Optimal Codes and Arcs for the Generalized Hamming Weights

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Abstract

This text contains some notes on the Griesmer bound. In particular, we give a geometric proof of the Griesmer bound for the generalized weights and show that a Solomon–Stiffler type construction attains it if the minimum distance is sufficiently large. We also determine the parameters of optimal binary codes for dimensions at most seven and the optimal ternary codes for dimensions at most five.

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1 Introduction

The Hamming weight of a codeword equals the size of its support and the minimum Hamming weight of a linear code is the minimum Hamming weight of the non-zero codewords. For a subcode the support us given by the set of all positions where at least one of the codewords in the subcode has a non-zero entry. With this, the rth generalized Hamming weight of a linear code is the size of the smallest support of an r-dimensional subcode. The generalized Hamming weights can be used to describe the cryptography performance of a linear code over the wire-tap channel of type II [29] and to determine the trellis complexity of the code [6, 9, 10, 18]. From a geometrical point of view the rth generalized Hamming weight of a linear code corresponds to the number of points outside of a subspace of codimension r, where the points are the one dimensional subspaces spanned by the columns of a generator matrix of the linear code, see e.g. [15, 27]. While one can easily find tables on the best known bounds for the parameters of linear codes with respect to the minimum Hamming distance for small parameters we were not able to find such a table for the rth generalized Hamming weight. The aim of this paper is to tabulate those numbers, where we will mainly use the geometric reformulation as multisets of points in projective spaces. In [2] the authors gave a construction for additive codes based on linear codes and their generalized Hamming weights. So, the constructions studied in this paper also give constructions for additive codes and the studied upper bounds show limitations for this specific construction. More precisely, for all sufficiently large distances a Griesmer type bound for additive codes can always be attained [20]. As we will see, this also holds for linear codes and the rth generalized Hamming weight, but the mentioned construction of additive codes results in optimal codes for a subset of the parameters only.

The remaining part of the paper is structured as follows. In Section 2 we present the necessary preliminaries. In Section 3 we analyze the Griesmer bound for the rth generalized Hamming distance. It turns out that a Griesmer code with respect to the Hamming distance also attains the Griesmer bound for the rth generalized Hamming distance. In Section 4, we determine the minimum possible lengths of $[n,k]_q$ codes with given minimum rth generalized Hamming weight d for some small parameters. We present our results in the geometric version, i.e. we determine the maximum number $m_q^{(r)}(k,w)$ of points in $\operatorname{PG}(k-1,q)$ such that each subspace of dimension r contains at most w points. We completely determine $m_2^{(r)}(k-1,w)$ for all $k \le 7$ and $m_3^{(r)}(k-1,w)$ for all $k \le 5$.

2 Preliminaries

Linear codes. A linear $[n,k]_q$ code C is a k-dimensional subspace of $\mathbb{F}_q^n$. For $c=(c_1,\dots,c_n)\in\mathbb{F}_q^n$ we call

$$\operatorname{supp}(c) := \{1 \le i \le n \,:\, c_i \neq 0\} \tag{1}$$

the *support* of c and $\operatorname{wt}(c)=|\operatorname{supp}(c)|$ its *weight*. More generally, for a vector subspace C in $\mathbb{F}_q^n$, we define its *support* as the set of all coordinate positions in which the vectors of C are not identically zero. In other words,

$$\operatorname{supp}(C) := \{1 \le i \le n \,:\, \exists c=(c_1,\dots,c_n)\in C, c_i \neq 0\}\,. \tag{2}$$

For two $\mathbb{F}_q$-vector spaces C, C' of $\mathbb{F}_q^n$ we write $C' \le C$ if C' is a subspace of C. The *rth generalized Hamming weight* of a linear code C [14, 19], denoted as $d_r(C)$, is the size of the smallest support of an r-dimensional subcode of C , i.e.

$$d_r(C) := \min\{|\operatorname{supp}(C')| \,:\, C' \le C, \dim(C') = r\}\,. \tag{3}$$

In particular, $d_1(C)$ is the minimum Hamming distance of the C. A linear code of length n, dimension k and rth generalized Hamming weight equal to d_r will be called an $[n, k, d_r]^{(r)}$-code. The sequence $(d_1(C), \ldots, d_k(C))$ is called the *weight hierarchy* of a linear $[n, k]_q$ code C. Clearly, $1 \leq d_1(C) < d_2 < \cdots < d_k(C) \leq n$ (cf. e.g. [29]).

Multisets of points in $\mathrm{PG}(\mathbf{k-1}, \mathbf{q})$. A *multiset* in $\mathrm{PG}(k-1, q)$ is a mapping $\mathcal{K}\colon \mathcal{P} \to \mathbb{N}_0$, from the pointset $\mathcal{P}$ of $\mathrm{PG}(r, q)$ to the set of non-negative integers, which assigns a multiplicity to each point of $\mathcal{P}$. This mapping is extended to any subset $\mathcal{Q}$ of $\mathcal{P}$ by $\mathcal{K}(\mathcal{Q}) = \sum_{P \in \mathcal{Q}} \mathcal{K}(P)$. The integer $\mathcal{K}(\mathcal{Q})$ is called the multiplicity of $\mathcal{Q}$.

A multiset $\mathcal{K}$ is called an *(n, w)-arc* (resp. *(n, u)-minihyper*), if $\mathcal{K}(\mathcal{P}) = n$, $\mathcal{K}(H) \leq w$ (resp. $\mathcal{K}(H) \geq u$) for each hyperplane H in $\mathrm{PG}(k-1, q)$, and there is a hyperplane H_0 with $\mathcal{K}(H_0) = w$ (resp. $\mathcal{K}(H_0) = u$).

Let $\mathcal{K}$ be a multiset in $\mathrm{PG}(k-1, q)$. We denote by w_r the maximal multiplicity of a r-dimensional subspace of $\mathrm{PG}(r, q)$ with respect to $\mathcal{K}$. In other words,

$$w_r := \max\left\{\mathcal{K}(S) \,:\, S \text{ is a subspace of } \mathrm{PG}(k-1, q), \dim(S) = r\right\}. \qquad (4)$$

Similarly, we denote by u_r the minimal multiplicity of a r-dimensional subspace of $\mathrm{PG}(r, q)$ with respect to $\mathcal{K}$. In other words,

$$u_r := \min\left\{\mathcal{K}(S) \,:\, S \text{ is a subspace of } \mathrm{PG}(k-1, q), \dim(S) = r\right\}. \qquad (5)$$

A multiset in $\mathrm{PG}(k-1, q)$ of multiplicity n is called an $(n, w_r)^{(r)}$-arc (resp $(n, u_r)^{(r)}$-minihyper) if each r-dimensional subspace of $\mathrm{PG}(k-1, q)$ has multiplicity at most w_r (resp. at least u_r), and there exists an r-dimensional subspace with this multiplicity.

It is known that there exists a one-to-one correspondence between the isomorphism classes of the linear $[n, k]_q$-codes of full length (no coordinate is identically zero in all codewords) and the classes of projectively equivalent multisets in $\mathrm{PG}(k-1, q)$, where $k \geq 2$. The correspondence can be described as follows. Let C be a linear code of full length with parameters $[n, k, d]_q$, and let $G = (g_1^T \cdots g_n^T)$, $g_i \in \mathbb{F}_q^k$, be a generator matrix of C. The columns g_i^T are considered as the homogeneous coordinates of points in $\mathrm{PG}(k-1, q)$. In this way, the generator matrix G is associated with an ordered n-tuple of points $(P_1, \ldots, P_n)$ in $\mathrm{PG}(k-1, q)$. This n-tuple defines a multiset $\mathcal{K}$ by $\mathcal{K}(P) := |\{i | P_i = P\}|$. Clearly,

$$w_r + d_{k-r-1} = n. \qquad (6)$$

This implies $w_0 < w_1 < \cdots < w_{k-1} = n$. If $s \geq w_0$ is an integer, then $\mathcal{K}' = s - \mathcal{K}$ is a multiset of cardinality $sv_k - n$ with $u_r = sv_{r+1} - w_r$, $r = 0, 1, \ldots, k-1$. Here $v_r = (q^r - 1)/(q-1)$, as usual.

Definition 1. *We denote by $m_q^{(r)}(k, w)$ the maximal multiplicity of a multiset in $\mathrm{PG}(k-1, q)$ such that every r-dimensional subspace has multiplicity at most w. In other words, $m_q^{(r)}(k, w)$ is defined as the largest n for which there exists an $(n, w)^{(r)}$-arc in $\mathrm{PG}(k-1, q)$.*

Double-counting directly gives:

Lemma 1. $m_q^{(r)}(k,w) \le \frac{v_k}{v_{k-r}} \cdot w$

Taking the union of two multisets of points gives:

Lemma 2. $m_q^{(r)}(k,w'_r+w''_r) \ge m_q^{(r)}(k,w'_r) + m_q^{(r)}(k,w''_r)$

In what follows we shall need multisets induced by a projection from a subspace. Let $\mathcal{K}$ be a multiset in $\mathrm{PG}(k-1,q)$. Fix an i-dimensional subspace δ in $\mathrm{PG}(k-1,q)$, with $\mathcal{K}(\delta)=t$. Let further π be a j-dimensional subspace in $\mathrm{PG}(k-1,q)$ of complementary dimension, i.e. $i+j=k-2$ and $\delta\cap\pi=\varnothing$. Define the projection $\varphi=\varphi_{\delta,\pi}$ from δ onto π by

$$\varphi\colon \begin{cases} \mathcal{P}\setminus\delta & \to \quad \pi \\ Q & \to \quad \pi\cap\langle\delta,Q\rangle. \end{cases} \tag{7}$$

As before, $\mathcal{P}$ denotes the set of all points of $\mathrm{PG}(k-1,q)$. Note that φ maps $(i+s)$-subspaces containing δ into $(s-1)$-subspaces in π. Denote by $\mathcal{P}'$ the set of all points in π. We define the induced multiset $\mathcal{K}^{\varphi}:\mathcal{P}'\to\mathbb{N}_0$ by

$$\mathcal{K}^{\varphi}(Q) \;=\; \sum_{P:\varphi(P)=Q} \mathcal{K}(P).$$

It is clear that For every subspace S in $\mathrm{PG}(k-1,q)$ that contains δ, it holds $\mathcal{K}^{\varphi}(\varphi(S))=\mathcal{K}(S)-t$. In particular, if $\mathcal{K}$ is a $(n,w_r)^{(r)}$-arc (resp. $(n,u_r)^{(r)}$-minihyper) then $\mathcal{K}^{\varphi}$ is an $(n-t,w_r-t)^{(r)}$-arc (resp. an $(n-t,u_r-t)^{(r)}$-minihyper).

3 The Griesmer bound

It is well known that the Griesmer bound [12]:

$$n \ge \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil =: g_q(k,d). \tag{8}$$

for the minimum length of an $[n,k]_q$ code with given minimum Hamming distance d is attained if d is sufficiently large. A similar estimate holds for the rth generalized Hamming distance. This result was proved in [15, 13]. The corresponding bound is called *the generalized Griesmer bound*. Similarly to the classical Griesmer bound, this result is of purely geometric nature.

Below, we give a geometric proof of the generalized Griesmer bound.

Theorem 1. *Let C be an $[n,k,d_r]^{(r)}$-code, where $1\le r\le k$. Then*

$$n \ge d_r + \sum_{i=1}^{k-r} \left\lceil \frac{d_r}{q^i v_r} \right\rceil. \tag{9}$$

Proof. The result is clearly true for $r = k$. It is enough to prove it for codes of full length, i.e. $n = d_k$.

Let C be an $[n, k, d_r]_q^{(r)}$ code and let $\mathcal{K}$ be the $(n, w_{k-r-1})^{(k-r-1)}$-arc in $\mathrm{PG}(k-1, q)$ associated with it. Consider a point of maximum multiplicity w_0. The total multiplicity of the remaining points is $n - w_0$, and there must be point among them of multiplicity at least $\dfrac{n - w_0}{v_k - 1}$. Hence

$$w_0 \geq \left\lceil \frac{n - w_0}{q v_{k-1}} \right\rceil .$$

This is the Griesmer inequality for $r = k - 1$ since $w_0 = n - d_{k-1}$.

Without loss of generality, we shall assume that C is not extendable in the following sense. Every point in $\mathrm{PG}(k-1, q)$ is contained in a maximal subspace of dimension r.

Fix a $(k-r-1)$-dimensional subspace S of maximal multiplicity, i.e. multiplicity w_{k-r-1}. There exist $v_{k+1} - v_{k-r}$ points in $\mathrm{PG}(k-1, q)$ outside S. Hence there is a point $P \notin S$ of multiplicity

$$t \geq \frac{n - w_{k-r-1}}{q^{k-r} v_r} = \frac{d_r}{q^{k-r} v_r}.$$

Consider a projection φ from P onto some hyperplane $H \cong \mathrm{PG}(k-2, q)$ with $P \notin H$. The induced arc $\mathcal{K}^{\varphi}$ has parameters $(n-t, w_r - t)^{(r)}$. The image of every subspace T through P is a subspace in H of dimension $\dim T - 1$. Moreover,

$$\mathcal{K}^{(\varphi)}(\varphi(T)) = \mathcal{K}(T) - t.$$

If we set $w'_i = \max_{T : \dim T = i} \mathcal{K}^{\varphi}(T)$, we have $w'_i \leq w_{i+1} - t$. If we also set $d'_i = n - t - w'_{k-i-2}$, we get $d_r \leq d'_r$. By the induction hypothesis

$$n' \geq d'_{r-1} + \sum_{i=1}^{k-r-1} \lceil \frac{d'_{r-1}}{q^i v_{r-1}} \rceil.$$

Now using $d_r \leq d'_r$, we obtain:

$$\begin{aligned} n' = n - t &= d'_r + \sum_{i=1}^{k-r-1} \left\lceil \frac{d'_i}{q^i v_r} \right\rceil \\ &= d_r + \sum_{i=1}^{k-r-1} \left\lceil \frac{d_i}{q^i v_r} \right\rceil \end{aligned}$$

This implies

$$\begin{aligned} n &\geq d_r + t + \sum_{i=1}^{k-r-1} \left\lceil \frac{d_i}{q^i v_r} \right\rceil \\ &\geq d_r + \left\lceil \frac{d_r}{q^{k-r} v_r} \right\rceil + \sum_{i=1}^{k-r-1} \left\lceil \frac{d_i}{q^i v_r} \right\rceil \\ &= d_r + \sum_{i=1}^{k-r} \left\lceil \frac{d_i}{q^i v_r} \right\rceil . \end{aligned}$$

□

We shall denote the left-hand side of (9) by $g_r^{(r)}(k, d_r)$. The following theorem is well-known (cf. e.g. [23])

Theorem 2. *Let $\mathcal{K}$ be an $(n, n-d)$-arc in* $\mathrm{PG}(k-1, q)$ *and let C be an $[n, k, d]_q$ code associated with $\mathcal{K}$, If $n = t + g_q(k, d)$, then*

$$w_j = t + \sum_{i=k-1-j}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil \tag{10}$$

Let $k \geq 1$ and let d be a positive integer. The integer d can be written uniquely as

$$d = \sigma q^{k-1} - \sum_{i=0}^{k-2} \varepsilon_i q^i \tag{11}$$

It is easily computed that

$$g_q^{(1)}(k, q) = \sigma v_k - \sum_{i=0}^{k-2} \varepsilon_i v_{i+1}. \tag{12}$$

Let C be an $[n = g_q(k, d), k, d]_q^{(1)}$-code and let $\mathcal{K}$ an arc $(n, n-d)$-arc in $\mathrm{PG}(k-1, q)$ associated with C. By Theorem 2,

$$w_{k-1-j} = \sigma v_{k-j} - \sum_{i=j}^{k-2} \varepsilon_i v_{i+1-j}. \tag{13}$$

Since the maximal multiplicity of a point with respect to $\mathcal{K}$ is σ, $\mathcal{K}$ can be represented as $\mathcal{K} = \sigma \chi_{\mathcal{P}} - \mathcal{F}$, where $\mathcal{F}$ is a minihyper with parameters

$$\left(\sum_{i=0}^{k-2} \varepsilon_i v_{i+1}, \sum_{i=0}^{k-2} \varepsilon_i v_i \right) \tag{14}$$

with maximal multiplicity of a point not exceeding σ. Minihypers with the above parameters, but without the restriction on the maximal point multiplicity, can always can be constructed as the sum of subspaces: ε_{k-2} hyperplanes, ε_{k-3} hyperlines, and so on, ε_1 lines and ε_0 points. Such minihypers are called *canonical.*

A canonical minihyper $\mathcal{F}$ in $\mathrm{PG}(k-1,q)$ is said to be of type $\sum_{i=1}^{k-2} \varepsilon_i[i-1]$ if it is the sum ε_{k-2} hyperplanes, ε_{k-3}-hyperlines, and so on ε_2 lines, and ε_1 points. In other words, $\mathcal{F} = \sum_i \chi_{S_i}$, where exactly ε_j of the subspaces S_i are of (projective) dimension $j-1$. The arc $\mathcal{K}$ obtained by subtracting $\mathcal{F}$ from s copies of $\mathrm{PG}(K-1,q)$ will be said to have type $s[k-1] - \sum_{i=1}^{k-2}[\varepsilon_i]$.

Minihypers with parameters

$$\left(\sum_{i=r}^{k-2} \varepsilon_i v_{i+1}, \sum_{i=r}^{k-2} \varepsilon_i v_{i+1-r}\right)^{(k-1-r)} \tag{15}$$

can be constructed in a similar fashion as canonical minihypers, i.e. as the sum of ε_{k-2} hyperplanes, ε_{k-3} hyperlines, and so on, ε_r subspaces of dimension r.

Theorem 3. *Let $\mathcal{F}$ be a minihyper in $\mathrm{PG}(k-1,q)$ with parameters given by (15), $0 \leq \varepsilon_{\leq} q-1$, and with maximal point multiplicity w_0. Then for every $\sigma \geq w_0$ the multise $\mathcal{K} = \sigma - \mathcal{F}$ is a (multi)arc in $\mathrm{PG}(k-1,q)$ with parameters*

$$\left(n = \sigma v_k - \sum_{i=r}^{k-2} \varepsilon_i v_{i+1}, w_{k-r-1} = \sigma v_{k-r} - \sum_{i=r}^{k-2} \varepsilon_i v_{i+1-r}\right)^{(k-1-r)}.$$

The code associated with $\mathcal{K}$ is a Griesmer code with respect to the rth generalized Hamming weight.

Proof. The parameters of $\mathcal{K}$ are obvious. Furthermore

$$d_r = n - w_{k-r-1} = v_r(\sigma q^{k-r} - \sum_{i=r}^{k-2} \varepsilon_i q^{i-r+1}).$$

Now it is a straightforward check that:

$$\begin{aligned}
g_q^{(r)}(k,r) &= d_r + \lceil \frac{d_r}{q v_r} \rceil + \cdots + \lceil \frac{d_r}{q^{k-r} v_r} \rceil \\
&= v_r(\sigma q^{k-r} - \sum_{i=r}^{k-2} \varepsilon_i q^{i-r+1}) + \\
&\quad \sigma q^{k-r-1} - \sum_{i=r}^{k-2} \varepsilon_i^{i-r} + \\
&\quad \sigma q^{k-r-2} - \sum_{i=r+1}^{k-2} \varepsilon_i^{i-r-1} + \ldots + \sigma \\
&= \sigma v_k - \sum_{i=r}^{k-2} \varepsilon_i v_{i+1} \\
&= n
\end{aligned}$$

□

If the maximal point multiplicity of $\mathcal{F}$ with parameters given by (15) is large, say at least $1+\sum_{i=r}^{k-2} i\varepsilon_i$, such minihypers can always be constructed. Thus we have the following theorem, which is well-known, and essentially due to Solomon and Stiffler [25].

Theorem 4. *If d_r is large enough Griesmer $[n = g_q^{(r)}(k, d_r)]_q^{(r)}$-codes do exist for all q, k and r.*

Next we are going to demonstrate that if a code attains the classical Griesmer bound, it attains also the Griesmer bound for all generalized weights.

Theorem 5. *Let C be an $[n = g_q(k, d), k, d]_q^{(1)}$-code where $d^{(1)} = d$ is given by (11). Then*

$$d_r(C) = v_r \cdot \left(\sigma q^{k-r} - \sum_{i=r}^{k-1} \varepsilon_{i-1} q^{i-r} \right) - \sum_{i=1}^{r-1} \varepsilon_{i-1} v_i,$$

and C attains the Griesmer bound for the rth generalized Hamming weight, i.e. $n = g_q^{(r)}(k, d_r)$.

Proof. By (13) and (6), and using the obvious $v_{i+r} - v_i = q^i v_r$, we get that

$$\begin{aligned} d_r &= n - w_{k-1-r} \\ &= \left(\sigma v_k - \sum_{i=1}^{k-1} \varepsilon_{i-1} v_i \right) - \left(\sigma v_{k-r} - \sum_{i=r}^{k-1} \varepsilon_{i-1} v_{i-r} \right) \\ &= \sigma(v_k - v_{k-r}) - \sum_{i=r}^{k-1} \varepsilon_{i-1}(v_i - v_{i-r}) - \sum_{i=1}^{r-1} \varepsilon_{i-1} v_i \\ &= v_r \cdot \left(\sigma q^{k-r} - \sum_{i=r}^{k-1} \varepsilon_{i-1} q^{i-r} \right) - \sum_{i=1}^{r-1} \varepsilon_{i-1} v_i. \end{aligned}$$

Let us note that

$$\sum_{i=1}^{r} \varepsilon_{i-1} v_i \le (q-1) \sum_{i=1}^{r} v_i < (q-1) v_r + v_r = q v_r.$$

Now it is a straightforward check that

$$g_q^{(r)}(k, d_r) = g^{(1)}(k, d_1).$$

□

4 Exact values

In this section, we tackle the problem finding the exact value of $m_q^{(r)}(k-1,w)$ for fixed k, r, w and q. It is in general hard to determine the values $m_q^{(r)}(k-1,w)$, but there is one easy case for which the result is obvious:

Proposition 1. *We have* $m_q^{(0)}(k-1,s) = sv_k$ *for each* $k \geq 1$ *and each* $s \geq 1$.

The case $r = 1$ is of special interest. This is the problem of determinig the largest size of a generalized cap – a (multi)set of points such that each line has multiplicity at most w. In particular if $w = 2$ this is the notorious maximal cap problem.

From the connection between the linear $[n,k]_q$ codes and the multisets of points in $\mathrm{PG}(k-1,q)$, we get that $m_q^{(r)}(k-1,w)$ is the largest integer n such that $n \geq g_q^{(k-r-1)}(k, n-w)$. This integer will be called the *Griesmer upper bound* for $m_q^{(r)}(k-1,w)$.

Another bound on $m_q^{(r)}(k-1)$ is the following. Set $s_r = w$ and $s_{r+i} = m_q^{(r+i-1)}(r+i, s_r)$, $i = 0, \ldots, k-1-r$. Then $m_q^{(r)}(k-1) \leq s_{k-1}$. We call this bound the *coding upper bound* for $m_q^{(r)}(k-1,w)$. In other words, the coding upper bound uses recursively the parameters of optimal multiarcs, which in turn can be obtained from the parameters of the optimal linear codes.

Example 1. *For* $m_2^{(4)}(6,21)$ *the Griesmer upper bound is* 81 *since*

$$81 \geq g_2^{(2)}(7,60) = 60 + \left\lceil \frac{60}{6} \right\rceil + \left\lceil \frac{60}{12} \right\rceil + \left\lceil \frac{60}{24} \right\rceil + \left\lceil \frac{60}{48} \right\rceil + \left\lceil \frac{60}{96} \right\rceil = 81,$$

while

$$82 < g_2^{(2)}(7,61) = 61 + \left\lceil \frac{61}{6} \right\rceil + \left\lceil \frac{61}{12} \right\rceil + \left\lceil \frac{61}{24} \right\rceil + \left\lceil \frac{61}{48} \right\rceil + \left\lceil \frac{61}{96} \right\rceil = 84.$$

The coding upper bound is 77. *Here* $s_4 = 21$, $s_5 = m_2^{(4)}(5,21) = 39$ *since there exists a binary* $[39,6,18]$*-code and there is no* $[40,6,19]$*-code (cf. Grassl's tables [11]). Furthermore,* $s_6 = m_2^{(5)}(6,39) = 77$ *since there exists a* $[77,6,38]_2$*-code and there is no* $[78,6,39]_2$*-code.*

In Subsection 4.1 we determine the exact values of $m_2^{(r)}(k-1,w)$ for all $k \leq 7$. We remark that $m_2^{(6)}(7,w)$ is completely known (by the fact that we know the optimal lengths of all binary 8-dimensional codes) while there only partial results for $m_2^{(7)}(8,w)$. However, the determination of $m_2^{(6)}(7,w)$ is scattered in many papers, so that we do not attempt to determine $m_2^{(r)}(8,w)$ here. In Subsection 4.2 we determine the exact values of $m_3^{(r)}(k-1,w)$ for all $k \leq 5$.

4.1 Exact values for $m_2^{(r)}(k-1,w)$

Proposition 2. *It holds:*

(a) $m_2^{(1)}(3,3t+2)=15t+8$,

(b) $m_2^{(1)}(3,3t+3)=15t+15$,

(c) $m_2^{(1)}(3,3t+4)=15t+16$,

for all $t\in N$.

Proof. The upper bounds are given by the Griesmer upper bound. Constructions for $m_2^{(1)}(3,2)\geq 8$, $m_2^{(1)}(3,3)\geq 15$, and $m_2^{(1)}(3,4)\geq 16$ are given by an affine solid, a solid, and a solid plus a point, respectively. Combining those examples with t copies of a solid yields the remaining upper bounds by Lemma 2. □

w	$m_2^{(1)}(3,w)$	construction	upper bound
2	8	affine solid	Griesmer upper bound
3	15	solid	Griesmer upper bound
4	16	solid plus a point	Griesmer upper bound

Table 1: Exact values for $m_2^{(1)}(3,w)$.

Corollary 1. $m_2^{(1)}(3,w)$ *is given by the Griesmer upper bound for all* $w\geq 2$.

Proposition 2 can be generalized:

Proposition 3. *For each* $r\geq 2$ *and each* $t\in\mathbb{N}$ *we have*

(a) $m_2^{(1)}(k-1,3t+2)=tv_k+2^{k-1}$,

(b) $m_2^{(1)}(k-1,3t+3)=(t+1)v_k$,

(c) $m_2^{(1)}(k-1,3t+4)=(t+1)v_k+1$.

Proof. Constructions for $m_2^{(1)}(k-1,2)\geq 2^{k-1}$, $m_2^{(1)}(k-1,3)\geq v_k$, and $m_2^{(1)}(k-1,2)\geq v_k+1$ are given by an affine $(k-1)$-space (type, a $(k-1)$-space, and an $(k-1)$-space plus a point, respectively. Combining those examples with t copies of a $(k-1)$-space yields the remaining upper bounds by Lemma 2.

The upper bounds are given by the Griesmer upper bound. More precisely, applying Theorem 1 with $d_{k-2}=tv_k+2^{k-1}-3t-2=(2t+1)\cdot 2v_{k-2}$ gives $n\geq(2t+1)\cdot 2v_{k-2}+(2t+1)+(t+1)=tv_k+2^{k-1}$ and applying Theorem 1 with $d_{k-2}=(2t+1)\cdot 2v_{k-2}+1$ gives $n\geq(2t+1)\cdot 2[r]_2+(2t+2)+(t+1)=tv_k+2^{k-1}+2$. The other two cases are treated in a similar fashion. □

Corollary 2. *For each $k \geq 3$ we have that $m_2^{(1)}(k-1,w)$ is given by the Griesmer upper bound for all $w \geq 2$.*

Lemma 3. *For $k \geq 5$ we have $m_2^{(k-3)}(k-1,k-2) = k+1$.*

Proof. A projective base (or frame) shows $m_2^{(k-3)}(k-1,k-2) \geq k+1$. From the Griesmer upper bound we conclude $m_2^{(k-3)}(k-2,k-2) \leq k$, so that we can assume that a multiset $\mathcal{M}$ of at least $k+1$ points contains the points spanned by the k unit vectors. Any further point in $\mathcal{M}$ spanned by $v \in \mathbb{F}_2^k$ then needs to have Hamming weight $k-1$ or k, since otherwise $k-1$ points would be contained in a subspace of codimension two. The sum of two different such vectors with Hamming weight k or $k-1$ has Hamming weight strictly less than $k-1$, so that we can find $k-1$ points in a subspace of codimension two. □

Note that ovoids imply $m_q^{(1)}(3,2) \geq q^2+1$.

Proposition 4. *We have*

(a) $m_2^{(2)}(4,7t+4) = 31t+16$,

(b) $m_2^{(2)}(4,7t+5) = 31t+17$,

(c) $m_2^{(2)}(4,7t+6) = 31t+24$,

(d) $m_2^{(2)}(4,7t+7) = 31t+31$,

(e) $m_2^{(2)}(4,7t+8) = 31t+32$,

(f) $m_2^{(2)}(4,7t+9) = 31t+33$,

(g) $m_2^{(2)}(4,7t+10) = 31t+40$

for all $t \in N$. Moreover, we have $m_2^{(2)}(4;3) = 6$.

Proof. Lemma 3 yields $m_2^{(2)}(4,3) = 6$. Constructions for $4 \leq s \leq 10$ are given by multisets of points with types $[4]-[3]$, $[4]-[3]+[0]$, $[4]-[2]$, $[4]$, $[4]+[0]$, $[4]+2[0]$, and $2[4]-[3]-[2]$, respectively. Combining those examples with t copies of a 4-space yields the remaining constructions by Lemma 2. The upper bounds for $m_2^{(2)}(4,w)$ are given by the Griesmer upper bound for all $w \geq 4$. □

Corollary 3. *$m_2^{(2)}(4,w)$ is given by the Griesmer upper bound for all $w \geq 4$.*

We can generalize Proposition 4 as follows:

Proposition 5. *For each $k \geq 4$ and all $w \geq 4$ we have that $m_2^{(2)}(k-1,w)$ is given by the Griesmer upper bound.*

w	$m_2^{(2)}(4,w)$	construction	upper bound
3	6	projective base	Lemma 3
4	16	$[4]-[3]$	Griesmer upper bound
5	17	plus point	Griesmer upper bound
6	24	$[4]-[2]$	Griesmer upper bound
7	31	$[4]$	Griesmer upper bound
8	32	plus point	Griesmer upper bound
9	33	plus point	Griesmer upper bound
10	40	$2[4]-[3]-[2]$	Griesmer upper bound

Table 2: Exact values for $m_2^{(2)}(4,w)$.

Proof. The Solomon–Stiffler constructions for $[k-1]-[k-2]$, $[k-1]-[k-3]$, $[k-1]$, and $2[k-1]-[k-2]-[k-3]$ give $m_2^{(k-4)}(k-1,4)\ge 2^{k-2}$, $m_2^{(k-4)}(k-1,6)\ge 3\cdot 2^{k-3}$, $m_2^{(2)}(k-1,7)\ge v_k$, and $m_2^{(2)}(k-1,10)\ge 5\cdot 2^{k-3}$. Adding points gives $m_2^{(2)}(k-1,5)\ge m_2^{(2)}(k-1,4)+1$, $m_2^{(2)}(k-1,8)\ge m_2^{(k-4)}(k-1,7)+1$, and $m_2^{(2)}(k-1,9)\ge m_2^{(2)}(k-1,7)+2$. For $w>10$ the lower bounds 2 given by $m_2^{(2)}(k-1,w)\ge m_2^{(2)}(k-1,w-7)+m_2^{(2)}(k-1,7)$.

The upper bounds are given by the Griesmer upper bound. More precisely, applying Theorem 1 with $d_{k-3}=(4t+2)\cdot 2v_{k-3}$ gives $n\ge (4t+2)\cdot 2v_{k-3}+(4t+2)+(2t+1)+(t+1)=t\cdot v_{+}2^{k-1}$; applying Theorem 1 with $d_r=(4t+3)\cdot 2v_{k-3}$ gives $n\ge (4t+3)\cdot 2v_{k-3}+(4t+3)+(2t+2)+(t+1)=t\cdot v_k+3\cdot 2^{k-2}$; applying Theorem 1 with $d_r=(4t+4)\cdot 2v_{k-3}$ gives $n\ge (4t+4)\cdot 2v_{k-3}+(4t+4)+(2t+2)+t=t\cdot v_k+v_k$, and applying Theorem 1 with $d_r=(4t+5)\cdot 2v_{k-3}$ gives $n\ge (4t+5)\cdot 2v_{k-3}+(4t+5)+(2t+3)+(t+2)=t\cdot v_k+5\cdot 2^{k-2}$. □

So, the only non-trivial value that is not determined yet is $m_2^{(2)}(k-1,3)$. The first values are given by $m_2^{(2)}(3,3)=5$ and $m_2^{(2)}(4,3)=6$.

w	$m_2^{(2)}(5,w)$	construction	upper bound
3	8	Lemma 4	Lemma 4
4	32	$[5]-[4]$	Griesmer upper bound
5	33	plus point	Griesmer upper bound
6	48	$[5]-[3]$	Griesmer upper bound
7	63	$[5]$	Griesmer upper bound
8	64	plus point	Griesmer upper bound
9	65	plus point	Griesmer upper bound
10	80	$2[5]-[4]-[3]$	Griesmer upper bound

Table 3: Exact values for $m_2^{(2)}(5,w)$.

Lemma 4. *We have* $n_2^{(2)}(5,3)=8$.

Proof. A feasible example is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Let $\mathcal{M}$ be a multiset of $n \geq 9$ points in $\operatorname{PG}(5,2)$ such that each plane contains at most three points. Since $m_2^{(2)}(5,3) = 6$ we assume w.l.o.g. that $\mathcal{M}$ contains the six points spanned by the six unit vectors. Clearly, the maximum point multiplicity is one and every additional point is spanned by a vector x with Hamming weight at least 4. Since $m_2^{(3)}(5,2;4) = 7$ we assume w.l.o.g. that $\mathcal{M}$ also contains the point spanned by $x = (1,1,1,1,0,0)^\top$. Let two further points be spanned by $y, z \in \mathbb{F}_2^6$. Since every plane contains at most three points we have $\operatorname{wt}(y), \operatorname{wt}(z) \geq 4$ and $d_H(x,z), d_H(x,y), d_H(y,z) \geq 4$. If $\operatorname{wt}(y) = 4$, then no such vector z exists, so that $\operatorname{wt}(y), \operatorname{wt}(z) \geq 5$, which contradicts $d_H(y,z) \geq 4$. □

Lemma 5. *We have $m_2^2(6,3) = 11$ and $m_2^{(2)}(7,3) = 17$.*

Proof. An example showing $m_2^{(2)}(6,3) \geq 11$ is given by

$$\begin{pmatrix} 1000000 & 11 & 10 \\ 0100000 & 11 & 01 \\ 0010000 & 10 & 10 \\ 0001000 & 10 & 01 \\ 0000100 & 01 & 10 \\ 0000010 & 01 & 01 \\ 0000001 & 00 & 11 \end{pmatrix}.$$

Let $\mathcal{M}$ be a multiset of $n \geq 12$ points in $\operatorname{PG}(6,2)$ such that each plane contains at most three points. Since $m_2^{(2)}(5,3) = 8$ we assume w.l.o.g. that $\mathcal{M}$ contains the seven points spanned by the seven unit vectors. Clearly, the maximum point multiplicity is one and every additional point is spanned by a vector x with Hamming weight at least 4. Via a small ILP computation we excluded $n \geq 12$. An example showing $m_2^{(2)}(7,3) \geq 17$ is given by

$$\begin{pmatrix} 10000000 & 1110 & 01111 \\ 01000000 & 1101 & 11101 \\ 00100000 & 1010 & 10011 \\ 00010000 & 1001 & 01110 \\ 00001000 & 0110 & 10101 \\ 00000100 & 0101 & 01011 \\ 00000010 & 0011 & 00111 \\ 00000001 & 0000 & 11111 \end{pmatrix}.$$

Again we can prescribe the eight points spanned by the unit vectors, so that any further point is spanned by a vector $x \in \mathbb{F}_2^8$ with Hamming weight at least 4. If there is no point spanned by a vector with Hamming weight 4, than a small ILP computations shows that the maximum cardinality is 16. So we can additionally prescribe an arbitrary point spanned by a vector with Hamming weight 4. Another ILP computation then shows that the maximum cardinality is 17. □

w	$m_2^{(2)}(6,w)$	construction	upper bound
3	11	Lemma 5	Lemma 5
4	64	$[6]-[5]$	Griesmer upper bound
5	65	plus point	Griesmer upper bound
6	96	$[6]-[4]$	Griesmer upper bound
7	127	$[6]$	Griesmer upper bound
8	128	plus point	Griesmer upper bound
9	129	plus point	Griesmer upper bound
10	160	$2[6]-[5]-[4]$	Griesmer upper bound

Table 4: Exact values for $m_2^{(2)}(6,w)$.

Lemma 6. *We have $m_2^{(3)}(5,11) = 38$.*

Proof. The lower bound is given by the unique $[38,6,18]_2$ code [5], which is a Griesmer code. A generator matrix is e.g. given by

$$\begin{pmatrix} 00000100000000000000001111111111111111 \\ 00001000000001111111110000000111111111 \\ 00010000001111100000111111000000001111 \\ 00100000110111100110001110011001110011 \\ 01000010110111101011011010101010010101 \\ 10000001111110101101011101001100110110 \end{pmatrix}.$$

Let $\mathcal{M}$ be a multiset of points in $\mathrm{PG}(5,2)$ with at most 11 points in each solid. Considering projections through a point and $m_2^{2)}(4,9) = 33$ implies $\mathcal{M}(P) \le 1$ for every point P. Since $m_2^{(3)}(4,11) = 21$ we have $\mathcal{M}(H) \le 21$ for every hyperplane H. Since $m_2^{(4)}(5,20) = 38$ we can assume the existence of a hyperplane $H^\star$ with $\mathcal{M}(H^\star) = 21$. There are exactly two $[21,5,10]_2$ codes, see e.g. [1, Theorem 6]. Prescribing these two possibilities for $H^\star$ a small ILP computation quickly shows $\#\mathcal{M} \le 38$. □

We remark that the Griesmer upper bound gives $m_2^{(3)}(5,11) \le 41$, while the coding upper bound yields $m_2^{(3)}(5,11) \le 39$ via $m_2^{(4)}(5,21) = 39$. Note that the complement of a hypothetical set of 39 points in $\mathrm{PG}(5,2)$ points with at most 11 points per solid is a multiset of points of cardinality 24 that blocks every solid at least four times. The union of a solid and a plane (plus two arbitrary points) gives such an example as a multiset but not as a set of points.

Proposition 6. *If $w \geq 12$ or $w \in \{6, 8, 9, 10\}$, then $m_2^{(3)}(5, w)$ is given by the Griesmer upper bound. Moreover, we have $m_2^{3)}(5, 4) = 7$, $m_2^{(3)}(5, 5) = 11$, $m_2^{(3)}(5, 7) = 19$, and $m_2^{(3)}(5, 11) = 38$.*

Proof. Using Lemma 3 and Lemma 6 we state that $m_2^{(3)}(5, 4) = 7$ and $m_2^{(3)}(5, 11) = 38$, respectively. We consider the following constructions

- types $t[5]$, $t[5] + [0]$, $t[5] + 2[0]$, $t[5] + 3[0]$, $t[5] - [4]$, $t[5] - [4] + [0]$, $t[5] - [4] + 2[0]$, $t[5] - [3]$, $t[5] - [3] + [0]$, and $t[5] - [2]$ for $t \geq 1$;
- types $t[5] - [4] - [3]$, $t[5] - [4] - [2]$, and $t[5] - [3] - [2]$ for $t \geq 2$; and
- type $t[5] - [4] - [3] - [2]$ for $t \geq 3$.

So, it remains to provide constructions for $w \in \{5, \ldots, 7, 19\}$. For $w = 7$ we add an arbitrary point to an example for $w = 6$. For $w \in \{5, 6, 19\}$ we provide explicit examples:

$$\begin{pmatrix} 00000100111 \\ 00001011111 \\ 00010001011 \\ 00100011100 \\ 01000011001 \\ 10000011010 \end{pmatrix}, \begin{pmatrix} 000000100001111111 \\ 000011000110001111 \\ 000100001110110011 \\ 001000011010011101 \\ 010000011101000111 \\ 100000010111101001 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 0001000000000000111111111111111100000000001111111111000000000001111111111 \\ 0010000001111111000000111111111100000011110000001111000000011110000001111 \\ 0000001110000111001110000011111010011001110001110011010011001110001110011 \\ 0000010110111000110010001100111101100001110001111100101100001110001111100 \\ 0100111011001001010110010101011000101010110110010101000101010110110010101 \\ 1000110011010011100100100110001001011100011010111001001011100011010111001 \end{pmatrix}.$$

The Griesmer upper bound is not attained for $w \in \{4, 5, 7, 11\}$ while it is for all other cases $w > 3$ For $w \in \{5, 7\}$ the coding upper bound is attained. □

The stored generator matrices of $[m_2^{(3)}(5, w), 6]_2$ codes in the database of *best known linear codes* (BKLC) in `Magma` give optimal examples for $w \in \{5, 6, 11, 19\}$. Note that for $w = 6$ we can take any $[18, 6, 8]_2$ Griesmer code and for $w = 19$ we can take any $[73, 6, 36]_2$ Griesmer code.

Lemma 7. *We have $m_2^{(4)}(6, 9) = 27$ and $m_2^{(4)}(6, 10) = 28$.*

Proof. An example showing $m_2^{(4)}(6, 9) \geq 27$ is given by

$$\begin{pmatrix} 000000100000000011111111111 \\ 000001000001111100000111111 \\ 000010001110111100011000011 \\ 000100010111000101111000101 \\ 001000011001001110101011100 \\ 010000001011011010110101010 \\ 100000010100110111010110001 \end{pmatrix},$$

so that adding an arbitrary point gives $m_2^{(4)}(6, 10) \geq 28$. For $w = 9$ the coding upper bound is attained. Next we consider a multiset $\mathcal{M}$ of points in $\mathrm{PG}(6, 2)$ such that every subspace of codimension two contains at most ten points. Starting from $m_2^{(5)}(6, 10) = 18$ we have used `LinCode` [4] to enumerate the two non-isomorphic $[18, 6, 8]_2$ codes. Prescribing the two possible configurations and an

w	$m_2^{(3)}(5,w)$	construction	upper bound
4	7	projective base	Lemma 3
5	11	BKLC/ILP	Coding upper bound
6	18	BKLC/ILP	Griesmer upper bound
7	19	plus point	Coding upper bound
8	32	$[5]-[4]$	Griesmer upper bound
9	33	plus point	Griesmer upper bound
10	34	plus point	Griesmer upper bound
11	38	BKLC/ILP	Lemma 6
12	48	$[5]-[3]$	Griesmer upper bound
13	49	plus point	Griesmer upper bound
14	56	$[5]-[2]$	Griesmer upper bound
15	63	$[5]$	Griesmer upper bound
16	64	plus point	Griesmer upper bound
17	65	plus point	Griesmer upper bound
18	66	plus point	Griesmer upper bound
19	73	BKLC/ILP	Griesmer upper bound
20	80	$2[5]-[4]-[3]$	Griesmer upper bound
21	81	plus point	Griesmer upper bound
22	88	$2[5]-[4]-[2]$	Griesmer upper bound
23	95	sum construction	Griesmer upper bound
24	96	plus point	Griesmer upper bound
25	97	plus point	Griesmer upper bound
26	104	$2[5]-[3]-[2]$	Griesmer upper bound
27	111	sum construction	Griesmer upper bound
28	112	plus point	Griesmer upper bound
29	119	sum construction	Griesmer upper bound
30	126	sum construction	Griesmer upper bound
31	127	plus point	Griesmer upper bound
32	128	plus point	Griesmer upper bound
33	129	plus point	Griesmer upper bound
34	136	$3[5]-[4]-[3]-[2]$	Griesmer upper bound

Table 5: Exact values for $m_2^{(3)}(5,w)$.

arbitrary point making the span 7-dimensional, we have used an ILP computation to conclude $\#\mathcal{M} \le 28$. In the following we assume $\mathcal{M}(H) \le 17$ for each hyperplane. Since $m_2^{(3)}(5,7) = 19$ we also assume $\mathcal{M}(P) \le 2$ for every point P. If there exists a solid S with $\mathcal{M}(S) \ge 7$, then we have $\#\mathcal{M} \le 7 \cdot 3 + 7 = 28$, so that we assume $\mathcal{M}(S) \le 6$ for every solid S. We have used `LinCode` [4] to enumerate the four non-isomorphic $[10,5,4]_2$ codes. Prescribing the four possible configurations and two points that make the span 7-dimensional, we have used ILP computations to conclude $\#\mathcal{M} \le 28$. □

Lemma 8. *We have* $m_2^{(4)}(6,20) = 71$.

Proof. An example showing $m_2^{(4)}(6,20) \ge 71$ is given by

$$\begin{pmatrix}
00000010000000000000000000000000000000111111111111111111111111111111111 \\
00000100000000000000111111111111111110000000000000000011111111111111111 \\
00001000000000111111111000000000111111110000000011111111000000000111111 11 \\
00010000111100001111100000111100001111000011110000111100000111100001111 \\
00100001001101110001100011001100110011001100110011001100111001100110011 \\
01000001010110110010101101010101010101011101010101010101010110101010101 \\
10000001111011010100110110100110010110011010011001011010001011001101001
\end{pmatrix}.$$

Next we consider a multiset $\mathcal{M}$ of points in $\mathrm{PG}(6,2)$ such that every subspace of codimension two contains at most 20 points. Starting from $m_2^{(4)}(5,20) = 38$ we have used `LinCode` [4] to construct the unique $[38,6,18]_2$ code [5]. Prescribing the corresponding unique configuration and a further point that makes the span 7-dimensional, we have used an ILP computation to conclude $\#\mathcal{M} \le 71$. Observing $m_2^{(5)}(6,37) = 71$ finishes the proof. □

Lemma 9. *We have* $m_2^{(4)}(6,22) = 82$ *and* $m_2^{(4)}(6,23) = 83$.

Proof. An example showing $m_2^{(4)}(6,22) \ge 82$ is given by each of the 11 $[82,7,40]_2$ Griesmer codes [5]. One generator matrix is e.g. given by

$$\begin{pmatrix}
0000001000000000000000000000000000000000000111111111111111111111111111111111111111 \\
0000010000000000000000000111111111111111111100000000000000000001111111111111111111 1 \\
0000100000001111111111110000000001111111111000000111111111111100000000011111111100 \\
0001000001110000001111110001111100000011111111110000000111110000011111000001111100 \\
0010000010110001110001111110011000011000001110100111100111001100110011001110001100 \\
0100000110101011001011001101010101001010011001111001101010110101010101010110010100 \\
1000000011101010101010101110100101001010101010101010101011011001011001101010100100
\end{pmatrix},$$

so that adding an arbitrary point gives $m_2^{(4)}(6,23) \ge 83$. For $s = 22$ the Griesmer upper bound is attained. Next we consider a multiset $\mathcal{M}$ of points in $\mathrm{PG}(6,2)$ such that every subspace of codimension two contains at most 23 points. Starting from $m_2^{(4)}(5,23) = 34$ we have used `LinCode` [4] to construct the unique $[45,6,22]_2$ code [5]. Prescribing the corresponding configuration and an arbitrary point making the span 7-dimensional, we have used an ILP computation to conclude $\#\mathcal{M} \le 83$. In the following we assume $\mathcal{M}(H) \le 44$ for each hyperplane. Since $m_2^{(3)}(5,20) = 80$ we also assume $\mathcal{M}(P) \le 1$ for every point P. We have used `LinCode` [4] to enumerate the unique $[44,6,21]_2$ code with maximum column multiplicity one. Prescribing the corresponding configuration and an arbitrary point making the span 7-dimensional, we have used an ILP computation to conclude $\#\mathcal{M} \le 83$. In the following we assume

$\mathcal{M}(H) \le 43$ for each hyperplane. If there exists a solid S with $\mathcal{M}(S) \ge 13$, then we have $\#\mathcal{M} \le 7 \cdot 10 + 13 = 83$, so that we assume $\mathcal{M}(S) \le 12$ for every solid S. We have used `LinCode` [4] to construct the unique $[23, 11, 5]_2$ code [24]. Prescribing the corresponding configuration and two points that make the span 7-dimensional, we have used an ILP computation to conclude $\#\mathcal{M} \le 83$. □

Proposition 7. *If $w \ge 24$ or $w \in \{7, 11, 14, 16, \ldots, 19, 22\}$, then $m_2^{(4)}(6, s)$ is given by the Griesmer upper bound. Moreover, we have $m_2^{(4)}(6, 5) = 8$, $m_2^{(4)}(6, 6) = 12$, $m_2^{(4)}(6, 8) = 20$, $m_2^{(4)}(6, 9) = 27$, $m_2^{(4)}(6, 10) = 28$, $m_2^{(4)}(6, 12) = 36$, ${}_2^{(4)}(6, 13) = 43$, $m_2^{(4)}(6, 15) = 51$, $m_2^{(4)}(6, 20) = 71$, $m_2^{(4)}(6, 21) = 75$, and $m_2^{(4)}(6, 23) = 83$.*

Proof. For $m_2^{(4)}(6, 9) = 27$ and $m_2^{(4)}(6, 10) = 28$ we refer to Lemma 7. For $m_2^{(4)}(6, 20) = 71$ we refer to Lemma 8. For $m_2^{(4)}(6, 22) = 82$ and $m_2^{(4)}(6, 23) = 83$ we refer to Lemma 9. We consider the following constructions

- types $t[6]$, $t[6] - [5]$, $t[6] - [4]$, $t[6] - [3]$, and $t[6] - [2]$ for $t \ge 1$;
- types $t[6]-[5]-[4]$, $t[6]-[5]-[3]$, $t[6]-[5]-[2]$, $t[6]-[4]-[3]$, $t[6]-[4]-[2]$, and $t[6] - [3] - [2]$ for $t \ge 2$;
- types $t[6] - [5] - [4] - [3]$, $t[6] - [5] - [4] - [2]$, $t[6] - [5] - [3] - [2]$, and $t[6] - [4] - [3] - [2]$ for $t \ge 3$; and
- type $t[6] - [5] - [4] - [3] - [2]$ for $t \ge 4$,

as well as adding up to four additional points to those constructions. By selecting the removed subspaces more carefully than by a chain, we can also have constructions for $[6] - [3] - [2]$, $2[6] - [5] - [3] - [2]$, $2[6] - [4] - [3] - [2]$, and $3[6] - [5] - [4] - [3] - [2]$, i.e. for $w \in \{27, 43, 51, 67\}$, which meet the Griesmer upper bound. The case $w = 5$ is treated in Lemma 3. For $w \in \{8, 12, 15, 38\}$ the best known construction can be obtained by adding an arbitrary point to a construction for $w - 1$. For $w \in \{6, 7, 11, 13, 14, 21\}$ we give explicit examples found by ILP searches.

$$\begin{pmatrix} 000000100111 \\ 000001001011 \\ 000010010011 \\ 000100011100 \\ 001000011111 \\ 010000001101 \\ 100000011001 \end{pmatrix}, \begin{pmatrix} 0000001000001111111 \\ 0000010001110001111 \\ 0000100110010010111 \\ 0001000011100110011 \\ 0010000010111100101 \\ 0100000101101101001 \\ 1000000011011011001 \end{pmatrix}, \begin{pmatrix} 00000010000000000000111111111111111 \\ 00000100000111111111000001111111111 \\ 00001000011000011111001110000011111 \\ 00010001111001100111010110001100001 \\ 00100000101110100011011010010101110 \\ 01000001011011111001100100101000110 \\ 10000000110011101010100111000111010 \end{pmatrix},$$

$$\begin{pmatrix} 0000001000000000000011111111111111111111111 \\ 0000010000011111111100000000011111111111111 \\ 0000100001100001111100001111100000001111111 \\ 0001000110101110011100110111100011110001111 \\ 0010000011010110001111001001100100110010111 \\ 0100000111011000100101010010101001010111011 \\ 1000000100101011100110111000110100010011101 \end{pmatrix},$$

$$\begin{pmatrix} 0000001000000000000000000111111111111111111111111100 \\ 0000010000000011111111111000000000001111111111111100 \\ 0000100111111110000000011110000000111100001111111100 \\ 0001000000001110000111011100111111000100010001111111 \\ 0010000000110010011001101101000110111011100100001111 \\ 0100000001010100111110000111011010010101101000010111 \\ 1000000010100101001010011010110100101111011000100111 \end{pmatrix},$$

$$\begin{pmatrix}
0000001000000000000000000000000000001111111111111111111111111111111111111\\
0000010000000000000000011111111111111111100000000000000000000111111111111111111\\
0000100000000000111111000000000111111111000000000111111111000000000111111111\\
00010000011111000111110001111100001111100001111100001111110000011110000011111\\
0010000001001110110011011000110011000110011000110011000111001110011001100111\\
0100000111100110101010110101101010101011011101011010101001011010110101010101001\\
10000001101010110100111010101100100110011010101100101110110001011001101 0011
\end{pmatrix},$$

For $w \in \{36, 37, 39\}$ we can take $[138, 7, 68]_2$, $[145, 7, 72]_2$, and $[153, 7, 76]_2$ Griesmer codes [28]. For $w \in \{6, 8, 12, 13, 15, 21\}$ the coding upper bound is attained. All other upper bounds are given by the Griesmer upper bound. □

The stored generator matrices of $[m_2^{(4)}(6, w), 7]_2$ codes in the database of *best known linear codes* (BKLC) in `Magma` give optimal examples for

$$w \in \{6, 7, 9, 11, 13, 14, 20, 21, 22, 36, 37, 39\}.$$

Lemma 10. *We have $m_2^{(3)}(6, 4) = 9$ and $m_2^{(3)}(6, 5) = 19$.*

Proof. Examples showing $m_2^{(3)}(6, 4) \geq 9$ and $m_2^{(3)}(6, 5) \geq 19$ are given by

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1\\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0\\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and

$$\begin{pmatrix}
1000000000111101101\\
0100000101010111001\\
0010000111100010011\\
0001000110111000110\\
0000100011011100011\\
0000010100100111110\\
0000001010010011111
\end{pmatrix},$$

respectively. For upper bounds are obtained by ILP computations prescribing the seven points generated by the unit vectors in $\mathbb{F}_2^7$. □

Lemma 11. *We have $m_2^{(3)}(6, 6) = 28$.*

Proof. An example showing $m_2^{(3)}(6, 6) \geq 28$ is given by

$$\begin{pmatrix}
0000000000001111110000111111\\
0000000001110001110111000111\\
0001111110000001110001001011\\
1110001110000000000011011101\\
0001110000000111111111000000\\
0110110110110110110000000000\\
1101101101101101100000000000
\end{pmatrix}.$$

Let $\mathcal{M}$ be a multiset of n points in $\mathrm{PG}(6, 2)$ such that every solid contains at most six points. We have used `LinCode` [4] to enumerate the two non-isomorphic $[18, 6, 8]_2$ codes. Prescribing the two possible configurations and an arbitrary

w	$m_2^{(4)}(6,w)$	construction	upper bound
5	8	projective base	Lemma 3
6	12	BKLC/ILP	Coding upper bound
7	19	BKLC/ILP	Griesmer upper bound
8	20	plus point	Coding upper bound
9	27	BKLC/ILP	Coding upper bound
10	28	plus point	Lemma 7
11	35	BKLC/ILP	Griesmer upper bound
12	36	plus point	Coding upper bound
13	43	BKLC/ILP	Coding upper bound
14	50	BKLC/ILP	Griesmer upper bound
15	51	plus point	Coding upper bound
16	64	$[6]-[5]$	Griesmer upper bound
17	65	plus point	Griesmer upper bound
18	66	plus point	Griesmer upper bound
19	67	plus point	Griesmer upper bound
20	71	BKLC/ILP	Lemma 8
21	75	BKLC/ILP	Coding upper bound
22	82	BKLC/ILP	Griesmer upper bound
23	83	plus point	Lemma 9
24	96	$[6]-[4]$	Griesmer upper bound
25	97	plus point	Griesmer upper bound
26	98	plus point	Griesmer upper bound
27	105	$[6]-[3]-[2]$	Griesmer upper bound
28	112	$[6]-[3]$	Griesmer upper bound
29	113	plus point	Griesmer upper bound
30	120	$[6]-[2]$	Griesmer upper bound
31	127	$[6]$	Griesmer upper bound
32	128	plus point	Griesmer upper bound
33	129	plus point	Griesmer upper bound
34	130	plus point	Griesmer upper bound
35	131	plus point	Griesmer upper bound
36	138	BKLC/ILP	Griesmer upper bound
37	145	BKLC/ILP	Griesmer upper bound
38	146	plus point	Griesmer upper bound
39	153	BKLC/ILP	Griesmer upper bound
40	160	$2[6]-[5]-[4]$	Griesmer upper bound
41	161	plus point	Griesmer upper bound
42	162	plus point	Griesmer upper bound
43	169	$2[6]-[5]-[3]-[2]$	Griesmer upper bound
44	176	$2[6]-[5]-[3]$	Griesmer upper bound
45	177	plus point	Griesmer upper bound

Table 6: Exact values for $m_2^{(4)}(6,w)$ – part 1.

w	$m_2^{(4)}(6,w)$	construction	upper bound
46	184	$2[6]-[5]-[2]$	Griesmer upper bound
47	191	sum construction	Griesmer upper bound
48	192	plus point	Griesmer upper bound
49	193	plus point	Griesmer upper bound
50	194	plus point	Griesmer upper bound
51	201	$2[6]-[4]-[3]-[2]$	Griesmer upper bound
52	208	$2[6]-[4]-[3]$	Griesmer upper bound
53	209	plus point	Griesmer upper bound
54	216	$2[6]-[4]-[2]$	Griesmer upper bound
55	223	sum construction	Griesmer upper bound
56	224	plus point	Griesmer upper bound
57	225	plus point	Griesmer upper bound
58	232	sum construction	Griesmer upper bound
59	239	sum construction	Griesmer upper bound
60	240	plus point	Griesmer upper bound
61	247	sum construction	Griesmer upper bound
62	254	sum construction	Griesmer upper bound
63	255	plus point	Griesmer upper bound
64	256	plus point	Griesmer upper bound
65	257	plus point	Griesmer upper bound
66	258	plus point	Griesmer upper bound
67	265	$3[6]-[5]-[4]-[3]-[2]$	Griesmer upper bound
68	272	$3[6]-[5]-[4]-[3]$	Griesmer upper bound
69	273	plus point	Griesmer upper bound
70	280	$3[6]-[5]-[4]-[2]$	

Table 7: Exact values for $m_2^{(4)}(6,w)$ – part 2.

point making the span 7-dimensional, we have used an ILP computation to conclude $\#\mathcal{M} \le 25$. In the following we assume $\mathcal{M}(H) \le 17$ for each hyperplane. We have used `LinCode` [4] to enumerate the three non-isomorphic $[17, 6, 7]_2$ codes. Prescribing the three possible configurations and an arbitrary point making the span 6-dimensional, we have used an ILP computation to conclude $\#\mathcal{M} \le 28$. In the following we assume $\mathcal{M}(H) \le 16$ for each hyperplane. We have used `LinCode` [4] to enumerate the four non-isomorphic $[10, 5, 4]_2$ codes. Prescribing the four possible configurations and two points that make the span 6-dimensional, we have used ILP computations to conclude $\#\mathcal{M} \le 28$. $\square$

Lemma 12. *We have* $m_2^{(3)}(6, 11) = 72$*.*

Proof. An example showing $m_2^{(3)}(6, 11) \ge 72$ is given by

$$\begin{pmatrix} 1000000110111000111010011100001100111010110100010101110010010111111111100 \\ 0100000101100100100111010010001010100111101110011111001011011100100000010 \\ 0010000011110110011011100011000011011110100101000111010100101110010000010 \\ 0001000000010001101000111101110010110111101001010100111001101111000100010 \\ 0000100000000110111111011100011100001100111100110101101010001011110001 0010 \\ 0000010000001001010001111110011101001110110100110001110110101011100001010 \\ 0000001111101001110110100100010101111101001000101110100100111100000000110 \end{pmatrix}.$$

Let $\mathcal{M}$ be a multiset of n points in $\mathrm{PG}(6, 2)$ such that every solid contains at most eleven points. We have $\mathcal{M}(H) \le m_2^{(3)}(5, 11) = 38$ for every hyperplane H, so that $\#\mathcal{M} \le m_2^{(5)}(6, 38) \le 72$. $\square$

Proposition 8. *If* $w \ge 12$ *or* $w \in \{8, 9, 10\}$*, then* $m_2^{(3)}(6, w)$ *is given by the Griesmer upper bound. Moreover, we have* $m_2^{(4)}(6, 4) = 9$*,* $m_2^{(4)}(6, 5) = 19$*,* $m_2^{(4)}(6, 6) = 28$*,* $m_2^{(4)}(6, 7) = 35$*, and* $m_2^{(4)}(6, 11) = 72$*.*

Proof. For $m_2^{(3)}(6, 4) = 9$ and $m_2^{(3)}(6, 5) = 19$ we refer to Lemma 10. For $m_2^{(3)}(6, 6) = 28$ we refer to Lemma 11. For $m_2^{(3)}(6, 11) = 72$ we refer to Lemma 12. We consider the following constructions

- types $t[6]$, $t[6] - [5]$, $t[6] - [4]$, and $t[6] - [3]$ for $t \ge 1$;
- types $t[6] - [5] - [4]$, $t[6] - [5] - [3]$, $t[6] - [5]$, $t[6] - [4] - [3]$, and $t[6] - [4]$ for $t \ge 2$,

as well as adding up to four additional points to those constructions. Examples showing $m_2^{(3)}(6, 7) \ge 35$ given by

$$\begin{pmatrix} 100101011010101101011101011000001111 \\ 010100010110110001110000010101110100 \\ 001101000001001000001100011101001111 \\ 000011001110001000001001011101101101 \\ 000000111110000100001001111110110110 \\ 000000000001111100000111111111000111 \\ 000000000000000011111111111111111000 \end{pmatrix}.$$

For $m_2^{(3)}(6, 19) \ge 145$ we can use a $[145, 7, 72]_2$ Griesmer code [28]. The upper bound $m_2^{(3)}(6, 7) \le 35$ is given by the coding upper bound. All other upper bounds are obtained from the Griesmer upper bound. $\square$

The stored generator matrices of $[m_2^{(3)}(6, w), 7]_2$ codes in the database of *best known linear codes* (BKLC) in `Magma` give optimal examples for $w \in \{5, 7, 19\}$.

w	$m_2^{(3)}(6, w)$	construction	upper bound
4	9	ILP	Lemma 10
5	19	BKLC/ILP	Lemma 10
6	28	ILP	Lemma 11
7	35	BKLC/ILP	Coding upper bound
8	64	$[6] - [5]$	Griesmer upper bound
9	65	plus point	Griesmer upper bound
10	66	plus point	Griesmer upper bound
11	72	Lemma 12	Lemma 12
12	96	$[6] - [4]$	Griesmer upper bound
13	97	plus point	Griesmer upper bound
14	112	$[6] - [3]$	Griesmer upper bound
15	127	$[6]$	Griesmer upper bound
16	128	plus point	Griesmer upper bound
17	129	plus point	Griesmer upper bound
18	130	plus point	Griesmer upper bound
19	145	BKLC/ILP	Griesmer upper bound
20	160	$2[6] - [5] - [4]$	Griesmer upper bound
21	161	plus point	Griesmer upper bound
22	176	$2[6] - [5] - [3]$	Griesmer upper bound
23	191	sum construction	Griesmer upper bound
24	192	sum construction	Griesmer upper bound
25	193	sum construction	Griesmer upper bound
26	208	$2[6] - [4] - [3]$	Griesmer upper bound

Table 8: Exact values for $m_2^{(3)}(6, w)$.

4.2 Exact values for $m_3^{(r)}(k-1,w)$

Proposition 9. *If $w \geq 3$, then $m_3^{(1)}(3,w)$ is given by the Griesmer upper bound. Moreover, we have $m_3^{(1)}(3,2) = 10$.*

Proof. The upper bound $m_3^{(1)}(3,2) \leq 10$ follows from the coding upper bound and all other upper bounds follow from the Griesmer upper bound. The existence of an ovoid in $\mathrm{PG}(3,3)$ yields $m_3^{(1)}(3,2) \geq 10$. A $[27,4,18]_3$ Griesmer code yields $m_3^{(1)}(3,3) \geq 27$. Note that Griesmer $[n,4,d]_3$ codes exist for all $d \geq 16$. □

We remark that we have $m_q^{(1)}(3,2) = q^2+1$ for all $q > 2$ [3, 22]. Moreover, we have $m_4^{(1)}(3,3) = 31$ and $m_5^{(1)}(3,3) = 44$ [8].

Lemma 13. *We have $m_3^{(2)}(4,4) = 20$.*

Proof. An example showing $m_3^{(2)}(4,4) \geq 20$ is given by

$$\begin{pmatrix} 10000220001102111221 \\ 01000002221001121111 \\ 00100212202022200211 \\ 00010202111222011010 \\ 00001111021120010111 \end{pmatrix}.$$

After prescribing the unique $[10,4;6]_3$ code a small ILP computation verifies $m_3^{(2)}(4,4) \leq 20$. □

Lemma 14. *We have $m_3^{(2)}(4,6) = 38$.*

Proof. An example showing $m_3^{(2)}(4,6) \geq 38$ is given by

$$\begin{pmatrix} 10000012010111221202110002001212201211 \\ 01000212122122010222212210101011010111 \\ 00100201200121120102201121120200101221 \\ 00010120120101221221201022020122110002 \\ 00001222222222222222201011122210222120 2] \end{pmatrix}.$$

After prescribing the three non-equivalent $[15,4;9]_3$ codes small ILP computations verify $m_3^{(2)}(4,6) \leq 38$. □

Lemma 15. *We have $m_3^{(2)}(4,11) = 91$.*

Proof. An example showing $m_3^{(2)}(4,6) \geq 38$ is given by the $[91,5,60]_3$ code in the database of *best known linear codes* (BKLC) in `Magma`. After prescribing the unique non-equivalent $[32,4;21]_3$ code an ILP computation verifies $m_3^{(2)}(4,11) \leq 91$. □

Proposition 10. *If $w \geq 18$ or $s \in \{7,9,10,12,13,14,15,16\}$, then $m_3^{(2)}(4,w)$ is given by the Griesmer upper bound. Moreover, we have $m_3^{(2)}(4,3) = 11$, $m_3^{(2)}(4,4) = 20$, $m_3^{(2)}(4,5) = 29$, $m_3^{(2)}(4,6) = 38$, $m_3^{(2)}(4,8) = 56$, $m_3^{(2)}(4,11) = 91$, and $m_3^{(2)}(4,17) \in \{143,\dots,146\}$.*

Proof. For $m_3^{(2)}(4,4) = 20$ we refer to Lemma 13 and for $m_3^{(2)}(4,6) = 38$ we refer to Lemma 14. For $m_3^{(2)}(4,11) = 91$ we refer to Lemma 15. For $w \in \{7,9,12,13,16\}$ the existence of $[55,5,36]_3$, $[81,5,54]_3$, $[108,5,72]_3$, $[121,5,81]_3$, and $[136,5,90]_3$ Griesmer codes yields the lower bounds. For $w \in \{10,14,15\}$ the lower bound is attained by adding arbitrary points. Also $m_3^{(2)}(4,8) \geq 56$ is given by adding a point. Since Griesmer $[n,5,d]_3$ codes do exist for all $d \geq 100$, $m_3^{(2)}(4,w)$ is given by the Griesmer upper bound for all $w \geq 18$. For $w \in \{3,4,5,6,11\}$ the stored generator matrices of $[n,5]_3$ codes in the database of *best known linear codes* (BKLC) in `Magma` yield $m_3^{(2)}(4,3) \geq 11$, $m_3^{(2)}(4,4) \geq 20$, $m_3^{(2)}(4,5) \geq 29$, $m_3^{(2)}(4,6) \geq 38$, and $m_3^{(2)}(4,11) \geq 91$, respectively.

The coding upper bound gives $m_3^{(2)}(4,3) \leq 11$, $m_3^{(2)}(4,5) \leq 29$, $m_3^{(2)}(4,8) \leq 56$, and $m_3^{(2)}(4,17) \leq 146$. All other upper bounds are given by the Griesmer upper bound. □

Proposition 11. *If $w \geq 3$, then $m_3^{(1)}(4,w)$ is given by the Griesmer upper bound. Moreover, we have $m_3^{(1)}(4,2) = 20$.*

Proof. For $m_3^{(1)}(4,2) = 20$ we refer e.g. to [21, 17].[1] The $[81,5,54]_3$ and the $[121,5,81]_3$ Griesmer codes give examples for $m_3^{(1)}(4,3) \geq 81$ and $m_3^{(1)}(4,4) \geq 121$, respectively. $m_3^{(1)}(4,5) \geq 122$ is obtained by adding an arbitrary point. The other lower bounds follow from the fact that Griesmer $[n,5,d]_3$ codes exist for all $d \geq 100$. The upper bounds are given by the Griesmer upper bound except for $w = 2$. □

We have $m_4^{(1)}(4,2) = 41$ [26, 7][2] and $m_3^{(1)}(5,2) = 56$ [16][3].

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