(cf. [10, Thm. 1.2]). Let X be an n-dimensional normal projective variety with klt singularities and such that K_X is ample. The following are equivalent:*

(1.1.1) *We have $X \cong \mathcal{D}/\Gamma$, where $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_m$ is a product of bounded symmetric domains of tube type such that $\operatorname{rk} \mathcal{D}_i$ divides $\dim \mathcal{D}_i$ for each $i = 1, \ldots, m$, and $\Gamma \subset \operatorname{Aut}(\mathcal{D})$ is a discrete cocompact subgroup whose action is free in codimension one.*

(1.1.2) *There is a semispecial tensor ψ on X, i.e. a nonzero section*

$$0 \neq \psi \in \mathrm{H}^0\big(X, \operatorname{Sym}^n(\Omega^1_X)(-K_X)[\otimes]\eta\big)$$

for some rank one reflexive sheaf η on X such that $\eta^{[2]} \cong \mathscr{O}_X$.

Here $[\otimes]$ denotes the reflexive tensor product and $(-)^{[k]}$ denotes the k-th reflexive tensor power.

Theorem 1.2 *(cf. [10, Thm. 1.3]). Let X be an n-dimensional normal projective variety with klt singularities and such that K_X is ample. The following are equivalent:*

(1.2.1) *We have $X \cong \mathcal{D}/\Gamma$, where $\mathcal{D}$ is a bounded symmetric domain of tube type and $\Gamma \subset \operatorname{Aut}(\mathcal{D})$ is a discrete cocompact subgroup whose action is free in codimension one.*

(1.2.2) *There is a slope zero tensor ψ on X, i.e. a nonzero section*

$$0 \neq \psi \in \mathrm{H}^0\big(X, \operatorname{Sym}^{mn}(\Omega^1_X)(-mK_X)\big)$$

for some positive integer $m > 0$.

Theorem 1.3 *(cf. [11, Thm. 1.2]). Let X be an n-dimensional normal projective variety with klt singularities and such that K_X is ample. The following are equivalent:*

(1.3.1) *We have $X \cong \mathcal{D}/\Gamma$, where $\mathcal{D}$ is a bounded symmetric domain none of whose irreducible factors is isomorphic to a unit ball $\mathbb{B}^m$ of dimension $m \geq 2$ and $\Gamma \subset \operatorname{Aut}(\mathcal{D})$ is a discrete cocompact subgroup whose action is free in codimension one.*

(1.3.2) *There is a tensor*

$$\sigma \in \mathrm{H}^0\big(X, \mathscr{E}nd\big(\mathcal{T}_X[\otimes]\Omega^1_X\big)\big) = \mathrm{H}^0\Big(X_{\mathrm{reg}}, \mathscr{E}nd\big(\mathcal{T}_{X_{\mathrm{reg}}} \otimes \Omega^1_{X_{\mathrm{reg}}}\big)\Big)$$

such that:

(1.3.3) *There is a point $p \in X_{\mathrm{reg}}$ and a splitting of the tangent space $T \coloneqq T_pX$ as*

$$T = T'_1 \oplus \cdots \oplus T'_m$$

such that the first Mok characteristic cone $\mathcal{CS}$ *of* σ_p*, defined as*

$$\mathcal{CS} = \left\{ t \in T \;\middle|\; \exists \alpha \in T^\vee \setminus \{0\} \colon t \otimes \alpha \in \ker(\sigma_p) \right\},$$

is not all of T *and* $\mathcal{CS}$ *splits into* m *irreducible components* $\mathcal{CS}'(j)$ *with*

(1.3.4) $\mathcal{CS}'(j) = T'_1 \times \cdots \times \mathcal{CS}'_j \times \cdots \times T'_m$*, where*

(1.3.5) $\mathcal{CS}'_j \subset T'_j$ *is the cone over a non-degenerate (i.e.* $\mathcal{CS}'_j$ *spans* T'_j*) smooth projective variety* $\mathcal{S}'_j$*, unless* $\mathcal{CS}'_j = \{0\}$ *and* $\dim T'_j = 1$.

We remark that although these results are formulated in terms of uniformization (i.e. without reference to the existence of a uVHS), the proofs are actually in the form promised above. The uniformization statements are then deduced by using Proposition 2.10 below. For example, instead of Theorem 1.3 we actually prove the following.

Theorem 1.4. *Let* X *be an* n*-dimensional normal projective variety with klt singularities and such that* K_X *is ample. The following are equivalent:*

(1.4.1) *The smooth locus* X_{reg} *admits a uniformizing VHS for a Hodge group* G_0 *of Hermitian type such that* $\mathfrak{g}_0$ *has no factors isomorphic to* $\mathfrak{su}(p,1)$ *with* $p \geq 2$.

(1.4.2) *There is a holomorphic tensor* $\sigma \in \mathrm{H}^0\big(X, \mathscr{E}nd\big(\mathcal{T}_X[\otimes]\Omega^1_X\big)\big)$ *satisfying the three conditions (1.3.3)–(1.3.5).*

Compactness of the holonomy group. As a key technical step in the proof, we establish the following result, which might be of independent interest. Recall that for a projective variety X with klt singularities and K_X ample, the smooth locus X_{reg} carries a Kähler–Einstein metric ω_{KE} [4,12]. It is relatively easy to deduce information about the restricted holonomy group H° of this metric from the existence of a tensor as in the above theorems, using the Bochner principle. Computing the full holonomy group H, however, is difficult, as it might even have infinitely many components. Theorem 1.5 overcomes this problem by showing that H is in fact compact. Since the tangent bundle $\mathcal{T}_{X_{\mathrm{reg}}}$ only admits a reduction of structure group to H, but not to H°, this statement is crucial in order to pass from the given X to the appropriate bounded symmetric domain $\mathcal{D}$ (roughly speaking, by considering $\mathcal{D} = {}^{G_0}\!/_{K_0}$ with $K_0 = H$).

Theorem 1.5 *(Holonomy cover). Let* X *be an* n*-dimensional normal projective variety with klt singularities and such that* K_X *is ample.*

(1.5.1) *The holonomy group* H *of* $(X_{\mathrm{reg}}, \omega_{\mathrm{KE}})$ *is compact.*

(1.5.2) *There is a finite quasi-étale Galois cover* $\gamma \colon Y \to X$ *such that the holonomy group of* $(Y_{\mathrm{reg}}, \omega_{\mathrm{KE},Y})$ *is connected.*

In the Ricci-flat case, analogous statements have been proved in [14, Thm. B] and [8, Thm. C].

Remark 1.6. It is claimed in [3, Prop. 10.114] that for symmetric spaces, the holonomy representation always has finite index inside its normalizer. This would easily imply Theorem 1.5, however [3] contains no proof and the reference given there [30, Cor. 7.2] only proves the weaker statement that the holonomy of a *compact* locally symmetric Riemannian manifold is compact. Unless X is smooth, X_{reg} is clearly not compact and not even complete [14, Prop. 4.2]. Therefore we cannot apply [30] in our situation. In fact, since our proof only works for Hermitian symmetric spaces, it does not imply that [3, Prop. 10.114] is valid for general Riemannian symmetric spaces (but we do not need this level of generality).

Remark 1.7. A few months after this paper was first posted on the arXiv, Di Scala [28, Thm. A] gave a complete proof of a more general version of Besse's claim. Even in the Hermitian case, his argument is different from ours, so now there are two different proofs of Theorem 1.5.

Sketch of proof. The proof of Theorem 1.4 is divided into five steps:

(1) We consider the singular Kähler–Einstein metric ω_{KE} on X. By passing to the holonomy cover (Theorem 1.5), we may assume that the holonomy group H of $(X_{\mathrm{reg}}, \omega_{\mathrm{KE}})$ is connected.
(2) We use the Bochner principle to show that each irreducible factor H_i of H is the holonomy of a bounded symmetric domain $\mathcal{D}_i$ not isomorphic to a unit ball $\mathbb{B}^m$ with $m \geq 2$.
(3) The tangent bundle $\mathcal{T}_{X_{\mathrm{reg}}}$ admits a reduction of structure group to H. In other words, the frame bundle of X_{reg} contains a principal H-bundle P_H. Letting $H_{\mathbb{C}}$ denote the complexification of H, the principal $H_{\mathbb{C}}$-bundle $P \coloneqq P_H \times_H H_{\mathbb{C}}$ provides a uniformizing system of Hodge bundles (P, ϑ) on X_{reg}.
(4) The holonomy bundle P_H is a metric on this system of Hodge bundles (in the sense of Definition 2.4) and makes it into a uniformizing VHS.
(5) The final step is to prove the converse, i.e. that we obtain a tensor satisfying (1.3.3)–(1.3.5) starting with a uniformizing VHS.

Further remarks on the proof. For the reader's benefit, we give some further explanation about the assumptions and proof techniques of the main theorems.

Remark 1.8 *(The klt assumption).* The assumption that X has "klt singularities" is a mildness assumption that originally comes from birational geometry. For the precise definition, we refer to [21, Def. 2.34]. It is used in our proof in several essential ways. First, it is necessary in order to ensure the existence of a Kähler–Einstein metric on X_{reg},

without which we could not even talk about holonomy groups. Second, a maximally quasi-étale cover (cf. Theorem 2.9 and Remark 1.10 below) will in general not exist for spaces with worse than klt singularities. Third, the Lipman–Zariski Conjecture (used in the proof of Proposition 2.10) is not known for general (non-klt) singularities.

That said, the klt assumption is very natural in our setting, since it applies for example to canonical models of varieties of general type (these have ample canonical divisor, but are not smooth in general) and since the spaces $X = \mathcal{D}/\Gamma$ we are characterizing have quotient (in particular, klt) singularities, at least a posteriori. Nevertheless, it would be interesting to weaken this assumption, e.g. to having log canonical singularities. But as explained above, this seems currently out of reach due to substantial technical difficulties.

Remark 1.9 *(Comparison to the smooth case).* We would like to highlight the differences to the proof in the setting where X is a complex manifold, as described in [11, pp. 570–571]. In both cases, the existence of a Kähler–Einstein metric on X is crucial – of course, in the smooth case this is due to Aubin and Yau [2,31]. Also, consideration of the holonomy of this metric is a common theme. However, in the smooth case we can apply the theorems of de Rham and Berger to the universal cover $\widetilde{X}$, while for singular X the maximally quasi-étale cover might be seen as a weak replacement for $\widetilde{X}$. Finally, in both cases the Bochner principle is used to obtain information about the holonomy group from the existence of a tensor. However, in the smooth case this information can be used to decompose $\widetilde{X}$, while we have to work with uniformizing variations of Hodge structure, as sketched above.

Remark 1.10 *(Maximally quasi-étale covers).* An important notion in our proofs is that of a *quasi-étale cover*. By this we mean a finite surjective map $Y \to X$ between normal varieties which is étale outside a codimension two subset (equivalently, by purity of branch locus, étale over the smooth locus of X). If X is klt, then it has a so-called *maximally quasi-étale cover* $\gamma\colon Y \to X$, whose main property is explained in Theorem 2.9. Its role in our proofs is that we aim to show that Y is smooth (using the Lipman–Zariski Conjecture) and then apply the classical uniformization result of Simpson to Y, which then descends to X in a suitable sense. This line of reasoning has been used previously by Shepherd-Barron, Wilson, Greb, Kebekus, Peternell, Taji and the authors [17–19,26,27].

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2. Preliminaries

2.1. Hodge groups and Hermitian symmetric spaces

We recall some basic facts about Hermitian symmetric spaces. This material can be found in [29] and [9].

Definition 2.1 *(Hodge group of Hermitian type).* A *Hodge group of Hermitian type* is a semisimple real algebraic group without compact factors G_0 whose complexified Lie algebra $\mathfrak{g} \coloneqq \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ carries a Hodge decomposition

$$\mathfrak{g} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$$

such that the Lie bracket of $\mathfrak{g}$ is compatible with the Hodge decomposition in the sense that $[\mathfrak{g}^{p,-p}, \mathfrak{g}^{q,-q}] \subset \mathfrak{g}^{p+q,-p-q}$ for all $p, q \in \{-1, 0, 1\}$. Furthermore the bilinear form on $\mathfrak{g}$ given by $(-1)^{p+1} \operatorname{tr}(\operatorname{ad}(U) \circ \operatorname{ad}(\overline{V}))$ on $\mathfrak{g}^{p,-p}$ must be positive definite.

Let K_0 denote the real subgroup of G_0 corresponding to the Lie algebra $\mathfrak{g}_0^{0,0} = \mathfrak{g}^{0,0} \cap \mathfrak{g}_0$. Then K_0 is the largest subgroup such that the adjoint action of K_0 on $\mathfrak{g}$ preserves the Hodge decomposition of $\mathfrak{g}$.

We note the following classical facts, which will be needed later.

Fact 2.2. *Let G_0 and K_0 be as above.*

(2.2.1) *K_0 is a maximal compact subgroup of G_0.*

(2.2.2) *The quotient $\mathcal{D} = {}^{G_0}\!/_{K_0}$ is a Hermitian symmetric space of non-compact type (= bounded symmetric domain). Moreover, every bounded symmetric domain can be expressed as such a quotient, by taking $G_0 = \operatorname{Aut}(\mathcal{D})$.*

(2.2.3) *Given $\mathcal{D}$, the groups G_0 and K_0 are uniquely determined up to isogeny. In particular, the Lie algebra $\mathfrak{g}$ is determined by $\mathcal{D}$.*

Let X be a smooth quasi-projective variety or a complex manifold, and let G_0, K_0 be as above. We denote by G and K the complexifications of the groups G_0 and K_0, respectively.

Definition 2.3 *(Uniformizing system of Hodge bundles).* A *uniformizing system of Hodge bundles for G_0* on X is a pair (P, ϑ), where P is a holomorphic principal K-bundle on X and ϑ is an isomorphism of vector bundles

$$\vartheta \colon \mathcal{T}_X \xrightarrow{\sim} P \times_K \mathfrak{g}^{-1,1} \tag{2.3.1}$$

such that $[\vartheta(u), \vartheta(v)] = 0$ for all local sections u, v of $\mathcal{T}_X$.

If (P, ϑ) is a uniformizing system of Hodge bundles on X, then we also have an isomorphism $\Omega^1_X \cong P \times_K \mathfrak{g}^{1,-1}$. Setting $\mathcal{E}^{i,-i} \coloneqq P \times_K \mathfrak{g}^{i,-i}$, we can form the direct sum

$$\mathcal{E} \coloneqq P \times_K \mathfrak{g} = \mathcal{E}^{-1,1} \oplus \mathcal{E}^{0,0} \oplus \mathcal{E}^{1,-1}.$$

There is a natural Higgs field $\mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$, which we also denote by ϑ, given by $u \mapsto (v \mapsto [\vartheta(v), u])$. It is easily checked that $\vartheta(\mathcal{E}^{i,-i}) \subset \mathcal{E}^{i-1,-i+1} \otimes \Omega^1_X$ and that $\vartheta \wedge \vartheta = 0$ (due to the Jacobi identity in $\mathfrak{g}$ and the fact that $[\vartheta(u), \vartheta(v)] = 0$ for all local vector fields u, v). So $\mathcal{E}$ is an honest system of Hodge bundles, which justifies the name.

Definition 2.4 *(Metric).* Let (P, ϑ) be as above. A *metric* h on P is a $\mathscr{C}^\infty$-reduction in structure group of P from K to K_0, i.e. a principal K_0-subbundle $P_H \subset P$. We then automatically have $P \cong P_H \times_{K_0} K$.

Recall the following well-known facts from the theory of principal bundles.

Proposition 2.5. *Let M be a smooth manifold, G a Lie group with Lie algebra $\mathfrak{g}$, and $f\colon P \to M$ a G-principal bundle.*

(2.5.1) *To give a connection on P (in the sense of a G-invariant horizontal subbundle of $\mathcal{T}_P$) is equivalent to giving a $\mathfrak{g}$-valued 1-form ω on P which is G-equivariant and whose restriction to the fibres of f equals the Maurer–Cartan form of G.*

(2.5.2) *If ω_1, ω_2 are two connections on P, then their difference $\omega_1 - \omega_2$ is a $\mathfrak{g}$-valued G-equivariant 1-form on P which is furthermore horizontal (i.e. it vanishes on the vertical subbundle of $\mathcal{T}_P$).*

(2.5.3) *To give a $\mathfrak{g}$-valued G-equivariant horizontal 1-form on P is equivalent to giving a map of vector bundles*

$$\mathcal{T}_M \longrightarrow P \times_G \mathfrak{g}.$$

Hence the set of all connections on P is an affine space over the vector space of $(P \times_G \mathfrak{g})$-valued 1-forms on M. □

We now revert to the notation used before Proposition 2.5. We borrow the following construction from [29, Section 8], which is necessary to define a uniformizing variation of Hodge structure.

Construction 2.6. Let X be a smooth quasi-projective variety or a complex manifold. Let (P, ϑ) be a uniformizing system of Hodge bundles on X equipped with a metric $P_H \subset P$, which we denote by h. If V is a polarized Hodge representation of G_0 in the sense of [29, p. 900], then h induces a Hermitian metric (in the usual sense) on the associated vector bundle $P \times_K V$. In particular, this applies to $V = \mathfrak{g}$. By abuse of notation, we denote again by h the Hermitian metric on $\mathcal{E} = P \times_K \mathfrak{g} = P_H \times_{K_0} \mathfrak{g}$.

There is a unique connection d_h on P_H which is compatible with the holomorphic structure of P [1]. We can push forward d_h to a connection on $R_H \coloneqq P_H \times_{K_0} G_0$, which we still denote by d_h. Let $\overline{\vartheta}\colon \mathcal{T}_X \to P \times_K \mathfrak{g}^{1,-1}$ be the conjugate of ϑ. Then $\vartheta + \overline{\vartheta}\colon \mathcal{T}_X \to P \times_K \mathfrak{g}_0$ and hence by Proposition 2.5, $D_h \coloneqq d_h + \vartheta + \overline{\vartheta}$ is a connection on R_H.

Definition 2.7 *(Uniformizing VHS).* A *uniformizing variation of Hodge structure (uVHS) for* G_0, where G_0 is a Hodge group of Hermitian type as in Definition 2.1, is a uniformizing system of Hodge bundles (P, ϑ) for G_0 together with a metric h such that the connection D_h is flat.

If (P, ϑ) is a uVHS, then D_h induces a flat connection on $\mathscr{E} = P \times_K \mathfrak{g} = R_H \times_{G_0} \mathfrak{g}$. This connection can also be constructed as follows: let d'_h be the Chern connection on $(\mathscr{E}, h)$, and let ϑ be the Higgs field on $\mathscr{E}$ as explained above. Let ϑ^h be the adjoint of ϑ with respect to h, i.e. we have

$$\langle \vartheta u, v\rangle_h = \langle u, \vartheta^h v\rangle_h$$

for all local sections u, v of $\mathscr{E}$. Then $D'_h \coloneqq d'_h + \vartheta + \vartheta^h$ is the flat connection induced by D_h.

Example 2.8 *(uVHS on bounded symmetric domains).* Let G_0 be a Hodge group of Hermitian type as in Definition 2.1, and let $K_0 \subset G_0$ be a maximal compact subgroup, which is unique up to conjugation. Recall that we denote by G and K the complexifications of G_0 and K_0, respectively, and let $\mathcal{D} = {}^{G_0}\!/_{K_0}$ be the associated bounded symmetric domain.

Then $G_0 \to \mathcal{D}$ is a principal K_0-bundle, which we denote by $P_{\mathcal{D},H}$. Set $P_{\mathcal{D}} \coloneqq P_{\mathcal{D},H} \times_{K_0} K$. Then there is an isomorphism $\vartheta_{\mathcal{D}}\colon \mathcal{T}_{\mathcal{D}} \xrightarrow{\sim} P_{\mathcal{D}} \times_K \mathfrak{g}^{-1,1}$, where K acts on $\mathfrak{g}^{-1,1} \subset \mathfrak{g}$ via the adjoint representation [9, Lemma 12.2.3]. Therefore $(P_{\mathcal{D}}, \vartheta_{\mathcal{D}})$ is a uniformizing system of Hodge bundles on $\mathcal{D}$.

The subbundle $P_{\mathcal{D},H}$ is a metric h on $(P_{\mathcal{D}}, \vartheta_{\mathcal{D}})$. The induced connection D_h on $R_H \coloneqq P_{\mathcal{D},H} \times_{K_0} G_0 \cong \mathcal{D} \times G_0$ is given by pulling back the Maurer–Cartan form ω_{G_0} on G_0 along the projection onto the second factor $R_H \to G_0$. From the Maurer–Cartan equation

$$\mathrm{d}\omega_{G_0} + \frac{1}{2}\big[\omega_{G_0}, \omega_{G_0}\big] = 0,$$

it follows that D_h is flat [9, Lemma 11.3.9]. This means that $(P_{\mathcal{D}}, \vartheta_{\mathcal{D}})$ together with the metric $P_{\mathcal{D},H}$ is a uniformizing VHS on $\mathcal{D}$.

We will use the following theorem from [16]. The cover $Y \to X$ given by the theorem is often referred to as a *maximally quasi-étale cover*.

Theorem 2.9 *(cf. [16, Thm. 1.14]). Let X be an n-dimensional normal quasi-projective variety with klt singularities. Then there exists a finite quasi-étale Galois cover $\gamma\colon Y \to X$ with the following property: for any flat, locally free, analytic sheaf $\mathscr{G}^\circ$ on Y_{reg}, there exists a flat, locally free, algebraic sheaf $\mathscr{G}$ on Y such that $\mathscr{G}^\circ$ is isomorphic to the analytification of $\mathscr{G}\big|_{Y_{\mathrm{reg}}}$.* □

The following proposition has already been mentioned in the introduction.

Proposition 2.10 *(cf. [29, Prop. 9.1]). Let X be an n-dimensional normal projective variety with klt singularities and such that K_X is ample. Let $\mathcal{D}$ be a bounded symmetric domain. The following are equivalent:*

(2.10.1) *We have $X \cong \mathcal{D}/\Gamma$, where $\Gamma \subset \mathrm{Aut}(\mathcal{D})$ is a discrete cocompact subgroup whose action is free in codimension one.*

(2.10.2) *X_{reg} admits a uniformizing VHS (P, ϑ) for some Hodge group G_0 of Hermitian type with $\mathcal{D} \cong G_0/K_0$.*

Proof. "(2.10.1) ⇒ (2.10.2)": Let $G_0 \coloneqq \mathrm{Aut}(\mathcal{D})$, then G_0 is a Hodge group of Hermitian type and $\mathcal{D} \cong G_0/K_0$. Consider the uniformizing VHS $(P_{\mathcal{D}}, \vartheta_{\mathcal{D}})$ on $\mathcal{D}$ given as in Example 2.8. Then in particular we have an isomorphism $\mathcal{T}_{\mathcal{D}} \cong G_0 \times_{K_0} \mathfrak{g}^{-1,1}$.

Let $\pi\colon \mathcal{D} \to X$ be the quotient map, and set $\mathcal{D}^\circ \coloneqq \pi^{-1}(X_{\mathrm{reg}})$. The complement of $\mathcal{D}^\circ$ in $\mathcal{D}$ has complex codimension at least two. Therefore the restriction $\pi\big|_{\mathcal{D}^\circ}\colon \mathcal{D}^\circ \to X_{\mathrm{reg}}$ is the universal cover of X_{reg}, and $X_{\mathrm{reg}} \cong \mathcal{D}^\circ/\Gamma$. In particular, note that $\pi_1(X_{\mathrm{reg}}) = \Gamma$.

Restrict $(P_{\mathcal{D}}, \vartheta_{\mathcal{D}})$ to a uniformizing VHS $(P_{\mathcal{D}^\circ}, \vartheta_{\mathcal{D}^\circ})$ on $\mathcal{D}^\circ$. Consider the action of Γ on G_0 given by multiplication on the left. This stabilizes $G_0\big|_{\mathcal{D}^\circ}$, hence the principal bundle $G_0\big|_{\mathcal{D}^\circ}$ is Γ-equivariant. Therefore it descends to X_{reg}. That is, there is a principal K_0-bundle P_0 on X_{reg} such that $\pi^* P_0 \cong G_0\big|_{\mathcal{D}^\circ}$ as principal K_0-bundles on $\mathcal{D}^\circ$. It follows from the proof of [29, Prop. 9.1] that the isomorphism $\vartheta_{\mathcal{D}^\circ}$ is also Γ-equivariant. Therefore on X_{reg} there is an isomorphism $\vartheta\colon \mathcal{T}_{X_{\mathrm{reg}}} \xrightarrow{\sim} P_0 \times_{K_0} \mathfrak{g}^{-1,1}$. Setting $P \coloneqq P_0 \times_{K_0} K$, we have $\vartheta\colon \mathcal{T}_{X_{\mathrm{reg}}} \xrightarrow{\sim} P \times_K \mathfrak{g}^{-1,1}$. Hence (P, ϑ) is a uniformizing system of Hodge bundles on X_{reg}.

We must show that the metric $P_0 \subset P$ makes (P, ϑ) into a uniformizing VHS. The connection D_h on $R \coloneqq P_0 \times_{K_0} G_0$ from Construction 2.6 pulls back under the map π to a connection $D_{\mathcal{D}^\circ}$ on the trivial bundle $R_{\mathcal{D}^\circ} \coloneqq G_0\big|_{\mathcal{D}^\circ} \times_{K_0} G_0$ on $\mathcal{D}^\circ$, and is constructed in the same way. Since $D_{\mathcal{D}^\circ}$ is flat, it follows that D_h is flat. Thus (P, ϑ) is a uniformizing VHS on X_{reg}, as desired.

"(2.10.2) ⇒ (2.10.1)": By the definition of a uniformizing VHS, P is a principal K-bundle, ϑ is an isomorphism $\mathcal{T}_{X_{\mathrm{reg}}} \xrightarrow{\sim} P \times_K \mathfrak{g}^{-1,1}$ and there is a principal K_0-bundle $P_H \subset P$ and a flat connection on the principal G_0-bundle $R_H \coloneqq P_H \times_{K_0} G_0$. Recall that a connection ω on a principal bundle induces a connection ∇_ω on any associated vector bundle and that there is a (local) formula expressing ∇_ω in terms of ω [9, Sec. 11.4, in particular Prop. 11.4.5]. In particular, if ω is flat, then also ∇_ω is flat. Applied to our

situation, this means that the flat connection on R_H induces a flat connection on the associated vector bundle

$$\mathscr{E} \coloneqq R_H \times_{G_0} \mathfrak{g} = P \times_K \mathfrak{g}$$

on X_{reg}, which contains $\mathcal{T}_{X_{\mathrm{reg}}}$ as a direct summand.

We now consider a maximally quasi-étale cover $\gamma\colon Y \to X$ as in Theorem 2.9. The pullback $\gamma^*\mathscr{E}$ is a flat locally free sheaf on $\gamma^{-1}(X_{\mathrm{reg}})$. By purity of branch locus and since γ is quasi-étale, we have $\gamma^{-1}(X_{\mathrm{reg}}) \subset Y_{\mathrm{reg}}$ and the complement of the former in the latter has codimension at least two. In particular, the induced map on fundamental groups $\pi_1\big(\gamma^{-1}(X_{\mathrm{reg}})\big) \to \pi_1(Y_{\mathrm{reg}})$ is an isomorphism. Therefore $\gamma^*\mathscr{E}$ extends to a flat locally free sheaf $\mathscr{G}^\circ$ on Y_{reg}. By Theorem 2.9, $\mathscr{G}^\circ$ extends further to a flat locally free sheaf $\mathscr{G}$ on all of Y, which we will call $\mathscr{E}_Y$. By reflexivity, the split surjection $\mathscr{E} \twoheadrightarrow \mathcal{T}_{X_{\mathrm{reg}}}$ extends to a split surjection $\mathscr{E}_Y \twoheadrightarrow \mathcal{T}_Y$. This means that $\mathscr{E}_Y$ contains $\mathcal{T}_Y$ as a direct summand. Therefore $\mathcal{T}_Y$ is also locally free, and Y is smooth by the Lipman–Zariski Conjecture for klt spaces [15].

We may now apply [29, Prop. 9.1] and conclude that the universal cover $\widetilde{Y}$ of Y is isomorphic to $\mathcal{D}$. This implies (2.10.1) by standard arguments involving Selberg's lemma, cf. the proof of [17, Thm. 1.3]. □

The following notions appear in the main results. They are taken from [10] and [11]. Let X be an n-dimensional normal projective variety with klt singularities and such that K_X is ample.

Definition 2.11. A *semispecial tensor* on X is a nonzero section

$$0 \neq \psi \in \mathrm{H}^0\big(X, \mathrm{Sym}^n(\Omega^1_X)(-K_X)[\otimes]\eta\big)$$

for some rank one reflexive sheaf η on X such that $\eta^{[2]} \cong \mathscr{O}_X$.

Definition 2.12. A *slope zero tensor* on X is a nonzero section

$$0 \neq \psi \in \mathrm{H}^0\big(X, \mathrm{Sym}^{mn}(\Omega^1_X)(-mK_X)\big)$$

for some positive integer $m > 0$.

Definition 2.13 *(First Mok characteristic cone).* Let T be a finite-dimensional complex vector space and $\sigma \in \mathrm{End}(T \otimes T^\vee)$. The *first Mok characteristic cone* $\mathcal{CS}$ of σ is defined as

$$\big\{t \in T \;\big|\; \exists \alpha \in T^\vee \setminus \{0\} \colon t \otimes \alpha \in \ker(\sigma)\big\}\,.$$

Table 1
Classification of irreducible Hermitian symmetric spaces.

[5]	[20]	Symmetric space	Dimension	Rank	Tube type
$\mathrm{I}_{p,q}$	A III	$\mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$	pq	$\min(p,q)$	$\Leftrightarrow p=q$
II_n	D III	$\mathrm{SO}^*(2n)/\mathrm{U}(n)$	$\frac{1}{2}n(n-1)$	$\lfloor n/2\rfloor$	$\Leftrightarrow n$ even
III_n	BD I ($q=2$)	$\mathrm{SO}_0(2,n)/\mathrm{SO}(2)\times\mathrm{SO}(n)$	n	$\min(n,2)$	yes
IV_n	C I	$\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$	$\frac{1}{2}n(n+1)$	n	yes
V	E III	$E_{6(-14)}/\mathrm{SO}(10)\times\mathrm{SO}(2)$	16	2	no
VI	E VII	$E_{7(-25)}/E_{6(-78)}\times\mathrm{SO}(2)$	27	3	yes

2.2. Classification of irreducible Hermitian symmetric spaces

For later reference, Table 1 summarizes the classification of irreducible Hermitian symmetric spaces and their most important invariants. This information can be found in [20] or [3] and in [13, p. 525].

Remark. In the first column, we have used the original numbering scheme by Cartan. Modern references such as [10] and [13] switch types III and IV.

Definition 2.14 *(Domains of tube type).* A bounded symmetric domain $\mathcal{D}$ is said to be *of tube type* if it is biholomorphic to a *tube domain* $\Omega = V \oplus \mathrm{i}\mathscr{C}$, where V is a real vector space and $\mathscr{C} \subset V$ is an open self-dual convex cone containing no lines.

Proposition 2.15. *The irreducible Hermitian symmetric spaces of rank one are exactly the unit balls $\mathbb{B}^n$ with $n \geq 1$. In particular, the only tube domain of rank one is the unit disc $\mathbb{B}^1$.*

Proof. According to Table 1, the rank equals one exactly in the following cases:

- $\mathrm{I}_{1,q}$ with $q \geq 1$, which gives $\mathbb{B}^q$
- II_2, which gives $\mathbb{B}^1$
- II_3, which gives $\mathbb{B}^3$ [20, p. 519, item (vii)]
- III_1, which gives $\mathbb{B}^1$
- IV_1, which gives $\mathbb{B}^1$

The second statement follows because $\mathbb{B}^n$ is not of tube type for $n \geq 2$ (also by Table 1). □

Proposition 2.16. *An irreducible Hermitian symmetric space G_0/K_0 is uniquely determined by its dimension and the Lie algebra $\mathfrak{k}_0$ of K_0.*

Table 2
Irreducible Hermitian symmetric spaces with dimensions and commutator subalgebra.

[5]	[20]	$\dim_{\mathbb{C}} {}^{G_0}\!/_{K_0}$	$\dim_{\mathbb{R}} \mathfrak{k}_0$	$[\mathfrak{k}_0, \mathfrak{k}_0]$
$\mathrm{I}_{p,q}$	A III	pq	$p^2 + q^2 - 1$	$\mathfrak{su}(p) \oplus \mathfrak{su}(q)$
II_n	D III	$\frac{1}{2}n(n-1)$	n^2	$\mathfrak{su}(n)$
III_n	BD I ($q = 2$)	n	$\frac{1}{2}n(n-1) + 1$	$\mathfrak{so}(n)$
IV_n	C I	$\frac{1}{2}n(n+1)$	n^2	$\mathfrak{su}(n)$
V	E III	16	46	$\mathfrak{so}(10)$
VI	E VII	27	79	$\mathfrak{e}_6$

Proof. From Table 2, we see that $[\mathfrak{k}_0, \mathfrak{k}_0]$ is always semisimple (this is clear because K_0 is compact), and it is not simple if and only if we are in case $\mathrm{I}_{p,q}$ with $p, q \geq 2$. From this observation, one can check that the only potential "collisions" are the following:

- $\mathrm{I}_{1,q}$ and II_n with $q = n = \frac{1}{2}n(n-1)$, i.e. $n = 3$. But II_3 is the unit ball $\mathbb{B}^3$ (see above), so they are actually isomorphic.
- $\mathrm{I}_{1,q}$ and IV_n with $q = n = \frac{1}{2}n(n+1)$, i.e. $n = 1$. But IV_1 is the unit ball $\mathbb{B}^1$.
- II_4 and III_6, which are isomorphic [20, p. 519, item (viii)].
- III_3 and IV_2, which are isomorphic [20, p. 519, item (ii)].

This proves the proposition. □

3. Compactness of the holonomy group

In this section we prove Theorem 1.5 (as Theorem 3.2). First we set up some notation.

Setup 3.1. Let X be an n-dimensional normal projective variety with klt singularities and ample canonical divisor K_X. Let ω_{KE} be as in [19, Setup 3.2]. Write g_{KE} for the associated Riemannian metric on X_{reg} and h_{KE} for the associated Hermitian metric on $\mathcal{T}_{X_{\mathrm{reg}}}$. Fix, once and for all, a smooth point $x \in X_{\mathrm{reg}}$, and consider the tangent space $V \coloneqq T_x X$ at that point. We write

$$H \coloneqq \mathrm{Hol}(X_{\mathrm{reg}}, g_{\mathrm{KE}}, x) \subset \mathrm{U}(V, h_{\mathrm{KE},x}) \cong \mathrm{U}(n) \quad \text{and}$$
$$H^\circ \coloneqq \mathrm{Hol}^\circ(X_{\mathrm{reg}}, g_{\mathrm{KE}}, x) \subset \mathrm{U}(V, h_{\mathrm{KE},x})$$

for the corresponding (restricted) holonomy group. Recall from [3, Cor. 10.41] and [19, Lemma 5.2] that there are decompositions

$$V = V_1 \oplus \cdots \oplus V_m \quad \text{and} \quad H^\circ = H_1^\circ \times \cdots \times H_m^\circ \tag{3.1.1}$$

such that for each $1 \leq i \leq m$, the factor H_i° acts irreducibly and non-trivially on V_i and trivially on V_j for $j \neq i$.

Theorem 3.2 *(Holonomy cover). Let X be as above.*

(3.2.1) *The holonomy group H of $(X_{\mathrm{reg}}, \omega_{\mathrm{KE}})$ is compact.*
(3.2.2) *There is a finite quasi-étale Galois cover $\gamma\colon Y \to X$ such that the holonomy group of $(Y_{\mathrm{reg}}, \omega_{\mathrm{KE},Y})$ is connected.*

Proof. Recall that the restricted holonomy group H° is the identity component of the full holonomy group H and that H° is compact [3, Cor. 10.48 and Thm. 10.51]. Also recall that for any Lie group G, its identity component G° is a normal subgroup. Applying this to H, we see that H is contained in the normalizer of H° inside the full unitary group, i.e. $H \subset N_{\mathrm{U}(V)}(H^\circ)$.

To prove the theorem, it is sufficient to show that the index of the identity component H° in H is finite, i.e. $[H : H^\circ] < \infty$. For then H is clearly compact (H° being compact), and the kernel of the natural surjection $\pi_1(X_{\mathrm{reg}}) \twoheadrightarrow {}^{H}\!/_{H^\circ}$ will yield the desired finite cover γ. To be more precise, said kernel is a finite index normal subgroup of $\pi_1(X_{\mathrm{reg}})$, so it corresponds to a finite étale Galois cover of X_{reg}. This can be extended to a finite quasi-étale Galois cover $\gamma\colon Y \to X$. The holonomy group of $(Y_{\mathrm{reg}}, \omega_{\mathrm{KE},Y})$ equals H° by [14, Cor. 4.8]. In particular, it is connected.

Since, as remarked above, we have $H \subset N_{\mathrm{U}(V)}(H^\circ)$, in order to show that the index $[H : H^\circ] < \infty$, it is sufficient to show that $N_{\mathrm{U}(V)}(H^\circ)$ is a finite extension of H°. By [19, Lemma 5.3], it is in turn sufficient to show that $N_{\mathrm{U}(V_i)}(H_i^\circ)$ is a finite extension of H_i° for each $1 \leq i \leq m$. We will therefore assume from now on that $m = 1$, i.e. that H° acts irreducibly on V.

According to [3, Cor. 10.92], either $H^\circ = \mathrm{U}(V)$ or $(X_{\mathrm{reg}}, g_{\mathrm{KE}})$ is locally symmetric. In the former case, the claim is clear, so we may assume that we are in the second case. That is, H° is the holonomy of an irreducible simply connected Hermitian symmetric space ${}^{G}\!/_{H^\circ}$.

The key fact in dealing with this situation is that the center $Z(H^\circ)$ is exactly one-dimensional. This follows from [22, App. 5, Lemma 2(3)] and [23, Ch. XI, Thm. 9.6(1)]. But it is also possible to verify this claim case by case, using the classification of symmetric spaces [3, Table 3 on p. 315].

By Schur's lemma, we have

$$Z(H^\circ) = \mathrm{U}(1) \cdot \mathrm{id}_V \coloneqq \left\{ \lambda \cdot \mathrm{id}_V \;\middle|\; |\lambda| = 1 \right\} \subset \mathrm{U}(V).$$

Let $C = C_{\mathrm{U}(V)}(H^\circ)$ be the centralizer of H°. By Proposition 3.3 and Proposition 3.7 below, it is sufficient to show that ${}^{C}\!/_{Z(H^\circ)}$ is finite. But by Schur's lemma again, $C = \mathrm{U}(1) \cdot \mathrm{id}_V$. So ${}^{C}\!/_{Z(H^\circ)}$ is even trivial. □

Proposition 3.3. *Let G be a compact connected Lie group. Then the outer automorphism group*

$$\mathrm{Out}(G) \coloneqq {}^{\mathrm{Aut}(G)}\!/_{\mathrm{Inn}(G)}$$

is finite if and only if the center $Z(G)$ has dimension at most one.

Proof. Let $k \coloneqq \dim Z(G)$. Since G is compact and connected, it has a finite cover $p\colon \widetilde{G} \to G$ with

$$\widetilde{G} = G' \times T,$$

where G' is a compact connected semisimple Lie group and $T \cong \mathbb{T}^k \coloneqq (S^1)^k$ is a k-dimensional torus [24, Thm. 4.29]. The kernel of p is a finite central subgroup $D \subset Z(\widetilde{G})$.

"$\Rightarrow$": Suppose that $k \geq 2$. Recall that an automorphism $\varphi \in \mathrm{Aut}(\widetilde{G})$ descends to an automorphism of G if and only if it stabilizes D, i.e. if $\varphi(D) = D$.

Claim 3.4. *The subgroup*

$$\mathrm{Aut}_D(T) \coloneqq \left\{\varphi \in \mathrm{Aut}(T) \;\middle|\; \mathrm{id}_{G'} \times \varphi \text{ stabilizes } D\right\}$$

has finite index in $\mathrm{Aut}(T)$.

Proof. Let D_T denote the projection of $D \subset \widetilde{G}$ to the second factor T. Consider the subgroup

$$H \coloneqq \left\{\varphi \in \mathrm{Aut}(T) \;\middle|\; \varphi \text{ fixes } D_T \text{ pointwise}\right\}.$$

Clearly, $H \subset \mathrm{Aut}_D(T)$ and hence it suffices to show that H has finite index in $\mathrm{Aut}(T)$. Since there are only finitely many torsion points of any given order in T, fixing a torsion point defines a finite index subgroup of $\mathrm{Aut}(T)$ by the orbit-stabilizer theorem. But D_T consists of finitely many torsion points of T. Therefore H is a finite intersection of finite index subgroups, hence itself has finite index. □

The automorphism group of T is $\mathrm{Aut}(T) = \mathrm{GL}(k, \mathbb{Z})$, which is infinite since $k \geq 2$. By Claim 3.4, also $\mathrm{Aut}_D(T)$ is infinite. We define a map

$$\mathrm{Aut}_D(T) \longrightarrow \mathrm{Out}(G) \tag{3.4.1}$$

by sending $\varphi \in \mathrm{Aut}_D(T)$ to the coset of φ_G, the automorphism of G induced by $\mathrm{id}_{G'} \times \varphi$. If φ_G is an inner automorphism, i.e. $\varphi_G = \mathrm{Int}_g$ for some $g \in G$, then so is $\mathrm{id}_{G'} \times \varphi = \mathrm{Int}_{\widetilde{g}}$, where $\widetilde{g} \in p^{-1}(g)$. Since inner automorphisms of $\widetilde{G}$ act trivially on T, we get $\varphi = \mathrm{id}_T$. This shows that the map (3.4.1) is injective, and hence $\mathrm{Out}(G)$ is infinite, as desired.

"$\Leftarrow$": Suppose $k \leq 1$. Write $p\colon \widetilde{G} = G' \times T \to G$ as before. Since G' is compact and semisimple, its fundamental group $\pi_1(G')$ is finite [24, Thm. 4.69]. We may and will therefore assume that G' is simply connected.

Claim 3.5. *There is an injective map* $\mathrm{Out}(G) \to \mathrm{Out}(\widetilde{G})$.

Proof. First we show that any $\varphi \in \mathrm{Aut}(G)$ lifts (uniquely) to some $\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G})$. For this, it is sufficient to show that the subgroup $\pi_1(\widetilde{G}) \subset \pi_1(G)$ is stabilized by any such φ. Note that $\pi_1(\widetilde{G}) = \pi_1(T) \cong \mathbb{Z}^k$, so we may assume that $k = 1$. Let $T_G = p(T) \subset G$, then $T_G \cong S^1$ and it suffices to show that T_G is stabilized by φ. Since $Z(G')$ is finite, $T = Z(\widetilde{G})^\circ$ is the identity component of the center of $\widetilde{G}$. This implies that $T_G = Z(G)^\circ$ because $p^{-1}(Z(G)) = Z(\widetilde{G})$. But $Z(G)^\circ$ is invariant under any automorphism of G.

Sending $\varphi \mapsto \widetilde{\varphi}$ defines a map $\mathrm{Aut}(G) \to \mathrm{Aut}(\widetilde{G})$. If $\varphi = \mathrm{Int}_g$ is inner, then so is the lift $\widetilde{\varphi} = \mathrm{Int}_{\widetilde{g}}$, where $\widetilde{g} \in p^{-1}(g)$. Conversely, if $\widetilde{\varphi} = \mathrm{Int}_{\widetilde{g}}$ is inner, then also $\varphi = \mathrm{Int}_{p(g)}$ is inner. Therefore the lifting map $\mathrm{Aut}(G) \to \mathrm{Aut}(\widetilde{G})$ induces a map $\mathrm{Out}(G) \to \mathrm{Out}(\widetilde{G})$, and the latter is injective. □

Recall that we want to show that $\mathrm{Out}(G)$ is finite. By Claim 3.5, it is sufficient to show that $\mathrm{Out}(\widetilde{G})$ is finite.

Claim 3.6. *We have* $\mathrm{Aut}(\widetilde{G}) = \mathrm{Aut}(G') \times \mathrm{Aut}(T)$.

Proof. Let $\varphi \in \mathrm{Aut}(\widetilde{G})$. Then φ is of the form

$$\varphi(g, z) = \big(\alpha(g) \cdot \gamma(z),\, \delta(g) \cdot \beta(z)\big),$$

where $\alpha\colon G' \to G'$, $\beta\colon T \to T$, $\gamma\colon T \to G'$ and $\delta\colon G' \to T$. We must show that γ and δ are trivial.

Since G' is semisimple, it is equal to its own commutator subgroup and so δ is trivial [24, Lemma 4.28]. Now consider $\gamma\colon T \to G'$. The image of γ must lie inside $C_{G'}(\alpha(G'))$, the centralizer of the image of α. But $\varphi(G' \times \{1\}) \subset G' \times \{1\}$ because δ is trivial. The same argument can be applied to φ^{-1}, so actually $\varphi(G' \times \{1\}) = G' \times \{1\}$. On the other hand, $\varphi(G' \times \{1\}) = \mathrm{im}(\alpha) \times \{1\}$, so α is surjective. So the image of γ lies in $Z(G')$, which is finite. But T is connected, thus γ is also trivial. □

Clearly, $\mathrm{Inn}(G' \times T) = \mathrm{Inn}(G') \times \mathrm{Inn}(T)$. By Claim 3.6, we get

$$\mathrm{Out}(\widetilde{G}) = \frac{\mathrm{Aut}(G') \times \mathrm{Aut}(T)}{\mathrm{Inn}(G') \times \mathrm{Inn}(T)} = \mathrm{Out}(G') \times \mathrm{Out}(T).$$

Since G' is semisimple and simply connected, $\mathrm{Out}(G')$ is finite: outer automorphisms correspond to the (finitely many) automorphisms of the Dynkin diagram of G', cf. [24, Thm. 7.8]. Likewise, $\mathrm{Out}(T) = \mathrm{Aut}(T) = \mathrm{GL}(k, \mathbb{Z})$ is finite because $k \leq 1$. So $\mathrm{Out}(\widetilde{G})$ is finite, too. □

Proposition 3.7. *Let G be a group, $H \subset G$ a subgroup, $N = N_G(H)$ the normalizer and $C = C_G(H)$ the centralizer of H in G. Then there is an exact sequence*

$$1 \longrightarrow {}^{C}\!/_{Z(H)} \longrightarrow {}^{N}\!/_{H} \longrightarrow \mathrm{Out}(H).$$

Proof. Each $n \in N$ defines an automorphism of H by conjugation:

$$\varphi(n)(h) = nhn^{-1}.$$

Then $\varphi\colon N \to \mathrm{Aut}(H)$ is a group homomorphism and $\varphi(H) \subset \mathrm{Inn}(H)$. Therefore φ induces a map

$$\psi\colon {}^{N}\!/_{H} \longrightarrow \mathrm{Out}(H).$$

An element $n \in N$ maps to the identity in $\mathrm{Out}(H)$ if and only if $\varphi(n) \in \mathrm{Inn}(H)$ if and only if there exists $h \in H$ such that

$$nxn^{-1} = hxh^{-1} \quad \text{for all } x \in H$$

if and only if $h^{-1}n \in C$ if and only if $n \in hC$. Thus $\varphi^{-1}\big(\mathrm{Inn}(H)\big) = HC$ (note that this is a subgroup). Therefore the kernel of ψ is

$$\ker(\psi) = {}^{HC}\!/_{H} \cong {}^{C}\!/_{C \cap H} = {}^{C}\!/_{Z(H)}.$$

This ends the proof. □

4. The restricted holonomy group

In this section we will deduce information about the restricted holonomy group H° from the existence of certain tensors on X, using the Bochner principle. As in [19, Lemma 5.4], the proofs are modeled on arguments in [10,11]. The only difference is that instead of considering the universal cover of X (which we clearly cannot do), we argue purely locally.

Lemma 4.1. *Let X be an n-dimensional normal projective variety with klt singularities and ample canonical divisor K_X. Consider the decomposition $H^\circ = H_1^\circ \times \cdots \times H_m^\circ$ of the restricted holonomy group, as in (3.1.1). Assume that X satisfies condition (1.1.2), i.e. there is a semispecial tensor ψ on X. Then each H_i° is the holonomy of an irreducible bounded symmetric domain $\mathcal{D}_i$ of tube type whose rank divides its dimension. Moreover, each $\mathcal{D}_i$ is determined uniquely by the tensor ψ.*

Proof. Using notation from Setup 3.1, consider the decomposition

$$V = T_x X = U_1' \oplus U_1'' \oplus U_2,$$

where:

- U_1' is the sum of all the V_i such that the corresponding H_i° is the holonomy of a bounded symmetric domain of tube type,

- U_1'' is the sum of the V_i such that H_i° is the holonomy of a bounded symmetric domain which is neither a ball nor of tube type,
- and U_2 is the sum of the V_i such that H_i° acts transitively.

Let $u_1, \dots, u_a; w_1, \dots, w_b; z_1, \dots, z_r$ be coordinates on U_1', U_1'' and U_2, respectively. Let $\mathrm{vol}_1' = \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_a$, $\mathrm{vol}_1'' = \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_b$, $\mathrm{vol}_2 = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_r$ be the corresponding volume forms. Locally near the smooth point $x \in X_{\mathrm{reg}}$, the reflexive sheaf η is a line bundle and we fix a trivialization of it. This means that the tensor ψ evaluated at x is a linear map $\bigwedge^n V^\vee \to \mathrm{Sym}^n V^\vee$. We write this map as

$$\psi_x = f(u, w, z) \cdot (\mathrm{vol}_1')^{-1} \wedge (\mathrm{vol}_1'')^{-1} \wedge (\mathrm{vol}_2)^{-1}, \tag{4.1.1}$$

where $f \in \mathrm{Sym}^n V^\vee$ is a homogeneous degree n polynomial on V. This is just a suggestive notation for the map sending $\mathrm{vol}_1' \wedge \mathrm{vol}_1'' \wedge \mathrm{vol}_2 \mapsto f$.

Claim 4.2. *The polynomial f only depends on u, i.e. $f(u, w, z) = f(u)$.*

Proof. By the Bochner principle for klt spaces [19, Cor. 3.4] and the holonomy principle, ψ_x is H°-invariant. In particular, its zero scheme $\{\psi_x = 0\} \subset \mathbb{P}V$ is also H°-invariant. But $\{\psi_x = 0\} = \{f = 0\}$, so we can apply [7, Prop. A.1] to f. We obtain that f does not depend on z. Then by [10, Cor. 2.2], f also does not depend on w. □

Claim 4.3. *We have $U_1'' = U_2 = 0$. That is, $V = U_1'$.*

Proof. Arguing by contradiction and assuming $U_1'' \neq 0$, let i be an index with $V_i \subset U_1''$. Recall that the group H_i° acts trivially on V_j for $j \neq i$. Then by Claim 4.2, it acts trivially on f. It also acts trivially on vol_1' and on vol_2. On other hand, it acts non-trivially on vol_1''. This is because H_i° is *not* contained in the special unitary group of V_i. Otherwise, the singular Kähler–Einstein metric ω_{KE} from Setup 3.1 would be Ricci-flat along the direction of V_i, contradicting the fact that $\mathrm{Ric}(\omega_{\mathrm{KE}}) = -\omega_{\mathrm{KE}}$ is negative definite. (Cf. also the proof of [19, Lemma 5.2]).

Putting together these observations with (4.1.1), we obtain that H_i° acts non-trivially on ψ_x, contradicting the H°-invariance of ψ_x. This contradiction shows that $U_1'' = 0$. An entirely similar argument shows that $U_2 = 0$. □

From Claim 4.3, it follows that each H_i° is the holonomy of some bounded symmetric domain of tube type $\mathcal{D}_i$. This proves the first part of the lemma's statement. The next claim proves the second part.

Claim 4.4. *For each $j = 1, \dots, m$, we have that $r_j \coloneqq \mathrm{rk}\, \mathcal{D}_j$ divides $n_j \coloneqq \dim \mathcal{D}_j$.*

Proof. Consider the Harish–Chandra realization of $\mathcal{D}_j$ as an open subset $0 \in \mathcal{D}_j \subset \mathbb{C}^{n_j}$ invariant under the group of rotations $S_j \coloneqq \left\{ \mathrm{e}^{\mathrm{i}t} I_{n_j} \;\middle|\; t \in \mathbb{R} \right\}$. This yields coordinates

$z_j = (z_{j,1}, \dots, z_{j,n_j})$ on $\mathcal{D}_j$, and we set $\mathrm{d}z_j \coloneqq \mathrm{d}z_{j,1} \wedge \cdots \wedge \mathrm{d}z_{j,n_j}$. We also set $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_m \subset \mathbb{C}^n$.

For each j, consider the isotropy subgroup $K_j \subset \mathrm{Aut}(\mathcal{D}_j)$ of $0 \in \mathcal{D}_j$. Note that by the above remark, we have $S_j \subset K_j$. Recall from [3, Prop. 10.79] that the holonomy group H_j° equals K_j. More precisely, we can identify V_j and $T_0\mathcal{D}_j = \mathbb{C}^{n_j}$ in such a way that the action of H_j° on V_j corresponds to the action of K_j on $T_0\mathcal{D}_j$. Doing this for all j, we can identify V and $T_0\mathcal{D} = \mathbb{C}^n$ in such a way that the action of H° on V corresponds to the action of $K \coloneqq K_1 \times \cdots \times K_m$ on $T_0\mathcal{D}$.

Under this identification, we consider ψ_x as a tensor on $T_0\mathcal{D}$ and the polynomial f from (4.1.1) as a polynomial on $\mathcal{D} \subset T_0\mathcal{D}$. Since ψ_x is K-invariant, f is K-semi-invariant. This means that there is a character $\chi\colon K \to \mathbb{C}^*$ such that for all $g \in K$ and $z \in \mathcal{D}$, we have $f(g \cdot z) = \chi(g) \cdot f(z)$. Applying [10, Cor. 2.2] to f, we write

$$\psi_x = C \cdot N_1^{\ell_1}(z_1) \cdots N_m^{\ell_m}(z_m) \cdot (\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m)^{-1}, \tag{4.4.1}$$

using notation as in (4.1.1) and with some nonzero constant $C \in \mathbb{C}^*$. Here the N_j are the polynomials on $\mathcal{D}_j$ given by [10, Thm. 2.1].

For each j, we know that:

- ψ_x is K_j-invariant, since it is even K-invariant.
- N_j is K_j-semi-invariant by [10, remark before Cor. 2.2].
- N_i is K_j-invariant for $i \neq j$, since K_j acts trivially on $\mathcal{D}_i$.
- $\mathrm{d}z_j$ is K_j-semi-invariant, since the space of volume forms is one-dimensional.
- $\mathrm{d}z_i$ is K_j-invariant for $i \neq j$, again since K_j acts trivially on $\mathcal{D}_i$.

Restrict the action of K_j to $S_j \subset K_j$, and note that $S_j \cong S^1$, the circle group. Since $N_j^{\ell_j}$ is homogeneous of degree $\ell_j r_j$, the action of S_j on $N_j^{\ell_j}$ is given by the character $t \mapsto t^{\ell_j r_j}$. Likewise, the action on $\mathrm{d}z_j$ is given by $t \mapsto t^{n_j}$. Thus the action on ψ_x is given by $t \mapsto t^{\ell_j r_j - n_j}$. We conclude by the S_j-invariance of ψ_x that $\ell_j r_j - n_j = 0$. Hence r_j divides n_j, as desired. □

To prove the last claim of the lemma, we may use the classification from [10, Thm. 2.3] as is, to conclude that the pairs $(\mathrm{rk}\,\mathcal{D}_j, \dim \mathcal{D}_j)$ uniquely determine each domain $\mathcal{D}_j$. □

Lemma 4.5. *Assume that X satisfies condition (1.2.2), i.e. there is a slope zero tensor ψ on X. Then each H_i° is the holonomy of an irreducible bounded symmetric domain $\mathcal{D}_i$ of tube type.*

Proof. The argument is very similar to the first half of the proof of Lemma 4.1. We will use notation from that proof and omit some details.

The slope zero tensor ψ evaluated at x can be written as

$$\psi_x = f(u, w, z) \cdot (\mathrm{vol}_1')^{-m} \wedge (\mathrm{vol}_1'')^{-m} \wedge (\mathrm{vol}_2)^{-m}, \tag{4.5.1}$$

where f is a homogeneous degree mn polynomial on V. As in Claim 4.2, we see that f only depends on u. Then as in Claim 4.3, we see that $U_1'' = U_2 = 0$. This means that each H_i° is the holonomy of an irreducible bounded symmetric domain of tube type. □

Lemma 4.6. *Assume that X satisfies condition (1.3.2). Then each H_i° is the holonomy of an irreducible bounded symmetric domain $\mathcal{D}_i$ not isomorphic to $\mathbb{B}^m$ for any $m \geq 2$. Moreover, each $\mathcal{D}_i$ is determined uniquely by the tensor σ.*

Proof. Since K_X is assumed to be ample, the tensor σ is parallel with respect to the Kähler–Einstein metric ω_{KE} on X_{reg} by the Bochner principle [19, Cor. 3.4]. In particular, the first Mok characteristic cone $\mathcal{CS}$ of σ is invariant under the full holonomy group H. Since the action of H on $V = T_xX$ is continuous, the restricted holonomy group H° is connected, and the set of irreducible components of $\mathcal{CS}$ is finite, it follows that the decomposition of $\mathcal{CS}$ into irreducible components is invariant under H°.

Since K_X is ample, each H_i° is either equal to $\mathrm{U}(V_i)$, or H_i° acts on V_i as the holonomy of an irreducible bounded symmetric domain $\mathcal{D}_i$ of rank ≥ 2. Since we work locally around the smooth point $x \in X_{\mathrm{reg}}$, the arguments in the proof of [11, Thm. 1.2] work verbatim. In particular, they show that the cones $\mathcal{CS}_i$ are just the origin when $\dim V_i = 1$, and they are the cones over the first Mok characteristic variety otherwise. This rules out the case $H_i^\circ = \mathrm{U}(V_i)$, for all i. In particular, no $\mathcal{D}_i$ can be a ball $\mathbb{B}^m$ with $m \geq 2$.

The tensor σ determines the first Mok characteristic varieties $\mathcal{S}_i^1$ (see [11, Def. 2.1]), and since $H_i^\circ = \mathrm{Hol}(\mathcal{D}_i, \omega_{\mathrm{Berg}})$ for all i, where the $\mathcal{D}_i$ are irreducible bounded symmetric domains, we have $\mathcal{S}_i^1 = \mathcal{S}^1(\mathcal{D}_i)$ for all i. From the claim in the proof of [11, Thm. 1.2], each irreducible bounded symmetric domain $\mathcal{D}_i$ is determined by the data $(\dim \mathcal{D}_i, \dim \mathcal{S}^1(\mathcal{D}_i))$. □

5. Local isometries

Assume that we are in the situation of Theorem 1.1, Theorem 1.2 or Theorem 1.3, and that we want to prove the implication from the second item (existence of a tensor) to the first one (uniformization). Applying the corresponding lemma from Section 4, we get irreducible bounded symmetric domains $\mathcal{D}_i$ such that each H_i° is the holonomy of $\mathcal{D}_i$. Set $\mathcal{D} \coloneqq \mathcal{D}_1 \times \cdots \times \mathcal{D}_m$ and let ω_{Berg} be the Bergman metric on $\mathcal{D}$. As an intermediate step towards uniformization, we prove the following statement.

Proposition 5.1. *The spaces $(X_{\mathrm{reg}}, \omega_{\mathrm{KE}})$ and $(\mathcal{D}, \omega_{\mathrm{Berg}})$ are locally isometric.*

Remark. Recall from [14, Prop. 4.3] that for any simply connected open subset $U \subset X_{\mathrm{reg}}$, the holonomy of $\left(U, \omega_{\mathrm{KE}}\big|_U\right)$ equals the restricted holonomy H° of $(X_{\mathrm{reg}}, \omega_{\mathrm{KE}})$. Since Proposition 5.1 is a local statement, this explains why it is sufficient to consider only H° in the proof (as opposed to the full holonomy group H).

Proof of Proposition 5.1. Pick an arbitrary point $x \in X_{\mathrm{reg}}$ and let $x \in U \subset X$ be a sufficiently small open neighborhood such that the decomposition (3.1.1) of T_xX is induced by a decomposition $U = U_1 \times \cdots \times U_m$, cf. [3, Thm. 10.38]. We want to show that U is isometric to an open subset of $\mathcal{D}$. For this, it is sufficient to show that for each $i = 1, \ldots, m$, the manifolds U_i and $\mathcal{D}_i$ are locally isometric. We already know that they have the same holonomy group, namely H_i°, and the same dimension, namely $\dim V_i$. After renumbering, we may assume that there is an m_0 such that $\dim U_i = 1$ if and only if $i \leq m_0$.

If $i \leq m_0$, then U_i is locally isometric to the unit disc with the Poincaré metric, which is nothing but $\mathcal{D}_i$. Otherwise, $\mathcal{D}_i$ is a symmetric space of rank ≥ 2: indeed, by Proposition 2.15, if $\operatorname{rk} \mathcal{D}_i = 1$, then $\mathcal{D}_i$ would be a ball $\mathbb{B}^n$ with $n \geq 2$, which is excluded in each case by the corresponding lemma in Section 4.

By [3, Thm. 10.90], the holonomy of $\mathcal{D}_i$ is *not* transitive on the unit sphere of its tangent space at the base point. Therefore also the holonomy of U_i is not transitive on the unit sphere. By said theorem again, U_i is an irreducible locally symmetric space of rank ≥ 2. By [20, Ch. IV, Theorem 5.1], we can embed U_i isometrically as an open subset of some (globally) symmetric space M_i. Since $\mathcal{D}_i$ and M_i have the same dimension and holonomy group (in particular, the same holonomy Lie algebra), Proposition 2.16 implies that they are isomorphic (in particular, isometric when equipped with the Bergman metric). This means that U_i and $\mathcal{D}_i$ are locally isometric, as desired. □

6. Proof of Theorem 1.4

In this section we prove Theorem 1.4. In Proposition 6.3, we show how to obtain a uVHS from a tensor whose first Mok characteristic cone satisfies the properties in the theorem. In Proposition 6.6, we deal with the converse direction. See [17, Sec. 5] for more details about variations of Hodge structures on klt spaces.

Remark 6.1. By [3, Prop. 10.79], each irreducible bounded symmetric domain $\mathcal{D}_i$ in Lemma 4.6 can be expressed as $\mathcal{D}_i = {}^{G_i^0}\!/_{K_i^0}$, where $K_i^0 = H_i^\circ = \mathrm{Hol}(\mathcal{D}_i, \omega_{\mathrm{Berg}})$ and H_i° acts on the tangent space $T_e\mathcal{D}_i$ via the adjoint representation. It follows that the irreducible holonomy factors H_i°, and hence the restricted holonomy group H°, are also determined by σ.

Lemma 6.2. *Let $\gamma\colon Y \to X$ be a finite quasi-étale cover, where X and Y are normal projective varieties with klt singularities. If Y_{reg} admits a uniformizing VHS for some Hodge group G_0 of Hermitian type then so does X_{reg}.*

Proof. Suppose Y_{reg} admits a uniformizing VHS corresponding to G_0. Then we know from Proposition 2.10 that Y is uniformized by $\mathcal{D} = {}^{G_0}\!/_{K_0}$. Set $Y^\circ \coloneqq \gamma^{-1}(X_{\mathrm{reg}})$. Since $Y^\circ \to X_{\mathrm{reg}}$ is étale, X_{reg} and Y° have the same universal cover $\mathcal{D}^\circ$, which is a big open

subset of $\mathcal{D}$. Moreover, $X_{\mathrm{reg}} \cong {\mathcal{D}^\circ}/{\Gamma}$, where $\Gamma \subset \operatorname{Aut}\mathcal{D}$ is a discrete cocompact subgroup isomorphic to $\pi_1(X_{\mathrm{reg}})$.

According to Example 2.8, there is a uniformizing VHS $(P_{\mathcal{D}}, \vartheta_{\mathcal{D}})$ on $\mathcal{D}$ corresponding to G_0, which restricts to a uniformizing VHS $(P_{\mathcal{D}}, \vartheta_{\mathcal{D}})\big|_{\mathcal{D}^\circ}$ on $\mathcal{D}^\circ$. Following the same arguments as in the proof of Proposition 2.10, we see that $(P_{\mathcal{D}}, \vartheta_{\mathcal{D}})\big|_{\mathcal{D}^\circ}$ is Γ-equivariant and descends to a uniformizing VHS on X_{reg}. □

We can now prove the implication "(1.4.2) ⇒ (1.4.1)".

Proposition 6.3. *If X admits a holomorphic tensor σ as in Theorem 1.4, then there is a uniformizing VHS (P, ϑ) on X_{reg} for a Hodge group G_0 of Hermitian type such that $\mathfrak{g}_0$ has no factors isomorphic to $\mathfrak{su}(p,1)$ with $p \geq 2$.*

Proof. Using notation from Setup 3.1, we know by Theorem 3.2 that H is compact and that there is a quasi-étale Galois cover $\gamma\colon Y \to X$ such that $\mathrm{Hol}(Y_{\mathrm{reg}}, g_{\mathrm{KE},Y})$ is connected. By Lemma 6.2, X_{reg} admits a uniformizing VHS if Y_{reg} does. Therefore, we may replace X by Y and assume without loss of generality that H is connected. In other words, we may assume that $H = H^\circ$.

Claim 6.4. *There is a uniformizing system of Hodge bundles (P, ϑ) on X_{reg}, together with a metric $P_H \subset P$.*

Proof of Claim 6.4. Recall that any hermitian vector bundle admits a reduction of structure group to its holonomy group. In particular, the tangent bundle of X_{reg} can be written as $\mathcal{T}_{X_{\mathrm{reg}}} \cong P_H \times_H V$, where P_H is the holonomy subbundle of the Chern connection on $\mathcal{T}_{X_{\mathrm{reg}}}$, the vector space V is the tangent space T_xX as in Setup 3.1, and H acts on V via the holonomy representation.

From Lemma 4.6 we have a decomposition $H = H^\circ = H_1^\circ \times \cdots \times H_m^\circ$, where $H_i^\circ = \mathrm{Hol}(\mathcal{D}_i, \omega_{\mathrm{Berg}})$ for $\mathcal{D}_i$ an irreducible bounded symmetric domain not isomorphic to a higher-dimensional ball for each $i = 1, \ldots, m$. Recall that V decomposes as $V = V_1 \oplus \cdots \oplus V_m$, where H_i° acts irreducibly on V_i and trivially on V_j for $j \neq i$.

By Remark 6.1, we have $\mathcal{D}_i = {G_i^0}\big/{K_i^0}$, where G_i^0 is a Hodge group of Hermitian type. Let G_i and K_i denote the complexifications of G_i^0 and K_i^0, respectively. Then $\mathfrak{g}_i := \mathrm{Lie}(G_i)$ admits a Hodge decomposition given by $\mathfrak{g}_i = \mathfrak{g}_i^{-1,1} \oplus \mathfrak{g}_i^{0,0} \oplus \mathfrak{g}_i^{1,-1}$. By Example 2.8, the tangent bundle of $\mathcal{D}_i$ can be written as

$$\mathcal{T}_{\mathcal{D}_i} \cong P_i \times_{K_i^0} \mathfrak{g}_i^{-1,1},$$

where P_i is the principal K_i^0-bundle $G_i^0 \to \mathcal{D}_i$ and K_i^0 acts on $\mathfrak{g}_i^{-1,1}$ via the adjoint representation. By [3, Prop. 10.79], the given action of H_i° on V_i is isomorphic to the adjoint action of K_i^0 on $\mathfrak{g}_i^{-1,1}$ under the above identifications. Since this action is via $\mathbb{C}$-linear maps, the adjoint representation of K_i^0 on V_i factors through the complexification K_i. It then follows that V_i and $\mathfrak{g}_i^{-1,1}$ are isomorphic as K_i-representations.

Let $\mathcal{D} \coloneqq \mathcal{D}_1 \times \cdots \times \mathcal{D}_m = {}^{G_0}\!/_{K_0}$, where $G_0 \coloneqq G_1^0 \times \cdots \times G_m^0$ and $K_0 \coloneqq K_1^0 \times \cdots \times K_m^0$. Note that $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$ has no factors isomorphic to $\mathfrak{su}(p,1)$ with $p \geq 2$ because no factor $\mathcal{D}_i$ is a higher-dimensional ball. Set $\mathfrak{g} \coloneqq \mathrm{Lie}(G)$ with G the complexification of G_0 and K the complexification of K_0. Consider the principal K-bundle $P \coloneqq P_H \times_{K_0} K$ on X_{reg}, where we have identified H and K_0. By the above discussion, we have a chain of isomorphisms

$$\mathcal{T}_{X_{\mathrm{reg}}} \cong P_H \times_{K_0} V \cong P \times_K V \cong \bigoplus_{i=1}^m P \times_{K_i} V_i \cong \bigoplus_{i=1}^m P \times_{K_i} \mathfrak{g}_i^{-1,1} \cong P \times_K \mathfrak{g}^{-1,1}.$$

This yields an isomorphism

$$\vartheta \colon \mathcal{T}_{X_{\mathrm{reg}}} \longrightarrow P \times_K \mathfrak{g}^{-1,1}$$

of vector bundles, which automatically satisfies $[\vartheta(u), \vartheta(v)] = 0$ for all local sections u, v of $\mathcal{T}_{X_{\mathrm{reg}}}$. Therefore, (P, ϑ) is a uniformizing system of Hodge bundles on X_{reg}, and $P_H \subset P$ is a metric. □

Claim 6.5. *The uniformizing system of Hodge bundles (P, ϑ) together with the metric P_H given in Claim 6.4 is in fact a uniformizing VHS.*

Proof of Claim 6.5. On the bounded symmetric domain $\mathcal{D}$ from above, there is by Example 2.8 a natural uniformizing system of Hodge bundles $(P_{\mathcal{D}}, \vartheta_{\mathcal{D}})$ given by $P_{\mathcal{D}} \coloneqq G_0 \times_{K_0} K$ and $\vartheta_{\mathcal{D}} \colon \mathcal{T}_{\mathcal{D}} \xrightarrow{\sim} P_{\mathcal{D}} \times_K \mathfrak{g}^{-1,1}$. The principal K-bundle $P_{\mathcal{D}}$ admits a metric $P_{\mathcal{D},H}$ given by $G_0 \subset P_{\mathcal{D}}$. This makes $(P_{\mathcal{D}}, \vartheta_{\mathcal{D}})$ together with $P_{\mathcal{D},H}$ into a uniformizing VHS.

It follows from Proposition 5.1 that X_{reg} and the domain $\mathcal{D}$ are locally isometric. More precisely, any point $p \in X_{\mathrm{reg}}$ has an analytic open neighborhood U such that there is an isometry $f \colon U \xrightarrow{\sim} U'$, where U' is an analytic open subset of $\mathcal{D}$. Let F_U be the frame bundle of $\mathcal{T}_U$ and $F_{U'}$ the frame bundle of $\mathcal{T}_{U'}$. Then f induces an isomorphism $\mathrm{d}f \colon F_U \xrightarrow{\sim} f^* F_{U'}$. Since f is an isometry, $\mathrm{d}f$ preserves the holonomy subbundles of U and U', that is, $\mathrm{d}f\big(P_H\big|_U\big) = f^*\big(P_{\mathcal{D},H}\big|_{U'}\big)$ as subbundles of $f^* F_{U'}$. Extending the structure group from K_0 to K, we see that $P\big|_U$ and $f^*\big(P_{\mathcal{D}}\big|_{U'}\big)$ are also isomorphic. We then get a commutative diagram

$$\begin{array}{ccc} \mathcal{T}_U & \xrightarrow{\quad\vartheta\quad} & P\big|_U \times_K \mathfrak{g}^{-1,1} \\ \wr\downarrow & & \downarrow\wr \\ f^*\mathcal{T}_{U'} & \xrightarrow{f^*\vartheta_{\mathcal{D}}} & f^*\big(P_{\mathcal{D}}\big|_{U'}\big) \times_K \mathfrak{g}^{-1,1}. \end{array}$$

This means that the uniformizing systems of Hodge bundles (P, ϑ) and $(P_{\mathcal{D}}, \vartheta_{\mathcal{D}})$ are locally isomorphic via $\mathrm{d}f$. Furthermore, as noted above the given metrics P_H and $P_{\mathcal{D},H}$ are compatible with this isomorphism.

Let $R \coloneqq P_H \times_{K_0} G_0$ be equipped with the connection D_h from Construction 2.6, and analogously define $R_{\mathcal{D}} \coloneqq P_{\mathcal{D},H} \times_{K_0} G_0$ with the connection $D_{\mathcal{D}}$. By construction, (R, D_h) is determined by the data given by (P, ϑ) and P_H, and analogously $(R_{\mathcal{D}}, D_{\mathcal{D}})$ is determined by $(P_{\mathcal{D}}, \vartheta_{\mathcal{D}})$ and $P_{\mathcal{D},H}$. More precisely, since $P_H\big|_U \cong f^* P_{\mathcal{D}}\big|_U$, we have $R\big|_U \cong P_H\big|_U \times_{K_0} G_0 \cong f^* P_{\mathcal{D},H}\big|_U \times_{K_0} G_0 \cong f^* R_{\mathcal{D}}\big|_U$ on the level of principal bundles. On the level of metrics, we have $D_h = d_h + \vartheta + \overline{\vartheta}$, where d_h is the connection on R induced by the unique connection on P_H compatible with the holomorphic structure on P, and analogously for $D_{\mathcal{D}}$. Since $(P, \vartheta)\big|_U \cong f^*\big((P_{\mathcal{D}}, \vartheta_{\mathcal{D}})\big|_U\big)$, it follows that there is an isomorphism $(R, D_h)\big|_U \cong f^*\big((R_{\mathcal{D}}, D_{\mathcal{D}})\big|_{U'}\big)$.

Recall that in order to prove the claim, we need to show that D_h is a flat connection. Since we already know that $D_{\mathcal{D}}$ is flat, it follows from the above isomorphism that $D_h\big|_U$ is flat. But flatness is a local property, hence it follows by varying $p \in X_{\mathrm{reg}}$ that D_h is in fact flat. Thus (P, ϑ) is a uniformizing VHS on X_{reg}. □

This ends the proof of Proposition 6.3. □

Now we prove the converse implication "(1.4.1) ⇒ (1.4.2)". This will complete the proof of Theorem 1.4.

Proposition 6.6. *Let X be as in Theorem 1.4, and suppose that X admits a uniformizing VHS (P, ϑ) corresponding to a Hodge group G_0 of Hermitian type such that $\mathfrak{g}_0$ has no factors isomorphic to $\mathfrak{su}(p, 1)$ with $p \geq 2$. Then there is a holomorphic tensor $\sigma \in \mathrm{H}^0\big(X, \mathscr{E}nd\big(\mathcal{T}_X[\otimes]\Omega^1_X\big)\big)$ satisfying properties (1.3.3)–(1.3.5).*

Proof. By assumption, there is a bounded symmetric domain $\mathcal{D} = G_0/K_0$ without higher-dimensional ball factors and an isomorphism $\vartheta\colon \mathcal{T}_{X_{\mathrm{reg}}} \xrightarrow{\sim} P \times_K \mathfrak{g}^{-1,1}$. Dualizing, we get $\Omega^1_{X_{\mathrm{reg}}} \cong P \times_K \mathfrak{g}^{1,-1}$. Let $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_m$ be the decomposition of $\mathcal{D}$ into irreducible bounded symmetric domains $\mathcal{D}_i = G_i^0 \big/ K_i^0$.

The Lie algebra $\mathfrak{g}$ decomposes into simple factors $\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i$, where $\mathfrak{g}_i$ is the complexified Lie algebra of G_i^0. From this we get $\mathfrak{g}^{p,-p} = \bigoplus_{i=1}^m \mathfrak{g}_i^{p,-p}$ for all $p \in \{-1, 0, 1\}$. The Lie bracket of each $\mathfrak{g}_i$ induces a surjective morphism

$$\rho_i\colon \mathfrak{g}_i^{-1,1} \otimes \mathfrak{g}_i^{1,-1} \longrightarrow \mathfrak{g}_i^{0,0}, \qquad u \otimes v \mapsto [u, v].$$

There is also a natural injective morphism in the other direction:

$$\tau_i\colon \mathfrak{g}_i^{0,0} \longrightarrow \mathfrak{g}_i^{-1,1} \otimes \mathfrak{g}_i^{1,-1} = \mathrm{Hom}\Big(\mathfrak{g}_i^{-1,1}, \mathfrak{g}_i^{-1,1}\Big), \qquad w \mapsto [w, -].$$

Setting $\sigma_i' \coloneqq \tau_i \circ \rho_i$, we obtain an endomorphism of $\mathfrak{g}_i^{-1,1} \otimes \mathfrak{g}_i^{1,-1}$. For each $1 \leq i \leq m$, we define σ_i as follows:

$$\sigma_i \coloneqq \begin{cases} \sigma_i' & \text{if } \dim \mathcal{D}_i > 1, \\ \mathrm{id}_{\mathfrak{g}_i^{-1,1} \otimes \mathfrak{g}_i^{1,-1}} & \text{if } \dim \mathcal{D}_i = 1. \end{cases}$$

Finally we set $\sigma \coloneqq \bigoplus_{i=1}^{m} \sigma_i$, an endomorphism of $\mathfrak{g}^{-1,1} \otimes \mathfrak{g}^{1,-1}$.

The endomorphism σ induces an endomorphism of $\mathcal{T}_{X_{\mathrm{reg}}} \otimes \Omega^1_{X_{\mathrm{reg}}}$, which by reflexivity extends to an endomorphism $\sigma \in \mathrm{H}^0\big(X, \mathscr{E}nd\big(\mathcal{T}_X[\otimes]\Omega^1_X\big)\big)$. It remains to verify that σ satisfies the three properties (1.3.3)–(1.3.5), for which we restrict to any point $p \in X_{\mathrm{reg}}$ and work again with $\sigma \in \mathrm{End}\big(\mathfrak{g}^{-1,1} \otimes \mathfrak{g}^{1,-1}\big)$.

By construction, σ_i coincides with the algebraic curvature tensor of an irreducible bounded symmetric domain defined in [25, p. 211] (paragraph preceding Lemma 2.9), for all i such that the rank of $\mathcal{D}_i$ is ≥ 2. It is then clear that σ coincides with the tensor defined in the first part of the proof of [11, Thm. 1.2]. Therefore, σ satisfies (1.3.3)–(1.3.5) by the same arguments therein. □

7. Proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.1. We prove both directions separately.

"(1.1.1) ⇒ (1.1.2)": We know from [10, proof of Thm. 1.2, p. 429] that on each $\mathcal{D}_i$, there is a special tensor Ψ_i semi-invariant under $\mathrm{Aut}(\mathcal{D}_i)$. (Here a special tensor is a semispecial tensor with η the trivial line bundle, and "semi-invariant" means invariant up to a character $\chi_i \colon \mathrm{Aut}(\mathcal{D}_i) \to \{\pm 1\}$.) Consider the special tensor $\Psi \coloneqq \mathrm{pr}_1^* \Psi_1 \otimes \cdots \otimes \mathrm{pr}_m^* \Psi_m$ on $\mathcal{D}$, where $\mathrm{pr}_i \colon \mathcal{D} \to \mathcal{D}_i$ are the projections. As in [19, proof of Thm. 6.1], it can be seen that Ψ descends to a semispecial tensor ψ on X.

"(1.1.2) ⇒ (1.1.1)": By Lemma 4.1, the existence of a semispecial tensor ψ implies that $\mathrm{Hol}(X_{\mathrm{reg}}, \omega_{\mathrm{KE}})$ is the holonomy of a bounded symmetric domain $\mathcal{D}$ whose irreducible factors are domains of tube type whose rank divides their dimension. Moreover, from Proposition 5.1 we have that X_{reg} and $\mathcal{D}$ are locally isometric. By the same arguments as in the proof of Proposition 6.3, it follows that X_{reg} admits a uniformizing VHS for $\mathrm{Aut}(\mathcal{D})$. We conclude from Proposition 2.10 that X is uniformized by $\mathcal{D}$. □

Proof of Theorem 1.2. The proof of Theorem 1.2 is very similar to the one of Theorem 1.1, and hence is omitted. □

Proof of Theorem 1.3. Theorem 1.3 is an immediate consequence of combining Theorem 1.4 and Proposition 2.10. □

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