MASTER THESIS

Spectral properties of the isochrone steady state of the Vlasov-Poisson system

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Introduction

In the endeavor to better understand our galaxy and the universe, mathematicians have tried to model galaxies and the universe for a long time. On the basis of Newton's law of gravity, one can try to model each stellar object as an individual point mass in space. However, in galaxies, for instance, the number of stars can be of the order of $\sim 10^{11}$ and it is no longer feasible to use this approach. For this reason, we interpret a galaxy as having a continuous mass distribution and therefore introduce the density on phase space $f = f(t, x, v)$. Here, $t \geq 0$ is the time variable, $x \in \mathbb{R}^3$ is the space variable, and $v \in \mathbb{R}^3$ is the velocity variable. If we want to know the amount of mass that is contained in a certain region of phase space at any given time t, we have to integrate $f(t, \cdot, \cdot)$ over this region.
By Newton's law, a particle at position x with velocity v that is exposed to the potential $U = U(t, x)$ satisfies the equations of motion

$$\dot{x} = v, \quad \dot{v} = -\partial_x U(t, x).$$

As collisions rarely happen, we can neglect them and therefore, f is constant along orbits, i.e. solutions to the equations of motion, which yields

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0.$$

We call this equation the Vlasov equation. It describes the motion and temporal change of the mass distribution induced by a given potential U. This potential is induced by the spatial mass density $\rho = \rho(t, x)$, governed by the Poisson equation

$$\Delta U = 4\pi\rho, \qquad \lim_{|x|\to\infty} U(t, x) = 0.$$

We couple the density on phase space f with the spatial mass density ρ through

$$\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v)\,\mathrm{d}v.$$

The model that consists of the three above equations and the boundary condition for U is called the Vlasov-Poisson system, which will be the subject of this thesis.

We are interested in steady states, i.e. time-independent solutions of the Vlasov-Poisson system. To be more precise, we will focus on a special type of steady state, namely the isochrone steady state. The isochrone model was introduced by the mathematician and astronomer Michel Hénon in [10], [11] and [12]. He derived the so-called isochrone potential, which is a spherically symmetric potential with the property that the period of a particle orbit only depends on its energy and is independent of its angular momentum. This derivation and some further discussions have also been carried out by James Binney in [2]. Additionally, the isochrone potential and a few of its properties can be found in [3].

In this thesis, we will start with some notations and with setting up the framework of spherical symmetry in Chapter 1. We will then be concerned with deriving and introducing the isochrone model. This will be done in Chapter 2. After that, in Chapter 3 and Chapter 4, we will examine its properties, including the period function that corresponds to periods of particle orbits. We will obtain that this period function is indeed independent of the angular momentum.
Chapter 5 deals with linearizing the Vlasov-Poisson system about the isochrone steady state, which will yield an operator $\mathcal{A}$ that we call the Antonov operator due to historical reasons. This operator will be properly defined in Chapter 6, where we will also collect and prove some operator properties.
In Chapter 7, we will prove the main result of this thesis: The spectrum of $\mathcal{A}$ is purely essential and equal to $[0,\infty[$. The proof consists of two main steps, which are to show that $[0,\infty[\subset \sigma_{\text{ess}}(\mathcal{A})$ and that $\sigma(\mathcal{A}) \subset [0,\infty[$. For the first part, we initially follow [9]. However, only a small part of this procedure works out in our case and we need to come up with an additional, new proof. A more detailed description of similarities and differences can be found in the beginning of Chapter 7. The proof of $\sigma(\mathcal{A}) \subset [0,\infty[$ relies on [7].
In Chapter 8, we describe future ambitions for using the results of this thesis to show linear, asymptotic stability of the isochrone steady state.

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1 Notations and spherical symmetry

Before we start with establishing and analyzing the isochrone model, we introduce the notations used in this thesis as well as the framework of spherical symmetry.

1.1 Notations

By $\mathbb{N}$, we denote the set containing all natural numbers, excluding 0, i.e. $\mathbb{N} = \mathbb{N} \setminus \{0\}$. Furthermore, $\mathbb{K}$ can either be $\mathbb{R}$ or $\mathbb{C}$. Considering $\mathbb{R}^n$, we denote the i-th unit vector by e_i. Moreover, for $x, v \in \mathbb{R}^n$, we use the notation

$$x \cdot v = \sum_{i=1}^{n} x_i v_i, \quad |x| = \sqrt{x \cdot x}$$

for the inner product of x and v and for the Euclidean norm of x, respectively. For $n = 3$, we denote the cross product of x and v by $x \times v$.
For $r > 0$ and $a \in \mathbb{R}^n$, the ball $B_r(a) \subset \mathbb{R}^n$ is defined by

$$B_r(a) := \{x \in \mathbb{R}^n \ \mid \ |x - a| < r\}.$$

For a subset $M \subset \mathbb{R}^n$, we define the indicator function 1_M through

$$1_M(x) := \begin{cases} 1, & x \in M, \\ 0, & x \notin M. \end{cases}$$

By λ, we denote the Lebesgue measure. For $p \geq 1$ and a measurable set $\Omega \subset \mathbb{R}^n$, we define $L^p(\Omega; \mathbb{K})$ as the set that contains all equivalence classes of functions $f : \Omega \to \mathbb{K}$ that are measurable and for which $|f|^p$ is integrable. Two functions are equivalent if they are equal almost everywhere on Ω. For the sake of simplicity, we denote the equivalence class of a function f by f again and consider some pointwise defined representative for pointwise operations. The corresponding norm is defined as

$$||f||_p := ||f||_{L^p} := ||f||_{L^p(\Omega)} := \left(\int_\Omega |f|^p \right)^{1/p}.$$

We define $L^1_{\mathrm{loc}}(\Omega; \mathbb{K})$ as the set containing all locally integrable functions, i.e. f is in $L^1_{\mathrm{loc}}(\Omega; \mathbb{K})$ if it is measurable and $f|_K$ is in $L^1(K; \mathbb{K})$ for every compact set $K \subset \Omega$. To define the Sobolev spaces $H^k(\Omega; \mathbb{K})$ for $k \in \mathbb{N}$, we additionally require that Ω is an open set. A function $f : \Omega \to \mathbb{K}$ is an element of $H^k(\Omega; \mathbb{K})$ if f and all of its weak derivatives up to k-th order are in $L^2(\Omega; \mathbb{K})$. The corresponding norm is

$$||f||_{H^k} := ||f||_{H^k(\Omega)} := \left(\sum_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq k} ||D^\alpha f||^2_{L^2(\Omega)} \right)^{1/2}.$$

For open sets $\Omega \subset \mathbb{R}^n$ and $k \in \mathbb{N}$, $C^k(\Omega;\mathbb{K})$ denotes the set containing all functions $f : \Omega \to \mathbb{K}$ that are k times differentiable in the classical sense with continuous derivatives. For $k = \infty$, f has to be differentiable arbitrarily often. An index c means that f is compactly supported in Ω. If $\Omega \subset \mathbb{R}$ is an interval, it must not be an open set. For the boundary points, we then consider one-sided limits.
For all function spaces, an index r denotes the restriction to spherically symmetric functions as defined in Section 1.2.
Generally, constants $C > 0$ may change in each step. We sometimes highlight special dependencies and if C depends on the parameter b that will be introduced in Chapter 2, we will always denote this by $C(b)$.
An integral without a specific integration area is to be understood as an integral over $\mathbb{R}^3$, i.e. "$\int = \int_{\mathbb{R}^3}$". Lastly, for sequences, we often write $(\cdot)$ instead of $(\cdot)_{k\in\mathbb{N}}$.

1.2 Spherical symmetry and (r, w, L)-coordinates

We start with defining spherical symmetry:

Definition 1.1. *A function $f : \mathbb{R}^3 \to \mathbb{K}$ is called spherically symmetric if*

$$f(Ax) = f(x), \quad A \in \mathrm{SO}(3), x \in \mathbb{R}^3.$$

A function $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{K}$ is called spherically symmetric if

$$g(Ax, Av) = g(x, v), \quad A \in \mathrm{SO}(3), (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

For functions $f : \mathbb{R}^3 \to \mathbb{K}$ or $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{K}$ that are only defined almost everywhere on $\mathbb{R}^3$ or $\mathbb{R}^3 \times \mathbb{R}^3$, respectively, we say that they are spherically symmetric if for all $A \in \mathrm{SO}(3)$,

$$f(Ax) = f(x) \text{ or } g(Ax, Av) = g(x, v),$$

for almost every $x \in \mathbb{R}^3$ or $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$, respectively, where the the sets of measure zero for the exceptions may depend on A.

For spherically symmetric L^1_{loc}-functions, we have the following lemma:

Lemma 1.2.

(i) Every spherically symmetric function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{K})$ can be expressed through the variable $r = |x|$, i.e. there exists a function $f^r :]0,\infty[\to \mathbb{K}$ such that $f(x) = f^r(|x|)$ f.a.e. $x \in \mathbb{R}^3$. The function f^r is uniquely defined a.e. on $]0,\infty[$.

(ii) Every spherically symmetric function $g \in L^1_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}^3;\mathbb{K})$ can be expressed through the variables $(r, w, L) = (|x|, (x \cdot v)/|x|, |x \times v|^2)$, i.e. there exists a function $g^r :]0,\infty[\times \mathbb{R} \times [0,\infty[\to \mathbb{K}$ such that $g(x, v) = g^r(r, w, L) = g^r(|x|, (x \cdot v)/|x|, |x \times v|^2)$ f.a.e. $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$. The function g^r is uniquely defined a.e. on $]0,\infty[\times \mathbb{R} \times [0,\infty[$.

For the sake of clarity, we will always denote the representative of a function f in r- or (r, w, L)-coordinates by f^r. In Chapter 3, we will introduce another set of coordinates whose corresponding functions will be denoted by f^θ.

Proof.
Proof of (i).
Let $f \in L^1_{\text{loc}}(\mathbb{R}^3;\mathbb{K})$ be spherically symmetric. We use a spherically the mollifier $J \in C_c^\infty(B_1(0);\mathbb{R})$, i.e. we have $\int_{\mathbb{R}^3} J = 1$. We define $J_k := k^3 J(k\cdot)$ for $k \in \mathbb{N}$, which are again spherically symmetric functions, and obtain that $f_k := J_k * \left(f 1_{B_N(0)}\right) \to f$ in $L^1(B_N(0);\mathbb{K})$ as $k \to \infty$ for all $N \in \mathbb{N}$ (see Remark (1) after Theorem 2.16 in [14]). The functions f_k are spherically symmetric due to

$$\begin{aligned} f_k(Ax) &= \int_{\mathbb{R}^3} J_k(Ax - y) f(y) 1_{B_N(0)}(y)\, \mathrm{d}y = \int_{\mathbb{R}^3} J_k(Ax - Ay) f(Ay) 1_{B_N(0)}(Ay)\, \mathrm{d}y \\ &= \int_{\mathbb{R}^3} J_k(x - y) f(y) 1_{B_N(0)}(y)\, \mathrm{d}y = f_k(x), \quad x \in \mathbb{R}^3, A \in \mathrm{SO}(3), \end{aligned}$$

note that $B_N(0)$ is spherically symmetric in the sense that $x \in B_N(0) \Leftrightarrow Ax \in B_N(0)$ for every $A \in \mathrm{SO}(3)$. The functions f_k are defined everywhere on $\mathbb{R}^3$ and it is therefore easy to obtain their radial representative $f_k^r = f_k(\cdot, 0, 0)$ on $[0, \infty[$. The next step is to obtain that $((\cdot)^2 f_k^r)$ is a Cauchy sequence in $L^1([0, N];\mathbb{K})$ for all $N \in \mathbb{N}$. To show this, we first prove that $((\cdot)^2 f_k^r)$ is in $L^1([0, N];\mathbb{K})$:

$$\begin{aligned} \left|\left|(\cdot)^2 f_k^r\right|\right|_{L^1([0,N])} &= \int_0^N \left|r^2 f_k^r(r)\right| \, \mathrm{d}r = \frac{1}{4\pi} \int_{B_N(0)} |f_k^r(|x|)| \, \mathrm{d}x \\ &= \frac{1}{4\pi} \int_{B_N(0)} |f_k(x)| \, \mathrm{d}x = \frac{1}{4\pi} \left|\left|f_k\right|\right|_{L^1(B_N(0))} < \infty. \end{aligned} \tag{1.1}$$

To obtain that $((\cdot)^2 f_k^r)$ is a Cauchy sequence in $L^1([0, N];\mathbb{K})$, we estimate $(\cdot)^2 f_k^r - (\cdot)^2 f_l^r$ instead of $(\cdot)^2 f_k^r$ as in (1.1). Since $f_k - f_l$ converges to 0 in $L^1(B_N(0);\mathbb{K})$ as $k, l \to \infty$, we infer that $(\cdot)^2 f_k^r - (\cdot)^2 f_l^r$ converges to 0 in $L^1([0, N];\mathbb{K})$ as $k, l \to \infty$ and hence, $((\cdot)^2 f_k^r)$ is a Cauchy sequence in $L^1([0, N];\mathbb{K})$. Consequently, there exists $g_N \in L^1([0, N];\mathbb{K})$ such that $(\cdot)^2 f_k^r \to g_N$ in $L^1([0, N];\mathbb{K})$ as $k \to \infty$. Due to the Riesz-Fischer theorem, the convergence also holds pointwise almost everywhere after extracting a subsequence. Therefore, we have that $g_N = g_L$ a.e. on $[0, \min\{N, L\}]$ for $N, L \in \mathbb{N}$, where it is possible that the set of measure zero is independent of N and L. For every $L \in \mathbb{N}$, we now define f^r piecewise a.e. on $]L-1, L]$ by $f^r(r) := r^{-2} g_L(r)$ and have that $f^r = (\cdot)^{-2} g_N$ a.e. on $[0, N]$ for $N \in \mathbb{N}$ due to the previous consideration. We find that

$$\begin{aligned} \int_{B_N(0)} |f_k(x) - f^r(|x|)| \, \mathrm{d}x &= \int_{B_N(0)} |f_k^r(|x|) - f^r(|x|)| \, \mathrm{d}x = 4\pi \int_0^N r^2 \left|f_k^r(r) - f^r(r)\right| \mathrm{d}r \\ &= 4\pi \int_0^N \left|r^2 f_k^r - g_N(r)\right| \, \mathrm{d}r \to 0 \text{ as } k \to \infty. \end{aligned}$$

Since L^1-limits are unique, we observe that $f = f^r(|\cdot|)$ almost everywhere on $B_N(0)$ for all $N \in \mathbb{N}$ and therefore almost everywhere on $\mathbb{R}^3$.

Proof of (ii).
Let $g \in L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3;\mathbb{K})$ be spherically symmetric. We again use a spherically symmetric mollifier $J \in C_c^\infty(B_1(0);\mathbb{R})$ (note that here, $B_1(0) \subset \mathbb{R}^6$) and define $J_k := k^6 J(k\cdot)$. As in the first part, we find that $g_k := J_k * \left(g 1_{B_N(0)}\right) \to g$ as $k \to \infty$ in $L^1(B_N(0);\mathbb{K})$ for all $N \in \mathbb{N}$ and that the functions g_k are spherically symmetric.

We define $g_k^r(r, w, L) := g_k((r, 0, 0), (w, \sqrt{L}/r, 0))$ for $(r, w, L) \in\,]0, \infty[\, \times \mathbb{R} \times [0, \infty[$, which implies that $g_k(x, v) = g_k^r(|x|, (x \cdot v)/|x|, |x \times v|^2)$ for $x \neq 0$. We show this in the end of this proof. Next, we define

$$M_N := \left\{ (r, w, L) \in\,]0, \infty[\, \times \mathbb{R} \times [0, \infty[\;\middle|\; ((r, 0, 0), (w, \sqrt{L}/r, 0)) \in B_N(0) \right\}$$

and claim that (g_k^r) is a Cauchy sequence in $L^1(M_N; \mathbb{K})$ for all $N \in \mathbb{N}$. We start by proving that $g_k^r \in L^1(M_N; \mathbb{K})$. For this, we use the same transformation as in (1.3); note that we already obtained the (r, w, L)-representative of g_k. Hence,

$$\begin{aligned}
||g_k^r||_{L^1(M_N)} &= \int_0^\infty \int_0^\infty \int_{\mathbb{R}} |g_k^r(r, w, L)|\, 1_{M_N}(r, w, L)\, \mathrm{d}w\, \mathrm{d}L\, \mathrm{d}r \\
&= \int_0^\infty \int_0^\infty \int_{\mathbb{R}} |g_k^r(r, w, L)|\, 1_{B_N(0)}((r, 0, 0), (w, \sqrt{L}/r, 0))\, \mathrm{d}w\, \mathrm{d}L\, \mathrm{d}r \\
&= \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_0^\infty \int_{\mathbb{R}} |g_k^r(|x|, w, L)|\, 1_{B_N(0)}((|x|, 0, 0), (w, \sqrt{L}/|x|, 0)) \frac{1}{|x|^2}\, \mathrm{d}w\, \mathrm{d}L\, \mathrm{d}x \\
&= \frac{1}{4\pi^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |g_k(x, v)|\, 1_{B_N(0)}(x, v)\, \mathrm{d}v\, \mathrm{d}x = \frac{1}{4\pi^2}\, ||g_k||_{L^1(B_N(0))} < \infty.
\end{aligned}$$

Inserting $g_k^r - g_l^r$ instead of g_k^r yields that (g_k^r) is a Cauchy sequence in $L^1(M_N; \mathbb{K})$ and therefore converges to some function $f_N \in L^1(M_N; \mathbb{K})$. Similar to part (i), we define $g^r := f_N$ on sets $M_N \setminus M_{N-1}$ and have that $g^r = f_N$ a.e. on M_N. We use the same transformation as before and find that

$$\begin{aligned}
&\int_{\mathbb{R}^6} \left| g_k(x, v) - g^r\left(|x|, (x \cdot v)/|x|, |x \times v|^2 \right) \right| 1_{B_N(0)}(x, v)\, \mathrm{d}(x, v) \\
&= 4\pi^2 \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} |g_k^r(r, w, L) - g^r(r, w, L)|\, 1_{M_N}(r, w, L)\, \mathrm{d}w\, \mathrm{d}L\, \mathrm{d}r \\
&= 4\pi^2 \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} |g_k^r(r, w, L) - f_N(r, w, L)|\, 1_{M_N}(r, w, L)\, \mathrm{d}w\, \mathrm{d}L\, \mathrm{d}r \to 0 \text{ as } k \to \infty.
\end{aligned}$$

The uniqueness of L^1-limits yields that $g(x, v) = g^r(|x|, (x \cdot v)/|x|, |x \times v|^2)$ f.a.e. $(x, v) \in B_N(0)$ for all $N \in \mathbb{N}$, thus, the identity holds a.e. on $\mathbb{R}^3 \times \mathbb{R}^3$.

To prove that $g_k(x, v) = g_k^r(|x|, (x \cdot v)/|x|, |x \times v|^2)$ for $x \neq 0$, let $x, v \in \mathbb{R}^3$ with $x \neq 0$. First, let $B \in \mathrm{SO}(3)$ be a rotation matrix such that $Bx = |x| e_1$. We then consider the point Bv and rotate it around the e_1-axis such that the third component vanishes and the second component is non-negative and call the corresponding rotation matrix $C \in \mathrm{SO}(3)$. If we define $A := C \cdot B$, then A is again in $\mathrm{SO}(3)$ with $Ax = |x| e_1$, $(Av)_2 \geq 0$ and $(Av)_3 = 0$. We furthermore observe that

$$\begin{aligned}
x \cdot v = (Ax) \cdot (Av) = |x| (Av)_1, \\
|x \times v|^2 = |x|^2 |v|^2 - (x \cdot v)^2 = |(Ax) \times (Av)|^2 = |x|^2 \left| (Av)_2 \right|^2,
\end{aligned}$$

which implies that

$$(Av)_1 = \frac{x \cdot v}{|x|}, \quad (Av)_2 = |(Av)_2| = \frac{|x \times v|}{|x|}.$$

Because of the spherical symmetry of g_k, we conclude that

$$\begin{aligned} g_k(x,v) &= g_k(Ax, Av) = g_k\left((|x|,0,0),((x\cdot v)/|x|, |x\times v|/|x|, 0)\right) \\ &= g_k^r\left(|x|, (x\cdot v)/|x|, |x\times v|^2\right). \end{aligned}$$

□

Remark 1. For spherically symmetric functions $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{K}$ that are defined everywhere, we obtain the (r, w, L)-representative $g^r :]0,\infty[\times \mathbb{R} \times [0,\infty[\to \mathbb{K}$ through

$$g^r(r,w,L) = g((r,0,0),(w,\sqrt{L}/r,0)).$$

Remark 2. If we consider Lemma 1.2, we immediately obtain that conversely, every function $f^r :]0,\infty[\to \mathbb{K}$ or $g^r :]0,\infty[\times \mathbb{R} \times [0,\infty[\to \mathbb{K}$ that is defined almost everywhere induces an almost everywhere defined, spherically symmetric function $f : \mathbb{R}^3 \to \mathbb{K}$ or $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{K}$, respectively.

For spherically symmetric functions on $\mathbb{R}^3 \times \mathbb{R}^3$, we have the following behavior of integrals when changing between (x, v)- and (r, w, L)-coordinates:

Lemma 1.3. *Let $g \in L^1_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{K})$ be spherically symmetric with (r, w, L)-representative $g^r :]0,\infty[\times \mathbb{R} \times [0,\infty[\to \mathbb{K}$. Then,*

$$\int_{\mathbb{R}^3} g(x,v)\,\mathrm{d}v = \frac{\pi}{|x|^2}\int_0^\infty \int_{\mathbb{R}} g^r(|x|,w,L)\,\mathrm{d}w\,\mathrm{d}L \qquad \text{f.a.e. } x\in\mathbb{R}^3, \tag{1.2}$$

and

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} g(x,v)\,\mathrm{d}(x,v) = 4\pi^2 \int_{]0,\infty[\times\mathbb{R}\times[0,\infty[} g^r(r,w,L)\,\mathrm{d}(r,w,L),$$

as well as

$$\int_{\mathbb{R}^3} v g(x,v)\,\mathrm{d}v = \pi\int_0^\infty \int_{\mathbb{R}} w g^r(|x|,w,L)\,\mathrm{d}w\,\mathrm{d}L \frac{x}{|x|^3} \qquad \text{f.a.e. } x\in\mathbb{R}^3,$$

provided that the integrals exist.

Proof. For the first and third identity, let $x \in \mathbb{R}^3 \setminus \{0\}$ such that $g^r(|x|, w, L)$ is defined f.a.e. $(w, L) \in \mathbb{R}\times[0,\infty[$. This holds true for almost every $x \in \mathbb{R}^3$ as g^r is defined almost everywhere on $]0,\infty[\times \mathbb{R} \times [0,\infty[$. For x chosen this way, let $A \in \mathrm{SO}(3)$ such that $Ax = |x|e_3$. Using the spherical symmetry of g, $\det A = 1$, and polar coordinates, we deduce that

$$\begin{aligned} \int_{\mathbb{R}^3} g(x,v)\,\mathrm{d}v &= \int_{\mathbb{R}^3} g(Ax,Av)\,\mathrm{d}v = \int_{\mathbb{R}^3} g(Ax,v)\,\mathrm{d}v \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^2} g^r\left(|x|, v_3, |x|^2(v_1^2+v_2^2)\right)\,\mathrm{d}(v_1,v_2)\,\mathrm{d}v_3 \\ &= 2\pi\int_{\mathbb{R}}\int_0^\infty g^r\left(|x|,w,|x|^2\tau^2\right)\tau\,\mathrm{d}\tau\,\mathrm{d}w = 2\pi\int_{\mathbb{R}}\int_0^\infty g^r(|x|,w,L)\frac{1}{2|x|^2}\,\mathrm{d}L\,\mathrm{d}w. \end{aligned} \tag{1.3}$$

In the last step, we substituted $L = L(\tau) = |x|^2\tau^2$. We hence proved the first identity.

For the third identity, we observe that

$$\int_{\mathbb{R}^3} vg(x,v)\,\mathrm{d}v = A^{-1}A\int_{\mathbb{R}^3} vg(Ax,Av)\,\mathrm{d}v = A^{-1}\int_{\mathbb{R}^3} Avg(Ax,Av)\,\mathrm{d}v = A^{-1}\int_{\mathbb{R}^3} vg(Ax,v)\,\mathrm{d}v,$$

where x and A are chosen as above. Omitting A^{-1} and only considering the first component of the integral, we use polar coordinates to find that

$$\begin{aligned}\left(\int_{\mathbb{R}^3} vg(Ax,v)\,\mathrm{d}v\right)_1 &= \int_{\mathbb{R}}\int_{\mathbb{R}^2} v_1 g^r\left(|x|, v_3, |x|^2(v_1^2+v_2^2)\right)\,\mathrm{d}(v_1,v_2)\,\mathrm{d}v_3 \\ &= \int_{\mathbb{R}}\int_0^\infty\int_0^{2\pi} r\cos\vartheta g^r\left(|x|, v_3, |x|^2r^2\right) r\,\mathrm{d}\vartheta\,\mathrm{d}r\,\mathrm{d}v_3 = 0\end{aligned}$$

since $\int_0^{2\pi}\cos\vartheta\,\mathrm{d}\vartheta = 0$. We find a similar expression for the second component with the slight difference that $\cos\vartheta$ is replaced by $\sin\vartheta$ and therefore, the second component vanishes as well. For the third component, similar to (1.3), we obtain that

$$\begin{aligned}\left(\int_{\mathbb{R}^3} vg(Ax,v)\,\mathrm{d}v\right)_3 &= \int_{\mathbb{R}^3} v_3 g^r\left(|x|, v_3, |x|^2(v_1^2+v_2^2)\right)\,\mathrm{d}v \\ &= \int_{\mathbb{R}} w\int_{\mathbb{R}^2} g^r\left(|x|^2, w, |x|^2(v_1^2+v_2^2)\right)\,\mathrm{d}(v_1,v_2)\,\mathrm{d}w \\ &= \int_{\mathbb{R}} w\int_0^\infty \frac{\pi}{|x|^2} g^r\left(|x|, w, L\right)\,\mathrm{d}L\,\mathrm{d}w.\end{aligned}$$

As a result,

$$\begin{aligned}\int_{\mathbb{R}^3} vg(x,v)\,\mathrm{d}v &= A^{-1}\left(\frac{\pi}{|x|^2}\int_{\mathbb{R}}\int_0^\infty wg^r(|x|,w,L)\,\mathrm{d}L\,\mathrm{d}we_3\right) \\ &= \frac{\pi}{|x|^2}\int_{\mathbb{R}}\int_0^\infty wg^r(|x|,w,L)\,\mathrm{d}L\,\mathrm{d}w\frac{1}{|x|}A^{-1}\left(|x|e_3\right) \\ &= \frac{\pi}{|x|^2}\int_{\mathbb{R}}\int_0^\infty wg^r(|x|,w,L)\,\mathrm{d}L\,\mathrm{d}w\frac{x}{|x|}.\end{aligned}$$

The first identity along with spherical coordinates yields that

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{R}^3} g(x,v)\,\mathrm{d}(x,v) &= \pi\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{1}{|x|^2}g^r(|x|,w,L)\,\mathrm{d}x\,\mathrm{d}w\,\mathrm{d}L \\ &= 4\pi^2\int_0^\infty\int_{\mathbb{R}}\int_0^\infty g^r(r,w,L)\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}L,\end{aligned}$$

proving the second identity. □

Remark. In Chapter 3, we will introduce the domain of definition for most functions that occur in this thesis, which will be the interior of the support of the density function on phase space corresponding to the isochrone steady state. We will denote this set by Ω_0. As this function will be spherically symmetric, it has a representative in (r, w, L)-coordinates and a corresponding set Ω_0^r. Integrals therefore change via

$$\int_{\Omega_0} g(x, v) \,\mathrm{d}(x, v) = 4\pi^2 \int_{\Omega_0^r} g^r(r, w, L) \,\mathrm{d}(r, w, L).$$

2 The Vlasov-Poisson system and the isochrone model

As mentioned in the introduction, we consider the gravitational Vlasov-Poisson system

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0, \tag{2.1}$$

$$\Delta U = 4\pi\rho, \quad \lim_{|x|\to\infty} U(t,x) = 0, \tag{2.2}$$

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,\mathrm{d}v. \tag{2.3}$$

We now turn to the isochrone model and note that, as mentioned earlier, it was introduced by Michel Hénon. For his derivation of the isochrone potential, we refer to [10]. The approach has been revisited by James Binney in [2].
For a fixed parameter $b > 0$, the isochrone potential is defined by

$$\begin{aligned} U_0(x) &:= -\frac{1}{b + \sqrt{|x|^2 + b^2}}, \quad x \in \mathbb{R}^3, \\ &= \frac{b - \sqrt{|x|^2 + b^2}}{|x|^2}, \quad x \in \mathbb{R}^3 \setminus \{0\}. \end{aligned}$$

This potential was derived particularly to have the property that the periods of particle orbits only depend on the particle's energy and not on its angular momentum. We will define this terminology later and will also see that this indeed holds true.

Given this time-independent potential U_0, one is now interested in finding suitable time-independent, non-negative, spherically symmetric functions f_0 and ρ_0 such that the triplet (U_0, f_0, ρ_0) is a steady state of the Vlasov-Poisson system, i.e. a time-independent solution of (2.1)-(2.3) in the sense of classical derivatives. Note that just like U_0, f_0 and ρ_0 may (and will) depend on b.
Because of the Poisson equation in (2.2), one can easily derive ρ_0 through the Laplace operator in spherical coordinates:

$$\begin{aligned} \rho_0^r(r) &= \frac{1}{4\pi}(\Delta U_0)^r(r) = \frac{1}{4\pi}\frac{1}{r^2}\partial_r\left(r^2\left(U_0^r\right)'\right)(r) \\ &= \frac{1}{4\pi}\left[\frac{3}{\sqrt{r^2+b^2}(b+\sqrt{r^2+b^2})^2} - \frac{r^2}{(r^2+b^2)^{3/2}(b+\sqrt{r^2+b^2})^2}\right. \\ &\qquad\left. - \frac{2r^2}{(r^2+b^2)(b+\sqrt{r^2+b^2})^3}\right], \quad r \geq 0. \end{aligned}$$

For $r \neq 0$, we can rewrite the above expression, simplifying it to

$$\rho_0^r(r) = \frac{1}{4\pi}\left[\frac{2b(r^2+b^2)^{3/2} - 3b^2r^2 - 2b^4}{r^4(r^2+b^2)^{3/2}}\right], \quad r > 0.$$

The function ρ_0^r, and therefore also ρ_0, is positive everywhere, which can easily be concluded by looking at the last representation and observing that both the numerator and denominator are positive for $r > 0$. For $r = 0$, we compute $\rho_0^r(0) = \frac{3}{16\pi}\frac{1}{b^3} > 0$.

We already observe that the resulting steady state–keeping in mind that the we still have to derive a suitable phase space density f_0–has an unbounded spatial support. This will later match the fact that the so-called cut-off energy E_0 is 0 for the isochrone case. If the cut-off energy would have been smaller than 0, we would have obtained a bounded spatial support.

Before being able to derive a matching phase space density function f_0, we define E and L as functions of $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$. One can interpret them as the energy and the square of the modulus of the angular momentum (we will use the convention of still calling it the angular momentum) of a particle located at position x with velocity v, corresponding to the given potential U_0. We define

$$E(x,v) := \frac{1}{2}|v|^2 + U_0(x),$$
$$L(x,v) := |x \times v|^2, \quad (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

To derive f_0, we first note that by Jeans' theorem, every spherically symmetric solution to the Vlasov-Poisson system that is independent of time can be expressed as a function of E and L (see [1]). As a matter of fact, in this case, we can make the ansatz that f_0 is a function only of E, i.e. $f_0 = \varphi \circ E$, where $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies

(i) $\int_{\mathbb{R}^3} (\varphi \circ E)(x,v)\,\mathrm{d}v = \rho_0(x)$ for $x \in \mathbb{R}^3$,

(ii) $\varphi(E) = 0$ for $E \in [0, \infty[$, and

(iii) $\varphi|_{]-\frac{1}{b},0]} \in C^1\left(]-\frac{1}{b},0]\right)$.

Considering these assumptions, we formally derive a function φ that satisfies all requirements. The derivation can be found after the following theorem.

Theorem 2.1. *For $E \in [-\frac{1}{2b}, 0[$, we consider the function that is (formally) derived from (i)-(iii):*

$$\begin{aligned}\varphi(E) &= \frac{1}{\sqrt{2}(2\pi)^3}\int_0^{-E} \frac{-2b^4t^5 + 10b^3t^4 - 20b^2t^3 + 24bt^2}{(1-bt)^5\sqrt{-E-t}}\,\mathrm{d}t \qquad (2.4)\\ &= \frac{1}{\sqrt{2}(2\pi)^3 b^{3/2}}\frac{\sqrt{-bE}}{(2(bE+1))^4}\Bigg[27 + 66bE + 320b^2E^2 + 240b^3E^3 + 64b^4E^4\\ &\qquad + 3\left(16b^2E^2 - 28bE - 9\right)\frac{\arcsin(\sqrt{-bE})}{\sqrt{-bE(1+bE)}}\Bigg].\end{aligned}$$

If we consider the above representation for $\varphi(E)$ on $]-\frac{1}{b}, 0[$ instead of $[-\frac{1}{2b}, 0[$ and extend it by 0 outside of $]-\frac{1}{b}, 0[$, then this resulting function satisfies (i)-(iii). In the following, φ shall be defined like this.

Derivation of φ. The basis of this derivation is the paper [5] of astrophysicist Arthur Stanley Eddington. We remark that what follows has to be understood as a formal derivation rather than a rigorous proof.

STEP 1: Relation between ρ_0^r and U_0^r.
For $r > 0$ the gravitational potential U_0^r can be written as $U_0^r(r) = \frac{b-\sqrt{r^2+b^2}}{r^2} \neq 0$, which implies that

$$r^2 = \frac{1}{U_0^r(r)^2} \frac{(b-\sqrt{r^2+b^2})^2}{r^2} = \frac{1}{U_0^r(r)^2} \left(\frac{2b^2 - 2b\sqrt{r^2+b^2}}{r^2} + 1 \right) = \frac{2bU_0^r(r)+1}{U_0^r(r)^2}. \tag{2.5}$$

Using this, we also derive that

$$\sqrt{r^2+b^2} = b - U_0^r(r) r^2 = b - \frac{1}{U_0^r(r)} \left(2bU_0^r(r)+1\right) = \frac{-1-bU_0^r(r)}{U_0^r(r)}, \quad r > 0. \tag{2.6}$$

Another way to express $\rho_0^r(r)$ for $r > 0$ is through $\rho_0^r(r) = \frac{1}{4\pi} \left[\frac{1}{r^2\sqrt{r^2+b^2}} + \frac{1}{(r^2+b^2)^{3/2}} + 2\frac{U_0^r(r)}{r^2} \right]$. Inserting (2.5) and (2.6) into this representation yields

$$\begin{aligned}
4\pi\rho_0^r(r) &= \frac{U_0^r(r)^2}{2bU_0^r(r)+1} \frac{U_0^r(r)}{(-1-bU_0^r(r))} + \frac{U_0^r(r)^3}{(-1-bU_0^r(r))^3} + \frac{2U_0^r(r)^3}{2bU_0^r(r)+1} \\
&= \frac{1}{(2bU_0^r(r)+1)(-1-bU_0^r(r))^3} \left(-2b^3U_0^r(r)^6 - 5b^2U_0^r(r)^5 - 2bU_0^r(r)^4\right) \\
&= \frac{-b^2U_0^r(r)^5 - 2bU_0^r(r)^4}{(-1-bU_0^r(r))^3}.
\end{aligned} \tag{2.7}$$

Consequently,

$$\rho_0^r(r) = K\left(U_0^r(r)\right), \quad r > 0, \tag{2.8}$$

where

$$K(U) := \frac{1}{4\pi} \frac{bU^4(bU+2)}{(1+bU)^3}, \quad U \in [-1/(2b), 0[.$$

Since K is also well-defined for $U = U_0^r(0) = -\frac{1}{2b}$, we can easily verify that (2.8) also holds for $r = 0$.

STEP 2: Expressing φ in terms of K.
Making use of assumption (i) and (ii), we find that

$$\begin{aligned}
\rho_0(x) &= \int_{\mathbb{R}^3} \varphi(E(x,v))\,\mathrm{d}v = \int_{\mathbb{R}^3} \varphi\left(U_0(x) + \frac{1}{2}|v|^2\right) \mathrm{d}v \\
&= 4\pi \int_0^\infty \varphi\left(U_0(x) + \frac{1}{2}s^2\right) s^2\,\mathrm{d}s = 4\pi \int_0^{\sqrt{-2U_0(x)}} \varphi\left(U_0(x) + \frac{1}{2}s^2\right) s^2\,\mathrm{d}s, \quad x \in \mathbb{R}^3.
\end{aligned}$$

Hence,

$$K(U_0^r(r)) \underset{(2.8)}{=} \rho_0^r(r) = 4\pi \int_0^{\sqrt{-2U_0^r(r)}} \varphi\left(U_0^r(r) + \frac{1}{2}s^2\right) s^2\,\mathrm{d}s, \quad r \geq 0.$$

Due to the fact that $U_0^r : [0,\infty[\to [-\frac{1}{2b}, 0[$ is bijective, we conclude that

$$K(U) = 4\pi \int_0^{\sqrt{-2U}} \varphi\left(U + \frac{1}{2}s^2\right) s^2 \,\mathrm{d}s, \quad U \in [-1/(2b), 0[.$$

Making use of assumption (iii) and the fundamental theorem of calculus, we observe that

$$\begin{aligned}
K'(U) &= 4\pi \left[\varphi\left(U + \frac{1}{2}\sqrt{-2U}^2\right) \sqrt{-2U}^2 \left(-\frac{1}{\sqrt{-2U}}\right) + \int_0^{\sqrt{-2U}} \varphi'\left(U + \frac{1}{2}s^2\right) s^2 \,\mathrm{d}s \right] \\
&= 4\pi \int_0^{\sqrt{-2U}} \varphi'\left(U + \frac{1}{2}s^2\right) s^2 \,\mathrm{d}s, \quad U \in [-1/(2b), 0[,
\end{aligned}$$

since $\varphi(0) = 0$ due to assumption (ii). Substituting $y = s^2$ and integrating by parts yields

$$\begin{aligned}
K'(U) &= 2\pi \int_0^{-2U} \varphi'\left(U + \frac{1}{2}y\right) \sqrt{y} \,\mathrm{d}y \\
&= 2\pi \left[\left(2\varphi\left(U + \frac{1}{2}y\right) \sqrt{y} \right) \Big|_{y=0}^{-2U} - \int_0^{-2U} 2\varphi\left(U + \frac{1}{2}y\right) \frac{1}{2} \frac{1}{\sqrt{y}} \,\mathrm{d}y \right] \\
&= -2\pi \int_0^{-2U} \varphi\left(U + \frac{1}{2}y\right) \frac{1}{\sqrt{y}} \,\mathrm{d}y, \quad U \in [-1/(2b), 0[.
\end{aligned}$$

The boundary terms vanish because of $\varphi(0) = 0$. Next, we yet again use a transformation, namely $\xi = \xi(y) = -U - \frac{1}{2}y$, in order to find that

$$\begin{aligned}
K'(U) &= 4\pi \int_{-U}^{0} \varphi(-\xi) \frac{1}{\sqrt{2(-\xi - U)}} \,\mathrm{d}\xi \\
&= -\frac{4\pi}{\sqrt{2}} \int_0^{-U} \varphi(-\xi) \frac{1}{\sqrt{-\xi - U}} \,\mathrm{d}\xi, \quad U \in [-1/(2b), 0[.
\end{aligned}$$

For the sake of simplicity, we define

$$\begin{aligned}
&\tilde{K}(U) := K'(-U), \quad U \in \,]0, 1/(2b)], \text{ and} \\
&\tilde{\varphi}(E) := \varphi(-E), \quad E \in \mathbb{R},
\end{aligned}$$

which then satisfy

$$\tilde{K}(U) = -\frac{4\pi}{\sqrt{2}} \int_0^U \tilde{\varphi}(\xi) \frac{1}{\sqrt{U - \xi}} \,\mathrm{d}\xi, \quad U \in \,]0, 1/(2b)].$$

In consequence of Abel's integral equation (see [6, Chapter 8, Equations (1) and (3)]),

$$\begin{aligned}
\frac{4\pi^2}{\sqrt{2}} \tilde{\varphi}(z) &= \frac{4\pi^2}{\sqrt{2}} \frac{1}{\pi} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z \frac{-\frac{\sqrt{2}}{4\pi} \tilde{K}(t)}{(z-t)^{1/2}} \,\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}z} \int_0^z \frac{K'(-t)}{(z-t)^{1/2}} \,\mathrm{d}t \\
&= -\frac{\mathrm{d}}{\mathrm{d}z} \left[-2K'(-t)\sqrt{z-t} \Big|_{t=0}^{z} - 2\int_0^z K''(-t)\sqrt{z-t} \,\mathrm{d}t \right] \\
&= 2\frac{\mathrm{d}}{\mathrm{d}z} \int_0^z K''(-t)\sqrt{z-t} \,\mathrm{d}t = 2\left(0 + \int_0^z K''(-t) \frac{1}{2\sqrt{z-t}} \,\mathrm{d}t \right) \\
&= \int_0^z K''(-t) \frac{1}{\sqrt{z-t}} \,\mathrm{d}t, \quad z \in \,]0, 1/(2b)],
\end{aligned}$$

where we again used integration by parts (the boundary terms vanish due to $K'(0) = 0$). As a result, we find that

$$\varphi(E) = \frac{\sqrt{2}}{4\pi^2} \int_0^{-E} \frac{K''(-t)}{\sqrt{-E-t}} \, \mathrm{d}t, \quad E \in [-1/(2b), 0[.$$

STEP 3: Computing the integral.
We differentiate twice and find that

$$K''(U) = \frac{1}{4\pi} \frac{2b^4U^5 + 10b^3U^4 + 20b^2U^3 + 24bU^2}{(1+bU)^5}, \quad E \in [-1/(2b), 0[,$$

which leads to

$$\varphi(E) = \frac{1}{\sqrt{2}(2\pi)^3} \int_0^{-E} \frac{-2b^4t^5 + 10b^3t^4 - 20b^2t^3 + 24bt^2}{(1-bt)^5\sqrt{-E-t}} \, \mathrm{d}t, \quad E \in [-1/(2b), 0[.$$

That this integral indeed leads to the desired representation of φ on $[-\frac{1}{2b}, 0[$ can be obtained by computing the integral. This is a lengthy process and we will therefore only outline the procedure: We substitute $u = u(t) = \sqrt{-E-t}$ without being concerned that the integrand is undefined at one of its boundary values as we interpret the integral $\int_0^{-E} \dots \mathrm{d}t$ as $\lim_{\varepsilon \to 0, \varepsilon > 0} \int_0^{-E-\varepsilon} \dots \mathrm{d}t$. After that, we perform a partial fraction decomposition and each occurring integral is then solvable using standard integration techniques. $\square$

We now turn to the proof of Theorem 2.1:

Proof of Theorem 2.1. We have to prove that (i), (ii) and (iii) hold. (ii) is obvious. To show (i), we make use of (ii) and spherical coordinates to find that

$$\int_{\mathbb{R}^3} \varphi(E(x,v)) \, \mathrm{d}v = \int_{B_{\sqrt{-2U_0(x)}}} \varphi\left(\frac{1}{2}|v|^2 + U_0(x)\right) \mathrm{d}v = 4\pi \int_0^{\sqrt{U_0(x)}} \varphi\left(\frac{1}{2}s^2 + U_0(x)\right) s^2 \, \mathrm{d}s$$

for $x \in \mathbb{R}^3$. Substituting $E = E(s) = \frac{1}{2}s^2 + U_0(x)$ yields that

$$\begin{aligned}
\int_{\mathbb{R}^3} \varphi(E(x,v)) \, \mathrm{d}v &= 4\pi \int_{U_0(x)}^0 \sqrt{2E - 2U_0(x)} \varphi(E) \, \mathrm{d}E \\
&\underset{(2.4)}{=} \frac{1}{2\pi^2} \int_{U_0(x)}^0 \sqrt{E - U_0(x)} \int_0^{-E} \frac{-2b^4t^5 + 10b^3t^4 - 20b^2t^3 + 24bt^2}{(1-bt)^5\sqrt{-E-t}} \, \mathrm{d}t \, \mathrm{d}E \\
&= \frac{1}{2\pi^2} \int_0^{-U_0(x)} \frac{-2b^4t^5 + 10b^3t^4 - 20b^2t^3 + 24bt^2}{(1-bt)^5} \int_{U_0(x)}^{-t} \frac{\sqrt{E - U_0(x)}}{\sqrt{-E-t}} \, \mathrm{d}E \, \mathrm{d}t.
\end{aligned}$$

Concerning the inner integral, we obtain that

$$F(E) := (U_0(x) + t) \arctan\left(\frac{\sqrt{-E-t}}{\sqrt{E - U_0(x)}}\right) - \sqrt{-E-t}\sqrt{E - U_0(x)}$$

is an antiderivative of $E \mapsto \sqrt{E-U_0(x)}/\sqrt{-E-t}$ on $]U_0(x),-t[$. We therefore compute that the inner integral is equal to

$$\begin{aligned}\int_{U_0(x)}^{-t} \frac{\sqrt{E-U_0(x)}}{\sqrt{-E-t}}\,\mathrm{d}E &= \lim_{\substack{a\to U_0(x), a>U_0(x)\\ b\to -t, b<-t}} \int_a^b \frac{\sqrt{E-U_0(x)}}{\sqrt{-E-t}}\,\mathrm{d}E \\ &= \lim_{\substack{a\to U_0(x), a>U_0(x)\\ b\to -t, b<-t}} \left(F(b)-F(a)\right) = 0-(U_0(x)+t)\,\frac{\pi}{2},\end{aligned}$$

and consequently,

$$\int_{\mathbb{R}^3} \varphi(E(x,v))\,\mathrm{d}v = \frac{1}{4\pi}\int_0^{-U_0(x)} (-U_0(x)-t)\frac{-2b^4t^5+10b^3t^4-20b^2t^3+24bt^2}{(1-bt)^5}\,\mathrm{d}t, \quad x\in\mathbb{R}^3.$$

We substitute $u=u(t)=1-bt$, which leads to an integrand that is of the form "polynomial in u divided by u^5", which is a standard integral that can be computed easily. After factorizing the resulting expression, we observe that

$$\int_{\mathbb{R}^3} \varphi(E(x,v))\,\mathrm{d}v = \frac{1}{4\pi}\frac{b^2U_0(x)^5+2bU_0(x)^4}{(bU_0(x)+1)^3}, \quad x\in\mathbb{R}^3,$$

which in turn is equal to $\rho_0(x)$ by (2.7). Note that this property was derived rigorously and we can therefore use it. Property (2.7) only holds for $r>0$, however, inserting $r=0$ yields that (2.7) is also true for $r=0$.

For (iii), we note that φ is obviously in $C^1\left(]-\frac{1}{b},0[\right)$. Since $\frac{\arcsin\left(\sqrt{-bE}\right)}{\sqrt{-bE(1+bE)}}\to 1$ as $E\to 0, E<0$ by l'Hôspital's rule, we observe that $\varphi(E)\to 0=\varphi(0)$ as $E\to 0, E<0$. For the derivative of φ , we define

$$C(b) := \sqrt{2}(2\pi)^3 b^{3/2} 2^4$$

and find that

$$\begin{aligned}C(b)\varphi'(E) = &\left(-\frac{b}{2}\frac{1}{\sqrt{-bE}(bE+1)^4} - 4b\frac{\sqrt{-bE}}{(bE+1)^5}\right)[...] \\ &+\frac{\sqrt{-bE}}{(bE+1)^4}\left(66b+640b^2E+720b^3E^2+256b^4E^3\right) \\ &+\frac{\sqrt{-bE}}{(bE+1)^4}\left((96b^2E-84b)\frac{\arcsin\left(\sqrt{-bE}\right)}{\sqrt{-bE(1+bE)}}\right) \\ &+\frac{\sqrt{-bE}}{(bE+1)^4}\left((48b^2E^2-84bE-27)\left(-\frac{b}{2}\frac{1}{-bE(1+bE)}\right.\right. \\ &\left.\left.+\frac{b+2b^2E}{2}\frac{\arcsin\left(\sqrt{-bE}\right)}{(-bE(1+bE))^{3/2}}\right)\right).\end{aligned}$$

If we consider the limit $E\to 0$ of this expression, we immediately obtain that the second summand of the first line as well as the second and third line converge to 0. In the last line, the term containing $\frac{2b^2E}{2}$ obviously vanishes as well. Hence,

$$\begin{aligned}\lim_{E\to 0}\big(C(b)\varphi'(E)\big) = \lim_{E\to 0}\Bigg[&-\frac{b}{2}\frac{1}{\sqrt{-bE}(bE+1)^4}[...]\\&+\frac{\sqrt{-bE}}{(bE+1)^4}(48b^2E^2-84bE-27)\left(-\frac{b}{2}\frac{1}{-bE(1+bE)}\right)\\&+\frac{\sqrt{-bE}}{(bE+1)^4}(48b^2E^2-84bE-27)\left(\frac{b}{2}\frac{\arcsin\left(\sqrt{-bE}\right)}{(-bE(1+bE))^{3/2}}\right)\Bigg].\end{aligned}$$

Here and in the following, all limits as $E \to 0$ have to be understood as $E \to 0, E < 0$. For the second and third line, it is obvious that the terms with factors 48 and -84 vanish as we take the limit. For the first line, we have

$$\begin{aligned}\frac{1}{\sqrt{-bE}}[...] =&\frac{27}{\sqrt{-bE}} - 66\sqrt{-bE} + 320(-bE)^{3/2} - 240(-bE)^{5/2} + 64(-bE)^{7/2}\\&+\left(48(-bE)^{3/2}+84\sqrt{-bE}-\frac{27}{\sqrt{-bE}}\right)\frac{\arcsin\left(\sqrt{-bE}\right)}{\sqrt{-bE(1+bE)}},\end{aligned}$$

thus, all terms except the ones with factors 27 vanish for $E \to 0$. Overall, we observe that

$$\begin{aligned}\lim_{E\to 0}\big(C(b)\varphi'(E)\big) &= \lim_{E\to 0}\Bigg(-\frac{27b}{2}\frac{1}{\sqrt{-bE}(1+bE)^4}+\frac{27b}{2}\frac{1}{\sqrt{-bE}(1+bE)^4}\frac{\arcsin\left(\sqrt{-bE}\right)}{\sqrt{-bE(1+bE)}}\\&\qquad+\frac{27b}{2}\frac{1}{\sqrt{-bE}(1+bE)^5}-\frac{27b}{2}\frac{1}{\sqrt{-bE}(1+bE)^5}\frac{\arcsin\left(\sqrt{-bE}\right)}{\sqrt{-bE(1+bE)}}\Bigg)\\&=\lim_{E\to 0}\Bigg(\frac{27b}{2}\frac{1}{\sqrt{-bE}(1+bE)^4}\left(-1+\frac{1}{1+bE}\right)\\&\qquad+\frac{27b}{2}\frac{1}{\sqrt{-bE}(1+bE)^4}\frac{\arcsin\left(\sqrt{-bE}\right)}{\sqrt{-bE(1+bE)}}\left(1-\frac{1}{1+bE}\right)\Bigg)\\&=\lim_{E\to 0}\Bigg(\frac{27b}{2}\frac{1}{\sqrt{-bE}(1+bE)^4}\left(\frac{-bE}{1+bE}\right)\\&\qquad+\frac{27b}{2}\frac{1}{\sqrt{-bE}(1+bE)^4}\frac{\arcsin\left(\sqrt{-bE}\right)}{\sqrt{-bE(1+bE)}}\left(\frac{bE}{1+bE}\right)\Bigg)\\&=0.\end{aligned}$$

□

Remark. Assumption (ii) can be interpreted as that the isochrone steady state shall not contain particles with positive or vanishing energy. Moreover, the values of φ for $E < -\frac{1}{2b}$ do not matter since $E(x, v) \geq U_0(x) \geq U_0(0) = -\frac{1}{2b}$. Lastly, we remark that due to $\lim_{E\to 0,E<0}\varphi'(E) = 0 = \lim_{E\to 0,E<0}\varphi(E)$ and $\varphi(E) = 0$ on $[0,\infty[$, we in fact have $\varphi \in C^1\left(\left]-\frac{1}{b},\infty\right[\right)$.

We define $f_0 := \varphi \circ E$ on $\mathbb{R}^3 \times \mathbb{R}^3$. Then, $f_0 \geq 0$ and (f_0, ρ_0, U_0) is a spherically symmetric steady state of the Vlasov-Poisson system, i.e. f_0, ρ_0 and U_0 are spherically symmetric, independent of time, and solve (2.1) - (2.3) in the sense of classical derivatives.

Proof. We have already obtained that ρ_0 and U_0 are spherically symmetric. The spherical symmetry of f_0 is obvious as it inherits the spherical symmetry of the energy function E. All functions are independent of time.

For the non-negativity of f_0, we need to show that $\varphi \geq 0$ on $[-\frac{1}{2b}, \infty[$. Because φ vanishes on $[0, \infty[$, the problem reduces to showing that $\varphi \geq 0$ on $[-\frac{1}{2b}, 0[$. On this set, we may use the integral representation of φ given in (2.4). For $E \in [-\frac{1}{2b}, 0[$ we have that $0 < -E$, which means that the lower bound of integration is smaller than the upper bound of integration, hence, it suffices to show that the integrand is non-negative. For the denominator, this is obvious. It therefore remains to show that the numerator is non-negative. For $t \in [0, \frac{1}{2b}]$, we define

$$\begin{aligned} h_b(t) :=& -2b^4t^5 + 10b^3t^4 - 20b^2t^3 + 24bt^2 \\ =& -2bt^2(bt-3)\,(bt(bt-2)+4)\,. \end{aligned} \tag{2.9}$$

The first two factors $-2bt^2$ and $(bt-3)$ are both non-positive and for the third one, we use that $bt(bt-2) \geq -|bt|\,|bt-2| \geq -\frac{1}{2}(2-bt) = \frac{bt}{2} - 1 \geq -1$, which makes the third factor positive. As a consequence, we have that $h_b \geq 0$ on $[0, \frac{1}{2b}]$. Additionally, since $h_b > 0$ on $]0, \frac{1}{2b}]$, we deduce from the integral representation that in fact,

$$\varphi > 0 \text{ on } [-1/(2b), 0[. \tag{2.10}$$

To obtain that f_0, ρ_0 and U_0 solve (2.1), (2.2), and (2.3) in the classical sense, first note that clearly, $U_0 \in C^2(\mathbb{R}^3)$. For the differentiability of f_0, we use that $\varphi \in C^1\left(]-\frac{1}{b}, \infty[\right)$ as obtained in the remark after the proof of Theorem 2.1. In addition to that, we have that $E \in C^1(\mathbb{R}^3 \times \mathbb{R}^3)$ with $E(\mathbb{R}^3 \times \mathbb{R}^3) \subset [-\frac{1}{2b}, \infty[$. As a result, we find that $f_0 = \varphi \circ E \in C^1(\mathbb{R}^3 \times \mathbb{R}^3)$. The Vlasov equation (2.1) is satisfied as

$$\begin{aligned} &\partial_t f_0(x,v) + v \cdot \partial_x f_0(x,v) - \partial_x U_0(x) \cdot \partial_v f_0(x,v) \\ &= 0 + v \cdot \big(\varphi'(E(x,v))\partial_x U_0(x)\big) - \partial_x U_0(x) \cdot \big(\varphi'(E(x,v))v\big) = 0, \quad (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3. \end{aligned}$$

The Poisson equation in (2.2) is satisfied since we derived ρ_0 from U_0 through the Poisson equation. Obviously, U_0 satisfies the boundary condition in (2.2). Equation (2.3) holds true as it is equal to (i) in Theorem 2.1 and the proof is therefore complete. □

3 Properties of the steady state and (θ, E, L)-coordinates

This chapter contains some properties of the steady state as well as the derivation of the period function and the deduction that it is a function only depending on E and being independent of L. Moreover, we will introduce (θ, E, L)-coordinates, which will prove to be very useful later.

A domain that is important throughout this thesis is the interior of the steady state support:

$$\begin{aligned}\Omega_0 &:= \left\{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; f_0(x,v) \neq 0\right\} \\ &= \left\{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; f_0(x,v) = \varphi(E(x,v)) > 0\right\} \\ &= \left\{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; E(x,v) < 0\right\}.\end{aligned}$$

The last equality results from (2.10) along with the fact that $\varphi = 0$ on $[0, \infty[$, as well as $E \geq -\frac{1}{2b}$ on $\mathbb{R}^3 \times \mathbb{R}^3$. The last characterization of Ω_0 means that the cut-off energy E_0 is 0 in our case. Since it is 0 and not smaller than 0, the spatial support of the steady state is unbounded. More precisely, it is equal to the whole space $\mathbb{R}^3$ since for every $x \in \mathbb{R}^3$, there exists some $v \in \mathbb{R}^3$ such that $(x,v) \in \Omega_0$, i.e. $f_0(x,v) \neq 0$. For example, $(x, 0) \in \operatorname{supp} f_0$ for all $x \in \mathbb{R}^3$. Nonetheless, the total mass of the steady state is finite. We have

$$\int_{\mathbb{R}^3} \rho_0(x)\,\mathrm{d}x = 1,$$

which we easily obtain using polar coordinates and the fact that U_0 solves the Poisson equation with right hand side $4\pi\rho_0$:

$$\begin{aligned}\int_{\mathbb{R}^3} \rho_0(x)\,\mathrm{d}x &= 4\pi \int_0^\infty r^2 \rho_0^r(r)\,\mathrm{d}r = 4\pi \int_0^\infty r^2 \frac{1}{4\pi}\frac{1}{r^2}\partial_r \left(r^2 \left(U_0^r\right)'\right)(r)\,\mathrm{d}r \\ &= \lim_{s\to\infty} \int_0^s \partial_r \left(r^2 \frac{r}{\sqrt{r^2+b^2}\left(b + \sqrt{r^2+b^2}\right)^2} \right) \mathrm{d}r \\ &= \lim_{s\to\infty} \frac{1}{\sqrt{1+\frac{b^2}{s^2}}\left(\frac{b}{s} + \sqrt{1+\frac{b^2}{s^2}}\right)^2} = 1.\end{aligned}$$

We also characterize Ω_0 in (r, w, L)-coordinates and denote this set by Ω_0^r. To do this, we first establish the representation of the energy function E in (r, w, L)-coordinates:

$$\begin{aligned}
E^r(r,w,L) &= E((r,0,0),(w,\sqrt{L}/r,0)) \\
&= \frac{1}{2}\left(w^2 + \frac{L}{r^2}\right) + U_0((r,0,0)) \\
&= \frac{1}{2}w^2 + \psi_L(r), \quad (r,w,L) \in\,]0,\infty[\, \times \mathbb{R} \times [0,\infty[,
\end{aligned}$$

where $\psi_L(r) := \frac{L}{2r^2} + U_0^r(r), r \in\,]0,\infty[, L \in [0,\infty[$, is the so-called effective potential. We now define

$$\begin{aligned}
\Omega_0^r :=& \{(r,w,L) \in\,]0,\infty[\, \times \mathbb{R} \times [0,\infty[\;\mid\; \exists (x,v) \in \Omega_0 : r = |x|, w = (x\cdot v)/|x|, L = L(x,v)\} \\
=& \{(r,w,L) \in\,]0,\infty[\, \times \mathbb{R} \times [0,\infty[\;\mid\; E^r(r,w,L) < 0\}\,.
\end{aligned}$$

Proof. If (r, w, L) is an element of the first set with corresponding $(x, v) \in \Omega_0$, we then find that $E^r(r, w, L) = E^r(|x|, (x \cdot v)/|x|, |x \times v|^2) = E(x, v) < 0$ and therefore, (r, w, L) is an element of the second set.
If (r, w, L) is an element of the second set, we define $x := (r, 0, 0)$ and $v := (w, \sqrt{L}/r, 0)$. Then, $E(x, v) = E((r, 0, 0), (w, \sqrt{L}/r, 0)) = E^r(r, w, L) < 0$, hence, (r, w, L) is an element of the first set. □

In the next section, we examine the effective potential and derive the period function for which we will observe that it indeed only depends on the particle energy.

3.1 The effective potential

For examining the effective potential, we follow [16, Theorem 4.1], however, since U_0 and therefore ψ_L is given explicitly, we are able to state the properties in a more explicit way. Although not all properties of the following theorem will be used in this thesis, we choose to be consistent with [16] and transfer the whole theorem to the isochrone case.

Theorem 3.1.

(i) *For any $L > 0$ there exists a unique $r_L > 0$ such that*

$$\min_{r\in]0,\infty[} \psi_L(r) = \psi_L(r_L) < 0,$$

namely,

$$r_L = \frac{1}{\sqrt{2}}\sqrt{L(4b+L) + (2b+L)\sqrt{L(4b+L)}}.$$

(ii) *For any $L > 0$ and $E \in\,]\psi_L(r_L), 0[$, there exist unique $r_\pm(E, L) > 0$ with*

$$r_-(E,L) < r_L < r_+(E,L)$$

such that $\psi_L(r_\pm(E, L)) = E$, namely,

$$r_\pm(E,L) = -\frac{1}{E}\sqrt{\left(b+\frac{L}{2}\right)E + \frac{1}{2} \pm \sqrt{b^2E^2 + \left(b+\frac{L}{2}\right)E + \frac{1}{4}}}.$$

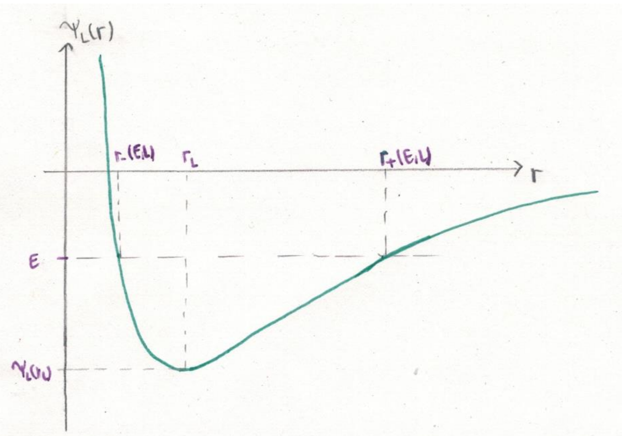
(*iii*) *For any $L > 0$ and $E \in]\psi_L(r_L), 0[$,*

$$r_+(E, L) < -\frac{1}{E}.$$

(*iv*) *For any $L > 0$, $E \in]\psi_L(r_L), 0[$, and $r \in [r_-(E, L), r_+(E, L)]$,*

$$E - \psi_L(r) \geq L \frac{(r_+(E, L) - r)\,(r - r_-(E, L))}{2r^2 r_-(E, L) r_+(E, L)}.$$

Before we start with the proof, we visualize the effective potential:



Proof.
Proof of (i).
For $L > 0$ fixed, we want to find the roots of the first derivative of

$$\psi_L(r) = \frac{L}{2r^2} + U_0^r(r) = \frac{L}{2r^2} + \frac{b - \sqrt{r^2 + b^2}}{r^2} = \frac{b + \frac{L}{2} - \sqrt{r^2 + b^2}}{r^2}$$

on $]0, \infty[$. The derivative reads

$$\psi_L'(r) = \frac{1}{r^3}\left(-\frac{r^2}{\sqrt{r^2 + b^2}} - 2b - L + 2\sqrt{r^2 + b^2}\right).$$

For $r > 0$, this yields that

$$\begin{aligned}
\psi_L'(r) = 0 &\Leftrightarrow r^4 - (4bL + L^2)r^2 - 4b^3 L - b^2 L^2 = 0 \\
&\Leftrightarrow r^2 = \frac{4bL + L^2}{2} + \sqrt{\frac{(4bL + L^2)^2}{4} + 4b^3 L + b^2 L^2} \\
&\qquad = \frac{1}{2}\left[L(4b + L) + (2b + L)\sqrt{L(4b + L)}\right] \\
&\Leftrightarrow r = \frac{1}{\sqrt{2}}\sqrt{L(4b + L) + (2b + L)\sqrt{L(4b + L)}} =: r_L.
\end{aligned}$$

Consequently, we have shown that r_L is the only root of ψ_L' in $]0,\infty[$. To show that ψ_L takes its minimum at $r = r_L$, we estimate

$$\begin{aligned}
\psi_L\left(r_L\right) &= \frac{1}{r_L^2}\left(b + \frac{L}{2} - \sqrt{\frac{1}{2}\left[L(4b+L) + (2b+L)\sqrt{L(4b+L)}\right] + b^2}\right) \\
&< \frac{1}{r_L^2}\left(b + \frac{L}{2} - \sqrt{\frac{1}{2}\left[L(4b+L)\right] + b^2}\right) \\
&< \frac{1}{r_L^2}\left(b + \frac{L}{2} - \sqrt{bL + \frac{L^2}{4} + b^2}\right) = 0.
\end{aligned}$$

Additionally, it is obvious that if we fix $L > 0$, $\lim_{r\to 0}\psi_L(r) = \infty$ and $\lim_{r\to\infty}\psi_L(r) = 0$. Since r_L is the only root of ψ_L' in $]0,\infty[$, we therefore infer that ψ_L takes its global minimum (exclusively) at $r = r_L$.

Proof of (ii).
Let $L > 0$ and $E \in]\psi_L(r_L), 0[$. We already obtained that ψ_L' only has one root at $r = r_L$. Due to continuity, the derivative of ψ_L therefore does not change its sign in either of the sets $]0, r_L[$ and $]r_L, \infty[$ and also never takes the value 0 on these sets which implies a strict monotonicity of ψ_L on both $]0, r_L[$ and $]r_L, \infty[$. Considering the convergence properties $\lim_{r\to 0}\psi_L(r) = \infty$ and $\lim_{r\to\infty}\psi_L(r) = 0$ as well as $\psi_L(r_L) < 0$ yields that ψ_L is strictly monotonically decreasing on $]0, r_L[$ and strictly monotonically increasing on $]r_L, \infty[$. As a consequence, we infer the existence and uniqueness of $r_\pm(E, L) > 0$ with $r_-(E, L) < r_L < r_+(E, L)$ and $\psi_L(r_\pm(E, L)) = E$.
To obtain the desired representation of $r_\pm(E, L)$, we observe the following equivalences for $r > 0$:

$$\begin{aligned}
(r = r_-(E,L) \vee r = r_+(E,L)) &\Leftrightarrow \psi_L(r) = E \\
&\Leftrightarrow \sqrt{r^2 + b^2} = b + \frac{L}{2} - r^2 E \\
&\Leftrightarrow r^2 + b^2 = E^2 r^4 + (-2bE - LE)r^2 + bL + \frac{L^2}{4} + b^2 \\
&\Leftrightarrow E^2 r^4 + (-2bE - LE - 1)r^2 + bL + \frac{L^2}{4} = 0. \qquad (3.1)
\end{aligned}$$

The equivalence in the third line is due to $E < 0$ and therefore $b + L/2 - r^2E > 0$. We infer that if we define

$$s_\pm := r_\pm(E, L)^2,$$

then $s_\pm$ are in $\mathbb{R}$ with $0 < s_- < s_+$ and they solve

$$\begin{aligned}
0 &\underset{(3.1)}{=} E^2(r_\pm(E,L)^2)^2 + (-2bE - LE - 1)r_\pm(E,L)^2 + bL + \frac{L^2}{4} \\
&= E^2 s_\pm^2 + (-2bE - LE - 1)s_\pm + bL + \frac{L^2}{4}.
\end{aligned}$$

At the same time, we know that $0 = E^2 s^2 + (-2bE - LE - 1)s + bL + \frac{L^2}{4}$ has exactly two solutions in $\mathbb{C}$ that are

$$
\begin{aligned}
a_\pm &= \frac{2bE + LE + 1}{2E^2} \pm \sqrt{\left(\frac{2bE + LE + 1}{2E^2}\right)^2 - \frac{1}{E^2}\left(bL + \frac{L^2}{4}\right)} \\
&= \frac{1}{E^2}\left(\left(b + \frac{L}{2}E\right) + \frac{1}{2} \pm \sqrt{b^2E^2 + \left(b + \frac{L}{2}\right)E + \frac{1}{4}}\right).
\end{aligned}
$$

Since $a_\pm$ are the only solutions in $\mathbb{C}$ but $s_\pm$ are also two different solutions, we infer that $a_\pm = s_\pm \in \mathbb{R}, > 0$. In particular, the radicand of the occurring square root has to be positive. Since $a_\pm > 0$, we can take the square root of it to obtain the well-defined representation

$$
r_\pm(E, L) = \sqrt{s_\pm} = \sqrt{a_\pm} = -\frac{1}{E}\sqrt{\left(b + \frac{L}{2}E\right) + \frac{1}{2} \pm \sqrt{b^2E^2 + \left(b + \frac{L}{2}\right)E + \frac{1}{4}}}.
$$

Proof of (iii).
Let $L > 0$ an $E \in]\psi_L(r_L), 0[$. Due to $\frac{L}{2r^2} > 0$,

$$
\psi_L(r_L) = \min_{r\in]0,\infty[}\left(\frac{L}{2r^2} + U_0^r(r)\right) \geq \inf_{r\in]0,\infty[} U_0^r(r) = U_0^r(0) = -\frac{1}{2b}.
$$

Thus, $E > -\frac{1}{2b}$, which means that $bE + 1/2 > 0$. We therefore obtain that

$$
\begin{aligned}
&\left(b + \frac{L}{2}\right)E + \frac{1}{2} + \sqrt{b^2E^2 + \left(b + \frac{L}{2}\right)E + \frac{1}{4}} \\
&= bE + \frac{1}{2}LE + \frac{1}{2} + \sqrt{\left(bE + \frac{1}{2}\right)^2 + \frac{1}{2}LE} \\
&< bE + \frac{1}{2} + \left|bE + \frac{1}{2}\right| = 2bE + 1 < 1.
\end{aligned}
$$

As a result,

$$
r_+(E, L) = -\frac{1}{E}\sqrt{\left(b + \frac{L}{2}\right)E + \frac{1}{2} + \sqrt{b^2E^2 + \left(b + \frac{L}{2}\right)E + \frac{1}{4}}} < -\frac{1}{E}.
$$

Proof of (iv).
Let $L > 0$ and $E \in]\psi_L(r_L), 0[$. We follow the proof in [16] and define

$$
\xi(r) := r\left[E - \psi_L(r) - L\frac{(r_+(E,L) - r)(r - r_-(E,L))}{2r^2 r_+(E,L) r_-(E,L)}\right], \quad r \in [r_-(E,L), r_+(E,L)].
$$

The first and second derivative of ξ read

$$
\xi'(r) = E - \psi_L(r) - r\psi_L'(r) - L\left(\frac{1}{2r^2} - \frac{1}{2r_+(E,L)r_-(E,L)}\right), \quad r \in [r_-(E,L), r_+(E,L)],
$$

and

$$\begin{aligned}\xi''(r) &= -2\psi_L'(r) - r\psi_L''(r) + \frac{L}{r^3} = -2\left(-\frac{L}{r^3} + (U_0^r)'(r)\right) - r\left(3\frac{L}{r^4} + (U_0^r)''(r)\right) + \frac{L}{r^3} \\ &= -r\left(\frac{1}{r^2}\partial_r\left[r^2\partial_r U_0^r(r)\right]\right) = -r\left(\Delta U_0\right)^r(r) \\ &= -r4\pi\rho_0^r(r) \leq 0, \quad r \in [r_-(E,L), r_+(E,L)].\end{aligned}$$

This yields that ξ is a concave function. With $\xi\left(r_-(E,L)\right) = 0 = \xi\left(r_+(E,L)\right)$, we conclude that $\xi \geq 0$ on $[r_-(E,L), r_+(E,L)]$, which implies the desired inequality. □

3.2 The characteristic system

In order to talk about periods of particle orbits, we need to consider the solutions to the characteristic system. It is convenient to do this in the (r, w, L)-coordinate system. We show that these solutions exists globally and furthermore examine their trajectories.

Global existence of solutions to the characteristic system

Let $(r, w, L) \in \mathring{\Omega}_0^r = \{(r, w, L) \in]0, \infty[\times \mathbb{R} \times]0, \infty[\mid E^r(r, w, L) < 0\}$. We consider the initial value problem for the characteristic system in (r, w, L)-coordinates:

$$\begin{pmatrix} \dot{R} \\ \dot{W} \\ \dot{L} \end{pmatrix} = \begin{pmatrix} W \\ -\psi_L'(R) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} R \\ W \\ L \end{pmatrix}(0) = \begin{pmatrix} r \\ w \\ L \end{pmatrix}. \tag{3.2}$$

Since the right hand side of the ODE in (3.2) is in $C^1(D)$ for $D := \left\{(R, W, L) \in \mathbb{R}^3 \mid R > 0\right\}$ (note that the definition of ψ_L also works for $L < 0$) and therefore Lipschitz continuous, we can apply the Picard-Lindelöf theorem to obtain that there exists a unique maximal solution $(R, W, L)(\cdot) = (R, W, L)(\cdot, r, w, L) : I \to \mathbb{R}^3$ to the initial value problem (3.2). Here, $I \subset \mathbb{R}$ is an interval with $0 \in I$. Furthermore, we immediately recognize that the differential equation for L is solved by $L(t) = L, t \in I$, which is why we did not choose different variables for the solution $L(t)$ and the initial value L.
To show that the solution exists globally, i.e. $I = \mathbb{R}$, we start by showing that the energy function E^r is a conserved quantity. For $t \in I$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E^r(R(t), W(t), L(t))\right) = \psi_{L(t)}'(R(t))\dot{R}(t) + W(t)\dot{W}(t) + 0 = 0.$$

Hence, if we define $\alpha := E^r(r, w, L) < 0$, we have

$$\frac{1}{2}W(t)^2 + \psi_L(R(t)) = E^r(R(t), W(t), L) = \alpha, \quad t \in I,$$

from which we infer that W is bounded:

$$W(t)^2 = 2\alpha - 2\psi_L(R(t)) \leq 2\alpha - 2\psi_L(r_L), \quad t \in I.$$

Additionally, due to $\psi_L(R(t)) \leq \alpha < 0$ and the behavior of $\psi_L(r)$ for $r \to 0$ and $r \to \infty$, we find that there exist constants $C_{1,2} > 0$ with $C_1 < R(t) < C_2$, $t \in I$. As a consequence, (R, W, L) is bounded on I and stays away from $\partial D = \left\{(R, W, L) \in \mathbb{R}^3 \mid R = 0\right\}$. Basic ODE theory then yields that the maximal solution exists globally, i.e. $I = \mathbb{R}$. As E^r is conserved, we also conclude that $(R, W, L)(t) \in \mathring{\Omega}_0^r$ for any $t \in \mathbb{R}$.

Trajectories of solutions

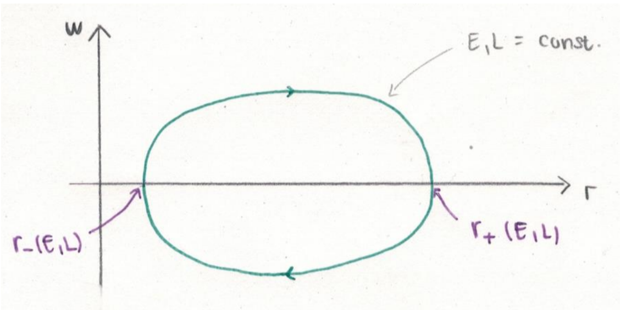
Let $(r, w, L) \in \mathring{\Omega}_0^r$ be fixed. We define $E := E^r(r, w, L)$ and consider the solution $(R, W, L)(\cdot) = (R, W, L)(\cdot, r, w, L) : \mathbb{R} \to \mathring{\Omega}_0^r$ of (3.2). We have

$$0 > E^r(r, w, L) = E = E^r(R(t), W(t), L) = \frac{1}{2}W(t)^2 + \psi_L(R(t)) \geq \psi_L(R(t)) \geq \psi_L(r_L)$$

for $t \in \mathbb{R}$ and therefore find that $R(t) \in [r_-(E, L), r_+(E, L)]$, using Theorem 3.1 (ii) and the monotonicity of ψ_L on $]0, r_L[$ and $]r_L, \infty[$. Moreover,

$$\dot{R}(t) = W(t) = \pm\sqrt{2E - 2\psi_L(R(t))}, \quad t \in \mathbb{R}. \tag{3.3}$$

Hence, $\dot{R}(t) \neq 0$ for $R(t) \in]r_-(E, L), r_+(E, L)[$. If $R(t) = r_-(E, L)$, then $\dot{R}(t) = 0$ and $\dot{W}(t) = -\psi_L'(r_-(E, L)) > 0$. Similarly, if $R(t) = r_+(E, L)$, then $\dot{R}(t) = 0$ and $\dot{W}(t) = -\psi_L'(r_+(E, L)) < 0$. These considerations imply that in (r, w)-space, the trajectory of the solution (R, W) is an ellipse-like curve:



3.3 The period function

In order to define and compute the period function, we need to make the following considerations: The Vlasov-Poisson system does not model single particles or point masses, but instead interprets globular clusters as having a continuous mass distribution. Therefore, strictly speaking, it is incorrect to talk about particles in the system or even periods of particle orbits. However, we can interpret solutions to the characteristic system as trajectories that small sample particles move on. That way, we can nonetheless talk about particle orbits and their periods.
Let us consider one of these particle curves, which is represented by a trajectory of the solution to the characteristic system. As seen above, E^r and L are constant along these trajectories and hence, every value $L > 0$ combined with a value $E \in]\psi_L(r_L), 0[$ represents a certain ellipse like curve in (r, w)-space. Since every trajectory corresponds to a particle and its path, we associate E and L with the (constant) energy and the angular momentum of a particle. Furthermore, we now ask the question of how long it takes for a particle to complete a full orbit in (r, w)-space. We call this time the period and express it in terms of the energy E and the angular momentum L. We will later obtain that the period function is actually independent of L, as intended.

To determine the period of a particle orbit, let the angular momentum of the particle $L > 0$, the energy of the particle $E \in]\psi_L(r_L), 0[$, and a solution $(R, W, L) : \mathbb{R} \to \mathbb{R}^3$ to the characteristic system with conserved energy E be fixed. To deduce how the time changes in dependence of the radius, we make the following considerations: If we look at the upper curve of the trajectory, we know that there exists an interval $]a, b[$ such that $R(a) = r_-(E, L)$, $R(b) = r_+(E, L)$ and $W > 0$ on $]a, b[$. R is differentiable on $[a, b]$ with derivative $\dot{R}(t) = W(t) > 0$ for $t \in]a, b[$. This additionally implies that R is strictly monotonically increasing on $[a, b]$. We can therefore apply the inverse function theorem to find that

$$\left(\left(R|_{[a,b]}\right)^{-1}\right)'(R(t)) = \frac{1}{\dot{R}(t)} \underset{(3.3)}{=} \frac{1}{\sqrt{2E - 2\psi_L(R(t))}}, \quad t \in]a, b[.$$

Since $R(]a, b[) =]r_-(E, L), r_+(E, L)[$ due to continuity,

$$\left(\left(R|_{[a,b]}\right)^{-1}\right)'(r) = \frac{1}{\sqrt{2E - 2\psi_L(r)}}, \quad r \in]r_-(E, L), r_+(E, L)[. \tag{3.4}$$

We interpret R^{-1} as the time that corresponds to a given r-value and therefore interpret $\left(R^{-1}\right)'$ as the rate with which time elapses at a given radius r. To obtain the total time that it takes the particle to move from $r_-(E, L)$ to $r_+(E, L)$, we integrate this elapse rate with respect to r over the region of integration $]r_-(E, L), r_+(E, L)[$:

$$\int_{r_-(E,L)}^{r_+(E,L)} \frac{1}{\sqrt{2E - 2\psi_L(r)}} \, \mathrm{d}r.$$

The time that it takes the particle to move from $r_+(E, L)$ to $r_-(E, L)$ can be derived analogously and we get the same integral. We therefore define the period function as follows:

Definition 3.2. *We define the period of a particle orbit as*

$$T(E, L) := 2 \int_{r_-(E,L)}^{r_+(E,L)} \frac{1}{\sqrt{2E - 2\psi_L(r)}} \, \mathrm{d}r,$$

where the particle corresponds to the angular momentum $L > 0$ and the energy $E \in]\psi_L(r_L), 0[$.

Since $r_-(E, L)$, $r_+(E, L)$ and $\psi_L(r_L)$ are given explicitly in our case, we are able to compute the integral and find that the period function T is independent of L:

Lemma 3.3. *The period function T is independent of L and*

$$T(E, L) = \frac{\pi}{\sqrt{2}} \frac{1}{|E|^{3/2}} =: T(E)$$

for all $L > 0$ and $E \in]\psi_L(r_L), 0[$. Since T is no longer dependent on L, we change its domain of definition to $\{E \in \mathbb{R} \mid \exists L > 0 : E \in]\psi_L(r_L), 0[\} =]-\frac{1}{2b}, 0[$.

Proof. We first note that the integrand has a singularity at both bounds of the region of integration. Therefore, in all of what follows, we have to interpret the integral as a limit. The first step is to substitute $s = s(r) = -\frac{1}{bU_0^T(r)} = 1 + \frac{1}{b}\sqrt{r^2 + b^2}$, which is equivalent to $r = b\sqrt{s(s-2)}$. We compute

$$\psi_L\left(b\sqrt{s(s-2)}\right) = \frac{b + \frac{L}{2} - \sqrt{b^2 s(s-2) + b^2}}{b^2 s(s-2)} = \frac{b + \frac{L}{2} - b(s-1)}{b^2 s(s-2)}.$$

The integrand $1/\sqrt{E-\psi_L(r)}$ with respect to s therefore transforms to

$$\frac{1}{\sqrt{E-\frac{b+\frac{L}{2}-b(s-1)}{b^2 s(s-2)}}}\frac{b(s-1)}{\sqrt{s(s-2)}}=\frac{b^2(s-1)}{\sqrt{b^2Es^2-2b^2Es-b-\frac{L}{2}+bs-b}}.$$

This yields

$$T(E,L)=\sqrt{2}\int_{s(r_-(E,L))}^{s(r_+(E,L))}\frac{b^2(s-1)}{\sqrt{b^2Es^2+(-2b^2E+b)s-2b-\frac{L}{2}}}\,\mathrm{d}s.$$

We define $x:=b^2E^2+\left(b+\frac{L}{2}\right)E+\frac{1}{4}$ to compute the bounds of integration:

$$\begin{aligned}
s_\pm:=s\left(r_\pm(E,L)\right)&=1+\frac{1}{b}\sqrt{\frac{1}{E^2}\left(\left(b+\frac{L}{2}\right)E+\frac{1}{2}\pm\sqrt{x}\right)+b^2}\\
&=1-\frac{1}{bE}\sqrt{\left(b+\frac{L}{2}\right)E+\frac{1}{2}\pm\sqrt{x}+b^2E^2}\\
&=1-\frac{1}{bE}\sqrt{x\pm\sqrt{x}+\frac{1}{4}}\\
&=1-\frac{1}{bE}\left|\frac{1}{2}\pm\sqrt{x}\right|\\
&=1-\frac{1}{bE}\left(\frac{1}{2}\pm\sqrt{x}\right).
\end{aligned}\tag{3.5}$$

To observe the last equality, we use that the explicit representations of $r_\pm(E,L)$ in Theorem 3.1 (ii) are well-defined and therefore,

$$0\leq\left(b+\frac{L}{2}\right)E+\frac{1}{2}\pm\sqrt{x}<\frac{1}{2}\pm\sqrt{x}$$

for $L>0$ and $E\in\,]\psi_L(r_L),0[$.

In order further simplify the period function integral, we compute the roots of the radicand $b^2Es^2+(-2b^2E+b)s-2b-\frac{L}{2}$:

$$\begin{aligned}
s_{1,2}&=1-\frac{1}{2bE}\pm\sqrt{\left(1-\frac{1}{2bE}\right)^2+\frac{2}{bE}+\frac{L}{2b^2E}}\\
&=1-\frac{1}{2bE}\mp\frac{1}{bE}\sqrt{b^2E^2-bE+\frac{1}{4}+2bE+\frac{L}{2}E}\\
&=1-\frac{1}{bE}\left(\frac{1}{2}\pm\sqrt{x}\right)=s_\pm.
\end{aligned}$$

Consequently, we find that

$$\begin{aligned}
T(E,L) &= \sqrt{2}\int_{s_-}^{s_+} \frac{b^2(s-1)}{\sqrt{b^2E(s-s_+)(s-s_-)}}\,\mathrm{d}s = \frac{\sqrt{2}b}{\sqrt{-E}}\int_{s_-}^{s_+} \frac{s-1}{\sqrt{(s_+-s)(s-s_-)}}\,\mathrm{d}s \\
&= \frac{\sqrt{2}b}{\sqrt{-E}}\left(\int_{s_-}^{s_+} \frac{s-\frac{1}{2}(s_++s_-)}{\sqrt{(s_+-s)(s-s_-)}}\,\mathrm{d}s + \int_{s_-}^{s_+} \frac{\frac{1}{2}(s_++s_-)-1}{\sqrt{(s_+-s)(s-s_-)}}\,\mathrm{d}s\right) \\
&= \frac{\sqrt{2}b}{\sqrt{-E}} \lim_{\substack{a_-\to s_-, a_->s_- \\ a_+\to s_+, a_+<s_+}} \left(\int_{a_-}^{a_+} \frac{s-\frac{1}{2}(s_++s_-)}{\sqrt{(s_+-s)(s-s_-)}}\,\mathrm{d}s + \int_{a_-}^{a_+} \frac{\frac{1}{2}(s_++s_-)-1}{\sqrt{(s_+-s)(s-s_-)}}\,\mathrm{d}s\right).
\end{aligned}$$

We examine each integral individually. To compute the first integral, we substitute $u = u(s) = (s_+-s)(s-s_-)$ and obtain that

$$\begin{aligned}
\int_{a_-}^{a_+} \frac{s-\frac{1}{2}(s_++s_-)}{\sqrt{(s_+-s)(s-s_-)}}\,\mathrm{d}s &= \int_{(s_+-a_-)(a_--s_-)}^{(s_+-a_+)(a_+-s_-)} \frac{-1}{2\sqrt{u}}\,\mathrm{d}u \\
&= \sqrt{(s_+-a_-)(a_--s_-)} - \sqrt{(s_+-a_+)(a_+-s_-)}.
\end{aligned}$$

If we take the limit, this term vanishes:

$$\lim_{\substack{a_-\to s_-, a_->s_- \\ a_+\to s_+, a_+<s_+}} \sqrt{(s_+-a_-)(a_--s_-)} - \sqrt{(s_+-a_+)(a_+-s_-)} = 0.$$

For the second integral, we start by observing that

$$(s_+-s)(s-s_-) = \left(\frac{s_+-s_-}{2}\right)^2\left[1-\left(\frac{2s-s_+-s_-}{s_+-s_-}\right)^2\right].$$

Inserting this into the integral as well as substituting $u = u(s) = \frac{2s-s_+-s_-}{s_+-s_-}$ yields that

$$\begin{aligned}
\int_{a_-}^{a_+} \frac{1}{\sqrt{(s_+-s)(s-s_-)}}\,\mathrm{d}s &= \frac{2}{s_+-s_-}\int_{a_-}^{a_+} \frac{1}{\sqrt{1-\left(\frac{2s-s_+-s_-}{s_+-s_-}\right)^2}}\,\mathrm{d}s \\
&= \frac{2}{s_+-s_-}\int_{\frac{2a_--s_+-s_-}{s_+-s_-}}^{\frac{2a_+-s_+-s_-}{s_+-s_-}} \frac{1}{\sqrt{1-u^2}}\frac{s_+-s_-}{2}\,\mathrm{d}u \\
&= \arcsin\left(\frac{2a_+-s_+-s_-}{s_+-s_-}\right) - \arcsin\left(\frac{2a_--s_+-s_-}{s_+-s_-}\right).
\end{aligned}$$

By taking the limit, we infer that

$$\begin{aligned}
&\lim_{\substack{a_-\to s_-, a_->s_- \\ a_+\to s_+, a_+<s_+}} \left[\arcsin\left(\frac{2a_+-s_+-s_-}{s_+-s_-}\right) - \arcsin\left(\frac{2a_--s_+-s_-}{s_+-s_-}\right)\right] \\
&= \arcsin(1) - \arcsin(-1) = \frac{\pi}{2} - \frac{-\pi}{2} = \pi.
\end{aligned}$$

As a result,

$$\begin{aligned}
T(E,L) &= \frac{\sqrt{2}b}{\sqrt{-E}}\left(0+\left(\frac{1}{2}(s_+ + s_-)-1\right)\pi\right)\\
&\underset{(3.5)}{=} \frac{\sqrt{2}\pi b}{\sqrt{-E}}\left(\frac{1}{2}\left(1-\frac{1}{2bE}-\frac{\sqrt{x}}{bE}+1-\frac{1}{2bE}+\frac{\sqrt{x}}{bE}\right)-1\right)\\
&= \frac{\sqrt{2}\pi b}{\sqrt{-E}}\left(-\frac{1}{2bE}\right) = \frac{\pi}{\sqrt{2}}\frac{1}{|E|^{3/2}}.
\end{aligned}$$

□

3.4 (θ, E, L)-coordinates

Based on the trajectories of solutions to the characteristic system, we get the idea to characterize the domain Ω_0 and spherically symmetric functions through another set of coordinates. The idea is the following: Consider some point (r, w, L). This point corresponds to an angular momentum value L and an energy value $E = E^r(r, w, L)$. We then solve (3.2) with initial values $(r_-(E, L), 0, L)$. Then, (r, w) lies on the trajectory in (r, w)-space. We introduce the variable $\theta \in [0, 1]$, which represents what fraction of the time of a full orbit it takes to get from $(r_-(E, L), 0)$ to (r, w) within the first rotation. Mathematically speaking:

Theorem 3.4. *The mapping*

$$\begin{aligned}
\Gamma : \,]0,1[\, \times \mathring{\Omega}_0^{EL} &\to \mathring{\Omega}_0^r \setminus \{(r,0,L)\in \mathring{\Omega}_0^r \mid r \le r_L\}\\
(\theta, E, L) &\mapsto (R, W, L)(\theta T(E), r_-(E,L), 0, L),
\end{aligned}$$

is bijective with inverse

$$\zeta : \mathring{\Omega}_0^r \setminus \{(r,0,L)\in \mathring{\Omega}_0^r \mid r \le r_L\} \to \,]0,1[\, \times \mathring{\Omega}_0^{EL},$$

$$(r,w,L) \mapsto \begin{cases} \left(\frac{1}{T(E^r(r,w,L))}\int_{r_-(E^r(r,w,L),L)}^{r} \frac{1}{\sqrt{2E^r(r,w,L)-2\psi_L(s)}}\,\mathrm{d}s, E^r(r,w,L), L\right), & w \ge 0,\\ \left(1-\frac{1}{T(E^r(r,w,L))}\int_{r_-(E^r(r,w,L),L)}^{r} \frac{1}{\sqrt{2E^r(r,w,L)-2\psi_L(s)}}\,\mathrm{d}s, E^r(r,w,L), L\right), & w < 0.\end{cases}$$

Here,

$$\begin{aligned}
\Omega_0^{EL} &:= \left\{(E,L)\in\mathbb{R}^2 \mid \exists (x,v)\in\Omega_0 : E = E(x,v), L = L(x,v)\right\}\\
&= \left\{(E,L)\in\mathbb{R}^2 \mid L\in[0,\infty[, E\in[\psi_L(r_L),0[\right\}, \quad \psi_0(r_0) := U_0(0) = -\frac{1}{2b},
\end{aligned}$$

and $(R, W, L)(\cdot, r_-(E, L), 0, L)$ is the solution to the initial value problem (3.2).

Proof. The proof is structured as follows: We first show that both functions Γ and ζ are well-defined. Then, we show that $\zeta \circ \Gamma = \mathrm{id}_{]0,1[\times\mathring{\Omega}_0^{EL}}$. This implies that ζ is surjective. After that, we prove that ζ is injective and therefore has an inverse function, which has to be Γ as $\zeta \circ \Gamma = \mathrm{id}_{]0,1[\times\mathring{\Omega}_0^{EL}}$.

Well-definedness of Γ.
Let $(\theta, E, L) \in]0,1[\times \mathring{\Omega}_0^{EL}$. We first notice that the initial value problem is well-posed since $(r_-(E,L), 0, L) \in \mathring{\Omega}_0^r$; note that for $\mathring{\Omega}_0^{EL}$, we have $L > 0$ and that $E^r(r_-(E,L), 0, L) = E < 0$ as well as $r_-(E,L) > 0$. We also have that $(R,W,L)(\theta T(E), r_-(E,L), 0, L) \in \mathring{\Omega}_0^r$ as observed in Section 3.2. It remains to show that $(R,W,L)(\theta T(E), r_-(E,L), 0, L))$ is not an element of $\{(r, 0, L) \in \mathring{\Omega}_0^r \mid r \leq r_L\}$. For this, we start by noticing that

$$\begin{aligned}&\{(r,0,L) \in \mathring{\Omega}_0^r \mid r \leq r_L\}\\ &= \{(r,0,L) \in \mathring{\Omega}_0^r \mid r = r_L\} \cup \{(r,0,L) \in \mathring{\Omega}_0^r \mid \exists E \in]\psi_L(r_L), 0[: r = r_-(E,L)\}.\end{aligned}$$

Now, suppose that there exists some $\tilde{L} > 0$ such that $(R,W,L)(\theta T(E), r_-(E,L), 0, L) = (r_{\tilde{L}}, 0, \tilde{L})$. Since $L(\theta T(E), r_-(E,L), 0, L) = L$, we find that $\tilde{L} = L$. As E^r is constant along characteristics,

$$E = E^r(r_-(E,L), 0, L) = E^r\left((R,W,L)(\theta T(E), r_-(E,L), 0, L)\right) = E^r\left(r_L, 0, L\right) = \psi_L(r_L),$$

which contradicts $(E,L) \in \mathring{\Omega}_0^{EL} = \left\{(E,L) \in \mathbb{R}^2 \mid L > 0, E \in]\psi_L(r_L), 0[\right\}$.
If we assume that there exists $\tilde{L} > 0$ and $\tilde{E} \in]\psi_{\tilde{L}}(r_{\tilde{L}}), 0[$ such that $\Gamma(\theta, E, L) = (r_-(\tilde{E}, \tilde{L}), 0, \tilde{L})$, we again find that $\tilde{L} = L$ and therefore

$$\tilde{E} = E^r(r_-(\tilde{E}, L), 0, L)) = E^r(\Gamma(\theta, E, L)) = E^r((R,W,L)(\theta T(E), r_-(E,L), 0, L)) = E.$$

Hence, $(R,W,L)(\theta T(E), r_-(E,L), 0, L) = (r_-(E,L), 0, L)$, which by the considerations in Section 3.2 and the definition of the period function is possible only if $\theta \in \mathbb{Z}$, which is contradictory to $\theta \in]0,1[$.

Well-definedness of ζ.
Let $(r, w, L) \in \mathring{\Omega}_0^r \setminus \{(r, 0, L) \in \mathring{\Omega}_0^r \mid r \leq r_L\}$. First, notice that $\zeta_3(r,w,L) = L > 0$. Furthermore, $\zeta_2(r,w,L) = E^r(r,w,L) \in [\psi_L(r_L), 0[$. Moreover, the value $\psi_L(r_L)$ can not be taken by $\zeta_2(r,w,L)$ since if it did, then $r = r_L$ and $w = 0$ would hold, which contradicts $(r,w,L) \notin \{(r,0,L) \in \mathring{\Omega}_0^r \mid r \leq r_L\}$. Consequently, $(\zeta_2, \zeta_3)(r,w,L) \in \mathring{\Omega}_0^{EL}$.
Next, we deduce that $r_-(E^r(r,w,L), L) \leq r \leq r_+(E^r(r,w,L), L)$. This is satisfied because if we assume that $r < r_-(E^r(r,w,L)))$, then the monotonicity properties of ψ_L imply that $\psi_L(r) > \psi_L(r_-(E^r(r,w,L), L)) = E^r(r,w,L) = \frac{1}{2}w^2 + \psi_L(r) \geq \psi_L(r)$, which is clearly a contradiction. Similarly, we can show that $r \leq r_+(E^r(r,w,L), L)$. Therefore, the integral occurring in the ζ-function is well-defined, which we infer from the well-definedness of the period function. Additionally, we find that

$$\begin{aligned}0 &\leq \int_{r_-(E^r(r,w,L),L)}^{r} \frac{1}{\sqrt{2E^r(r,w,L) - 2\psi_L(s)}}\,\mathrm{d}s\\ &\leq \int_{r_-(E^r(r,w,L),L)}^{r_+(E^r(r,w,L),L)} \frac{1}{\sqrt{2E^r(r,w,L) - 2\psi_L(s)}}\,\mathrm{d}s = \frac{1}{2}T(E^r(r,w,L)).\end{aligned}$$

This leads to $\zeta_1(r,w,L) \in [0,1]$.
To exclude $\zeta_1(r,w,L) = 0$, we first consider that this is only possible for $w \geq 0$ and $\int_{r_-(E^r(r,w,L),L)}^{r} \frac{1}{\sqrt{2E^r(r,w,L)-2\psi_L(s)}}\,\mathrm{d}s = 0$. This yields that $r = r_-(E^r(r,w,L), L)$ which implies that

$$\psi_L(r) = \psi_L(r_-(E^r(r,w,L), L)) = E^r(r,w,L) = \frac{1}{2}w^2 + \psi_L(r).$$

Consequently, $w = 0$, and therefore, $(r, w, L) = (r_-(E^r(r, w, L), L), 0, L)$. This is a contradiction to $(r, w, L) \notin \{(r, 0, L) \in \mathring{\Omega}_0^r \mid r \leq r_L\}$.
To exclude $\zeta_1(r, w, L) = 1$, we note that this is only possible for $w < 0$ and if the integral $\int_{r_-(E^r(r,w,L),L)}^{r} \frac{1}{\sqrt{2E^r(r,w,L)-2\psi_L(s)}} \, \mathrm{d}s$ is equal to 0. The same arguments as above lead to $w = 0$, which contradicts $w < 0$.

Proof of $\zeta \circ \Gamma = \mathrm{id}_{]0,1[\times \mathring{\Omega}_0^{EL}}$.
Let $(\theta, E, L) \in]0, 1[\times \mathring{\Omega}_0^{EL}$. We obviously have

$$\zeta_2(\Gamma(\theta, E, L)) = E^r((R, W, L)(\theta T(E), r_-(E, L), 0, L)) = E^r(r_-(E, L), 0, L) = E$$

and

$$\zeta_3(\Gamma(\theta, E, L)) = \Gamma_3(\theta, E, L) = L(\theta T(E), r_-(E, L), 0, L) = L.$$

For the first component of $\zeta(\Gamma(\theta, E, L))$, we distinguish between $\theta \in]0, \frac{1}{2}]$ and $\theta \in]\frac{1}{2}, 1[$. Values for θ in $]0, \frac{1}{2}]$ mean that $\theta T(E) \in]0, \frac{1}{2}T(E)]$. If we remember the definition of the period function and its proof, this corresponds to the upper half of a rotation in (r, w)-space if we start at $(r_-(E, L), 0)$, and therefore, $W(\theta T(E), r_-(E, L), 0, L) \geq 0$. We use (3.4) and find that

$$\begin{aligned}
\zeta_1(\Gamma(\theta, E, L)) &= \frac{1}{T(E)} \int_{r_-(E,L)}^{R(\theta T(E), r_-(E,L),0,L)} \frac{1}{\sqrt{2E - 2\psi_L(s)}} \, \mathrm{d}s \\
&= \frac{1}{T(E)} \int_{r_-(E,L)}^{R(\theta T(E), r_-(E,L),0,L)} \left(\left(R|[0, \frac{1}{2}T(E)] \right)^{-1} \right)' (s) \, \mathrm{d}s \\
&= \frac{1}{T(E)} (\theta T(E) - 0) = \theta.
\end{aligned}$$

Note that since we only have the existence of the derivative of the inverse of the restriction of R and the identity (3.4) on $]r_-(E, L), r_+(E, L)[$, we technically have to work with limits in order to apply the fundamental theorem of calculus. However, the continuity of the inverse of R on $[r_-(E, L), r_+(E, L)]$ ensures that the argument still works out.
As $\theta \in]\frac{1}{2}, 1[$ corresponds to the lower half of the rotation, we have $W(\theta T(E), r_-(E, L), 0, L) \leq 0$. Using the same argumentation as for (3.4), we deduce that

$$\left(\left(R|_{[\frac{1}{2}T(E), T(E)]} \right)^{-1} \right)' = -\frac{1}{2E - 2\psi_L(r)}, \quad r \in]r_-(E, L), r_+(E, L)[.$$

Consequently,

$$\begin{aligned}
\zeta_1(\Gamma(\theta, E, L)) &= 1 - \frac{1}{T(E)} \int_{r_-(E,L)}^{R(\theta T(E), r_-(E,L),0,L)} - \left(\left(R|_{[\frac{1}{2}T(E), T(E)]} \right)^{-1} \right)' (s) \, \mathrm{d}s \\
&= 1 - \frac{1}{T(E)} (-\theta T(E) + T(E)) = \theta.
\end{aligned}$$

ζ is injective.
Let $(r, w, L), (\tilde{r}, \tilde{w}, \tilde{L}) \in \mathring{\Omega}_0^r \setminus \{(r, 0, L) \in \mathring{\Omega}_0^r \mid r \leq r_L\}$ with $\zeta(r, w, L) = \zeta(\tilde{r}, \tilde{w}, \tilde{L})$. We immediately observe that $\tilde{L} = L$ and $E^r(r, w, L) = E^r(\tilde{r}, \tilde{w}, L) =: E$, hence,

$$\frac{1}{2}w^2 + \psi_L(r) = \frac{1}{2}\tilde{w}^2 + \psi_L(\tilde{r}) = E. \tag{3.6}$$

If $w \geq 0$ and $\tilde{w} \geq 0$ or $w < 0$ and $\tilde{w} < 0$, then $\zeta_1(r, w, L) = \zeta_1(\tilde{r}, \tilde{w}, L)$ leads to

$$\frac{1}{T(E)} \int_{r_-(E,L)}^{r} \frac{1}{\sqrt{2E - 2\psi_L(s)}} \, \mathrm{d}s = \frac{1}{T(E)} \int_{r_-(E,L)}^{\tilde{r}} \frac{1}{\sqrt{2E - 2\psi_L(s)}} \, \mathrm{d}s,$$

which can only hold true if $r = \tilde{r}$. Equation (3.6) then yields that $|w| = |\tilde{w}|$ which is equivalent to $w = \tilde{w}$ in this case.
If $w \geq 0$ and $\tilde{w} < 0$, then

$$[0, 1/2] \ni \frac{1}{T(E)} \int_{r_-(E,L)}^{r} \ldots \, \mathrm{d}s = \zeta_1(r, w, L) = \zeta_1(\tilde{r}, \tilde{w}, \tilde{L}) = 1 - \frac{1}{T(E)} \int_{r_-(E,L)}^{\tilde{r}} \ldots \, \mathrm{d}s \in [1/2, 1].$$

This implies that both integral terms have to be equal to $T(E)/2$ which is only possible if the upper bounds of the region of integration r and $\tilde{r}$ are equal to $r_+(E, L)$. Equation (3.6) then yields that $w = \tilde{w} = 0$, which contradicts $\tilde{w} < 0$. As a result, $w \geq 0$ and $\tilde{w} < 0$ is not possible and renaming the variables yields that $w < 0$ and $\tilde{w} \geq 0$ is not possible as well. Overall, we conclude that $(r, w, L) = (\tilde{r}, \tilde{w}, \tilde{L})$, i.e. ζ is injective and the proof is complete. □

Before introducing representative functions in (θ, E, L)-coordinates, we define the corresponding domain as

$$\Omega_0^\theta := [0, 1] \times \Omega_0^{EL}.$$

Note that the domain of definition of Γ is equal to Ω_0^θ except for a set of measure zero. Now, let $g \in L^1_{\mathrm{loc}}(\Omega_0; \mathbb{K})$ be a spherically symmetric function (interpreted as being extended by 0 onto $\mathbb{R}^3 \times \mathbb{R}^3$) with (r, w, L)-representative $g^r : \Omega_0^r \to \mathbb{K}$. We then define the (θ, E, L)-representative g^θ by

$$g^\theta(\theta, E, L) := g^r(\Gamma(\theta, E, L)) \tag{3.7}$$

for $(\theta, E, L) \in]0, 1[\times \mathring{\Omega}_0^{EL}$ with $\Gamma(\theta, E, L) \notin N$ where N is the null set containing the points where g^r is undefined. As a consequence, (3.7) is defined for almost every $(\theta, E, L) \in \Omega_0^\theta$ since Γ and ζ map null sets to null sets (see Lemma 3.6). Due to that, we conversely find

$$g^r(r, w, L) = g^\theta(\Gamma^{-1}(r, w, L)) = g^\theta(\zeta(r, w, L)), \quad \text{f.a.e. } (r, w, L) \in \Omega_0^r. \tag{3.8}$$

In order to prove that Γ and ζ map null sets to null sets, we show the following:

Theorem 3.5. *The functions Γ and ζ from Theorem 3.4 are C^1-diffeomorphisms.*

Proof. Due to Theorem 3.4, we only have to show that Γ and ζ are C^1-functions on their corresponding domains of definition.

For Γ, basic ODE theory yields that $(R, W, L) \in C^1(\mathbb{R} \times \mathring{\Omega}_0^r)$. Due to the explicit representations of $T(E)$ and $r_-(E, L)$ (see Lemma 3.3 and Theorem 3.1), we find that $T, r_- \in C^1(\mathring{\Omega}_0^{EL})$ if we interpret T as a function on $\mathring{\Omega}_0^{EL}$. This yields that $\Gamma \in C^1(]0,1[\times \mathring{\Omega}_0^{EL})$.
The idea for ζ is to use the inverse function theorem. It suffices to prove that the determinant of the Jacobian matrix of Γ does not vanish on $]0,1[\times \mathring{\Omega}_0^{EL}$. For $(\theta, E, L) \in\,]0,1[\times \mathring{\Omega}_0^{EL}$, the Jacobian matrix $D\Gamma(\theta, E, L)$ reads

$$\begin{pmatrix} \dot{R}(...)T(E) & \dot{R}(...)\theta T'(E) + \partial_r R(...)\partial_E r_-(E,L) & * \\ \dot{W}(...)T(E) & \dot{W}(...)\theta T'(E) + \partial_r W(...)\partial_E r_-(E,L) & * \\ \dot{L}(...)T(E) & \dot{L}(...)\theta T'(E) + \partial_r L(...)\partial_E r_-(E,L) & \partial_r L(...)\partial_L r_-(E,L) + \partial_L L(...) \end{pmatrix}$$

where $(...) = (\theta T(E), r_-(E, L), 0, L)$. Since L is constant and independent of the r-value of the initial condition, the last row reads $(0, 0, 1)$. This leads to

$$\begin{aligned} \det(D\Gamma(\theta, E, L)) &= T(E)\partial_E r_-(E,L) \left[\dot{R}(...)\partial_r W(...) - \dot{W}(...)\partial_r R(...)\right] \\ &= T(E)\partial_E r_-(E,L) \left[W(...)\partial_r W(...) + \psi_L'(R(...))\partial_r R(...)\right]. \end{aligned}$$

We examine the bracket [...] on $\mathbb{R} \times \mathring{\Omega}_0^r$. To do this, we again consider the initial value problem (3.2) and denote the right hand side by $f = f(\tilde{R}, \tilde{W}, \tilde{L}) := (\tilde{W}, -\psi_{\tilde{L}}'(\tilde{R}), 0)$ on $\mathring{\Omega}_0^r$. Its Jacobian matrix is

$$Df(\tilde{R}, \tilde{W}, \tilde{L}) = \begin{pmatrix} 0 & 1 & 0 \\ -\psi_{\tilde{L}}''(\tilde{R}) & 0 & 1/\tilde{R}^3 \\ 0 & 0 & 0 \end{pmatrix}.$$

ODE theory yields that

$$\partial_t \begin{pmatrix} \partial_r R & \partial_w R & \partial_L R \\ \partial_r W & \partial_w W & \partial_L W \\ \partial_r L & \partial_w L & \partial_L L \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\psi_L''(R) & 0 & 1/R^3 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_r R & \partial_w R & \partial_L R \\ \partial_r W & \partial_w W & \partial_L W \\ \partial_r L & \partial_w L & \partial_L L \end{pmatrix}$$

on $\mathbb{R} \times \mathring{\Omega}_0^r$, as well as $\partial_r R(0, r, w, L) = 1$ and $\partial_r W(0, r, w, L) = 0$. We again note that the last row of the last matrix reads $(0, 0, 1)$. We therefore find that $\partial_t \partial_r R = \partial_r W$ and $\partial_t \partial_r W = -\psi_L''(R)\partial_r R$ on $\mathbb{R} \times \mathring{\Omega}_0^r$. Hence,

$$\partial_t \left[W\partial_r W + \psi_L'(R)\partial_r R\right] = \dot{W}\partial_r W + W\left(-\psi_L''\partial_r R\right) + \psi_L''(R)\dot{R}\partial_r R + \psi_L'(R)\partial_r W = 0$$

on $\mathbb{R} \times \mathring{\Omega}_0^r$, using that $\dot{R} = W$ and $\dot{W} - \psi_L'(R)$. Consequently, we observe that

$$\begin{aligned} &W(t,r,w,L)\partial_r W(t,r,w,L) + \psi_L'(R(t,r,w,L))\partial_r R(t,r,w,L) \\ &= W(0,r,w,L)\partial_r W(0,r,w,L) + \psi_L'(R(0,r,w,L))\partial_r R(0,r,w,L) \\ &= \psi_L'(r) \end{aligned}$$

for $(t, r, w, L) \in \mathbb{R} \times \mathring{\Omega}_0^r$. As a result,

$$\begin{aligned} \det(D\Gamma(\theta, E, L)) &= T(E)\partial_E r_-(E,L)\psi_L'(r_-(E,L)) \\ &= T(E)\partial_E\left(\psi_L(r_-(E,L))\right) \\ &= T(E)\partial_E(E) = T(E), \quad (\theta, E, L) \in\,]0,1[\times \mathring{\Omega}_0^{EL}. \end{aligned} \tag{3.9}$$

As T never vanishes, we infer by the inverse function theorem that ζ is a C^1-function. $\square$

We now prove that Γ and ζ map null sets to null sets:

Lemma 3.6. *The functions Γ and ζ from Theorem 3.4 map null sets to null sets.*

Proof. First, let $N^r \subset \mathring{\Omega}_0^r \setminus \{(r,0,L) \in \mathring{\Omega}_0^r \mid r \le r_L\}$ be a null set. We find that

$$\begin{aligned} 0 &\le T(-1/(2b))\lambda\left(\zeta(N^r)\right) \\ &\le \int_{\Omega_0^\theta} T(E) 1_{\zeta(N^r)}(\theta,E,L)\,\mathrm{d}(\theta,E,L) = \int_{\Omega_0^\theta} T(E) 1_{N^r}(\Gamma(\theta,E,L))\,\mathrm{d}(\theta,E,L) \end{aligned}$$

as $(\theta,E,L) \in \zeta(N^r) \Leftrightarrow \Gamma(\theta,E,L) \in N^r$. We use the transformation

$$]0,1[\, \times \mathring{\Omega}_0^{EL} \ni (\theta,E,L) \mapsto (r,w,L) := \Gamma(\theta,E,L) \in \mathring{\Omega}_0^r \setminus \{(r,0,L) \in \mathring{\Omega}_0^r \mid r \le r_L\}, \tag{3.10}$$

which is a C^1-diffeomorphism by Theorem 3.5. Equation (3.9) yields that

$$\int_{\Omega_0^\theta} T(E) 1_{N^r}(\Gamma(\theta,E,L))\,\mathrm{d}(\theta,E,L) = \int_{\Omega_0^r} 1_{N^r}(r,w,L)\,\mathrm{d}(r,w,L) = \lambda(N^r) = 0.$$

Note that Ω_0^θ and the domain of definition of Γ only differ from each other by a set of measure 0, the same holds for Ω_0^r and the domain of definition of ζ. Therefore, $\lambda(\zeta(N^r)) = 0$.

Conversely, if $N^\theta \subset\,]0,1[\, \times \mathring{\Omega}_0^{EL}$ is a null set, we use the same transformation to obtain that

$$\begin{aligned} 0 &= \int_{\Omega_0^\theta} T(E) 1_{N^\theta}(\theta,E,L)\,\mathrm{d}(\theta,E,L) = \int_{\Omega_0^\theta} T(E) 1_{N^\theta}(\zeta(\Gamma(\theta,E,L)))\,\mathrm{d}(\theta,E,L) \\ &= \int_{\Omega_0^r} 1_{N^\theta}(\zeta(r,w,L))\,\mathrm{d}(r,w,L) = \int_{\Omega_0^r} 1_{\Gamma(N^\theta)}(r,w,L)\,\mathrm{d}(r,w,L) = \lambda(\Gamma(N^\theta)) \end{aligned}$$

as $\zeta(r,w,L) \in N^\theta \Leftrightarrow (r,w,L) \in \Gamma(N^\theta)$. □

The transformation (3.10) that we used in the previous proof will now be used for switching between the different coordinate representations in integrals:

Lemma 3.7. *Let $g \in L^1_{\mathrm{loc}}(\Omega_0; \mathbb{K})$ be a spherically symmetric function with (r,w,L)-representative $g^r : \Omega_0^r \to \mathbb{K}$ and (θ,E,L)-representative $g^\theta : \Omega_0^\theta \to \mathbb{K}$. Then,*

$$\int_{\Omega_0^r} g^r(r,w,L)\,\mathrm{d}(r,w,L) = \int_{\Omega_0^\theta} T(E) g^\theta(\theta,E,L)\,\mathrm{d}(\theta,E,L),$$

provided that the integrals exist.

Proof. We use the transformation (3.10) and (3.9), to find that

$$\begin{aligned} \int_{\Omega_0^\theta} T(E) g^\theta(\theta,E,L)\,\mathrm{d}(\theta,E,L) &\underset{(3.7)}{=} \int_{\Omega_0^\theta} T(E) g^r(\Gamma(\theta,E,L))\,\mathrm{d}(\theta,E,L) \\ &= \int_{\Omega_0^r} g^r(r,w,L)\,\mathrm{d}(r,w,L). \end{aligned}$$

□

Remark. Combining Lemma 1.3 and Lemma 3.7 yields that integrals change via

$$\text{"}\,\mathrm{d}x\,\mathrm{d}v = 4\pi^2\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}L = 4\pi^2 T(E)\,\mathrm{d}\theta\,\mathrm{d}E\,\mathrm{d}L\text{"}. \tag{3.11}$$

4 Properties of φ

In the first lemma of this chapter, we obtain an important monotonicity property of φ and also examine its behavior for $E \to 0$.

Lemma 4.1.

(i) *φ is strictly monotonically decreasing on $[-\frac{1}{2b}, 0]$ and $\varphi' < 0$ on $[-\frac{1}{2b}, 0[$.*

(ii) *$\varphi(E) \to 0$ for $E \to 0, E < 0$, as fast as $|E|^{5/2}$:*

$$\frac{\varphi(E)}{|E|^{5/2}} \to C_1(b), \quad E \to 0, E < 0,$$

for some constant $C_1(b) > 0$.

(iii) *There exist constants $C_2(b), C_3(b) > 0$ such that*

$$C_2(b)|E|^{5/2} \leq \varphi(E) \leq C_3(b)|E|^{5/2}, \quad E \in [-1/(2b), 0].$$

(iv) *$\varphi'(E) \to 0$ for $E \to 0, E < 0$, as fast as $|E|^{3/2}$:*

$$\frac{\varphi'(E)}{|E|^{3/2}} \to C_4(b), \quad E \to 0, E < 0,$$

for some constant $C_4(b) < 0$.

(v) *There exist constants $C_5(b), C_6(b) > 0$ such that*

$$C_5(b)|E|^{3/2} \leq |\varphi'(E)| \leq C_6(b)|E|^{3/2}, \quad E \in [-1/(2b), 0].$$

Proof.
Proof of (i).
We define

$$g(E) := \varphi(-E) \underset{(2.4)}{=} \alpha \int_0^E K_b(t) \frac{1}{\sqrt{E-t}} \, \mathrm{d}t, \quad E \in \,]0, 1/(2b)],$$

where

$$\alpha = \frac{1}{\sqrt{2}(2\pi)^3} \text{ and } K_b(t) := \frac{h_b(t)}{(1-bt)^5}, \quad t \in [0, 1/(2b)],$$

see (2.9) for the definition of h_b. K_b is monotonically increasing on $[0, \frac{1}{2b}]$ since $K_b'(t) = \frac{h_b'(t)(1-bt)+5bh_b(t)}{(1-bt)^6}$, where the numerator is equal to $12b^2t^2 + 48bt \geq 0$. We express g through

$$g(E) = \lim_{r \to 0, r > 0} \alpha \int_0^{E-r} K_b(t) \frac{1}{\sqrt{E-t}} \, \mathrm{d}t, \quad E \in \,]0, 1/(2b)].$$

Staying away from the singularity of the integrand now enables us to substitute $\tau = \tau(t) = E - t$ and we observe that

$$g(E) = \lim_{r\to 0, r>0} \alpha \int_r^E K_b(E-\tau)\frac{1}{\sqrt{\tau}}\,\mathrm{d}\tau, \quad E \in\,]0, 1/(2b)].$$

Next, let $E \in\,]0, \frac{1}{2b}]$, $h \geq 0$ with $E + h \in\,]0, \frac{1}{2b}]$, and $0 < r < E$. Using the monotonicity of K_b and the non-negativity of the integrand, we estimate

$$\int_r^{E+h} K_b(E+h-\tau)\frac{1}{\sqrt{\tau}}\,\mathrm{d}\tau \geq \int_r^{E+h} K_b(E-\tau)\frac{1}{\sqrt{\tau}}\,\mathrm{d}\tau \geq \int_r^{E} K_b(E-\tau)\frac{1}{\sqrt{\tau}}\,\mathrm{d}\tau.$$

Consequently, $g(E+h) \geq g(E)$ for all $E \in\,]0, \frac{1}{2b}]$ and $h \geq 0$ with $E+h \in\,]0, \frac{1}{2b}]$. We therefore conclude that g is monotonically increasing on $]0, \frac{1}{2b}]$, which means that φ is monotonically decreasing on $[-\frac{1}{2b}, 0[$. This implies that $\varphi' \leq 0$ on $]-\frac{1}{2b}, 0[$ and therefore also $\varphi'(-1/(2b)) \leq 0$ due to continuity. Computing the derivative of φ yields that it also does not vanish on $[-\frac{1}{2b}, 0[$, which implies that $\varphi' < 0$ on $[-\frac{1}{2b}, 0[$. This leads to φ being strictly monotonically decreasing on $[-\frac{1}{2b}, 0]$.

Proof of (ii).
This proof is straightforward: One shows that the fraction $\varphi(E)/|E|^{5/2}$ converges to a constant by applying l'Hôspital's rule twice while carefully handling the various occurring terms. In the following, limits $E \to 0$ are to be interpreted as $E \to 0, E < 0$, and we use that $\frac{\arcsin(\sqrt{-bE})}{\sqrt{-bE}} \to 1$ as $E \to 0$. We find that

$$\begin{aligned}
&\lim_{E\to 0}\left(\sqrt{2}(2\pi)^3 b^{3/2} 2^4 \frac{1}{\sqrt{b}}\frac{\varphi(E)}{|E|^{5/2}}(bE+1)^4\right) \\
&= \lim_{E\to 0}\left(\frac{1}{(-E)^{5/2}}\Bigg[27\sqrt{-E} - 66b(-E)^{3/2} + 320b^2(-E)^{5/2} - 240b^3(-E)^{7/2} + 64b^4(-E)^{9/2}\right. \\
&\qquad\qquad \left. + \left(48b^2E^2 - 84bE - 27\right)\frac{\arcsin\left(\sqrt{-bE}\right)}{\sqrt{b}\sqrt{1+bE}}\Bigg]\right) \\
&= 320b^2 + 48b^2 + \lim_{E\to 0}\left(\frac{1}{(-E)^{5/2}}\left[27\sqrt{-E} - 66(-E)^{3/2} + (-84bE-27)\frac{\arcsin\left(\sqrt{-bE}\right)}{\sqrt{b}\sqrt{1+bE}}\right]\right) \\
&\underset{\text{l'H.}}{=} 368b^2 + \lim_{E\to 0}\left(\frac{1}{-\frac{5}{2}(-E)^{3/2}}\Bigg[-\frac{27}{2}\frac{1}{\sqrt{-E}} + 99b\sqrt{-E} - 84\sqrt{b}\frac{\arcsin\left(\sqrt{-bE}\right)}{\sqrt{1+bE}}\right. \\
&\qquad\qquad \left. + \left(-84\sqrt{b}E - \frac{27}{\sqrt{b}}\right)\left(-\frac{b}{2}\right)\left(\frac{1}{\sqrt{-bE}(1+bE)} + \frac{\arcsin\left(\sqrt{-bE}\right)}{(1+bE)^{3/2}}\right)\Bigg]\right).
\end{aligned}$$

Now, note that the expression [...] converges to 0 as $E \to 0$ due to

$$-\frac{27}{2}\frac{1}{\sqrt{-E}} + \frac{27}{2}\sqrt{b}\frac{1}{\sqrt{-bE}(1+bE)} = \frac{27}{2}\frac{\sqrt{-E}b}{1+bE}.$$

As a consequence,

$$
\begin{aligned}
&\lim_{E\to 0}\left(\sqrt{2}(2\pi)^3 b^{3/2} 2^4 \frac{1}{\sqrt{b}}\frac{\varphi(E)}{|E|^{5/2}}(bE+1)^4\right)\\
&\underset{\text{l'H.}}{=} 368b^2 + \lim_{E\to 0}\left(\frac{1}{\frac{15}{4}\sqrt{-E}}\left[-\frac{27}{4}\frac{1}{(-E)^{3/2}} - \frac{99}{2}b\frac{1}{\sqrt{-E}}\right.\right.\\
&\qquad - 84\sqrt{b}\left(-\frac{b}{2}\right)\left(\frac{1}{\sqrt{-bE}(1+bE)} + \frac{\arcsin\left(\sqrt{-bE}\right)}{(1+bE)^{3/2}}\right)\\
&\qquad + 42b^{3/2}\left(\frac{1}{\sqrt{-bE}(1+bE)} + \frac{\arcsin\left(\sqrt{-bE}\right)}{(1+bE)^{3/2}}\right)\\
&\qquad \left.\left. + \left(42b^{3/2}E + \frac{27}{2}\sqrt{b}\right)\frac{b}{2}\left(\frac{1}{(-bE)^{3/2}(1+bE)} - \frac{3}{\sqrt{-bE}(1+bE)^2} - \frac{3\arcsin\left(\sqrt{-bE}\right)}{(1+bE)^{5/2}}\right)\right]\right)\\
&= 368b^2 + \lim_{E\to 0}\left(\frac{4}{15}\left[-\frac{27}{4}\frac{1}{E^2} + \frac{99}{2}\frac{b}{E} - \frac{42b}{E(1+bE)} + 42b^2\frac{\arcsin\left(\sqrt{-bE}\right)}{\sqrt{-bE}(1+bE)^{3/2}}\right.\right.\\
&\qquad\qquad - \frac{42b}{E(1+bE)} + 42b^2\frac{\arcsin\left(\sqrt{-bE}\right)}{\sqrt{-bE}(1+bE)^{3/2}}\\
&\qquad\qquad + \frac{21b}{E(1+bE)} + \frac{63b^2}{(1+bE)^2} + 63b^{5/2}\frac{\sqrt{-E}\arcsin(\sqrt{-bE})}{(1+bE)^{5/2}}\\
&\qquad\qquad \left.\left. + \frac{27}{4}\frac{1}{E^2(1+bE)} + \frac{81}{4}b\frac{1}{E(1+bE)^2} - \frac{81}{4}b^2\frac{\arcsin(\sqrt{-bE})}{\sqrt{bE}(1+bE)^{5/2}}\right]\right)\\
&= 368b^2 + \frac{4}{15}\left(42b^2 + 42b^2 + 63b^2 + 0 - \frac{81}{4}b^2\right)\\
&\quad + \lim_{E\to 0}\left(\frac{4}{15}\left[-\frac{27}{4}\frac{1}{E^2} + \frac{99}{2}\frac{b}{E} - \frac{42b}{E(1+bE)} - \frac{42b}{E(1+bE)} + \frac{21b}{E(1+bE)}\right.\right.\\
&\qquad\qquad \left.\left. + \frac{27}{4}\frac{1}{E^2(1+bE)} + \frac{81}{4}b\frac{1}{E(1+bE)^2}\right]\right)\\
&= \frac{2009}{5}b^2 + \lim_{E\to 0}\left(\frac{4}{15}\frac{1}{E^2(1+bE)^2}\left[-\frac{27}{4}(1+bE)^2 + \frac{99}{2}bE(1+bE)^2\right.\right.\\
&\qquad\qquad \left.\left. + (-42b-42b+21b)E(1+bE) + \frac{27}{4}(1+bE) + \frac{81}{4}bE\right]\right)\\
&= \frac{2009}{5}b^2 + \lim_{E\to 0}\left(\frac{4}{15}\frac{1}{E^2(1+bE)^2}\left[\left(-\frac{27}{4} + \frac{27}{4}\right) + \left(-\frac{27}{2} + \frac{99}{2} - 63 + \frac{27}{4} + \frac{81}{4}\right)bE\right.\right.\\
&\qquad\qquad \left.\left. + \left(-\frac{27}{4} + 99 - 63\right)b^2E^2 + \frac{99}{2}b^3E^3\right]\right)\\
&= \frac{2009}{5}b^2 + \frac{39}{5}b^2 = \frac{2048}{5}b^2.
\end{aligned}
$$

In conclusion,

$$\lim_{E\to 0, E<0} \frac{\varphi(E)}{|E|^{5/2}} = \frac{2048}{5} b^2 \frac{\sqrt{b}}{\sqrt{2}(2\pi)^3 b^{3/2} 2^4} = \frac{16}{5\sqrt{2}\pi^3} b.$$

Proof of (iii).
Using (*ii*), we infer that there exists some $0 < \delta < \frac{1}{2b}$ with

$$\frac{C_1(b)}{2} \leq \frac{\varphi(E)}{|E|^{5/2}} \leq 2C_1(b), \quad E \in]-\delta, 0[.$$

Furthermore, the mapping $[-\frac{1}{2b}, -\delta] \ni E \mapsto \frac{\varphi(E)}{|E|^{5/2}} \in]0,\infty[$ is continuous, which implies that it takes its minimum as well as its maximum for some values in $[-\frac{1}{2b}, -\delta]$ and therefore stays away from 0 and is bounded from above. As a consequence, there exist constants $C_2(b), C_3(b) > 0$ such that

$$C_2(b) \leq \frac{\varphi(E)}{|E|^{5/2}} \leq C_3(b), \quad E \in [-1/(2b), 0[,$$

which leads to both inequalities in (*iii*). Note that both inequalities are trivial for $E = 0$ as $\varphi(E) = 0$.

Proof of (iv).
This proof can be done analogously to the proof of (*ii*) except for the fact that one has to compute the derivative first. However, since it is even lengthier and does not give any new mathematical insights, we will not carry it out here.

Proof of (v).
We conclude this property analogously to the proof of (*iii*) by using (*iv*). Note that $\varphi' \in C\left([-\frac{1}{2b}, 0]\right)$ and $\varphi'(0) = 0$ due to Theorem 2.1. $\square$

In addition to its monotonicity property and its behavior at $E = 0$, φ also fulfills the following integral estimates:

Lemma 4.2.

(*i*) *For all* $x \in \mathbb{R}^3$,

$$\int_{\mathbb{R}^3} \left|\varphi'(E(x,v))\right| 1_{\Omega_0}(x,v)\,\mathrm{d}v \leq \left(\frac{12}{\pi^2} + \frac{27}{8\pi}\right)\frac{1}{b^2}.$$

(*ii*) *For all* $r \in]0,\infty[$,

$$\frac{1}{r^2}\int_0^\infty \int_{\mathbb{R}} w^2 \left|\varphi'(E^r(r,w,L))\right| 1_{\Omega_0^r}(r,w,L)\,\mathrm{d}w\,\mathrm{d}L = \frac{1}{\pi}\rho_0^r(r) \leq \frac{3}{16\pi^2}\frac{1}{b^3}.$$

(*iii*) *For all* $\gamma > 1$,

$$\int_{\Omega_0} \left|\varphi'(E(x,v))\right|^\gamma \mathrm{d}(x,v) = C(b,\gamma) < \infty.$$

Remark. The integral in (iii) is not infinite if we choose $\gamma = 1$. This can be obtained by doing a similar proof as for $\gamma > 1$ with applying the first inequality in Lemma 4.1 (v) instead of the second one.

Proof. We first note that the integrals are well-defined since on Ω_0 and Ω_0^r, $E \in [-\frac{1}{2b}, 0[$ and $E^r \in [\frac{1}{2b}, 0[$, respectively. On this set, the derivative of φ exists, see Theorem 2.1 (iii).

Proof of (i).
Let $x \in \mathbb{R}^3$ be fixed. Considering the indicator function 1_{Ω_0}, the integration area is

$$\left\{v \in \mathbb{R}^3 \;\mid\; E(x,v) < 0\right\} = \left\{v \in \mathbb{R}^3 \;\middle|\; \frac{1}{2}|v|^2 + U_0(x) < 0\right\} = \left\{v \in \mathbb{R}^3 \;\middle|\; |v| < \sqrt{-2U_0(x)}\right\}.$$

Thus, using a transformation to spherical coordinates, we find that

$$\begin{aligned}\int_{\mathbb{R}^3} \left|\varphi'(E(x,v))\right| 1_{\Omega_0}(x,v)\,\mathrm{d}v &= \int_{B_{\sqrt{-2U_0(x)}}(0)} \left|\varphi'\left(\frac{1}{2}|v|^2 + U_0(x)\right)\right| \mathrm{d}v \\ &= 4\pi \int_0^{\sqrt{-2U_0(x)}} s^2 \left|\varphi'\left(\frac{1}{2}s^2 + U_0(x)\right)\right| \mathrm{d}s.\end{aligned}$$

We substitute $E = E(s) = \frac{1}{2}s^2 + U_0(x)$ and observe that

$$\int_{\mathbb{R}^3} \left|\varphi'(E(x,v))\right| 1_{\Omega_0}(x,v)\,\mathrm{d}v = 4\pi \int_{U_0(x)}^0 \sqrt{2E - 2U_0(x)}\,\left|\varphi'(E)\right| \mathrm{d}E. \tag{4.1}$$

The radicand can be estimated by $2E - 2U_0(x) \leq -2U_0(x) \leq -2U_0(0) = -2(-1/(2b)) = 1/b$ for $E \in [U_0(x), 0]$, resulting in

$$\begin{aligned}\int_{\mathbb{R}^3} \left|\varphi'(E(x,v))\right| 1_{\Omega_0}(x,v)\,\mathrm{d}v &\leq \frac{4\pi}{b^{1/2}} \int_{U_0(x)}^0 \left|\varphi'(E)\right| \mathrm{d}E \leq -\frac{4\pi}{b^{1/2}} \int_{U_0(0)}^0 \varphi'(E)\,\mathrm{d}E \\ &= -\frac{4\pi}{b^{1/2}} \left(\varphi(0) - \varphi\left(-\frac{1}{2b}\right)\right) = \frac{4\pi}{b^{1/2}} \varphi\left(-\frac{1}{2b}\right) \\ &= \frac{4\pi}{b^{1/2}} \left(\frac{3}{\pi^3 b^{3/2}} + \frac{27}{32\pi^2 b^{3/2}}\right),\end{aligned}$$

where we used that φ' is negative on $[-\frac{1}{2b}, 0[$ as obtained in Lemma 4.1 (i).

Proof of (ii).
We first comment that if we only wanted to prove the inequality, the proof of (ii) could be much shorter than the following by using (i). However, we want to have the identity with $\rho_0^r(r)$ as we use it in Chapter 7 for the proof of the positivity of the operator $\mathcal{A}$.

Let $r \in]0, \infty[$. Since $\varphi' < 0$ on $[-\frac{1}{2b}, 0[$ (see Lemma 4.1 (i)) and since $\Omega_0^r \setminus \mathring{\Omega}_0^r$ is a null set, we obtain that

$$\begin{aligned}&\frac{1}{r^2} \int_0^\infty \int_{\mathbb{R}} w^2 \left|\varphi'(E^r(r,w,L))\right| 1_{\Omega_0^r}(r,w,L)\,\mathrm{d}w\,\mathrm{d}L \\ &= -\frac{1}{r^2} \int_0^\infty \int_{\mathbb{R}} 1_{\mathring{\Omega}_0^r}(r,w,L) w^2 \varphi'\left(E^r(r,w,L)\right) \mathrm{d}w\,\mathrm{d}L.\end{aligned}$$

As $E^r(r,w,L)<0$ on $\mathring{\Omega}_0^r$, we have $\frac{1}{2}w^2+\psi_L(r)<0$, which is equivalent to

$$\psi_L(r)<0 \ \wedge \ w\in\,]-\sqrt{-2\psi_L(r)},\sqrt{-2\psi_L(r)}[.$$

Therefore,

$$\begin{aligned}&\frac{1}{r^2}\int_0^\infty\int_{\mathbb{R}} w^2\left|\varphi'\left(E^r(r,w,L)\right)\right| 1_{\Omega_0^r}(r,w,L)\,\mathrm{d}w\,\mathrm{d}L\\ &=-\frac{1}{r^2}\int_{\{L\in]0,\infty[\,\mid\,\psi_L(r)<0\}}\int_{-\sqrt{-2\psi_L(r)}}^{\sqrt{-2\psi_L(r)}} w^2\varphi'\left(E^r(r,w,L)\right)\,\mathrm{d}w\,\mathrm{d}L.\end{aligned}$$

Since $w\mapsto\varphi\left(E^r(r,w,L)\right)$ is in $C^1([-\sqrt{-2\psi_L(r)},\sqrt{-2\psi_L(r)}])$ due to Theorem 2.1 (*iii*) with $\partial_w\left[\varphi\left(E^r(r,w,L)\right)\right]=w\varphi'\left(E^r(r,w,L)\right)$, we can perform integration by parts for the w-integral to find that

$$\begin{aligned}&\int_{-\sqrt{-2\psi_L(r)}}^{\sqrt{-2\psi_L(r)}} w^2\varphi'\left(E^r(r,w,L)\right)\,\mathrm{d}w=\int_{-\sqrt{-2\psi_L(r)}}^{\sqrt{-2\psi_L(r)}} w\partial_w\left[\varphi\left(E^r(r,w,L)\right)\right]\,\mathrm{d}w\\ &=w\varphi\left(E^r(r,w,L)\right)\Big|_{w=-\sqrt{-2\psi_L(r)}}^{\sqrt{-2\psi_L(r)}}-\int_{-\sqrt{-2\psi_L(r)}}^{\sqrt{-2\psi_L(r)}}\varphi\left(E^r(r,w,L)\right)\,\mathrm{d}w.\end{aligned}$$

The boundary term vanishes because $E^r(r,-\sqrt{-2\psi_L(r)},L)=0=E^r(r,\sqrt{-2\psi_L(r)},L)$ and $\varphi(0)=0$. Consequently,

$$\begin{aligned}&\frac{1}{r^2}\int_0^\infty\int_{\mathbb{R}} w^2\left|\varphi'\left(E^r(r,w,L)\right)\right| 1_{\Omega_0^r}(r,w,L)\,\mathrm{d}w\,\mathrm{d}L\\ &=\frac{1}{r^2}\int_{\{L\in]0,\infty[\,\mid\,\psi_L(r)<0\}}\int_{-\sqrt{-2\psi_L(r)}}^{\sqrt{-2\psi_L(r)}}\varphi\left(E^r(r,w,L)\right)\,\mathrm{d}w\,\mathrm{d}L\\ &=\frac{1}{r^2}\int_0^\infty\int_{\mathbb{R}}\varphi\left(E^r(r,w,L)\right)1_{\mathring{\Omega}_0^r}(r,w,L)\,\mathrm{d}w\,\mathrm{d}L.\end{aligned}$$

Since on $\mathring{\Omega}_0^r$, $E^r\in[-\frac{1}{2b},0[$, and since $\varphi=0$ on $[0,\infty[$, we can omit the indicator function $1_{\mathring{\Omega}_0^r}$ and therefore,

$$\begin{aligned}&\frac{1}{r^2}\int_0^\infty\int_{\mathbb{R}} w^2\left|\varphi'\left(E^r(r,w,L)\right)\right| 1_{\Omega_0^r}(r,w,L)\,\mathrm{d}w\,\mathrm{d}L=\frac{1}{r^2}\int_0^\infty\int_{\mathbb{R}}\varphi\left(E^r(r,w,L)\right)\,\mathrm{d}w\,\mathrm{d}L\\ &=\frac{1}{\pi}\int_{\mathbb{R}^3}\varphi\left(E\left((r,0,0),v\right)\right)\,\mathrm{d}v=\frac{1}{\pi}\int_{\mathbb{R}^3}f_0\left((r,0,0),v\right)\,\mathrm{d}v\\ &=\frac{1}{\pi}\rho_0((r,0,0))\end{aligned}$$

by (1.2). Note that for functions that are defined everywhere, Lemma 1.3 holds everywhere, which is easily obtained with the same proof of Lemma 1.3 as for functions that are only defined almost everywhere. The inequality now follows from

$$\frac{1}{\pi}\rho_0((r,0,0))=\frac{1}{\pi}\rho_0^r(r)\le\frac{1}{\pi}\frac{1}{4\pi}\frac{3}{\sqrt{r^2+b^2}\left(b+\sqrt{r^2+b^2}\right)^2}\le\frac{1}{4\pi^2}\frac{3}{b(2b)^2}\le\frac{3}{16\pi^2}\frac{1}{b^3}.$$

Proof of (iii).
Let $\gamma > 1$. We start as in the proof of (*i*) with $|\varphi'|^\gamma$ instead of $|\varphi'|$. Therefore, by (4.1), we find that

$$\begin{aligned}\int_{\Omega_0} \left|\varphi'(E(x,v))\right|^\gamma \,\mathrm{d}(x,v) &= 4\pi \int_{\mathbb{R}^3} \int_{U_0^r(|x|)}^0 \sqrt{2E - 2U_0^r(|x|)}\, \left|\varphi'(E)\right|^\gamma \,\mathrm{d}E\,\mathrm{d}x \\ &\leq 16\pi^2\sqrt{2}C_6(b)^\gamma \int_0^\infty r^2 \int_{U_0^r(r)}^0 \sqrt{E - U_0^r(r)}\,|E|^{\frac{3}{2}\gamma} \,\mathrm{d}E\,\mathrm{d}r. \qquad (4.2)\end{aligned}$$

For the inequality, we use polar coordinates as well as Lemma 4.1 (*v*). Next, we consider the E-integral and substitute $u = u(E) = \sqrt{E - U_0^r(r)}$. Note that technically, since the square root is not differentiable in 0, we have to interpret the E-integral as $\lim_{\alpha\to 0,\alpha>0} \int_{U_0^r(r)+\alpha}^0 \dots \mathrm{d}E$. If we do so, we obtain that

$$\int_{U_0^r(r)}^0 \sqrt{E - U_0^r(r)}\,|E|^{\frac{3}{2}\gamma} \,\mathrm{d}E = \int_0^{\sqrt{-U_0^r(r)}} u\left(-u^2 - U_0^r(r)\right)^{\frac{3}{2}\gamma} 2u\,\mathrm{d}u.$$

We substitute again, this time $v = v(u) = \arcsin\left(u/\sqrt{-U_0^r(r)}\right)$. As before, we interpret the integral as a limit. Doing this, we observe that

$$\begin{aligned}&\int_{U_0^r(r)}^0 \sqrt{E - U_0^r(r)}\,|E|^{\frac{3}{2}\gamma} \,\mathrm{d}E \\ &= 2\int_0^{\pi/2} \left(-U_0^r(r)\right)\sin(v)^2 \left(U_0^r(r)\sin(v)^2 - U_0^r(r)\right)^{\frac{3}{2}\gamma} \cos(v)\sqrt{-U_0^r(r)}\,\mathrm{d}v \\ &= 2\left(-U_0^r(r)\right)^{\frac{3}{2}+\frac{3}{2}\gamma} \int_0^{\pi/2} \sin(v)^2\cos(v)^{3\gamma+1}\,\mathrm{d}v.\end{aligned}$$

The last integral is finite since both the region of integration as well as the integrand are bounded. Hence, it is equal to some value $0 < C(\gamma) < \infty$. Inserting this into (4.2) results in

$$\int_{\Omega_0} \left|\varphi'(E(x,v))\right|^\gamma \,\mathrm{d}(x,v) \leq C(b,\gamma) \int_0^\infty r^2 \left(-U_0^r(r)\right)^{\frac{3}{2}(\gamma+1)} \,\mathrm{d}r.$$

We split the integral and consider the regions of integration $[0,1]$ and $[1,\infty]$. For $[0,1]$, we have

$$\int_0^1 r^2 \left(-U_0^r(r)\right)^{\frac{3}{2}(\gamma+1)} \,\mathrm{d}r = \int_0^1 r^2 \left|U_0^r(r)\right|^{\frac{3}{2}(\gamma+1)} \,\mathrm{d}r \leq \int_0^1 \left|U_0^r(0)\right|^{\frac{3}{2}(\gamma+1)} \,\mathrm{d}r = \left(\frac{1}{2b}\right)^{\frac{3}{2}(\gamma+1)}.$$

For the region $[1,\infty]$, we use $\sqrt{r^2+b^2} - b \leq r$ to obtain that

$$\begin{aligned}\int_1^\infty r^2 \left(-U_0^r(r)\right)^{\frac{3}{2}(\gamma+1)} \,\mathrm{d}r &= \int_1^\infty r^2 \left(\frac{\sqrt{r^2+b^2} - b}{r^2}\right)^{\frac{3}{2}(\gamma+1)} \,\mathrm{d}r \\ &\leq \int_1^\infty r^2 \left(\frac{1}{r}\right)^{\frac{3}{2}(\gamma+1)} \,\mathrm{d}r = \int_1^\infty \frac{1}{r^{\frac{3}{2}\gamma - \frac{1}{2}}} \,\mathrm{d}r < \infty\end{aligned}$$

as $\frac{3}{2}\gamma - \frac{1}{2} > \frac{3}{2} - \frac{1}{2} = 1$ since $\gamma > 1$. Hence, the assertion follows. □

We complete this chapter with two remaining properties that will be useful for the proofs in Chapter 7.

Lemma 4.3.

(i) φ' *is bounded on* $[-\frac{1}{2b}, 0]$*, i.e. there exists some constant* $C(b) > 0$ *such that*

$$\left|\varphi'(E)\right| \leq C(b), \quad E \in [-1/(2b), 0].$$

(ii) $\varphi \in C^\infty\left(\left]-\frac{1}{b}, 0\right[\right)$*.*

Proof. Assertion (i) immediately follows from $\varphi \in C^1\left(\left]-\frac{1}{b}, \infty\right[\right)$. Assertion (ii) is obvious since all functions occurring in the second representation of φ in Theorem 2.1 are differentiable arbitrarily often on $\left]-\frac{1}{b}, 0\right[$. □

5 The linearized Vlasov-Poisson system

In this chapter, we will linearize the Vlasov-Poisson system about the isochrone steady state and then formally derive an operator–the so-called Antonov operator–from the linearized system. The rigorous definition of the Antonov operator will be given in Chapter 6.

We consider the isochrone steady state (f_0, ρ_0, U_0). Assume that we have a perturbation $(\delta f, \delta\rho, \delta U)$, which means that $(f_0 + \delta f, \rho_0 + \delta\rho, U_0 + \delta U)$ must be a (time-dependent) solution to the Vlasov-Poisson system (2.1)-(2.3). For the Vlasov equation, we get

$$\begin{aligned} 0 &= \partial_t(f_0 + \delta f) + v \cdot \partial_x(f_0 + \delta f) - \partial_x(U_0 + \delta U) \cdot \partial_v(f_0 + \delta f) \\ &= \partial_t \delta f + v \cdot \partial_x \delta f - \partial_x U_0 \cdot \partial_v \delta f - \partial_x \delta U \cdot \partial_v f_0 - \partial_x \delta U \cdot \partial_v \delta f. \end{aligned}$$

We linearize the system, which means that we neglect the term that is non-linear in the perturbation such that the system becomes linear in the perturbation. We therefore omit the term $\partial_x \delta U \cdot \partial_v \delta f$ in the Vlasov equation. We obtain the linearized Vlasov-Poisson system for the perturbation:

$$\partial_t \delta f + \mathcal{D}\delta f - \partial_x \delta U \cdot \partial_v f_0 = 0, \tag{5.1}$$

$$\Delta \delta U = 4\pi \delta\rho, \qquad \lim_{|x|\to\infty} \delta U(t,x) = 0, \tag{5.2}$$

$$\delta\rho(t,x) = \int_{\mathbb{R}^3} \delta f(t,x,v)\, \mathrm{d}v, \tag{5.3}$$

where the operator $\mathcal{D}$ is defined as

$$\mathcal{D} := v \cdot \partial_x - \partial_x U_0 \cdot \partial_v.$$

Before we start with the derivation, it is convenient to define the following expressions for some function $g = g(t,x,v)$:

$$\rho_g := \int_{\mathbb{R}^3} g(\cdot,\cdot,v)\, \mathrm{d}v, \quad j_g := \int_{\mathbb{R}^3} v g(\cdot,\cdot,v)\, \mathrm{d}v, \quad U_g := -\frac{1}{|x|} * \rho_g,$$

where the convolution is meant only with respect to the x-variable. Now, let $(\delta f, \delta\rho, \delta U)$ be a solution to (5.1) - (5.3). We split the phase space density function δf into its parts that are even and odd in v, i.e. $\delta f = \delta f_+ + \delta f_-$, where

$$\delta f_\pm(t,x,v) := \frac{1}{2}\left(\delta f(t,x,v) \pm \delta f(t,x,-v)\right).$$

We define $g(t,x,v) := \delta f(t,x,-v)$ and the Vlasov equation (5.1) evaluated at $(t,x,-v)$ then yields that

$$\begin{aligned} 0 &= \partial_t g(t,x,v) - v \cdot \partial_x g(t,x,v) + \partial_x U_0(x) \cdot \partial_v g(t,x,v) - \partial_x \delta U(t,x) \cdot \partial_v \delta f_0(x,-v) \\ &= \partial_t g(t,x,v) - \mathcal{D} g(t,x,v) - \partial_x \delta U(t,x) \cdot \partial_v \delta f_0(x,-v). \end{aligned}$$

Adding and subtracting this to/from the Vlasov equation (5.1) leads to

$$\partial_t(\delta f + g) + \mathcal{D}(\delta f - g) - \partial_x \delta U \cdot (\partial_v f_0 + \partial_v f_0(\cdot, -\cdot)) = 0,$$
$$\partial_t(\delta f - g) + \mathcal{D}(\delta f + g) - \partial_x \delta U \cdot (\partial_v f_0 - \partial_v f_0(\cdot, -\cdot)) = 0.$$

Since f_0 is even in v, we have that $\partial_v f_0(\cdot, -\cdot) = -\partial_v f_0$. Furthermore, we note that $\delta f + g = 2\delta f_+$ and $\delta f - g = 2\delta f_-$. Additionally, we examine δU and observe that

$$\begin{aligned}\delta U(t,\cdot) &= -\frac{1}{|\cdot|} * \delta\rho(t,\cdot) = -\frac{1}{|\cdot|} * \left(\int_{\mathbb{R}^3} \delta f_+(t,\cdot,v)\,\mathrm{d}v + \int_{\mathbb{R}^3} \delta f_-(t,\cdot,v)\,\mathrm{d}v\right)\\ &= -\frac{1}{|\cdot|} * \left(\int_{\mathbb{R}^3} \delta f_+(t,\cdot,v)\,\mathrm{d}v\right) = U_{\delta f_+}(t,\cdot).\end{aligned}$$

All of this results in

$$\partial_t \delta f_+ + \mathcal{D}\delta f_- = 0, \tag{5.4}$$
$$\partial_t \delta f_- + \mathcal{D}\delta f_+ - \partial_x U_{\delta f_+} \cdot \partial_v f_0 = 0. \tag{5.5}$$

We differentiate (5.5) with respect to t and make use of (5.4) to obtain that

$$\begin{aligned}0 &= \partial_t^2 \delta f_- + \mathcal{D}(\partial_t \delta f_+) - \partial_x U_{\partial_t \delta f_+} \cdot \partial_v f_0\\ &= \partial_t^2 \delta f_- - \mathcal{D}^2 \delta f_- + \partial_x U_{\mathcal{D}\delta f_-} \cdot \partial_v f_0.\end{aligned} \tag{5.6}$$

To examine the last term of this equation, we first deduce that

$$\mathrm{div} j_{\delta f_-} = \sum_{i=1}^{3} \partial_{x_i} \int_{\mathbb{R}^3} v_i \delta f_-(\cdot,\cdot,v)\,\mathrm{d}v = \int_{\mathbb{R}^3} v \cdot \partial_x \delta f_-(\cdot,\cdot,v)\,\mathrm{d}v.$$

At the same time, we find that

$$\rho_{\mathcal{D}\delta f_-} = \int_{\mathbb{R}^3} \mathcal{D}\delta f_-(\cdot,\cdot,v)\,\mathrm{d}v = \int_{\mathbb{R}^3} v \cdot \partial_x \delta f_-(\cdot,\cdot,v)\,\mathrm{d}v - \partial_x U_0 \cdot \int_{\mathbb{R}^3} \partial_v \delta f_-(\cdot,\cdot,v)\,\mathrm{d}v.$$

We assume that δf_- is compactly supported with respect to the v-variable and hence, the last term vanishes. We justify this assumption with the fact that f_0 is compactly supported with respect to v and we will consider the derived operators on a function space that will consist of functions with domain of definition Ω_0, a set that is bounded in v. Consequently, we have that $\mathrm{div} j_{\delta f_-} = \rho_{\mathcal{D}\delta f_-}$, which yields that

$$U_{\mathcal{D}\delta f_-} = -\frac{1}{|\cdot|} * \rho_{\mathcal{D}\delta f_-} = -\frac{1}{|\cdot|} * \mathrm{div} j_{\delta f_-}.$$

This motivates the definition

$$(\mathcal{B}g)(x,v) := \partial_v f_0(x,v) \cdot \partial_x \int_{\mathbb{R}^3} \frac{1}{|x-y|} \mathrm{div} j_g(y)\,\mathrm{d}y$$

for a function $g = g(x,v)$. For functions that depend on t, this definition is to be understood in the sense that it holds for every fixed t. All of the previous considerations lead to the fact that (5.6) implies

$$0 = \partial_t^2 \delta f_- - \mathcal{D}^2 \delta f_- - \mathcal{B}\delta f_- = \partial_t^2 \delta f_- + \mathcal{A}\delta f_-,$$

where we define the operator $\mathcal{A}$ by

$$\mathcal{A} := -\mathcal{D}^2 - \mathcal{B}.$$

For historical reasons, we call $\mathcal{A}$ the Antonov operator and we will study its spectrum in the following chapters. To justify the definition of the operator $\mathcal{B}$ in the next chapter, we will now further examine $\mathcal{B}$ for spherically symmetric functions.

If g is a spherically symmetric function with (r, w, L)-representative g^r, then Lemma 1.3 leads to

$$j_g(x) = \int_{\mathbb{R}^3} v g(x, v) \,\mathrm{d}v = \frac{x}{|x|} \tilde{J}_g(|x|), \tag{5.7}$$

where we define

$$\tilde{J}_g(r) := \frac{\pi}{r^2} \int_0^\infty \int_{\mathbb{R}} w g^r(r, w, L) \,\mathrm{d}w \,\mathrm{d}L.$$

Here and in the following, we omit possible dependencies on t. We now note that j_g does not have to be spherically symmetric, but from (5.7), we can easily obtain that $\mathrm{div} j_g$ is spherically symmetric and consequently,

$$\begin{aligned}\partial_x \int_{\mathbb{R}^3} \frac{1}{|x-y|} \mathrm{div} j_g(y) \,\mathrm{d}y &= \partial_x \left(\frac{4\pi}{|x|} \int_0^{|x|} s^2 \left(\mathrm{div} j_g\right)^r(s) \,\mathrm{d}s + 4\pi \int_{|x|}^\infty s \left(\mathrm{div} j_g\right)^r(s) \,\mathrm{d}s \right) \\ &= -\frac{4\pi}{|x|^2} \int_0^{|x|} s^2 \left(\mathrm{div} j_g\right)^r(s) \,\mathrm{d}\frac{x}{|x|} \\ &= -\frac{x}{|x|^3} \int_{B_{|x|}(0)} \mathrm{div} j_g(y) \,\mathrm{d}y.\end{aligned}$$

We use the divergence theorem and (5.7) to find that

$$\begin{aligned}\int_{B_{|x|}(0)} \mathrm{div} j_g(y) \,\mathrm{d}y &= \int_{S_{|x|}(0)} j_g(y) \cdot \frac{y}{|y|} \,\mathrm{d}S(y) = \int_{S_{|x|}(0)} \left(\frac{y}{|y|} \tilde{J}_g(|y|) \right) \cdot \frac{y}{|y|} \,\mathrm{d}S(y) \\ &= \tilde{J}_g(|x|) \lambda(S_{|x|}(0)) = 4\pi |x|^2 \tilde{J}_g(|x|).\end{aligned}$$

This results in

$$\left(\mathcal{B} g\right)(x, v) = -4\pi \partial_v f_0(x, v) \cdot \frac{x}{|x|} \tilde{J}_g(|x|) = -4\pi \varphi'(E(x, v)) v \cdot \frac{x}{|x|} \tilde{J}_g(|x|).$$

It is easy to verify that $\mathcal{B} g$ is again spherically symmetric with (r, w, L)-representative

$$\begin{aligned}\left(\mathcal{B} g\right)^r(r, w, L) &= -4\pi \varphi'(E^r(r, w, L)) w \tilde{J}_g(r) \\ &= 4\pi \left|\varphi'(E^r(r, w, L))\right| w \frac{\pi}{r^2} \int_0^\infty \int_{\mathbb{R}} \tilde{w} g^r(r, \tilde{w}, \tilde{L}) \,\mathrm{d}\tilde{w} \,\mathrm{d}\tilde{L} \\ &= 4\pi^2 \left|\varphi'(E^r(r, w, L))\right| \frac{w}{r^2} \int_0^\infty \int_{\mathbb{R}} \tilde{w} g^r(r, \tilde{w}, \tilde{L}) \,\mathrm{d}\tilde{w} \,\mathrm{d}\tilde{L},\end{aligned}$$

which completes this chapter.

6 The operators

In order to rigorously define the operators that we derived in the previous chapter, we define two Hilbert spaces:

Definition 6.1. *For $g \in L^1_{\text{loc}}(\Omega_0;\mathbb{C})$ we define*

$$||g||_H := \left(\int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} \, |g(x,v)|^2 \, \mathrm{d}(x,v) \right)^{1/2}.$$

The complex Hilbert space H is then defined by

$$H := \left\{ g \in L^1_{\text{loc}}(\Omega_0;\mathbb{C}) \text{ spherically symmetric } \mid ||g||_H < \infty \right\},$$

with corresponding inner product

$$\langle g,h \rangle_H := \int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} g(x,v)\overline{h(x,v)} \, \mathrm{d}(x,v), \quad g,h \in H.$$

We furthermore define

$$H^{\text{odd}} := \{ g \in H \mid g \text{ is odd in } v \},$$

which also is a complex Hilbert space with the same inner product and norm.

Before proving that H and H^{odd} are Hilbert spaces, we give the following definition such that H^{odd} is well-defined:

Definition and Lemma 6.2. *Let $g \in H$ with representative g^r in (r,w,L)-coordinates. We say that g is even (odd) in v if $g(x,v) = (-)g(x,-v)$ f.a.e. $(x,v) \in \Omega_0$. Analogously, we say that g^r is even (odd) in w if $g^r(r,w,L) = (-)g^r(r,-w,L)$ f.a.e. $(r,w,L) \in \Omega_0^r$.*
g is even (odd) in v if and only if g^r is even (odd) in w.

Proof. We only prove the assertion for even functions as the proof for odd functions works analogously. Hence, let $g \in H$ be even in v. This implies that for almost every $(x,v) \in \Omega_0$,

$$g^r \left(|x|, \frac{x \cdot v}{|x|}, |x \times v|^2 \right) = g(x,v) = g(x,-v) = g^r \left(|x|, -\frac{x \cdot v}{|x|}, |x \times v|^2 \right). \tag{6.1}$$

Assume that g^r is not even in w. This leads to the existence of a measurable set $M^r \subset \Omega_0^r$ with $\lambda(M^r) > 0$ and $g^r(r,w,L) \neq g^r(r,-w,L)$ for $(r,w,L) \in M^r$. We then define $M := \left\{ (x,v) \in \Omega_0 \,\middle|\, (|x|, (x \cdot v)/|x|, |x \times v|^2) \in M^r \right\}$, which is measurable, and Lemma 1.3 yields that

$$\lambda(M) = \int_{\Omega_0} 1_M(x,v) \, \mathrm{d}(x,v) = 4\pi^2 \int_{\Omega_0^r} 1_{M^r}(r,w,L) \, \mathrm{d}(r,w,L) = 4\pi^2 \lambda(M^r) > 0. \tag{6.2}$$

We also know that for $g^r(|x|, (x\cdot v)/|x|, |x \times v|^2) \neq g^r(|x|, -(x\cdot v)/|x|, |x \times v|^2)$ for $(x,v) \in M$, which contradicts (6.1). Hence, the assumption was wrong and we conclude that g^r is even in w.

Now, let g^r be even in w. Consequently,

$$g(x,v) = g^r\left(|x|, \frac{x\cdot v}{|x|}, |x \times v|^2\right) = g^r\left(|x|, -\frac{x\cdot v}{|x|}, |x \times v|^2\right) = g(x,-v)$$

for almost every $(x,v) \in \Omega_0$, and as a consequence, g is even in v. For the second equality, note that the calculation in (6.2) also implies that null sets $M^r \subset \Omega_0^r$ correspond to null sets $M \subset \Omega_0$. $\square$

Proof that H and H^{odd} are Hilbert spaces. Since H obviously is a complex vector space and $\langle \cdot, \cdot \rangle_H$ is an inner product, it remains to show that H is complete. For this, let $(g_k) \subset H$ be a Cauchy sequence. We then define $f_k := \frac{1}{\sqrt{|\varphi'(E)|}} g_k \in L^2(\Omega_0; \mathbb{C})$ for $k \in \mathbb{N}$. Then,

$$||f_k - f_l||_2 = ||g_k - g_l||_H \to 0 \text{ as } k, l \to \infty,$$

which means that (f_k) is a Cauchy sequence in $L^2(\Omega_0; \mathbb{C})$ and therefore converges to some function $f \in L^2(\Omega_0; \mathbb{C})$. By the Riesz-Fischer theorem, the convergence also holds pointwise almost everywhere on Ω_0 after extracting a subsequence and consequently, f inherits the spherical symmetry of the functions f_k. We now define $g := \sqrt{|\varphi'(E)|} f$ which is spherically symmetric and in H. We then obtain that H is complete because of

$$||g_k - g||_H = ||f_k - f||_2 \to 0 \text{ as } k \to \infty.$$

The fact that H^{odd} is a Hilbert space follows from it being a closed subspace of H. The closedness can be obtained as follows: Let $(g_k) \subset H^{\mathrm{odd}}$ with $g_k \to g \in H$ in H. Therefore, $|\varphi'(E)|^{-1/2} g_k \to |\varphi'(E)|^{-1/2} g$ in $L^2(\Omega_0; \mathbb{C})$, which by the Riesz-Fischer theorem implies that $g_k \to g$ pointwise a.e. on Ω_0 after extracting a subsequence. Hence,

$$g(x,-v) = \lim_{k\to\infty} g_k(x,-v) = -\lim_{k\to\infty} g_k(x,v) = -g(x,v) \quad \text{f.a.e. } (x,v) \in \Omega_0,$$

note that the countable union of null sets is again a null set. Consequently, g is odd in v and therefore an element of H^{odd}, yielding the closedness of H^{odd} in H. $\square$

Remark 1. Definition 1.1 for spherical symmetry only considers functions defined on $\mathbb{R}^3 \times \mathbb{R}^3$. We therefore generally think of functions defined on smaller domains as being extended by 0 onto the whole space. Hence, spherical symmetry in Definition 6.1 is meant to be tested with the extended function. Note that the domain Ω_0 is spherically symmetric in the sense that for all $A \in \mathrm{SO}(3)$, $(x,v) \in \Omega_0 \Leftrightarrow (Ax, Av) \in \Omega_0$.

Remark 2. The convention that functions are extended by 0 onto the whole space shall apply to all functions in all coordinate systems.

Remark 3. We interpret H as a complex Hilbert space. Although this sounds counterintuitive as we are trying to model the motion of globular clusters, it is necessary for the spectral analysis that will be performed in Chapter 7.

We note that the even and the odd subspace of H are orthogonal:

Lemma 6.3. *For $g, h \in H$ with g being even in v and h being odd in v,*

$$\langle g, h \rangle_H = 0.$$

Proof. We use the transformation $(x, v) \mapsto (x, -v)$ and compute

$$\begin{aligned}\langle g, h \rangle_H &= \int_{\Omega_0} \frac{1}{|\varphi'(E(x,-v))|} g(x,-v)\overline{h(x,-v)}\,\mathrm{d}(x,v) \\ &= \int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} g(x,v)(-1)\overline{h(x,v)}\,\mathrm{d}(x,v) = -\langle g, h \rangle_H.\end{aligned}$$

Note that the null sets in the definition of v-parity do not matter due to the integration. $\square$

6.1 Definitions of operators and properties

In this section, we rigorously define the operators $\mathcal{D}$, $\mathcal{D}^2$, $\mathcal{B}$ and $\mathcal{A}$. We start with the classical definition of the operator $\mathcal{D}$ that is called the transport operator as it is associated to the transport induced by the stationary Vlasov equation.

For $g \in C^1(\Omega_0, \mathbb{K})$, we define

$$\mathcal{D}g(x,v) := v \cdot \partial_x g(x,v) - \partial_x U_0(x) \cdot \partial_v g(x,v), \quad (x,v) \in \Omega_0. \tag{6.3}$$

We generalize the transport operator $\mathcal{D}$ for spherically symmetric L^1_{loc}-functions:

Definition 6.4. *For $g \in L^1_{\mathrm{loc}}(\Omega_0; \mathbb{K})$ spherically symmetric, we say that $\mathcal{D}g$ exists weakly if there exists $\mu \in L^1_{\mathrm{loc}}(\Omega_0; \mathbb{K})$ spherically symmetric such that for all $\xi \in C^1_{c,r}(\Omega_0; \mathbb{R})$,*

$$\int_{\Omega_0} \frac{1}{|\varphi'(E)|} g \mathcal{D}\xi = -\int_{\Omega_0} \frac{1}{|\varphi'(E)|} \mu \xi.$$

In this case, we define $\mathcal{D}g := \mu$. Moreover, we define the domain of $\mathcal{D}$ and $\mathcal{D}^2$ by

$$\begin{aligned} D(\mathcal{D}) &:= \{ g \in H \mid \mathcal{D}g \text{ exists weakly and } \mathcal{D}g \in H \}, \\ D(\mathcal{D}^2) &:= \{ g \in D(\mathcal{D}) \mid \mathcal{D}g \in D(\mathcal{D}) \}. \end{aligned}$$

Remark 1. It is not relevant whether one chooses the test functions $\xi \in C^1_{c,r}(\Omega_0)$ to be real-valued or complex-valued. By splitting complex functions into their real and imaginary part, one can obtain that both definitions are equivalent.

Remark 2. It is easy to obtain that $g \in D(\mathcal{D}) \Leftrightarrow \operatorname{Re} g, \operatorname{Im} g \in D(\mathcal{D})$ and in this case, $\mathcal{D}(\operatorname{Re} g) = \operatorname{Re} \mathcal{D}g$ and $\mathcal{D}(\operatorname{Im} g) = \operatorname{Im} \mathcal{D}g$.

Remark 3. $\mathcal{D}g$ is uniquely defined a.e. on Ω_0: If $g \in L^1_{\mathrm{loc}}(\Omega_0; \mathbb{K})$ with $\mathcal{D}g$ and $\widetilde{\mathcal{D}g}$ weakly, we have

$$\int_{\Omega_0} \frac{1}{|\varphi'(E)|} (\mathcal{D}g - \widetilde{\mathcal{D}g})\xi = 0, \quad \xi \in C^1_{c,r}(\Omega_0; \mathbb{R}).$$

We want to apply the Dubois-Reymond lemma. To deal with the restriction to spherically symmetric test functions, we use Lemma 1.3 to find that

$$\int_{\mathring{\Omega}_0^r} \frac{1}{|\varphi'(E^r(r,w,L))|} \Big((\mathcal{D}g)^r\,(r,w,L) - (\widetilde{\mathcal{D}g})^r(r,w,L) \Big)\, \xi^r(r,w,L) = 0, \quad \xi \in C^1_{c,r}(\Omega_0;\mathbb{R}).$$

Any function $\varrho^r \in C_c^\infty(\mathring{\Omega}_0^r;\mathbb{R})$ can be expressed as the radial representation of some function $\varrho \in C^\infty_{c,r}(\Omega_0;\mathbb{R})$, simply take $\varrho(x,v) := \varrho^r(|x|,(x\cdot v)/|x|,|x\times v|^2)$. If we define ϱ this way, the compact support of ϱ^r in $\mathring{\Omega}_0^r$ ensures the compact support of ϱ in Ω_0 and the fact that $\operatorname{supp}\varrho^r$ stays away from $r=0$ is crucial for the differentiability of ϱ. We therefore find that $(\mathcal{D}g)^r = (\widetilde{\mathcal{D}g})^r$ a.e. on $\mathring{\Omega}_0^r$ by the Dubois-Reymond lemma. Using the integral transformation in Lemma 1.3 then yields that $\mathcal{D}g = \widetilde{\mathcal{D}g}$ a.e. on Ω_0.

Lemma 6.5. *Definition 6.4 is an extension of the definition in* (6.3) *on* $C^1_{c,r}(\Omega_0;\mathbb{K})$*. This means that for all* $g \in C^1_{c,r}(\Omega_0;\mathbb{K})$*,* $\mathcal{D}g$ *exists weakly in the sense of Definition 6.4 and the weak and classical definitions of* $\mathcal{D}g$ *coincide. Consequently,* $C^1_{c,r}(\Omega_0;\mathbb{K}) \subset D(\mathcal{D})$*.*

Proof. Let $g \in C^1_{c,r}(\Omega_0;\mathbb{K})$ with corresponding $\mathcal{D}g$ in the classical sense. The spherical symmetries of g and U_0 ensure that $\mathcal{D}g$ is spherically symmetric as well. Now, consider a test function $\xi \in C^1_{c,r}(\Omega_0;\mathbb{R})$. In the following, we will perform integration by parts. Note that the $\frac{1}{|\varphi'(E)|}$-term is not problematic as we have Lemma 4.1 (*i*), Lemma 4.3 (*ii*) and we stay away from $E=0$ due to the compact support of ξ in Ω_0. Also note that since g is complex-valued, we technically have to split the integrals into their real and imaginary parts, but as this gives the same result, we ignore it here. Lastly, we note that the compact support of ξ also ensures that the occurring boundary terms vanish. Hence,

$$\begin{aligned}
\int_{\Omega_0} \frac{1}{|\varphi'(E)|} g\mathcal{D}\xi &= -\int_{\Omega_0} \frac{1}{\varphi'\left(\frac{1}{2}|v|^2+U_0(x)\right)} g(x,v)\left(v\cdot\partial_x\xi(x,v) - \partial_x U_0(x)\cdot\partial_v\xi(x,v)\right)\mathrm{d}(x,v)\\
&= \int_{\Omega_0} \partial_x\left(\frac{1}{\varphi'\left(\frac{1}{2}|v|^2+U_0(x)\right)} g(x,v)\right)\cdot v\xi(x,v)\,\mathrm{d}(x,v)\\
&\quad - \int_{\Omega_0} \partial_v\left(\frac{1}{\varphi'\left(\frac{1}{2}|v|^2+U_0(x)\right)} g(x,v)\right)\cdot \partial_x U_0(x)\xi(x,v)\,\mathrm{d}(x,v)\\
&= \cancel{-\int_{\Omega_0} \frac{\varphi''(E(x,v))}{|\varphi'(E(x,v))|^2} g(x,v)\partial_x U_0(x)\cdot v\xi(x,v)\,\mathrm{d}(x,v)}\\
&\quad + \int_{\Omega_0} \frac{1}{\varphi'(E(x,v))}\partial_x g(x,v)\cdot v\xi(x,v)\,\mathrm{d}(x,v)\\
&\quad \cancel{+ \int_{\Omega_0} \frac{\varphi''(E(x,v))}{|\varphi'(E(x,v))|^2} g(x,v) v\cdot\partial_x U_0(x)\xi(x,v)\,\mathrm{d}(x,v)}\\
&\quad - \int_{\Omega_0} \frac{1}{\varphi'(E(x,v))}\partial_v g(x,v)\cdot\partial_x U_0(x)\xi(x,v)\,\mathrm{d}(x,v) = -\int_{\Omega_0}\frac{1}{|\varphi'(E)|}\mathcal{D}g\xi.
\end{aligned}$$

The uniqueness of the weak definition of $\mathcal{D}$ now implies that the weak and the classical definitions of $\mathcal{D}g$ coincide. □

Remark. It is clear from Lemma 6.5 that $C^\infty_{c,r}(\Omega_0;\mathbb{K}) \subset D(\mathcal{D}^2)$.

The next step is to establish a useful approximation result for functions in $D(\mathcal{D})$.

Proposition 6.6. *For every real-valued function $g \in D(\mathcal{D})$ there exists a sequence $(g_k)_{k\in\mathbb{N}} \subset C^\infty_{c,r}(\Omega_0;\mathbb{R})$ with $g_k^r \in C_c^\infty(\mathring{\Omega}_0^r;\mathbb{R})$ such that*

$$g_k \to g \text{ and } \mathcal{D}g_k \to \mathcal{D}g \text{ in } H \text{ as } k \to \infty.$$

Proof. We refer to [16, Proposition 2]. We remark that the sets S_0 and S_0^r in [16] correspond to Ω_0 and $\mathring{\Omega}_0^r$. We furthermore remark that in [16], $E_0 < 0$ holds, contrasting $E_0 = 0$ in our case. In [16], this leads to a boundedness of both S_0 and S_0^r, whereas in our case, Ω_0 is only bounded in v and $\mathring{\Omega}_0^r$ is only bounded in w. However, the proof of Proposition 2 as well as used assertions in [16] do not make use of $E_0 < 0$ or the boundedness of S_0 and S_0^r and therefore, the proof can be adopted directly from [16]. □

By splitting a complex-valued function $g \in D(\mathcal{D})$ into its real and imaginary part and approximating both parts separately via Proposition 6.6, we obtain the following corollary:

Corollary 6.7. *For every $g \in D(\mathcal{D})$ there exists a sequence $(g_k)_{k\in\mathbb{N}} \subset C^\infty_{c,r}(\Omega_0;\mathbb{C})$ with $g_k^r \in C_c^\infty(\mathring{\Omega}_0^r;\mathbb{C})$ such that*

$$g_k \to g \text{ and } \mathcal{D}g_k \to \mathcal{D}g \text{ in } H \text{ as } k \to \infty.$$

We want to prove that $\mathcal{D}$ is skew-adjoint and $\mathcal{D}^2$ is self-adjoint as densely defined operators on their domains of definition. We therefore show that their domains are dense in the corresponding Hilbert spaces:

Lemma 6.8. *$D(\mathcal{D})$ and $D(\mathcal{D}^2)$ are densely contained in H. Furthermore, $D(\mathcal{D}) \cap H^{\mathrm{odd}}$ and $D(\mathcal{D}^2) \cap H^{\mathrm{odd}}$ are densely contained in H^{odd}.*

Proof. For the density of $D(\mathcal{D})$ and $D(\mathcal{D}^2)$ in H, it suffices to show that $C^\infty_{c,r}(\Omega_0;\mathbb{C})$ is dense in H since $C^\infty_{c,r}(\Omega_0;\mathbb{C}) \subset D(\mathcal{D}^2) \subset D(\mathcal{D})$. To prove this, let $g \in H$. By the definition of $||\cdot||_H$ and Lemma 1.3, $h^r := |\varphi'(E^r)|^{1/2} g^r$ is in $L^2(\mathring{\Omega}_0^r;\mathbb{C})$. As a consequence, there exists a sequence $(h_k^r) \subset C_c^\infty(\mathring{\Omega}_0^r;\mathbb{C})$ such that $h_k^r \to h^r$ in $L^2(\mathring{\Omega}_0^r;\mathbb{C})$ as $k \to \infty$. Next, we define $h_k(x,v) := h_k^r(|x|, (x\cdot v)/|x|, |x\times v|^2)$, which are the representatives in (x,v)-coordinates. These functions are in $C^\infty_{c,r}(\Omega_0;\mathbb{C})$; we saw this argument in Remark 3 after Definition 6.4. We then define $g_k := \sqrt{|\varphi'(E)|}h_k$ which again is in $C^\infty_{c,r}(\Omega_0;\mathbb{C})$ since E is a spherically symmetric function that is differentiable arbitrarily often and additionally, φ' is differentiable arbitrarily often on $]-\frac{1}{2b}, 0[$ and stays away from 0 since E stays away from 0 on $\operatorname{supp} h_k$, see Lemma 4.1 (i) and Lemma 4.3 (ii). The sequence (g_k) approximates g, which leads to the density of $C^\infty_{c,r}(\Omega_0;\mathbb{C})$ and therefore of $D(\mathcal{D})$ and $D(\mathcal{D}^2)$ in H:

$$||g - g_k||_H = ||h - h_k||_{L^2(\Omega_0)} \underset{(3.11)}{=} C\,||h^r - h_k^r||_{L^2(\mathring{\Omega}_0^r)} \to 0 \text{ as } k \to \infty.$$

For the second part, we start by noticing that $C^\infty_{c,r}(\Omega_0;\mathbb{C}) \subset D(\mathcal{D}^2)$ implies that all $C^\infty_{c,r}(\Omega_0;\mathbb{C})$-functions that are odd in v are in $D(\mathcal{D}^2) \cap H^{\mathrm{odd}} \subset D(\mathcal{D}) \cap H^{\mathrm{odd}}$ and it therefore suffices to show that these functions are dense in H^{odd}. For that purpose, let $g \in H^{\mathrm{odd}}$. By the first part of the proof, there exists a sequence $(\tilde{g}_k) \subset C^\infty_{c,r}(\Omega_0;\mathbb{C})$ such that $\tilde{g}_k \to g$ in H as $k \to \infty$. We define $g_k(x,v) := \frac{1}{2}(\tilde{g}_k(x,v) - \tilde{g}_k(x,-v))$ for $(x,v) \in \Omega_0$, which then still is a function in

$C^\infty_{c,r}(\Omega_0;\mathbb{C})$ and in addition to that is also odd in v. We obtain that

$$\begin{aligned}||g-g_k|| &= \frac{1}{2}\,||g(x,v)-g(x,-v)-\tilde g_k(x,v)+\tilde g_k(x,-v)||_H \\ &\le \frac{1}{2}\,||g-\tilde g_k||_H + \frac{1}{2}\left(\int_{\Omega_0}\frac{1}{|\varphi'(E(x,v))|}\,|g(x,-v)-\tilde g_k(x,-v)|^2\,\mathrm{d}(x,v)\right)^{1/2} \\ &= \frac{1}{2}\,||g-\tilde g_k||_H + \frac{1}{2}\,||g-\tilde g_k||_H \to 0 \text{ as } k\to\infty,\end{aligned}$$

where we used the measure-preserving transformation $(x,-v)\mapsto(x,v)$; note that $E(x,v) = E(x,-v)$. Hence, the proof is complete. □

We now prove the skew-adjointness and the self-adjointness of the operators:

Theorem 6.9.

(i) $\mathcal{D}: D(\mathcal{D})\to H$ *is linear and skew-adjoint as a densely defined operator on* H*.*

(ii) $\mathcal{D}^2: D\left(\mathcal{D}^2\right)\to H$ *is linear and self-adjoint as a densely defined operator on* H*.*

(iii) $\mathcal{D}^2: D\left(\mathcal{D}^2\right)\cap H^{\mathrm{odd}}\to H^{\mathrm{odd}}$ *is linear and self-adjoint as a densely defined operator on* H^{odd}*.*

Proof. Obviously, all operators are linear. For the rest of the proof, we follow [16].

Proof of (i).
There are two properties that have to be shown: the skew-symmetry of $\mathcal{D}$ and $D(\mathcal{D}^*) = D(\mathcal{D})$. For the skew-symmetry, let $f,g\in D(\mathcal{D})$ and let $(f_k)\subset C^\infty_{c,r}(\Omega_0;\mathbb{C})$ be an approximating sequence of f according to Corollary 6.7. Since $\mathrm{Re}\,f_k$ and $\mathrm{Im}\,f_k$ are admissible test functions for Definition 6.4, we find that

$$\begin{aligned}\langle\mathcal{D}g,f_k\rangle_H &= \int_{\Omega_0}\frac{1}{|\varphi'(E)|}\mathcal{D}g\overline{f_k} = \int_{\Omega_0}\frac{1}{|\varphi'(E)|}\mathcal{D}g\mathrm{Re}\,f_k - i\int_{\Omega_0}\frac{1}{|\varphi'(E)|}\mathcal{D}g\mathrm{Im}\,f_k \\ &= -\int_{\Omega_0}\frac{1}{|\varphi'(E)|}g\mathcal{D}(\mathrm{Re}\,f_k) + i\int_{\Omega_0}\frac{1}{|\varphi'(E)|}g\mathcal{D}(\mathrm{Im}\,f_k) \\ &= -\int_{\Omega_0}\frac{1}{|\varphi'(E)|}g\mathcal{D}(\overline{f_k}) = -\int_{\Omega_0}\frac{1}{|\varphi'(E)|}g(\overline{\mathcal{D}f_k}) = -\langle g,\mathcal{D}f_k\rangle_H,\quad k\in\mathbb{N}.\end{aligned}$$

In addition to that, both sides of the equation converge:

$$|\langle\mathcal{D}g,f_k\rangle_H-\langle\mathcal{D}g,f\rangle_H| = |\langle\mathcal{D}g,f_k-f\rangle_H| \le ||\mathcal{D}g||_H\,||f_k-f||_H\to 0 \text{ as } k\to\infty,$$

and

$$|\langle g,\mathcal{D}f_k\rangle_H-\langle g,\mathcal{D}f\rangle_H| = |\langle g,\mathcal{D}f_k-\mathcal{D}f\rangle_H| \le ||g||_H\,||\mathcal{D}f_k-\mathcal{D}f||_H\to 0 \text{ as } k\to\infty.$$

As a result, we conclude that $\langle\mathcal{D}g,f\rangle_H = -\langle g,\mathcal{D}f\rangle_H$, and therefore, $\mathcal{D}$ is skew-symmetric. To prove that $D(\mathcal{D}^*) = D(\mathcal{D})$, we start with

$$g\in D(\mathcal{D}^*) = \{g\in H \mid \langle\mathcal{D}\cdot,g\rangle_H \text{ is continuous on } D(\mathcal{D})\}\,.$$

Since $D(\mathcal{D})$ is dense in H, we can extend the operator $\langle \mathcal{D}\cdot, g\rangle_H$ to an operator $\phi \in H'$ where H' is the dual space of H. The Riesz representation theorem now yields that there exists $h \in H$ such that $\phi = \langle \cdot, h\rangle_H$. Consequently,

$$\langle \mathcal{D}f, g\rangle_H = \phi(f) = \langle f, h\rangle_H, \quad f \in D(\mathcal{D}).$$

This implies that for all $\xi \in C^1_{c,r}(\Omega_0;\mathbb{R}) \subset D(\mathcal{D})$,

$$\int_{\Omega_0} \frac{1}{|\varphi'(E)|} g\mathcal{D}\xi = \langle g, \mathcal{D}\xi\rangle_H = \overline{\langle \mathcal{D}\xi, g\rangle}_H = \overline{\langle \xi, h\rangle}_H = \langle h, \xi\rangle_H = \int_{\Omega_0} \frac{1}{|\varphi'(E)|} h\xi,$$

which means that $\mathcal{D}g$ exists weakly and $\mathcal{D}g = -h \in H$. As a consequence, $g \in D(\mathcal{D})$.
On the other hand, if $g \in D(\mathcal{D})$, the skew-symmetry of $\mathcal{D}$ leads to

$$|\langle \mathcal{D}f, g\rangle_H| = |\langle f, \mathcal{D}g\rangle_H| \leq ||\mathcal{D}g||_H \, ||f||_H, \quad f \in D(\mathcal{D}),$$

implying that $\langle \mathcal{D}\cdot, g\rangle_H$ is bounded and therefore continuous on $D(\mathcal{D})$. Thus, $g \in D(\mathcal{D}^*)$.

Proof of (*ii*).
We apply von Neumann's theorem (see [15, Theorem X.25]). The assumptions are satisfied as an adjoint operator is always closed, hence, $\mathcal{D} = -\mathcal{D}^*$ is closed. The theorem yields that $\mathcal{D}^*\mathcal{D}$ is self-adjoint as an operator on $\{g \in D(\mathcal{D}) \mid \mathcal{D}g \in D(\mathcal{D}^*)\}$. Since $\mathcal{D}^*\mathcal{D} = -\mathcal{D}^2$ and $\{g \in D(\mathcal{D}) \mid \mathcal{D}g \in D(\mathcal{D}^*)\} = \{g \in D(\mathcal{D}) \mid \mathcal{D}g \in D(\mathcal{D})\} = D(\mathcal{D}^2)$, we find that $\mathcal{D}^2$ is self-adjoint as an operator that is densely defined on $D(\mathcal{D}^2)$.

Proof of (*iii*).
In order to prevent confusion, we denote the restricted operator by $\mathcal{D}^2_{\mathrm{odd}}$. We need to show symmetry and that $D((\mathcal{D}^2_{\mathrm{odd}})^*) = D(\mathcal{D}^2_{\mathrm{odd}})$. The symmetry immediately follows from (*ii*). For the equivalence of the domains of definition, we first prove that $D((\mathcal{D}^2_{\mathrm{odd}})^*) = D((\mathcal{D}^2)^*) \cap H^{\mathrm{odd}}$. Let g be an element of

$$\begin{aligned} D((\mathcal{D}^2_{\mathrm{odd}})^*) &= \{g \in H^{\mathrm{odd}} \mid \langle \mathcal{D}^2_{\mathrm{odd}}\cdot, g\rangle_H \text{ is continuous on } D(\mathcal{D}^2_{\mathrm{odd}})\} \\ &= \{g \in H^{\mathrm{odd}} \mid \langle \mathcal{D}^2\cdot, g\rangle_H \text{ is continuous on } D(\mathcal{D}^2) \cap H^{\mathrm{odd}}\}. \end{aligned}$$

Furthermore, let $f \in D(\mathcal{D}^2)$. We split f into its parts that are even and odd in v. i.e. $f = f_+ + f_-$, where

$$f_\pm(x,v) := \frac{1}{2}\left(f(x,v) \pm f(x,-v)\right) \qquad \text{f.a.e. } (x,v) \in \Omega_0.$$

f_+ and f_- are again in $D(\mathcal{D}^2)$ (see below for a proof) and we therefore observe that

$$\left|\langle \mathcal{D}^2 f, g\rangle_H\right| \leq \left|\langle \mathcal{D}^2 f_+, g\rangle_H\right| + \left|\langle \mathcal{D}^2 f_-, g\rangle_H\right| = \left|\langle \mathcal{D}^2 f_-, g\rangle_H\right|,$$

using Lemma 6.3 and that $\mathcal{D}^2 f_+$ is even in v, which will be proven later in Lemma 6.14 (*iii*); note that the proof of Lemma 6.14 (*iii*) solely relies on the representation of $\mathcal{D}^2$ in (θ, E, L)-coordinates. Since f_- is odd in v, we can make use of the continuity of $\langle \mathcal{D}^2\cdot, g\rangle_H$ on $D(\mathcal{D}^2) \cap H^{\mathrm{odd}}$ to find that

$$\begin{aligned} \left|\langle \mathcal{D}^2 f, g\rangle_H\right| &\leq C\,||f_-||_H \leq C\left(\frac{1}{2}\,||f||_H + \frac{1}{2}\left(\int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|}\,|f(x,-v)|^2 \,\mathrm{d}(x,v)\right)^{1/2}\right) \\ &= C\left(\frac{1}{2}\,||f||_H + \frac{1}{2}\,||f||_H\right) = C\,||f||_H, \quad f \in D(\mathcal{D}^2), \end{aligned}$$

where we used the measure-preserving transformation $(x, v) \mapsto (x, -v)$ and that $E(x, v) = E(x, -v)$. As a consequence, $\langle \mathcal{D}^2 \cdot, g \rangle_H$ is continuous on $D(\mathcal{D}^2)$. Hence, $g \in D((\mathcal{D}^2)^*) \cap H^{\text{odd}}$. On the other hand, if $g \in D((\mathcal{D}^2)^*) \cap H^{\text{odd}}$, it is obvious that $g \in D((\mathcal{D}^2_{\text{odd}})^*)$.
Overall, we have

$$D((\mathcal{D}^2_{\text{odd}})^*) = D((\mathcal{D}^2)^*) \cap H^{\text{odd}} \underset{(ii)}{=} D(\mathcal{D}^2) \cap H^{\text{odd}} = D(\mathcal{D}^2_{\text{odd}}),$$

and conclude that the restricted operator $\mathcal{D}^2 : D(\mathcal{D}^2) \cap H^{\text{odd}} \to H^{\text{odd}}$ is self-adjoint.

Proof of $\big(f \in D(\mathcal{D}^2) \Rightarrow f_+, f_- \in D(\mathcal{D}^2)\big)$.
Let $f \in D(\mathcal{D}^2)$. It suffices to show that $g := f(\cdot, -\cdot) \in D(\mathcal{D}^2)$. It is easy to obtain $g \in H$ using the transformation $(x, v) \mapsto (x, -v)$. Next, let $\xi \in C^1_{c,r}(\Omega_0; \mathbb{R})$, we again use the transformation $(x, v) \mapsto (x, -v)$ and obtain that

$$\begin{aligned}
0 &= \int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} \left[f(x,v)\mathcal{D}\xi(x,v) + \mathcal{D}f(x,v)\xi(x,v) \right] \mathrm{d}(x,v) \\
&= \int_{\Omega_0} \frac{1}{|\varphi'(E(x,-v))|} \left[f(x,-v)\mathcal{D}\xi(x,-v) + \mathcal{D}f(x,-v)\xi(x,-v) \right] \mathrm{d}(x,v) \\
&= \int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} \left[-g(x,v)\mathcal{D}\left(\xi(\cdot,-\cdot)\right)(x,v) + \mathcal{D}f(x,-v)\xi(x,-v) \right] \mathrm{d}(x,v),
\end{aligned}$$

which means that $\mathcal{D}g$ exists weakly with $\mathcal{D}g = -\mathcal{D}f(\cdot, -\cdot) \in H$ and hence, $g \in D(\mathcal{D})$. Note that the mapping $C^1_{c,r}(\Omega_0; \mathbb{R}) \ni \xi \mapsto \xi(\cdot, -\cdot) \in C^1_{c,r}(\Omega_0; \mathbb{R})$ is bijective which follows immediately from the composition of the map with itself being the identity function.
As we only used $f \in D(\mathcal{D})$ so far, we can apply the same argument to $\mathcal{D}f \in D(\mathcal{D})$ and conclude that $\mathcal{D}f(\cdot, -\cdot) \in D(\mathcal{D})$. Consequently, $\mathcal{D}g = -\mathcal{D}f(\cdot, -\cdot) \in D(\mathcal{D})$ and therefore, $g \in D(\mathcal{D}^2)$. □

For classically differentiable functions, we obtain the following representation of $\mathcal{D}$ in (r, w, L)-coordinates:

Lemma 6.10. *For* $g \in C^1_r(\Omega_0; \mathbb{K})$,

$$\begin{aligned}
(\mathcal{D}g)^r(r,w,L) &= \partial_w E^r(r,w,L)\partial_r g^r(r,w,L) - \partial_r E^r(r,w,L)\partial_w g^r(r,w,L) \\
&= w\partial_r g^r(r,w,L) - \psi_L'(r)\partial_w g^r(r,w,L), \quad (r,w,L) \in \mathring{\Omega}_0^r.
\end{aligned}$$

Proof. Let $g \in C^1_r(\Omega_0; \mathbb{K})$. Since g is defined everywhere on Ω_0, we can express the (r, w, L)-representative of g via $g^r(r, w, L) = g((r, 0, 0), (w, \sqrt{L}/r, 0))$ for $(r, w, L) \in \Omega_0^r$. g being a C^1-function then yields that $g^r \in C^1(\mathring{\Omega}_0^r; \mathbb{K})$, note that $L = 0$ has to be excluded as the square root is not differentiable at 0.
Since g is a C^1-function, $\mathcal{D}g$ is also defined everywhere on Ω_0 and we have

$$\begin{aligned}
(\mathcal{D}g)^r(r,w,L) = &\,\mathcal{D}g((r,0,0),(w,\sqrt{L}/r,0)) \\
= &\,(w,\sqrt{L}/r,0) \cdot \partial_x g((r,0,0),(w,\sqrt{L}/r,0)) \\
&- \partial_x U_0((r,0,0)) \cdot \partial_v g((r,0,0),(w,\sqrt{L}/r,0)) \quad (r,w,L) \in \Omega_0^r.
\end{aligned}$$

To compute the right hand side, let $(x,v) \in \Omega_0$ with $x \neq 0$ and $x \times v \neq 0$. We observe that

$$\begin{aligned}\partial_x g(x,v) &= \partial_x \left(g^r \left(|x|, \frac{x \cdot v}{|x|}, |x \times v|^2 \right) \right) \\ &= \partial_r g^r(...) \frac{x}{|x|} + \partial_w g^r(...) \left(\frac{v}{|x|} - \frac{x \cdot v}{|x|^3} x \right) + \partial_L g^r(...) \left(2|v|^2 x - 2(x \cdot v) v \right).\end{aligned}$$

As a result,

$$\begin{aligned}\partial_x g((r,0,0),(w,\sqrt{L}/r,0)) = &\left(\partial_r g^r(r,w,L) + 2\frac{L}{r} \partial_L g^r(r,w,L) \right) e_1 \\ &+ \left(\frac{\sqrt{L}}{r^2} \partial_w g^r(r,w,L) - 2w\sqrt{L} \partial_L g^r(r,w,L) \right) e_2,\end{aligned}$$

for $(r,w,L) \in \mathring{\Omega}_0^r$ (note that this implies that $(r,0,0) \neq 0$ and $(r,0,0) \times (w,\sqrt{L}/r,0) = (0,0,\sqrt{L}) \neq 0$). Analogously, we compute

$$\begin{aligned}\partial_v g(x,v) &= \partial_v \left(g^r \left(|x|, \frac{x \cdot v}{|x|} |x \times v|^2 \right) \right) \\ &= \partial_w g^r(...) \frac{x}{|x|} + \partial_L g^r(...) \left(2|x|^2 v - 2(x \cdot v) x \right)\end{aligned}$$

for $(x,v) \in \Omega_0$ with $x \neq 0$ and $x \times v \neq 0$ to find that for $(r,w,L) \in \mathring{\Omega}_0^r$,

$$\partial_v g((r,0,0),(w,\sqrt{L}/r,0)) = \partial_w g^r(r,w,L) e_1 + 2\sqrt{L} r \partial_L g^r(r,w,L) e_2.$$

For U_0 we obtain that $\partial_x U_0((r,0,0)) = (U_0^r)'(r) e_1$. Combining all equations, we conclude that for $(r,w,L) \in \mathring{\Omega}_0^r$, $(\mathcal{D}g)^r(r,w,L)$ is equal to

$$\begin{aligned}&\begin{pmatrix} w \\ \frac{\sqrt{L}}{r} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_r g^r(...) + 2\frac{L}{r} \partial_L g^r(...) \\ \frac{\sqrt{L}}{r^2} \partial_w g^r(...) - 2w\sqrt{L} \partial_L g^r(r,w,L) \\ 0 \end{pmatrix} - \begin{pmatrix} (U_0^r)'(r) \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_w g^r(...) \\ 2\sqrt{L} r \partial_L g^r(...) \\ 0 \end{pmatrix} \\ &= w \partial_r g^r(r,w,L) + \left(\frac{L}{r^3} - (U_0^r)'(r) \right) \partial_w g^r(r,w,L) + \left(2w\frac{L}{r} - 2w\frac{L}{r} \right) \partial_L g^r(r,w,L) \\ &= w \partial_r(r,w,L) - \psi_L'(r) \partial_w g^r(r,w,L).\end{aligned}$$

□

As obtained in Chapter 5, the Antonov operator $\mathcal{A}$ consists of two operators, and with $\mathcal{D}^2$, we have covered one of those two. We will now define the second operator that occurs in $\mathcal{A}$.

Definition 6.11. *For $g \in H$, we define $\mathcal{B}g$ using (r,w,L)-coordinates:*

$$(\mathcal{B}g)^r(r,w,L) := 4\pi^2 \left| \varphi'(E^r(r,w,L)) \right| \frac{w}{r^2} J(g)(r), \quad f.a.e\ (r,w,L) \in \Omega_0^r,$$

where

$$J(g)(r) := \int_0^\infty \int_{\mathbb{R}} w g^r(r,w,L) \, \mathrm{d}w \, \mathrm{d}L.$$

We consider $\mathcal{B}$ as an operator on H and obtain the following properties:

Lemma 6.12. *$\mathcal{B} : H \to H$ is well-defined, linear, bounded and symmetric.*

Proof. As $\mathcal{B}g$ is defined through (r, w, L)-coordinates, we immediately obtain that $\mathcal{B}g$ is spherically symmetric. In the following, let all functions be extended by 0 outside of Ω_0^r. We use Lemma 4.2 (ii) and the Cauchy-Schwarz inequality to obtain that

$$
\begin{aligned}
||\mathcal{B}g||_H^2 &\underset{(3.11)}{=} 4\pi^2 \int_{\Omega_0^r} \frac{1}{|\varphi'(E^r(r,w,L))|} \left|4\pi^2 \left|\varphi'(E^r(r,w,L))\right| \frac{w}{r^2} J(g)(r)\right|^2 \mathrm{d}(r,w,L) \\
&= (2\pi)^6 \int_0^\infty |J(g)(r)|^2 \frac{1}{r^2}\frac{1}{r^2} \int_0^\infty \int_{\mathbb{R}} w^2 \left|\varphi'(E^r(r,w,L))\right| \mathrm{d}w\,\mathrm{d}L\,\mathrm{d}r \\
&\le C \frac{1}{b^3} \int_0^\infty \frac{1}{r^2} |J(g)(r)|^2 \,\mathrm{d}r \\
&= C \frac{1}{b^3} \int_0^\infty \frac{1}{r^2} \left| \int_0^\infty \int_{\mathbb{R}} w \left|\varphi'(E^r(r,w,L))\right|^{1/2} \frac{g^r(r,w,L)}{|\varphi'(E^r(r,w,L))|^{1/2}} \mathrm{d}w\,\mathrm{d}L \right|^2 \mathrm{d}r \\
&\le C \frac{1}{b^3} \int_0^\infty \frac{1}{r^2} \left(\int_0^\infty \int_{\mathbb{R}} w^2 \left|\varphi'(E^r(...))\right| \mathrm{d}w\,\mathrm{d}L \right) \left(\int_0^\infty \int_{\mathbb{R}} \frac{|g^r(...)|^2}{|\varphi'(E^r(...))|} \mathrm{d}w\,\mathrm{d}L \right) \mathrm{d}r \\
&\le C \frac{1}{b^6} \int_{\Omega_0^r} \frac{1}{|\varphi'(E^r(r,w,L))|} |g^r(r,w,L)|^2 \,\mathrm{d}(r,w,L) \\
&\underset{(3.11)}{=} C \frac{1}{b^6} ||g||_H^2 < \infty.
\end{aligned}
$$

Consequently, $\mathcal{B}g \in H$ which means that $\mathcal{B} : H \to H$ is well-defined, and we also deduce that $\mathcal{B}$ is bounded. The linearity of $\mathcal{B}$ is trivial since $J(\cdot)$ is linear. Lastly, we have to show that $\mathcal{B}$ is a symmetric operator. For this, let $g, h \in H$. Then,

$$
\begin{aligned}
\langle \mathcal{B}g, h\rangle_H &\underset{(3.11)}{=} 4\pi^2 \int_{\Omega_0^r} \frac{1}{|\varphi'(E^r(r,w,L))|} (\mathcal{B}g)^r(r,w,L) \overline{h^r(r,w,L)} \,\mathrm{d}(r,w,L) \\
&= 4\pi^2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \frac{1}{|\varphi'(E^r(...))|} 4\pi^2 \left|\varphi'(E^r(...))\right| \frac{w}{r^2} J(g)(r) \overline{h^r(r,w,L)} \,\mathrm{d}w\,\mathrm{d}L\,\mathrm{d}r \\
&= (2\pi)^4 \int_0^\infty \frac{1}{r^2} J(g)(r) \int_0^\infty \int_{\mathbb{R}} w \overline{h^r(r,w,L)} \,\mathrm{d}w\,\mathrm{d}L\,\mathrm{d}r \\
&= (2\pi)^4 \int_0^\infty \frac{1}{r^2} J(g)(r) J(\overline{h})(r) \,\mathrm{d}r \qquad (6.4) \\
&= \overline{(2\pi)^4 \int_0^\infty \frac{1}{r^2} J(\overline{g})(r) J(h)(r) \,\mathrm{d}r} \underset{(6.4)}{=} \overline{\langle \mathcal{B}h, g\rangle_H} = \langle g, \mathcal{B}h\rangle_H.
\end{aligned}
$$

□

We additionally consider $\mathcal{B}$ as an operator on H^{odd} and obtain the same properties:

Lemma 6.13. *$\mathcal{B} : H^{\mathrm{odd}} \to H^{\mathrm{odd}}$ is well-defined, linear, bounded and symmetric.*

Proof. The linearity, boundedness and symmetry immediately follow from Lemma 6.12 as H^{odd} is a subspace of H and therefore has the same corresponding inner product and norm. For the well-definedness, note that for $g \in H^{\mathrm{odd}} \subset H$, $\mathcal{B}g \in H$ also results from Lemma 6.12. The v-parity of $\mathcal{B}g$ can be inferred from the following lemma. □

Lemma 6.14.

(i) *$\mathcal{B}g$ is odd in v for every $g \in H$.*

(ii) *If $g \in H$ is even in v, then $\mathcal{B}g = 0$ in H (i.e. almost everywhere).*

(iii) *$\mathcal{D}^2$ preserves v-parity, i.e. if $g \in D(\mathcal{D}^2)$ is even (odd) in v, then $\mathcal{D}^2 g$ is even (odd) in v.*

Proof.
Proof of (i).
Let $g \in H$. Then, for almost every $(r, w, L) \in \Omega_0^r$,

$$\begin{aligned}(\mathcal{B}g)^r(r, -w, L) &= 4\pi^2 \left|\varphi'\left(E^r(r, -w, L)\right)\right| \frac{-w}{r^2} J(g)(r) \\ &= -4\pi^2 \left|\varphi'\left(E^r(r, w, L)\right)\right| \frac{w}{r^2} J(g)(r) = -\left(\mathcal{B}g\right)^r(r, w, L),\end{aligned}$$

because E^r is even in w. Hence, $(\mathcal{B}g)^r$ is odd in w. Due to Lemma 6.2, this is equivalent to $\mathcal{B}g$ being odd on v.

Proof of (ii).
Let $g \in H$ be even in v. By Lemma 6.2, this implies that g^r is even in w. Consequently, for almost every $r \in]0, \infty[$, using a transformation $w \mapsto -w$, we find that

$$J(g)(r) = \int_0^\infty \int_{\mathbb{R}} -w g^r(r, -w, L) \,\mathrm{d}w \,\mathrm{d}L = -\int_0^\infty \int_{\mathbb{R}} w g^r(r, w, L) \,\mathrm{d}w \,\mathrm{d}L = -J(g)(r).$$

By the definition of the operator $\mathcal{B}$, $J(g) = 0$ almost everywhere on $]0, \infty[$ implies that $(\mathcal{B}g)^r = 0$ almost everywhere on Ω_0^r. In particular, $\mathcal{B}g = 0$ in H.

Proof of (iii).
This proof will become much easier once we examine the operators $\mathcal{D}$ and $\mathcal{D}^2$ in (θ, E, L)-coordinates. For this reason, we postpone it for now and will come back to it at the end of Section 6.2. □

Definition 6.15. *We define the Antonov operator $\mathcal{A}$ by*

$$\mathcal{A} : D(\mathcal{A}) := D(\mathcal{D}^2) \to H, \quad \mathcal{A} := -\mathcal{D}^2 - \mathcal{B}.$$

The Antonov operator is self-adjoint, which we need for the spectral analysis that will be carried out in the next chapter.

Lemma 6.16. *The operator $\mathcal{A} : D(\mathcal{A}) \to H$ is self-adjoint as a densely defined operator on H. If we interpret $\mathcal{A}$ as an operator on $D(\mathcal{A}) \cap H^{\text{odd}}$, then $\mathcal{A} : D(\mathcal{A}) \cap H^{\text{odd}} \to H^{\text{odd}}$ is self-adjoint as a densely defined operator on H^{odd}.*

Proof. Since $\mathcal{D}^2 : D(\mathcal{D}^2) \to H$ and $\mathcal{D}^2 : D(\mathcal{D}^2) \cap H^{\text{odd}} \to H^{\text{odd}}$ are self-adjoint by Theorem 6.9 and since $\mathcal{B} : H \to H$ and $\mathcal{B} : H^{\text{odd}} \to H^{\text{odd}}$ are linear, bounded and symmetric due to Lemma 6.12 and 6.13, the self-adjointness in both cases follows from the Kato-Rellich theorem, see [13, Theorem 13.5]. □

6.2 Domain and operator properties in (θ, E, L)-coordinates

In this section, we will use (θ, E, L)-coordinates to characterize $D(\mathcal{D})$ and $D(\mathcal{D}^2)$, as well as $\mathcal{D}$ and $\mathcal{D}^2$ in a more convenient way.

Lemma 6.17. *For all $g \in C^1_r(\Omega_0; \mathbb{K})$,*

$$(\mathcal{D}g)^\theta(\theta, E, L) = \frac{1}{T(E)} \partial_\theta g^\theta(\theta, E, L), \quad (\theta, E, L) \in]0,1[\times \mathring{\Omega}_0^{EL},$$

and for all $g \in C^2_r(\Omega_0; \mathbb{K})$,

$$\left(\mathcal{D}^2 g\right)^\theta(\theta, E, L) = \frac{1}{T(E)^2} \partial_\theta^2 g^\theta(\theta, E, L), \quad (\theta, E, L) \in]0,1[\times \mathring{\Omega}_0^{EL}.$$

Proof. First, let $g \in C^1_r(\Omega_0; \mathbb{K})$. We use (3.7) and Lemma 6.10 to obtain that

$$\begin{aligned}(\mathcal{D}g)^\theta(\theta, E, L) &\underset{(3.7)}{=} (\mathcal{D}g)^r(\Gamma(\theta, E, L)) \\ &= \Gamma_2(\theta, E, L)\partial_r g^r(\Gamma(\theta, E, L)) - \psi'_{\Gamma_3(\theta,E,L)}(\Gamma_1(\theta, E, L))\partial_w g^r(\Gamma(\theta, E, L))\end{aligned}$$

for $(\theta, E, L) \in]0,1[\times \mathring{\Omega}_0^{EL}$. Equation (3.7) holds everywhere as we deal with functions that are defined everywhere in this case. We now examine the right hand side. Due to $g \in C^1_r(\Omega_0; \mathbb{K})$, we obtained in the proof of Lemma 6.10 that $g^r \in C^1(\mathring{\Omega}_0^r; \mathbb{K})$. Along with Theorem 3.5, we find that $g^\theta = g^r \circ \Gamma :]0,1[\times \mathring{\Omega}_0^{EL} \to \mathbb{K}$ is differentiable and thus,

$$\frac{1}{T(E)} \partial_\theta g^\theta = \frac{1}{T(E)} \left[\partial_r g^r(\Gamma)\partial_\theta \Gamma_1 + \partial_w g^r(\Gamma)\partial_\theta \Gamma_2 + \partial_L g^r(\Gamma)\partial_\theta \Gamma_3\right]$$

on $]0,1[\times \mathring{\Omega}_0^{EL}$. As a result,

$$\begin{aligned}\frac{1}{T(E)} \partial_\theta g^\theta(\theta, E, L) &= \partial_r g^r(\Gamma(\theta, E, L))\Gamma_2(\theta, E, L) - \partial_w g^r(\Gamma(\theta, E, L))\psi'_{\Gamma_3(\theta,E,L)}(\Gamma_1(\theta, E, L)) \\ &= (\mathcal{D}g)^\theta(\theta, E, L), \quad (\theta, E, L) \in]0,1[\times \mathring{\Omega}_0^{EL}.\end{aligned}$$

For the second assertion, let $g \in C^2_r(\Omega_0; \mathbb{K})$. Applying the first assertion to $\mathcal{D}g \in C^1_r(\Omega_0; \mathbb{K})$ yields the desired property; the existence of $\partial_\theta^2 g^\theta$ is a consequence of the first assertion: $\mathcal{D}g \in C^1_r(\Omega_0; \mathbb{K})$ leads to the existence of $\partial_\theta (\mathcal{D}g)^\theta$, and since $(\mathcal{D}g)^\theta = \frac{1}{T}\partial_\theta g^\theta$, we conclude that $\partial_\theta^2 g^\theta$ exists. □

To establish the characterizations of $D(\mathcal{D})$ and $D(\mathcal{D}^2)$ using (θ, E, L)-coordinates, we define two auxiliary function spaces:

Definition 6.18. *We define*

$$H^1_\theta := \left\{ y \in H^1\left(]0,1[; \mathbb{C}\right) \mid y(0) = y(1) \right\},$$

and

$$H^2_\theta := \left\{ y \in H^2\left(]0,1[; \mathbb{C}\right) \mid y(0) = y(1) \wedge \dot{y}(0) = \dot{y}(1) \right\}.$$

The boundary conditions have to hold for the continuous or the differentiable representative, respectively, note that $H^1\left(]0,1[; \mathbb{C}\right) \hookrightarrow C([0,1]; \mathbb{C})$ and $H^2(]0,1[; \mathbb{C}) \hookrightarrow C^1([0,1]; \mathbb{C})$.

We establish a new characterization of the domain $D(\mathcal{D})$ and the operator $\mathcal{D}$:

Lemma 6.19. *The domain of $\mathcal{D}$ has the following representation:*

$$D(\mathcal{D}) = \Bigg\{ g \in H \;\Big|\; g^\theta(\cdot, E, L) \in H^1_\theta \text{ f.a.e. } (E,L) \in \Omega_0^{EL},$$

$$\text{and } \int_{\Omega_0^\theta} \frac{T(E)^{-1}}{|\varphi'(E)|} \left| \partial_\theta g^\theta(\theta, E, L) \right|^2 \mathrm{d}(\theta, E, L) < \infty \Bigg\}.$$

For $g \in D(\mathcal{D})$,

$$(\mathcal{D}g)^\theta(\theta, E, L) = \frac{1}{T(E)} \partial_\theta g^\theta(\theta, E, L) \quad \text{f.a.e. } (\theta, E, L) \in \Omega_0^\theta.$$

Proof. We follow the proof of Lemma 5.2 in [9] with the slight difference that some arguments will be carried out in more detail.

For the inclusion "$\subset$", let $g \in D(\mathcal{D})$. Obviously, $g \in H$. Let $\varrho \in C_c^\infty(]0,1[;\mathbb{R})$ and $\chi \in C_c^\infty(\mathring{\Omega}_0^{EL};\mathbb{R})$. We define $\xi^\theta(\theta, E, L) := \varrho(\theta)\chi(E,L)$ for $(\theta, E, L) \in]0,1[\times \mathring{\Omega}_0^{EL}$, which is a function in $C_c^\infty(]0,1[\times \mathring{\Omega}_0^{EL};\mathbb{R})$. On $\mathring{\Omega}_0^r \setminus \{(r,0,L) \in \mathring{\Omega}_0^r \mid r \le r_L\}$, we have that the (r,w,L)-representative of ξ is in C^1 as on that set, $\xi^r = \xi^\theta \circ \zeta$ and ζ is a C^1-diffeomorphism by Theorem 3.5. Furthermore, $\operatorname{supp} \xi^r \subset \Gamma\left(\operatorname{supp} \xi^\theta\right)$, which is compact in $\mathring{\Omega}_0^r \setminus \{(r,0,L) \in \mathring{\Omega}_0^r \mid r \le r_L\}$ since $\operatorname{supp} \xi^\theta$ is compact in $]0,1[\times \mathring{\Omega}_0^{EL}$. We can therefore extend ξ^r by 0 onto $\mathring{\Omega}_0^r$ and have that $\xi^r \in C_c^1(\mathring{\Omega}_0^r;\mathbb{R})$. Finally, the (x,v)-representative ξ is in $C^1_{c,r}(\Omega_0;\mathbb{R})$, see Remark 3 after Definition 6.4, and hence, it is an admissible test function for the weak definition of $\mathcal{D}$. Along with Lemma 6.17 and (3.11), this yields

$$\begin{aligned}
\int_{\Omega_0^{EL}} \frac{\chi(E,L)}{|\varphi'(E)|} \int_0^1 \dot{\varrho}(\theta) g^\theta(\theta, E, L)\,\mathrm{d}\theta\,\mathrm{d}(E,L) &= \int_{\Omega_0^\theta} \frac{1}{|\varphi'(E)|} \partial_\theta \xi^\theta(\theta,E,L) g^\theta(\theta,E,L)\,\mathrm{d}(\theta,E,L) \\
&= \int_{\Omega_0^\theta} \frac{T(E)}{|\varphi'(E)|} (\mathcal{D}\xi)^\theta(\theta,E,L) g^\theta(\theta,E,L)\,\mathrm{d}(\theta,E,L) \\
&= \frac{1}{4\pi^2} \int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} \mathcal{D}\xi(x,v) g(x,v)\,\mathrm{d}(x,v) \\
&= -\frac{1}{4\pi^2} \int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} \xi(x,v) \mathcal{D}g(x,v)\,\mathrm{d}(x,v) \\
&= -\int_{\Omega_0^\theta} \frac{T(E)}{|\varphi'(E)|} \xi^\theta(\theta,E,L) (\mathcal{D}g)^\theta(\theta,E,L)\,\mathrm{d}(\theta,E,L) \\
&= -\int_{\Omega_0^{EL}} \frac{\chi(E,L)}{|\varphi'(E)|} T(E) \int_0^1 \varrho(\theta) (\mathcal{D}g)^\theta(\theta,E,L)\,\mathrm{d}\theta\,\mathrm{d}(E,L).
\end{aligned}$$

As this holds for all $\chi \in C_c^\infty(\mathring{\Omega}_0^{EL};\mathbb{R})$, we infer by the Dubois-Reymond lemma that for all $\varrho \in C_c^\infty(]0,1[;\mathbb{R})$, there exists a null set $N_\varrho \subset \Omega_0^{EL}$ with

$$\int_0^1 \dot{\varrho}(\theta) g^\theta(\theta,E,L)\,\mathrm{d}\theta = -T(E) \int_0^1 \varrho(\theta) (\mathcal{D}g)^\theta(\theta,E,L)\,\mathrm{d}\theta, \quad (E,L) \in \Omega_0^{EL} \setminus N_\varrho. \tag{6.5}$$

To apply the definition of weak derivatives, the null set has to become independent of ϱ which can in fact be achieved. For this, we first note that since g and $\mathcal{D}g$ are in H and therefore $||g||_H, ||\mathcal{D}g||_H < \infty$, we use (3.11) and obtain that there exists a null set $N_1 \subset \Omega_0^{EL}$ such that

$$g^\theta(\cdot, E, L), (\mathcal{D}g)^\theta(\cdot, E, L) \in L^2(]0,1[;\mathbb{C}), \quad (E, L) \in \Omega_0^{EL} \setminus N_1. \tag{6.6}$$

Next, we use that $C_c^\infty(]0,1[;\mathbb{R})$ has a dense, countable subset with respect to $||\cdot||_{H^1}$; this can be obtained by taking a dense, countable subset of $H_0^1(]0,1[;\mathbb{R})$, whose elements can be approximated by sequences in $C_c^\infty(]0,1[;\mathbb{R})$. Joining all of these sequences results in a countable set that is dense in $H_0^1(]0,1[;\mathbb{R})$ and a subset of $C_c^\infty(]0,1[;\mathbb{R})$. Hence, let $\{\varrho_n\} \subset C_c^\infty(]0,1[;\mathbb{R})$ be dense in $C_c^\infty(]0,1[;\mathbb{R})$ with respect to $||\cdot||_{H^1}$. We define $N_2 := \cup_{n\in\mathbb{N}} N_{\varrho_n}$ and observe that for all $n \in N$,

$$\int_0^1 \dot{\varrho}_n(\theta) g^\theta(\theta, E, L)\,\mathrm{d}\theta = -T(E)\int_0^1 \varrho_n(\theta)\,(\mathcal{D}g)^\theta(\theta, E, L)\,\mathrm{d}\theta, \quad (E, L) \in \Omega_0^{EL} \setminus N_2. \tag{6.7}$$

If we take any function $\varrho \in C_c^\infty(]0,1[;\mathbb{R})$ and approximate it by some sequence $(\varrho_j) \subset \{\varrho_n\}$ with respect to $||\cdot||_{H^1}$, (6.6), (6.7), and the Cauchy-Schwarz inequality imply that

$$\begin{aligned}
\int_0^1 \dot{\varrho}(\theta) g^\theta(\theta, E, L)\,\mathrm{d}\theta &= \lim_{j\to\infty}\int_0^1 \dot{\varrho}_j(\theta) g^\theta(\theta, E, L)\,\mathrm{d}\theta \\
&= -\lim_{j\to\infty} T(E)\int_0^1 \varrho_j(\theta)\,(\mathcal{D}g)^\theta(\theta, E, L)\,\mathrm{d}\theta \\
&= -T(E)\int_0^1 \varrho(\theta)\,(\mathcal{D}g)^\theta(\theta, E, L)\,\mathrm{d}\theta, \quad (E, L) \in \Omega_0^{EL} \setminus (N_1 \cup N_2),
\end{aligned}$$

and the null set $N_1 \cup N_2$ does not depend on ϱ. Consequently, we have for all $(E, L) \in \Omega_0^{EL} \setminus (N_1 \cup N_2)$ that $g^\theta(\cdot, E, L)$ is weakly differentiable on $]0,1[$ with

$$\partial_\theta g^\theta(\cdot, E, L) = T(E)\,(\mathcal{D}g)^\theta(\cdot, E, L),$$

which is in $L^2(]0,1[;\mathbb{C})$ and therefore, $g^\theta \in H^1(]0,1[;\mathbb{C})$. Additionally, by (3.11),

$$\begin{aligned}
&\int_{\Omega_0^\theta} \frac{T(E)^{-1}}{|\varphi'(E)|}\left|\partial_\theta g^\theta(\theta, E, L)\right|^2 \mathrm{d}(\theta, E, L) = \int_{\Omega_0^\theta} \frac{T(E)}{|\varphi'(E)|}\left|(\mathcal{D}g)^\theta(\theta, E, L)\right|^2 \mathrm{d}(\theta, E, L) \\
&= \frac{1}{4\pi^2}\int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|}\left|\mathcal{D}g(x,v)\right|^2 \mathrm{d}(x,v) = \frac{1}{4\pi^2}\,||\mathcal{D}g||_H^2 < \infty.
\end{aligned}$$

It remains to show that $g^\theta(\cdot, E, L)$ fulfills the boundary condition such that $g^\theta(\cdot, E, L) \in H_\theta^1$ f.a.e. $(E, L) \in \Omega_0^{EL}$. We leave χ as it is, but consider $\varrho = 1_{[0,1]}$. The (r, w, L)-representative ξ^r is

$$\xi^r(r, w, L) = \xi^\theta(\zeta(r, w, L)), \quad (r, w, L) \in \mathring{\Omega}_0^r.$$

Note that we can interpret ζ as a function on $\mathring{\Omega}_0^r$. If we do so, then ζ is in $C^1(\mathring{\Omega}_0^r;\mathbb{R})$ with $\zeta(\mathring{\Omega}_0^r) \subset$ "$[0,1[\times[\psi_L(r_L), 0[\times]0,\infty[$". On this set, ξ^θ is in C^∞ (extended by 0) and therefore, $\xi^r \in C^1(\mathring{\Omega}_0^r)$. In addition to that, we have that $\operatorname{supp}\chi \subset \mathring{\Omega}_0^{EL}$. Hence, ξ^r vanishes for $(E^r(r, w, L), L) = (\zeta_2(r, w, L), \zeta_3(r, w, L)) \notin \operatorname{supp}\chi \subset \mathring{\Omega}_0^{EL}$. This implies that the support of ξ^r is in $\mathring{\Omega}_0^r$. Thus, $\xi^r \in C_c^1(\mathring{\Omega}_0^r;\mathbb{R})$, which implies that $\xi \in C_{c,r}^1(\Omega_0;\mathbb{R})$ is a admissible test function for the weak definition of $\mathcal{D}$. We therefore find that (6.5) also holds for $\varrho = 1_{[0,1]}$, and consequently,

$$\begin{aligned} 0 &= -\int_0^1 \dot{\varrho}(\theta) g^\theta(\theta, E, L)\,\mathrm{d}\theta = \int_0^1 \varrho(\theta)\partial_\theta g^\theta(\theta, E, L)\,\mathrm{d}\theta \\ &= \int_0^1 \partial_\theta g^\theta(\theta, E, L)\,\mathrm{d}\theta = g^\theta(1, E, L) - g^\theta(0, E, L) \end{aligned}$$

for the continuous representative of $g^\theta(\cdot, E, L)$ by the weak form of the fundamental theorem of calculus (see [4, Theorem 8.2]) f.a.e. $(E, L) \in \Omega_0^{EL}$. This completes the first inclusion.

For the inclusion "$\supset$", let g be an element of the set on the right hand side. Furthermore, let $\xi \in C^1_{c,r}(\Omega_0; \mathbb{R})$. The (r, w, L)-representative $\xi^r(r, w, L) = \xi((r, 0, 0), (w, \sqrt{L}/r, 0))$ then is in $C^1(\mathring{\Omega}_0^r; \mathbb{R})$. By Theorem 3.5, $\xi^\theta = \xi^r \circ \Gamma \in C^1(]0, 1[\times \mathring{\Omega}_0^{EL}; \mathbb{R})$. In fact, Γ and therefore ξ^θ can be defined for $\theta \in [0, 1]$ with $\Gamma \in C([0, 1] \times \mathring{\Omega}_0^{EL}; \mathbb{R})$ and hence, $\xi^\theta \in C([0, 1] \times \mathring{\Omega}_0^{EL}; \mathbb{R})$ with

$$\xi^\theta(0, E, L) = \xi^r(\Gamma(0, E, L)) = \xi^r(\Gamma(1, E, L)) = \xi^\theta(1, E, L), \quad (E, L) \in \mathring{\Omega}_0^{EL}.$$

This property along with $g^\theta(\cdot, E, L) \in H^1_\theta$ f.a.e. $(E, L) \in \Omega_0^{EL}$ yields that the boundary terms for the weak version of integration by parts in the following computation vanish:

$$\begin{aligned} \int_{\Omega_0} \frac{1}{|\varphi'(E)|} g\mathcal{D}\xi &\underset{(3.11)}{=} 4\pi^2 \int_{\Omega_0^\theta} \frac{T(E)}{|\varphi'(E)|} g^\theta(\theta, E, L)\,(\mathcal{D}\xi)^\theta\,(\theta, E, L)\,\mathrm{d}(\theta, E, L) \\ &= 4\pi^2 \int_{\Omega_0^{EL}} \frac{1}{|\varphi'(E)|} \int_0^1 g^\theta(\theta, E, L)\partial_\theta \xi^\theta(\theta, E, L)\,\mathrm{d}\theta\,\mathrm{d}(E, L) \\ &= -4\pi^2 \int_{\Omega_0^{EL}} \frac{1}{|\varphi'(E)|} \int_0^1 \partial_\theta g^\theta(\theta, E, L)\xi^\theta(\theta, E, L)\,\mathrm{d}\theta\,\mathrm{d}(E, L) \\ &\underset{(3.11)}{=} -\int_{\Omega_0} \frac{1}{|\varphi'(E)|} \frac{1}{T(E)} \left(\partial_\theta g^\theta\right)^{(x,v)} \xi, \end{aligned}$$

using Lemma 6.17 for ξ, where $(\cdot)^{(x,v)}$ is the representative in (x, v)-coordinates. Hence, $\mathcal{D}g$ exists weakly with $(\mathcal{D}g)^\theta = \frac{1}{T(E)}\partial_\theta g^\theta$ a.e. on Ω_0^θ, which implies that

$$\begin{aligned} ||\mathcal{D}g||_H^2 &\underset{(3.11)}{=} 4\pi^2 \int_{\Omega_0^\theta} \frac{T(E)}{|\varphi'(E)|} \left|(\mathcal{D}g)^\theta\,(\theta, E, L)\right|^2 \mathrm{d}(\theta, E, L) \\ &= 4\pi^2 \int_{\Omega_0^\theta} \frac{T(E)^{-1}}{|\varphi'(E)|} \left|\partial_\theta g^\theta(\theta, E, L)\right|^2 \mathrm{d}(\theta, E, L) < \infty. \end{aligned}$$

As a result, we obtain that $g \in D(\mathcal{D})$, which completes the proof. $\square$

Similarly, we obtain the following representation for the domain $D(\mathcal{D}^2)$ and the operator $\mathcal{D}^2$:

Lemma 6.20. *The domain of $\mathcal{D}^2$ can be expressed through*

$$D(\mathcal{D}^2) = \Bigg\{ g \in H \;\Big|\; g^\theta(\cdot, E, L) \in H^2_\theta \text{ f.a.e. } (E,L) \in \Omega_0^{EL}, \\ \text{and } \int_{\Omega_0^\theta} \frac{T(E)^{1-2j}}{|\varphi'(E)|} \left| \partial_\theta^j g^\theta(\theta, E, L) \right|^2 \mathrm{d}(\theta, E, L) < \infty \text{ for } j = 1, 2. \Bigg\}.$$

For $g \in D(\mathcal{D}^2)$,

$$\left(\mathcal{D}^2 g\right)^\theta (\theta, E, L) = \frac{1}{T(E)^2} \partial_\theta^2 g^\theta(\theta, E, L) \quad \text{f.a.e. } (\theta, E, L) \in \Omega_0^\theta.$$

Proof. This lemma is immediately obtained by applying Lemma 6.19 twice. Note that for $g \in D(\mathcal{D}^2)$, we have $g \in D(\mathcal{D})$ and $\mathcal{D}g \in D(\mathcal{D})$ and can use Lemma 6.19 for both g and $\mathcal{D}g$. For g that is an element of the set on the right hand side of this lemma, we have that g is an element of the set on the right hand side in Lemma 6.19 and therefore also in $D(\mathcal{D})$. We then use that $\mathcal{D}g$ is again an element of the set on the right hand side in Lemma 6.19. □

We will now turn to the proof of Lemma 6.14 (*iii*) that we postponed.

Proof of Lemma 6.14 (iii).
Let $g \in D(\mathcal{D}^2)$ be even in v, which implies that g^r is even in w by Lemma 6.2. We now consider the solutions of (3.2) and use that the elapse rate of the time is independent of the sign of w. Hence, if we start at a certain point (r, w) on the curve and want to get to $(r, -w)$, it takes as long to get from (r, w) to $(r_\pm(...), 0)$ as it takes to get from $(r_\pm(...), 0)$ to $(r, -w)$. We therefore infer that

$$\begin{aligned}(\Gamma_1, -\Gamma_2, \Gamma_3)(\theta, E, L) &= (R, -W, L)(\theta T(E), r_-(E, L), 0, L) \\ &= (R, W, L)((1-\theta)T(E), r_-(E, L), 0, L) = \Gamma(1-\theta, E, L)\end{aligned}$$

for almost every $(\theta, E, L) \in \Omega_0^\theta$, note that Γ is not defined everywhere on Ω_0^θ. Consequently,

$$\begin{aligned}g^\theta((1-\theta), E, L) &= g^r(\Gamma(1-\theta, E, L)) = g^r((\Gamma_1, -\Gamma_2, \Gamma_3)(\theta, E, L)) \\ &= g^r((\Gamma_1, \Gamma_2, \Gamma_3)(\theta, E, L)) = g^\theta(\theta, E, L), \quad \text{f.a.e. } (\theta, E, L) \in \Omega_0^\theta.\end{aligned}$$

By Lemma 6.20,

$$\begin{aligned}\left(\mathcal{D}^2 g\right)^\theta (1-\theta, E, L) &= \frac{1}{T(E)^2} \partial_\theta^2 g^\theta(1-\theta, E, L) = \frac{1}{T(E)^2} \partial_\theta^2 \Big(g^\theta(1-\theta, E, L) \Big) \\ &= \frac{1}{T(E)^2} \partial_\theta^2 \Big(g^\theta(\theta, E, L) \Big) = \left(\mathcal{D}^2 g\right)^\theta (\theta, E, L), \quad \text{f.a.e. } (\theta, E, L) \in \Omega_0^\theta.\end{aligned}$$

Using the definition of the function ζ,

$$\begin{aligned}\left(\mathcal{D}^2 g\right)^r (r, -w, L) &= \left(\mathcal{D}^2 g\right)^\theta (\zeta(r, -w, L)) = \left(\mathcal{D}^2 g\right)^\theta ((1-\zeta_1, \zeta_2, \zeta_3)(r, w, L)) \\ &= \left(\mathcal{D}^2 g\right)^\theta (\zeta(r, w, L)) = \left(\mathcal{D}^2 g\right)^r (r, w, L), \quad \text{f.a.e. } (r, w, L) \in \Omega_0^r,\end{aligned}$$

where we used that $\Gamma = \zeta^{-1}$ maps null sets onto null sets. As a result, we obtain that $\left(\mathcal{D}^2 g\right)^r$ is even in w which corresponds to $\mathcal{D}^2 g$ being even in v.
If we want to prove the assertion with oddness in v, it works analogously with inserting minus signs in the right places. □

7 The spectrum of the Antonov operator

This chapter will deal with the spectrum of the Antonov operator $\mathcal{A}$. More precisely, we are going to show that the spectrum of $\mathcal{A}$ is purely essential and equal to the non-negative real line. This will be done by proving the following inclusions:

$$[0, \infty[\subset \sigma_{\mathrm{ess}}(\mathcal{A}) \subset \sigma(\mathcal{A}) \subset [0, \infty[.$$

This result seems promising to be a helpful step for the stability analysis of the isochrone steady state, see Chapter 8.

The proof in Section 7.1 is based on [9, Theorem 5.7]. In this paper, the authors prove that the spectrum of the operator $(-\mathcal{D}^2)$ is purely essential and equal to a set that is characterized by the occurring values of the period function. Let us call this set M, which in our case will turn out to be $[0, \infty[$. They then continue to show that the operator $(-\mathcal{B})$ is $(-\mathcal{D}^2)$-compact, which yields that the essential spectra of $\mathcal{A}$ and $(-\mathcal{D}^2)$ are equal as a result of the Weyl theorem. Hence, they were able to obtain that the essential spectrum of $\mathcal{A}$ is equal to M. However, the latter part of this procedure strongly relies on the spatial boundedness of the domain Ω_0, a property that is invalid in our case. To deal with this issue, we proceed as follows:

1) First, we prove that $[0, \infty[$ is a subset of the essential spectrum of $\mathcal{A}$. To do this, we choose the same Weyl sequence as in [9, Theorem 5.7] and try to apply the first part of the proof, where it was shown that $M \subset \sigma_{\mathrm{ess}}(-\mathcal{D}^2)$, to the operator $\mathcal{A}$ instead of $-\mathcal{D}^2$. Since $\mathcal{A} = -\mathcal{D}^2 - \mathcal{B}$, the main part of the problem is to come up with a proof to show that the sequence we obtain by applying $\mathcal{B}$ to the Weyl sequence converges to 0 in H (after extracting a subsequence).

2) To obtain that the spectrum of $\mathcal{A}$ is a subset of $[0, \infty[$, we show that $\mathcal{A}$ is a positive operator which yields the desired property. This proof makes use of smooth approximations g_k for functions $g \in D(\mathcal{A})$ and the main part of the proof is do deduce that $\langle \mathcal{A} g_k, g_k \rangle_H \geq 0$, for which we follow [7, Theorem 1.1].

7.1 Proof of $[0, \infty[\subset \sigma_{\mathrm{ess}}(\mathcal{A})$

Let $k^* \in \mathbb{N}$ and $(E^*, L^*) \in \mathring{\Omega}_0^{EL}$ be arbitrary, but fixed, and let us define

$$\lambda^* := \left(\frac{2\pi k^*}{T(E^*)} \right)^2 .$$

Our aim is to show that λ^* is an element of the essential spectrum of $\mathcal{A}$. For the proof, we use a spectral theory theorem:

Theorem 7.1. ***Weyl's criterion***
Let $A : D(A) \to H$ be a densely defined operator on a Hilbert space H. Additionally, let A be self-adjoint and $\lambda \in \mathbb{R}$. Then, $\lambda \in \sigma_{\text{ess}}(A)$ if and only if there exists a Weyl sequence for λ and A.
A Weyl sequence for λ and A is a sequence $(u_j) \subset D(A)$ which satisfies the following properties:

(*i*) $||u_j||_H = 1$ *for* $j \in \mathbb{N}$,

(*ii*) $(A - \lambda \mathrm{id})u_j \to 0$ *in* H *as* $j \to \infty$, *and*

(*iii*) $u_j \rightharpoonup 0$ *in* H *as* $j \to \infty$.

Proof. For a proof, we refer to [13, Chapter 7]. □

Of course, we now want to find a Weyl sequence for λ^* and $\mathcal{A}$. As mentioned in the beginning of this chapter, we use the same Weyl sequence candidate as in [9, Theorem 5.7]. In order to do this, we first need to define some auxiliary functions:

For $j \in \mathbb{N}$, let $\chi_j : \mathbb{R}^2 \to \mathbb{R}$ with

(α) $\operatorname{supp} \chi_j \subset \mathring{\Omega}_0^{EL} \cap B_{\frac{1}{j}}(E^*, L^*)$ and

(β) $\int_{\Omega_0^{EL}} \chi_j(E, L)^2 \, \mathrm{d}(E, L) = \frac{1}{2\pi^2}$.

We are then able to obtain a suitable Weyl sequence for λ^* and $\mathcal{A}$:

Theorem 7.2. *For $j \in \mathbb{N}$, let $\chi_j : \mathbb{R}^2 \to \mathbb{R}$ be defined as above. We define functions $g_j : \Omega_0 \to \mathbb{R}$ through (θ, E, L)-coordinates as*

$$g_j^\theta(\theta, E, L) := \sqrt{\frac{|\varphi'(E)|}{T(E)}} \chi_j(E, L) \sin(2\pi k^* \theta), \quad (\theta, E, L) \in \Omega_0^\theta, j \in \mathbb{N}.$$

Then, after extracting a subsequence, (g_j) is a Weyl sequence for λ^* and $\mathcal{A}$.

Proof. This proof consists of four main parts: First, we need to verify that the functions g_j are elements of the domain of $\mathcal{A}$. Furthermore, we need to check that the three properties (*i*)-(*iii*) of the characterization of a Weyl sequence are fulfilled, at least for a subsequence.

Verification of $g_j \in D(\mathcal{A}) = D(\mathcal{D}^2)$
In order to show this, we make use of the characterization of the domain of $D(\mathcal{D}^2)$ in (θ, E, L)-coordinates (Lemma 6.20). Let $j \in \mathbb{N}$ in the following. First, we obtain that $g_j \in H$ by

$$\begin{aligned} ||g_j||_H^2 &\underset{(3.11)}{=} 4\pi^2 \int_{\Omega_0^{EL}} \frac{T(E)}{|\varphi'(E)|} \int_0^1 \left| g_j^\theta(\theta, E, L) \right|^2 \mathrm{d}\theta \, \mathrm{d}(E, L) \\ &= 4\pi^2 \int_{\Omega_0^{EL}} \chi_j(E, L)^2 \int_0^1 \sin(2\pi k^* \theta)^2 \, \mathrm{d}\theta \, \mathrm{d}(E, L) \\ &= 2\pi^2 \int_{\Omega_0^{EL}} \chi_j(E, L)^2 \, \mathrm{d}(E, L) \underset{(\beta)}{=} 1, \end{aligned} \tag{7.1}$$

where we used that k^* is a natural number which yields that the θ-integral is equal to $\frac{1}{2}$.
Next, we have to show that $g_j^\theta(\cdot,E,L)\in H_\theta^2$ for almost every $(E,L)\in\Omega_0^{EL}$. Fortunately, there is no need to be concerned with any exceptions on sets of measure zero since g_j^θ is a function that is defined everywhere on Ω_0^θ and the required property will hold for all $(E,L)\in\Omega_0^{EL}$. In fact, the required property is even more trivial since $g_j^\theta(\cdot,E,L):[0,1]\to\mathbb{R}$ is classically differentiable. Because θ only occurs as an argument in the sin-function, $g_j^\theta(\cdot,E,L)$ and its first two derivatives with respect to θ are bounded and therefore in $L^2(]0,1[;\mathbb{R})$ for all $(E,L)\in\Omega_0^{EL}$. Hence, $g_j^\theta(\cdot,E,L)\in H^2(]0,1[;\mathbb{R})$. The properties $g_j^\theta(0,E,L)=g_j^\theta(1,E,L)$ and $\partial_\theta g_j^\theta(0,E,L)=\partial_\theta g_j^\theta(1,E,L)$ are easily obtained by inserting the corresponding values. Therefore, $g_j^\theta(\cdot,E,L)\in H_\theta^2$ for all $(E,L)\in\Omega_0^{EL}$.
Lastly, we have to verify that the following integral expression is finite:

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega_0^\theta}\frac{T(E)^{1-2i}}{|\varphi'(E)|}\left|\partial_\theta^i g_j^\theta(\theta,E,L)\right|^2\mathrm{d}(\theta,E,L)\\
&=\int_{\Omega_0^\theta}\Bigg(\frac{1}{T(E)\,|\varphi'(E)|}\frac{|\varphi'(E)|}{T(E)}\chi_j(E,L)^2(2\pi k^*)^2\cos(2\pi k^*\theta)^2\\
&\qquad+\frac{1}{T(E)^3\,|\varphi'(E)|}\frac{|\varphi'(E)|}{T(E)}\chi_j(E,L)^2(2\pi k^*)^4\sin(2\pi k^*\theta)^2\Bigg)\mathrm{d}(\theta,E,L)\\
&\le C(k^*)\int_{\Omega_0^{EL}}\chi_j(E,L)^2\left(\frac{1}{T(E)^2}+\frac{1}{T(E)^4}\right)\int_0^1 1\,\mathrm{d}\theta\,\mathrm{d}(E,L)\le C(b,k^*)<\infty.
\end{aligned}$$

We used that on Ω_0^{EL}, $\frac{1}{T(E)^2}+\frac{1}{T(E)^4}=C(|E|^3+|E|^6)\le C(b)$ and we also used (β). In conclusion, g_j is an element of $D(\mathcal{D}^2)=D(\mathcal{A})$.

Verification of (i)
We already obtained this in (7.1).

Verification of (iii)
We skip (ii) for now and show (iii) first. Let ϕ be an element of the dual space of H. The Riesz representation theorem then yields the existence of a function $f\in H$ such that $\phi=\langle\cdot,f\rangle_H$. Consequently, using (3.11), we find that

$$\begin{aligned}
|\phi(g_j)|&=|\langle g_j,f\rangle_H|=\left|\langle g_j,1_{\operatorname{supp}g_j}f\rangle_H\right|\le||g_j||_H||1_{\operatorname{supp}g_j}f||_H\\
&\le 2\pi\left(\int_{\Omega_0^\theta}\frac{T(E)}{|\varphi'(E)|}1_{\operatorname{supp}\chi_j}(E,L)\left|f^\theta(\theta,E,L)\right|^2\mathrm{d}(\theta,E,L)\right)^{1/2}.
\end{aligned}$$

Here, we also used the Cauchy-Schwarz inequality as well as (i) and also the fact that $1_{\operatorname{supp}g_j}(\theta,E,L)\le 1_{\operatorname{supp}\chi_j}(E,L)$ for $(\theta,E,L)\in\Omega_0^\theta$. We now consider the last expression, which is equal to the H-norm of f if we neglect the indicator function. The integrand without the indicator function is therefore integrable and serves as a dominating function for Lebesgue's convergence theorem. Additionally, since $\operatorname{supp}\chi_j\subset B_{1/j}(E^*,L^*)$, the integrand converges to 0 pointwise almost everywhere on Ω_0^θ as $j\to\infty$. As a result of Lebesgue's dominated convergence theorem, we thus observe that $\phi(g_j)\to 0$ as $j\to\infty$.

Verification of (ii)
We must show that– at least after extracting a subsequence–$(\mathcal{A} - \lambda^*\mathrm{id})\, g_j \to 0$ in H as $j \to \infty$. To start, we estimate

$$||(\mathcal{A} - \lambda^*\mathrm{id})\, g_j||_H = \left|\left|-\mathcal{D}^2 g_j - \mathcal{B} g_j - \lambda^* g_j\right|\right|_H \leq \left|\left|-\mathcal{D}^2 g_j - \lambda^* g_j\right|\right|_H + ||\mathcal{B} g_j||_H . \tag{7.2}$$

We examine both summands on the right hand side separately and obtain their convergence to 0. For the first summand, this holds true without extracting a subsequence, whereas for the second one, we extract a subsequence before observing the convergence.

Let us consider the first summand in (7.2). Using a transformation to (θ, E, L)-coordinates (see (3.11)) and Lemma 6.20 yields that

$$\begin{aligned}
\left|\left|-\mathcal{D}^2 g_j - \lambda^* g_j\right|\right|_H^2 &= 4\pi^2 \int_{\Omega_0^\theta} \frac{T(E)}{|\varphi'(E)|} \left| -\frac{1}{T(E)^2} \partial_\theta^2 g_j^\theta(\theta, E, L) - \lambda^* g_j^\theta(\theta, E, L) \right|^2 \mathrm{d}(\theta, E, L) \\
&= 4\pi^2 \int_{\Omega_0^\theta} \frac{T(E)}{|\varphi'(E)|} \left| \frac{(2\pi k^*)^2}{T(E)^2} g_j^\theta(\theta, E, L) - \frac{(2\pi k^*)^2}{T(E^*)^2} g_j^\theta(\theta, E, L) \right|^2 \mathrm{d}(\theta, E, L) \\
&= C(k^*) \int_{\Omega_0^{EL}} \left| \frac{1}{T(E)^2} - \frac{1}{T(E^*)^2} \right|^2 \chi_j(E, L)^2 \int_0^1 \sin(2\pi k^* \theta)^2 \,\mathrm{d}\theta \,\mathrm{d}(E, L).
\end{aligned}$$

Considering the factor containing the difference of the period functions, note that for $(E, L) \in \operatorname{supp} \chi_j \subset B_{\frac{1}{j}}(E^*, L^*) \cap \mathring{\Omega}_0^{EL} \subset B_{\frac{1}{j}}(E^*, L^*) \cap \left( ]-\frac{1}{2b}, 0[\times]0, \infty[\right)$,

$$\left| \frac{1}{T(E)^2} - \frac{1}{T(E^*)^2} \right| = C \left| (-E)^3 - (-E^*)^3 \right| = C\, |E^* - E| \left| E^2 + EE^* + (E^*)^2 \right| \leq C \frac{1}{j} \frac{3}{4b^2}.$$

Along with the θ-integral being equal to $\frac{1}{2}$, this results in

$$\left|\left|-\mathcal{D}^2 g_j - \lambda^* g_j\right|\right|_H^2 \leq C(b, k^*) \frac{1}{j^2} \int_{\Omega_0^{EL}} \chi_j(E, L)^2 \,\mathrm{d}(E, L) \underset{(\beta)}{=} C(b, k^*) \frac{1}{j^2} \to 0 \text{ as } j \to \infty.$$

The approach for the second summand is lengthier and requires us to introduce some auxiliary functions which is done in the appendix in Definition and Theorem 9.2 . For the sake of clarity, we define them here once more and collect some known and new results.

STEP 1: Auxiliary functions and properties.
For $j \in \mathbb{N}$, we define

$$\rho_j := \rho_{\mathcal{D} g_j} = \int_{\mathbb{R}^3} \mathcal{D} g_j(\cdot, v) \,\mathrm{d}v \text{ on } \mathbb{R}^3$$

and

$$U_j := U_{\mathcal{D} g_j} = -\frac{1}{|\cdot|} * \rho_{\mathcal{D} g_j} = -\frac{1}{|\cdot|} * \rho_j \text{ on } \mathbb{R}^3.$$

These functions satisfy the following properties:

(a) ρ_j and U_j are well-defined for $j\in\mathbb{N}$.

(b) $\rho_j \in L^1\cap L^2(\mathbb{R}^3;\mathbb{R})$ with $||\rho_j||_1 \le C(b)\,||\rho_j||_2 \le \tilde{C}(b)\,||\mathcal{D}g_j||_H$ for $j\in\mathbb{N}$, where the constants $C(b),\tilde{C}(b)>0$ are independent of j.

(c) There exists $R(b)>0$ such that for all $j\in\mathbb{N}$ and almost every $w\in\mathbb{R}$ and $L\in[0,\infty[$,

$$\operatorname{supp}\rho_j \subset B_{R(b)}(0) \text{ and } \operatorname{supp} g_j^r(\cdot,w,L)\subset[0,R(b)[.$$

(d) $U_j \in H^2(\mathbb{R}^3;\mathbb{R})$ with $||U_j||_{H^2(\mathbb{R}^3)} \le C(b)\,||\rho_j||_{L^2(\mathbb{R}^3)}$ for $j\in\mathbb{N}$, where the constant $C(b)>0$ is independent of j.

(e) In the weak sense, for all $j\in\mathbb{N}$,

$$\left(U_j^r\right)'(r) = \frac{4\pi^2}{r^2}J\left(g_j\right)(r) \quad \text{f.a.e. } r\in\,]0,\infty[.$$

Proof. Assertion (a) immediately follows from Theorem 9.2 (i) with $\mathcal{D}g_j\in D(\mathcal{D})\subset H$ instead of f. Furthermore, Theorem 9.2 (iii) yields that $\rho_j\in L^2(\mathbb{R}^3)$ with $||\rho_j||_2\le\tilde{C}(b)\,||\mathcal{D}g_j||_H$ for some constant $\tilde{C}(b)>0$ that is independent of $j\in\mathbb{N}$ (since in Theorem 9.2, all constants are independent of f). Assertion (e) follows from (ix) in Theorem 9.2. For the proof of the first inequality in (b) and for (d), we prove (c) first:

Proof of (c).
Since (E^*,L^*) is an element of the interior of Ω_0^{EL}, we infer that there exists $N_0\in\mathbb{N}$ such that $\overline{B_{\frac{1}{N_0}}(E^*,L^*)}\subset\mathring{\Omega}_0^{EL}=\left\{(E,L)\in\mathbb{R}^2 \mid L\in\,]0,\infty[, E\in\,]\psi_L(r_L),0[\right\}$. Hence, there exists $\varepsilon_0>0$ with $B_{\frac{1}{N_0}}(E^*,L^*)\subset\left\{(E,L)\in\mathbb{R}^2 \mid L\in\,]0,\infty[, E\in\,]\psi_L(r_L),-\varepsilon_0[\right\}$. We now deduce that for all $j\ge N_0$,

$$\operatorname{supp}\chi_j \underset{(\alpha)}{\subset} B_{\frac{1}{j}}(E^*,L^*)\subset B_{\frac{1}{N_0}}(E^*,L^*)\subset\left\{(E,L)\in\mathbb{R}^2 \mid L\in\,]0,\infty[, E\in\,]\psi_L(r_L),-\varepsilon_0[\right\}.$$

This implies that $E<-\varepsilon_0$ for all $j\ge N_0$ and $(E,L)\in\operatorname{supp}\chi_j$, which then in turn holds for all $j\ge N_0$ and $(\theta,E,L)\in\operatorname{supp} g_j^\theta$.
For $j<N_0$ we know that $\operatorname{supp}\chi_j\subset\mathring{\Omega}_0^{EL}$, and consequently, there exists $\varepsilon_j>0$ such that $E<-\varepsilon_j$ for all $(\theta,E,L)\in\operatorname{supp} g_j^\theta$.
If we now define $\varepsilon:=\min\{\varepsilon_j \mid j=0,...,N_0-1\}>0$, we conclude that $E<-\varepsilon$ for all $(\theta,E,L)\in\operatorname{supp} g_j^\theta$ and $j\in\mathbb{N}$, and we note that ε is independent of $j\in\mathbb{N}$.

To show the support property of g_j^r, we use (3.8) and obtain that

$$g_j^r(r,w,L)=0 \text{ f.a.e. } (r,w,L)\in\Omega_0^r \text{ with } E^r(r,w,L)=\zeta_2(r,w,L)\ge-\varepsilon.$$

Since $E^r(r,w,L)=\frac{1}{2}w^2+\frac{L}{2r^2}+U_0^r(r)\ge U_0^r(r)$, we find that

$$g_j^r(r,w,L)=0 \text{ f.a.e. } (r,w,L)\in\Omega_0^r \text{ with } U_0^r(r)\ge-\varepsilon.$$

The monotonicity of $U_0^r:[0,\infty[\,\to[-\frac{1}{2b},0[$ implies that U_0^r has an inverse and $U_0^r(r)\ge-\varepsilon$ if and only if $r\ge\left(U_0^r\right)^{-1}(-\varepsilon)$. Note that we can make ε smaller such that $-\varepsilon\in[-\frac{1}{2b},0[$. Consequently,

$$g_j^r(r,w,L)=0 \text{ f.a.e. } (r,w,L)\in\Omega_0^r \text{ with } r\ge\left(U_0^r\right)^{-1}(-\varepsilon). \tag{7.3}$$

For the support property of ρ_j, our first aim is to show that (7.3) also holds for $(\mathcal{D}g_j)^r$ instead of g_j^r. For this, let $\xi \in C_{c,r}^\infty(\Omega_0;\mathbb{R})$. The weak definition of $\mathcal{D}$ and Lemma 1.3 yield that

$$0 = \int_{\Omega_0} \frac{1}{|\varphi'(E)|} \left(g_j \mathcal{D}\xi + \mathcal{D}g_j \xi\right) = 4\pi^2 \int_{\Omega_0^r} \frac{1}{|\varphi'(E^r)|} \left(g_j^r \left(\mathcal{D}\xi\right)^r + \left(\mathcal{D}g_j\right)^r \xi^r\right).$$

Next, we define $M^r := \{(r,w,L) \in \mathring{\Omega}_0^r \mid r > (U_0^r)^{-1}(-\varepsilon)\}$, which is a measurable and open set. We now consider test functions $\varrho^r \in C_c^\infty(M^r;\mathbb{R})$. We know that every such function is also in $C_c^\infty(\mathring{\Omega}_0^r;\mathbb{R})$ and therefore, there exists some $\xi \in C_{c,r}^\infty(\Omega_0;\mathbb{R})$ such that $\xi^r = \varrho^r$, see Remark 3 after Definition 6.4. If we consider only these test functions, we find that

$$0 = \int_{\Omega_0^r} \frac{1}{|\varphi'(E^r)|} \left(\mathcal{D}g_j\right)^r \varrho^r, \quad \varrho^r \in C_c^\infty(M^r;\mathbb{R}),$$

as the first term vanishes due to $\operatorname{supp}\left(\mathcal{D}\varrho\right)^r \subset \operatorname{supp}\varrho^r \subset M^r$ and (7.3). Using the Dubois-Reymond lemma, we obtain that $\left(\mathcal{D}g_j\right)^r = 0$ almost everywhere on M^r. Lemma 1.3 implies that

$$\begin{aligned} 0 &= 4\pi^2 \int_{\Omega_0^r} \left|\left(\mathcal{D}g_j\right)^r(r,w,L)\right| 1_{M^r}(r,w,L)\,\mathrm{d}(r,w,L) \\ &= \int_{\Omega_0} |\mathcal{D}g_j(x,v)|\, 1_{\left\{|x|>\left(U_0^r\right)^{-1}(-\varepsilon)\right\}}(x,v)\,\mathrm{d}(x,v), \end{aligned}$$

therefore, $\mathcal{D}g_j(x,v) = 0$ f.a.e. $(x,v) \in \Omega_0$ with $|x| > \left(U_0^r\right)^{-1}(-\varepsilon)$. Consequently, $\rho_j(x) = 0$ f.a.e. $x \in \mathbb{R}^3$ with $|x| > \left(U_0^r\right)^{-1}(-\varepsilon)$.

Proof of the first inequality in (b).
The first inequality in (b) now results from the compact support of ρ_j and Hölder's inequality with constant $C(b) = \lambda\left(B_{R(b)}(0)\right)^{1/2}$ (independent of $j \in \mathbb{N}$):

$$\int_{\mathbb{R}^3} |\rho_j(x)|\,\mathrm{d}x = \int_{\mathbb{R}^3} |\rho_j(x)|\, 1_{B_{R(b)}(0)}(x)\,\mathrm{d}x \le ||\rho_j||_2\, \lambda\left(B_{R(b)}(0)\right)^{1/2}, \quad j \in \mathbb{N}.$$

Proof of (d).
For (d), we start with two estimates for $|U_j(x)|$. For the first one, let $x \in \mathbb{R}^3$. We observe that

$$\begin{aligned} |U_j(x)| &\le \int_{|x-y|\le 1} \frac{|\rho_j(y)|}{|x-y|}\,\mathrm{d}y + \int_{|x-y|>1} \frac{|\rho_j(y)|}{|x-y|}\,\mathrm{d}y \\ &\le \left(\int_{|x-y|\le 1} \frac{1}{|x-y|^2}\,\mathrm{d}y\right)^{1/2} \left(\int_{|x-y|\le 1} |\rho_j(y)|^2\,\mathrm{d}y\right)^{1/2} + \int_{\mathbb{R}^3} |\rho_j(y)|\,\mathrm{d}y \\ &\underset{(b)}{\le} \left(4\pi \int_0^1 \frac{1}{r^2} r^2\,\mathrm{d}r\right)^{1/2} ||\rho_j||_2 + C(b)\,||\rho_j||_2 = C(b)\,||\rho_j||_2\,, \quad x \in \mathbb{R}^3. \end{aligned} \tag{7.4}$$

For the second inequality, we consider $x \in \mathbb{R}^3$ with $|x| > 2R(b)$. Using Theorem 9.2 (x), we find that

$$\begin{aligned} |U_j(x)| &\underset{(c)}{=} \left| \int_{B_{R(b)}(0)} \frac{\rho_j(y)}{|x-y|} \,\mathrm{d}y \right| = \left| \int_{B_{R(b)}(0)} \frac{\rho_j(y)}{|x-y|} \,\mathrm{d}y - \int_{B_{R(b)}(0)} \frac{\rho_j(y)}{|x|} \,\mathrm{d}y \right| \\ &\le \int_{B_{R(b)}(0)} |\rho_j(y)| \frac{||x| - |x-y||}{|x|\,|x-y|} \,\mathrm{d}y. \end{aligned}$$

For $|x| > 2R(b)$ and $|y| \le R(b)$, we know that $||x| - |x-y|| \le |x - x + y| \le R(b)$. The fact that $|x-y| \ge R(b)$ leads to $2|x-y| \ge |x-y| + R(b) \ge |x-y| + |y| \ge |x|$. Hence,

$$|U_j(x)| \le \int_{B_{R(b)}(0)} |\rho_j(y)| \frac{R(b)}{|x|\,(|x|/2)} \,\mathrm{d}y \le \frac{2R(b)}{|x|^2} \,||\rho_j||_1 , \quad |x| > 2R(b). \tag{7.5}$$

The inequalities (7.4) and (7.5) yield that

$$\begin{aligned} \int_{\mathbb{R}^3} |U_j(x)|^2 \,\mathrm{d}x &\le \int_{|x|\le 2R(b)} C(b)^2 \,||\rho_j||_2^2 \,\mathrm{d}x + \int_{|x|>2R(b)} \frac{4R(b)^2}{|x|^4} \,||\rho_j||_1^2 \,\mathrm{d}x \\ &\underset{(b)}{\le} C(b)\,||\rho_j||_2^2 + C(b)\,||\rho_j||_2^2 \int_{|x|>2R(b)} \frac{1}{|x|^4} \,\mathrm{d}x = C(b)\,||\rho_j||_2^2 . \end{aligned}$$

Thus, $U_j \in L^2(\mathbb{R}^3;\mathbb{R})$ with $||U_j||_2 \le C(b)\,||\rho_j||_2$. Basic potential theory yields that in the weak sense, $\Delta U_j = 4\pi\rho_j$. Combining the properties that $U_j \in L^2(\mathbb{R}^3;\mathbb{R})$ and $\Delta U_j = 4\pi\rho_j \in L^2(\mathbb{R}^3;\mathbb{R})$ implies that $U_j \in H^2(\mathbb{R}^3;\mathbb{R})$. As a consequence,

$$\widehat{D_\alpha U_j} = (2\pi i \cdot)^\alpha \widehat{U_j}, \quad \alpha \in \mathbb{N}_0^3, |\alpha| \le 2,$$

where $\widehat{(\ldots)}$ denotes the Fourier transform. Therefore, by Plancherel,

$$\begin{aligned} ||\partial_i \partial_k U_j||_2 &= \left|\left| \widehat{\partial_i \partial_k U_j} \right|\right|_2 = C \left|\left| \xi_i \xi_k \widehat{U_j}(\xi) \right|\right|_2 \le C \left|\left| |\xi|^2 \,\widehat{U_j}(\xi) \right|\right|_2 \\ &= C \left|\left| \widehat{\Delta U_j} \right|\right|_2 = ||\Delta U_j||_2 = C\,||\rho_j||_2 , \quad i,k \in \{1,2,3\}. \end{aligned}$$

For the first derivatives, we have

$$\begin{aligned} ||\partial_i U_j||_2^2 &= \left|\left| \widehat{\partial_i U_j} \right|\right|_2^2 = C \left|\left| \xi_i \widehat{U_j}(\xi) \right|\right|_2^2 \le C \left|\left| |\xi| \,\widehat{U_j}(\xi) \right|\right|_2^2 \\ &= C \int_{|\xi|\le 1} |\xi|^2 \left| \widehat{U_j}(\xi) \right|^2 \mathrm{d}\xi + C \int_{|\xi|>1} |\xi|^2 \left| \widehat{U_j}(\xi) \right|^2 \mathrm{d}\xi \\ &\le C \int \left| \widehat{U_j}(\xi) \right|^2 \mathrm{d}\xi + C \int |\xi|^4 \left| \widehat{U_j}(\xi) \right|^2 \mathrm{d}\xi \\ &= C \left|\left| \widehat{U_j} \right|\right|_2^2 + C \left|\left| |\xi|^2 \,\widehat{U_j}(\xi) \right|\right|_2^2 \\ &= C\,||U_j||_2^2 + C\,||\rho_j||_2^2 \le C(b)\,||\rho_j||_2^2 , \quad i \in \{1,2,3\}. \end{aligned}$$

In summary, we observe that $||U_j||_{H^2(\mathbb{R}^3)} \le C(b)\,||\rho_j||_2$ and the proof is complete. $\square$

STEP 2: Boundedness of $(\mathcal{D}g_j)$ in H.
We obtain this property using that $(\mathcal{D}g_j)^\theta(\theta,E,L) = \frac{1}{T(E)}\partial_\theta g_j^\theta(\theta,E,L)$ due to Lemma 6.19, as well as using that on Ω_0^{EL}, $\frac{1}{T(E)^2} = C|E|^3 \leq C(b)$:

$$\begin{aligned}||\mathcal{D}g_j||_H^2 &\underset{(3.11)}{=} 4\pi^2 \int_{\Omega_0^{EL}} \frac{T(E)}{|\varphi'(E)|} \int_0^1 \left|(\mathcal{D}g_j)^\theta(\theta,E,L)\right|^2 \mathrm{d}\theta\,\mathrm{d}(E,L) \\ &= 4\pi^2 \int_{\Omega_0^{EL}} \frac{1}{T(E)^2}\chi_j(E,L)^2(2\pi k^*)^2 \int_0^1 \cos(2\pi k^*\theta)^2\,\mathrm{d}\theta\,\mathrm{d}(E,L) \\ &\leq C(b,k^*) \int_{\Omega_0^{EL}} \chi_j(E,L)^2\,\mathrm{d}(E,L) \underset{(\beta)}{=} C(b,k^*), \quad j\in\mathbb{N},\end{aligned}$$

where the constant $C(b,k^*)$ does not depend on $j \in \mathbb{N}$. Therefore, we conclude that $(\mathcal{D}g_j)$ is bounded in H.

STEP 3: Existence of $(j_k) \subset (j)$ s.t. (∇U_{j_k}) is a Cauchy sequence in $L^2(\mathbb{R}^3;\mathbb{R})$.
We start by combining some estimates. For $j \in \mathbb{N}$,

$$||U_j||_{H^2} \underset{(d)}{\leq} C(b)\,||\rho_j||_2 \underset{(b)}{\leq} C(b)\,||\mathcal{D}g_j||_H \underset{\text{step 2}}{\leq} C(b,k^*),$$

which implies that $(U_j) \subset H^2(\mathbb{R}^3;\mathbb{R})$ is bounded. Now, the fact that (c) holds will play a major role, since this allows us to restrict the domain of the Sobolev space which is crucial for the following compact embedding. Using that

$$\left(U_j|_{B_{R(b)+1}(0)}\right) \subset H^2\left(B_{R(b)+1}(0);\mathbb{R}\right) \text{ is bounded,}$$

the compact embedding $H^2\left(B_{R(b)+1}(0);\mathbb{R}\right) \Subset H^1\left(B_{R(b)+1}(0);\mathbb{R}\right)$ yields the existence of a subsequence $(j_k) \subset (j)$ such that

$$\left(U_{j_k}|_{B_{R(b)+1}(0)}\right) \subset H^1\left(B_{R(b)+1}(0);\mathbb{R}\right) \text{ is convergent.}$$

In particular,

$$\left(\nabla\left(U_{j_k}|_{B_{R(b)+1}(0)}\right)\right) \subset L^2\left(B_{R(b)+1}(0);\mathbb{R}\right) \text{ is convergent.}$$

Due to Lemma 9.1,

$$\partial_i U_{j_k}(x) = \left(U_{j_k}^r\right)'(|x|)\,\frac{x_i}{|x|} \underset{(e)}{=} \frac{4\pi^2}{|x|^2}\int_0^\infty \int_{\mathbb{R}} w g_{j_k}^r(|x|,w,L)\,\mathrm{d}w\,\mathrm{d}L\,\frac{x_i}{|x|} \underset{(c)}{=} 0$$

for almost every $x \in \mathbb{R}^3 \backslash B_{R(b)}(0)$. As a consequence, we can drop the restriction to $B_{R(b)+1}(0)$ and find that

$$(\nabla U_{j_k}) \subset L^2(\mathbb{R}^3;\mathbb{R}) \text{ is convergent,}$$

in particular, (∇U_{j_k}) is a Cauchy sequence in $L^2(\mathbb{R}^3;\mathbb{R})$. Let $(j_k) \subset (j)$ be the subsequence obtained in this step in the following.

STEP 4: $(\mathcal{B}g_{j_k})$ *is a Cauchy sequence in* H.
We use (r,w,L)-coordinates and property (e) to observe that

$$\begin{aligned}
\|\mathcal{B}g_{j_k}-\mathcal{B}g_{j_l}\|_H^2 &\underset{(3.11)}{=} 4\pi^2\int_{\Omega_0^r}\frac{1}{|\varphi'(E^r(r,w,L))|}\left|(\mathcal{B}g_{j_k})^r(r,w,L)-(\mathcal{B}g_{j_l})^r(r,w,L)\right|^2 \mathrm{d}(r,w,L)\\
&\underset{(e)}{=} 4\pi^2\int_{\Omega_0^r}\left|\varphi'(E^r(r,w,L))\right| w^2\left|\left(U_{j_k}^r\right)'(r)-\left(U_{j_l}^r\right)'(r)\right|^2 \mathrm{d}(r,w,L)\\
&= 4\pi^2\int_0^\infty\left|\left(U_{j_k}^r\right)'(r)-\left(U_{j_l}^r\right)'(r)\right|^2\int_0^\infty\int_{\mathbb{R}} w^2\left|\varphi'(E^r(...))\right| 1_{\Omega_0^r}(...)\,\mathrm{d}w\,\mathrm{d}L\,\mathrm{d}r\\
&\le C(b)\int_0^\infty r^2\left|\left(U_{j_k}^r\right)'(r)-\left(U_{j_l}^r\right)'(r)\right|^2 \mathrm{d}r,
\end{aligned}$$

using Lemma 4.2 (ii). Transforming from spherical coordinates to Cartesian ones, we deduce that

$$\begin{aligned}
\|\mathcal{B}g_{j_k}-\mathcal{B}g_{j_l}\|_H^2 &\le C(b)\int_{\mathbb{R}^3}\left|\left(U_{j_k}^r\right)'(|x|)-\left(U_{j_l}^r\right)'(|x|)\right|^2 \mathrm{d}x\\
&= C(b)\int_{\mathbb{R}^3}\left|\nabla U_{j_k}(x)-\nabla U_{j_l}(x)\right|^2 \mathrm{d}x \to 0 \text{ as } k,l\to\infty,
\end{aligned}$$

because of

$$\left|\left(U_{j_k}^r\right)'(|x|)-\left(U_{j_l}^r\right)'(|x|)\right| = \left|\left(U_{j_k}^r-U_{j_l}^r\right)'(|x|)\right| = \left|\nabla\left(U_{j_k}-U_{j_l}\right)(x)\right|$$

by Lemma 9.1. As a consequence, we conclude that $(\mathcal{B}g_{j_k})\subset H$ is a Cauchy sequence.

STEP 5: $\mathcal{B}g_{j_k}\to 0$ *in* H *as* $k\to\infty$.
Combining the property that $(\mathcal{B}g_{j_k})$ is a Cauchy sequence in H and the fact that H is a Hilbert space, we infer that there exists some function $g\in H$ such that $\mathcal{B}g_{j_k}\to g$ in H as $k\to\infty$. Extracting another subsequence, we additionally observe a pointwise convergence almost everywhere: Since the Hilbert space H was defined as a weighted L^2-space, the convergence of $\mathcal{B}g_{j_k}\to g$ in H is equivalent to the the convergence of $|\varphi'(E^r)|^{-1/2}(\mathcal{B}g_{j_k})^r\to|\varphi'(E^r)|^{-1/2}g^r$ in $L^2(\Omega_0^r;\mathbb{C})$ as $k\to\infty$, using (3.11). The Riesz-Fischer theorem then yields the existence of a subsequence $(j_{k_l})\subset(j_k)$ such that $|\varphi'(E^r)|^{-1/2}\left(\mathcal{B}g_{j_{k_l}}\right)^r\to|\varphi'(E^r)|^{-1/2}g^r$ pointwise almost everywhere on Ω_0^r as $l\to\infty$. Hence, there exists a null set $N_1\subset\Omega_0^r$ such that

$$\left(\mathcal{B}g_{j_{k_l}}\right)^r(r,w,L)\to g^r(r,w,L),\quad (r,w,L)\in\Omega_0^r\setminus N_1. \tag{7.6}$$

Recalling the definition of $\mathcal{B}$, we find there exists a null set $N_2\subset\Omega_0^r$ such that for all $l\in\mathbb{N}$,

$$\left(\mathcal{B}g_{j_{k_l}}\right)^r(r,w,L)=4\pi^2\left|\varphi'(E^r(r,w,L))\right|\frac{w}{r^2}J\left(g_{j_{k_l}}\right)(r),\quad (r,w,L)\in\Omega_0^r\setminus N_2. \tag{7.7}$$

Note that N_2 can be chosen such that it is independent of l by taking the union of all corresponding null sets. We additionally require that $\{(r,w,L)\in\Omega_0^r \mid w=0\}\subset N_2$. This along with $\varphi'(E^r)\neq 0$ on Ω_0^r, (7.6), and (7.7) implies that

$$J\left(g_{j_{k_l}}\right)(r) \text{ is covergent },\quad (r,w,L)\in\Omega_0^r\setminus(N_1\cup N_2). \tag{7.8}$$

We now claim that $\int_0^\infty |J(g_j)(r)|\,\mathrm{d}r \to 0$ as $j \to \infty$. For $(r,w,L) \in \Omega_0^r$, we know that $0 > E^r(r,w,L) = \frac{1}{2}w^2 + \frac{L}{2r^2} + U_0^r(r)$ and therefore, $w^2 < -\frac{L}{2r^2} - 2U_0^r(r) \leq 0 - 2U_0^r(0) = \frac{1}{b}$. As a consequence, we have

$$\begin{aligned}
\int_0^\infty |J(g_j)(r)|\,\mathrm{d}r &\leq \int_0^\infty \int_0^\infty \int_{\mathbb{R}} |w|\left|g_j^r(r,w,L)\right| \mathrm{d}w\,\mathrm{d}L\,\mathrm{d}r \\
&\leq C(b) \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \left|g_j^r(r,w,L)\right| \mathrm{d}w\,\mathrm{d}L\,\mathrm{d}r \\
&\underset{(3.11)}{=} C(b) \int_{\Omega_0^{EL}} T(E)\sqrt{\frac{|\varphi'(E)|}{T(E)}}\,|\chi_j(E,L)| \int_0^1 |\sin(2\pi k^*\theta)|\,\mathrm{d}\theta\,\mathrm{d}(E,L) \\
&\leq C(b) \int_{\Omega_0^{EL}} \sqrt{T(E)}\sqrt{|\varphi'(E)|}\,|\chi_j(E,L)|\,\mathrm{d}(E,L).
\end{aligned}$$

Next, we estimate $\sqrt{T(E)}$ in the following way: For $(E,L) \in \operatorname{supp}\chi_j$, E is bounded away from 0 by a constant $C(b)$ independent of j; we obtained this in the proof of (c). We therefore find that $\sqrt{T(E)} = C\frac{1}{|E|^{3/4}} \leq C(b)$ on $\operatorname{supp}\chi_j$. $\sqrt{|\varphi'(E)|}$ can be estimated by another constant $C(b)$ due to Lemma 4.3 (i). This leads to

$$\begin{aligned}
\int_0^\infty |J(g_j)(r)|\,\mathrm{d}r &\leq C(b) \int_{\Omega_0^{EL}} |\chi_j(E,L)|\,\mathrm{d}(E,L) \\
&\underset{(\alpha)}{=} C(b) \int_{\Omega_0^{EL}} 1_{B_{\frac{1}{j}}(E^*,L^*)}(E,L)\,|\chi_j(E,L)|\,\mathrm{d}(E,L) \\
&\leq C(b)\lambda\left(B_{\frac{1}{j}}(E^*,L^*)\right)^{1/2}\left(\int_{\Omega_0^{EL}} \chi_j(E,L)^2\,\mathrm{d}(E,L)\right)^{1/2} \\
&\underset{(\beta)}{=} C(b)\frac{1}{j} \to 0 \text{ as } j \to \infty,
\end{aligned}$$

where we used Hölder's inequality. We use this property and Fatou's lemma to deduce that

$$0 \leq \int_0^\infty \liminf_{l\to\infty} \left|J\left(g_{j_{k_l}}\right)(r)\right|\,\mathrm{d}r \leq \liminf_{l\to\infty} \int_0^\infty \left|J\left(g_{j_{k_l}}\right)(r)\right|\,\mathrm{d}r = 0,$$

which implies that

$$\int_0^\infty \liminf_{l\to\infty} \left|J\left(g_{j_{k_l}}\right)(r)\right|\,\mathrm{d}r = 0.$$

Therefore, the integrand $\liminf_{l\to\infty}\left|J\left(g_{j_{k_l}}\right)(r)\right|$ vanishes on $]0,\infty[\setminus N_r$ for some set of measure zero $N_r \subset\ ]0,\infty[$. We then define $N_3 := \{(r,w,L) \in \Omega_0^r \mid r \in N_r\}$. This set is measurable and again a null set by Fubini's theorem, which results in

$$\liminf_{l\to\infty} \left|J\left(g_{j_{k_l}}\right)(r)\right| = 0, \quad (r,w,L) \in \Omega_0^r \setminus N_3. \tag{7.9}$$

To combine our observations, we start by defining $N = N_1 \cup N_2 \cup N_3$, which is a null set, and then, (7.8) and (7.9) lead to

$$J\left(g_{j_{k_l}}\right)(r) \to 0 \text{ as } l \to \infty, \quad (r,w,L) \in \Omega_0^r \setminus N.$$

Due to (7.6) and (7.7), this means that $g^r(r,w,L) = 0$ for $(r,w,L) \in \Omega_0^r \setminus N$. Since N was a set of measure zero, $g = 0$ in H which implies that $\mathcal{B}g_{j_k} \to 0$ in H as $k \to \infty$, completing the proof:
We have shown that in (7.2), both summands on the right hand side converge to zero after extracting the subsequence $(j_k) \subset (j)$. Thus, all required properties of the definition of a Weyl sequence have been proven and consequently, (g_{j_k}) is a Weyl sequence for λ^* and $\mathcal{A}$. $\square$

Theorem 7.2 and Weyl's criterion (Theorem 7.1) imply that $\lambda^* = \left(\frac{2\pi k^*}{T(E^*)}\right)^2 \in \sigma_{\text{ess}}(\mathcal{A})$. Because $k^* \in \mathbb{N}$ and $(E^*,L^*) \in \mathring{\Omega}_0^{EL}$ were arbitrary, we find that

$$\begin{aligned}\sigma_{\text{ess}}(\mathcal{A}) &\supset \left\{\left(\frac{2\pi k^*}{T(E^*)}\right)^2 \,\middle|\, k^* \in \mathbb{N}, (E^*,L^*) \in \mathring{\Omega}_0^{EL}\right\} \\ &= \left\{8(k^*)^2\,|E^*|^3 \,\middle|\, k^* \in \mathbb{N}, E^* \in \,]-1/(2b),0[\right\} = \,]0,\infty[.\end{aligned}$$

Since the essential spectrum of an operator is a closed set, $0 \in \sigma_{\text{ess}}(\mathcal{A})$ must hold as well, which completes the proof of the desired theorem:

Theorem 7.3. $[0,\infty[\, \subset \sigma_{\text{ess}}(\mathcal{A})$.

7.2 Proof of $\sigma(\mathcal{A}) \subset [0,\infty[$

As already mentioned, the idea is to show that $\mathcal{A}$ is a positive operator, which is defined as follows.

Definition 7.4. *Let H be a Hilbert space and let $A : D(A) \to H$ be a self-adjoint operator, densely defined on H. We say that A is positive if $\langle Ag, g\rangle_H \geq 0$ for all $g \in D(A)$.*

We want to show that $\mathcal{A}$ is a positive operator because we can then apply the following theorem.

Theorem 7.5. *Let H be a Hilbert space and let $A : D(A) \to H$ be a self-adjoint operator, densely defined on H. Then A is positive if and only if $\sigma(A) \subset [0,\infty[$.*

Proof. For a proof, we refer to [13, Proposition 5.12]. $\square$

Using the same techniques as in [7, Lemma 1.1], we deduce the first result for showing the positivity of $\mathcal{A}$:

Theorem 7.6. *For all $g \in C^\infty_{c,r}(\Omega_0;\mathbb{R})$ odd in v with $g^r \in C_c^\infty(\mathring{\Omega}_0^r;\mathbb{R})$,*

$$\langle \mathcal{A}g, g\rangle_H \geq 0.$$

To prove this theorem, we first show an auxiliary result:

Lemma 7.7. *Let g be a function as in Theorem 7.6, and let us define*

$$\begin{aligned}\mu(r,w,L) &:= \frac{1}{rw}g^r(r,w,L), \quad (r,w,L) \in \,]0,\infty[\, \times (\mathbb{R}\setminus\{0\}) \times \,]0,\infty[, \\ \mu(r,0,L) &:= \frac{1}{r}\partial_w g^r(r,0,L), \quad r,L \in \,]0,\infty[.\end{aligned}$$

Then,

$$\mu \in C^1(]0,\infty[\, \times \mathbb{R} \times \,]0,\infty[).$$

Proof. The assertion follows since $g^r \in C_c^\infty(\mathring{\Omega}_0^r)$ (which means that $g^r \in C^\infty(]0,\infty[\times \mathbb{R} \times]0,\infty[)$ as a function extended by zero) and since g is odd in v which is equivalent to g^r being odd in w. To obtain this, let us first note that the only problematic situation is the case $w \to 0$. For showing continuity and differentiability in $w = 0$, we first observe some properties of g^r that result from its w-parity. Due to

$$\begin{aligned} g^r(r,w,L) &= -g^r(r,-w,L), \\ \partial_w^2 g^r(r,w,L) &= -\partial_w^2 g^r(r,-w,L), \\ \partial_r g^r(r,w,L) &= -\partial_r g^r(r,-w,L), \text{ and} \\ \partial_L g^r(r,w,L) &= -\partial_L g^r(r,-w,L), \quad (r,w,L) \in]0,\infty[\times \mathbb{R} \times]0,\infty[, \end{aligned}$$

we find that

$$g^r(r,0,L) = \partial_w^2 g^r(r,0,L) = \partial_r g^r(r,0,L) = \partial_L g^r(r,0,L) = 0, \quad r,L \in]0,\infty[. \tag{7.10}$$

Using Taylor's theorem and (7.10), we observe that

$$g^r(r,w,L) = \partial_w g^r(r,0,L)w + \frac{1}{6}\partial_w^3 g^r(r,\xi(w),L)w^3 \tag{7.11}$$

for all $(r,w,L) \in]0,\infty[\times \mathbb{R} \times]0,\infty[$ and some $|\xi(w)| \le |w|$. Here, ξ can also depend on r and L which will not be an issue. In the following, let $r, L \in]0,\infty[$ be fixed. We start by showing the continuity of μ in $w = 0$:

$$\begin{aligned} \lim_{w\to 0, w\neq 0} \mu(r,w,L) &\underset{(7.10)}{=} \frac{1}{r} \lim_{w\to 0, w\neq 0} \left(\frac{1}{w} \left(g^r(r,w,L) - g^r(r,0,L) \right) \right) \\ &= \frac{1}{r}\partial_w g^r(r,0,L) = \mu(r,0,L). \end{aligned}$$

For the differentiability, we first consider the derivative with respect to r:

$$\begin{aligned} \partial_r \mu(r,0,L) &= \lim_{h\to 0, h\neq 0} \left(\frac{\mu(r+h,0,L) - \mu(r,0,L)}{h} \right) \\ &= \lim_{h\to 0, h\neq 0} \frac{1}{h} \left(\left(\frac{1}{r} - \frac{h}{r(r+h)} \right) \partial_w g^r(r+h,0,L) - \frac{1}{r}\partial_w g^r(r,0,L) \right) \\ &= \frac{1}{r} \lim_{h\to 0, h\neq 0} \frac{1}{h} \left(\partial_w g^r(r+h,0,L) - \partial_w g^r(r,0,L) \right) - \lim_{h\to 0, h\neq 0} \frac{1}{r(r+h)} \partial_w g^r(r+h,0,L) \\ &= \frac{1}{r}\partial_r\partial_w g^r(r,0,L) - \frac{1}{r^2}\partial_w g^r(r,0,L). \end{aligned}$$

The r-derivative is continuous, which can be obtained as follows:

$$\begin{aligned} \lim_{w\to 0, w\neq 0} \partial_r \mu(r,w,L) &= \lim_{w\to 0, w\neq 0} \left(\frac{1}{rw}\partial_r g^r(r,w,L) - \frac{1}{r^2 w} g^r(r,w,L) \right) \\ &\underset{(7.10)}{=} \lim_{w\to 0, w\neq 0} \left(\frac{1}{r} \frac{\partial_r g^r(r,w,L) - \partial_r g^r(r,0,L)}{w} - \frac{1}{r^2} \frac{g^r(r,w,L) - g^r(r,0,L)}{w} \right) \\ &= \frac{1}{r}\partial_w\partial_r g^r(r,0,L) - \frac{1}{r^2}\partial_w g^r(r,0,L) \\ &= \partial_r \mu(r,0,L). \end{aligned}$$

Next, we consider the derivative with respect to w:

$$\begin{aligned}
\partial_w \mu(r,0,L) &= \lim_{h\to 0, h\neq 0} \frac{\mu(r,h,L)-\mu(r,0,L)}{h} \\
&= \lim_{h\to 0, h\neq 0} \frac{1}{h}\left(\frac{1}{rh} g^r(r,h,L) - \frac{1}{r}\partial_w g^r(r,0,L)\right) \\
&\underset{(7.11)}{=} \lim_{h\to 0, h\neq 0} \frac{1}{h}\left[\frac{1}{rh}\left(\partial_w g^r(r,0,L)h + \frac{1}{6}\partial_w^3 g^r(r,\xi(h),L)h^3\right) - \frac{1}{r}\partial_w g^r(r,0,L)\right] \\
&= \lim_{h\to 0, h\neq 0} \frac{h}{6r}\partial_w^3 g^r(r,\xi(h),L) \\
&= 0\cdot \partial_w^3 g^r(r,0,L) = 0
\end{aligned}$$

since $|\xi(h)| \leq |h| \to 0$ as $h \to 0$. For the continuity of this derivative, we again use $|\xi(w)| \leq |w|$ and observe that

$$\begin{aligned}
\lim_{w\to 0, w\neq 0} \partial_w \mu(r,w,L) &= \lim_{w\to 0, w\neq 0}\left(\frac{1}{rw}\partial_w g^r(r,w,L) - \frac{1}{rw^2} g^r(r,w,L)\right) \\
&\underset{(7.11)}{=} \lim_{w\to 0, w\neq 0}\left[\frac{1}{rw}\partial_w g^r(r,w,L) - \frac{1}{rw^2}\left(\partial_w g^r(r,0,L)w + \frac{1}{6}\partial_w^3 g^r(r,\xi(w),L)w^3\right)\right] \\
&= \lim_{w\to 0, w\neq 0}\left[\frac{1}{rw}\left(\partial_w g^r(r,w,L) - \partial_w g^r(r,0,L)\right) - \frac{w}{6r}\partial_w^3 g^r(r,\xi(w),L)\right] \\
&= \frac{1}{r}\partial_w^2 g^r(r,0,L) - 0\cdot\partial_w^3 g^r(r,0,L) \\
&\underset{(7.10)}{=} 0 = \partial_w \mu(r,0,L).
\end{aligned}$$

Last, we consider the L-derivative:

$$\begin{aligned}
\partial_L \mu(r,0,L) &= \lim_{h\to 0, h\neq 0} \frac{\mu(r,0,L+h)-\mu(r,0,L)}{h} \\
&= \lim_{h\to 0, h\neq 0} \frac{1}{h}\left(\frac{1}{r}\partial_w g^r(r,0,L+h) - \frac{1}{r}\partial_w g^r(r,0,L)\right) \\
&= \frac{1}{r}\partial_L\partial_w g^r(r,0,L).
\end{aligned}$$

To finish the proof, it remains to show that this derivative is continuous:

$$\begin{aligned}
\lim_{w\to 0, w\neq 0} \partial_L \mu(r,w,L) &= \lim_{w\to 0, w\neq 0} \frac{1}{rw}\partial_L g^r(r,w,L) \\
&\underset{(7.10)}{=} \lim_{w\to 0, w\neq 0} \frac{1}{rw}\left(\partial_L g^r(r,w,L) - \partial_L g^r(r,0,L)\right) \\
&= \frac{1}{r}\partial_w\partial_L g^r(r,0,L) \\
&= \partial_L \mu(r,0,L).
\end{aligned}$$

□

Also in preparation for the proof of Theorem 7.6, we define the Poisson bracket:

Definition 7.8. *For $f, g \in C^1(\Omega_0; \mathbb{R})$, we define the Poisson bracket of f and g as*

$$\{f, g\} := \partial_x f \cdot \partial_v g - \partial_x g \cdot \partial_v f \qquad \text{on } \Omega_0.$$

Remark. If we choose $f = E$ and g spherically symmetric with $g^r \in C^1(\mathring{\Omega}_0^r; \mathbb{R})$, notice that $\{E, g\} = -\mathcal{D}g$ and we find by Lemma 6.10 that

$$\{E, g\}^r = -(\mathcal{D}g)^r = \partial_r E^r \partial_w g^r - \partial_w E^r \partial_r g^r \qquad \text{on } \mathring{\Omega}_0^r. \tag{7.12}$$

We will now prove Theorem 7.6:

Proof of Theorem 7.6. Let g fulfill the assumptions and let μ be corresponding to g according to Lemma 7.7. By (6.4) and the skew-adjointness of $\mathcal{D}$, we observe that

$$\begin{aligned}\langle \mathcal{A}g, g\rangle_H &= \langle -\mathcal{D}^2 g, g\rangle_H - \langle \mathcal{B}g, g\rangle_H \\ &= \langle -\mathcal{D}g, -\mathcal{D}g\rangle_H - (2\pi)^4 \int_0^\infty \frac{1}{r^2} |J(g)(r)|^2 \,\mathrm{d}r \\ &= ||\mathcal{D}g||_H^2 - 4\pi^3 \int_{\mathbb{R}^3} \frac{1}{|x|^4} |J(g)(|x|)|^2 \,\mathrm{d}x.\end{aligned} \tag{7.13}$$

We now consider the second term. Using Cauchy-Schwarz, we find that

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{1}{|x|^4} |J(g)(|x|)|^2 \,\mathrm{d}x &= \int_{\mathbb{R}^3} \frac{1}{|x|^4} \left| \int_0^\infty \int_{\mathbb{R}} \frac{|\varphi'(E^r(|x|, w, L))|^{1/2}}{|\varphi'(E^r(|x|, w, L))|^{1/2}} w g^r(|x|, w, L) \,\mathrm{d}w \,\mathrm{d}L \right|^2 \mathrm{d}x \\ &\leq \int_{\mathbb{R}^3} \frac{1}{|x|^4} \left[\int_0^\infty \int_{\mathbb{R}} 1_{\Omega_0^r}(|x|, w, L) \left|\varphi'(E^r(|x|, w, L))\right| w^2 \,\mathrm{d}w \,\mathrm{d}L \right] \cdot \\ &\qquad \left[\int_0^\infty \int_{\mathbb{R}} \frac{|g^r(|x|, w, L)|^2}{|\varphi'(E^r(r, w, L))|} \,\mathrm{d}w \,\mathrm{d}L \right] \mathrm{d}x.\end{aligned}$$

The first bracket on the right hand side is equal to $(|x|^2/\pi)\rho_0(x)$ (see Lemma 4.2 (*ii*)), while the second bracket can be transformed into a v-integral via $\mathrm{d}w \,\mathrm{d}L = |x|^2/\pi \,\mathrm{d}v$, see (1.2). This results in

$$\begin{aligned}4\pi^3 \int_{\mathbb{R}^3} \frac{1}{|x|^4} |J(g)(|x|)|^2 \,\mathrm{d}x &\leq 4\pi^3 \int_{\mathbb{R}^3} \frac{1}{|x|^4} \frac{|x|^2}{\pi} \rho_0(x) \frac{|x|^2}{\pi} \int_{\mathbb{R}^3} \frac{|g(x, v)|^2}{|\varphi'(E(x, v))|} \,\mathrm{d}v \,\mathrm{d}x \\ &= 4\pi \int_{\Omega_0} \frac{1}{|\varphi'(E(x, v))|} |g(x, v)|^2 \rho_0(x) \,\mathrm{d}(x, v).\end{aligned}$$

Inserting this into (7.13) yields

$$\begin{aligned}\langle \mathcal{A}g, g\rangle_H &\geq \int_{\Omega_0} \frac{1}{|\varphi'(E(x, v))|} |\mathcal{D}g(x, v)|^2 \,\mathrm{d}(x, v) - 4\pi \int_{\Omega_0} \frac{1}{|\varphi'(E(x, v))|} g(x, v)^2 \rho_0(x) \,\mathrm{d}(x, v) \\ &= \int_{\Omega_0} \frac{1}{|\varphi'(E)|} \left[|\{E, g\}|^2 - 4\pi g^2 \rho_0 \right].\end{aligned} \tag{7.14}$$

We now investigate the term with the Poisson bracket $\{E, g\}$. In what follows, there will be an abuse of notation for the sake of simplicity: We will not always distinguish between the

representatives in (x,v)- and (r,w,L)-coordinates. The meaning should be clear from the context.
We use that for $r \neq 0$, $\mu(r,w,L) = \frac{1}{rw} g^r(r,w,L)$ and thus, $g^r(r,w,L) = rw\mu(r,w,L)$. This equation also holds for $r=0$ since $g^r(r,0,L)=0$ which results from g being odd on v and therefore g^r being odd in w. Consequently, also using (7.12), we obtain on $\mathring{\Omega}_0^r$ that

$$\begin{aligned}
\{E,g\}^r &= \partial_r E^r \partial_w g^r - \partial_w E^r \partial_r g^r \\
&= \partial_r E^r (r\mu + rw\partial_w \mu) - \partial_w E^r (w\mu + rw\partial_r \mu) \\
&= rw(\partial_r E^r \partial_w \mu - \partial_w E^r \partial_r \mu) + \mu(\partial_r E^r r - \partial_w E^r w) \\
&= rw\{E,\mu\}^r + \mu\{E,rw\}^r.
\end{aligned}$$

By using this result as well as

$$2\mu\{E,\mu\}^r = 2\mu(\partial_r E^r \partial_w \mu - \partial_w E^r \partial_r \mu) = \partial_r E^r \partial_w(\mu^2) - \partial_w E^r \partial_r(\mu^2) = \{E,\mu^2\}^r,$$

we find that

$$\begin{aligned}
|\{E,g\}^r|^2 &= (r,w)^2 \left|\{E,\mu\}^r\right|^2 + 2rw\mu\{E,\mu\}^r\{E,rw\}^r + \mu^2 \left|\{E,rw\}^r\right|^2 \\
&= (r,w)^2 \left|\{E,\mu\}^r\right|^2 + rw\{E,\mu^2\}^r\{E,rw\}^r + \mu^2 \left|\{E,rw\}^r\right|^2 \\
&= (r,w)^2 \left|\{E,\mu\}^r\right|^2 + \left\{E, \mu^2 rw\{E,rw\}^r\right\}^r - \mu^2 rw \left\{E,\{E,rw\}^r\right\}^r. \qquad (7.15)
\end{aligned}$$

For the last equation, note that

$$\begin{aligned}
&\left\{E,\mu^2 rw\{E,rw\}^r\right\}^r - \mu^2 rw\left\{E,\{E,rw\}^r\right\}^r \\
&= \partial_r E^r \left[2\mu\partial_w\mu rw\{E,rw\}^r + \mu^2 r\{E,rw\}^r + \cancel{\mu^2 rw\partial_w\{E,rw\}^r}\right] \\
&\quad - \partial_w E^r \left[2\mu\partial_r\mu rw\{E,rw\}^r + \mu^2 w\{E,rw\}^r + \cancel{\mu^2 rw\partial_r\{E,rw\}^r}\right] \\
&\quad - \mu^2 rw\left[\cancel{\partial_r E^r\partial_w\{E,rw\}^r} - \cancel{\partial_w E^r\partial_r\{E,rw\}^r}\right] \\
&= \{E,rw\}^r\left[\partial_r E^r 2\mu\partial_w\mu rw + \partial_r E^r \mu^2 r - \partial_w E^r 2\mu\partial_r\mu rw - \partial_w E^r\mu^2 w\right] \\
&= \{E,rw\}^r\left[(rw)\left(\partial_r E^r\partial_w(\mu^2) - \partial_w E^r\partial_r(\mu^2)\right) + \mu^2\left(\partial_r E^r\partial_w(rw) - \partial_w E^r\partial_r(rw)\right)\right] \\
&= rw\{E,rw\}^r\{E,\mu^2\}^r + \mu^2\left|\{E,rw\}^2\right|^2.
\end{aligned}$$

The next step is to examine each term in (7.15) separately. We recall that in (7.14), the terms from (7.15) will be integrated after being multiplied with $|\varphi'(E)|^{-1}$.

First term of (7.15). For the first term in (7.15), we only note that

$$\frac{1}{|\varphi'(E^r(r,w,L))|}(r,w)^2\left|\{E,\mu\}^r(r,w,L)\right|^2 \geq 0, \quad (r,w,L)\in\mathring{\Omega}_0^r.$$

Second term of (7.15). The second term in (7.15) turns out to be zero after integrating with the weight $|\varphi'(E)|^{-1}$. In order to observe this, we start by defining an auxiliary function

$$F(s) := \int_{-\frac{1}{2b}}^{s} \frac{1}{\varphi'(\tau)}\,\mathrm{d}\tau, \quad s\in[-1/(2b),0[.$$

This function is well-defined and in $C^2\left([-\frac{1}{2b},0[\right)$ due to the following argument: By Lemma 4.3 (ii) and Lemma 4.1 (i), we know that $\varphi \in C^2\left(]-\frac{1}{b},0[\right)$ with $\varphi' \neq 0$ on $[-\frac{1}{2b},0[$. Hence, F is well-defined and differentiable by the fundamental theorem of calculus with $F'(s) = \frac{1}{\varphi'(s)}$, $s \in [-\frac{1}{2b},0[$, which again is a $C^1\left([-\frac{1}{2b},0[\right)$-function.
Since $E \in C^2(\Omega_0)$ with $E(\Omega_0) \subset [-\frac{1}{2b},0[$ and $E^r \in C^2(\mathring{\Omega}_0^r)$ with $E^r(\Omega_0^r) \subset [-\frac{1}{2b},0[$, we conclude that $F\circ E \in C^2(\Omega_0;\mathbb{R})$ and $(F\circ E)^r = F\circ E^r \in C^2(\mathring{\Omega}_0^r;\mathbb{R})$.
We now define $G^r(r,w,L) := \mu^2(r,w,L)rw\,\{E,rw\}^r\,(r,w,L)$ for $(r,w,L) \in \mathring{\Omega}_0^r$ which is a $C^1(\mathring{\Omega}_0^r;\mathbb{R})$-function. Using (7.12), we obtain that

$$\begin{aligned}
\{F\circ E, G\}^r &= \partial_r(F\circ E^r)\partial_w G^r - \partial_w(F\circ E^r)\partial_r G^r \\
&= F'\circ E^r\left(\partial_r E^r \partial_w G^r - \partial_w E^r \partial_r G^r\right) \\
&= \frac{1}{\varphi'(E^r)}\{E,G\}^r = -\frac{1}{|\varphi'(E^r)|}\left\{E,\mu^2 rw\{E,rw\}^r\right\}^r \qquad \text{on } \mathring{\Omega}_0^r.
\end{aligned}$$

If we integrate the second term in (7.15) with multiplied weight $|\varphi'(E)|^{-1}$, transform the (x,v)-integral into an (r,w,L)-integral via (3.11) and omit the $4\pi^2$-factor, we have

$$\begin{aligned}
&\int_{\Omega_0^r}\frac{1}{|\varphi'(E^r)|}\left\{E,\mu^2 rw\{E,rw\}^r\right\}^r \mathrm{d}(r,w,L) = -\int_{\Omega_0^r}\{F\circ E,G\}^r\,\mathrm{d}(r,w,L) \\
&= -\int_0^\infty\int_{\mathbb{R}}\int_0^\infty \partial_r(F\circ E^r)\partial_w G^r\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}L + \int_0^\infty\int_{\mathbb{R}}\int_0^\infty \partial_w(F\circ E^r)\partial_r G^r\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}L \\
&=: I_1 + I_2.
\end{aligned}$$

Before we handle I_1 and I_2, notice that $\operatorname{supp} G^r \subset \operatorname{supp}\mu \subset \operatorname{supp} g^r \subset \mathring{\Omega}_0^r$. This yields the existence of $\varepsilon > 0$ such that

$$\operatorname{supp} G^r \subset \Omega_\varepsilon^r := \{(r,w,L) \in \mathring{\Omega}_0^r \mid E^r(r,w,L) < -\varepsilon\}. \tag{7.16}$$

The integrals I_1 and I_2 can therefore be interpreted as integrals over the set Ω_ε^r.

For I_1, let $r,L>0$ be arbitrary, but fixed, such that there exists $w\in\mathbb{R}$ with $(r,w,L)\in\Omega_\varepsilon^r$. For all such w, it must hold that $-\varepsilon > E^r(r,w,L) = \frac{1}{2}w^2 + \psi_L(r)$, which means that $|w| < \sqrt{-2\varepsilon - 2\psi_L(r)}$. The radicand is positive since we assumed that $(r,w,L)\in\Omega_\varepsilon^r$. If we consider only the w-integral for these fixed r,L, we have to integrate over $J_1(r,L) :=]-\sqrt{-2\varepsilon-2\psi_L(r)}, \sqrt{-2\varepsilon-2\psi_L(r)}[=:]-j_1(r,L), j_1(r,L)[$. Due to $E^r(r,w,L) = \frac{1}{2}w^2 + \psi_L(r) \leq \frac{1}{2}j_1(r,L)^2 + \psi_L(r) = E^r(r,j_1(r,L),L) = -\varepsilon < 0$ for $w \in \overline{J_1(r,L)}$, we conclude that $F\circ E^r(r,\cdot,L) \in C^2(\overline{J_1(r,L)})$ which allows us to perform integration by parts for the w-integral. We observe that

$$\int_{-j_1(r,L)}^{j_1(r,L)} \partial_r(F\circ E^r)\partial_w G^r\,\mathrm{d}w = -\int_{-j_1(r,L)}^{j_1(r,L)} \partial_w\partial_r(F\circ E^r)G^r\,\mathrm{d}w.$$

The boundary term $\partial_r(F\circ E^r)(r,\cdot,L)G^r(r,\cdot,L)|_{-j_1(r,L)}^{j_1(r,L)}$ vanishes since $E^r(r,-j_1(r,L),L) = E^r(r,j_1(r,L),L) = -\varepsilon$ which by (7.16) implies that $G^r(r,-j_1(r,L),L) = 0 = G_r(r,j_1(r,L),L)$. In conclusion,

$$I_1 = \int_0^\infty\int_{\mathbb{R}}\int_0^\infty \partial_w\partial_r(F\circ E^r)G^r\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}L.$$

For I_2, let $w \in \mathbb{R}$ and $L > 0$ be arbitrary, but fixed, such that there exists $r > 0$ with $(r,w,L) \in \Omega_\varepsilon^r$. For all such r, it must hold that $\psi_L(r) < -\varepsilon - \frac{1}{2}w^2$. Due to the monotonicity properties of ψ_L, this is valid as long as $r \in J_2(w,L) :=]r_-(-\varepsilon - \frac{1}{2}w^2, L), r_+(-\varepsilon - \frac{1}{2}w^2, L)[=:]j_2(w,L), j_3(w,L)[$. Note that $0 > -\varepsilon - \frac{1}{2}w^2 > \psi_L(r) \geq \psi_L(r_L)$ which implies that $r_-(...)$ and $r_+(...)$ are well-defined. Because of $E^r(j_2(w,L),w,L) = -\varepsilon = E^r(j_3(w,L),w,L)$ and since for $r \in \overline{J_2(w,L)}$, $E^r(r,w,L) = \frac{1}{2}w^2 + \psi_L(r) \leq \frac{1}{2}w^2 + \psi_L(j_2(r,L)) = -\varepsilon < 0$, we conclude that $F \circ E^r(\cdot,w,L) \in C^2(\overline{J_2(w,L)})$ which enables us to perform integration by parts for the r-integral, yielding

$$\int_{j_2(w,L)}^{j_3(w,L)} \partial_w(F \circ E^r)\partial_r G^r \,\mathrm{d}r = -\int_{j_2(w,L)}^{j_3(w,L)} \partial_r\partial_w(F \circ E^r)G^r \,\mathrm{d}r.$$

Here again, the boundary term $\partial_w(F \circ E^r)(\cdot,w,L)G^r(\cdot,w,L)|_{j_2(w,L)}^{j_3(w,L)}$ vanishes because of $E^r(j_2(w,L),w,L) = -\varepsilon = E^r(j_3(w,L),w,L)$, which by (7.16) implies that $G^r(j_2(w,L),w,L) = 0 = G^r(j_3(w,L),w,L)$. Consequently,

$$I_2 = -\int_0^\infty \int_{\mathbb{R}} \int_0^\infty \partial_r\partial_w(F \circ E^r)G^r \,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}L.$$

Combining I_1 and I_2, we find that

$$\begin{aligned}
&\int_{\Omega_0^r} \frac{1}{|\varphi'(E^r)|} \left\{E, \mu^2 rw\{E,rw\}^r\right\}^r \,\mathrm{d}(r,w,L) \\
&= \int_0^\infty \int_{\mathbb{R}} \int_0^\infty \left[\partial_w\partial_r(F \circ E^r)G^r - \partial_r\partial_w(F \circ E^r)G^r\right] \mathrm{d}r\,\mathrm{d}w\,\mathrm{d}L = 0
\end{aligned}$$

since we can switch the r- and w-derivatives due to $F \circ E^r \in C^2(\mathring{\Omega}_0^r)$.

Third term of (7.15). For the third term of (7.15), we use (7.12) to obtain that

$$\begin{aligned}
\{E,rw\}^r(r,w,L) &= \partial_r E^r(r,w,L)\partial_w(rw) - \partial_w E^r(r,w,L)\partial_r(rw) \\
&= \left(-\frac{L}{r^3} + (U_0^r)'(r)\right) r - w^2 \\
&= r\,(U_0^r)'(r) - \frac{L}{r^2} - w^2, \quad (r,w,L) \in \mathring{\Omega}_0^r.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
&\{E,\{E,rw\}\}^r(r,w,L) \\
&= \partial_r E^r(r,w,L)\partial_w\{E,rw\}^r(r,w,L) - \partial_w E^r(r,w,L)\partial_r\{E,rw\}^r(r,w,L) \\
&= \left(-\frac{L}{r^3} + (U_0^r)'(r)\right)(-2w) - w\left(\partial_r\left(r\,(U_0^r)'(r)\right) + 2\frac{L}{r^3}\right) \\
&= -2w\,(U_0^r)'(r) - w\partial_r\left(r\,(U_0^r)'(r)\right), \quad (r,w,L) \in \mathring{\Omega}_0^r.
\end{aligned} \tag{7.17}$$

Additionally, using the Laplacian in spherical coordinates, we find that

$$4\pi\rho_0^r(r) = (\Delta U_0)^r(r) = \frac{1}{r^2}\partial_r\left(r \cdot r\,(U_0^r)'(r)\right) = \frac{1}{r^2}\left(r\,(U_0^r)'(r) + r\partial_r\left(r\,(U_0^r)'(r)\right)\right)$$

for $r \in]0,\infty[$, which implies that

$$\partial_r \left(r \left(U_0^r \right)' (r) \right) = 4\pi r \rho_0^r(r) - \left(U_0^r \right)' (r), \quad r \in]0,\infty[.$$

Consequently, inserting this into (7.17),

$$\begin{aligned} \{E, \{E, rw\}\}^r (r,w,L) &= -2w \left(U_0^r \right)' (r) - 4\pi r w \rho_0^r(r) + w \left(U_0^r \right)' (r) \\ &= -rw \left(4\pi \rho_0^r(r) + \frac{1}{r} \left(U_0^r \right)' (r) \right), \quad (r,w,L) \in \mathring{\Omega}_0^r. \end{aligned}$$

Therefore, the third term in (7.15) can be expressed as

$$\begin{aligned} -\mu^2(r,w,L) rw \{E, \{E, rw\}^r\}^r (r,w,L) &= \mu^2(r,w,L)(rw)^2 \left(4\pi \rho_0^r(r) + \frac{1}{r} \left(U_0^r \right)' (r) \right) \\ &= g^r(r,w,L)^2 \left(4\pi \rho_0^r(r) + \frac{1}{r} \left(U_0^r \right)' (r) \right) \end{aligned}$$

for $(r,w,L) \in \mathring{\Omega}_0^r$.

Gathering all of the information about the terms in (7.15), we obtain that the equation in (7.14) can be estimated by

$$\begin{aligned} \langle \mathcal{A}g, g \rangle_H &\geq 4\pi^2 \int_{\Omega_0^r} \frac{1}{|\varphi'(E^r)|} \left[0 + 0 + 4\pi \rho_0^r (g^r)^2 + \frac{1}{r} \left(U_0^r \right)' (r) (g^r)^2 - 4\pi \rho_0^r (g^r)^2 \right] \mathrm{d}(r,w,L) \\ &= 4\pi^2 \int_{\Omega_0^r} \frac{1}{|\varphi'(E^r)|} \frac{1}{r} \left(U_0^r \right)' (r) (g^r)^2 \, \mathrm{d}(r,w,L) \geq 0 \end{aligned}$$

since $r > 0$ on Ω_0^r and $(U_0^r)'$ is positive on $]0,\infty[$. This completes the proof. □

The next step is to get rid of the parity assumption with respect to v that we demanded of g in Theorem 7.6.

Lemma 7.9. *For all $g \in C_{c,r}^\infty(\Omega_0; \mathbb{R})$ with $g^r \in C_c^\infty(\mathring{\Omega}_0^r; \mathbb{R})$,*

$$\langle \mathcal{A}g, g \rangle_H \geq 0.$$

Proof. Let g be a function satisfying the assumptions. For $(x,v) \in \Omega_0$, we define the even and the odd part of g (with respect to v) by

$$g_\pm(x,v) := \frac{1}{2} \left(g(x,v) \pm g(x,-v) \right),$$

note that $(x,v) \in \Omega_0$ is equivalent to $(x,-v) \in \Omega_0$. As the name suggests, g_+ is even in v and g_- is odd in v. Both functions g_+ and g_- are still in $C_{c,r}^\infty(\Omega_0; \mathbb{R})$ with $g_+^r, g_-^r \in C_c^\infty(\mathring{\Omega}_0^r; \mathbb{R})$. We split the inner product as follows:

$$\begin{aligned} \langle \mathcal{A}g, g \rangle_H &= \langle \mathcal{A}(g_+ + g_-), (g_+ + g_-) \rangle_H \\ &= \langle \mathcal{A}g_+, g_+ \rangle_H + \langle \mathcal{A}g_+, g_- \rangle_H + \langle \mathcal{A}g_-, g_+ \rangle_H + \langle \mathcal{A}g_-, g_- \rangle_H. \end{aligned} \tag{7.18}$$

Since g_+ is even in v, we deduce that $\mathcal{B}g_+ = 0$, see Lemma 6.14 (*ii*). Consequently, we find that in (7.18), the first term is non-negative due to

$$\langle \mathcal{A}g_+, g_+ \rangle_H = \langle -\mathcal{D}^2 g_+, g_+ \rangle_H + \langle -\mathcal{B}g_+, g_+ \rangle_H = \langle -\mathcal{D}g_+, -\mathcal{D}g_+ \rangle_H = ||\mathcal{D}g_+||_H^2 \geq 0,$$

using the skew-adjointness of $\mathcal{D}$.
The second term in (7.18) vanishes. To find this, we first observe that since $\mathcal{D}^2$ preserves v-parity, see Lemma 6.14 (*iii*), $-\mathcal{D}^2 g_+$ is even in v. Using Lemma 6.3, this leads to

$$\langle \mathcal{A}g_+, g_- \rangle_H = \langle -\mathcal{D}^2 g_+, g_- \rangle_H + \langle -\mathcal{B}g_+, g_- \rangle_H = \langle -\mathcal{D}^2 g_+, g_- \rangle_H = 0.$$

The orthogonality is also used for the third term in (7.18). By using the preservation of the v-parity of $\mathcal{D}^2$, we find that $\mathcal{D}^2 g_-$ is odd in v. By Lemma 6.14 (*i*), $\mathcal{B}g_-$ is odd in v as well, which implies that $\mathcal{A}g_- = -\mathcal{D}^2 g_- - \mathcal{B}g_-$ turns out to be odd in v. Hence we conclude that $\langle \mathcal{A}g_-, g_+ \rangle_H = 0$ by Lemma 6.3.
For the last term in (7.18), we make use of Theorem 7.6. g_- satisfies the assumptions, which implies that

$$\langle \mathcal{A}g_-, g_- \rangle_H \geq 0.$$

As a consequence, we find that $\langle \mathcal{A}g, g \rangle_H \geq 0$. □

We now show the non-negativity of the Antonov operator $\mathcal{A}$.

Theorem 7.10. *The operator $\mathcal{A} : D(\mathcal{A}) \to H$ is positive, i.e.*

$$\langle \mathcal{A}g, g \rangle_H \geq 0 \textit{ for all } g \in D(\mathcal{A}).$$

Proof. We start this proof with showing that for any function $g \in D(\mathcal{A})$, $\langle \mathcal{A}g, g \rangle_H$ is real-valued. Note that this is not trivial since functions in $D(\mathcal{A})$ are in general complex-valued. The self-adjointness of $\mathcal{A}$ yields

$$\langle \mathcal{A}g, g \rangle_H = \langle g, \mathcal{A}g \rangle_H = \overline{\langle \mathcal{A}g, g \rangle_H} \in \mathbb{R}. \tag{7.19}$$

Now, let $g \in D(\mathcal{A}) = D(\mathcal{D}^2)$. We split g into its real and imaginary part by $g = f + ih$, where $f = \operatorname{Re} g$ and $h = \operatorname{Im} g$. This implies that both f and h are real-valued functions in $D(\mathcal{D}^2)$ and can therefore be approximated according to Proposition 6.6. Hence, there exist sequences $(f_k), (h_k) \subset C^\infty_{c,r}(\Omega_0; \mathbb{R})$ with $(f_k^r), (h_k^r) \subset C^\infty_c(\mathring{\Omega}_0^r; \mathbb{R})$ such that $f_k \to f, \mathcal{D}f_k \to \mathcal{D}f, h_k \to h$ and $\mathcal{D}h_k \to \mathcal{D}h$ in H as $k \to \infty$. For these functions, we know from Lemma 7.9 that

$$\langle \mathcal{A}f_k, f_k \rangle_H, \langle \mathcal{A}h_k, h_k \rangle_H \geq 0, \quad k \in \mathbb{N}. \tag{7.20}$$

The first step in showing the non-negativity of the Antonov operator is to deduce the convergence

$$\langle \mathcal{A}g, g \rangle_H = \lim_{k \to \infty} \langle \mathcal{A}(f_k + ih_k), (f_k + ih_k) \rangle_H. \tag{7.21}$$

We take a closer look at the inner product on the right hand side by splitting it:

$$\begin{aligned} \langle \mathcal{A}(f_k + ih_k), (f_k + ih_k) \rangle_H &= \langle \mathcal{A}f_k, f_k \rangle_H + \langle \mathcal{A}f_k, ih_k \rangle_H + \langle i\mathcal{A}h_k, f_k \rangle_H + \langle i\mathcal{A}h_k, ih_k \rangle_H \\ &= \langle \mathcal{A}f_k, f_k \rangle_H - i\langle \mathcal{A}f_k, h_k \rangle_H + i\langle \mathcal{A}h_k, f_k \rangle_H + \langle \mathcal{A}h_k, h_k \rangle_H. \end{aligned}$$

By (7.19), the expression has to be real-valued, and since f_k and h_k are real-valued, so are $\mathcal{A}f_k$ and $\mathcal{A}h_k$, which implies that the inner products on the right hand side are all real-valued. As a consequence, the second and third term on the right hand side cancel out. We can perform an analogous argument for $\langle \mathcal{A}g, g\rangle_H = \langle \mathcal{A}(f+ih), (f+ih)\rangle_H$ and therefore, it is sufficient for (7.21) to show that

$$\langle \mathcal{A}f, f\rangle_H + \langle \mathcal{A}h, h\rangle_H = \lim_{k\to\infty} \left(\langle \mathcal{A}f_k, f_k\rangle_H + \langle \mathcal{A}h_k, h_k\rangle_H\right).$$

We consider each term individually and show that $\langle \mathcal{A}f_k, f_k\rangle_H \to \langle \mathcal{A}f, f\rangle$ and $\langle \mathcal{A}h_k, h_k\rangle_H \to \langle \mathcal{A}h, h\rangle_H$ as $k \to \infty$. By Cauchy-Schwarz,

$$\begin{aligned}
&|\langle \mathcal{A}f, f\rangle_H - \langle \mathcal{A}f_k, f_k\rangle_H| \\
&\le |\langle \mathcal{A}f, f\rangle_H - \langle \mathcal{A}f, f_k\rangle_H| + |\langle \mathcal{A}f, f_k\rangle_H - \langle \mathcal{A}f_k, f_k\rangle_H| \\
&= |\langle \mathcal{A}f, f - f_k\rangle_H| + |\langle \mathcal{A}(f - f_k), f_k\rangle_H| \\
&\le ||\mathcal{A}f||_H \, ||f - f_k||_H + \left|\langle -\mathcal{D}^2(f - f_k), f_k\rangle_H\right| + |\langle \mathcal{B}(f - f_k), f_k\rangle_H| \\
&\le ||\mathcal{A}f||_H \, ||f - f_k||_H + |\langle -\mathcal{D}(f - f_k), -\mathcal{D}f_k\rangle_H| + ||\mathcal{B}(f - f_k)||_H \, ||f_k||_H \\
&\le ||\mathcal{A}f||_H \, ||f - f_k||_H + ||\mathcal{D}f_k - \mathcal{D}f_k||_H \, ||\mathcal{D}f_k||_H + C(b) \, ||f - f_k||_H \, ||f_k||_H \,.
\end{aligned}$$

We used the boundedness of the operator $\mathcal{B} : H \to H$. All three terms on the right hand side converge to 0 as $k \to \infty$ since $f_k \to f$ and $\mathcal{D}f_k \to \mathcal{D}f$ in H as $k \to \infty$.
The same approach can be done for h_k and h instead of f_k and f. This results in (7.21) holding true. We now show that the expression on the right hand side of (7.21) is non-negative for all $k \in \mathbb{N}$, which completes this proof. In order to do so, we again use that $\langle \mathcal{A}(f_k + ih_k), (f_k + ih_k)\rangle_H = \langle \mathcal{A}f_k, f_k\rangle_H + \langle \mathcal{A}h_k, h_k\rangle_H$ since the imaginary part has to vanish. By (7.20), both terms are non-negative, and consequently, we infer that all elements of the sequence on the right hand side in (7.21) are non-negative which leads to $\langle \mathcal{A}g, g\rangle_H \ge 0$. □

Theorem 7.5 and Theorem 7.10 immediately result in the following theorem.

Theorem 7.11. $\sigma(\mathcal{A}) \subset [0, \infty[$.

7.3 Main result

We are now able to infer the main results of this thesis.

Theorem 7.12. *The spectrum of the Antonov operator $\mathcal{A} : D(\mathcal{A}) \to H$ is purely essential and equal to the non-negative real line, i.e.*

$$\sigma(\mathcal{A}) = \sigma_{\text{ess}}(\mathcal{A}) = [0, \infty[.$$

Proof. The assertion follows from combining Theorem 7.3 and Theorem 7.11 along with the fact that the essential spectrum of an operator is a subset of the spectrum:

$$[0, \infty[\subset \sigma_{\text{ess}}(\mathcal{A}) \subset \sigma(\mathcal{A}) \subset [0, \infty[.$$

□

The same property also holds if we restrict the operator $\mathcal{A}$ to functions that are odd in v:

Theorem 7.13. *The spectrum of the Antonov operator* $\mathcal{A} : D(\mathcal{A}) \cap H^{\mathrm{odd}} \to H^{\mathrm{odd}}$ *is purely essential and equal to the non-negative real line, i.e.*

$$\sigma(\mathcal{A}) = \sigma_{\mathrm{ess}}(\mathcal{A}) = [0, \infty[.$$

Proof. We can easily show that the inclusions $[0,\infty[\subset \sigma_{\mathrm{ess}}(\mathcal{A}) \subset \sigma(\mathcal{A}) \subset [0,\infty[$ also hold if we consider H^{odd} instead of H. Note that by Lemma 6.16, $\mathcal{A} : D(\mathcal{A}) \cap H^{\mathrm{odd}} \to H^{\mathrm{odd}}$ is self-adjoint.
For the first inclusion, we show that the functions defined in Theorem 7.2 are odd in v. Then, Theorem 7.2 also holds for the restricted Antonov operator since H^{odd} has the same inner product and norm as H. Note that for property (*iii*) of Weyl sequences, the Riesz representation theorem yields that functions ϕ that are an element of the dual space of H^{odd} can be represented by H^{odd}-functions which are again in H and hence, the proof for (*iii*) works analogously.
For the v-parity of a function g_j from Theorem 7.2, we use (3.8) to find that for almost every $(r, w, L) \in \Omega_0^r$,

$$g_j^r(r, -w, L) = g_j^\theta(\zeta(r, -w, L)) = g_j^\theta(1 - \zeta_1(r, w, L), \zeta_2(r, w, L), \zeta_3(r, w, L)),$$

where we used that $E^r(r, -w, L) = E^r(r, w, L)$. Since θ only occurs in the sin-term of g_j^θ and $\sin(2\pi k^*(1-\theta)) = -\sin(2\pi k^* \theta)$, we infer that

$$g_j^r(r, -w, L) = -g_j^\theta(\zeta(r, w, L)) = -g_j^r(r, w, L) \quad \text{f.a.e. } (r, w, L) \in \Omega_0^r.$$

By Lemma 6.2, we conclude that g_j is odd on v and therefore, Theorem 7.2 holds for the restricted Antonov operator. Analogously to Theorem 7.3, we find that $[0,\infty[\subset \sigma_{\mathrm{ess}}(\mathcal{A})$.
For the third inclusion, Theorem 7.10 immediately yields that $\langle \mathcal{A} g, g \rangle_H \geq 0$ for $g \in D(\mathcal{A}) \cap H^{\mathrm{odd}}$ as $D(\mathcal{A}) \cap H^{\mathrm{odd}} \subset D(\mathcal{A})$. Theorem 7.5 then results in $\sigma(\mathcal{A}) \subset [0,\infty[$, completing this proof. $\square$

8 Outlook

Having studied the spectrum of the Antonov operator $\mathcal{A}$, we aim to obtain linear, asymptotic stability of the isochrone steady state in a suitable sense. One idea to achieve this is to try to apply the techniques from [8] to the isochrone case. It is unclear whether this will work out and if so, how many adjustments have to be made. However, having examined the spectrum of $\mathcal{A}$, we know that $\sigma(\mathcal{A}) = \sigma_{\text{ess}}(\mathcal{A}) = [0, \infty[$, and therefore, if eigenvalues of $\mathcal{A}$ exist, they have to be in $[0, \infty[$. This is a crucial step since in [8], the authors assume that there exists an embedded eigenvalue in $[0, \infty[$ and deduce a contradiction with this assumption.
The potential theory that can be found in the appendix is–at least to a large extent–a preparation for the approach of applying [8].

9 Appendix: Potential theory

We start with an auxiliary lemma.

Lemma 9.1. *Let $U \in L^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{K})$ be a spherically symmetric function with radial representative $U^r \in L^1_{\mathrm{loc}}(]0,\infty[;\mathbb{K})$. Additionally, let U and U^r be weakly differentiable with derivatives ∇U and $(U^r)'$ on $\mathbb{R}^3$ and $]0,\infty[$, respectively. Then,*

$$\nabla U(x) = (U^r)'(|x|)\frac{x}{|x|} \qquad \text{f.a.e. } x \in \mathbb{R}^3.$$

Proof. Let $i \in \{1,2,3\}$ and $k,l \in \{1,2,3\} \setminus \{i\}$ with $k \neq l$. Due to the weak differentiability, we have that

$$\forall \chi \in C_c^\infty(]0,\infty[;\mathbb{R}) : \int_0^\infty \chi(r)\,(U^r)'(r)\,\mathrm{d}r = -\int_0^\infty \chi'(r) U^r(r)\,\mathrm{d}r \tag{9.1}$$

and

$$\forall \phi \in C_c^\infty(\mathbb{R}^3;\mathbb{R}) : \int_{\mathbb{R}^3} \phi(x)\partial_i U(x)\,\mathrm{d}x = -\int_{\mathbb{R}^3} \partial_i\phi(x)U(x)\,\mathrm{d}x. \tag{9.2}$$

Next, let $\phi \in C_c^\infty\left(\mathbb{R}^3 \setminus \{x_i = 0\};\mathbb{R}\right)$ be arbitrary, but fixed. Then,

$$\int_{\mathbb{R}^3} \phi(x)\partial_i U(x)\,\mathrm{d}x \underset{(9.2)}{=} -\int_{\mathbb{R}^3} \partial_i\phi(x)U(x)\,\mathrm{d}x = -\int_{\mathbb{R}^2}\int_{\mathbb{R}} \partial_i\phi(x)U^r(|x|)\,\mathrm{d}x_i\,\mathrm{d}(x_k,x_l). \tag{9.3}$$

We fix $x_k, x_l \in \mathbb{R}$ and consider the inner integral. For $r \in \mathbb{R}$, we define

$$\chi(r) := \phi(x_k e_k + x_l e_l + r e_i).$$

The choice of ϕ yields that $\chi \in C_c^\infty(\mathbb{R}\setminus\{0\};\mathbb{R})$ and $\chi'(r) = \partial_i\phi(x_k e_k + x_l e_l + r e_i)$ for $r \in \mathbb{R}\setminus\{0\}$. The inner integral of the right hand side of (9.3) is therefore equal to

$$\int_{-\infty}^{-\varepsilon} \chi'(r)U^r\left(\sqrt{x_k^2+x_l^2+r^2}\right)\mathrm{d}r + \int_{\varepsilon}^{\infty} \chi'(r)U^r\left(\sqrt{x_k^2+x_l^2+r^2}\right)\mathrm{d}r$$

for some $\varepsilon > 0$ such that $\chi = 0$ on $[-\varepsilon,\varepsilon]$. As r stays away from 0, we can substitute $\tau = \tau(r) = \sqrt{x_k^2+x_l^2+r^2}$ to obtain that

$$\begin{aligned}
\int_{\mathbb{R}} \partial_i\phi(x)U^r(|x|)\,\mathrm{d}x_i &= \int_{\infty}^{\sqrt{x_k^2+x_l^2+\varepsilon^2}} \chi'\left(-\sqrt{\tau^2-x_k^2-x_l^2}\right)U^r(\tau)\frac{-\tau}{\sqrt{\tau^2-x_k^2-x_l^2}}\,\mathrm{d}\tau \\
&\quad + \int_{\sqrt{x_k^2+x_l^2+\varepsilon^2}}^{\infty} \chi'\left(\sqrt{\tau^2-x_k^2-x_l^2}\right)U^r(\tau)\frac{\tau}{\sqrt{\tau^2-x_k^2-x_l^2}}\,\mathrm{d}\tau \\
&= \int_{\sqrt{x_k^2+x_l^2+\varepsilon^2}}^{\infty} \tilde{\chi}'(\tau)U^r(\tau)\,\mathrm{d}\tau,
\end{aligned}$$

where

$$\tilde{\chi}(\tau) = \chi\left(\sqrt{\tau^2 - x_k^2 - x_l^2}\right) - \chi\left(-\sqrt{\tau^2 - x_k^2 - x_l^2}\right), \quad \tau \geq \sqrt{x_k^2 + x_l^2 + \varepsilon^2}.$$

If we furthermore define $\tilde{\chi}(\tau) = 0$ for $0 < \tau < \sqrt{x_k^2 + x_l^2 + \varepsilon^2}$, then $\tilde{\chi} \in C_c^\infty(]0, \infty[; \mathbb{R})$ as ε was chosen such that $\chi(r) = 0$ for $-\varepsilon \leq r \leq \varepsilon$. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_i \phi(x) U^r(|x|) \,\mathrm{d}x_i = \int_0^\infty \tilde{\chi}'(\tau) U^r(\tau) \,\mathrm{d}\tau \underset{(9.1)}{=} -\int_0^\infty \tilde{\chi}(\tau) \left(U^r\right)'(\tau) \,\mathrm{d}\tau \\
&= -\int_{\sqrt{x_k^2+x_l^2+\varepsilon^2}}^\infty \left[\chi\left(\sqrt{\tau^2 - x_k^2 - x_l^2}\right) - \chi\left(-\sqrt{\tau^2 - x_k^2 - x_l^2}\right)\right] \left(U^r\right)'(\tau) \,\mathrm{d}\tau.
\end{aligned}$$

We split the integral and substitute $r = r(\tau) = \sqrt{\tau^2 - x_k^2 - x_l^2}$ and $r = r(\tau) = -\sqrt{\tau^2 - x_k^2 - x_l^2}$, respectively, which results in

$$\begin{aligned}
\int_{\mathbb{R}} \partial_i \phi(x) U^r(|x|) \,\mathrm{d}x_i &= -\int_0^\infty \chi(r) \left(U^r\right)'\left(\sqrt{x_k^2 + x_l^2 + r^2}\right) \frac{r}{\sqrt{x_k^2 + x_l^2 + r^2}} \,\mathrm{d}r \\
&\quad + \int_0^{-\infty} \chi(r) \left(U^r\right)'\left(\sqrt{x_k^2 + x_l^2 + r^2}\right) \frac{r}{\sqrt{x_k^2 + x_l^2 + r^2}} \,\mathrm{d}r \\
&= -\int_{\mathbb{R}} \phi(x_k e_k + x_l e_l + r e_i) \left(U^r\right)'\left(\sqrt{x_k^2 + x_l^2 + r^2}\right) \frac{r}{\sqrt{x_k^2 + x_l^2 + r^2}} \,\mathrm{d}r.
\end{aligned}$$

Inserting this into (9.3) yields

$$\int_{\mathbb{R}^3} \phi(x) \partial_i U(x) \,\mathrm{d}x = \int_{\mathbb{R}^3} \phi(x) \left(U^r\right)'(|x|) \frac{x_i}{|x|} \,\mathrm{d}x.$$

Since $\phi \in C_c^\infty\left(\mathbb{R}^3 \setminus \{x_i = 0\}; \mathbb{R}\right)$ was arbitrary, the Dubois-Reymond lemma implies that

$$\partial_i U(x) = \left(U^r\right)'(|x|) \frac{x_i}{|x|}$$

almost everywhere on $\mathbb{R}^3 \setminus \{x_i = 0\}$ and therefore almost everywhere on $\mathbb{R}^3$. Since $i \in \{1, 2, 3\}$ was arbitrary as well, the proof is complete. □

We now collect the following properties of potentials that are derived from functions of the Hilbert space H. We remark that not all properties are needed for this thesis, however, the analysis of the induced potentials is an important step towards using the techniques from [8] to show linear, asymptotic stability in a suitable sense, see Chapter 8.

Definition and Theorem 9.2. *Let $f \in H$. We define $\rho_f := \int_{\mathbb{R}^3} f(\cdot, v)\,\mathrm{d}v$ as well as $U_f := -\frac{1}{|\cdot|} * \rho_f$, both on $\mathbb{R}^3$. Then the following properties hold:*

(i) *ρ_f and U_f are well-defined, i.e. the integrals exist almost everywhere on $\mathbb{R}^3$.*

(ii) *ρ_f and U_f are spherically symmetric.*

(iii) *For all $p \in]1,2]$, $\rho_f \in L^p\left(\mathbb{R}^3;\mathbb{C}\right)$ with $||\rho_f||_p \leq C(b,p)\,||f||_H$, where the constant $C(b,p) > 0$ is independent of f.*

(iv) *For all $r \in]3,\infty[$, $U_f \in L^r\left(\mathbb{R}^3;\mathbb{C}\right)$ with $||U_f||_r \leq C(r)\,||\rho_f||_{\frac{3r}{3+2r}} \leq \tilde{C}(b,r)\,||f||_H$, where the constants $C(r), \tilde{C}(b,r) > 0$ are independent of f.*

(v) *For almost every $x \in \mathbb{R}^3$,*

$$U_f(x) = -\frac{4\pi}{|x|}\int_0^{|x|} s^2 \rho_f^r(s)\,\mathrm{d}s - 4\pi\int_{|x|}^{\infty} s\rho_f^r(s)\,\mathrm{d}s,$$

hence, for almost every $r \in]0,\infty[$,

$$U_f^r(r) = -\frac{4\pi}{r}\int_0^{r} s^2 \rho_f^r(s)\,\mathrm{d}s - 4\pi\int_{r}^{\infty} s\rho_f^r(s)\,\mathrm{d}s.$$

(vi) *∇U_f exists in the weak sense and for all $r \in]\frac{3}{2},6]$, $\nabla U_f \in L^r\left(\mathbb{R}^3;\mathbb{C}\right)$ with $||\nabla U_f||_r \leq C(r)\,||\rho_f||_{\frac{3r}{3+r}} \leq \tilde{C}(b,r)\,||f||_H$, where the constants $C(r), \tilde{C}(b,r) > 0$ are independent of f. Furthermore,*

$$\nabla U_f(x) = \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3}\rho_f(y)\,\mathrm{d}y \qquad \text{f.a.e. } x \in \mathbb{R}^3.$$

(vii) *U_f^r is weakly differentiable on $]0,\infty[$ with*

$$\left(U_f^r\right)'(r) = \frac{4\pi}{r^2}\int_0^r s^2\rho_f^r(s)\,\mathrm{d}s \qquad \text{f.a.e } r \in]0,\infty[.$$

(viii) *For almost every $x \in \mathbb{R}^3$,*

$$\nabla U_f(x) = \left(U_f^r\right)'(|x|)\,\frac{x}{|x|} = \frac{4\pi}{|x|^2}\int_0^{|x|} s^2\rho_f^r(s)\,\mathrm{d}s\frac{x}{|x|}.$$

(ix) *If we additionally require that $f \in D(\mathcal{D})$, then,*

$$\left(U_{\mathcal{D}f}^r\right)'(r) = \frac{4\pi^2}{r^2}J(f)(r) \qquad \text{f.a.e } r \in]0,\infty[.$$

(x) *If we additionally require that $f \in D(\mathcal{D})$, then,*

$$\int_{\mathbb{R}^3} \rho_{\mathcal{D}f}(x)\,\mathrm{d}x = 0.$$

Proof. Assertion (i) will follow from the proof of (iii) and (iv), since if the integrals occurring in these statements exist, the integrals defining ρ_f and U_f exist almost everywhere by Fubini's theorem. Assertion (ii) is trivial since f is spherically symmetric.

Proof of (iii).
Let $p \in]1,2[$ and $q \in]2,\infty[$ with $\frac{1}{p}+\frac{1}{q}=1$, i.e. $q=\frac{p}{p-1}$. We now use Hölder's inequality for the v-integral and that on Ω_0, $E(x,v)<0$ and therefore $|v|^2 < -2U_0(x) \le -2U_0(0) = \frac{1}{b}$. We obtain that

$$\begin{aligned}
\int_{\mathbb{R}^3} |\rho_f(x)|^p \,\mathrm{d}x &= \int \left| \int f(x,v) 1_{\Omega_0}(x,v)\,\mathrm{d}v \right|^p \mathrm{d}x \\
&\le \int \left(\left(\int |f(x,v)|^p \,\mathrm{d}v \right)^{1/p} \left(\int 1_{\Omega_0}(x,v)\,\mathrm{d}v \right)^{1/q} \right)^p \mathrm{d}x \\
&\le \left(\frac{4\pi}{3} \frac{1}{b^{3/2}} \right)^{p/q} \int\int |f(x,v)|^p \,\mathrm{d}v\,\mathrm{d}x \\
&= \left(\frac{4\pi}{3} \frac{1}{b^{3/2}} \right)^{p-1} \int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|^{p/2}} |f(x,v)|^p \left|\varphi'(E(x,v))\right|^{p/2} \mathrm{d}(x,v).
\end{aligned}$$

We again use Hölder's inequality, this time with exponents $2/p$ and $2/(2-p)$, and find that

$$\begin{aligned}
\int_{\mathbb{R}^3} |\rho_f(x)|^p \,\mathrm{d}x \le \left(\frac{4\pi}{3} \frac{1}{b^{3/2}} \right)^{p-1} &\left(\int_{\Omega_0} \frac{1}{|\varphi'(E(x,v))|} |f(x,v)|^2 \,\mathrm{d}(x,v) \right)^{p/2} \\
&\cdot \left(\int_{\Omega_0} \left|\varphi'(E(x,v))\right|^{p/(2-p)} \mathrm{d}(x,v) \right)^{(2-p)/2}.
\end{aligned}$$

The latter integral is finite due to Lemma 4.2 (iii) and we observe that

$$\int_{\mathbb{R}^3} |\rho_f(x)|^p \,\mathrm{d}x \le C(b,p)||f||_H^p.$$

For $p=2$, we use Hölder's inequality with exponents $1/2$, which yields

$$\begin{aligned}
\int_{\mathbb{R}^3} |\rho_f(x)|^2 \,\mathrm{d}x &\le \int \left(\int \frac{1}{|\varphi'(E(x,v))|^{1/2}} |f(x,v)| \left|\varphi'(E(x,v))\right|^{1/2} \mathrm{d}v \right)^2 \mathrm{d}x \\
&\le \int \left(\left(\int \frac{|f(x,v)|^2}{|\varphi'(E(x,v))|} \,\mathrm{d}v \right)^{1/2} \left(\int \left|\varphi'(E(x,v)) 1_{\Omega_0}(x,v)\right| \mathrm{d}v \right)^{1/2} \right)^2 \mathrm{d}x \\
&\le C\frac{1}{b^2} \,||f||_H^2 ,
\end{aligned}$$

using Lemma 4.2 (i).

Proof of (iv).
Let $r \in]3,\infty[$. Then, there exists $p \in]1,\frac{3}{2}[$ such that $\frac{1}{p}+\frac{1}{3}=1+\frac{1}{r}$, i.e. $p=\frac{3r}{3+2r}$. Due to [14, (9) in Section 4.3], we find that $U_f \in L^r(\mathbb{R}^3;\mathbb{C})$ with

$$||U_f||_r \le C(r)\,||\rho_f||_{\frac{3r}{3+2r}} \underset{(iii)}{\le} \tilde{C}(b,r)\,||f||_H .$$

Proof of (v).
To show this identity, we consider $x \in \mathbb{R}^3 \setminus \{0\}$ such that $U_f(x)$ exists, i.e. that the integral $\int_{\mathbb{R}^3} \frac{\rho_f(y)}{|x-y|}\,\mathrm{d}y$ exists. Using the spherical symmetry of ρ_f, i.e. $\rho_f(y) = \rho_f^r(|y|)$ f.a.e. $y \in \mathbb{R}^3$, a transformation with $A \in \mathrm{SO}(3)$ such that $A\frac{x}{|x|} = e_3$, and spherical coordinates yields the assertion.

Proof (vi).
For a proof of the existence of ∇U_f as well as the desired representation, we refer to [14, Theorem 6.21]. Now, let $r \in \,]\frac{3}{2}, 6]$. Then, there exists $p \in \,]1,2]$ such that $\frac{1}{p} + \frac{2}{3} = 1 + \frac{1}{r}$, i.e. $p = \frac{3r}{3+r}$. Using the integral representation of ∇U_f, [14, (9) in Section 4.3] then yields that $\nabla U_f \in L^r(\mathbb{R}^3;\mathbb{C})$ with

$$||\nabla U_f||_r \leq C(r)\,||\rho_f||_{\frac{3r}{3+r}} \underset{(iii)}{\leq} C(b,r)\,||f||_H\,.$$

Proof of (vii).
By (v),

$$U_f^r(r) = -\frac{4\pi}{r}\int_0^r s^2\rho_f^r(s)\,\mathrm{d}s - 4\pi\int_r^1 s\rho_f^r(s)\,\mathrm{d}s - 4\pi\int_1^\infty s\rho_f^r(s)\,\mathrm{d}s =: U_1(r) + U_2(r) + U_3(r)$$

f.a.e. $r \in \,]0,\infty[$. We investigate the weak differentiability for each function U_1, U_2, U_3 separately. For this, let $\phi \in C_c^\infty(]0,\infty[;\mathbb{R})$ be a test function.

For U_1, we find that $]0,\infty[\, \ni r \mapsto g(r) := \int_0^r s^2\rho_f^r(s)\,\mathrm{d}s$ is weakly differentiable with derivative $]0,\infty[\, \ni r \mapsto r^2\rho_f^r(r) =: g'(r)$, see [4, Theorem 8.2]. Therefore,

$$\int_0^\infty g(r)\chi'(r)\,\mathrm{d}r = -\int_0^\infty g'(r)\chi(r)\,\mathrm{d}r, \quad \chi \in C_c^\infty(]0,\infty[;\mathbb{R}).$$

This especially holds for $\chi(r) := \frac{1}{r}\phi(r)$ and as a consequence,

$$\begin{aligned}\int_0^\infty \frac{4\pi}{r}g'(r)\phi(r) &= -4\pi\int_0^\infty g(r)\left(-\frac{1}{r^2}\phi(r) + \frac{1}{r}\phi'(r)\right)\mathrm{d}r \\ &= \int_0^\infty \frac{4\pi}{r^2}g(r)\phi(r)\,\mathrm{d}r + \int_0^\infty U_1(r)\phi'(r)\,\mathrm{d}r.\end{aligned}$$

Since $\phi \in C_c^\infty(]0,\infty[;\mathbb{R})$ was arbitrary, this implies that U_1 is weakly differentiable on $]0,\infty[$ with

$$U_1'(r) = \frac{4\pi}{r^2}g(r) - \frac{4\pi}{r}g'(r) = \frac{4\pi}{r^2}\int_0^r s^2\rho_f^r(s)\,\mathrm{d}s - 4\pi r\rho_f^r(r) \qquad \text{f.a.e. } r \in \,]0,\infty[.$$

Again by [4, Theorem 8.2], we we find that U_2 is weakly differentiable on $]0,\infty[$ with $U_2'(r) = 4\pi r\rho_f^r(r)$ f.a.e. $r \in \,]0,\infty[$. Since U_3 is independent of r, it is differentiable with vanishing derivative.

In conclusion, U_f^r is weakly differentiable on $]0,\infty[$ with

$$\begin{aligned}\left(U_f^r\right)'(r) &= U_1'(r)+U_2'(r)+U_3'(r) = \frac{4\pi}{r^2}\int_0^r s^2\rho_f^r(s)\,\mathrm{d}s - 4\pi r\rho_f^r(r) + 4\pi r\rho_f^r(r) + 0\\ &= \frac{4\pi}{r^2}\int_0^r s^2\rho_f^r(s)\,\mathrm{d}s \qquad \text{f.a.e. } r\in\,]0,\infty[.\end{aligned}$$

Proof of (viii).
This is an immediate result of (vi) and (vii) using Lemma 9.1.

Proof of (ix).
For now, we first restrict ourselves to $f\in D(\mathcal{D})$ being real-valued. Let (f_k) be the sequence of smooth, approximating functions as in Proposition 6.6. We start by showing that the assertion holds for all f_k:

We first use (1.2) to find that

$$\rho_{\mathcal{D}f_k}^r(x) = \int_{\mathbb{R}^3}\mathcal{D}f_k(x,v)\,\mathrm{d}v = \frac{\pi}{|x|^2}\int_0^\infty\int_{\mathbb{R}}\left(\mathcal{D}f_k\right)^r(|x|,w,L)\,\mathrm{d}w\,\mathrm{d}L \qquad \text{f.a.e. } x\in\mathbb{R}^3.$$

We use this and the fact that by $(viii)$, $U_{\mathcal{D}f_k}^r$ is weakly differentiable on $]0,\infty[$ with

$$\begin{aligned}&\left(U_{\mathcal{D}f_k}^r\right)'(r) = \frac{4\pi^2}{r^2}\int_0^r\int_0^\infty\int_{\mathbb{R}}\left(\mathcal{D}f_k\right)^r(s,w,L)\,\mathrm{d}w\,\mathrm{d}L\,\mathrm{d}s\\ &= \frac{4\pi^2}{r^2}\left[\int_0^\infty\int_{\mathbb{R}} w\int_0^r\partial_r f_k^r(s,w,L)\,\mathrm{d}s\,\mathrm{d}w\,\mathrm{d}L - \int_0^r\int_0^\infty\psi_L'(s)\int_{\mathbb{R}}\partial_w f_k^r(s,w,L)\,\mathrm{d}w\,\mathrm{d}L\,\mathrm{d}s\right]\end{aligned}$$

f.a.e. $r\in\,]0,\infty[$, where we used Lemma 6.10. For the inner integral of the first summand, we use the fundamental theorem of calculus (note that $f_k^r\in C_c^\infty(\mathring{\Omega}_0^r;\mathbb{R})$) to find that

$$\int_0^r\partial_r f_k^r(s,w,L)\,\mathrm{d}s = f_k^r(r,w,L) - f_k^r(0,w,L) = f_k^r(r,w,L).$$

The second boundary term vanishes as the support of f_k^r is in $\mathring{\Omega}_0^r$ and therefore stays away from $r=0$. For the inner integral of the second summand, we again use the fundamental theorem of calculus after interpreting the integral as a limit. The integral then vanishes due to the support of f_k^r, note that Ω_0^r is bounded in w. In summary, we observe that

$$\left(U_{\mathcal{D}f_k}^r\right)'(r) = \frac{4\pi^2}{r^2}\int_0^\infty\int_{\mathbb{R}} wf_k^r(r,w,L)\,\mathrm{d}w\,\mathrm{d}L = \frac{4\pi^2}{r^2}J(f_k)(r) \qquad \text{f.a.e. } r\in\,]0,\infty[,$$

and hence,

$$\left(\mathcal{B}f_k\right)^r(r,w,L) = \left(U_{\mathcal{D}f_k}^r\right)'(r)w\left|\varphi'(E^r(r,w,L))\right| \qquad \text{f.a.e. } (r,w,L)\in\Omega_0^r. \tag{9.4}$$

We now show the convergence of both sides. For the left hand side, we use that $\mathcal{B}$ is linear and bounded to obtain that

$$||\mathcal{B}f_k - \mathcal{B}f||_H \le C(b)\,||f_k - f||_H \to 0 \text{ as } k\to\infty.$$

For the right hand side, we first show that $\Omega_0^r \ni (r,w,L) \mapsto \left(U_{\mathcal{D}f}^r\right)'(r)w\,|\varphi'(E^r(r,w,L))| =: F^r(r,w,L)$ is in H. Using (3.11) and Lemma 4.2 (*ii*) yields

$$\begin{aligned}
||F||_H^2 &= 4\pi^2 \int_{\Omega_0^r} \frac{1}{|\varphi'(E^r(r,w,L))|} \left|\left(U_{\mathcal{D}f}^r\right)'(r)w\left|\varphi'(E^r(r,w,L))\right|\right|^2 \mathrm{d}(r,w,L) \\
&= 4\pi^2 \int_0^\infty \left|\left(U_{\mathcal{D}f}^r\right)'(r)\right|^2 \int_0^\infty \int_{\mathbb{R}} w^2 \left|\varphi'(E^r(r,w,L))\right| 1_{\Omega_0^r}(r,w,L)\,\mathrm{d}w\,\mathrm{d}L\,\mathrm{d}r \\
&\leq C(b) \int_0^\infty \left|\left(U_{\mathcal{D}f}^r\right)'(r)\right|^2 r^2\,\mathrm{d}r \\
&= C(b) \int_{\mathbb{R}^3} \left|\left(U_{\mathcal{D}f}^r\right)'(|x|)\right|^2 \mathrm{d}x \\
&\underset{(viii)}{=} C(b) \int_{\mathbb{R}^3} \left|\nabla U_{\mathcal{D}f}(x)\right|^2 \mathrm{d}x \underset{(vi)}{\leq} C(b)\,||\mathcal{D}f||_H^2 < \infty.
\end{aligned}$$

We can now obtain the convergence of the right hand side of (9.4). Estimating in the same way as above but with $\left(U_{\mathcal{D}f_k}^r\right)' - \left(U_{\mathcal{D}f}^r\right)'$ instead of $\left(U_{\mathcal{D}f}^r\right)'$ results in

$$\begin{aligned}
&\int_{\Omega_0^r} \frac{1}{|\varphi'(E^r(r,w,L))|} \left|\left(U_{\mathcal{D}f_k}^r\right)'(r)w\left|\varphi'(E^r(r,w,L))\right| - \left(U_{\mathcal{D}f}^r\right)'(r)w\left|\varphi'(E^r(r,w,L))\right|\right|^2 \mathrm{d}(r,w,L) \\
&\leq C(b) \int_{\mathbb{R}^3} \left|\nabla U_{\mathcal{D}(f_k-f)}(x)\right|^2 \mathrm{d}x \underset{(vi)}{\leq} C(b)\,||\mathcal{D}(f_k-f)||_H \to 0 \text{ as } k\to\infty.
\end{aligned}$$

We have obtained that both sides of (9.4) converge in H and we therefore find that

$$(\mathcal{B}f)^r(r,w,L) = \left(U_{\mathcal{D}f}^r\right)'(r)w\left|\varphi'(E^r(r,w,L))\right|$$

in H and therefore f.a.e. $(r,w,L) \in \Omega_0^r$. By the definition of $\mathcal{B}$, this leads to the desired property for real-valued functions. For complex-valued functions, we split them into their real part and their imaginary part and then obtain the desired property. Note that both sides of the equality that we want to show are linear in f.

Proof of (x).
Let $f \in D(\mathcal{D})$. Lemma 6.19 then implies that f.a.e. $(E,L) \in \Omega_0^{EL}$, $f^\theta(\cdot,E,L) \in H_\theta^1$. For those (E,L), let $\tilde{f}_{EL}$ be the continuous representative. We then find that

$$\begin{aligned}
\int_{\mathbb{R}^3} \rho_{\mathcal{D}f}(x)\,\mathrm{d}x &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathcal{D}f(x,v)\,\mathrm{d}v\,\mathrm{d}x \underset{(3.11)}{=} 4\pi^2 \int_{\Omega_0^\theta} T(E)\,(\mathcal{D}f)^\theta(\theta,E,L)\,\mathrm{d}(\theta,E,L) \\
&= \int_{\Omega_0^{EL}} \int_0^1 \partial_\theta f^\theta(\theta,E,L)\,\mathrm{d}\theta\,\mathrm{d}(E,L),
\end{aligned}$$

where we used the representation of $\mathcal{D}$ given in Lemma 6.19. We now consider the inner integral. Due to [4, Theorem 8.2], we can compute that for almost every $(E,L) \in \Omega_0^{EL}$,

$$[0,1] \ni \tau \mapsto v(\tau) := \int_0^\tau \partial_\theta f^\theta(\theta,E,L)\,\mathrm{d}\theta$$

is continuous and weakly differentiable with derivative $\partial_\theta f^\theta(\cdot,E,L)$. This implies that $\tilde{f}_{EL}$ and v have the same weak derivative and therefore only differ by a constant $c \in \mathbb{R}$ (this holds everywhere on $[0,1]$ due to the continuity of both functions). Hence, $\tilde{f}_{EL} = v + c$. We determine c by using that $v(0) = 0$ and therefore, $c = \tilde{f}_{EL}(0)$. Thus,

$$\int_0^1 \partial_\theta f^\theta(\theta, E, L)\,\mathrm{d}\theta = v(1) = \tilde{f}_{EL}(1) - c = \tilde{f}_{EL}(1) - \tilde{f}_{EL}(0) = 0$$

due to the definition of H^1_θ. As this holds for almost every $(E, L) \in \Omega_0^{EL}$, we find that

$$\int_{\mathbb{R}^3} \rho_{\mathcal{D}f}(x)\,\mathrm{d}x = 0.$$

$\square$

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