*The RAND Journal of Economics*

**ORIGINAL ARTICLE** OPEN ACCESS

# Auctioning Off Budgets in Procurement

Claudio Rottner

Faculty of Law, Business & Economics, University of Bayreuth, Bayreuth95447, Germany

**Correspondence:** Claudio Rottner (Claudio.Rottner@uni-bayreuth.de)

**Received:** 24 October 2025 | **Accepted:** 24 October 2025

**Keywords:** reverse auction | multi-unit auction | procurement | pay-as-bid

## ABSTRACT

This article investigates a multi-unit pay-as-bid procurement auction where the auctioneer fixes total spending and maximizes the quantity procured with their predetermined, secret budget. Previous literature has analyzed fixed-quantity auctions, where the traded quantity is fixed but unknown to the bidders. Compared to such auctions, budget auctions lower the auctioneer's costs by introducing an additional interaction between a bidder's bids; bidders not only weigh a higher profit margin on a unit against a lower probability of supplying that unit; a higher margin on some unit also reduces the probability that the budget suffices to procure more units from the bidder.

**JEL Classification:** C72, D44, H57

## 1 | Introduction

In 2020, the Austrian telecommunications regulatory authorities (TKK and RTR) used a reverse auction to procure mobile network coverage for remote areas that had previously lacked an adequate network infrastructure (RTR 2020). In the pay-as-bid auction, each mobile network operator submitted a set of bids, each of which specified the total subsidy they demanded for providing coverage to a certain number of municipalities (TKK 2019). The auctioneer selected the combination of the different bidders' bids that maximized the number of municipalities to which coverage would be provided, subject to the constraint that the total subsidy did not exceed a secret, predetermined budget (TKK 2019).

The Austrian spectrum auction can be considered the real-world equivalent of the model in this article. The model describes a novel strategic effect in bidding behavior. Due to the auctioneer's budget cap, there is an interaction between a bidder's bids on different units: A high bid on some unit reduces the residual budget. Thus, fewer of the bidder's bids can be considered. Such a link between bids on different units does not have a correspondence in previous literature.

This article derives the symmetric equilibrium in a budget auction when bidders have the same cost function and share a stochastic belief about the unknown budget of the auctioneer. Previous literature (e.g., Holmberg 2009; Pycia and Woodward 2023) has considered the case where the auctioneer fixes the traded quantity, which is unknown to the bidders. In a pay-as-bid procurement auction in such a setting, when bidders raise their bids, they trade off the additional margin on a particular unit (price effect) with a decreased probability of realizing the margin on this unit (quantity effect). By contrast, increasing the profit margin on some unit also hurts the chances of realizing the profit margin on *other* units in the budget auction. Higher bids decrease the residual budget and therefore the probability that bids on higher quantities can be served without exceeding the auctioneer's budget (budget effect). Consequently, the auctioneer's costs in a budget auction are strictly lower than in a comparable fixed-quantity auction, that is, a fixed-quantity auction with the same distribution of realized demand in terms of quantities. The cost advantage of the budget auction is most pronounced when traditional fixed-quantity auctions suffer from soft competition, that is, when there are few bidders whose costs exhibit strong diseconomies of scale.

This article proceeds as follows. Section 2 discusses related literature. Section 3.1 first explains the basic model setup for the fixed-quantity and the budget auction. For ease of exposition, it then briefly discusses the symmetric equilibrium in the fixed-quantity auction translating the model from Pycia and Woodward (2023) to the procurement context. Afterward, I characterize the symmetric equilibrium in the strategically more complex budget auction. Section 4 shows that the budget auction comes at a lower cost to the auctioneer than a comparable fixed-quantity auction. Section 5 discusses some of this article's limitations before Section 6 concludes.

## 2 | Related Literature

Traditionally, auction theory assumes that the traded quantity is fixed in advance. In the "single-unit", or rather, "'single-winner" context, the analysis of auctions with an endogenous traded quantity dates back at least to Hansen (1988). Dasgupta and Spulber (1989) demonstrate that variable-quantity auctions are optimal under plausible assumptions, both in cases with a single winner and multiple winners. However, in contrast to typical real-world auctions or budget auctions, implementing these optimal auctions requires the auctioneer to possess extensive knowledge regarding the bidders' cost structure. Budget auctions, as a specific form of auctions with a variable traded quantity, have been discussed in the single-winner context by Dastidar (2008), Deck and Wilson (2008), and Liu and Parlour (2014).

However, this article is better understood against the backdrop of the literature explicitly characterizing equilibria in multi-unit (multi-winner) auctions. Burkett and Woodward (2020b) are a notable outlier in this literature because they consider a *discrete* uniform-price auction with *uncertainty over individual bidder types*. The rest of this literature, including this article, rather follows Klemperer and Meyer (1989) in analyzing multi-winner auctions with a continuous traded quantity and bidders with an identical, known cost structure. Specifically, Klemperer and Meyer (1989) consider a uniform-price procurement auction in which the bidders share a belief about the random, downward-sloping demand function. They explicitly characterize the potentially infinite number of symmetric equilibria. They show that the equilibrium bidding functions lie in a band that bounds them from above and below. If one were to adapt their model such that demand is completely inelastic, that is, the traded quantity is fixed in advance, this would correspond to the uniform-price equivalent of the model of a pay-as-bid fixed-quantity auction in this article. In such a model, the infinitely many symmetric equilibria can be arbitrarily unattractive to the auctioneer. These equilibria are called the "'low-revenue equilibria" of the uniform price auction. Their occurrence was first observed by Wilson (1979).

For the pay-as-bid auction, Holmberg (2009) derives the symmetric equilibrium for the reverse auction with completely inelastic demand. Just like Klemperer and Meyer (1989), he assumes that bidders have an identical, known cost function and share a belief over the distribution of demand. However, for his model to work, he requires that demand surpasses the suppliers' production capacity with a positive probability, which limits the applicability of his results. Pycia and Woodward (2023) relax this assumption in the context of a forward auction. Their model forms the fixed-quantity baseline to which I compare the budget auction in this article.

Pycia and Woodward (2023) also consider an endogenous traded quantity in their analysis of random reserve prices. Random reserve prices punish high bids by increasing the probability that the increased bid will not be considered. This reinforces the quantity effect and is, therefore, possible in both fixed-quantity and budget auctions. Thus, the insights into secret reserve prices complement the analysis in this article, which focuses on a novel strategic effect deterring high bids by reducing the probability that bids on other units are accepted.

For the uniform-price auction, some more articles consider a variable traded quantity besides Klemperer and Meyer (1989). These models address the issue of low-revenue equilibria by allowing the auctioneer to restrict the traded quantity after observing the bids (e.g., Lengwiler 1999; Back and Zender 2001; McAdams 2007).[1] Like in this article, the threat to reduce the traded quantity if bids are too unattractive to the auctioneer prompts the bidders to reduce bid shading. However, the auction designs from this literature can only be applied in the real world under exceptional circumstances. Consider McAdams (2007), for example. In his model, the auctioneer adjusts supply *at will* ex-post, that is, does not make *any* commitment regarding the traded quantity, and the bidders reveal their type *truthfully* as a consequence.

The auctioneer could exploit this design by learning the bidders' types without trading any significant quantity and then extracting rents from the bidders in trade outside the auction. According to Rothkopf et al. (1990), the bidders' unwillingness to reveal their type is one of the main reasons for the scarcity of second-price auctions in the real world. This concern is mitigated in the pay-as-bid setting considered here. Furthermore, the auctioneer in this article does commit to a budget, although the specific amount is not disclosed. Bergemann and Hörner (2018) and Allen et al. (2024) suggest that such opacity regarding parts of the auction design is not unusual in real-world auctions.

This raises the question of how the auctioneer can credibly commit when the bidders cannot observe this commitment. Tillio et al. (2016) argue that credible commitment is possible if the auctioneer is indifferent between the available alternatives. However, this seems unlikely in most applications and is certainly not the case for the budget auction. This suggests other mechanisms by which the auctioneer can make their private commitment credible exist. These include trust, as in the case of the Austrian coverage auction, and trusted intermediaries, such as eBay, for secret reserve prices on the platform. Theoretically, there are also trustless solutions. For example, the auctioneer could publish the budget in an encrypted document before and the encryption key after the auction. However, it seems unlikely that procurement agencies would opt for such a solution. Equally theoretically, uncertainty over the budget could also stem from tying the budget to a random but ex-post-verifiable outcome, for example, proceeds from future auctions of emission allowances. Why the auctioneer might *want* to implement an uncertain budget is discussed in Section 5.

## 3 | Model

### 3.1 | Setup

The auctioneer wants to procure multiple units of a homogeneous good using a pay-as-bid auction. The main claim of this article is that the auctioneer is better off when doing so using a budget auction than when using a fixed-quantity design, which has been analyzed by Holmberg (2009) and Pycia and Woodward (2023).

#### 3.1.1 | Bidders

For both auction designs, assume that all of the $n \geq 2$ bidders have the same marginal cost $c(x) > 0$ for producing the $x$-th unit with $0 < c'(x) < \infty$.[2] The bidders know each others' costs, whereas the auctioneer does not. The auction rules require each bidder to specify a weakly positive, twice differentiable, finite, and increasing bidding function $b_i(x)$, which denotes the marginal payment bidder $i$ demands for providing their $x$-th unit. I assume that the submitted bid functions are smooth in the sense that they have *finite* first and second derivatives for $x > 0$.[3] The inverse of $b_i(x)$ is denoted by $\beta_i(\cdot)$. The auctioneer pays bidder $i$ $B_i(x) = \int_0^{x_i} b_i(y)\,dy$ when buying $x_i$ units from them. I refer to $B_i(x)$ as bidder $i$'s cumulative bids. The pay-off of bidder $i$ when assigned quantity $x$ is

$$\pi_i(x; B_i) = B_i(x) - \int_0^x c(y)\,dy.$$

#### 3.1.2 | Fixed-Quantity Auction

In the fixed-quantity auction, the auctioneer procures $X$ units, whatever the costs of doing so are. The auctioneer minimizes the total payments to bidders subject to the constraint that $X$ units are procured. In line with previous literature, assume that the bidders do not know the quantity the auctioneer wants to procure but believe that it is distributed on some interval $[0, \bar{X}]$ according to the distribution function $G(X)$ with $G'(X) = g(X) > 0$ and $\bar{X} < \infty$. This setup for the fixed-quantity auction is the reverse auction equivalent of the model from Pycia and Woodward (2023).

#### 3.1.3 | Budget Auction

In the budget auction, the auctioneer intends to procure as many units of the good as possible with the fixed budget $A$. Given the submitted bids, the auctioneer solves the following optimization problem to determine the quantity $x_i$ they buy from bidder $i$:

$$\begin{aligned} \max_{x_1,\ldots,x_n} \quad & \textstyle\sum_{i=1}^n x_i \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^n B_i(x_i) \leq A \\ & \forall i : x_i \geq 0. \end{aligned}$$

Assume that each bidder submits bids at least up to the quantity that the auctioneer wants to assign to them. The first-order conditions of this problem imply that in an interior solution with $x_i > 0$ for all bidders, all bidding functions must have the same value at their respective chosen quantity. Refer to this unique value as the market (clearing) price. Consequently, with increasing bids, the auctioneer procures all units for which the demanded price is lower or equal to the market price.

The auctioneer's budget is unknown to the bidders. They only share a common belief that the auctioneer's budget follows a twice-differentiable cumulative distribution $F(A)$ on $[0, \bar{A}]$ with density $f(A) > 0$, $\bar{A} < \infty$ and $f(\bar{A}) < \infty$, as well as $f'(\bar{A}) < \infty$. The bidders' beliefs are rational in the sense that $A \in [0, \bar{A}]$.

### 3.2 | Symmetric Equilibrium in the Fixed-Quantity Auction

Pycia and Woodward (2023) have extensively analyzed the fixed-quantity auction. Proposition 1 summarizes their result on equilibrium behavior.

**Proposition 1.** *(**Pycia and Woodward** 2023) The symmetric equilibrium in a pay-as-bid fixed-quantity auction, in which $n$ suppliers with costs $c(x)$ believe that the demanded quantity is distributed according to the distribution function $G(\cdot)$ on $[0, \bar{X}]$, is described by*

$$b(x) = c(x) + [1 - G(nx)]^{\frac{1-n}{n}} \int_x^{\bar{X}/n} c'(y)\,[1 - G(ny)]^{\frac{n-1}{n}}\,dy.$$

*Bids are strictly above cost for all but the last unit and equal cost for the last unit.*

To better understand the differences between the fixed-quantity and the budget auction regarding equilibrium behavior, the following lays out the crucial steps in the derivation of Proposition 1.

As bids are increasing, there is a unique market clearing price $p$, which ensures that demand equals supply, that is, $\sum_i \beta_i(p) = X$, and all units for which the bids are below $p$ are supplied. The quantity $x_i$ supplied by player $i$ is implicitly defined by $p = b_i(x_i)$. Following Pycia and Woodward (2023), the probability that player $i$ supplies weakly less than $x$ units given some set of strategies $\boldsymbol{b} = (b_i, \boldsymbol{b_{-i}})$ is

$$\text{Prob.}(x_i \leq x; \boldsymbol{b}) = G\left(x + \sum_{j \neq i} \beta_j(b_i(x))\right).$$

Bidder $i$ winning some quantity $x$ implies a market clearing price of $b_i(x)$, which means that the other bidders supply $\sum_{j \neq i} \beta_j(b_i(x))$ units. Consequently, overall demand must not exceed $x + \sum_{j \neq i} \beta_j(b_i(x))$ if player $i$ is to win $x$ units or less. As shown by Pycia and Woodward (2023), this probability distribution allows us to write the expected profits of bidder $i$ as

$$\max_{b_i(x)} \quad \mathbb{E}[\pi_i] = \int_0^{\bar{x}_i} [b_i(x) - c(x)] \left[1 - G\left(x + \sum_{j \neq i} \beta_j(b_i(x))\right)\right] dx, \qquad (1)$$

where $\bar{x}_i = \inf\{x : G(\cdot) = 1\}$. The intuitive interpretation of Equation (1) is that the profit margin on a particular unit must be weighted by the probability with which it is realized. Due to the pay-as-bid nature of the auction, the profit margin on some quantity $x$ is realized whenever the bidder wins $x$ or more units.

As there is (abstracting away from the monotonicity constraint on bids) no interaction between the bids on different quantities, the first-order condition to the bidder's problem (1) is obtained by point-wise maximization of the integrand. Holmberg (2009) and Pycia and Woodward (2023) derive the first-order condition as

$$\left(\underbrace{1 - G(X)}_{\text{Price Effect}}\right) - \underbrace{[b_i(x) - c(x)]\, g(X) \sum_{j \neq i} \beta_j'(b_i(x))}_{\text{Quantity Effect}} = 0, \tag{2}$$

where the market clearing condition $x_i + \sum_{j \neq i} \beta_j(b_i(x)) = X$ has been used.[4]

The first-order condition in Equation (2) demonstrates the trade-off between price and quantity effect. The first term represents the positive effect of raising bids. In all cases where the bidder wins more than $x$ units, they benefit from additional revenue if they increase $b_i(x)$ (price effect). The second term shows that an increase in the bid on unit $x$ reduces the probability that the corresponding profit margin is realized (quantity effect). This effect constitutes the competition aspect of the auction. If a bidder demands a higher payment for some unit, the auctioneer considers more of the other bidders' bids first. This reduces the probability that the bidder gets to supply the unit in question.

Given that the quantity assigned to bidder $i$ is strictly increasing in total demand $X$, Equation (2) allows us to gain further insights into optimal bidding behavior. As long as total demand is lower than $\bar{X}$, the first term in Equation (2) is positive, such that the first-order condition can only be satisfied if the corresponding bids are above marginal costs. By the same argument, for the highest quantity ever won, which is the quantity $\bar{x}_i$ won when $X = \bar{X}$, it must be that the corresponding bid equals marginal cost, as $1 - G(\bar{X}) = 0$. Intuitively, the price effect of increasing the bid on the last unit is zero, as the additional margin is only realized when $X = \bar{X}$. By contrast, the risk of losing the (profitable) unit when increasing the bid remains.

Thus, we obtain the terminal condition $b_i(\bar{x}_i) = c(\bar{x}_i)$. Taken together with symmetry, that is, $b_i(x) = b_j(x)\, \forall i, j$, this allows us to write the symmetric equilibrium, based on Equation (2), as

$$b(x) = (n-1)[1 - G(nx)]^{\frac{1-n}{n}} \int_x^{\bar{x}} [1 - G(ny)]^{-1/n} g(ny) c(y)\, dy \tag{3}$$

where $\bar{x}_i = \bar{x} = \bar{X}/n$. Integration by parts results in the formulation of Proposition 1.

## 3.3 | Symmetric Equilibrium in the Budget Auction

Next, consider the symmetric equilibrium in the budget auction.

### 3.3.1 | Bidder's Problem

Given a strategy profile $\boldsymbol{B} = (B_i, \boldsymbol{B}_{-i})$, for bidder $i$ to win $x$ units or less, it must be that the budget suffices at most to pay for the $x$ units of bidder $i$ and the $\sum_{j \neq i} \beta_j(b_i(x))$ units that the other bidders will supply in this situation, that is,

$$\text{Prob.}\,(x_i \leq x; \boldsymbol{B}) = F\left(B_i(x) + \sum_{j \neq i} B_j(\beta_j(b_i(x)))\right).$$

As before, the bidder's expected profit is given by weighting the profit margin on some unit $x$ with the probability that the bidder wins at least $x$ units. The problem of bidder $i$ is

$$\max_{B_i(x)} \mathbb{E}[\pi_i] = \int_0^{\bar{x}_i} \underbrace{[b_i(x) - c(x)]\left[1 - F\left(B_i(x) + \sum\nolimits_{j \neq i} B_j(\beta_j(b_i(x)))\right)\right]}_{\equiv \Psi(B_i, b_i, x)} dx \tag{4a}$$

$$B(0) = 0 \tag{4b}$$

$$B_i(\bar{x}_i) = \underbrace{\bar{A} - \sum_{j \neq i} B_j(\beta_j(b_i(\bar{x}_i)))}_{\equiv \phi(b_i, \bar{x}_i)}. \tag{4c}$$

The bidder's best response problem comes with two boundary conditions. First, by construction, cumulative bids start at zero. Second, the interval of relevant quantities is endogenously determined by the considered bidder's bids. As bids are increasing, the highest quantity a bidder can win is the quantity the bidder wins when the auctioneer auctions off the highest possible budget $\bar{A}$.

### 3.3.2 | Strategic Effects

A closer look at Equation (4a) reveals that the bid on some unit $x$ is chosen according to three considerations:

1. Its effect on the profit margin on unit $x$ (price effect).
2. Its effect on the probability of realizing the profit margin on unit $x$ (quantity effect).
3. Its effect on the probability of winning units $y > x$ (budget effect).[5]

The price effect is reflected in the first bracket of the integrand in (4a). The quantity effect is described by the second summand in the distribution function. It reflects competition in the sense that it determines the order in which the different bidders' bids are considered by the auctioneer. As was seen in Section 3.2, the trade-off between price and quantity effect characterizes the equilibrium in the fixed-quantity auction. The budget effect, which is reflected in the $B_i(x)$ in the distribution function, has no correspondence in the fixed-quantity auction. To make the distinction between the effects clear, think of bids and cumulative bids as independent variables. In this case, demanding a higher total compensation does not contribute to profits. However, it reduces the residual budget and thus makes it less likely that the budget suffices to buy further units from the bidder. Interestingly, in contrast to the quantity effect, this negative effect of raising bids is not a competition effect. It is not about the relative attractiveness of the different bidders' bids. Rather, higher cumulative bids of a bidder shift the probability of winning higher quantities downward independently of what the other bidders do.

### 3.3.3 | Equilibrium Bidding Behavior

As the bidder's problem from (4a) includes an interaction between the bids on different units, the first-order condition for this variational calculus problem is (see, e.g., Chiang 1992)

$$\frac{\partial \Psi}{\partial B_i} - \frac{d}{dx}\frac{\partial \Psi}{\partial b_i} \overset{!}{=} 0. \tag{5}$$

Roughly speaking, $\partial\Psi/\partial B_i$ represents the budget effect, $\partial\Psi/\partial b_i$ is the balance of price and quantity effect, and the $d/dx$ accounts for the relation between bids and cumulative bids. The relevant derivatives can be found in Appendix A.1. Using symmetry, that is, $b(x) = b_i(x) = b_j(x)\ \forall i, j$, the Euler equation emerges as the third-order differential equation[6]

$$b''(x) = \frac{[b'(x)]^2}{b(x)[b(x)-c(x)]}\left[3b(x) - \frac{(n-2)}{(n-1)}c(x) - \frac{c'(x)}{b'(x)}b(x) + n\frac{f'(nB(x))}{f(nB(x))}[b(x)-c(x)]\frac{[b(x)]^2}{b'(x)}\right]. \tag{6}$$

The two boundary conditions in (4b) and (4c) eliminate two degrees of freedom. The initial condition (4b) removes the first degree of freedom. The second piece of information lies in the terminal condition from (4c). For the symmetric case, this implies

$$B(\bar{x}) = \frac{\bar{A}}{n}. \tag{7}$$

However, Equation (7) only pins down the level of cumulative bids at the last unit but does not tell us what the last unit is. Appendix A.2 shows that the corresponding transversality condition implicitly defines the last unit through $b_i(\bar{x}_i) = c(\bar{x}_i)$. Equivalently, for the symmetric case, where $\bar{x} \equiv \bar{x}_i = \bar{x}_j$, we have

$$b(\bar{x}) = c(\bar{x}). \tag{8}$$

Equation (8) says that the last unit should be chosen in such a manner that all opportunities to realize additional profits are exhausted; that is, there is no positive margin on the last unit. Put differently, of the three strategic effects discussed earlier, only the price effect favors higher bids. However, as the weight bidders attach to the profit margin on the last unit is zero, it is absent for the last unit. The only effect active on the last unit is the quantity effect. Thus, the last unit should be offered at marginal cost.

Finally, Appendix A.2 shows that the smoothness assumption on $b(x)$ implies

$$b'(\bar{x}) = \frac{n-1}{2n-1}c'(\bar{x}). \tag{9}$$

Any increasing continuation of the bidding function beyond $\bar{x}$ stabilizes the equilibrium.[7] Proposition 2 summarizes the findings on the symmetric equilibrium.

**Proposition 2.** *The symmetric equilibrium in a pay-as-bid budget auction, in which n suppliers with costs $c(x)$ believe that the budget is distributed according to the distribution function $F(\cdot)$ on $[0, \bar{A}]$, is described by Equations (4b) and (6)–(9). Bids are increasing and strictly above cost for all but the last unit. Bids equal costs for the last unit.*

In addition to the previously discussed, Proposition 2 claims that bids are above marginal cost and satisfy the monotonicity constraint imposed in the model setup. The corresponding proofs and the second-order conditions are relegated to Appendices A.3 and A.4, respectively. Generally, there is no closed form for the symmetric equilibrium.

### 3.3.4 | Example

The following example, which allows for a closed-form representation, will later serve to illustrate the cost advantage of the budget auction. Consider the case where $n = 2$, $c(x) = \alpha x + \gamma$, and $A \sim U[0, \bar{A}]$ with $\alpha, \gamma, \bar{A} > 0$. The symmetric equilibrium is

$$b(x) = \frac{\sqrt{36\gamma^2 + 60\alpha\bar{A}} + 9\gamma}{15} + \frac{\alpha}{3}x.$$

The second-order condition for this example is discussed in Appendix A.4.

## 4 | Cost Advantage of the Budget Auction

Auctioning off a secret budget instead of an unknown quantity introduces an additional consideration into the bidders' decision. Under both auction designs, when raising bids, the bidders need to weigh additional profits on a unit (price effect) against a lower probability of realizing them (quantity effect). However, in the budget auction, increasing the bid on some quantity has the additional adverse effect of making it less likely to win higher quantities, as the auctioneer runs out of budget "'earlier" (budget effect). Therefore, intuitively, the budget auction should be more attractive to the auctioneer.

Testing this hypothesis requires fully specifying the distributions in which bidders believe under the two auction designs, respectively. Luckily, the bidders' knowledge of equilibrium bids allows them to seamlessly move between expectations over total spending and the procured quantity. Given their equilibrium bids in the budget auction, the bidders can map every potential budget into a quantity, which the auctioneer procures with that budget in total. Thus, the bidders can directly translate the probability distribution over budgets, which describes their beliefs in a budget auction, into a "'corresponding" probability distribution over the number of units procured by the auctioneer. This corresponding distribution describes the bidders' belief in a comparable fixed-quantity auction.

For the case of symmetric bidders, the number of units $X$ the auctioneer procures in a budget auction with a given budget is implicitly given as $A = n\,B^B(X/n)$ (where the superscript $B$ denotes the budget auction). Thus, the probability that the auctioneer procures less than $X$ units is equivalent to the probability that the budget does not exceed $nB^B(X/n)$. Therefore, the bidders believe that the auctioneer's demand follows the distribution

$$\tilde{G}(X) \equiv F\left(n\,B^B(X/n)\right) \tag{10}$$

in the corresponding fixed-quantity auction. These corresponding auctions can also be generalized to bidders who do not bid the same in equilibrium because they are asymmetric. In this case, we can write the budget needed to procure a given total quantity through a budget auction as a function of the quantity $x_i$ supplied by bidder $i$. We then get

$$\tilde{G}(X) \equiv F\left( B_i^B(x_i) + \sum_{j \neq i} B_j^B(\beta_j^B(b_i^B(x_i))) \right) \tag{11}$$

with the implicit definitions $A = B_i^B(x_i) + \sum_{j\neq i} B_j^B(\beta_j^B(b_i^B(x_i)))$ and $x_i = X - \sum_{j\neq i} \beta_j^B(b_i^B(x_i))$. This gives $\tilde{g}(X) = f(\cdot)\frac{\partial A}{\partial x_i}\frac{\partial x_i}{\partial X} = f(\cdot)b_i^B(x_i)$ because a marginal change in the procured quantity must be scaled by the corresponding payment. Comparing equilibrium bids in a budget auction with the corresponding fixed-quantity auction yields two results. First, directly formalizing the intuition from the previous section, when fixing the other bidders' bids, a bidder's best response is to bid higher in in the fixed-quantity auction than in the budget auction (Proposition 3). Second, for symmetric bidders, this also entails lower equilibrium bids in the budget auction (Proposition 4).

**Proposition 3.** *Consider a fixed-quantity auction with demand normalized according to Equation (11). Fix the bids of bidders $j \neq i$ in the fixed-quantity auction at the level of their equilibrium bids in the corresponding budget auction, that is, $b_j(x) = b_j^B(x)$. Then, some bidder i's best response is to bid $b_i(x) > b_i^B(x)$ for $x < \bar{x}_i$.*

*Proof.* Consider the best response $b_i^B(x)$ of bidder $i$ in the budget auction. It has them bidding truthfully on the last unit they receive (see Appendix A.2). With reference to Equation (A2), this means that $\frac{\partial \Psi}{\partial b_i^B}\Big|_{x_i=\bar{x}_i} = 0$. This formalizes the intuition that the optimal bid equalizes price and quantity effect because the budget effect is absent for the last unit. However, the bidder's best response has him bid below the level that would balance these two effects for all other units, that is, $\forall x < \bar{x}_i : \frac{\partial \Psi}{\partial b_i^B} > 0$. Formally, the bidder's best response entails bidding above costs for these units (see Appendix A.3). Thus, there is a budget effect in the sense that $\forall x < \bar{x}_i : \frac{\partial \Psi}{\partial B_i^B} < 0$, as can be seen from Equation (A1). Therefore, by the first-order condition (5), $\forall x < x_i : \frac{d}{dx}\frac{\partial \Psi}{\partial b_i^B} < 0$. It follows that along the equilibrium path $\forall x < \bar{x}_i : \frac{\partial \Psi}{\partial b_i^B} > 0$. The normalization of demand implies that $\frac{\partial \Psi}{\partial b_i^B}$ is equivalent to the derivative in the first-order condition of the bidder's problem in the fixed-quantity auction if evaluated at $b_i(x) = b_i^B(x)\, \forall i$. Intuitively, the normalization ensures that the density attached to every unit is the same under both auction designs such that price and quantity effects are identical. If the bidder's problem in the fixed-quantity auction is well behaved, that is, concave in $b_i$, the level of their best response bids is above $b_i^B(x)$. □

Although Proposition 3 demonstrates the incentive wedge between comparable budget and fixed-quantity auctions, it does not have direct implications for the equilibrium outcomes. Proposition 4 entails those at the cost of restricting attention to symmetric equilibria. It shows that equilibrium bids in a budget auction, denoted by $b^B(x)$, are lower than the equilibrium bids, $b^{FQ}(x)$, in a comparable fixed-quantity auction, where demand is normalized according to Equation (10).

**Proposition 4.** *If demand is normalized according to (10), bids in the symmetric equilibrium are lower in the budget auction than in the comparable fixed-quantity auction:*

$$b^B(x) \leq b^{FQ}(x) \quad \forall x \in [0, \bar{x}],$$

*where the inequality is strict for $x \in [0, \bar{x})$ and the superscripts FQ and B denote the fixed-quantity and budget auction, respectively.*

The result of Proposition 4 is very strong in that it postulates that *every single unit* is less expensive in the budget auction.

The equality of bids for the last unit follows from the fact that the maximum number of units an individual bidder supplies is identical between the corresponding auctions by construction. As bidders do not have a positive margin on the last unit under both auction designs, the bids on the last unit are identical in the corresponding auctions.[8] The proof of the remainder of Proposition 4 is relegated to Appendix B. A sufficient condition for higher equilibrium bids in the fixed-quantity auction is that $\frac{\partial \Psi}{\partial b_i}\Big|_{B_i=B_j=B^B, b_i=b_j=b^B} > 0$ for $x < \bar{x}$. This means that in a hypothetical situation without budget effect, bids should be higher than in the equilibrium of the budget auction. As the budget effect pulls toward lower bids, this is the case. Formally, the proof of this is analogous to the one in Proposition 3.

### 4.1 | Example

Let us revisit the example from Section 3.3. Equilibrium bids in the budget auction with $n = 2$, $c(x) = x$, and $A \sim U[0, 1]$ are $B^B(x) = 1/6\, x^2 + \sqrt{60}/15\, x$. The probability that less than $X$ units are procured, that is, the corresponding distribution in the fixed-quantity auction is[9]

$$\tilde{G}(X) = \text{Prob.}(A \leq 2\, B^B(\frac{X}{2})) = \frac{1}{12}X^2 + \frac{\sqrt{60}}{15}X.$$

The left panel of Figure 1 plots the difference between the bids in the two auction designs, that is, $b^{FQ}(x) - b^B(x)$. Bids converge in the quantity supplied. Intuitively, the budget effect, which explains the difference between bids in the first place, becomes less relevant with increasing quantities. Both margins and the weight attached to them decrease in the quantity supplied. Consequently, the budget effect gives less and less reason to lower bids to increase the probability of realizing the margins on higher units as the quantity increases.

The right panel of Figure 1 shows the auctioneer's total cost of procurement in the budget auction as a share of the total procurement costs in the corresponding fixed-quantity auction, that is, $\frac{n\, B^B(X/n)}{n\, B^{FQ}(X/n)}$. In this example, the procurement costs in the budget auction are between roughly 96% and 99% of the procurement costs in the fixed-quantity auction.

Evidently, the size of the cost advantage of the budget auction varies with the model parameters. In particular, we are inter-

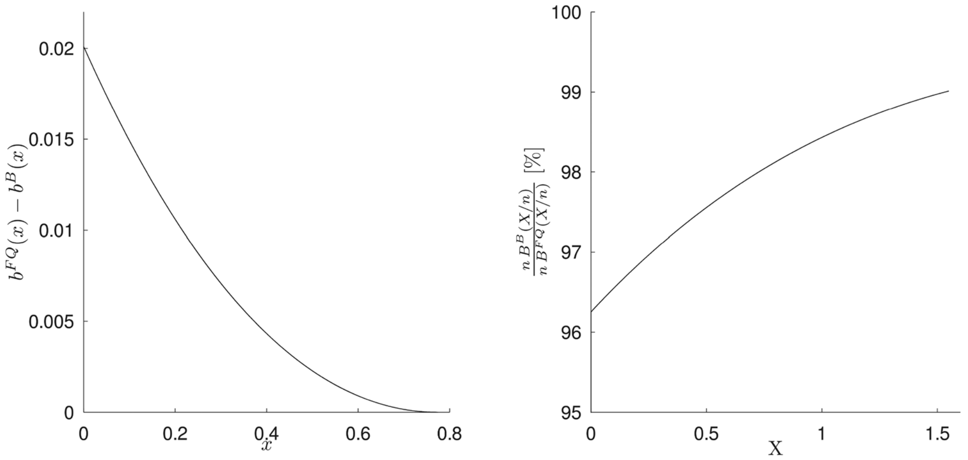


**FIGURE 1** | The difference between the equilibrium bids in the fixed-quantity auction and the equilibrium bids in the budget auction to which it corresponds ($n = 2$, $c(x) = x$ and $A \sim U[0,1]$). On the left: difference in bids, on the right: total costs of procurement in the budget auction as a share of the total costs in the corresponding fixed-quantity auction.

ested in the impact that some specific model parameter, say $n$, has on the relation of total procurement costs under the two auction designs; that is, we would like to know $\frac{d}{dn}\frac{B^B(x)}{B^{FQ}(x)}$. This requires gauging the relative size of $\frac{d}{dn}B^B(x)$ and $\frac{d}{dn}B^{FQ}(x)$ for $\forall x < \bar{x}$. However, the established approaches to doing analytic comparative "'statics" on the equilibrium paths do not allow to do so. Even though Oniki (1973) generalizes comparative statics via implicit differentiation of the first-order condition to the dynamic context, his approach only allows *signing* the relevant expressions by drawing a phase diagram of the corresponding system of linear differential equations. Similarly, the primal-dual methodology, comprehensively set out in Caputo (2005), uses the information contained in the second-order conditions to *sign* the effect of parameter changes on the optimal paths *integrated over the entire planning horizon*, that is, $\int_0^{\bar{x}} \frac{d}{dn}B^{B/FQ}(x)\,dx$ in the case at hand. Thus, these approaches do not permit extracting the necessary information on the relative size of the impact of a parameter between the two auction designs in the absence of a closed-form solution. Consequently, the following only provides suggestive evidence on the economic contexts favorable to the budget auction based on numeric calculation.

The budget auction reduces competition problems faced by fixed-quantity auctions. Put differently, the cost advantage of the budget auction should be particularly large when fixed-quantity auctions suffer from soft competition. The general intuition is that bids in the budget auction are reduced to increase the probability of realizing margins on other units. As a consequence, bids are reduced most strongly when the margins on these other units are high, that is, competition is soft. Specifically, the budget auction should have a stronger cost advantage over the fixed-quantity auction when there are few bidders whose cost structure exhibits strong diseconomies of scale.

Figure 2 illustrates these points based on numerical calculation. The findings are robust to various specifications regarding the form of costs and the underlying distribution. To numerically construct a measure of the budget auction's cost advantage, we first need to determine equilibrium bids in the budget auction. To do this, we can guess an arbitrary value for $\bar{x}$, which fixes $B(\bar{x})$, $b(\bar{x})$, $b'(\bar{x})$ and $b''(\bar{x})$ by Equations (7), (8), (9), and (A8), respectively. Using the Euler equation from (6), we can develop the solution "'backward," that is, calculate bids for ever lower quantities. The correct choice of $\bar{x}$ is identified by forcing $B(0) = 0$. We can then numerically determine the distribution $\tilde{G}(X)$, and equilibrium bids in the corresponding fixed-quantity auction.

The left panel of Figure 2 plots the cost advantage, that is, $\frac{n\,B^B(X/n)}{n\,B^{FQ}(X/n)}$, for the previous example not only for the case of two (solid line) but also four and ten bidders. As can be seen, the fewer bidders there are, the lower the number of units the auctioneer can procure with a given budget.[10] Under both auction designs, a decreasing number of players softens competition and thus increases margins. Increased margins mean that the budget effect becomes more relevant. Therefore, the relative cost advantage of the budget auction increases when there are fewer bidders.

For the same reason, stronger diseconomies of scale increase the cost advantage of the budget auction, as can be seen in the right panel of Figure 2.[11] Again, the solid line corresponds to the previous example, and solutions are obtained numerically. To understand why diseconomies of scale increase profit margins, first consider the case of a flat marginal cost curve. In this case, the considered auctions would collapse into Bertrand competition with bids competed down to cost. It is only when marginal costs are increasing that bidders can realize profits because other bidders who already supply a larger quantity cannot exert full competitive pressure, as they are in a less favorable cost position. Extending this logic, the faster costs increase, that is, the larger the diseconomies of scale, the higher the profit margins bidders can realize. As margins increase, the budget effect becomes more relevant, and the cost advantage of budget auctions increases.

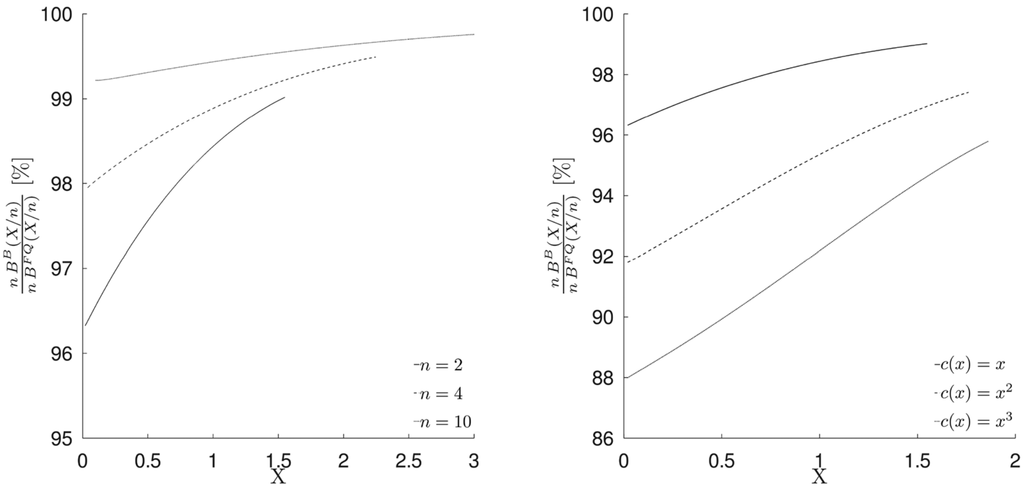


**FIGURE 2** | Total costs of procurement in the budget auction as a share of total costs in the corresponding fixed-quantity auction. On the left: comparison for a budget auction with $c(x) = x$, $A \sim U[0, 1]$, and a differing number of players. On the right, the comparison is shown for a budget auction with $n = 2$, $A \sim U[0, 1]$, and differing cost structures.

## 5 | Limitations

### 5.1 | Uncertainty of the Budget

There are two reasons for introducing uncertainty regarding the budget. From a theoretical point of view, without uncertainty, the equilibrium from Proposition 2 only persists under the assumption that the bidders bid below their costs on units they do not supply in equilibrium. As this is weakly dominated by an otherwise unchanged strategy with above-cost bids for off-equilibrium quantities, such an equilibrium does not seem plausible in real auctions. As the distribution converges to a point mass on $\bar{A}$, bids converge to flat bids at level $c(\bar{x})$. The reason is that there is no cost of making the own bids on the infra-marginal units the bidder might lose, less attractive, as the density attached to these units is zero. However, the bidders have an incentive to deviate upward in this case. Consider the incentive of a bidder $i$ to marginally raise their constant bids from $b_i = c(\bar{x}_i)$. Their profits for a constant $b_i \geq c(\bar{x}_i)$ are $\pi_i = A - \sum_{j\neq i} B_j(\beta_j(b_i)) - C(\frac{A-\sum_{j\neq i} B_j(\beta_j(b_i))}{b_i})$, where $A - \sum_{j\neq i} B_j(\beta_j(c(\bar{x}_i))) \equiv A_i$. If the other bidders bid weakly above costs for $x > \bar{x}_i$, the deviating bidder's profits are differentiable for a marginal upward deviation in $b_i$. Then, the benefit of marginally deviating upward is

$$\frac{\partial \pi_i}{\partial b_i}\Big|_{b_i=c(A_i/b_i)} = \frac{\partial A_i}{\partial b_i} - c\left(\frac{A_i}{b_i}\right)\left(\frac{\partial A_i}{\partial b_i}\frac{1}{b_i} - \frac{A_i}{(b_i)^2}\right) = \frac{A_i}{b_i} > 0.$$

Intuitively, the bidder only loses a non-profitable marginal unit but gains a margin on all other units. It is easy to verify that the same holds for fixed-quantity auctions.

From a practical point of view, some literature suggests that clear focal points make collusive behavior more likely (see, for example, Knittel and Stango 2003; Cramton and Ockenfels 2017). This is often a major concern of practitioners. An unknown budget can prevent collusion by eliminating the clear focal point of evenly splitting the budget. However, if procurement agencies are generally unable or unwilling to create uncertainty over the budget, the results of this article remain primarily theoretical.

### 5.2 | Uncertainty of the Bidders' Cost

The importance of an uncertain budget for the model hinges on the assumption of common knowledge of costs among the bidders. If their costs are private information, a symmetric equilibrium in pure strategies likely persists with a known budget, and collusion is harder as well. Consequently, an uncertain budget may not be necessary. Not considering private costs is the main limitation of the model in this article. However, there is reason to believe that this does alter the strategic effect of budget caps.

The budget effect probably carries over to a model with private costs and a known budget: A bidder raising their bids curbs the total demanded quantity and, thus, the quantity the bidder can win. Fundamentally, the bidders' problems share the same structure in both models, whether the probability of winning a certain number of units corresponds to the auctioneer's budget being sufficiently high or the other bidders not being too competitive.

### 5.3 | Optimal Mechanism with Individual Bidder Types

A model with individual bidder types and a known budget also allows for the analysis of the optimal mechanism for procuring the highest possible quantity with a given budget in a natural setup. Such a perspective complements the findings of this article. First, this article is not concerned with the *optimal* mechanism but with the strategic differences between budget and fixed-quantity auctions. Second, it does not model the auctioneer's uncertainty over the bidders' costs explicitly. Thus, it does not allow meaningfully answering the question of optimality.

If there is a continuum of bidder types, with high types having unambiguously higher costs than low types, and it is common knowledge that types represent independent draws from some distribution, the optimal mechanism for maximizing the procured quantity with a given budget can be derived analogously to Dasgupta and Spulber (1989). Participation constraints and incentive compatibility have standard implications, with no information rents accruing to the least efficient type and information rents reflecting the cost advantage of more efficient types. However, the auctioneer always spends the same amount, that is, the entire budget, as this allows them to procure a higher quantity. Consequently, when all bidders are of the least efficient type, demand is not distorted: as would be the case under perfect information, the auctioneer only pays for production costs and spends the entire budget. Thus, the same quantity as under perfect information is procured. By contrast, the bidders always make profits on inframarginal units in the budget auction.

### 5.4 | Endogenous Quality

The model also abstracts from quality considerations. These could weaken the result of Proposition 4 if the more competitive environment of the budget auction entailed a stronger incentive for bidders to lower product quality and costs. However, the marginal increase in expected profits a bidder can realize by lowering their costs is identical in both auction designs due to the dynamic equivalent of the standard envelope theorem (see Caputo 1990). As profits are maximized with respect to bids, a decrease in costs does not affect profits through bids. It only increases profits to the degree that it reduces costs for the (unchanged) equilibrium quantity. Thus, with the normalization of demand in Proposition 4, reducing costs is equally attractive under both designs.

### 5.5 | Preferences of the Auctioneer

Lower unit costs in the budget auction do not imply that the auctioneer necessarily prefers the budget auction. The article of Weitzman (1974) gives insights into how the preferences of the auctioneer, which are not modeled explicitly in this article, matter. In his model, the state either fixes a level of emissions or their price. The market outcome regarding the other variable is unknown to the state when it decides on its policy, as the costs of market participants are uncertain. In the model of Weitzman (1974), market participants do not behave strategically. Consequently, the costs of reaching a given level of emissions are identical under both policy options. Nonetheless, the state normally prefers one option: By fixing the level of emissions or their price for all realizations of costs, the state generally misses the optimal outcome in any given situation. Roughly speaking, the state should fix the price if the optimal price of emissions is almost identical for all cost realization, whereas the optimal level of emissions varies strongly and vice versa. An auctioneer who chooses between a budget and a fixed-quantity auction faces a similar problem. They fix either total spending or the traded quantity, and the not-fixed variable varies with the bidders' unknown costs. The insights of Weitzman (1974) imply that an auctioneer who wants to spend roughly the same amount (procure the same quantity) for any realization of the bidders' cost by tendency prefers a budget (fixed-quantity) auction. For example, an auctioneer who intrinsically values spending exactly their budget (for example, because of internal budgeting) tends to prefer a budget auction. In contrast to the model of Weitzman (1974), an auctioneer choosing between a budget and a fixed-quantity auction must also account for the strategic differences between the two. As the budget auction allows procuring a given quantity at lower total costs, it gains in attractiveness compared to a setting without strategic effects for most plausible preferences of the auctioneer.

## 6 | Conclusion

This article has considered a pay-as-bid procurement auction, in which the auctioneer maximizes the number of procured units given a secret budget constraint. In line with previous literature, the analysis assumed that the bidders are symmetric and share a common belief concerning the distribution of the auctioneer's budget. Deriving the symmetric equilibrium of the budget auction revealed that the budget auction introduces additional strategic complexity in comparison to the case of an uncertain but fixed quantity. In the budget auction, bidders not only weigh higher profit margins against a lower probability of realizing these same margins; higher bids also negatively influence the probability of winning later units. If a bidder demands a higher payment for some quantity, this reduces the auctioneer's residual budget such that it is less likely that the auctioneer's budget suffices to procure higher quantities from the bidder, which lowers bids. Thus, the budget cap leads to a link between bids in the bidders' problem, which has no correspondence in auctions with a fixed traded quantity. Due to this effect, bids in the budget auction are lower than in an auction with uncertain quantity, in which bidders believe in the same distribution of procured units as in the equilibrium of the budget auction.

**Acknowledgments**

I am grateful to Stefan Seifert for sparking my interest in the topic of this article and providing invaluable feedback. I am also deeply indebted to Fabian Herweg, Alessandro Kadner-Graziano, Christoph Kretschmer, three anonymous referees, and the editor Alessandro Bonatti for their extremely helpful comments and suggestions on earlier versions of this work. All remaining errors and shortcomings are entirely my own responsibility.

Open access funding enabled and organized by Projekt DEAL.

**Endnotes**

[1]Other possible solutions have been explored, for example, by Kremer and Nyborg (2004), Damianov (2005), and Burkett and Woodward (2020a).

[2]It only makes sense for the auctioneer to conduct a multi-unit auction, that is, source from multiple bidders, if marginal costs are increasing on some relevant interval.

[3]This is important because in most cases, the symmetric equilibrium in the budget auction can only be obtained numerically, which is impossible with an infinite slope.

[4]Due to both technical reasons and the fact that Holmberg (2009) considers electricity markets, he analyzes a somewhat different case, where demand exceeds production capacities with a positive probability. The model of Pycia and Woodward (2023) needs to be adapted to the case of a procurement auction.

[5]Technically, this formulation is somewhat imprecise, as a *single* bid does not influence a bidder's cumulative bids and, therefore, does not influence the probability of winning $y > x$ units.

[6] Normally, the Euler equation is a second-order differential equation. However, simultaneously determining all bidders' mutual best responses adds a degree of freedom.

[7] Off-equilibrium quantities are only relevant, if a deviating player reduces the maximum quantity they win. This would either require $b_i(\bar{x}_i) > c(\bar{x}_i)$, or $b_i(\bar{x}_i) = c(\bar{x}_i) < b_j(\bar{x}_j) = c(\bar{x}_j)$, where $i \neq j$ and $\bar{x}_j > \bar{x}_i$. In either case, the bidder could profitably win further units.

[8] The convenient property that the relevant part of the bidding function spans the same interval also explains the choice of the budget auction as the baseline for the comparison.

[9] Bids in the corresponding fixed-quantity auction are $b^{FQ}(x) = \frac{1}{2}x - \frac{\sqrt{15}}{5} + \frac{27\sqrt{3}\arccos(\frac{\sqrt{15}}{45}(5x+2\sqrt{15}))}{2\sqrt{225-75x^2-60\sqrt{15}x}}$.

[10] This is due to increased competition and total social costs for producing a given quantity decreasing with more producers.

[11] Stronger diseconomies of scale here are interpreted as "'more convex" cost functions. In the example, this is formalized analogously to utility functions with constant relative risk aversion. The finding regarding bidders' cost is not driven by the more convex cost functions resulting in lower costs for relevant quantities.

[12] The strategy $\hat{b}_i(x)$ is of course inadmissible as a solution because it has jump discontinuities at $x_2$ and $x_3$. However, the *optimal solution* is continuous, as can easily be seen from setting the bidder's problem up as a problem of optimal control, which would allow jump discontinuities in $b_i(x)$.

[13] Note that this proof is out of proper logical order.

## References


Allen, J., R. Clark, B. Hickman, and E. Richert. 2024. "Resolving Failed Banks: Uncertainty, Multiple Bidding and Auction Design." *Review of Economic Studies* 91: 1201–1242.

Back, K., and J. F. Zender. 2001. "Auctions of Divisible Goods with Endogenous Supply." *Economics Letters* 73: 29–34.

Bergemann, D., and J. Hörner. 2018. "Should First-Price Auctions Be Transparent?" *American Economic Journal: Microeconomics* 10: 177–218.

Burkett, J., and K. Woodward. 2020a. "Reserve Prices Eliminate Low Revenue Equilibria in Uniform Price Auctions." *Games and Economic Behavior* 121: 297–306.

Burkett, J., and K. Woodward. 2020b. "Uniform Price Auctions with a Last Accepted Bid Pricing Rule." *Journal of Economic Theory* 185: 104954.

Caputo, M. R. 1990. "Comparative Dynamics via Envelope Methods in Variational Calculus." *The Review of Economic Studies* 57: 689–697.

Caputo, M. R. 2005. *Foundations of Dynamic Economic Analysis: Optimal Control Theory and Applications* Cambridge University Press.

Chiang, A. C. 1992. *Elements of Dynamic Optimization* Waveland Press.

Cramton, P., and A. Ockenfels. 2017. "The German 4G Spectrum Auction: Design and Behaviour." *The Economic Journal* 127: F305–F324.

Damianov, D. S. 2005. "The Uniform Price Auction with Endogenous Supply." *Economics Letters* 88: 152–158.

Dasgupta, S., and D. F. Spulber. 1989. "Managing Procurement Auctions." *Information Economics and Policy* 4: 5–29.

Dastidar, K. G. 2008. "On Procurement Auctions with Fixed Budgets." *Research in Economics* 62: 72–91.

Deck, C. A., and B. J. Wilson. 2008. "Fixed Revenue Auctions: Theory and Behavior." *Economic Inquiry* 46: 342–354.

Hansen, R. G. 1988. "Auctions with Endogenous Quantity." *The RAND Journal of Economics* 19: 44–58.

Holmberg, P. 2009. "Supply Function Equilibria of Pay-as-bid Auctions." *Journal of Regulatory Economics* 36: 154–177.

Klemperer, P. D., and M. A. Meyer. 1989. "Supply Function Equilibria in Oligopoly under Uncertainty." *Econometrica* 57: 1243–1277.

Knittel, C. R., and V. Stango. 2003. "Price Ceilings as Focal Points for Tacit Collusion: Evidence from Credit Cards." *American Economic Review* 93: 1703–1729.

Kremer, I., and K. G. Nyborg. 2004. "Underpricing and Market Power in Uniform Price Auctions." *The Review of Financial Studies* 17: 849–877.

Lengwiler, Y. 1999. "The Multiple Unit Auction with Variable Supply." *Economic Theory* 14: 373–392.

Liu, T., and C. Parlour. 2014. "Raising Money." Unpublished.

McAdams, D. 2007. "Adjustable Supply in Uniform Price Auctions: Non-Commitment as a Strategic Tool." *Economics Letters* 95: 48–53.

Oniki, H. 1973. "Comparative Dynamics (Sensitivity Analysis) in Optimal Control Theory." *Journal of Economic Theory* 6: 265–283.

Pycia, M., and K. Woodward. 2023. "Auctions of Homogeneous Goods: A Case for Pay-as-Bid." Working Paper. Accessed September 14, 2023. https://sites.econ.uzh.ch/pycia/pycia-woodward-paba.pdf.

Rothkopf, M. H., T. J. Teisberg, and E. P. Kahn. 1990. "Why Are Vickrey Auctions Rare?" *Journal of Political Economy* 98: 94–109.

RTR. 2020. "5G-Auktion macht nahezu flächendeckende Mobilfunk-Breitbandversorgung in Zukunft möglich." Accessed November 21, 2022. https://www.rtr.at/TKP/presse/pressemitteilungen/pressemitteilungen/pinfo11092020tkp.de.html.

Tillio, A. d., N. Kos, and M. Messner. 2016. "The Design of Ambiguous Mechanisms." *The Review of Economic Studies* 84: 237–276.

TKK. 2019. "Tender Document in the Procedure for Awarding Spectrum in the 700, 1500 and 2100 MHz Ranges." Accessed December 16, 2022. https://www.rtr.at/TKP/was_wir_tun/telekommunikation/spectrum/procedures/Multibandauktion_700-1500-2100MHz_2020/FRQ5G_2020_tender_document.de.html.

Weitzman, M. L. 1974. "Prices vs. Quantities." *The Review of Economic Studies* 41: 477–491.

Wilson, R. 1979. "Auctions of Shares." *The Quarterly Journal of Economics* 93: 675–689.

## Appendix A: Proof of Proposition 2

### A.1 | First-Order Condition

The relevant derivatives for the first-order condition given in (5) are

$$\frac{\partial \Psi}{\partial B_i} = -[b_i(x) - c(x)] f(B_i(x) + \sum_{j \neq i} B_j(\beta_j(b_i(x)))), \tag{A1}$$

$$\begin{aligned} \frac{\partial \Psi}{\partial b_i} = \quad & 1 - F(B_i(x) + \textstyle\sum_{j \neq i} B_j(\beta_j(b_i(x)))) \\ & -[b_i(x) - c(x)] f(B_i(x) + \textstyle\sum_{j \neq i} B_j(\beta_j(b_i(x)))) \textstyle\sum_{j \neq i} b_i(x) \beta_j'(b_i(x)), \end{aligned} \tag{A2}$$

and

$$\begin{aligned} \frac{d}{dx}\left(\frac{\partial \Psi}{\partial b_i}\right) = & -f(\cdot)\left[b_i(x) + \sum_{j \neq i} b_i(x)\, \beta_j'(b_i(x))\, b_i'(x)\right] \\ & -\left[b_i'(x) - c'(x)\right]\, f(\cdot) \sum_{j \neq i} b_i(x)\, \beta_j'(b_i(x)) \\ & -[b_i(x) - c(x)] \left\{ f'(\cdot) \left[b_i(x) + \sum_{j \neq i} b_i(x)\, \beta_j'(b_i(x))\, b_i'(x)\right] \right. \\ & \times \sum_{j \neq i} b_i(x)\, \beta_j'(b_i(x)) + f(\cdot) \sum_{j \neq i} b_i'(x)\, \beta_j'(b_i(x)) \\ & \left. + f(\cdot) \sum_{j \neq i} b_i(x)\, \beta_j''(b_i(x))\, b_i'(x) \right\}. \end{aligned} \tag{A3}$$

Searching for a symmetric equilibrium with $b(x) = b_i(x) = b_j(x) \forall j$, furthermore gives $\beta_j'(b_i(x)) = \frac{1}{b'(x)}$ and $\beta_j''(b_i(x)) = -\frac{b''(x)}{[b'(x)]^3}$. Using these symmetry conditions together with the derivatives in Equations (A1) and (A3) in the necessary condition from Equation (5) gives Equation (6).

### A.2 | Definitizing the (numeric) Solution to the Characteristic Differential Equation

**Definitizing $B(\bar{x})$ and $b(\bar{x})$**: The highest quantity a given bidder can win, is determined according to the terminal curve $B_i(\bar{x}_i) = \phi(b_i, \bar{x}_i)$. Based on the general transversality condition for the endpoint of a variational problem, the Euler-Lagrange equation of the considered bidder's best response problem from Equation (5) is completed by $B_i(\bar{x}_i) = \phi(b_i, \bar{x}_i)$ and $[\Psi(\cdot) + (\frac{d}{d\bar{x}_i}\phi(\cdot) - b_i(x))\frac{\partial \Psi}{\partial b_i}]_{x=\bar{x}_i} = 0$ (see e.g., Chiang 1992). These conditions translate to

$$B_i(\bar{x}_i) + \sum_{j \neq i} B_j\left(\beta_j(b_i(\bar{x}_i))\right) = \bar{A} \tag{A4}$$

and given Equation (A4)

$$\left(\sum_{j \neq i} b_i(\bar{x}_i)\, \beta_j'(b_i(\bar{x}_i))\, b_i'(\bar{x}_i) + b_i(x)\right) [b_i(\bar{x}_i) - c(\bar{x}_i)]\, f(\bar{A}) \sum_{j \neq i} b_i(\bar{x}_i)\, \beta_j'(b_i(x)) = 0. \tag{A5}$$

By assumption $f(\bar{A}) > 0$ and bids of all players are positive and increasing, such that Equation (A5) implies that

$$b_i(\bar{x}_i) = c(\bar{x}_i). \tag{A6}$$

For the symmetric equilibrium, the transversality conditions from Equations (A4) and (A6) give Equations (7) and (8).

**Definitizing $b'(\bar{x})$**: Given that for a response of $i$ to be optimal, it must be that $b_i(\bar{x}_i) = c(\bar{x}_i)$, the first-order condition of bidder $i$'s problem obtained by using Equations (A1) and (A3) in Equation (5) simplifies substantially. Making use of the assumptions that $f(\bar{A})$, $f'(\bar{A})$, as well as the derivatives of the bidding functions, are finite, several terms vanish such that the simplified first-order condition emerges as

$$b_i'(\bar{x}_i) = \frac{c'(\bar{x}_i)}{2} - \frac{1}{2\sum_{j \neq i} \beta_j'(c(\bar{x}_i))}. \tag{A7}$$

As, in equilibrium, this condition must hold for every player, we can use symmetry to obtain the formulation in Equation (9).

**Definitizing** $b''(\bar{x})$: The formulation of $b''(\bar{x})$, as given in (6), cannot be used for the numerical approximation of the symmetric equilibrium, as it is indeterminate. This can be seen by substituting the expression for $b'(\bar{x})$ from (9) and noting that $b(\bar{x}) - c(\bar{x}) = 0$. Applying Hôpital's rule to (6) yields

$$b''(\bar{x}) = \frac{[b'(\bar{x})]^2}{b(\bar{x})\,[b'(\bar{x})-2c'(\bar{x})]} \Bigg[3b'(\bar{x}) - \frac{(2n-3)}{(n-1)}\,c'(\bar{x}) - c''(\bar{x})\,\frac{b(\bar{x})}{b'(\bar{x})} + n\,\frac{f'(nB(\bar{x}))}{f(nB(\bar{x}))}\,[b'(\bar{x}) - c'(\bar{x})]\,\frac{[b(\bar{x})]^2}{b'(\bar{x})}\Bigg]. \tag{A8}$$

## A.3 | Properties of Equilibrium Bids

**Bids are Above Marginal Costs**: Consider a bidder $i$ with bidding function $b_i(x)$ who, on some arbitrarily small, relevant interval $[x_1, x_2]$ of strictly positive length, bids weakly below marginal costs. Furthermore, assume that the bidder also has a weakly positive margin on the interval $[x_2, x_3]$ with $x_3 - x_2 = x_2 - x_1$ and $x_3 \le \bar{x}_i$. Consider the alternative bidding function

$$\hat{b}_i(x) = \begin{cases} b_i(x) + \varepsilon & \text{for } x \in [x_1, x_2] \\ b_i(x) - \delta\varepsilon & \text{for } x \in (x_2, x_3] \\ b_i(x) & \text{else,} \end{cases}$$

where $\delta \ge 1$ and $x_3 = [(1+\delta)x_2 - x_1)]/\delta$. For an appropriate choice of $\delta$ the bidder would profit from choosing a positive $\varepsilon$. Because both $b_i(x)$ and $B_i(x)$ are unchanged outside the interval $[x_1, x_3]$, the marginal effect on profits of increasing $\varepsilon$ from 0 is

$$\begin{aligned} \frac{\partial \mathbb{E}[\pi_i]}{\partial \varepsilon}\Big|_{\varepsilon=0} = &\int_{x_1}^{x_2} \Bigg[1 - F\Bigg(B_i(x) + \sum_{j\neq i} B_j\big(\beta_j(b_i(x))\big)\Bigg)\Bigg]\,dx - \int_{x_2}^{x_3(\delta)} \delta[1 - F(\cdot)]\,dx \\ &- \int_{x_1}^{x_2} \underbrace{[b_i(x) - c(x)]\,f(\cdot)\Bigg[(x - x_1) + \sum_{j\neq i} b_i(x)\beta_j'(b_i(x))\Bigg]}_{\le 0}\,dx \\ &+ \int_{x_2}^{x_3(\delta)} [b_i(x) - c(x)]\,f(\cdot)\,\delta \sum_{j\neq i} b_i(x)\beta_j'(b_i(x))\,dx \\ &- \int_{x_2}^{x_3(\delta)} [b_i(x) - c(x)]\,f(\cdot)\,[(x_2 - x_1) - \delta(x - x_2)]\,dx. \end{aligned} \tag{A9}$$

For high enough $\delta$, $\frac{\partial \mathbb{E}[\pi_i]}{\partial \varepsilon}|_{\varepsilon=0} > 0$. To see this, note that the first three lines are each (weakly) positive. The first line is positive because $\int_{x_1}^{x_2} 1 - F(\cdot)dx > (x_2 - x_1)[1 - F(\cdot)]_{x=x_2} = \delta(x_3 - x_2)[1 - F(\cdot)]_{x=x_2} > \int_{x_2}^{x_3} \delta[1 - F(\cdot)]dx$. The second line is positive due to the weakly negative margins in the interval. Lastly, all terms in the integrand in the third line are positive. The integral in the fourth line is decreasing in $\delta$ and converges to zero as $\delta \to \infty$. It follows that for high enough $\delta$, the expression in (A9) is positive and there is a profitable deviation from $b_i(x)$.[12] This argument also holds if the bidder does not bid above costs for $x > x_2$. As a consequence, the profitability of a deviation to $\hat{b}_i(x)$ shows that a bidder's best response cannot include below-cost bidding.

**Monotonicity of Bids**: Equation (9) establishes that $b'(\bar{x}) > 0$. Against the backdrop of Equation (B2) the slope of equilibrium bids for $x \in [0, \bar{x})$ can be written as[13]

$$b'(x) = b'_{FQ}(x) - \Delta b'(x). \tag{A10}$$

Given Equations (B1) and (B3), we can write

$$b'(x) = (n-1)\,[1 - F(\cdot)]^{-1}\,f(\cdot)\,b(x)\,[b(x) - c(x)] - (n-1)\,[1 - F(\cdot)]^{\frac{1-n}{n}}\,\Delta\tilde{b}'(x) > 0. \tag{A11}$$

It was shown above that $b(x) > c(x)$ for $x \in [0, \bar{x})$ and Appendix B shows that $\Delta\tilde{b}'(x) < 0$ for $x \in [0, \bar{x})$. Thus, Equation (A11) establishes the monotonicity of bids.

## A.4 | Second-Order Condition

**General Case**: The most straightforward way to ensure that the symmetric equilibrium candidate described by Equations (6)–(9) does indeed constitute an equilibrium is to make sure that the integrand of a unilaterally deviating bidder's profit function is globally concave. Although, depending on the case at hand, weaker conditions might be given, I will state the conditions for the strict concavity of $\Psi(\cdot)$. For the deviating bidder $i$, the integrand of their expected profits is

$$\Psi(B_i, b_i, x) = [b_i(x) - c(x)]\,[1 - F(B_i(x) + (n-1)\,B(\beta(b_i(x)))]. \tag{A12}$$

Strict global concavity requires that for any $B_i$ and $b_i$ the corresponding quadratic form is negative definite. In particular, it must be the case that

$$\frac{\partial^2\Psi}{\partial(b_i)^2} < 0 \text{ and } \frac{\partial^2\Psi}{\partial(b_i)^2} \times \frac{\partial^2\Psi}{\partial(B_i)^2} - \left(\frac{\partial^2\Psi}{\partial b_i\,\partial B_i}\right)^2 < 0 \tag{A13}$$

for all $(B_i, b_i)$, where

$$\frac{\partial^2\Psi}{\partial(b_i)^2} = -2(n-1) f(\cdot) b_i(x) \beta'(\cdot) - [b_i(x) - c(x)] f'(\cdot) \left[(n-1) b_i(x) \beta'(\cdot)\right]^2 - (n-1)[b_i(x) - c(x)] f(\cdot)\left(\beta'(\cdot) + b_i(x)\beta''(\cdot)\right),$$

$$\frac{\partial^2\Psi}{\partial(B_i)^2} = -[b_i(x) - c(x)] f'(\cdot) \text{ and } \frac{\partial^2\Psi}{\partial b_i\, \partial B_i} = -f(\cdot) - (n-1)[b_i(x) - c(x)] f'(\cdot) b_i(x) \beta'(\cdot).$$

The condition of global concavity can be weakened, as any maximum of a bidder's expected profits can only be located in the area where $b_i(x) > c(x)$.

**Example from Section 3.3**: Assume that the continuation of the bidding function for off-equilibrium quantities is differentiable and weakly concave. The problem satisfies concavity given that only cases with $b_i(x) - c(x)$ need to be considered. In this case $\frac{\partial^2\Psi}{\partial(b_i)^2} < 0$. Furthermore, due to the uniform distribution $\frac{\partial^2\Psi}{\partial(B_i)^2} = 0$ and $\frac{\partial^2\Psi}{\partial b_i\, \partial B_i} = -1$. Thus, the second-order condition is satisfied in the example.

### Appendix B: Proof of Proposition 4

The main text establishes that $b^{FQ}(\bar{x}) = b^B(\bar{x})$. It remains to be shown that $b^{FQ}(x) > b^B(x)$ for $x \in [0, \bar{x})$. Based on Equations (3) and (10), bids in the corresponding fixed-quantity auction are

$$b^{FQ}(x) = (n-1)\left[1 - F\left(n B^B(x)\right)\right]^{\frac{1-n}{n}} \int_x^{\bar{x}} [1 - F(\cdot)]^{-1/n} f(\cdot)\, b^B(y)\, c(y)\, dy. \tag{B1}$$

Noting that bids in the budget auction can be written as

$$b^B(x) = (n-1)\left[1 - F\left(n B^B(x)\right)\right]^{\frac{1-n}{n}} \mu(x) \int_x^{\bar{x}} [1 - F(\cdot)]^{-1/n} f(\cdot)\, b^B(y)\, b^B(y)\, dy,$$

$$\text{where } \mu(x) = \frac{b^B(x)}{b^B(x) + \left[1 - F\left(n B^B(x)\right)\right]^{\frac{1-n}{n}} \int_x^{\bar{x}} \left[1 - F\left(n B^B(y)\right)\right]^{\frac{n-1}{n}} b^{B\prime}(y)\, dy},$$

the difference in bids, $\Delta b(x) = b^{FQ}(x) - b^B(x)$, is

$$\Delta b(x) = (n-1)[1 - F(\cdot)]^{\frac{1-n}{n}} \int_x^{\bar{x}} [1 - F(\cdot)]^{-1/n} f(\cdot)\, b^B(y) [c(y) - \mu(x) b^B(y)]\, dy. \tag{B2}$$

Given that we are considering $x < \bar{x}$, the hypothesis that $\Delta b(x) > 0$ is equivalent to

$$\Delta\tilde{b}(x) = \int_x^{\bar{x}} \left[1 - F\left(n B^B(y)\right)\right]^{-1/n} f\left(n B^B(y)\right) b^B(y)\, [c(y) - \mu(x) b^B(y)]\, dy > 0. \tag{B3}$$

Because $\lim_{x \to \bar{x}} \Delta\tilde{b}(x) = 0$, a sufficient condition for inequality (B3) and by extension $\Delta b(x) > 0$ for $x < \bar{x}$ to hold is that $\Delta\tilde{b}'(x) < 0$, that is,

$$\Delta\tilde{b}'(x) = [1 - F(\cdot)]^{-1/n} f(\cdot)\, b^B(x) \left[\mu(x) b^B(x) - c(x)\right] - \mu'(x) \int_x^{\bar{x}} [1 - F(\cdot)]^{-1/n} f(\cdot)\, [b^B(y)]^2\, dy < 0. \tag{B4}$$

Some manipulation reveals that inequality (B4) is equivalent to

$$b^B(x) - c(x) - \frac{1}{(n-1)} \frac{1 - F\left(n B^B(x)\right)}{f(n B^B(x))} \frac{b^{B\prime}(x)}{b^B(x)} < 0. \tag{B5}$$

A comparison with Equation (A2) shows that inequality (B5) is equivalent to $\left.\frac{\partial\Psi}{\partial b_i}\right|_{B_i = B_j = B^B, b_i = b_j = b^B} > 0$. Given that bids are above costs (see Appendix A.3), Equation (A1) reveals that $\frac{\partial\Psi}{\partial B_i} < 0$ for $x \in [0, \bar{x})$ if evaluated along the equilibrium path. Thus, the first-order condition from (5) implies that $\frac{d}{dx}\frac{\partial\Psi}{\partial b_i} < 0$ for $x < \bar{x}$ in equilibrium. Using the terminal conditions from Equations (7) and (8) in (A2) gives that, evaluated at equilibrium bids, $\left.\frac{\partial\Psi}{\partial b_i}\right|_{x = \bar{x}} = 0$. It follows that $\left.\frac{\partial\Psi}{\partial b_i}\right|_{B_i = B_j = B^B, b_i = b_j = b^B} > 0$, that is, inequality (B5) holds.