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Joachim Frenkler

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Joachim Frenkler[a)]

AFFILIATIONS

Fakultät für Mathematik, Physik und Informatik, Universität Bayreuth, D-95440 Bayreuth, Germany

[a)] Author to whom correspondence should be addressed: joachim.frenkler@uni-bayreuth.de

ABSTRACT

We link the QUMOND theory with the Helmholtz–Weyl decomposition and introduce a new formula for the gradient of the Mondian potential using singular integral operators. This approach allows us to demonstrate that, under very general assumptions on the mass distribution, the Mondian potential is well-defined, once weakly differentiable, with its gradient given through the Helmholtz–Weyl decomposition. Furthermore, we establish that the gradient of the Mondian potential is an L^p vector field. These findings lay the foundation for a rigorous mathematical analysis of various issues within the realm of QUMOND. Further, we prove that the once weakly differentiable Mondian potential solves a second-order partial differential equation in distribution sense. Thus, the question arises whether the potential has second-order derivatives. We affirmatively answer this question in the situation of spherical symmetry, although our investigation reveals that the regularity of the second derivatives is weaker than anticipated. We doubt that a similarly general regularity result can be proven without symmetry assumptions. In conclusion, we explore the implications of our results for numerous problems within the domain of QUMOND.

I. INTRODUCTION

About 40 years ago Milgrom[1] proposed MOND (MOdified Newtonian Dynamics), a non-linear modification of Newton's law of gravity motivated by profound challenges in astrophysics. The basic MOND paradigm introduces a critical acceleration $a_0 = 1.2 \times 10^{-10}$ m s^{-2}, stating that the real gravitational acceleration g_{real} of an object and its acceleration g_N expected from Newtonian gravity are related as follows:

$$\begin{aligned} g_{real} &\approx \sqrt{a_0 g_N} \qquad && \text{if } g_N \ll a_0, \\ g_{real} &\approx g_N && \text{if } g_N \gg a_0. \end{aligned}$$

Thus, in the regime of large accelerations, MOND predicts behavior consistent with Newtonian gravity. However, at extremely low accelerations MOND predicts that g_{real} is proportional to the square root of the acceleration g_N expected from Newtonian physics. Through its single modification, MOND provides explanations for numerous astrophysical phenomena, many of which were predicted *a priori* by MOND.[2] While MOND very effectively describes dynamics on the scales of galaxies, it faces more serious problems on scales slightly larger than the solar system. Recent debates have emerged regarding whether the data from GAIA[3,4] on wide binary stars supports MOND[5,6] or contradicts it.[7]

The present paper focuses on mathematical questions, analysing the equations that are used to describe Mondian physics. We study in detail whether these equations are well posed, introduce a new formula for the Mondian gravitational field using the Helmholtz–Weyl decomposition, and analyze the regularity of the Mondian potential and its derivatives.

When considering to replace Newton's law of gravity by Mondian gravity, one is tempted to replace the Newtonian field ∇U_ρ^N, which corresponds to some density ρ on $\mathbb{R}^3$, by

$$\nabla U_\rho^N + \lambda\left(\left|\nabla U_\rho^N\right|\right)\nabla U_\rho^N. \tag{1}$$

In this formula we simply added a collinear correction term to the Newtonian field. In view of the basic MOND paradigm, one demands that $\lambda : [0,\infty) \to [0,\infty)$ is such that

$$\lambda(u) \approx \sqrt{a_0}/\sqrt{u} \qquad \text{if } u \ll a_0,$$
$$\lambda(u) \approx 0 \qquad \text{if } u \gg a_0.$$

Then we get

$$\left|\nabla U_\rho^N + \lambda\left(\left|\nabla U_\rho^N\right|\right)\nabla U_\rho^N\right| \approx \left|\lambda\left(\left|\nabla U_\rho^N\right|\right)\nabla U_\rho^N\right| \approx \sqrt{a_0\left|\nabla U_\rho^N\right|}, \qquad \text{if } \left|\nabla U_\rho^N\right| \ll a_0,$$
$$\left|\nabla U_\rho^N + \lambda\left(\left|\nabla U_\rho^N\right|\right)\nabla U_\rho^N\right| \approx \left|\nabla U_\rho^N\right|, \qquad \text{if } \left|\nabla U_\rho^N\right| \gg a_0.$$

But with this simple implementation one runs into a problem. The field (1) is in general not the gradient of some potential, thus leading to a loss of classical conservation laws of physics like conservation of momentum [Ref. 8, Sec. 6]. A more refined approach is required. Milgrom[9] proposed a theory, which is called QUMOND (QUasi linear formulation of MOND), where the Mondian potential U_ρ^M is defined as the solution of the partial differential equation (PDE)

$$\operatorname{div}\left(\nabla U_\rho^M\right) = \operatorname{div}\left(\nabla U_\rho^N + \lambda\left(\left|\nabla U_\rho^N\right|\right)\nabla U_\rho^N\right). \tag{2}$$

Is the above PDE well posed? Milgrom provided an explicit formula for its solution U_ρ^M, namely

$$U_\rho^M(x) = U_\rho^N(x) + \frac{1}{4\pi}\int \lambda\left(\left|\nabla U_\rho^N(y)\right|\right)\nabla U_\rho^N(y)\cdot\left(\frac{x-y}{|x-y|^3} + \frac{y}{|y|^3}\right)\mathrm{d}y, \quad x \in \mathbb{R}^3. \tag{3}$$

But is this U_ρ^M well defined? And if yes, which regularity properties does it have? These questions we answer in the present paper. To do so we develope a new mathematical foundation for the QUMOND theory using the Helmholtz–Weyl decomposition. Simply put, the Helmholtz–Weyl decomposition states that every well-behaved vector field v in $\mathbb{R}^3$ that vanishes at infinity can be uniquely decomposed into an irrotational vector field plus a solenoidal vector field. While the solenoidal field has a vector potential, the irrotational field has a scalar potential U, and U satisfies the PDE

$$\operatorname{div}(\nabla U) = \operatorname{div}(v). \tag{4}$$

Comparing the PDEs (2) and (4), we see that we can identify v with the vector field (1) and the potential U with U_ρ^M. Thus, the vector field ∇U_ρ^M should be the irrotational part of the vector field (1).

In this paper we use the Helmholtz–Weyl decomposition in the form proven by Galdi[10] for L^p vector fields[11] and introduce a new explicit formula for the irrotational part of a vector field on $\mathbb{R}^3$ using singular integral operators. These operators are used to derive a new, explicit expression for the Mondian gravitational field ∇U_ρ^M too. This new formulation is very useful to analyze the PDE (2) and the regularity properties of ∇U_ρ^M. It enables us to prove the following theorem:

Theorem I.1. *For every density ρ on $\mathbb{R}^3$ that has finite mass and is an L^p function for some $p > 1$, the corresponding Mondian potential U_ρ^M – defined as in (3) – is well defined and once weakly differentiable with ∇U_ρ^M being the irrotational part of the vector field (1) in the sense of the Helmholtz–Weyl decomposition. $\nabla U_\rho^M = \nabla U_\rho^N + \nabla U_\rho^\lambda$ can be decomposed into an L^q vector field ∇U_ρ^N plus an L^r vector field ∇U_ρ^λ with $q > 3/2$ and $r > 3$. The potential U_ρ^M solves the PDE (2) in distribution sense*

Further in this paper, we analyze second derivatives of U_ρ^M. Under the additional assumptions that ρ is bounded and spherically symmetric, we prove that U_ρ^M is twice weakly differentiable and $D^2U_\rho^M$ is an L^r function. Using handwaving arguments, one would expect that this should hold for $1 < r < 6$. But this is wrong. It is only possible to prove that $D^2U_\rho^M \in L^r(\mathbb{R}^3)$ for $1 < r < 2$ and this result is really optimal. Through counterexamples, we show that it is impossible to achieve such a regularity result for $r > 2$. This is a surprising fact and it is due to the square root appearing above in the basic MOND paradigm. We discuss why achieving similarly general regularity results for the second derivatives of U_ρ^M without assuming spherical symmetry seems doubtful.

The regularity results for U_ρ^M, ∇U_ρ^M and $D^2U_\rho^M$ presented in this paper are essential for addressing further important questions with mathematical rigor. For example they enable us to examine whether initial value problems using Mondian gravity are well-posed, whether corresponding solutions conserve energy, or what the stability properties of stationary solutions are.

The outline of this paper is as follows. In Sec. II, we analyze Newtonian potentials, a prerequisite for analysing Mondian potentials later on, and we introduce the singular integral operators that are important for the rest of the paper. In Sec. III, we study the Helmholtz–Weyl theory from Ref. 10 and provide a new expression for the irrotational part of a vector field using the singular integral operators defined previously. In Sec. IV, we bring together the QUMOND theory from Ref. 9 and our new knowledge about the Helmholtz–Weyl decomposition to prove Theorem I.1. In Sec. V, we analyze the (non-)existence of second derivatives of the Mondian potential. In Sec. VI, we discuss how the results of this paper can be applied to many problems in QUMOND.

II. NEWTONIAN POTENTIALS

In this paper Newtonian potentials will play an important role in two different ways. On the one hand when we have a certain mass distribution with density ρ then U^N_ρ is the Newtonian gravitational potential that belongs to the density ρ. On the other hand in the QUMOND theory we must understand how to decompose a vector field v in its irrotational and its solenoidal part. Here the Newtonian potentials of the three components v_i of the vector field play an important role. This we treat in Sec. III. Since we need Newtonian potentials of both densities ρ and components v_i of vector fields, we use the "neutral" letter f for the source term of the Newtonian potential in the present section.

Given a measurable function $f : \mathbb{R}^3 \to \mathbb{R}$, the corresponding Newtonian gravitational potential U^N_f is given by

$$U^N_f(x) = -G \int \frac{f(y)}{|x-y|} \mathrm{d}y, \quad x \in \mathbb{R}^3, \tag{5}$$

provided the convolution integral exists. Since the concrete value of the gravitational constant G does not affect our analysis we set it to unity. Next we want to introduce some useful singular integral operators. For $\epsilon > 0$ and $i, j = 1, 2, 3$, we define

$$T^\epsilon_{ij} f(x) := -\int_{|x-y|>\epsilon} \left[3\frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} - \frac{\delta_{ij}}{|x-y|^3} \right] f(y) \mathrm{d}y, \quad x \in \mathbb{R}^3,$$

provided that the convolution integral on the right hand side exists. Since

$$\partial_{x_i} \partial_{x_j} \frac{1}{|x|} = 3\frac{x_i x_j}{|x|^5} - \frac{\delta_{ij}}{|x|^3}, \quad x \neq 0,$$

the limit of $T^\epsilon_{ij} f$ for $\epsilon \to 0$ plays an important role in understanding the second derivatives of the Newtonian potential U^N_f. Further, it plays an important role for the Helmholtz–Weyl decomposition as we will see below and hence for the QUMOND theory. In the following two propositions we study this limit.

Proposition II.1. For every $\epsilon > 0$ and $f \in C^1_c(\mathbb{R}^3)$

$$T^\epsilon_{ij} f \in C(\mathbb{R}^3)$$

and the limit

$$T_{ij} f := \lim_{\epsilon \to 0} T^\epsilon_{ij} f$$

exists in $L^\infty(\mathbb{R}^3)$. In particular

$$T_{ij} f \in C(\mathbb{R}^3).$$

Proposition II.2. Let $1 < p < \infty$. There is a $C_p > 0$ such that for every $\epsilon > 0$ and $f \in L^p(\mathbb{R}^3)$

$$\| T^\epsilon_{ij} f \|_p \leq C_p \| f \|_p$$

and the limit

$$T_{ij} f := \lim_{\epsilon \to 0} T^\epsilon_{ij} f$$

exists in $L^p(\mathbb{R}^3)$ with

$$\| T_{ij} f \|_p \leq C_p \| f \|_p.$$

Thus, $T_{ij} : L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$ is a bounded, linear operator.

Proof of Proposition II.1 and II.2. The statements follow quite directly from the literature. To apply the results from the literature, we have to verify that

$$\Omega_{ij}(x) := 3\frac{x_i x_j}{|x|^2} - \delta_{ij}, \quad x \in \mathbb{R}^3, x \neq 0,$$

satisfies the following four assumptions:

1. Ω_{ij} must be homogeneous of degree 0, i.e., $\Omega_{ij}(\delta x) = \Omega_{ij}(x)$ for all $\delta > 0$, $x \neq 0$. This is obviously true.
2. Ω_{ij} must satisfy the cancellation property

$$\int_{|x|=1} \Omega_{ij}(x) \mathrm{d}S(x) = 0.$$

If $i \neq j$, this is obviously true. If $i = j$ this is also true, since

$$\int_{|x|=1} \Omega_{ii}(x)\mathrm{d}S(x) = 3\int_{|x|=1} x_i^2 \mathrm{d}S(x) - 4\pi$$
$$= \int_{|x|=1} |x|^2 \,\mathrm{d}S(x) - 4\pi = 0.$$

3. Ω_{ij} must be bounded on $\{|x| = 1\}$. This is obviously true since Ω_{ij} is continuous on $\mathbb{R}^3\backslash\{0\}$.
4. Ω_{ij} must satisfy the following smoothness property: For

$$w(\delta) := \sup_{\substack{|x-x'|<\delta \\ |x|=|x'|=1}} |\Omega_{ij}(x) - \Omega_{ij}(x')|$$

must hold

$$\int_0^1 \frac{w(\delta)}{\delta}\mathrm{d}\delta < \infty.$$

This is true since for $x, x' \in \mathbb{R}^3$ with $|x| = |x'| = 1$ and $|x - x'| < \delta$ we have

$$|\Omega_{ij}(x) - \Omega_{ij}(x')| = 3|x_i x_j - x_i' x_j'| \leq 3|x_i||x_j - x_j'| + 3|x_j'||x_i - x_i'| \leq 6\delta.$$

Now Proposition II.2 follows directly from Ref. 12, Chap. II, Theorem 3 and Proposition II.1 follows from Ref. 13, Satz 2.2. In the formulation of the theorem in Ref. 13, Dietz does not mention the continuity of the $T_{ij}^{\epsilon} f$, but studying her proof carefully one sees that she has proven the Hölder continuity of $T_{ij}^{\epsilon} f$ under the assumption that $\mathrm{supp} f \subset B_1 := \{|x| < 1\}$. This holds obviously also for every $f \in C_c^1(\mathbb{R}^3)$ after a suitable scaling. If however one is interested solely in the continuity of $T_{ij}^{\epsilon} f$, like we here in this paper, one could also simply apply the transformation $y \mapsto x - y$ in the definition of $T_{ij}^{\epsilon} f$ and use standard results to deduce that $T_{ij}^{\epsilon} f$ is continuous.

□

Next we formulate regularity results for the Newtonian potential. Note that we have set the gravitational constant G to unity.

Lemma II.3. Let $f \in C_c^{1+n}(\mathbb{R}^3)$, $n \in \mathbb{N}_0$. Then the following holds

(a) *The Newtonian potential $U_f^N \in C^{2+n}(\mathbb{R}^3)$. Its first derivative is given by*

$$\partial_{x_i} U_f^N = U_{\partial_{y_i} f}^N, \quad i = 1, 2, 3,$$

which, using integration by parts, can be written as

$$\nabla U_f^N(x) = \int \frac{x - y}{|x - y|^3} f(y)\mathrm{d}y, \quad x \in \mathbb{R}^3.$$

The second derivative of U_f^N is given by

$$\partial_{x_i}\partial_{x_j} U_f^N = T_{ij} f + \delta_{ij}\frac{4\pi}{3} f,$$

where $i, j = 1, 2, 3$.

(b) *For every $R > 0$ there is a $C > 0$ such that*

$$\|U_f^N\|_\infty + \|\nabla U_f^N\|_\infty \leq C\|f\|_\infty.$$

and

$$\|D^2 U_f^N\|_\infty \leq C(\|f\|_\infty + \|\nabla f\|_\infty)$$

provided $\mathrm{supp} f \subset B_R$ where B_R denotes the open ball about zero with radius $R > 0$.

(c) *U_f^N is the unique solution of*

$$\Delta U_f^N = 4\pi f, \quad \lim_{|x|\to\infty} U_f^N(x) = 0.$$

in $C^2(\mathbb{R}^3)$

Proof. It is proven in Ref. 14, Lemma P1 that $U_f^N \in C^2(\mathbb{R}^3)$ if $f \in C_c^1(\mathbb{R}^3)$ and that the formulae for the first derivatives hold. If $f \in C_c^{1+n}(\mathbb{R}^3)$ with $n \geq 1$, it follows directly from

$$\partial_{x_i} U_f^N = U_{\partial_{y_i} f}^N$$

that $U_f^N \in C^{2+n}(\mathbb{R}^3)$. To prove (a) it remains to verify the formula for the second derivatives. For every $x \in \mathbb{R}^3$ we have

$$\begin{aligned}\partial_{x_i}\partial_{x_j} U_f^N(x) &= \partial_{x_i} U_{\partial_{y_j} f}^N(x) = \int \frac{x_i - y_i}{|x-y|^3}\partial_{y_j} f(y)\mathrm{d}y \\ &= -\int \frac{y_i}{|y|^3}\partial_{y_j}(f(x-y))\mathrm{d}y = \int \partial_{y_i}(|y|^{-1})\partial_{y_j}(f(x-y))\mathrm{d}y.\end{aligned}$$

Dominated convergences and integration by parts then yield

$$\begin{aligned}\partial_{x_i}\partial_{x_j} U_f^N(x) &= \lim_{\epsilon\to 0}\int_{|y|>\epsilon} \partial_{y_i}(|y|^{-1})\partial_{y_j}(f(x-y))\mathrm{d}y \\ &= \lim_{\epsilon\to 0}\left(T_{ij}^{\epsilon} f(x) + \int_{|y|=\epsilon}\frac{y_i y_j}{|y|^4} f(x-y)\mathrm{dS}(y)\right);\end{aligned}$$

observe that the normal on $\{|y| = \epsilon\}$ is pointing inward and that there is no boundary term at infinity due to the compact support of f. $T_{ij}^{\epsilon} f$ converges uniformly to $T_{ij} f$ after Proposition II.1. If $i \neq j$ then

$$\left|\int_{|y|=\epsilon}\frac{y_i y_j}{|y|^4} f(x-y)\mathrm{dS}(y)\right| = \left|\int_{|y|=\epsilon}\frac{y_i y_j}{|y|^4}(f(x-y) - f(x))\mathrm{dS}(y)\right| \leq 4\pi\|\nabla f\|_\infty \epsilon.$$

Hence the boundary term vanishes. If $i = j$ then

$$\int_{|y|=\epsilon}\frac{y_i^2}{|y|^4} f(x-y)\mathrm{dS}(y) = \int_{|y|=\epsilon}\frac{y_i^2}{|y|^4}(f(x-y) - f(x))\mathrm{dS}(y) + f(x)\int_{|y|=\epsilon}\frac{y_i^2}{|y|^4}\mathrm{dS}(y).$$

As above the first term vanishes, however, the second one evaluates to $4\pi f(x)/3$. In total we get

$$\partial_{x_i}\partial_{x_j} U_f^N(x) = T_{ij} f(x) + \delta_{ij}\frac{4\pi}{3} f(x).$$

Let us turn to (b). Since $\operatorname{supp} f \subset B_R$ and f is bounded, one sees directly that

$$\|U_f^N\|_\infty + \|\nabla U_f^N\|_\infty \leq C\|f\|_\infty.$$

That

$$\|D^2 U_f^N\|_\infty \leq C(\|f\|_\infty + \|\nabla f\|_\infty),$$

is proven in Ref. 14, Lemma P1.

It remains to show (c). It is stated in Ref. 14, Lemma P1 that U_f^N is the unique solution of

$$\Delta U_f^N = 4\pi f, \quad \lim_{|x|\to\infty} U_f^N(x) = 0, \tag{6}$$

however the proof is omitted. So let us briefly summarize the proof of this well known fact. Since

$$\sum_{i=1}^{3}\left(3\frac{x_i^2}{|x|^5} - \frac{1}{|x|^3}\right) = 0,$$

we have

$$\sum_{i=1}^{3} T_{ii} f = 0.$$

Thus

$$\Delta U_f^N = \sum_{i=1}^{3}\partial_{x_i}^2 U_f^N = \sum_{i=1}^{3}\left(T_{ii} f + \frac{4\pi}{3} f\right) = 4\pi f.$$

The asymptotic behavior of $U_f^N(x)$ for $|x| \to \infty$ follows from the compact support of f. That U_f^N is the unique solution of the PDE (6) follows from the strong maximum principle (Ref. 15, Theorem 2.2). □

Lemma II.4. Let $f \in L^1 \cap L^p(\mathbb{R}^3)$ for a $1 < p < \infty$. Then $U_f^N \in L^1_{loc}(\mathbb{R}^3)$ exists, is twice weakly differentiable and the formulae for ∇U_f^N and $\partial_{x_i}\partial_{x_j} U_f^N$ from Lemma II.3 and the following estimates hold

(a) *If $1 < p < \frac{3}{2}$ and $3 < r < \infty$ with $\frac{1}{3} + \frac{1}{p} = 1 + \frac{1}{r}$ then*

$$\|U_f^N\|_r \le C_{p,r}\|f\|_p.$$

(b) *If $1 < p < 3$ and $\frac{3}{2} < s < \infty$ with $\frac{2}{3} + \frac{1}{p} = 1 + \frac{1}{s}$ then*

$$\|\nabla U_f^N\|_s \le C_{p,s}\|f\|_p.$$

(c) *For every $1 < p < \infty$*

$$\|D^2 U_f^N\|_p \le C_p\|f\|_p.$$

Proof. Reference 15, Theorem 9.9 implies that $U_f^N \in L^1_{loc}(\mathbb{R}^3)$ is well defined, twice weakly differentiable and that the estimate for $D^2 U_f^N$ holds. If ∇U_f^N is defined by the formula from Lemma II.3, Ref. 12, Chap. V, Sec. 1.2, Theorem 1 implies that $U_f^N \in L^r(\mathbb{R}^3)$ and $\nabla U_f^N \in L^s(\mathbb{R}^3)$ with the desired estimates provided $p < 3/2$ and $p < 3$ respectively.

It remains to check that the weak derivatives ∇U_f^N and $D^2 U_f^N$ are indeed given by the formulae from Lemma II.3. Consider the first derivatives and take $\phi \in C_c^\infty(\mathbb{R}^3)$. The Hardy–Littlewood–Sobolev inequality (Ref. 16, Theorem 4.3) allows us to use Fubini:

$$\int U_f^N(x)\partial_{x_i}\phi(x)\mathrm{d}x = -\iint \frac{f(y)\partial_{x_i}\phi(x)}{|x-y|}\mathrm{d}x\,\mathrm{d}y.$$

Now Lemma II.3 implies

$$\begin{aligned}\int U_f^N(x)\partial_{x_i}\phi(x)\mathrm{d}x &= \int f(y)U^N_{\partial_{x_i}\phi}(y)\mathrm{d}y = \int f(y)\partial_{y_i}U_\phi^N(y)\mathrm{d}y\\ &= \iint f(y)\frac{y_i - x_i}{|y-x|^3}\phi(x)\mathrm{d}x\,\mathrm{d}y\\ &= -\int\left(\int \frac{x_i - y_i}{|x-y|}f(y)\mathrm{d}y\right)\phi(x)\mathrm{d}x\\ &= -\int \partial_{x_i}U_f^N(x)\phi(x)\mathrm{d}x.\end{aligned}$$

So the weak gradient of U_f^N is given by the formula for ∇U_f^N from Lemma II.3.

Let $1 < p < \infty$. We study the second derivatives and take $(f_k) \subset C_c^1(\mathbb{R}^3)$ such that

$$f_k \to f \quad \text{in } L^p(\mathbb{R}^3) \text{ for } k \to \infty.$$

Then Hölder, integration by parts and Lemma II.3 give

$$\begin{aligned}\int U_f^N \partial_{x_i}\partial_{x_j}\phi\,\mathrm{d}x &= \lim_{k\to\infty}\int U_{f_k}^N\partial_{x_i}\partial_{x_j}\phi\,\mathrm{d}x\\ &= \lim_{k\to\infty}\int\left(T_{ij}f_k + \delta_{ij}\frac{4\pi}{3}f_k\right)\phi\,\mathrm{d}x\\ &= \int\left(T_{ij}f + \delta_{ij}\frac{4\pi}{3}f\right)\phi\,\mathrm{d}x.\end{aligned}$$

Thus the weak second derivatives of U_f^N are given by the same formula as in Lemma II.3. □

In the situation of spherical symmetry there is a second formula for the Newtonian field ∇U_f^N, which often is quite useful.

Lemma II.5. Let $1 < p < 3$ and $f \in L^1 \cap L^p(\mathbb{R}^3)$, ≥ 0 be spherically symmetric. Then

$$\nabla U_f^N(x) = \frac{M(r)}{r^2}\frac{x}{r}$$

for a.e. $x \in \mathbb{R}^3$ *with* $r = |x|$ *and*

$$M(r) := \int_{B_r} f(x)\mathrm{d}x = 4\pi \int_0^r s^2 f(s)\mathrm{d}s$$

denoting the mass inside the ball with radius r.

Proof. This lemma was first proven by Newton[17] in a similar version. Below we give a proof of our modern version using L^p theory. Assume that f would be continuous and compactly supported. Then $M \in C^1([0,\infty))$ with

$$M'(r) = 4\pi r^2 f(r), \quad r \geq 0.$$

Further

$$|M(r)| \leq \|f\|_1$$

and

$$|M(r)| \leq \frac{4\pi}{3}\|f\|_\infty r^3$$

for $r \geq 0$. For $x \in \mathbb{R}^3$ and $r = |x|$

$$U(x) := -\int_r^\infty \frac{M(s)}{s^2}\mathrm{d}s$$

is well defined. If $r = |x| > 0$, U is continuously differentiable with

$$\nabla U(x) = \frac{M(r)}{r^2}\frac{x}{r}.$$

Since

$$|\nabla U(x)| \leq \frac{4\pi}{3}\|f\|_\infty r$$

we have

$$\nabla U \in C(\mathbb{R}^3).$$

Further

$$\partial_{x_i}\partial_{x_j} U(x) = 4\pi f(r)\frac{x_i x_j}{r^2} - 3M(r)\frac{x_i x_j}{r^5} + \frac{M(r)}{r^3}\delta_{ij}, \quad r > 0.$$

Since f is continuous,

$$\frac{M(r)}{r^3} = \frac{4\pi}{3}\frac{1}{\mathcal{L}(B_r)}\int_{B_r} f\mathrm{d}x \to \frac{4\pi}{3}f(0)$$

and

$$\left|4\pi f(r) - \frac{3M(r)}{r^3}\right| = 4\pi\left|f(r) - \frac{1}{\mathcal{L}(B_r)}\int_{B_r} f\mathrm{d}x\right| \to 0$$

for $r \to 0$. Hence

$$\partial_{x_i}\partial_{x_j} U \in C(\mathbb{R}^3)$$

for every $i, j = 1, 2, 3$. Thus

$$U \in C^2(\mathbb{R}^3).$$

Further

$$\Delta U = 4\pi f$$

and

$$\lim_{|x|\to\infty}|U(x)| \leq \lim_{|x|\to\infty}\frac{\|f\|_1}{|x|} = 0.$$

Since by Lemma II.3 U_f^N is a solution of this PDE, too, and this solutions is unique

$$U_f^N = U$$

and

$$\nabla U_f^N(x) = \nabla U(x) = \frac{M(r)}{r^2}\frac{x}{r}, \quad x \in \mathbb{R}^3. \tag{7}$$

If now $f \in L^1 \cap L^p(\mathbb{R}^3)$, we take a sequence $(f_n) \subset C_c(\mathbb{R}^3)$ of spherically symmetric densities such that

$$f_n \to f \quad \text{in } L^p(\mathbb{R}^3) \text{ for } n \to \infty.$$

By Lemma II.4

$$\nabla U_{f_n}^N \to \nabla U_f^N \quad \text{in } L^s(\mathbb{R}^3) \text{ for } n \to \infty \tag{8}$$

where $s > 3/2$ with $1/p + 2/3 = 1 + 1/s$. Set

$$M_n(r) := \int_{B_r} f_n \,\mathrm{d}x, \quad r \geq 0.$$

Then for every $r \geq 0$

$$|M_n(r) - M(r)| \leq \|f_n - f\|_p \|1_{B_r}\|_{p/(p-1)}.$$

Hence for all $0 < S < R$

$$M_n \to M \quad \text{uniformly on } B_R \text{ for } n \to \infty$$

and

$$\frac{M_n(r)}{r^2}\frac{x}{r} \to \frac{M(r)}{r^2}\frac{x}{r} \text{ uniformly on } \{S < |x| < R\} \text{ for } n \to \infty.$$

Together with (7) and (8) this implies that for a.e. $x \in \mathbb{R}^3$

$$\nabla U_f^N(x) = \frac{M(r)}{r^2}\frac{x}{r}.$$

□

Later on, we will make regular use of the following statement.

Lemma II.6. If $f, h \in L^{6/5}(\mathbb{R}^3)$ then

$$-\frac{1}{8\pi}\int \nabla U_f^N \cdot \nabla U_h^N \mathrm{d}x = \frac{1}{2}\int U_f^N h \,\mathrm{d}x = -\frac{1}{2}\iint \frac{f(y)h(x)}{|x-y|}\mathrm{d}x\,\mathrm{d}y.$$

Proof. $\nabla U_f^N, \nabla U_h^N \in L^2(\mathbb{R}^3)$ and $U_f^N \in L^6(\mathbb{R}^3)$ according to Lemma II.4. Thus the first two integrals are well defined. By the Hardy–Littlewood–Sobolev inequality (Ref. 16, Theorem 4.3) also the third integral is well defined. If $f, h \in C_c^\infty(\mathbb{R}^3)$, integration by parts and $\Delta U_h^N = 4\pi h$ give the above equalities of the integrals. Since $C_c^\infty(\mathbb{R}^3) \subset L^{6/5}(\mathbb{R}^3)$ is dense and all three integrals above are continuous maps from $L^{6/5} \times L^{6/5} \to \mathbb{R}$, the above equalities hold for all $f, h \in L^{6/5}(\mathbb{R}^3)$. □

III. IRROTATIONAL VECTOR FIELDS

As stated in Theorem I.1, we want to prove that the gradient of the Mondian potential U_ρ^M is the irrotational part of the vector field $\nabla U_\rho^N + \lambda(|\nabla U_\rho^N|)\nabla U_\rho^N$. In this section we specify what we mean with the "irrotational part of a vector field." To do so, we make use of the singular integral operators T_{ij} introduced in the previous section about Newtonian potentials.

Definition III.1 (Irrotational part of a vector field). Let $1 < p < \infty$ and $v \in L^p(\mathbb{R}^3)$ be a vector field. For $i = 1, 2, 3$ we define

$$H_i v := \frac{1}{4\pi}\sum_{j=1}^3 T_{ij} v_j + \frac{1}{3} v_i.$$

We call the vector field $Hv \in L^p(\mathbb{R}^3)$ the irrotational part of v. Observe that H is a bounded linear operator on $L^p(\mathbb{R}^3)$ according to Proposition II.2.

We will see in Theorem III.4 that Hv is indeed the irrotational part of v in the sense of the Helmholtz–Weyl decomposition:

Notion. We denote both the space of p-integrable functions on $\mathbb{R}^3$ and the space of vector fields on $\mathbb{R}^3$ having p-integrable components by $L^p(\mathbb{R}^3)$. In expressions like "$Hv \in L^p(\mathbb{R}^3)$" or "$\nabla U_\rho^N \in L^q(\mathbb{R}^3)$," it is evident from the context that we talk about vector fields. In ambiguous situations, we will always write "Let $v \in L^p(\mathbb{R}^3)$ be a vector field" if we talk about a vector field v, or "Let $\rho \in L^p(\mathbb{R}^3)$" if we talk about a function ρ.

Theorem III.2 (Helmholtz–Weyl decomposition). *For every vector field $v \in L^p(\mathbb{R}^3)$, $1 < p < \infty$, exist uniquely determined vector fields $v_1 \in L^p_{irr}(\mathbb{R}^3)$ and $v_2 \in L^p_{sol}(\mathbb{R}^3)$ such that*

$$v = v_1 + v_2,$$

where the two subspaces $L^p_{irr}(\mathbb{R}^3)$ and $L^p_{sol}(\mathbb{R}^3)$ of $L^p(\mathbb{R}^3)$ are defined as follows:

$$L^p_{irr}(\mathbb{R}^3) := \Big\{ w \in L^p(\mathbb{R}^3) \text{ a vector field such that a potential } U \in W^{1,p}_{loc}(\mathbb{R}^3) \text{ exists with } w = \nabla U \Big\},$$
$$L^p_{sol}(\mathbb{R}^3) := \Big\{ w \in L^p(\mathbb{R}^3) \text{ a vector field such that } \operatorname{div} w = 0 \text{ weakly} \Big\}.$$

Remark. The space $W^{1,p}_{loc}(\mathbb{R}^3)$ denotes the Sobolev space of scalar functions U on $\mathbb{R}^3$ that are once weakly differentiable and that are locally integrable if taken to the power p, i.e., for each compact domain K the integral $\int_K |U|^p \mathrm{d}x$ is finite. Further, also the gradient of U must be locally integrable if taken to the power p, but observe that in the definition of L^p_{irr} we additionally demanded that the gradient shall be an L^p vector field not only on every compact domain but on the entire space $\mathbb{R}^3$.

Proof. In Ref. 10, Theorem III.1.2 it is proven that the Helmholtz–Weyl decomposition in the sense of Ref. 10, Eq. III.1.5 holds. This form of the theorem makes use of a different definition of the space L^p_{sol}. However, in Ref. 10, Theorem III.2.3 it is proven that our definition here coincides with the definition used in Ref. 10, Theorem III.1.2. □

Let us study how the Helmholtz–Weyl decomposition looks like for smooth vector fields with compact support.

Lemma III.3. Let $v \in C^2_c(\mathbb{R}^3)$ be a vector field. For every $1 < p < \infty$

$$Hv = \frac{1}{4\pi} \nabla \left(\sum_{j=1}^{3} \partial_{x_j} U^N_{v_j} \right) \in C^1 \cap L^p_{irr}(\mathbb{R}^3)$$

is the uniquely determined, irrotational part of the vector field v according to the Helmholtz–Weyl decomposition. Further

$$\operatorname{div} Hv = \operatorname{div} v \text{ and } \operatorname{rot} Hv = 0.$$

Proof. From Definition III.1 follows directly that $Hv \in L^p(\mathbb{R}^3)$ for every $1 < p < \infty$. Since by Lemma II.3

$$U^N_{v_j} \in C^3(\mathbb{R}^3), \quad j = 1, 2, 3,$$

we have also

$$Hv = \frac{1}{4\pi} \nabla \left(\sum_{j=1}^{3} \partial_{x_j} U^N_{v_j} \right) \in C^1(\mathbb{R}^3).$$

In particular $Hv \in L^p_{irr}(\mathbb{R}^3)$. Since Hv is a gradient, its rotation is zero. For the divergence we have with Lemma II.3

$$\operatorname{div} Hv = \frac{1}{4\pi} \sum_{j=1}^{3} \Delta U^N_{\partial_{y_j} v_j} = \sum_{j=1}^{3} \partial_{x_j} v_j = \operatorname{div} v.$$

Further we get

$$v_2 := v - Hv \in C^1 \cap L^p(\mathbb{R}^3)$$

for every $1 < p < \infty$ with $\operatorname{div}(v_2) = 0$ classically. Hence $v_2 \in L^p_{sol}(\mathbb{R}^3)$. □

We are particularly interested in the Helmholtz–Weyl decomposition of vector fields $v \in L^p(\mathbb{R}^3)$. If $p < 3$, we can show that $\partial_{x_j} U^N_{v_j}$ exists and, using the same formula as in Lemma III.3, it is easy to deduce that in this situation, too, the Helmholtz–Weyl decomposition of v is given by $Hv + (v - Hv)$. If $p \geq 3$ however (and this is the case of special interest in Mondian physics), the integral

$$\partial_{x_j} U^N_{v_j} = \int \frac{x_j - y_j}{|x - y|^3} v_j(y) \mathrm{d}y$$

does not necessarily converge. Nevertheless, also in this situation the Helmholtz–Weyl decomposition of v is given by $Hv + (v - Hv)$ but we can no longer make use of the formula from Lemma III.3. The key ingredients to prove this explicit form of the Helmholtz–Weyl decomposition for $v \in L^p(\mathbb{R}^3)$ is that $L^p_{irr}(\mathbb{R}^3)$ and $L^p_{sol}(\mathbb{R}^3)$ are closed subsets of $L^p(\mathbb{R}^3)$. For $L^p_{sol}(\mathbb{R}^3)$ this is proven in Ref. 10. For $L^p_{irr}(\mathbb{R}^3)$, Ref. 10 leaves this as an exercise to the reader (Exercise III.1.2). This exercise can be solved using Poincaré's inequality. Using that $L^p_{sol}(\mathbb{R}^3)$ and $L^p_{irr}(\mathbb{R}^3)$ are closed subsets, leads to the following theorem:

Theorem III.4 (Explicit Helmholtz–Weyl decomposition). *Let $1 < p < \infty$ and $v \in L^p(\mathbb{R}^3)$ be a vector field. Then the uniquely determined Helmholtz–Weyl decomposition of v is given by*

$$v = Hv + (v - Hv)$$

with

$$Hv \in L^p_{irr}(\mathbb{R}^3) \quad \textit{and} \quad v - Hv \in L^p_{sol}(\mathbb{R}^3).$$

Proof. We can approximate the vector field $v \in L^p(\mathbb{R}^3)$ with a sequence of vector fields $(v_k) \subset C^2_c(\mathbb{R}^3)$. By Lemma III.3 the Helmholtz–Weyl decomposition of v_k is given by

$$v_k = Hv_k + (v_k - Hv_k)$$

with

$$Hv_k \in L^p_{irr}(\mathbb{R}^3) \quad \text{and} \quad v_k - Hv_k \in L^p_{sol}(\mathbb{R}^3).$$

By Ref. 10, Exercise III.1.2, $L^p_{irr}(\mathbb{R}^3)$ is a closed subset of $L^p(\mathbb{R}^3)$. By Ref. 10, Theorem III.2.3, $L^p_{sol}(\mathbb{R}^3)$ is the closure of the set

$$\{v \in C^\infty_c(\mathbb{R}^3) \text{ with } \operatorname{div} v = 0\}$$

with respect to the L^p-norm on $\mathbb{R}^3$. Hence, also $L^p_{sol}(\mathbb{R}^3)$ is a closed subset of $L^p(\mathbb{R}^3)$. Since $v_k \to v$ in $L^p(\mathbb{R}^3)$ for $k \to \infty$ and H is a bounded, linear operator on L^p vector fields, also $Hv_k \to Hv$ in $L^p(\mathbb{R}^3)$. Thus,

$$Hv \in L^p_{irr}(\mathbb{R}^3) \quad \text{and} \quad v - Hv \in L^p_{sol}(\mathbb{R}^3)$$

and

$$v = Hv + (v - Hv)$$

is the uniquely determined Helmholtz–Weyl decomposition of v. □

Thus the vector field Hv as defined in Definition III.1 is indeed the irrotational part of the vector field v in the sense of the Helmholtz–Weyl decomposition. Before we close this section we prove two useful lemmas. First: For spherically symmetric vector fields the Helmholtz–Weyl decomposition is trivial.

Lemma III.5. Let $1 < p < \infty$. Then for every spherically symmetric vector field $v \in L^p(\mathbb{R}^3)$

$$Hv = v.$$

Proof. Let $v \in L^p(\mathbb{R}^3)$ be a spherically symmetric vector field. There exists a sequence $(v_k) \subset C^\infty_c(\mathbb{R}^3)$ of spherically symmetric vector fields with

$$v_k \to v \quad \text{in } L^p(\mathbb{R}^3) \text{ for } k \to \infty.$$

Since the v_k are spherically symmetric

$$\operatorname{rot} v_k = 0.$$

Hence, by standard results for vector calculus, there exist potentials $(U_k) \subset C^\infty(\mathbb{R}^3)$ such that for every $k \in \mathbb{N}$

$$v_k = \nabla U_k,$$

in particular $v_k \in L^p_{irr}(\mathbb{R}^3)$ and the uniqueness of the Helmholtz–Weyl decomposition implies

$$Hv_k = v_k.$$

Since $H : L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$ is continuous

$$Hv = v.$$

□

And last in this section we prove the useful fact that the operator H is symmetric.

Lemma III.6. Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, and let $v \in L^p(\mathbb{R}^3)$, $w \in L^q(\mathbb{R}^3)$ be vector fields. Then

$$\int v \cdot Hw \, \mathrm{d}x = \int Hv \cdot w \, \mathrm{d}x.$$

Proof. Assume that $v \in L^1 \cap L^p(\mathbb{R}^3)$ and $w \in L^1 \cap L^q(\mathbb{R}^3)$. Since $v, w \in L^1(\mathbb{R}^3)$ we can apply Fubini and get that for every $\epsilon > 0$ and $i, j = 1, 2, 3$

$$\begin{aligned} \int T_{ij}^{\epsilon} v_j \; w_i \, \mathrm{d}x &= - \iint_{|x-y|>\epsilon} \partial_{x_i} \partial_{x_j} \left(\frac{1}{|x-y|} \right) v_j(y) \mathrm{d}y \; w_i(x) \mathrm{d}x \\ &= - \int v_j(y) \int_{|x-y|>\epsilon} \partial_{y_i} \partial_{y_j} \left(\frac{1}{|x-y|} \right) w_i(x) \mathrm{d}x \, \mathrm{d}y \\ &= \int v_j \; T_{ij}^{\epsilon} w_i \, \mathrm{d}y. \end{aligned}$$

Hence by Hölder

$$\begin{aligned} \int Hv \cdot w \, \mathrm{d}x &= \frac{1}{4\pi} \sum_{i,j=1}^{3} \lim_{\epsilon \to 0} \int T_{ij}^{\epsilon} v_j \; w_i \, \mathrm{d}x + \frac{1}{3} \sum_{i=1}^{3} \int v_i w_i \, \mathrm{d}x \\ &= \frac{1}{4\pi} \sum_{i,j=1}^{3} \lim_{\epsilon \to 0} \int v_j \; T_{ij}^{\epsilon} w_i \, \mathrm{d}x + \frac{1}{3} \sum_{i=1}^{3} \int v_i w_i \, \mathrm{d}x \\ &= \int v \cdot Hw \, \mathrm{d}x. \end{aligned}$$

Since $L^1 \cap L^p(\mathbb{R}^3) \subset L^p(\mathbb{R}^3)$ and $L^1 \cap L^q(\mathbb{R}^3) \subset L^q(\mathbb{R}^3)$ are dense, and H is continuous, it follows that for every $v \in L^p(\mathbb{R}^3)$ and $w \in L^q(\mathbb{R}^3)$

$$\int Hv \cdot w \, \mathrm{d}x = \int v \cdot Hw \, \mathrm{d}x.$$

□

IV. MONDIAN POTENTIALS

In Ref. 9, Milgrom introduced the QUMOND theory where the Mondian potential U_ρ^M belonging to some density ρ is given as the solution of the PDE (2). Milgrom gave an explicit formula for the solution of this PDE - our Eq. (3). Here in this paper we take another way to approach the Mondian potential U_ρ^M. This new approach enables us to place the entire QUMOND theory on a more robust, mathematical foundation. We define

$$\nabla U_\rho^M \text{ is the irrotational part of } \nabla U_\rho^N + \lambda(|\nabla U_\rho^N|) \nabla U_\rho^N.$$

The theory of the previous section guarantees that the field ∇U_ρ^M is indeed the gradient of some potential U_ρ^M. Further, it guarantees that ∇U_ρ^M is an L^p vector field for some $p > 1$. To bring this new definition of ∇U_ρ^M together with the theory from Ref. 9, we have to take a closer look on the potential U_ρ^M. How do we get an explicit formula for it? In Lemma III.3 we have seen that for a vector field

$$v \in C_c^2(\mathbb{R}^3)$$

Hv is the gradient of

$$\frac{1}{4\pi} \sum_{j=1}^{3} \partial_{x_j} U_{v_j}^N. \tag{9}$$

But if $\rho \in L^1 \cap L^p(\mathbb{R}^3)$ for a $1 < p < 3$, then

$$\nabla U_\rho^N \in L^q(\mathbb{R}^3)$$

for a $3/2 < q < \infty$. If now $\lambda(\sigma) \approx \sqrt{a_0}/\sqrt{\sigma}$ then

$$v := \lambda\big(|\nabla U_\rho^N|\big) \nabla U_\rho^N \in L^{2q}(\mathbb{R}^3)$$

with $2q > 3$. But then $\partial_{x_j} U^N_{v_j}$ is not well defined; for this it would be necessary that v is an L^p vector field for some $p < 3$. However we can still recover a slight variation of Lemma III.3. If we compare Eq. (9) with Eq. (3), which is the definition of the Mondian potential in Ref. 9, we see that both are quite similar. We prove that Eq. (3) really yields a well defined potential and that its gradient is given by ∇U^M_ρ as defined above.

For better readability we decompse U^M_ρ and write $U^M_\rho = U^N_\rho + U^\lambda_\rho$ where the gradient of U^N_ρ is ∇U^N_ρ and the gradient of U^λ_ρ shall be $H(\lambda(|\nabla U^N_\rho|)\nabla U^N_\rho)$. In the next lemma we analyze U^λ_ρ.

Lemma IV.1. Assume that $\lambda : (0,\infty) \to (0,\infty)$ is measurable and that there is $\Lambda > 0$ such that $\lambda(\sigma) \le \Lambda/\sqrt{\sigma}$, for every $\sigma > 0$ (thus the function λ remains in its physically motivated regime). Let $\rho \in L^1 \cap L^p(\mathbb{R}^3)$ for a $1 < p < 3$ and let $3/2 < q < \infty$ with $2/3 + 1/p = 1 + 1/q$. Set

$$U^\lambda_\rho(x) := \frac{1}{4\pi}\int \lambda\big(|\nabla U^N_\rho(y)|\big)\nabla U^N_\rho(y)\cdot\left(\frac{x-y}{|x-y|^3} + \frac{y}{|y|^3}\right)\mathrm{d}y, \quad x \in \mathbb{R}^3.$$

Then

$$U^\lambda_\rho \in W^{1,2q}_{loc}(\mathbb{R}^3)$$

and

$$\nabla U^\lambda_\rho = H\big(\lambda\big(|\nabla U^N_\rho|\big)\nabla U^N_\rho\big) \in L^{2q}(\mathbb{R}^3).$$

Remark. The formula for U^λ_ρ above is the same one as in Eq. (3) and it is almost the same one as (9). The only difference is the term "$+y/|y|^3$," which guarantees that the integral is finite.

Proof. Let $R > 0$. First we prove that for

$$I(x,y) := \lambda\big(|\nabla U^N_\rho(y)|\big)\nabla U^N_\rho(y)\cdot\left(\frac{x-y}{|x-y|^3} + \frac{y}{|y|^3}\right), \quad x,y \in \mathbb{R}^3,$$

it holds

$$\iint_{|x|\le R} |I(x,y)|\,\mathrm{d}x\,\mathrm{d}y < \infty. \tag{10}$$

Let p, q be as stated above and let r be the dual exponent of $2q$. Since $3 < 2q < \infty$,

$$1 < r < \frac{3}{2}.$$

Then

$$\begin{aligned}\iint_{|x|\le R, |y|\le 2R} |I(x,y)|\,\mathrm{d}x\,\mathrm{d}y &\le \Lambda \iint_{|x|\le R, |y|\le 2R} |\nabla U^N_\rho(y)|^{1/2}\left(\frac{1}{|x-y|^2} + \frac{1}{|y|^2}\right)\mathrm{d}x\,\mathrm{d}y \\ &\le 2\Lambda\,\mathcal{L}(B_R)\,\|\nabla U^N_\rho\|_q^{1/2}\left\|\frac{1}{|y|^2}\right\|_{L^r(B_{3R})} < \infty.\end{aligned}$$

Next observe that for all $y \in \mathbb{R}^3\backslash\{0\}$, $i, j = 1, 2, 3$

$$\left|\partial_{y_i}\frac{y_j}{|y|^3}\right| = \left|\frac{\delta_{ij}}{|y|^3} - 3\frac{y_i y_j}{|y|^5}\right| \le \frac{4}{|y|^3}.$$

Thus for $x, y \in \mathbb{R}^3$ with $|x| \le R$, $|y| > 2R$ holds

$$\left|\frac{x_j - y_j}{|x-y|^3} - \frac{y_j}{|y|^3}\right| = \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\frac{y_j - s x_j}{|y - sx|^3}\mathrm{d}s\right| \le R\int_0^1 \frac{\mathrm{d}s}{|y-sx|^3}.$$

Since for all $0 \le s \le 1$

$$|y - sx| \ge \frac{|y|}{2},$$

we estimate further

$$\left|\frac{x_j - y_j}{|x-y|^3} - \frac{y_j}{|y|^3}\right| \le \frac{8R}{|y|^3}.$$

Hence

$$\iint_{|x|\le R,|y|>2R} |I(x,y)|\mathrm{d}x\,\mathrm{d}y \le C\iint_{|x|\le R,|y|>2R} |\nabla U_\rho^N(y)|^{1/2}\frac{1}{|y|^3}\mathrm{d}y$$
$$\le C\mathcal{L}(B_R)\|\nabla U_\rho^N\|_q^{1/2}\left\|\frac{1}{|y|^3}\right\|_{L^r(\{|y|>2R\})}$$
$$< \infty.$$

Thus Ref. 10 holds. Fubini then implies that

$$U_\rho^\lambda \in L^1_{loc}(\mathbb{R}^3).$$

Hence the potential U_ρ^λ is well defined. It remains to prove that

$$\nabla U_\rho^\lambda = H\big(\lambda\big(|\nabla U_\rho^N|\big)\nabla U_\rho^N\big).$$

We write shortly

$$v := \frac{1}{4\pi}\lambda\big(|\nabla U_\rho^N|\big)\nabla U_\rho^N \in L^{2q}(\mathbb{R}^3).$$

Let $\phi \in C_c^\infty(\mathbb{R}^3)$. Then

$$\int U_\rho^\lambda \partial_{x_i}\phi\,\mathrm{d}x = \iint v(y)\cdot\left(\frac{x-y}{|x-y|^3}+\frac{y}{|y|^3}\right)\partial_{x_i}\phi(x)\mathrm{d}y\,\mathrm{d}x.$$

Thanks to Ref. 10 we can apply Fubini and, since

$$\int \partial_{x_i}\phi(x)\mathrm{d}x = 0,$$

we have

$$\int U_\rho^\lambda \partial_{x_i}\phi\,\mathrm{d}x = -\int v\cdot\nabla U^N_{\partial_{x_i}\phi}\mathrm{d}y = -\sum_{j=1}^3\int v_j\partial_{y_i}\partial_{y_j}U_\phi^N\,\mathrm{d}y.$$

Lemma II.3 and Proposition II.2 imply

$$\int U_\rho^\lambda \partial_{x_i}\phi\,\mathrm{d}x = -\sum_{j=1}^3\lim_{\epsilon\to 0}\int v_j\left(T_{ij}^\epsilon\phi + \delta_{ij}\frac{4\pi}{3}\phi\right)\mathrm{d}y.$$

As in the Proof of Lemma III.6 we have

$$\int v_j\ T_{ij}^\epsilon\phi\,\mathrm{d}y = \int T_{ij}^\epsilon v_j\ \phi\,\mathrm{d}y.$$

Hence

$$\int U_\rho^\lambda \partial_{x_i}\phi\,\mathrm{d}x = -4\pi\int\left(\frac{1}{4\pi}\sum_{j=1}^3 T_{ij}v_j + \frac{1}{3}v_i\right)\phi\,\mathrm{d}y$$
$$= -4\pi\int H_i v\ \phi\,\mathrm{d}y.$$

Thus

$$\nabla U_\rho^\lambda = 4\pi H v = H\big(\lambda\big(|\nabla U_\rho^N|\big)\nabla U_\rho^N\big).$$

In particular, the Helmholtz–Weyl decomposition Theorem III.4 implies

$$U_\rho^\lambda \in W^{1,2q}_{loc}(\mathbb{R}^3)$$

and

$$\nabla U_\rho^\lambda \in L^{2q}(\mathbb{R}^3).$$

□

Taking Lemmas IV.1 and II.4 together gives

Corollary IV.2. Let λ be as in Lemma IV.1 and $\rho \in L^1 \cap L^p(\mathbb{R}^3)$ for some $p > 1$. Then

$$U_\rho^M := U_\rho^N + U_\rho^\lambda$$

is well defined and once weakly differentiable. The gradient ∇U_ρ^M can be decomposed into an L^q plus an L^r vector field, where ∇U_ρ^N is an L^q vector field for some $q > 3/2$ and ∇U_ρ^λ is an L^r vector field for some $r > 3$.

Proof. When $\rho \in L^1 \cap L^p(\mathbb{R}^3)$ it follows from the interpolation formula that $\rho \in L^{p'}$ for every $1 < p' < p$, in particular for $p' < 3$. Thus Lemmas II.4 and IV.1 imply that ∇U_ρ^N and ∇U_ρ^λ are both well defined and that $\nabla U_\rho^N \in L^q(\mathbb{R}^3)$ for some $q > 3/2$ and $\nabla U_\rho^\lambda \in L^r(\mathbb{R}^3)$ for some $r > 3$. □

To summarize, we have now proven that the potential U_ρ^M as given by Eq. (3) is really well defined. It is once weakly differentiable and its gradient ∇U_ρ^M is the irrotational part of the vector field $\nabla U_\rho^N + \lambda(|\nabla U_\rho^N|)\nabla U_\rho^N$ in the sense of the Helmholtz–Weyl decomposition. Further, when we write $\nabla U_\rho^M = \nabla U_\rho^N + \nabla U_\rho^\lambda$, we have that ∇U_ρ^N is an L^p vector field for some $p > 3/2$ and that ∇U_ρ^λ is an L^p vector field for some $p > 3$. Lastly, from the definition of the operator H (Definition III.1) we have a new explicit formula for ∇U_ρ^λ using singular integral operators. And this new formula is not only useful to analyze the regularity of ∇U_ρ^λ as done above but it is also very useful to verify that U_ρ^M is really a solution of the PDE (2) from the introduction. This we do in the last lemma of this section.

Lemma IV.3. Let λ be as in Lemma IV.1 and $\rho \in L^1 \cap L^p(\mathbb{R}^3)$ for some $p > 1$, then U_ρ^M solves the PDE

$$\operatorname{div}\left(\nabla U_\rho^M\right) = \operatorname{div}\left(\nabla U_\rho^N + \lambda\left(|\nabla U_\rho^N|\right)\nabla U_\rho^N\right)$$

in distribution sense, i.e., for every $\phi \in C_c^\infty(\mathbb{R}^3)$

$$\int \nabla U_\rho^M \cdot \nabla\phi \,\mathrm{d}x = \int \left(\nabla U_\rho^N + \lambda\left(|\nabla U_\rho^N|\right)\nabla U_\rho^N\right) \cdot \nabla\phi \,\mathrm{d}x.$$

Proof. We have

$$\nabla U_\rho^M = H\left(\nabla U_\rho^N + \lambda\left(|\nabla U_\rho^N|\right)\nabla U_\rho^N\right).$$

Since ∇U_ρ^N is already a gradient, we get $H(\nabla U_\rho^N) = \nabla U_\rho^N$. Thus

$$\begin{aligned}\int \nabla U_\rho^M \cdot \nabla\phi \,\mathrm{d}x &= \int H\left(\nabla U_\rho^N + \lambda\left(|\nabla U_\rho^N|\right)\nabla U_\rho^N\right) \cdot \nabla\phi \,\mathrm{d}x \\ &= \int \left[\nabla U_\rho^N + H\left(\lambda\left(|\nabla U_\rho^N|\right)\nabla U_\rho^N\right)\right] \cdot \nabla\phi \,\mathrm{d}x.\end{aligned}$$

The operator H is symmetric (Lemma III.6) and hence

$$\int \nabla U_\rho^M \cdot \nabla\phi \,\mathrm{d}x = \int \nabla U_\rho^N \cdot \nabla\phi \,\mathrm{d}x + \int \lambda\left(|\nabla U_\rho^N|\right)\nabla U_\rho^N \cdot H(\nabla\phi)\mathrm{d}x.$$

$H(\nabla\phi) = \nabla\phi$ since $\nabla\phi$ is already a gradient. Thus it follows that

$$\begin{aligned}\int \nabla U_\rho^M \cdot \nabla\phi \,\mathrm{d}x &= \int \nabla U_\rho^N \cdot \nabla\phi \,\mathrm{d}x + \int \lambda\left(|\nabla U_\rho^N|\right)\nabla U_\rho^N \cdot \nabla\phi \,\mathrm{d}x \\ &= \int \left(\nabla U_\rho^N + \lambda\left(|\nabla U_\rho^N|\right)\nabla U_\rho^N\right) \cdot \nabla\phi \,\mathrm{d}x.\end{aligned}$$

This means that U_ρ^M solves the PDE

$$\operatorname{div}\left(\nabla U_\rho^M\right) = \operatorname{div}\left(\nabla U_\rho^N + \lambda\left(|\nabla U_\rho^N|\right)\nabla U_\rho^N\right)$$

in distribution sense. □

Taking all statements of the lemmas proven in this section together implies that Theorem I.1 from the introduction holds.

V. SECOND DERIVATIVES OF MONDIAN POTENTIALS

In the previous section we have shown that in QUMOND the potential U_ρ^M, which corresponds to some density ρ on $\mathbb{R}^3$, is well defined, once weakly differentiable and solves the second order PDE (2) in distribution sense. Does U_ρ^M also have second order derivatives?

Consider a density $\rho \in L^1 \cap L^\infty(\mathbb{R}^3)$ with finite support. Such a density is a reasonable model for the distribution of mass in systems like globular clusters or galaxies. If we have such a density the interpolation formula and Lemma II.4 tell us that the second derivatives of the corresponding Newtonian potential are L^p-functions:

$$D^2 U_\rho^N \in L^p(\mathbb{R}^3)$$

for every $p \in (1, \infty)$. Therefore

$$\nabla U_\rho^N \in C^{0,\alpha}(\mathbb{R}^3)$$

for every $\alpha \in (0,1)$ by Morrey's inequality; the space $C^{0,\alpha}(\mathbb{R}^3)$ denotes the space of bounded and Hölder continuous functions with Hölder exponent α on $\mathbb{R}^3$. Under quite general assumptions (see Lemma V.2) the function

$$\mathbb{R}^3 \ni u \mapsto \lambda(|u|)u \in \mathbb{R}^3$$

is Hölder continuous with Hölder exponent $\frac{1}{2}$. Thus

$$\lambda(|\nabla U_\rho^N|)\nabla U_\rho^N \in C^{0,\beta}(\mathbb{R}^3)$$

for every $\beta \in (0, \frac{1}{2})$. Assuming for simplicity that we are in spherical symmetry, we have by Lemma III.5

$$\nabla U_\rho^\lambda = \lambda(|\nabla U_\rho^N|)\nabla U_\rho^N \in C^{0,\beta}(\mathbb{R}^3).$$

Taking a second look on Morrey's inequality one could now expect that

$$D^2 U_\rho^\lambda \in L^p(\mathbb{R}^3)$$

for all $1 < p < 6$. But this expectation proves deceptive. Why? Let us remain in the situation of spherical symmetry and for $\rho = \rho(x)$ spherically symmetric study the divergence of

$$\nabla U_\rho^M = \nabla U_\rho^N + \lambda(|\nabla U_\rho^N|)\nabla U_\rho^N.$$

In view of Lemma II.5

$$\lambda(|\nabla U_\rho^N(x)|)\nabla U_\rho^N(x) = \frac{\sqrt{M(r)}}{r}\frac{x}{r}, \quad r = |x|,$$

where for convenience we assumed $\lambda(\sigma) = 1/\sqrt{\sigma}$, $\sigma > 0$. So

$$\begin{aligned}\operatorname{div}(\nabla U_\rho^M(x)) &= \Delta U_\rho^N(x) + \frac{1}{r^2}(r\sqrt{M(r)})' \\ &= 4\pi\rho(r) + \frac{\sqrt{M(r)}}{r^2} + \frac{\sqrt{M(r)}'}{r}.\end{aligned}$$

ρ is just fine, the second term can be controlled as expected above, but the third one will cause problems. We prove the following

Proposition V.1. Let $R > 0$, $1 < p, q < \infty$ and $\rho \in C^1 \cap L^p(\mathbb{R}^3)$, ≥ 0, spherically symmetric. Then

$$\left\| \frac{\sqrt{M(r)}}{r^2} \right\|_{L^q(B_R)} \leq C\|\rho\|_p^{1/2}$$

if $1 < q < 6$ and $p > 3q/(6-q)$ with $C = C(p,q,R) > 0$, and

$$\left\| \frac{\sqrt{M(r)}'}{r} \right\|_{L^q(B_R)} \leq C\|\rho\|_p^{1/2}$$

if $1 < q < 2$ and $p > q/(2-q) + q$ with $C = C(p,q,R) > 0$. With $\sqrt{M(r)}'$ we denote the function

$$\sqrt{M(r)}' := \begin{cases} \dfrac{2\pi r^2\rho(r)}{\sqrt{M(r)}}, & \text{if } M(r) > 0 \\ 0, & \text{if } M(r) = 0. \end{cases}$$

Proof. Let $1 < q < 6$ and $1 < p < \infty$ with $p > 3q/(6-q)$. For $r \geq 0$

$$M(r) = \int_{B_r} \rho(x)\mathrm{d}x \leq \|\rho\|_p \|1_{B_r}\|_{p/(p-1)} \leq C\|\rho\|_p r^{3-3/p}.$$

Thus

$$\left\| \frac{\sqrt{M(r)}}{r^2} \right\|_{L^q(B_R)}^q = \int_{B_R} M(r)^{q/2} r^{-2q}\,\mathrm{d}x \leq C\|\rho\|_p^{q/2} \int_{B_R} r^{\frac{3q}{2}-\frac{3q}{2p}-2q}\,\mathrm{d}x.$$

Since

$$\frac{3q}{2} - \frac{3q}{2p} - 2q > -3 \Leftrightarrow 3 - \frac{q}{2} > \frac{3q}{2p} \Leftrightarrow p > \frac{3q}{6-q}$$

we have

$$\left\| \frac{\sqrt{M(r)}}{r^2} \right\|_{L^q(B_R)} \leq C\|\rho\|_p^{1/2}.$$

Now we turn to the second estimate. Let $1 < q < 2$, $p > q/(2-q) + q$ and $r_0 \geq 0$ be such that $M(r_0) = 0$ and $M(r) > 0$ for all $r > r_0$. Since $\rho \in C^1(\mathbb{R}^3)$, $M(r) \in C^1\left([0,\infty)\right)$ with

$$M'(r) = \frac{\mathrm{d}}{\mathrm{d}r} 4\pi \int_0^r s^2 \rho(s)\mathrm{d}s = 4\pi r^2 \rho(r).$$

Hence $\sqrt{(M(r))} \in C^1((r_0,\infty))$ with

$$\sqrt{M(r)}' = \frac{2\pi r^2 \rho(r)}{\sqrt{M(r)}}, \qquad r > r_0.$$

Assume that $R > r_0$, then

$$\begin{aligned}\left\| \frac{\sqrt{M(r)}'}{r} \right\|_{L^q(B_R\setminus B_{r_0})}^q &= \int_{B_R\setminus B_{r_0}} \left(\frac{2\pi r \rho(r)}{\sqrt{M(r)}} \right)^q \mathrm{d}x \\ &\leq (2\pi R)^q \int_{B_R\setminus B_{r_0}} \rho(r)^\alpha \frac{\rho(r)^{q-\alpha}}{M(r)^{q/2}}\mathrm{d}x\end{aligned}$$

with

$$\alpha := \frac{p}{p-1}(q-1).$$

Obviously $\alpha > 0$, and further $\alpha < q$ since

$$\alpha = \frac{p}{p-1}(q-1) < q \Leftrightarrow 1 - \frac{1}{q} < 1 - \frac{1}{p} \Leftrightarrow q < p.$$

Now we apply Hölder's inequality and get

$$\left\| \frac{\sqrt{M(r)}'}{r} \right\|_{L^q(B_R\setminus B_{r_0})}^q \leq C\|\rho\|_p^\alpha \left\| \frac{\rho(r)^{q-\alpha}}{M(r)^{q/2}} \right\|_{L^{p/(p-\alpha)}(B_R\setminus B_{r_0})};$$

note that $0 < \alpha < q < p$. Since

$$(q-\alpha)\frac{p}{p-\alpha} = 1 \Leftrightarrow q - \alpha = 1 - \frac{\alpha}{p} \Leftrightarrow q - 1 = \alpha\left(1 - \frac{1}{p}\right) \Leftrightarrow \alpha = \frac{p}{p-1}(q-1),$$

we have

$$\begin{aligned}\left\| \frac{\rho(r)^{q-\alpha}}{M(r)^{q/2}} \right\|_{L^{p/(p-\alpha)}(B_R\setminus B_{r_0})} &= \left(\int_{B_R\setminus B_{r_0}} \rho(r) M(r)^{-pq/(2p-2\alpha)}\,\mathrm{d}x \right)^{(p-\alpha)/p} \\ &= C\left[\int_{r_0}^R \left(M(r)^{1-pq/(2p-2\alpha)} \right)' \mathrm{d}r \right]^{(p-\alpha)/p};\end{aligned}$$

here we have used that $pq/(2p-2\alpha) < 1$ since

$$\begin{aligned}\frac{pq}{2p-2\alpha} < 1 &\Leftrightarrow \frac{q}{2} < 1 - \frac{\alpha}{p} = 1 - \frac{q-1}{p-1}\\ &\Leftrightarrow \frac{2-q}{2} > \frac{q-1}{p-1}\\ &\Leftrightarrow p > 1 + \frac{2(q-1)}{2-q} = \frac{2-q+2q-2}{2-q} = \frac{q}{2-q}.\end{aligned}$$

Thus

$$\left\| \frac{\sqrt{M(r)}'}{r} \right\|^q_{L^q(B_R\setminus B_{r_0})} \le C\|\rho\|_p^{\alpha}\|\rho\|_{L^1(B_R)}^{(p-\alpha)/p-q/2}.$$

Since

$$\|\rho\|_{L^1(B_R)} \le C\|\rho\|_p$$

and

$$\begin{aligned}\frac{1}{q}\left(\alpha + \frac{p-\alpha}{p} - \frac{q}{2}\right) &= \frac{q-1}{q}\frac{p}{p-1} + \frac{1}{q}\left(1 - \frac{q-1}{p-1}\right) - \frac{1}{2}\\ &= \frac{(q-1)p + (p-q)}{q(p-1)} - \frac{1}{2}\\ &= \frac{pq-q}{q(p-1)} - \frac{1}{2}\\ &= \frac{1}{2},\end{aligned}$$

we finally have

$$\left\| \frac{\sqrt{M(r)}'}{r} \right\|_{L^q(B_R\setminus B_{r_0})} \le C\|\rho\|_q^{1/2}.$$

□

Now we can analyze derivatives of $\lambda(|\nabla U_\rho^N|)\nabla U_\rho^N$. However, before we do so, we need to strengthen the assumptions on λ that we made in Lemma IV.1. This we do in the next lemma, where we take a look on the derivative λ'. The assumptions of the next lemma imply that λ has the same regularity as in Lemma IV.1 and that additionally the function $\mathbb{R}^3 \ni u \mapsto \lambda(|u|)u$ is Hölder continuous.

Lemma V.2. Assume that $\lambda \in C^1((0,\infty))$, $\lambda(\sigma) \to 0$ *as* $\sigma \to \infty$ *and there is* $\Lambda > 0$ *such that* $-\Lambda/(2\sigma^{3/2}) \le \lambda'(\sigma) \le 0$, *for* $\sigma > 0$. *Then*

$$\lambda(\sigma) \le \Lambda/\sqrt{\sigma}$$

for every $\sigma > 0$ *(as in Lemma IV.1) and there is a* $C > 0$ *such that for all* $u, v \in \mathbb{R}^3$

$$|\lambda(|u|)u - \lambda(|v|)v| \le C|u-v|^{1/2}$$

with $\lambda(|u|)u = 0$ *if* $u = 0$.

We postpone the proof of this Lemma to the Appendix and return our attention to the analysis of the second derivatives of U_ρ^M. Using Proposition V.1 we can control L^q-norms of the derivatives of the Mondian part

$$\lambda(|\nabla U_\rho^N|)\nabla U_\rho^N$$

of the field ∇U_ρ^M provided $1 < q < 2$ and $\rho \ge 0$ is spherically symmetric.

Lemma V.3. Let $1 < q < 2$, $p > \frac{q}{2-q} + q$, $R > 0$ *and* $\rho \in L^1 \cap L^p(\mathbb{R}^3)$, ≥ 0, *spherically symmetric. Assume that* λ *is as in Lemma V.2, then*

$$\lambda(|\nabla U_\rho^N|)\nabla U_\rho^N \in W^{1,q}_{loc}(\mathbb{R}^3)$$

with

$$\left\|\nabla\left[\lambda(|\nabla U_\rho^N|)\nabla U_\rho^N\right]\right\|_{L^q(B_R)} \le C\|\rho\|_p^{1/2}$$

where $C = C(p,q,R) > 0$.

Proof. Since we are in spherical symmetry, Lemma II.5 gives

$$\lambda(|\nabla U_\rho^N|)\nabla U_\rho^N = \lambda\left(\frac{M(r)}{r^2}\right)\frac{M(r)}{r^2}\frac{x}{r}, \quad x\in\mathbb{R}^3,\ r=|x|;$$

for better readability we suppress the x-argument on the left side. Using the abbreviation

$$\tilde\lambda(\sigma) = \lambda(\sigma)\sigma, \quad \sigma \ge 0,$$

we have

$$\lambda(|\nabla U_\rho^N|)\nabla U_\rho^N = \tilde\lambda\left(\frac{M(r)}{r^2}\right)\frac{x}{r}.$$

Thanks to Lemma V.2

$$|\tilde\lambda(\sigma) - \tilde\lambda(\tau)| \le C|\sigma-\tau|^{1/2}, \quad \sigma,\tau \ge 0, \tag{11}$$

for a $C > 0$ where

$$\tilde\lambda(0) = 0.$$

From this lemma follows further

$$0 \le \tilde\lambda(\sigma) \le \Lambda\sqrt{\sigma}, \quad \sigma \ge 0. \tag{12}$$

Using the bounds for λ and λ' we get

$$|\tilde\lambda'(\sigma)| \le |\lambda'(\sigma)|\sigma + \lambda(\sigma) \le \frac{C}{\sqrt{\sigma}}, \quad \sigma > 0, \tag{13}$$

for a $C > 0$. Thanks to Eq. (12), for every $R > 0$ holds

$$\left\|\lambda(|\nabla U_\rho^N|)\nabla U_\rho^N\right\|_{L^q(B_R)}^q \le \Lambda^q\int_{B_R}\left(\frac{\sqrt{M(r)}}{r}\right)^q \mathrm{d}x \le C\|\rho\|_1^{q/2}.$$

Next we approximate ρ by smooth densities ρ_n and study the (weak) derivatives of $\lambda(|\nabla U_{\rho_n}^N|)\nabla U_{\rho_n}^N$. Let $(\rho_n) \subset C_c^1(\mathbb{R}^3)$ be a sequence of spherically symmetric densities such that

$$\rho_n \to \rho \quad \text{strongly in } L^1(\mathbb{R}^3) \text{ and } L^p(\mathbb{R}^3) \text{ for } n\to\infty.$$

As above $\lambda(|\nabla U_{\rho_n}^N|)\nabla U_{\rho_n}^N \in L^q_{loc}(\mathbb{R}^3)$. Denote by

$$M_n(r) = \int_{B_r}\rho_n\,\mathrm{d}x, \quad r \ge 0,$$

the mass of ρ_n inside the ball with radius r. Then $M_n \in C^1(\mathbb{R}^3)$ with

$$\nabla(M_n(r)) = M_n'(r)\frac{x}{r} = 4\pi\rho_n(r)rx.$$

Let $r_n \ge 0$ be such that $M_n(r_n) = 0$ and $M_n(r) > 0$ for all $r > r_n$. Then

$$\lambda(|\nabla U_{\rho_n}^N|)\nabla U_{\rho_n}^N \in C^1(\mathbb{R}^3\setminus\{|x| = r_n\})$$

with

$$\partial_{x_i}\left[\lambda(|\nabla U_{\rho_n}^N|)\partial_{x_j}U_{\rho_n}^N\right] = 0 \tag{14}$$

if $|x| < r_n$, and

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$$\partial_{x_i}\big[\lambda(|\nabla U^N_{\rho_n}|)\partial_{x_j}U^N_{\rho_n}\big] = \partial_{x_i}\left(\tilde\lambda\left(\frac{M_n(r)}{r^2}\right)\frac{x_j}{r}\right)$$
$$= \tilde\lambda'\left(\frac{M_n(r)}{r^2}\right)M_n'(r)\frac{x_ix_j}{r^4}$$
$$- 2\tilde\lambda'\left(\frac{M_n(r)}{r^2}\right)M_n(r)\frac{x_ix_j}{r^5} \tag{15}$$
$$+ \tilde\lambda\left(\frac{M_n(r)}{r^2}\right)\left(\frac{\delta_{ij}}{r} - \frac{x_ix_j}{r^3}\right)$$

if $|x| > r_n$ and $i, j = 1, 2, 3$. Denote by

$$\partial_{x_i}\big[\lambda(|\nabla U^N_{\rho_n}|)\partial_{x_j}U^N_{\rho_n}\big]$$

the functions that are pointwise almost everywhere on $\mathbb{R}^3$ defined by Eqs. (14) and (15); the only region where the $\partial_{x_i}[\ldots]$ are not defined by Eqs. (14) and (15) is $\{|x| = r_n\}$ where they might have a discontinuity. Using Eqs. (12) and (13) we get for $|x| > r_n$

$$\big|\partial_{x_i}\big[\lambda(|\nabla U^N_{\rho_n}|)\partial_{x_j}U^N_{\rho_n}\big]\big| \le C\left(\frac{M_n'(r)}{2\sqrt{M_n(r)}}\frac{1}{r} + \frac{\sqrt{M_n(r)}}{r^2}\right)$$
$$= C\left(\frac{\sqrt{M_n(r)}'}{r} + \frac{\sqrt{M_n(r)}}{r^2}\right).$$

Since $p > q/(2-q) + q$ and

$$\frac{q}{2-q} = \frac{3q}{6-3q} > \frac{3q}{6-q},$$

we can apply Proposition V.1 and get for every $R > 0$

$$\big\|\partial_{x_i}\big[\lambda(|\nabla U^N_{\rho_n}|)\partial_{x_j}U^N_{\rho_n}\big]\big\|_{L^q(B_R)} \le C\|\rho_n\|_p^{1/2}. \tag{16}$$

Now we prove that the functions given by Eqs. (14) and (15) are indeed the weak derivatives of $\lambda(|\nabla U^N_{\rho_n}|)\nabla U^N_{\rho_n}$. For every $\phi \in C_c^\infty(\mathbb{R}^3)$

$$\int \lambda(|\nabla U^N_{\rho_n}|)\partial_{x_j}U^N_{\rho_n}\partial_{x_i}\phi\,dx = \lim_{s\searrow r_n}\int_{\{|x|\ge s\}}\lambda(|\nabla U^N_{\rho_n}|)\partial_{x_j}U^N_{\rho_n}\partial_{x_i}\phi\,dx$$
$$= -\int \partial_{x_i}\big(\lambda(|\nabla U^N_{\rho_n}|)\partial_{x_j}U^N_{\rho_n}\big)\phi\,dx$$
$$+ \lim_{s\searrow r_n}\int_{\{|x|=s\}}\lambda(|\nabla U^N_{\rho_n}|)\partial_{x_j}U^N_{\rho_n}\phi\frac{x_i}{|x|}\,dS(x).$$

If $r_n = 0$, we use

$$\big|\lambda(|\nabla U^N_{\rho_n}|)\nabla U^N_{\rho_n}\big| \le \frac{\Lambda\sqrt{M_n(r)}}{r} \le \frac{\Lambda\|\rho\|_1^{1/2}}{r}$$

and get

$$\left|\int_{\{|x|=s\}}\lambda(|\nabla U^N_{\rho_n}|)\partial_{x_j}U^N_{\rho_n}\phi\frac{x_i}{|x|}\,dS(x)\right| \le Cs \to 0 \quad \text{for } s \to 0.$$

If $r_n > 0$, we use

$$\big|\lambda(|\nabla U^N_{\rho_n}|)\nabla U^N_{\rho_n}\big| \le \frac{\Lambda}{r_n}\left(\int_{r_n<|x|<s}\rho_n\,dx\right)^{1/2} \to 0 \quad \text{for } s \to r_n,$$

and get, too, that the boundary term in the above integration by parts vanishes. Hence, the Eqs. (14) and (15) pointwise almost everywhere defined functions are indeed the weak derivatives of

$$\lambda(|\nabla U^N_{\rho_n}|)\nabla U^N_{\rho_n} \in W^{1,q}_{loc}(\mathbb{R}^3).$$

It remains to prove that

$$\lambda(|\nabla U^N_{\rho}|)\partial_{x_j}U^N_{\rho} \in W^{1,q}_{loc}(\mathbb{R}^3)$$

and that the estimate Eq. (16) holds with ρ_n replaced by ρ. Using Eq. (11) and Hölder we have for $R > 0$

$$\begin{aligned}\|\lambda(|\nabla U_{\rho_n}^N|)\nabla U_{\rho_n}^N - \lambda(|\nabla U_\rho^N|)\partial_{x_j} U_\rho^N\|_{L^1(B_R)} &= \int_{B_R}\left|\tilde{\lambda}\left(\frac{M(r)}{r^2}\right) - \tilde{\lambda}\left(\frac{M_n(r)}{r^2}\right)\right|\mathrm{d}x \\ &\le C\int_{B_R}\frac{|M(r)-M_n(r)|^{1/2}}{r}\mathrm{d}x \\ &\le C\left(\int_{B_R}|M_n(r)-M(r)|\mathrm{d}x\right)^{1/2} \\ &\le C\|\rho_n - \rho\|_1^{1/2}.\end{aligned}$$

Thus

$$\lambda(|\nabla U_{\rho_n}^N|)\nabla U_{\rho_n}^N \to \lambda(|\nabla U_\rho^N|)\partial_{x_j} U_\rho^N \quad \text{strongly in } L^1(B_R) \text{ for } n\to\infty.$$

Since

$$\|\rho_n\|_p \le C$$

independent of $n \in \mathbb{N}$, Eq. (16) implies that there is a subsequence [again denoted by (ρ_n)] such that

$$\partial_{x_i}\left[\lambda(|\nabla U_{\rho_n}^N|)\partial_{x_j} U_{\rho_n}^N\right] \rightharpoonup V \quad \text{weakly in } L^q(\mathbb{R}^3) \text{ for } n\to\infty.$$

V is the weak derivative of

$$\lambda(|\nabla U_\rho^N|)\partial_{x_j} U_\rho^N$$

with respect to x_i and hence

$$\lambda(|\nabla U_\rho^N|)\nabla U_\rho^N \in W_{loc}^{1,q}(\mathbb{R}^3).$$

Since the L^q-norm is weakly lower semi-continuous, Eq. (16) implies

$$\begin{aligned}\|V\|_{L^q(B_R)} &\le \liminf_{n\to\infty}\left\|\partial_{x_i}\left[\lambda(|\nabla U_{\rho_n}^N|)\partial_{x_j} U_{\rho_n}^N\right]\right\|_{L^q(B_R)} \\ &\le C\lim_{n\to\infty}\|\rho_n\|_p^{1/2} \\ &= C\|\rho\|_p^{1/2}.\end{aligned}$$

□

Thus, in spherical symmetry it follows from Lemma V.3 (and Lemma II.4) that the Mondian potential U_ρ^M is always twice weakly differentiable. But, as we have argued in the introduction to this section, one might expect from a naïve argumentation that

$$D^2 U_\rho^M \in L^q(\mathbb{R}^3)$$

for $1 < q < 6$ if

$$\rho \in L^1 \cap L^\infty(\mathbb{R}^3).$$

However, in Lemma V.3 we were only able to prove an estimate of the type

$$\|D^2 U_\rho^M\|_q \le C\|\rho\|_p^{1/2}$$

if $1 < q < 2$. In the following Lemma we show that this estimate is indeed optimal; there is no such estimate if $q > 2$. The subsequent Lemma will further show that it is unlikely that any such estimate can be proven if we drop the assumption of spherical symmetry.

Lemma V.4. Let $\lambda(\sigma) = 1/\sqrt{\sigma}$, $\sigma > 0$. Then there is a sequence of spherically symmetric densities $(\rho_n) \subset L^1 \cap L^\infty(\mathbb{R}^3)$ such that for all $n \in \mathbb{N}$ $\rho_n \ge 0$, $\operatorname{supp}\rho_n \subset B_2$ *and $\|\rho_n\|_\infty \le 1$, but*

$$\|D^2 U_{\rho_n}^M\|_q \to \infty \qquad \text{for } n\to\infty$$

if $2 < q < 6$.

Remark V.5. The idea behind the Proof of Lemma V.4 is the following: In ΔU_ρ^M appears the term

$$\frac{\sqrt{M(r)}'}{r} = \frac{2\pi r\rho(r)}{\sqrt{M(r)}} = \frac{2\pi\rho(r)}{\sqrt{N(r)}}\frac{1}{\sqrt{r}}$$

where we have introduced the notion

$$N(r) := \frac{1}{r^3}M(r) = \frac{1}{r^3}\int_{B_r}\rho(x)\mathrm{d}x.$$

Reference 18, Satz 7.7 implies that for a.e. $y \in \mathbb{R}^3$

$$\frac{1}{r^3}\int_{B_r(y)}\rho(x)\mathrm{d}x \to \frac{4\pi}{3}\rho(y) \quad \text{for } r \to 0.$$

So we could expect that

$$\frac{\sqrt{M(r)}'}{r} \approx \frac{\sqrt{3\pi}\rho(r)}{\sqrt{\rho(0)}}\frac{1}{\sqrt{r}} \quad \text{for } r > 0 \text{ small.}$$

Assuming for the moment that $\rho(0) > 0$ and that $\|\rho\|_\infty < \infty$ this would guarantee that $\|\sqrt{M(r)}'/r\|_q$ is bounded for all $1 < q < 6$. Together with Proposition V.1 this would give us a bound for $\|D^2 U_\rho^M\|_q$ for all $1 < q < 6$. However, the pointwise representation of an L^p-function ρ is tricky:

Let us take an open set $\Omega_n \subset [0,2]$ such that for all $\epsilon > 0$

$$\mathcal{L}(\Omega_n \cap [0,\epsilon]) \approx \frac{\epsilon}{n}$$

and set

$$\rho_n(r) := 1_{\Omega_n}(r).$$

Then there is no well defined value of $\rho(0)$ and we get

$$N(r) \approx \frac{C}{n} \quad \text{for } r > 0 \text{ small}$$

with a constant $C > 0$ independent of n. Thus

$$\frac{\sqrt{M(r)}'}{r} \approx \frac{2\pi}{\sqrt{C}}\sqrt{n}1_{\Omega_n}(r)\frac{1}{\sqrt{r}} \quad \text{for } r > 0 \text{ small,}$$

and when we send $n \to \infty$ this is unbounded in L^q for $2 < q < 6$.

Proof of Lemma V.4. For $n \in \mathbb{N}$ set

$$\Omega_n := \bigcup_{i=0}^{\infty}\left[2^{-i}, \left(1+\frac{1}{n}\right)2^{-i}\right)$$

and define

$$\rho_n(r) := \frac{1}{4\pi}1_{\Omega_n}(r), \quad r \geq 0.$$

Denote by

$$M_n(r) := \int_{B_r}\rho_n\,\mathrm{d}x$$

the mass of ρ_n inside the ball with radius $r \geq 0$. Let $n, j \in \mathbb{N}$, then

$$\begin{aligned}M_n(2^{-j+1}) &= \sum_{i=j}^{\infty}\int_{2^{-i}}^{(1+1/n)2^{-i}} r^2\,\mathrm{d}r \leq \sum_{i=j}^{\infty}\frac{1}{n}2^{-i}(2^{-i+1})^2 = \frac{4}{n}\sum_{i=j}^{\infty}\left(\frac{1}{8}\right)^i \\ &= \frac{4}{n}\left(\frac{1}{1-1/8} - \frac{1-(1/8)^j}{1-1/8}\right) = \frac{4}{n}\left(\frac{1}{8}\right)^j\frac{8}{7} = \frac{C_0}{n}(2^{-j})^3.\end{aligned}$$

Let $r \in [2^{-j}, 2^{-j+1})$ for a $j \geq 0$. Then

$$M_n(r) \leq M_n(2^{-j+1}) \leq \frac{C_0}{n}(2^{-j})^3 \leq \frac{C_0}{n}r^3.$$

Thus

$$N_n(r) := \frac{1}{r^3} M_n(r) \le \frac{C_0}{n}$$

and

$$\frac{\rho_n(r)}{\sqrt{N_n(r)}} \ge \frac{\sqrt{n}}{4\pi\sqrt{C_0}} 1_{\Omega_n}(r). \tag{17}$$

Let now $2 < q < 6$, then

$$\|D^2 U^M_{\rho_n}\|_q \ge C\|\Delta U^M_{\rho_n}\|_q = C\left\| 4\pi\rho_n(r) + \frac{\sqrt{M_n(r)}}{r^2} + \frac{\sqrt{M_n(r)}'}{r} \right\|_q .$$

Since $\rho_n, M_n \ge 0$ and M_n is monotonic increasing

$$\|D^2 U^M_{\rho_n}\|_q \ge C\left\| \frac{\sqrt{M_n(r)}'}{r} \right\|_q = C\left\| \frac{r\rho_n(r)}{\sqrt{M_n(r)}} \right\|_q = C\left\| \frac{\rho_n(r)}{\sqrt{N_n(r)}} r^{-1/2} \right\|_q .$$

Now we use the estimate Eq. (17) and get

$$\begin{aligned} \|D^2 U^M_{\rho_n}\|_q &\ge C\sqrt{n}\left\| r^{-1/2} 1_{\Omega_n}(r) \right\|_q \\ &= C\sqrt{n}\left(\int_{\Omega_n} r^{2-q/2}\, \mathrm{d}r \right)^{1/q} \\ &= C\sqrt{n}\left(\sum_{i=0}^{\infty} \int_{2^{-i}}^{(1+1/n)2^{-i}} r^{2-q/2}\, \mathrm{d}r \right)^{1/q} . \end{aligned}$$

For $2 < q \le 4$ we have

$$\sum_{i=0}^{\infty} \int_{2^{-i}}^{(1+1/n)2^{-i}} r^{2-q/2}\, \mathrm{d}r \ge \sum_{i=0}^{\infty} \frac{1}{n} 2^{-i} (2^{-i})^{2-q/2} = \frac{1}{n} \sum_{i=0}^{\infty} (2^{-3+q/2})^i$$

and for $4 < q < 6$

$$\sum_{i=0}^{\infty} \int_{2^{-i}}^{(1+1/n)2^{-i}} r^{2-q/2}\, \mathrm{d}r \ge \sum_{i=0}^{\infty} \frac{1}{n} 2^{-i} (2^{-i+1})^{2-q/2} = \frac{1}{n} 2^{2-q/2} \sum_{i=0}^{\infty} (2^{-3+q/2})^i .$$

Hence

$$\|D^2 U^M_{\rho_n}\|_q \ge C n^{1/2 - 1/q},$$

and this is divergent if $q > 2$. □

So it is not possible for any $q > 2$ to prove an estimate of the form

$$\|D^2 U^M_\rho\|_{L^q(B_R)} \le C\|\rho\|_p^{1/2}$$

even if ρ is spherically symmetric (and non-negative). Will the situation get even worse if we leave spherical symmetry?

Let us look at the difficulties that one can encounter. $D^2 U^M_\rho$ causes difficulties when $\nabla U^N_\rho(x) = 0$ for an $x \in \mathbb{R}^3$ because then

$$\lambda\left(\left|\nabla U^N_\rho(x+y)\right|\right)\left|\nabla U^N_\rho(x+y)\right| = \left|\nabla U^N_\rho(x+y)\right|^{1/2} \approx C\sqrt{y}$$

if $|y|$ is small and $\lambda(\sigma) = 1/\sqrt{\sigma}$ for $\sigma > 0$. Consider now the following, non-spherically symmetric case: For every $n \in \mathbb{N}$ place a point mass at position

$$x_n = (1 - 1/n, 0, 0).$$

Then for every $n \in \mathbb{N}$ there is $0 < \alpha_n < 1$ such that for

$$y_n = \alpha_n x_n + (1 - \alpha_n) x_{i+1}$$

we have

$$\nabla U^N(y_n) = 0;$$

U^N denotes the Newtonian gravitational potential created by all the masses at the points x_n. So for every $n \in \mathbb{N}$ $D^2 U^N(y_n)$ will cause difficulties.

The exact treatment of such a non-spherically symmetric case is difficult. Can we perhaps mimic the above difficulties in spherical symmetry? The answer is yes, if we do not demand that ρ has to be non-negative. Then the next lemma shows that it is no more possible for any $1 \le p, q \le \infty$ to prove an estimate of the form

$$\|D^2 U_\rho^M\|_{L^q(B_R)} \le C\|\rho\|_p^{1/2}.$$

Lemma V.6. Let $\lambda(\sigma) = 1/\sqrt{\sigma}$, $\sigma > 0$. Then there exists a $\rho \in L^1 \cap L^\infty(\mathbb{R}^3)$, spherically symmetric, which takes positive and negative values, such that

$$\nabla U_\rho^M \notin W_{loc}^{1,1}(\mathbb{R}^3).$$

Proof. For $n \in \mathbb{N}$ set

$$a_n := \sum_{i=1}^n \frac{2}{i^2}$$

and let m_n be the center between a_n and a_{n+1}, i.e.,

$$m_n := a_n + \frac{1}{(n+1)^2}.$$

Then $a_1 = 2$ and

$$a_n \to \frac{\pi^2}{3} < 4 \quad \text{for } n \to \infty.$$

Set $M(r) := 0$ if $r \in [0, 2)$ or $r \in [\pi^2/3, \infty)$. If $r \in [2, \pi^2/3)$ set

$$M(r) := \begin{cases} \alpha & \text{if } r \in [a_n, m_n) \text{ and } r = a_n + \alpha \\ 1/(n+1)^2 - \alpha & \text{if } r \in [m_n, a_{n+1}) \text{ and } r = m_n + \alpha. \end{cases}$$

Then M is continuous and

$$M(a_n) = 0, \qquad M(m_n) = \frac{1}{(n+1)^2}. \tag{18}$$

Set $\rho(r) := 0$ if $r \in [0, 2)$ or $r \in [\pi^2/3, \infty)$. If $r \in [2, \pi^2/3)$ set

$$\rho(r) := \begin{cases} 1/(4\pi r^2) & \text{if } r \in [a_n, m_n) \\ -1/(4\pi r^2) & \text{if } r \in [m_n, a_{n+1}). \end{cases}$$

Then $\rho \in L^1 \cap L^\infty(\mathbb{R}^3)$. Further for $r \ge 0$

$$M'(r) = 4\pi r^2 \rho(r)$$

and thus

$$M(r) = \int_0^r 4\pi s^2 \rho(s) \mathrm{d}s = \int_{B_r} \rho \, \mathrm{d}x.$$

In view of Eq. (18)

$$\nabla U_\rho^N(x) = \frac{M(r)}{r^2} \frac{x}{r}$$

will have a zero for all $x = (a_n, 0, 0)$, $n \in \mathbb{N}$. Let us see how this troubles the second derivatives of the Mondian potential:

As in the introduction to this section we have

$$\operatorname{div}(\nabla U_\rho^M(x)) = \rho(x) + \frac{1}{r^2}(r\sqrt{M(r)})'$$

for $r = |x| > 0$. But

$$\frac{1}{r^2}(r\sqrt{M(r)})' \notin L^1(B_4)$$

since

$$\int_{B_4}\left|\frac{1}{r^2}(r\sqrt{M(r)})'\right|\mathrm{d}x = 4\pi\int_0^4\left|(r\sqrt{M(r)})'\right|\mathrm{d}r$$
$$= 4\pi\sum_{i=1}^{\infty}\left[\int_{a_n}^{m_n}(r\sqrt{M(r)})'\,\mathrm{d}r - \int_{m_n}^{a_{n+1}}(r\sqrt{M(r)})'\,\mathrm{d}r\right]$$
$$= 8\pi\sum_{i=1}^{\infty} m_n\sqrt{M(m_n)} \geq 8\pi\sum_{i=1}^{\infty}\frac{1}{n+1} = \infty.$$

Hence

$$\operatorname{div}(\nabla U_\rho^M) \notin L^1(B_4)$$

and

$$\nabla U_\rho^M \notin W^{1,1}_{loc}(\mathbb{R}^3).$$

□

Since the density ρ constructed in Lemma V.6 mimics the difficulties that one can encounter in a situation without symmetry assumptions, we suspect that it is impossible to prove the existence of weak, integrable derivatives of ∇U_ρ^M for general $\rho \in L^1 \cap L^\infty(\mathbb{R}^3)$, ≥0. Thus the assumption of spherical symmetry in Lemma V.3 seems indeed to be necessary if one wants to prove that U_ρ^M is twice weakly differentiable.

VI. DISCUSSION

We have conducted an extensive analysis of the QUMOND theory, focusing initially on the gradient ∇U_ρ^M of the Mondian potential instead of directly studying the potential U_ρ^M. Our investigation reveals that this gradient is the irrotational part of the vector field (1) in the sense of the Helmholtz–Weyl decomposition. This assures that ∇U_ρ^M is an L^p vector field and indeed the weak gradient of a potential. Our findings show that the corresponding potential is given by Eq. (3), which was introduced by Milgrom,[9] and that it is well defined.

These results were attained through a careful examination of the operator H responsible for extracting the irrotational part of a vector field. We developed a new, explicit expression for this operator using singular integral operators. Using the operator H also significantly aided in demonstrating that the Mondian potential solves the PDE (2) in distribution sense. Thus by linking the QUMOND theory with the Helmholtz–Weyl decomposition, we established a robust mathematical foundation for QUMOND.

Furthermore, we investigated second-order derivatives of the Mondian potential U_ρ^M. Under the additional assumption of spherical symmetry, we proved that the Mondian potential is twice weakly differentiable. However, the regularity of the second derivatives was found to be weaker than anticipated. Additionally, we illustrated why proving a similarly general regularity result for the second derivatives without symmetry assumptions seems impossible.

Our findings can be applied to many problems in QUMOND. For instance, in an accompanying paper,[19] we establish the stability of a large class of spherically symmetric models. The perturbations permitted are still confined to spherical symmetry and removing this restriction draws heavily upon the results presented in this paper, a discussion of which is provided in the accompanying work. Moreover, our results can be applied to analyze initial value problems. Recent work by Carina Keller in her master's thesis demonstrates the existence of global weak solutions to the initial value problem for the collisionless Boltzmann equation. Her result is limited to spherically symmetric solutions. Generalizing it to solutions devoid of symmetry restrictions necessitates the use of the theory presented here and a further generalization of it: We have to use that the operator H also preserves Hölder continuity. This is work in progress.

Our research contributes to the investigation of solutions to the initial value problem for the collisionless Boltzmann equation in yet another way. Building upon the theory from DiPerna and Lions,[20] we have established that weak Lagrangian solutions conserve energy. This unpublished result, not imposing any symmetry restrictions, heavily relies on the results proven in this paper. Further, the question of whether every Eulerian solution of the collisionless Boltzmann equation is also a Lagrangian one, and vice versa, is of considerable interest. DiPerna and Lions[20] have shown that this equivalence holds if the Mondian potential has second-order weak derivatives. Thus, our findings confirm this equivalence for spherically symmetric solutions, but cast doubt on extending this conclusion to nonsymmetric scenarios.

In summary, with QUMOND now placed on a robust mathematical foundation, it is possible to analyze many interesting yet unsolved questions with mathematical rigor.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Joachim Frenkler: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: HÖLDER CONTINUITY OF $\mathbb{R}^3 \ni u \mapsto \lambda(|u|)u$

We omitted the Proof of Lemma V.2 and we give it now in the Appendix.

Proof of Lemma V.2. Let $\sigma > 0$, then

$$\lambda(\sigma) = -\int_\sigma^\infty \lambda'(s)\mathrm{d}s \le \frac{\Lambda}{2}\int_\sigma^\infty \frac{\mathrm{d}s}{s^{3/2}} = \frac{\Lambda}{\sqrt{\sigma}}$$

as desired. Further, the function $\lambda(|u|)u$ is continuously differentiable on $\mathbb{R}^3\backslash\{0\}$, and for $u \in \mathbb{R}^3$, $u \ne 0$, holds

$$D(\lambda(|u|)u) = \lambda(|u|)E_3 + \lambda'(|u|)\frac{uu^T}{|u|}$$

where E_3 denotes the identity matrix of dimension 3. Using the bounds for λ and λ', we have

$$|D(\lambda(|u|)u)| \le \frac{C}{\sqrt{|u|}}.$$

Let now $u, v \in \mathbb{R}^3$ be such that for all $t \in [0,1]$

$$w_t := v + t(u - v)$$

is different from zero. Then

$$|\lambda(|u|)u - \lambda(|v|)v| \le \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}t}(\lambda(|w_t|)w_t)\right|\mathrm{d}t \le C\int_0^1 \frac{|u-v|^{1/2}}{|w_t|^{1/2}}\mathrm{d}t|u-v|^{1/2}.$$

Set

$$a := \frac{v}{|u-v|} \quad \text{and} \quad b := \frac{u-v}{|u-v|}$$

then $|b| = 1$ and we have

$$\int_0^1 \frac{|u-v|^{1/2}}{|w_t|^{1/2}}\mathrm{d}t = \int_0^1 \frac{\mathrm{d}t}{|a+tb|^{1/2}} \le 2\int_0^{1/2}\frac{\mathrm{d}s}{\sqrt{s}} < \infty.$$

Thus for a.e. $u, v \in \mathbb{R}^3\backslash\{0\}$

$$|\lambda(|u|)u - \lambda(|v|)v| \le C|u-v|^{1/2}.$$

By continuity this holds for all $u, v \ne 0$ and due to the Hölder continuity this holds for all $u, v \in \mathbb{R}^3$. □

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