Divisible Codes

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Abstract A linear code over $\mathbb{F}_q$ with the Hamming metric is called Δ-divisible if the weights of all codewords are divisible by Δ. They have been introduced by Harold Ward a few decades ago [206]. Applications include subspace codes, partial spreads, vector space partitions, and distance optimal codes. The determination of the possible lengths of projective divisible codes is an interesting and comprehensive challenge.

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1 Introduction

A *linear code* C of *length* n is a subspace of the vector space $\mathbb{F}_q^n$ of n-tuples with entries in the finite field $\mathbb{F}_q$, where the field size q is a prime power p^m. The *(Hamming) weight* $\mathrm{wt}(\mathbf{c})$ of each *codeword* $\mathbf{c} \in C$ is the number of non-zero coordinates of $\mathbf{c}$, i.e., $\mathrm{wt}(\mathbf{c}) := \#\{1 \le i \le n \,:\, c_i \neq 0\}$. With this, the *Hamming distance* between two codewords $\mathbf{c}$ and $\mathbf{c}'$ is given by $\mathrm{d_H}(\mathbf{c}, \mathbf{c}') = \mathrm{wt}(\mathbf{c} - \mathbf{c}')$. In other words, the Hamming distance counts the number of coordinates that differ between two codewords. A linear code C is called Δ-*divisible* iff the weights of all codewords are divisible by Δ.[1] Note that every linear code is 1-divisible, so that one mostly considers the cases $\Delta > 1$ only. If $\Delta = 2$ or $\Delta = 4$ we also speak of *even* or *doubly-even* codes, respectively.

Example 1.1. *The first order binary (generalized) Reed–Muller code* $\mathrm{RM}_2(4, 1)$ *of length* $2^4 = 16$ *given by the generator matrix*

$$\begin{pmatrix} 1111111111111111 \\ 1111111100000000 \\ 1111000011110000 \\ 1100110011001100 \\ 1010101010101010 \end{pmatrix},$$

has weight enumerator $1 + 30x^8 + 1x^{16}$, *i.e., the code is* 8*-divisible.*

1.1 An introductory application

Consider binary vectors of length 9, i.e., elements of $\mathbb{F}_2^9$. The span $\langle v_1, \ldots, v_r \rangle$ of a sequence of those vectors forms a subspace, i.e., a subset of $\mathbb{F}_2^9$ that is closed under addition and scalar multiplication. For the vectors

$$\begin{aligned} \mathbf{v}^1 &= (1,0,0,0,1,1,1,0,0), \\ \mathbf{v}^2 &= (1,1,0,0,0,1,0,1,1), \\ \mathbf{v}^3 &= (0,1,0,0,1,0,1,1,1), \text{ and} \\ \mathbf{v}^4 &= (0,0,0,1,0,1,1,0,0) \end{aligned}$$

the set

$$\left\langle \mathbf{v}^1, \ldots, \mathbf{v}^4 \right\rangle := \left\{ \sum_{i=1}^{4} \lambda_i \mathbf{v}^i \,:\, \lambda_i \in \mathbb{F}_2 \, \forall 1 \le i \le 4 \right\}$$

[1] A non-linear code is called Δ-divisible if every distance between a pair of codewords is divisible by Δ. The study of divisible codes is also of interest in other metrics besides the Hamming metric, see e.g. [192] for rank metric codes.

consists of 8 elements and is a 3-dimensional subspace, i.e., it admits a basis of size 3 and contains 2^3 elements. Note that we are using row vectors for the elements of $\mathbb{F}_2^9$.

Exercise 1.2. *Compute a basis of* $\langle \mathbf{v}^1, \ldots, \mathbf{v}^4 \rangle$ *using the Gaussian elimination algorithm (over* $\mathbb{F}_2$*).*

Note that each non-empty subspace S (of $\mathbb{F}_2^9$) contains the all-zero vector $\mathbf{0}$. Now we want to consider the following packing question: Do there exist 20 four-dimensional and 30 three-dimensional subspaces in $\mathbb{F}_2^9$ such that their pairwise intersection is trivial, i.e., the intersection consists just of the zero vector $\mathbf{0}$?

In order to answer this question we first observe that each k-dimensional subspace of $\mathbb{F}_2^9$, where $0 \le k \le 9$, consists of exactly $2^k - 1$ non-zero vectors. So, the 20 four-dimensional and the 30 three-dimensional subspaces cover exactly

$$20 \cdot \left(2^4 - 1\right) + 30 \cdot \left(2^3 - 1\right) = 510$$

of the 511 non-zero vectors in $\mathbb{F}_2^9$. In other words, there would be exactly one uncovered non-zero vector $\mathbf{u}$. This does not yield a contradiction directly, but we may consider the set of covered non-zero vectors $\mathbf{v}$ that satisfy $\mathbf{v}\mathbf{h}^\top = 0$ for some (row-) vector $\mathbf{h} \in \mathbb{F}_2^9 \backslash \{\mathbf{0}\}$. For an arbitrary four-dimensional subspace S and an arbitrary three-dimensional subspace E we have

$$\left| \left\{ \mathbf{v} \in S \,:\, \mathbf{v} \in \mathbb{F}_2^9 \backslash \{\mathbf{0}\}, \mathbf{v}\mathbf{h}^\top = 0 \right\} \right| \in \{7, 15\} \tag{1.1}$$

and

$$\left| \left\{ \mathbf{v} \in E \,:\, \mathbf{v} \in \mathbb{F}_2^9 \backslash \{\mathbf{0}\}, \mathbf{v}\mathbf{h}^\top = 0 \right\} \right| \in \{3, 7\}. \tag{1.2}$$

Exercise 1.3. *For* $1 \le k \le n$ *let* S *be a* k*-dimensional subspace of* $\mathbb{F}_q^n$ *and* $\mathbf{h} \in \mathbb{F}_q^n \backslash \{\mathbf{0}\}$*. Show that the set* $\left\{ \mathbf{v} \in S \,:\, \mathbf{v} \in \mathbb{F}_q^n, \mathbf{v}\mathbf{h}^\top = 0 \right\}$ *is a subspace of dimension* k *or* $k-1$*.*

From (1.1) and (1.2) we can conclude that the number of non-zero vectors v that satisfy $\mathbf{v}\mathbf{h}^\top = 0$ and are covered by one of the $20 + 30 = 50$ subspaces is congruent to 3 modulo 4. Thus, the total number of covered non-zero vectors satisfying $\mathbf{v}\mathbf{h}^\top = 0$ is even, so that the number of uncovered non-zero vectors being perpendicular to $\mathbf{h}$ is odd. Since $\mathbf{u}$ is the unique non-zero vector that is not contained in one of the 50 subspaces, we have $\mathbf{u}\mathbf{h}^\top = 0$ for all $\mathbf{h} \in \mathbb{F}_2^9 \backslash \{\mathbf{0}\}$. This implies $\mathbf{u} = \mathbf{0}$, which is a contradiction. Thus, no such 20 four-dimensional and 30 three-dimensional subspaces can exist in $\mathbb{F}_2^9$.

While our argument and example is rather ad-hoc, something more general is hiding behind the scenes. The problem is an existence question for so-called vector space partitions. The set of covered non-zero vectors can be associated with a linear code C_0 of (effective) length 510 and the complement, i.e., the set of uncovered non-zero vectors, can be associated with a linear code C_1 of (effective) length 1. As we will see in Lemma 3.12 and Lemma 4.16, both codes C_0 and C_1 have to be 4-divisible. However, there is no 4-divisible binary linear code of effective length 1. In other words, the non-existence of Δ-divisible codes with a certain effective length certifies the non-existence of a vector space partition of a certain type. For the details on vector space partitions we refer to Section 10 and for non-existence results for divisible codes we refer to Section 4 and Section 6.

2 Preliminaries

Let $C \subseteq \mathbb{F}_q^n$ be a linear code over $\mathbb{F}_q$. If C is a k-dimensional subspace, we say that C is an $[n,k]_q$*-code*. The number k is called the *dimension* of C and n its *length*. Note that $n \ge k$ and that we will assume $k \ge 1$ in the following. If $q = 2$, $q = 3$, or $q = 4$, we speak of a *binary*, a *ternary*, or a *quaternary* code, respectively. The *support* $\operatorname{supp}(\mathbf{x})$ of a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ is the set of indices of the non-zero coordinates, i.e., $\operatorname{supp}(\mathbf{x}) := \{1 \le i \le n : x_i \neq 0\}$. With this, we have $\operatorname{wt}(\mathbf{c}) = \#\operatorname{supp}(\mathbf{c})$ for each codeword $\mathbf{c} \in C$. The number $\#C$ of codewords of C is given by q^k. Given a basis $\mathbf{g}^1, \ldots, \mathbf{g}^k \in \mathbb{F}_q^n$ of an $[n,k]_q$-code C we call the matrix

$$G = \begin{pmatrix} \mathbf{g}^1 \\ \vdots \\ \mathbf{g}^k \end{pmatrix} = \begin{pmatrix} g_1^1 & g_2^1 & \cdots & g_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ g_1^k & g_2^k & \cdots & g_n^k \end{pmatrix} \in \mathbb{F}_q^{k \times n}$$

a *generator matrix* of C, where $\mathbf{g}^i = \left(g_1^i, \ldots, g_n^i\right) \in \mathbb{F}_q^n$ for all $1 \le i \le k$. An example is given by

$$G = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{F}_3^{3 \times 6}, \tag{2.1}$$

where we denote the elements of $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ by $\{0, 1, \ldots, p-1\}$ if the field size equals a prime p. If $q = p^m$ with $m > 1$, then for each irreducible polynomial f of degree m over $\mathbb{F}_q$ we have $\mathbb{F}_q \cong \mathbb{F}_q[x]/f$. As representatives we choose polynomials of degree at most $m-1$ with coefficients in $\{0, 1, \ldots, p-1\}$.

Exercise 2.1. *Verify that each $[n,k]_q$-code admits $\prod_{i=0}^{k-1}\left(q^k - q^i\right)$ different bases, i.e., different generator matrices.*

Applying any sequence of row operations of the Gaussian elimination algorithm to G gives another generator matrix of G. For our example in (2.1) the Gaussian elimination algorithm gives the generator matrix

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\operatorname{Aut}\left(\mathbb{F}_q^n\right)$ be the group of semilinear transformations of $\mathbb{F}_q^n$ that leave the Hamming distance invariant. For each transformation $\mu \in \operatorname{Aut}\left(F_q^n\right)$ we can find a permutation π of the set $\{1, \ldots, n\}$, non-zero field elements $a_i \in \mathbb{F}_q \backslash \{0\}$, where $1 \le i \le n$, and a field automorphism α of $\mathbb{F}_q$ such that

$$\mu((x_1, \ldots, x_n)) = \left(\alpha\left(a_1 x_{\pi(1)}\right), \alpha\left(a_2 x_{\pi(2)}\right), \ldots, \alpha\left(a_n x_{\pi(n)}\right)\right) \tag{2.2}$$

for all $(x_1,\dots,x_n)\in\mathbb{F}_q^n$. Two codes $C,C'\subseteq\mathbb{F}_q^n$ are said to be *equivalent* or *isomorphic* if a transformation $\mu\in\operatorname{Aut}\bigl(\mathbb{F}_q^n\bigr)$ exists such that $\mu(C)=C'$. The *automorphism group* $\operatorname{Aut}(C)$ of a code $C\subseteq\mathbb{F}_q^n$ is the group

$$\operatorname{Aut}(C):=\left\{\mu\in\operatorname{Aut}\bigl(\mathbb{F}_q^n\bigr)\,:\,\mu(C)=C\right\}. \tag{2.3}$$

Note that for the binary field we only have to consider permutations of the set $\{1,\dots,n\}$ of coordinate positions. So, by applying row operations and column permutations we can conclude that for each $[n,k]_q$-code C there exists a generator matrix G of an equivalent code C' with generator matrix G' whose leftmost part is a $k\times k$ *unit-matrix* I_k. Such a matrix G' is called *systematic generator matrix.* In our example, generated by the matrix in Equation (2.1), a systematic generator matrix is given by

$$G'=\begin{pmatrix}1&0&0&1&2&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{pmatrix}. \tag{2.4}$$

Note that the third column of the generator matrix G in (2.1), or the sixth column of the generator matrix G' in (2.4), is the zero vector $\mathbf{0}$. The number n_{eff} of non-zero column vectors in a generator matrix G of an $[n,k]_q$-code C is called the *effective length* of C. By $\operatorname{supp}(C):=\cup_{\mathbf{c}\in C}\operatorname{supp}(\mathbf{c})$ we denote the support of a code C, so that $\#\operatorname{supp}(C)=n_{\text{eff}}(C)$. If $n_{\text{eff}}=n$, then C is also called *spanning* or of *full length.* In our example we have effective length $n_{\text{eff}}=5$.

The *minimum (Hamming) distance* of a linear code C is given by

$$d(C)=\min\{\mathrm{d_H}(\mathbf{c},\mathbf{c}')\,:\,\mathbf{c},\mathbf{c}'\in C,\mathbf{c}\neq\mathbf{c}'\}=\min\{\operatorname{wt}(\mathbf{c})\,:\,\mathbf{c}\in C\}. \tag{2.5}$$

An $[n,k,d]_q$-code is an $[n,k]_q$-code with minimum distance d. If the weights of all non-zero codewords of an $[n,k]_q$-code C are contained in $W=\{w_1,\dots,w_l\}$, then we speak of an $[n,k,W]_q$-code. By $A_w(C)\in\mathbb{N}_0$ we denote the number of codewords of weight w in C, where $0\le w\le n$. So, we have $A_w(C)=0$ for all $0<w<d(C)$. The sequence of all weights can be summarized in the *homogeneous weight enumerator*

$$\overline{W}_C(x,y)=\sum_{w=0}^{n}A_w(C)x^wy^{n-w} \tag{2.6}$$

of C. Setting $y=1$ we obtain the *weight enumerator*

$$W_C(x)=\sum_{w=0}^{n}A_w(C)x^w. \tag{2.7}$$

Exercise 2.2. *Let C and C' be isomorphic codes. Verify $W_C(x)=W_{C'}(x)$ and $\overline{W}_C(x,y)=\overline{W}_{C'}(x,y)$, so that particularly we have $d(C)=d(C')$.*

Exercise 2.3. *Let C be an $[n,k]_q$-code and C' be an $[n',k]_q$-code with $n'\le n$ that arises from C by removing some all-zero coordinates. Verify $W_C(x)=W_{C'}(x)$ and $\overline{W}_C(x,y)=\overline{W}_{C'}(x,y)$, so that in particular we have $d(C)=d(C')$.*

The orthogonal complement

$$C^{\perp} := \left\{\mathbf{y} \in \mathbb{F}_q^n \,:\, \langle \mathbf{x}, \mathbf{y}\rangle = 0 \text{ for all } \mathbf{x} \in C\right\} \tag{2.8}$$

of an $[n,k]_q$-code C, with respect to the standard inner product

$$\langle \mathbf{x}, \mathbf{y}\rangle := \sum_{i=1}^{n} x_i y_i \tag{2.9}$$

is called the *dual code* of C. Note that $C^{\perp}$ is an $[n, n-k]_q$-code and $\operatorname{Aut}(C) = \operatorname{Aut}\big(C^{\perp}\big)$ since $\langle \mu(\mathbf{x}), \mathbf{y}\rangle = \big\langle \mathbf{x}, \mu^{-1}(\mathbf{y})\big\rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$ and all $\mu \in \operatorname{Aut}\big(\mathbb{F}_q^n\big)$. The *dual minimum distance* $d^{\perp}$ is the minimum distance of the dual code. We call a linear code *projective* iff $d^{\perp} \ge 3$.

Exercise 2.4. *Let C be an $[n,k]_q$-code. Prove that $d^{\perp}(C) = 1$ iff the effective length of C is strictly smaller than n. Moreover, we have $d^{\perp}(C) = 2$ if a generator matrix G of C does not contain a zero column but two linearly dependent columns.*

If $C \subseteq C^{\perp}$, then C is called *self-orthogonal* and *self-dual* if $C = C^{\perp}$.

Exercise 2.5. *Show that*

(a) every binary self-orthogonal linear code is even;

(b) every doubly-even binary linear code is self-orthogonal;

(c) every self-dual ternary linear code is 3*-divisible.*

2.1 The MacWilliams Equations and the Linear Programming Method

The homogeneous weight enumerator $\overline{W}_C(x,y)$ of a linear code C over $\mathbb{F}_q$ and the homogeneous weight enumerator $\overline{W}_{C^{\perp}}(x,y)$ of its dual code $C^{\perp}$ are related by the so-called *MacWilliams identity* [175]

$$\overline{W}_{C^{\perp}}(x,y) = |C|^{-1} \cdot \overline{W}_C(y-x, y+(q-1)x). \tag{2.10}$$

So, given the complete *weight distribution* (A_i) of C, the weight distribution (B_i) of the dual code $C^{\perp}$ with $B_i(C) = A_i(C^{\perp}) \in \mathbb{N}_0$ is uniquely determined. We have

$$\sum_{j=0}^{n-i} \binom{n-j}{i} A_j = q^{k-i} \cdot \sum_{j=0}^{i} \binom{n-j}{n-i} B_j \tag{2.11}$$

for all $0 \le i \le n$, see e.g. [173, Lemma 2.2]. If we restrict the range of i to $0 \le i < t$, then we speak of the *first t MacWilliams equations*. Solving the equation system for the B_i gives:

Theorem 2.6. *(MacWilliams Equations, see [175])*
For an $[n,k,d]_q$-code C we have

$$\sum_{j=0}^{n} K_i(j)A_j(C) = q^k B_i(C) \tag{2.12}$$

for $0 \le i \le n$, where

$$K_i(j) := \sum_{s=0}^{i} (-1)^s \binom{n-j}{i-s}\binom{j}{s}(q-1)^{i-s} \tag{2.13}$$

are the Krawtchouck polynomials *(here j is considered as variable of a polynomial).*

There are lots of ways how to state the MacWilliams Equations. Another common representation are the so-called *power moments* [190]. For the binary case $q = 2$ and the first five MacWilliams equations they are spelled out in:

Exercise 2.7. *The weight distributions (A_i) and (B_i) of an $[n,k]_2$-code and its dual code satisfy*

$$\sum_{i=1}^{n} A_i = 2^k - 1 \tag{2.14}$$

$$\sum_{i=1}^{n} iA_i = 2^{k-1}(n - B_1) \tag{2.15}$$

$$\sum_{i=1}^{n} i^2 A_i = 2^{k-1}(B_2 - nB_1 + n(n+1)/2) \tag{2.16}$$

$$\sum_{i=1}^{n} i^3 A_i = 2^{k-2}\left(3(B_2 n - B_3) - (3n^2+3n-2)/2 \cdot B_1 + n^2(n+3)/2\right) \tag{2.17}$$

$$\begin{aligned}\sum_{i=1}^{n} i^4 A_i = {} & 2^{k-4}\big(4!(B_4 - nB_3) + 4(3n^2+3n-4)B_2 - 4(n^3+3n^2-9n+7)B_1 \\ & + (n^4+6n^3+3n^2-2n)\big).\end{aligned} \tag{2.18}$$

In our context we have several additional conditions on the A_i and B_i. First note that we have $A_0 = B_0 = 1$ in general, $B_1 = 0$ iff the code is of full length, and $B_2 = 0$ iff the code is projective. Δ-divisibility implies $A_i = 0$ for all $i \in \mathbb{N}$ with $i \not\equiv 0 \pmod{\Delta}$. Via residual codes, see Lemma 3.4 and the discussion thereafter, and non-existence results for the effective lengths of divisible codes, see Section 4 and Section 7, we can also exclude additional weights in many situations. Since the A_i and B_i are counts, they are integral. Moreover, the fact that scalar multiples of codewords are codewords again imply that also $A_i/(q-1)$ and $B_i/(q-1)$ are integers, cf. Exercise 2.21. More sophisticated extra conditions are discussed in Section 14.

(Integer) Linear programming method

By "the" *linear programming method* (for linear codes) we understand the application of linear programs certifying the non-existence of linear codes, cf. [61]. In general, a *linear program* consists of a set of real *variables* x_i, where some of them may be assumed to be non-negative, and a set of linear non-strict *constraints*, i.e., "$\le$", "$\ge$", or "$=$". Additionally, there is a linear *target function* that is either maximized or minimized. Specially structured forms are e.g. given by

$$\max \left\{ \mathbf{c}^\top \mathbf{x} \,:\, A\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge 0 \right\}$$

or

$$\max \left\{ \mathbf{c}^\top \mathbf{x} \,:\, A\mathbf{x} \le \mathbf{b}, D\mathbf{x} = \mathbf{d}, \mathbf{x} \ge 0 \right\}.$$

We remark that every linear program, LP for short, can be reformulated into e.g. the first form, possibly including a change of variables. Those linear programs can be solved efficiently in terms of the number of variables, the number of constraints, and the order of magnitude of the occurring coefficients. Choosing $\mathbf{c} = \mathbf{0}$ we can treat the question whether a linear inequality system admits a solution as an optimization problem. We say that an LP is *infeasible* if there exists no solution satisfying all constraints. If some of the variables are assumed to be integral, we speak of an integer linear program, (ILP) for short. While LPs can be solved in polynomial time, solving ILPs is NP hard.

In our context we choose the A_i, B_i as variables and the MacWilliams equations as constraints. Also the mentioned additional conditions can be formulated in this setting. Of course the length n, the dimension k, and the field size q have to be specified. If such an LP does not admit a real-valued solution we say that the non-existence of a linear code with corresponding parameters is certified by the linear programming method. If we assume $A_i/(q-1)$ and $B_i/(q-1)$ to be integers, then we speak of the integer linear programming method (for linear codes). Of course, this setting allows a lot of variations, so that there is no precise definition of "the" (integer) linear programming method for linear codes.

For more details on the application of linear programming in coding theory we refer to e.g. [20].

Coefficients of LPs for linear codes can grow very quickly

Even for moderate parameters the coefficients of the Krawtchouck polynomials, see Equation (2.13), can grow very quickly. This causes severe numerical problems when computing with limited precision. Note that while the coefficients in e.g. Equation (2.11) are a bit smaller, this advantage is quickly used up when a solution algorithm has performed some changes of basis. Some implementations with unbounded precision are available, see e.g. the computer algebra system `Maple` or the non-commercial solver `SCIP` for mixed integer programming.. However, computation times significantly increase when using long number arithmetic.

In order to keep the number of constraints small and to partially avoid the mentioned numerical issues we will mainly use the first t MacWilliams equations only, where t is rather small. Based on experimental evidence we remark that choosing $t \in \{3, 4, 5\}$ gives the same implication on non-existence as larger value of t in almost all cases.

Example 2.8. *No projective* 8*-divisible* $[52,10]_2$*-code exists since solving the first four MacWilliams equations for* $\{A_8, A_{16}, A_{24}, A_{32}\}$ *gives*

$$\begin{aligned} A_8 &= 10 + A_{40} + 4A_{48} + \frac{1}{4}B_3 \\ A_{16} &= -28 - 4A_{40} - 15A_{48} - \frac{3}{4}B_3 \\ A_{24} &= 790 + 6A_{40} + 20A_{48} + \frac{3}{4}B_3 \\ A_{32} &= 251 - 4A_{40} - 10A_{48} - \frac{1}{4}B_3, \end{aligned}$$

so that $A_{16} \le -28 < 0$*, which is a contradiction.*

Later on we will observe that no projective 4-divisible binary linear codes of lengths 4 or 12 exist, so that we may additionally use $A_{48} = 0$ and $A_{40} = 0$.

Example 2.9. *Let* C *be a projective* $[41,6]_2$*-code whose non-zero weights are contained in* $\{20, 24, 26, 40\}$*. Here, the first four MacWilliams equations imply*

$$\begin{aligned} B_3 &= \frac{470}{3} - \frac{280}{3}A_{40} \\ A_{20} &= \frac{158}{3} - \frac{28}{3}A_{40} \\ A_{24} &= 5 + 35A_{40} \\ A_{26} &= \frac{16}{3} - \frac{80}{3}A_{40}. \end{aligned}$$

However, $A_{26} \ge 0$ *yields* $A_{40} = 0$*, so that* $A_{26} = \frac{16}{3} \notin \mathbb{N}_0$*, which is a contradiction.*

The context of that example is that for field size $q = 2$, dimension $k = 6$, and minimum distance $d = 20$ the *Griesmer bound* is not attained, see e.g. [13].

Exercise 2.10. *Prove that an even* $[41,6,20]_2$*-code is projective and has non-zero weights in* $\{20, 24, 26, 40\}$ *only.*

In general we can determine lower and upper bounds for any linear combination of the A_i and B_i by using some subset of the MacWilliams equations. Adding integer rounding cuts sometimes gives tighter bounds:

Example 2.11. *In this example we want to show that each even* $[13,5,6]_2$*-code satisfies* $B_1 = 0$*,* $B_2 = 0$*,* $2 \le B_3 \le 4$*,* $23 \le A_6 \le 24$*,* $3 \le A_8 \le 6$*,* $1 \le A_{10} \le 4$*, and* $0 \le A_{12} \le 1$*. To this end we consider the following linear program based on the first four MacWilliams equations:*

$$\begin{aligned} \max B_1 & \quad \textit{subject to} \\ A_6 + A_8 + A_{10} + A_{12} &= 31 \\ 6A_6 + 8A_8 + 10A_{10} + 12A_{12} + 16B_1 &= 208 \\ 36A_6 + 64A_8 + 100A_{10} + 144A_{12} + 208B_1 - 16B_2 &= 1456 \\ 216A_6 + 512A_8 + 1000A_{10} + 1728A_{12} + 2176B_1 - 312B_2 + 24B_3 &= 10816. \end{aligned}$$

The (unique) optimal solution, computed with `Maple`*, is given by*

$$B_1 = \frac{3}{8}, B_2 = 0, B_3 = 0, A_6 = \frac{109}{4}, A_8 = 0, A_{10} = \frac{13}{4}, A_{12} = \frac{1}{2}$$

so that, in general, $B_1 \le \left\lfloor \frac{3}{8} \right\rfloor = 0$, i.e., we can assume $B_1 = 0$. With this additional equation, maximizing B_2, B_3, A_6, A_8, A_{10}, and A_{12} gives $B_2 \le \left\lfloor \frac{18}{17} \right\rfloor = 1$, $B_3 \le 4$, $A_6 \le \left\lfloor \frac{437}{17} \right\rfloor = 25$, $A_8 \le 6$, $A_{10} \le \left\lfloor \frac{11}{2} \right\rfloor = 5$, and $A_{12} \le \left\lfloor \frac{20}{13} \right\rfloor = 1$, respectively. Adding the tightened upper bounds, i.e., those for B_2, A_6, A_{10}, and A_{12}, maximizing B_2 again yields $B_2 \le \left\lfloor \frac{6}{7} \right\rfloor = 0$, so that $B_2 = 0$. Another iteration yields $B_3 \le 4$, $A_6 \le 24$, $A_8 \le 6$, $A_{10} \le 4$, and $A_{12} \le 1$. Similarly we obtain $B_3 \ge 2$, $A_6 \ge 23$, $A_8 \ge 3$, $A_{10} \ge 1$, and $A_{12} \ge 0$ by minimizing the variables. All these final lower and upper bounds for the variables can indeed by attained as shown in the subsequent example.

Example 2.12. *The non-negative integral solutions $(B_1, B_2, B_3, A_8, A_{10}, A_{12})$ of the first four MacWilliams equations of an even $[13, 5, 6]_2$-code are given by*

$$(0, 0, 4, 24, 3, 4, 0) \ \textit{and} \ (0, 0, 2, 23, 6, 1, 1).$$

To this end we solve the four equations for $\{B_3, A_6, A_8, A_{10}\}$:

$$\begin{aligned} B_3 &= 4 - 2A_{12} - 8B_1 - 3B_2 \\ A_6 &= 24 - A_{12} + 10B_1 + 2B_2 \\ A_8 &= 3 + 3A_{12} - 12B_1 - 4B_2 \\ A_{10} &= 4 - 3A_{12} + 2B_1 + 2B_2. \end{aligned}$$

From $B_3 \ge 0$ we conclude $B_1 = 0$ and $B_2 \in \{0, 1\}$. If $B_2 = 1$, then $B_3 \ge 0$ implies $A_{12} = 0$, so that $A_8 = -1 < 0$. Thus, we have $B_2 = 0$ and $A_{10} \ge 0$ implies $A_{12} \in \{0, 1\}$, which gives the two solutions stated above. The MacWilliams transforms of the corresponding weight distributions $(A_i)_{0 \le i \le 13}$ are given by

$$(B_i)_{0 \le i \le 13} = (1, 0, 0, 4, 30, 57, 36, 36, 57, 30, 4, 0, 0, 1)$$

and

$$(B_i)_{0 \le i \le 13} = (1, 0, 0, 2, 40, 39, 46, 46, 39, 40, 2, 0, 0, 1).$$

Of course, the latter does not show that both such codes exist, but is shows that we cannot conclude a contradiction using the linear programming method with all MacWilliams equations.

While the above examples indicate that one eventually have to deal with a few details in the computations, we would like to remark that it is always possible to hide the linear programming computations in mathematical non-existence proofs:

Exercise 2.13. *Use some arbitrary textbook on linear programming in order to show the following facts:*

- *The* Farkas' lemma *or* the Fourier–Motzkin elimination *algorithm yield a constructive certificate for the infeasibility of an LP or a linear inequality system, respectively.*
- *The* LP duality theorem *and the solution of the* dual linear program *can be used to compute multipliers for the constraints of the original LP whose (scaled) sum gives a tight bound for the optimum value of the target value or shows infeasibility if a given feasibility problem is reformulated as the minimization of the violation of the constraints.*

Example 2.14. *The first four MacWilliams equations for a projective* $[52,9]_2$*-code are given by*

$$\begin{aligned}
A_8 + A_{16} + A_{24} + A_{32} + A_{40} + A_{48} &= 511 && (2.19)\\
44A_8 + 36A_{16} + 28A_{24} + 20A_{32} + 12A_{40} + 4A_{48} &= 13260 && (2.20)\\
946A_8 + 630A_{16} + 378A_{24} + 190A_{32} + 66A_{40} + 6A_{48} &= 168402 && (2.21)\\
13244A_8 + 7140A_{16} + 3276A_{24} + 1140A_{32} + 220A_{40} + 4A_{48} &= 1392300 + 64B_3, && (2.22)
\end{aligned}$$

so that a linear program for the minimization of the violation reads

$$\begin{aligned}
\min x \quad & \quad \textit{subject to}\\
A_8 + A_{16} + A_{24} + A_{32} + A_{40} + A_{48} + x &\ge 511\\
A_8 + A_{16} + A_{24} + A_{32} + A_{40} + A_{48} - x &\le 511\\
44A_8 + 36A_{16} + 28A_{24} + 20A_{32} + 12A_{40} + 4A_{48} + x &\ge 13260\\
44A_8 + 36A_{16} + 28A_{24} + 20A_{32} + 12A_{40} + 4A_{48} - x &\le 13260\\
946A_8 + 630A_{16} + 378A_{24} + 190A_{32} + 66A_{40} + 6A_{48} + x &\ge 168402\\
946A_8 + 630A_{16} + 378A_{24} + 190A_{32} + 66A_{40} + 6A_{48} - x &\le 168402\\
-64B_3 + 13244A_8 + 7140A_{16} + 3276A_{24} + 1140A_{32} + 220A_{40} + 4A_{48} + x &\ge 1392300\\
-64B_3 + 13244A_8 + 7140A_{16} + 3276A_{24} + 1140A_{32} + 220A_{40} + 4A_{48} - x &\le 1392300.
\end{aligned}$$

Numbering the dual variables corresponding to the constraints of the above LP by $c_1, \dots, c_8$*, the optimal solution of the corresponding dual LP is given by* $c_2 = -0.919540$*,* $c_3 = 0.077176$*, and* $c_6 = -0.003284$ *with an optimal target value of* 0.42036124795*. Using a suitable continued fractions approximation we obtain the multipliers* $m_1 := -\frac{80}{87}$*,* $m_2 := \frac{47}{609}$*, and* $m_3 := -\frac{2}{609}$ *(as rational approximations for the floating points values of* c_2*,* c_3*, and* c_6*, respectively). With this,* m_1 *times Equation (2.19) plus* m_2 *times Equation (2.20) plus* m_3 *times Equation (2.21) gives*

$$-\frac{128}{203}A_8 - \frac{128}{609}A_{16} - \frac{128}{609}A_{40} - \frac{128}{203}A_{48} = \tfrac{256}{203} > 0,$$

which is a contradiction since $A_8, A_{16}, A_{40}, A_{48} \ge 0$*.*

In a mathematical proof we may just state the multipliers without justification or details of their computation. This also allows us to give rigor conclusions from numerical computations, i.e., compute multipliers with limited numerical precision, round them to some reasonably close rationals, and verify the final inequality with exact arithmetic, cf. Example 2.14.

In Section 6 we will draw several analytical conclusions from the linear programming method that do not rely on floating-point computations at all.

2.2 Geometric description of linear codes

Our next aim is to briefly describe linear codes from a geometric point of view. For further details we refer the interested reader to [65]. So, let $V \simeq \mathbb{F}_q^v$ be a v-dimensional vector space over $\mathbb{F}_q$. We call each i-dimensional subspace of V an *i-space*. As a shorthand, we use the geometric terms *points*, *lines*, *planes*, and *hyperplanes* for 1-, 2-, 3-, and $(v-1)$-spaces, respectively. A $(v-j)$-space is also called a (sub-)space of *codimension* j, where $0 \le j \le v$. In the special case of a space of codimension 2, i.e., a $(v-2)$-space, we also speak of *hyperlines*. Since two different 1-dimensional subspaces generate a unique 2-dimensional subspace, two different points are on exactly one common line, which partially motivates the use of the geometric language. Here we use the *algebraic dimension* and not the *geometric dimension*, which is one less.[1] The only exception is the notion of the $(v-1)$-dimensional *projective geometry* $\operatorname{PG}(v-1,q)$ associated with $\mathbb{F}_q^v$. There are $v-1$ types of geometric objects ranging from points (1-spaces) to hyperplanes $(v-1)$-spaces. By $\mathcal{P}$ we denote the set of points and by $\mathcal{H}$ we denote the set of hyperplanes whenever the dimension v of the ambient space and the field size q are clear from the context. Each point $P \in \mathcal{P}$ can be written as a 1-space

$$P = \left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_v \end{pmatrix} \right\rangle_q,$$

where $(x_1, \dots, x_v) \in \mathbb{F}_q^v \backslash \mathbf{0}$, or using projective coordinates $(x_1 : x_2 : \dots : x_v)$, where

$$(tx_1 : tx_2 : \dots : tx_v) = (x_1 : x_2 : \dots : x_v)$$

for all $t \in \mathbb{F}_q \backslash \{0\}$. Since the orthogonal complement of a $(v-1)$-space is a 1-space, we have similar notations for hyperplanes.

Number of subspaces

By $\begin{bmatrix} V \\ k \end{bmatrix}$ we denote the set of all k-spaces in V and by $\begin{bmatrix} v \\ k \end{bmatrix}_q$ their cardinality $\#\begin{bmatrix} V \\ k \end{bmatrix}$. For integers $0 \le k \le v$ we have,

$$\begin{bmatrix} v \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^{v-i}-1}{q^{k-i}-1}. \tag{2.23}$$

For other values of k we set $\begin{bmatrix} v \\ k \end{bmatrix}_q = 0$ by convention.

Exercise 2.15. *Prove Equation (2.23) by counting ordered bases of subspaces.*

[1] Points are 0-dimensional geometric objects and lines are 1-dimensional geometric objects, while we prefer to say that 1-spaces have (algebraic) dimension 1 and 2-spaces have (algebraic) dimension 2.

Using the notation $[v]_q := \frac{q^v-1}{q-1}$ and $[v]_q! := \prod_{i=1}^{v}[i]_q$ we can write

$$\begin{bmatrix} v \\ k \end{bmatrix}_q = \frac{[v]_q!}{[k]_q! \cdot [v-k]_q!}, \tag{2.24}$$

which motivates that the numbers $\begin{bmatrix} v \\ k \end{bmatrix}_q$ are also called *q-binomial* or *Gaussian binomial coefficients*. As they count the number of k-spaces contained in a v-space, they are a *q-analog* of the binomial coefficients $\binom{v}{k}$ which count the number of k-sets contained in a v-set. Here, a t-set is a set of cardinality t and we have $\lim_{q\to 1} \begin{bmatrix} v \\ k \end{bmatrix}_q = \binom{v}{k}$. An important instance of Equation (2.23) is given by

$$\#\mathcal{P} = \begin{bmatrix} v \\ 1 \end{bmatrix}_q = \begin{bmatrix} v \\ v-1 \end{bmatrix}_q = \#\mathcal{H} = \frac{q^v-1}{q-1} = [v]_q. \tag{2.25}$$

Exercise 2.16. *Verify* $\lim_{q\to 1} \begin{bmatrix} v \\ k \end{bmatrix}_q = \binom{v}{k}$,

$$\begin{bmatrix} v \\ k \end{bmatrix}_q = \begin{bmatrix} v \\ v-k \end{bmatrix}_q \quad \textit{and} \quad \begin{bmatrix} v \\ k \end{bmatrix}_q = q^k \begin{bmatrix} v-1 \\ k \end{bmatrix}_q + \begin{bmatrix} v-1 \\ k-1 \end{bmatrix}_q = \begin{bmatrix} v-1 \\ k \end{bmatrix}_q + q^{v-k} \begin{bmatrix} v-1 \\ k-1 \end{bmatrix}_q.$$

Multisets of points

A *multiset* $\mathcal{M}$ of points in $\mathrm{PG}(v-1,q)$ is a mapping $\mathcal{M}\colon \mathcal{P} \to \mathbb{N}_0$. For each point $P \in \mathcal{P}$ the integer $\mathcal{M}(P)$ is called the *multiplicity* of P and it counts how often point P is contained in the multiset. If $\mathcal{M}(P) \in \{0,1\}$ for all $P \in \mathcal{P}$ we also speak of a set instead of a multiset (of points).

Example 2.17. *For the list of points*

$$\left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

in $\mathrm{PG}(2,3)$ *a representation as a multiset* $\mathcal{M}$ *is given by* $\mathcal{M}(\langle(1,0,1)^\top\rangle) = 1$, $\mathcal{M}(\langle(1,2,0)^\top\rangle) = 3$, $\mathcal{M}(\langle(0,0,1)^\top\rangle) = 1$, *and* $\mathcal{M}(P) = 0$ *for all other points* P *in* $\mathcal{P}$. *Note that* $\langle(2,1,0)^\top\rangle = \langle(1,2,0)^\top\rangle$.

For the ease of a canonical representation of a point $\langle(x_1,\dots,x_v)^\top\rangle$ we will assume that the first non-zero value $x_i \in \mathbb{F}_q$ is equal to 1. The *cardinality* of the multiset is defined as

$$\#\mathcal{M} = \sum_{P\in\mathcal{P}} \mathcal{M}(P). \tag{2.26}$$

More generally, we set $\mathcal{M}(\mathcal{Q}) := \sum_{P\in\mathcal{Q}} \mathcal{M}(P)$ for each subset $\mathcal{Q} \subseteq \mathcal{P}$, i.e., we extend the mapping $\mathcal{M}$ additively. For each subspace S in $\mathrm{PG}(v-1,q)$ we also use the notation $\mathcal{M}(S)$ interpreting the points in S as a subset of $\mathcal{P}$. We also write $\mathcal{P}\backslash S$ for the set of points that are not contained in a subspace S. Choosing $S = \mathrm{PG}(v-1,q)$ we have $\mathcal{M}(S) = \#\mathcal{M}$, i.e., another expression for the cardinality of $\mathcal{M}$. We say that a subspace

S is *empty* (with respect to $\mathcal{M}$) if $\mathcal{M}(S) = 0$ and we say that $\mathcal{M}$ is *empty* if $\#\mathcal{M} = 0$. We also extend the notion of multiplicity from points to arbitrary subsets $\mathcal{Q} \subseteq \mathcal{P}$. For i-spaces $\mathcal{Q}$ of multiplicity m we speak of m-points, m-lines, m-planes, or m-hyperplanes in the cases where $i = 1$, $i = 2$, $i = 3$, or $i = v - 1$, respectively. The *support* $\operatorname{supp}(\mathcal{M})$ of a multiset of points $\mathcal{M}$ is the set of points of strictly positive multiplicity. We call $\mathcal{M}$ *spanning* if the 1-spaces in $\operatorname{supp}(\mathcal{M})$ span $\mathbb{F}_q^v$. In other words if no hyperplane has multiplicity $\#\mathcal{M}$. In Example 2.17 we have $\#\mathcal{P} = 13$, $\#\mathcal{M} = 5$, and the support of $\mathcal{M}$ has cardinality 3. Moreover, $\mathcal{M}$ is spanning

Main correspondence between linear codes and multisets of points

Now we describe the main correspondence between $[n, k]_q$-codes C with effective length n and spanning multisets $\mathcal{M}$ of points in $\mathrm{PG}(k-1, q)$. Let G be an arbitrary generator matrix of C. Due to the condition on the effective length n_{eff} of C, the matrix G does not contain a zero column. So, we can construct a multiset of points $P_1, \ldots, P_n$ in $\mathrm{PG}(k-1, q)$ by assigning to each column $\mathbf{x} \in \mathbb{F}_q^k$ of G the point $\langle \mathbf{x} \rangle_q \in \mathcal{P}$. In the other direction we can use a generator $\mathbf{x} \in \mathbb{F}_q^k \backslash \{\mathbf{0}\}$ of each point $\langle \mathbf{x} \rangle_q$ of the multiset as a column, with the corresponding multiplicity, of a generator matrix G of C.

This geometric description allows us to read off the code parameters from the multiset $\mathcal{M}$ of points in $\mathrm{PG}(k-1, q)$. The subsequent theorem shows how to determine the weight distribution of C from $\mathcal{M}$. To this end, we observe that the codewords of C are the $\mathbb{F}_q$-linear combinations of the rows of a generator matrix G of C. Let $\mathbf{g}^i = \left(g_1^i, \ldots, g_n^i\right) \in \mathbb{F}_q^n$ denote the ith row of G, so that each codeword $\mathbf{c} \in C$ has the form $\mathbf{c} = h_1 g^1 + h_2 g^2 + \cdots + h_k g^k$ and is uniquely determined by $\mathbf{h} = (h_1, \ldots, h_k) \in \mathbb{F}_q^k$. For a fixed coordinate $1 \le j \le n$, corresponding to the point P_j, the vector $\mathbf{c}$ has entry 0 in coordinate j exactly if

$$c_j \;=\; h_1 g_j^1 + h_2 g_j^2 + \cdots + h_k g_j^k \;=\; 0. \tag{2.27}$$

The coefficients h_i, collected in $\mathbf{h}$, of this linear equation define a hyperplane $H \in \mathcal{H}$. In other words, we have $c_j = 0$ iff the point P_j is contained in the hyperplane H. The above reasoning implies the following correspondence between linear codes and multisets of points:

Theorem 2.18. *Let C be a spanning $[n, k]_q$-code, G be a generator matrix of C, and $\mathcal{M}$ be the corresponding multiset of points in $\mathrm{PG}(k-1, q)$ (as described above). For each non-zero $\mathbf{h} = (h_1, \ldots, h_k) \in \mathbb{F}_q^k \backslash \{\mathbf{0}\}$ let $\mathbf{h}^\perp$ characterize the hyperplane $H \in \mathcal{H}$, which consists of all $\mathbf{y} = (y_1, \ldots, y_k)$ with $\langle \mathbf{h}, \mathbf{y} \rangle = 0$. Then, the weight of the codeword $\mathbf{c} = \sum_{i=1}^k h_i g^i$ is given by*

$$\mathrm{wt}(\mathbf{c}) = \sum_{P \in \mathcal{P}, P \notin H} \mathcal{M}(P) = \mathcal{M}(\mathcal{P} \backslash H) = n - \mathcal{M}(H). \tag{2.28}$$

The minimum Hamming distance is given by

$$d(C) = \min\{\mathcal{M}(\mathcal{P} \backslash H) \,:\, H \in \mathcal{H}\} = n - \max\{\mathcal{M}(H) \,:\, H \in \mathcal{H}\}. \tag{2.29}$$

In other words, the weight $\mathrm{wt}(\mathbf{c})$ of a codeword $\mathbf{c} \in C$ equals the number of points of $\mathcal{M}$ (counted with multiplicities) that are not contained in the hyperplane $H = \mathbf{h}^\perp$ associated to $\mathbf{c}$. We remark that if we start with a (non-empty) multiset $\mathcal{M}$ of points in $\mathrm{PG}(k-1,q)$, then the corresponding code C has dimension k iff $\mathcal{M}$ is spanning. The rank of the constructed matrix G would be strictly smaller than k otherwise.

We call two multisets of points isomorphic if the corresponding codes are. The set of automorphisms of $\mathrm{PG}(v-1,q)$ preserving the $\le$-ordering of subspaces is given by be natural action of $\mathrm{P\Gamma L}(\mathbb{F}_q^v)$ if $v \ge 3$. This famous result is called "Fundamental Theorem of Projective Geometry", see e.g. [2].

Some codes have very nice descriptions using the geometric language.

Example 2.19. *Let $\mathcal{M}$ be the (multi-)set in $\mathrm{PG}(k-1,q)$ defined by $\mathcal{M}(P) = 1$ for all $P \in \mathcal{P}$, where $k \ge 2$. It corresponds to the (projective) $\left[[k]_q, k, q^{k-1}\right]_q$* simplex code. *The minimum distance follows from the fact that each hyperplane $H \in \mathcal{H}$ contains $[k-1]_q = \frac{q^{k-1}-1}{q-1}$ points from $\mathcal{P}$, so that $\mathcal{M}(\mathcal{P}\backslash H) = [k]_q - [k-1]_q = q^{k-1}$. Since the weights of all non-zero codewords are equal to q^{k-1}, the code is q^{k-1}-divisible.*

Divisibility for multisets of points

We call a multiset of points Δ-divisible iff the corresponding linear code C is Δ-divisible. Note that this is equivalent to

$$\mathcal{M}(H) \equiv \#\mathcal{M} \pmod{\Delta} \tag{2.30}$$

for all hyperplanes $H \in \mathcal{H}$ if $\mathcal{M}$ is in $\mathrm{PG}(v-1,q)$ with $v \ge 2$. If $v = 1$, then $\dim(C) = 1$ and the condition is equivalent to $\#\mathcal{M} \equiv 0 \pmod{\Delta}$.[2]

With respect to $[n, \le k, d]_q$-codes,[3] Equation (2.29) motivates the following geometric notion. A multiset $\mathcal{K}$ of points in $\mathrm{PG}(k-1,q)$ is an (n,s)-*arc* if $\mathcal{K}(\mathcal{P}) = n$, $\mathcal{K}(H) \le s$ for every hyperplane $H \in \mathcal{H}$, and there exists a hyperplane $H_0 \in \mathcal{H}$ with $\mathcal{K}(H_0) = s$. If the last condition is skipped, we speak of an $(n, \le s)$-arc. The relation between d and s is given by $s = n - d$. The dimension of the subspace spanned by the points in the support of $\mathcal{K}$ is called dimension $\dim(\mathcal{K})$ of $\mathcal{K}$. The corresponding linear code has dimension $\dim(\mathcal{K})$.

There is an analog of the weight distribution of linear codes for arcs.

Definition 2.20. *Let $\mathcal{M}$ be an $(n, \le s)$-arc in $\mathrm{PG}(k-1,q)$. The* spectrum *of $\mathcal{M}$ is the vector $\mathbf{a} = (a_0, \ldots, a_s) \in \mathbb{N}_0^{s+1}$, where*

$$a_i = \#\left\{H \in \mathcal{H} \,:\, \mathcal{M}(H) = i\right\} \tag{2.31}$$

for $0 \le i \le s$.

[2] Note that if a multiset of points $\mathcal{M}$ with $\dim(\mathcal{M}) = 1$ is embedded in $\mathrm{PG}(v-1,q)$ with $v \ge 2$, then we have $\mathcal{M}(H) \in \{0, \#\mathcal{M}\}$ for all hyperplanes and there indeed exists a hyperplane with $\mathcal{M}(H) = 0$. See also Lemma 3.1 stating that the dimension of the ambient space is irrelevant.

[3] An $[n, \le k, d]_q$-code is an $[n, k', d]_q$-code where $1 \le k' \le k$. We also use the "$\le$"- or "$\ge$"-notation for other parameters.

Exercise 2.21. *Let $\mathcal{M}$ be a k-dimensional multiset of n points in $\mathrm{PG}(k-1,q)$ and C be the corresponding $[n,k]_q$-code. Show that*

$$A_i(C) = (q-1)\cdot a_{n-i}(\mathcal{M}) \tag{2.32}$$

for all $1 \le i \le n$.

In the case of a Δ-divisible multiset of points $a_i > 0$ implies $i \equiv n \pmod{\Delta}$. The analog of the first three MacWilliams equations are the so-called *standard equations*:

Lemma 2.22. *The spectrum $\mathbf{a} = (a_0,\dots,a_s)$ of an $(n,\le s)$-arc $\mathcal{M}$ in $\mathrm{PG}(k-1,q)$, where $k \ge 2$, satisfies*

$$\sum_{i=0}^{s} a_i = [k]_q \tag{2.33}$$

$$\sum_{i=0}^{s} i a_i = n\cdot[k-1]_q \tag{2.34}$$

$$\sum_{i=0}^{s} \binom{i}{2} a_i = \binom{n}{2}\cdot[k-2]_q + q^{k-2}\cdot\sum_{i\ge 2}\binom{i}{2}\lambda_i, \tag{2.35}$$

where λ_j denotes the number of points $P \in \mathcal{P}$ with $\mathcal{M}(P) = j$ for all $j \in \mathbb{N}$.

Exercise 2.23. *Prove Lemma 2.22 by double counting hyperplanes, incidences between points and hyperplanes, and incidences between pairs of points and hyperplanes. Show that the three standard equations are indeed equivalent to the first three MacWilliams equations assuming a code of full length, i.e. $B_1 = 0$.*

Exercise 2.24. *Let $\mathcal{M}$ be a multiset of points in $\mathrm{PG}(k-1,q)$ and $\mathcal{M}'$ be an embedding in $\mathrm{PG}(v-1,q)$ with $v > k$. Compute the spectrum $\mathbf{a}'$ of $\mathcal{M}'$ from the spectrum $\mathbf{a}$ of $\mathcal{M}$.*

Define the sum of two multisets $\mathcal{K}'$ and $\mathcal{K}''$ in the same geometry $\mathrm{PG}(k-1,q)$ by $(\mathcal{K}'+\mathcal{K}'')(P) = \mathcal{K}'(P) + \mathcal{K}''(P)$ for all points $P \in \mathcal{P}$. With the aid of so-called characteristic functions we can describe more sophisticated constructions in a compact manner. So, given a set of points $\mathcal{Q} \subseteq \mathcal{P}$, we denote by $\chi_{\mathcal{Q}} \colon \mathcal{P} \to \{0,1\}$ the *characteristic function* of $\mathcal{Q}$, i.e., $\chi_{\mathcal{Q}}(P) = 1$ if $P \in \mathcal{Q}$ and $\chi_{\mathcal{Q}}(P) = 0$ otherwise. If J is a j-space in $\mathrm{PG}(k-1,q)$, where $1 \le j \le k$, then we write χ_J for the characteristic function of the points contained in J.

First-order Reed–Muller codes a.k.a. affine k-spaces

Example 2.25. *Let H be a hyperplane in $V = \mathrm{PG}(k-1,q)$, where $k \ge 2$. Then $\mathcal{K} = \chi_V - \chi_H = \chi_{\mathcal{P}} - \chi_H$ is a $\left(q^{k-1}, q^{k-2}\right)$-arc that corresponds to a $\left[q^{k-1}, k, q^{k-1} - q^{k-2}\right]_q$-code.*

We remark that $\mathcal{K}$ is an affine geometry $\mathrm{AG}(k-1,q)$ and that the corresponding code is a first-order Reed-Muller code $\mathrm{RM}_q(k-1,1)$ of length q^{k-1}. We also call the (multi-)set of points an *affine k-space.*

In general we have:

Lemma 2.26. *Let $\mathcal{Q}_1,\dots,\mathcal{Q}_l \subseteq \mathcal{P}$ be multisets of points and $m_1,\dots,m_l \in \mathbb{Q}$. If*

$$\sum_{i=1}^{l} m_i \cdot \mathcal{Q}_i(P) \in \mathbb{N}_0 \tag{2.36}$$

for each $P \in \mathcal{P}$, then

$$\mathcal{M} = m_1 \cdot \mathcal{Q}_1 + m_2 \cdot \mathcal{Q}_2 + \cdots + m_l \cdot \mathcal{Q}_l \tag{2.37}$$

defines a multiset of points in $\operatorname{PG}(k-1,q)$.

Exercise 2.27. *Prove Lemma 2.26 and show that the code defined in Example 2.25 is q^{k-2}-divisible.*

Exercise 2.28. *Let $\mathcal{Q}_1,\dots,\mathcal{Q}_l \subseteq \mathcal{P}$ be multisets of points that are Δ-divisible. Sow that the multiset $\mathcal{M} = \sum_{i=1}^{l} \mathcal{Q}_i$ is Δ-divisible and that the multiset $t \cdot \mathcal{Q}_i$ is $t\Delta$-divisible for each integer $t \ge 1$.*

If $\mathcal{M}$ is a multiset of points in $\operatorname{PG}(k-1,q)$, then we can embed $\operatorname{PG}(k-1,q)$ in a k-space of $\operatorname{PG}(v-1,q)$ for each $v \ge k$ and naturally obtain a multiset $\mathcal{M}'$ of points in $\operatorname{PG}(v-1,q)$. If C is the linear code corresponding to $\mathcal{M}$ and C' the linear code corresponding to $\mathcal{M}'$, then C and C' are isomorphic and the (effective) lengths of C, C' equal $\#\mathcal{M} = \#\mathcal{M}'$ and the dimensions equal $\dim(\mathcal{M}) = \dim(\mathcal{M}')$. So, the union of the 20 solids and 30 planes considered in Subsection 1.1 yields a multiset $\mathcal{M}$ of points in $\operatorname{PG}(8,2)$ that is $\min\{2^{4-1}, 2^{3-1}\} = 4$-divisible. Thus, also the corresponding binary linear code is 4-divisible. We will consider the complementary multiset of points and its corresponding linear code in the next section.

By $\gamma_0(\mathcal{M})$ we denote the *maximum point multiplicity* of a given multiset of points $\mathcal{M}$ in $\operatorname{PG}(v-1,q)$, i.e., we have $\mathcal{M}(P) \le \gamma_0(\mathcal{M})$ for all $P \in \mathcal{P}$ and there exists a point $Q \in \mathcal{P}$ with $\mathcal{M}(Q) = \gamma_0(\mathcal{M})$. If $\gamma_0(\mathcal{M}) = 1$, then we also speak of a set of points instead a multiset of points. Clearly we have $\gamma_0(\mathcal{M}) = 0$ iff $\mathcal{M}$ is empty, i.e., $\#\mathcal{M} = 0$. We say that $\mathcal{M}$ is *proper* iff there exists a point $P \in \mathcal{P}$ with $\mathcal{M}(P) = 0$. Otherwise $\mathcal{M}$ has full support.

Exercise 2.29. *Show that for a given multiset of points $\mathcal{M}$ in* $\operatorname{PG}(v-1,q)$ *we have $\gamma_0(\mathcal{M}) = 1$ iff the corresponding linear code C is projective.*

The analog of the point multiplicity $\mathcal{M}(P)$ of a point P for the corresponding linear code C is the number of columns g^i of a generator matrix with $\langle g^i \rangle = P$. Here we may also speak of the (maximum) column multiplicity.

3 Basic results for $\triangle$-divisible multisets of points

As already observed, each multiset of points $\mathcal{M}$ in $\mathrm{PG}(v-1,q)$ can be embedded in a larger ambient space $\mathrm{PG}(v'-1,q)$, where $v'>v$. We can also embed in a smaller ambient space as long as the dimension is at least $\dim(\mathcal{M})$. As readily computed, the dimension of the ambient space is not really relevant for the notion of Δ-divisibility.

Lemma 3.1. *Let $V_1 < V_2$ be $\mathbb{F}_q$-vector spaces and $\mathcal{M}$ a multiset of points in V_1. Then $\mathcal{M}$ is Δ-divisible in V_1 iff $\mathcal{M}$ is Δ-divisible in V_2 (using the natural continuation of the characteristic function $\mathcal{M}(P)=0$ for all $P\in\mathcal{P}(V_2)\backslash\mathcal{P}(V_1)$).*

So, we will also speak of a Δ-divisible multiset of points $\mathcal{M}$ over $\mathbb{F}_q$ without specifying the dimension of the ambient space. (Of course we have to assume that the dimension of the ambient space is at least $\dim(\mathcal{M})$.)

As observed by Harold Ward, it is not necessary to consider all positive integers Δ when studying Δ-divisible codes.

Theorem 3.2. *([206, Theorem 1]) Let C be a Δ-divisible $[n,k]_q$-code with $k\ge 1$ and $\gcd(\Delta,q)=1$. Then C is equivalent to a code obtained by taking a linear code C' over $\mathbb{F}_q$, repeating each coordinate Δ times, and appending enough 0 entries to make a code whose length is that of C.*

Proof. The statement is clearly true for $k=1$, so that we assume $k\ge 2$. Let $\mathcal{M}$ be the corresponding multiset of points in $\mathrm{PG}(k-1,q)$. If $k\ge 3$ consider an arbitrary subspace S of codimension 2 and the $q+1$ hyperplanes $H_0,\dots,H_q$ containing S. We have

$$(q+1)\#\mathcal{M}\equiv\sum_{i=0}^{q}\mathcal{M}(H_i)=\#\mathcal{M}+q\cdot\mathcal{M}(S)\pmod{\Delta},$$

so that $\gcd(\Delta,q)=1$ implies that also the restriction $\mathcal{M}|_{H_i}$ of $\mathcal{M}$ to hyperplane H_i is Δ-divisible, i.e., we have $\mathcal{M}(S)\equiv\mathcal{M}(H_i)\equiv\#\mathcal{M}$ for every hyperplane S of H_i. Thus, it suffices to consider the case $k=2$ where we have $\mathcal{M}(P)\equiv\#\mathcal{M}\pmod{\Delta}$. From $\#\mathcal{M}=\sum_{P\in\mathcal{P}}\mathcal{M}(P)\equiv(q+1)\#\mathcal{M}\pmod{\Delta}$ and $\gcd(\Delta,q)=1$ we then conclude $\mathcal{M}(P)\equiv\#\mathcal{M}\equiv 0\pmod{\Delta}$, i.e., the multiplicity of every point is divisible by Δ. $\square$

s dividing Δ while $\gcd(s,q)=1$ implies s-fold repetition

Given an arbitrary positive integer Δ and a field size $q=p^m$, we can uniquely write $\Delta=s\cdot t$, where $s,t\in\mathbb{N}$, $\gcd(s,q)=1$, and t divides a sufficiently large power of p, i.e., there exist an integer e with $t=p^e$. From Theorem 3.2 we conclude that each

Δ-divisible $[n,k]_q$ code C arises from a q^r-divisible $[n',k]_q$-code C', where $r=\frac{e}{m}$ and $n's \le n$, by repeating each coordinate in C' exactly s times and adding $n-n's$ zero entries. Thus, it is sufficient to study q^r-divisible codes over $\mathbb{F}_q$, where q^r is a power of the characteristic p of the finite field $\mathbb{F}_q$.

Exercise 3.3. *Show that no projective* $[54,6,\{24,27,30\}]_2$*-code exists.*

Divisibility inherits

Assume that q^r divides Δ and that $\mathcal{M}$ is a Δ-divisible multiset of points in $\mathrm{PG}(v-1,q)$ with $v\ge 3$. If W is a subspace of codimension 2, then there are $q+1$ hyperplanes $H_1,\dots,H_{q+1}$ through W, i.e., hyperplanes in $\mathrm{PG}(v-1,q)$ that fully contain the subspace W. Counting points yields

$$\sum_{i=1}^{q+1}\mathcal{M}(H_i)=q\cdot\mathcal{M}(W)\;+\;\#\mathcal{M}\equiv(q+1)\#\mathcal{M}\pmod{\Delta},\tag{3.1}$$

so that

$$q\cdot\#\mathcal{M}(H_i)\equiv q\cdot\#\mathcal{M}\equiv q\cdot\mathcal{M}(W)\pmod{\Delta},$$

which is equivalent to

$$\#\mathcal{M}(H_i)\equiv\#\mathcal{M}\equiv\mathcal{M}(W)\pmod{\Delta/q}\tag{3.2}$$

if $r\ge 1$, i.e., $\Delta/q\in\mathbb{N}$. By induction over j we can easily prove:

Lemma 3.4. *Let $\mathcal{M}$ be a Δ-divisible multiset of points in $V\simeq\mathrm{PG}(v-1,q)$, where q^r divides Δ, and $U\neq\langle\mathbf{0}\rangle$ be a subspace of V of codimension $0\le j\le r$. Then, the restriction $\mathcal{M}|_U$ is a q^{r-j}-divisible multiset in the $v-j$-dimensional vector space U.*

So, e.g. when restricting a given multiset of points $\mathcal{M}$ over $\mathbb{F}_q$ to a hyperplane H, the multiplicity goes down by at most a factor q. Of course, if Δ is the maximum possible divisibility of $\mathcal{M}$, then the maximum possible divisibility of $\mathcal{M}|_H$ divides Δ and is at least 1. The converse of Lemma 3.4 is not true in general:

Example 3.5. *Let $\mathcal{M}$ in $\mathrm{PG}(5,q)$ be given by $q\cdot\chi_{\mathbf{e}_1}+q\cdot\chi_{\mathbf{e}_2}+q\cdot\chi_{\mathbf{e}_3}+q\cdot\chi_{\mathbf{e}_4}$, where $\mathbf{e}_i$ denotes the ith unit vector. I.e., we consider a multiset $\mathcal{M}$ given by four q-fold points whose span is 4-dimensional. With this, we have $\#\mathcal{M}=4q$ and $\mathcal{M}$ is q-divisible. For any hyperplane H that contains $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ but not $\mathbf{e}_4$ we have $\mathcal{M}(H)=3q$. Thus, $\mathcal{M}$ is not Δ-divisible for any $\Delta>q$. However, we even have $\mathcal{M}(L)\equiv\#\mathcal{M}\pmod{q}$ for any line L.*

Exercise 3.6. *Let $\mathcal{M}$ be a multiset of points in $\mathrm{PG}(v-1,p^h)$ and $1\le s\le v-1$ be an integer. Show that $\mathcal{M}(S)\equiv\#\mathcal{M}\pmod{p^r}$ for each s-space S in $\mathrm{PG}(v-1,p^h)$ implies $\mathcal{M}(T)\equiv\#\mathcal{M}\pmod{p^r}$ for each t-space T in $\mathrm{PG}(v-1,p^h)$, where $s\le t\le v-1$ and r are positive integers.*

For the special case $j = 1$, we can easily translate Lemma 3.4 to the language of linear codes. To this end we need a little more notation. Let C be an $[n,k]_q$-code. For an arbitrary index set $I \subseteq \{1,\dots,n\}$ and an arbitrary codeword $\mathbf{c} \in C$ by $\mathbf{c}_I$ we denote the $|I|$-tuple that consists of the entries c_i with $i \in I$; $\mathbf{c}_I$ is also called restricted codeword. For an arbitrary but fixed codeword $\tilde{\mathbf{c}} \in C$ we set $I := \{1,\dots,n\}\backslash \operatorname{supp}(\tilde{\mathbf{c}})$ as abbreviation. With this we can define the so-called *residual code* of C with respect to $\tilde{\mathbf{c}}$ by

$$\operatorname{Res}(C;\tilde{\mathbf{c}}) := \{\mathbf{c}_I \,:\, \mathbf{c} \in C\}\,.$$

This code of length $|I|$ is Δ/q-divisible as $\mathcal{M}|_H$ is Δ/q-divisible, where $\mathcal{M}$ is the multiset of points corresponding to C and H is the hyperplane corresponding to the codeword $\tilde{\mathbf{c}}$. In this latter and special form, Lemma 3.4 can be found in [209, Lemma 13].

Exercise 3.7. *Let C be a q^r-divisible $[n,k]_q$-code with $r > k-1$. Show that C arises from a q^{k-1}-divisible $[n',k]_q$-code C' by repeating each non-zero coordinate q^{r-k+1} times and adding a suitable number of zero coordinates. So, in particular we have that the effective length of C is divisible by q^{r-k+1}, cf. [207, Theorem 1.3].*

To this end, let $c^1,\dots,c^k$ be arbitrary codewords of C and $I^{\mathbf{x}} \subseteq \{1,\dots,n\}$ be defined as the set of all indices $1 \le i \le$ such that $c^j_i = x_i$ for all $1 \le j \le k$, where $\mathbf{x} \in \{0,1,\dots,q-1\}^k$ is arbitrary. By eventually considering an isomorphic code assume the following normalization criterion: For each $1 \le j \le k$ and each $i \in \{1,\dots,n\}\backslash \operatorname{supp}(\langle c^1,\dots,c^{j-1}\rangle)$ we have $c^j_i \in \{0,1\}$. With this, show that $\#I^{\mathbf{x}}$ is divisible by q^{r-k+1} if $\mathbf{x} \neq \mathbf{0}$.

An easy averaging argument

Let $\mathcal{M}$ be a non-empty multiset of points in $\operatorname{PG}(v-1,q)$, where $v \ge 2$. Since each point is contained in $\begin{bmatrix}v-1\\v-2\end{bmatrix}_q = \begin{bmatrix}v-1\\1\end{bmatrix}_q = [v-1]_q$ hyperplanes, see Exercise 3.11, and there are $\begin{bmatrix}v\\v-1\end{bmatrix}_q = \begin{bmatrix}v\\1\end{bmatrix}_q = [v]_q$ hyperplanes in total, the average number of points per hyperplanes is given by

$$\frac{\sum_{H\in\mathcal{H}} \mathcal{M}(H)}{|\mathcal{H}|} = \frac{\#\mathcal{M}\cdot[v-1]_q}{[v]_q} = \frac{\#\mathcal{M}\cdot[v-1]_q}{q[v-1]_q+1} = \frac{\#\mathcal{M}}{q+\frac{1}{[v-1]_q}} < \frac{\#\mathcal{M}}{q}. \tag{3.3}$$

Choosing a hyperplane $H \in \mathcal{H}$ that minimizes $\mathcal{M}(H)$ we obtain:

Lemma 3.8. *Let $\mathcal{M}$ be a non-empty multiset of points in $\operatorname{PG}(v-1,q)$. If $v \ge 2$, then there exists a hyperplane $H \in \mathcal{H}$ with $\mathcal{M}(H) < \frac{\#\mathcal{M}}{q}$.*

Exercise 3.9. *Let $\mathcal{M}$ be a proper multiset of points in $\operatorname{PG}(v-1,q)$ with $v \ge 2$. Show $\min\{\mathcal{M}(H) \,:\, H \in \mathcal{H}\} \le \frac{[v-2]_q}{[v-1]_q}\cdot\#\mathcal{M}$.*

The non-geometric coding counterpart of Lemma 3.8 is the well-known existence of a codeword of weight $> \frac{q-1}{q}\cdot n_{\text{eff}}$. For a refinement we refer to the *Hamada bound* in Theorem 8.3.

Example 3.10. *From Lemma 3.8 we can directly conclude that there is no 2-divisible multiset of points of cardinality 1 over $\mathbb{F}_2$. Note that due to $1 \not\equiv 0 \pmod 2$, there cannot* *be such a multiset in* PG$(1-1,2)$. *Now assume that* $\mathcal{M}$ *is a* 4*-divisible multiset of points of cardinality* 9 *over* $\mathbb{F}_2$. *Since* $9 \not\equiv 0 \pmod 4$, *we conclude that the dimension* v *of the ambient space of* $\mathcal{M}$ *is at least* 2. *Lemma 3.8 guarantees the existence of a hyperplane* H *with* $\mathcal{M}(H) < \#\mathcal{M}/q = 9/2$. *Since* $\mathcal{M}(H) \equiv \#\mathcal{M} \pmod 4$, *we have* $0 < \mathcal{M}(H) \le 1$. *So, due to Lemma 3.4 we have that the restricted arc* $\mathcal{M}|_H$ *is* 2*-divisible and has cardinality* 1. *Thus, there is no* 4*-divisible multiset of points of cardinality* 9 *over* $\mathbb{F}_2$.

Exercise 3.11. *Let* S *be an* s*-space in* PG$(v-1,q)$ *and* t *be an integer with* $s \le t \le v$. *Show that the number of* t*-spaces that contain* S *is given by* ${\genfrac{[}{]}{0pt}{}{v-s}{t-s}}_q$.

Example 2.19 and Exercise 2.27 directly give:

Lemma 3.12. *Let* $\mathcal{U}$ *be a multiset of subspaces of* PG$(v-1,q)$ *and* $\mathcal{M} = \uplus_{U\in\mathcal{U}} \genfrac{[}{]}{0pt}{}{U}{1}$ *the* associated multiset of points. *(In the expression* $\biguplus_{U\in\mathcal{U}}$, *the subspace* U *is repeated according to its multiplicity in the multiset* $\mathcal{U}$.*) Let* k *be the smallest dimension among the subspaces in* $\mathcal{U}$. *If* $k \ge 1$, *then the multiset* $\mathcal{M}$ *is* q^{k-1}*-divisible.*

So the multiset of points given by the points of the 20 solids and 30 planes from Subsection 1.1 is Δ-divisible for $\Delta = 2^{3-1} = 4$.

The kernel of the incidence matrix between points and k-spaces

Let $A \in \{0,1\}^{[v]_q \times {\genfrac{[}{]}{0pt}{}{v}{k}}_q}$ be the incidence matrix between the $[v]_q$ points P and the ${\genfrac{[}{]}{0pt}{}{v}{k}}_q$ k-spaces K in PG$(v-1,q)$, i.e., the entries of A are given by $a_{P,K} = 1$ iff $P \le K$. For any multiset $\mathcal{K}$ of k-spaces in PG$(v-1,q)$ let $\mathbf{x} \in \mathbb{N}^{{\genfrac{[}{]}{0pt}{}{v}{k}}_q}$ be the corresponding counting vector. With this, the vector $A\mathbf{x} \in \mathbb{N}^{[v]_q}$ is in one-to-one correspondence to the multiset $\mathcal{M}$ of points associated to $\mathcal{K}$. Let $\mathbf{y} \in \mathbb{R}^{[v]_q}$ be an element of the cokernel of A, which is the kernel of $A^\top$, i.e., $A^\top \mathbf{y} = \mathbf{0}$. So, we have $\mathbf{y}^\top A = \mathbf{0}$, $\mathbf{y}^\top(A\mathbf{x}) = \mathbf{0}$, and the necessary condition $\mathbf{y}^\top \mathbf{z} = \mathbf{0}$ for any candidate vector $\mathbf{z} \in \mathbb{N}^{[v]_q}$ for $\mathcal{M}$. In other words, the kernel of the incidence matrix $A^\top$ gives necessary conditions whether a given multiset $\mathcal{M}$ of points can be decomposed into a multiset of k-spaces. (As in Lemma 3.12, we can also consider the situation of subspaces K with $\dim(K) \ge k$ instead of $\dim(K) = k$.)

Now observe that we have some freedom over which domain we compute the kernel and interprete the corresponding necessary conditions $\mathbf{y}^\top \mathbf{z} = \mathbf{0}$. Over the real or rational numbers the kernel of $A^\top$ is trivial so that we don't get any non-trivial conditions $\mathbf{y}^\top \mathbf{z} = \mathbf{0}$. For $k = v-1$, i.e. hyperplanes, this observation follows from Exercise 3.25 and we can use Exercise 3.24 for the general case. If p is the characteristic of $\mathbb{F}_q$, then the domain $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ is a natural choice. To this end we remark that p-rank of the incidence matrix between points and k-spaces can be explicitly computed using the famous *Hamada formula* [100], see also [174, 200] for ancestors. Here we want to start with a more general perspective first. We are interested in the set of multisets of points that can be decomposed into a multiset of k-spaces, i.e., non-negative integer combinations of the rows of $A^\top$. As said, over $\mathbb{Q}$ the row space of $A^\top$ is $\mathbb{Q}^{[v]_q}$. Over $\mathbb{Z}$ we can use the structure theorem for finitely generated modules over a principal ideal domain R – the *invariant factor decomposition* to be more precise. For every non-zero matrix $B \in R^{m\times n}$ there exist invertible matrices $S \in R^{m\times m}$ and $T \in R^{n\times n}$ such that $D = SBT \in R^{m\times n}$ has zero entries outside of its main diagonal $d_1, \dots, d_{\min(m,n)}$, where we additionally have $d_i \,|\, d_{i+1}$ for all $1 \le i < \min(m, n)$. This is the *Smith normal form* of B and the d_i are called *invariant factors*, *elementary divisors*, or *invariants*. The matrices S and T can be algorithmically obtained by recursively applying invertible row and column operations to B till it reaches the desired diagonal form. Now we choose $R = \mathbb{Z}$ and set $M = \left\{x^\top B : x \in \mathbb{Z}^m\right\}$, $M' = \left\{x^\top D : x \in \mathbb{Z}^m\right\}$. Setting $d_i := 0$ for $\min\{m, n\} < i \le n$ we have $M' = d_1\mathbb{Z} \times d_2\mathbb{Z} \times \cdots \times d_n\mathbb{Z} \subseteq \mathbb{Z}^{1\times n}$. Using the convention that $i \equiv j \pmod 0$ iff $i = j$ for all $i, j \in \mathbb{Z}$, we can also write

$$M' = \left\{z' \in \mathbb{Z}^{1\times m} \,:\, z'\mathbf{e}_i \equiv 0 \pmod{d_i}\ \forall 1 \le i \le n\right\}, \tag{3.4}$$

where $\mathbf{e}_i$ denote the unit vectors. Using that $x^\top B = z$ is equivalent to $\left(x^\top S^{-1}\right) D = zT$ we obtain

$$M = \left\{z \in \mathbb{Z}^{1\times m} \,:\, z\left(T\mathbf{e}_i\right) \equiv 0 \pmod{d_i}\ \forall 1 \le i \le n\right\}, \tag{3.5}$$

i.e., we can read off the conditions for elements in M from the columns of T. For a brief description of the Smith normal form in our context we refer to [149, Section 4].

Example 3.13. *Let* $B = \begin{pmatrix} 1 & 1 & 5 & 7 \\ 2 & 8 & 10 & 20 \\ 3 & 3 & 45 & 51 \end{pmatrix}$ *and M be the $\mathbb{Z}$-modul generated by the rows of B. The Smith normal form of B is given by*

$$D := SBT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 30 & 0 \end{pmatrix} \quad \textit{with} \quad S = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad T = \begin{pmatrix} 1 & -1 & -6 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

We have

$$M' = \left\{(z_1, z_2, z_3, z_4) \in \mathbb{Z}^{1\times 4} \,:\, z_2 \equiv 0 \pmod 6, z_3 \equiv 0 \pmod{30}, z_4 = 0\right\},$$

for the $\mathbb{Z}$-modul M' generated by the rows of D and

$$\begin{aligned} M = \big\{ &(z_1, z_2, z_3, z_4) \in \mathbb{Z}^{1\times 4} \,:\, -z_1 + z_2 \equiv 0 \pmod 6, -6z_1 - z_2 + z_4 \equiv 0 \pmod{30}, \\ &z_1 + z_2 + z_3 - z_4 = 0\big\}. \end{aligned}$$

The conditions can as well be split up to the different occurring prime powers and we can modify the coefficients according to the respective moduli:

$$\begin{aligned} z_1 + z_2 &\equiv 0 \pmod 2 & z_1 + z_2 + z_3 + 2z_4 &\equiv 0 \pmod 3 \\ z_2 + z_4 &\equiv 0 \pmod 2 & 4z_1 + 4z_2 + z_4 &\equiv 0 \pmod 5 \\ z_1 + z_2 + z_3 + z_4 &\equiv 0 \pmod 2 & z_1 + z_2 + z_3 + 4z_4 &\equiv 0 \pmod 5 \\ 2z_1 + z_2 &\equiv 0 \pmod 3 & z_1 + z_2 + z_3 - z_4 &= 0 \\ 2z_2 + z_4 &\equiv 0 \pmod 3 & & \end{aligned}$$

The last condition $\pmod p$ *for* $p \in \{2,3,5\}$ *is superfluous since it is implied by* $z_1 + z_2 + z_3 - z_4 = 0$. *We can easily generate more superfluous conditions like e.g.* $z_1 + z_3 \equiv 0 \pmod 2$ *or* $z_3 \equiv 0 \pmod 5$ *by considering the span of our constraints. Further irredundant descriptions of* M*, i.e. a description as a tuple of integers where we cannot drop one of the conditions, are e.g. given by*

$$\begin{aligned} M \;=\; & \big\{ (z_1,z_2,z_3,z_4) \in \mathbb{Z}^{1\times 4} \,:\, z_1+z_2 \equiv 0 \pmod 2, z_1+z_3 \equiv 0 \pmod 2, z_3 \equiv 0 \pmod 5, \\ & z_2+z_3 \equiv 0 \pmod 3, z_1+z_2+z_4 \equiv 0 \pmod 3, z_1+z_2+z_3-z_4 = 0 \big\} \end{aligned}$$

or

$$\begin{aligned} M \;=\; & \big\{ (z_1,z_2,z_3,z_4) \in \mathbb{Z}^{1\times 4} \,:\, 5z_1+5z_2+2z_3 \equiv 0 \pmod{10}, 3z_1+2z_2+5z_3 \equiv 0 \pmod 6, \\ & z_1+z_2+z_4 \equiv 0 \pmod 3, z_1+z_2+z_3-z_4 = 0 \big\}. \end{aligned}$$

The span of our original set of constraints as well as the Chinese remainder theorem *allow a lot of freedom in the choice of a suitable (irredundant) description of* M*. We only need to ensure that the span of the used constraints coincide with the original one.*

Theorem 3.14. *(E.g. [51, Theorem 3.1].) Let* A *be the incidence matrix of points and* k*-spaces in* $\mathrm{PG}(v-1,q)$ *and let* p *be the characteristic of* $\mathbb{F}_q$*. Then, the invariant factors of* $A^\top$ *are all* p*-powers except the last, which is a* p*-power times* $[k]_q$*.*

Example 3.15. *Let* B *be the incidence matrix between lines and points in* $\mathrm{PG}(2,3)$ *and* T *be the column transformation matrix of the Smith normal form of* B*, i.e.*

$$B = \begin{pmatrix} 1&0&0&1&0&0&1&0&0&1&0&0&0\\ 1&0&0&0&1&0&0&0&1&0&1&0&0\\ 1&0&0&0&0&1&0&1&0&0&0&1&0\\ 0&1&0&0&1&0&0&1&0&1&0&0&0\\ 0&1&0&0&0&1&1&0&0&0&1&0&0\\ 0&1&0&1&0&0&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1&1&0&0&0\\ 0&0&1&1&0&0&0&1&0&0&1&0&0\\ 0&0&1&0&1&0&1&0&0&0&0&1&0\\ 1&1&1&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&1&1&1&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1&1&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1&0&0&0&0&0&0&0&-2&-1&-1&-2&-3\\ 0&1&0&0&0&0&0&-1&-1&-1&-2&-1&-11\\ 0&0&1&0&0&0&0&-2&0&-1&0&0&-11\\ 0&0&0&1&0&0&0&-2&0&-2&-1&0&-7\\ 0&0&0&0&1&0&0&0&-2&-2&0&-1&-3\\ 0&0&0&0&0&1&0&-1&-1&-2&-2&-2&-3\\ 0&0&0&0&0&1&1&-1&-1&-3&-3&-3&-7\\ 0&0&0&0&0&0&0&1&0&0&0&0&-3\\ 0&0&0&0&0&0&0&0&1&0&0&0&-3\\ 0&0&0&0&0&0&1&0&0&0&-1&-1&-7\\ 0&0&0&0&0&0&0&0&0&0&1&0&-3\\ 0&0&0&0&0&0&0&0&0&0&0&1&-3\\ 0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

Here five invariant factors of B *equal* 3*, the last equals* 12*, and the first seven equal* 1*. So, for* $z = (z_1,\ldots,z_{13}) \in \mathbb{Z}^{1\times[3]_3}$ *we have* $z \in M := \big\{x^\top B \,:\, x \in \mathbb{Z}^{[3]_3}\big\}$ *iff*

$$\begin{aligned} 2z_2+z_3+z_4+2z_6+2z_7+z_8 &\equiv 0 \pmod 3,\\ z_1+2z_2+z_5+2z_6+2z_7+z_9 &\equiv 0 \pmod 3,\\ 2z_1+2z_2+2z_3+z_4+z_5+z_6 &\equiv 0 \pmod 3,\\ 2z_1+z_2+2z_4+z_6+2z_{10}+z_{11} &\equiv 0 \pmod 3,\\ z_1+2z_2+2z_5+z_6+2z_{10}+z_{12} &\equiv 0 \pmod 3,\\ z_2+z_3+2z_4+2z_7+2z_{10}+z_{13} &\equiv 0 \pmod 3,\\ \sum_{i=1}^{13} z_i &\equiv 0 \pmod 4, \end{aligned}$$

where we have broken up the condition of the last column of T *modulo* 12 *into two conditions modulo* 3 *and modulo* 4*, respectively.*

Since $[k]_q$ is coprime to $p = \operatorname{char}(\mathbb{F}_q)$, Theorem 3.14 suggest to split the implied conditions into several $\bmod p^l$ conditions (for possibly different values of l) and a single $\bmod [k]_q$ condition. The latter just resembles the fact that we have $\#\mathcal{M} \equiv 0 \pmod{[k]_q}$ for a multiset of points arising as a union of k-space in $\operatorname{PG}(v-1,q)$, noting that this isn't true any more if we consider multisets of subspaces with dimension at least k and assume $v > k$. The $\bmod 3$ conditions in Example 3.15 may be a bit harder to guess. We observe that the sum of the first two conditions is equivalent to $\sum_{i=1}^{9} z_i \equiv 0 \pmod 3$, so that it might be worthwhile to look for different (irredundant) descriptions of M. Also in the generic case of an arbitrary integer matrix B the Chinese remainder theorem allows us to restrict our considerations to $\bmod p^l$ conditions. We now formalize the underlying objects in terms of codes over integer residue rings, see e.g. [24].

Definition 3.16. *Let p be a prime, l be a positive integer, and $B \in R^{m\times n}$ a matrix where $R = \mathbb{Z}$ or $R = \mathbb{Z}/p^l\mathbb{Z}$. The* (linear) $\mathbb{Z}_{p^l}$-code *C of B is given by the row span of B w.r.t. $\mathbb{Z}/p^l\mathbb{Z}$. The matrix B is called a* generator matrix *of C. The* dual code *$C^\perp$ consists of all row vectors that are orthogonal to all elements in C (w.r.t. $\mathbb{Z}/p^l\mathbb{Z}$). We also call $C^\perp$ the* $\mathbb{Z}_{p^l}$-kernel *of B.*

Example 3.17. *Let C be the $\mathbb{Z}_3$-code of the incidence matrix between lines and points in* $\operatorname{PG}(2,3)$ *B as in Example 3.15. Generator matrices of C and its dual code $C^\perp$ are e.g. given by*

$$\begin{pmatrix} 1&0&0&0&0&1&0&1&0&0&0&1&0\\ 0&1&0&0&0&1&0&2&2&0&1&0&2\\ 0&0&1&0&0&1&0&0&1&0&2&2&2\\ 0&0&0&1&0&2&0&1&2&0&2&1&1\\ 0&0&0&0&1&2&0&2&1&0&1&2&0\\ 0&0&0&0&0&0&1&1&1&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1&0&0&0&0&1&0&1&0&2&2&0&2\\ 0&1&0&0&0&1&0&2&2&0&1&0&2\\ 0&0&1&0&0&1&0&0&1&0&2&2&2\\ 0&0&0&1&0&2&0&1&2&2&1&0&0\\ 0&0&0&0&1&2&0&2&1&0&1&2&0\\ 0&0&0&0&0&0&1&1&1&2&2&2&0 \end{pmatrix},$$

respectively. Clearly, we have $\dim(C) + \dim(C^\perp) = [3]_3 = 13$ *since C and $C^\perp$ are vector spaces. Now consider the incidence matrix*

$$B' = \begin{pmatrix} 0&1&1&0&1&1&0&1&1&0&1&1&1\\ 0&1&1&1&0&1&1&1&0&1&0&1&1\\ 0&1&1&1&1&0&1&0&1&1&1&0&1\\ 1&0&1&1&0&1&1&0&1&0&1&1&1\\ 1&0&1&1&1&0&0&1&1&1&0&1&1\\ 1&0&1&0&1&1&1&1&0&1&1&0&1\\ 1&1&0&1&1&0&1&1&0&0&1&1&1\\ 1&1&0&0&1&1&1&0&1&1&0&1&1\\ 1&1&0&1&0&1&0&1&1&1&1&0&1\\ 0&0&0&1&1&1&1&1&1&1&1&1&0\\ 1&1&1&0&0&0&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&0&0&0&1&1&1&0\\ 1&1&1&1&1&1&1&1&1&0&0&0&0 \end{pmatrix}$$

between affine planes and points in $\operatorname{PG}(2,3)$. *By C' we denote the corresponding $\mathbb{Z}_3$-code and observe that every row of B' is orthogonal to every row of B w.r.t. $\mathbb{Z}/3\mathbb{Z}$, i.e. $C' \subseteq C^\perp$. By e.g. computing the Hermite normal form of B' we can verify* $\dim(C') = 6$*, so that indeed $C' = C^\perp$. In the context of Example 3.15 this means that we can replace the six* $\bmod 3$-*conditions by* $B'z \equiv 0 \pmod 3$*, which corresponds to thirteen single* $\bmod 3$-*conditions. Of course we can also select six of these such that the corresponding rows of B' generate C'.*

Lemma 3.18. *For positive integers k, k', v with $l := k + k' - v - 1 \geq 1$ the incidence vector of an affine k'-space is contained in the $\mathbb{Z}_{q^l}$-kernel of the incidence matrix of k-spaces and points in* $\operatorname{PG}(v-1, q)$.

Proof. We describe an affine k'-space A' by an k'-space S' and an $(k'-1)$-space L' with $L' \leq S'$, i.e. $\chi_{A'} = \chi_{S'} - \chi_{L'}$ or $A' = S' \backslash L'$. Now let S be an arbitrary k-space and $I := S' \cap S$, so that $i := \dim(I) \geq k + k' - v$. If $i \leq 0$ or $I \leq L'$, then we have $\#(A' \cap S) = 0$ and $\#(A' \cap S) = q^{i-1}$ otherwise. Thus, we have $\#(A' \cap S) \equiv 0 \pmod{q^l}$ in all cases. □

Note that the statement for $l = 0$ is trivially true.

Example 3.19. *Let B be the incidence matrix between planes and points in* $\operatorname{PG}(3,2)$ *and T be the column transformation matrix of the Smith normal form of B, i.e.*

$$B = \begin{pmatrix}
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0\\
1&0&0&1&1&0&0&1&1&0&0&1&0&1&0\\
1&0&1&0&0&1&0&1&0&1&0&1&1&0&0\\
1&0&0&1&0&1&1&0&0&1&1&0&0&1&0\\
0&1&0&1&0&1&0&1&1&0&1&0&1&0&0\\
0&1&1&0&0&1&1&0&1&0&0&1&0&1&0\\
0&1&0&1&1&0&1&0&0&1&0&1&1&0&0\\
0&1&1&0&1&0&0&1&0&1&1&0&0&1&0\\
1&1&0&0&1&1&0&0&1&1&0&0&0&0&1\\
1&1&0&0&0&0&1&1&0&0&1&1&0&0&1\\
0&0&1&1&0&0&1&1&1&1&0&0&0&0&1\\
0&0&1&1&1&1&0&0&0&0&1&1&0&0&1\\
1&1&1&1&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1
\end{pmatrix} \quad \textit{and} \quad T = \begin{pmatrix}
1&0&0&0&0&0&0&0&-1&0&0&-1&-3&-3&-13\\
0&1&0&0&0&0&-1&-1&-1&-1&-1&-3&-1&-1&-6\\
0&0&1&0&0&-1&0&-1&0&-1&-1&-3&-1&-2&-20\\
0&0&0&1&0&-1&-1&0&0&0&-1&-1&-3&-2&-27\\
0&0&0&1&1&-1&-1&-1&0&0&-1&-4&-5&-2&-41\\
0&0&0&0&0&1&0&0&0&0&0&0&-1&0&-6\\
0&0&0&0&0&0&1&0&0&0&0&0&-1&-1&-6\\
0&0&0&0&0&0&0&1&0&0&0&0&-1&-1&-13\\
0&0&0&0&1&0&0&0&1&0&0&-3&-4&-1&-34\\
0&0&0&0&0&0&0&0&1&1&0&-1&-2&-1&-13\\
0&0&0&0&0&0&0&0&0&1&1&-1&-2&-2&-13\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&-6\\
0&0&0&0&0&0&0&0&0&0&1&0&0&-1&-13\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&-6\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}.$$

Here six invariant factors of B equal 2, *three equal* 4, *the last equals* 28, *and the first five equal* 1. *Generator matrices for the $\mathbb{Z}_4$-code C of B and its dual code $C^{\perp}$ are e.g. given by*

$$\begin{pmatrix}
\mathit{1}&0&0&1&0&1&1&0&0&1&1&0&0&1&0\\
0&\mathit{1}&0&1&0&1&0&1&0&1&0&1&0&1&1\\
0&0&\mathit{1}&1&0&0&1&1&0&0&1&1&1&1&0\\
0&0&0&\mathit{2}&0&0&0&2&0&0&0&2&0&2&0\\
0&0&0&0&\mathit{1}&1&1&1&0&0&0&0&1&1&1\\
0&0&0&0&0&\mathit{2}&0&2&0&0&0&0&0&2&2\\
0&0&0&0&0&0&\mathit{2}&2&0&0&0&0&0&0&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\mathit{1}&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&\mathit{2}&0&2&0&2&2\\
0&0&0&0&0&0&0&0&0&0&\mathit{2}&2&0&0&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&\mathit{2}&2&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix} \quad \textit{and} \quad \begin{pmatrix}
\mathit{1}&0&0&1&0&1&1&0&1&0&0&1&1&0&1\\
0&\mathit{1}&0&1&0&1&0&1&0&1&0&1&0&1&1\\
0&0&\mathit{1}&1&0&0&1&1&0&0&1&1&1&1&0\\
0&0&0&\mathit{2}&0&0&0&2&0&0&0&2&0&2&0\\
0&0&0&0&\mathit{1}&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&0&\mathit{2}&0&2&0&0&0&0&0&2&2\\
0&0&0&0&0&0&\mathit{2}&2&0&0&0&0&2&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\mathit{2}&0&0&2&2&0&2\\
0&0&0&0&0&0&0&0&0&\mathit{2}&0&2&0&2&2\\
0&0&0&0&0&0&0&0&0&0&\mathit{2}&2&2&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

with invariant factors $[1,1,1,1,1,2,2,2,2,2,2,0,0,0,0]$ *and* $[1,1,1,1,2,2,2,2,2,2,0,0,0,0,0]$, *which we abbreviate as* $1^5 2^6 0^4$ *and* $1^4 2^6 0^5$. *The rows of the stated generator matrix for $C^{\perp}$ correspond to ten* mod 4*-conditions, where six can be rewritten to* mod 2*-conditions. We remark that the four rows with a* 1 *as leading coefficient are incidence vectors of affine solids and the six rows with a* 2 *as leading coefficient are twice the incidence vector of an affine plane.*

Definition 3.20. *Let $B \in R^{m \times n}$ be a non-empty matrix and C be the generated $\mathbb{Z}_{p^l}$-code, where p is a prime, l a positive integer, and $R = \mathbb{Z}$ or $R = \mathbb{Z}/p^l\mathbb{Z}$. Let $d_1, \ldots d_n$ be the invariant factors of B w.r.t. $\mathbb{Z}/p^l\mathbb{Z}$, $k_i := \#\left\{1 \leq j \leq n \,:\, d_j = p^i\right\}$ for $0 \leq i \leq l-1$, and $k_l := \#\left\{1 \leq j \leq n \,:\, d_j = 0\right\}$. Then $[k_0, \ldots, k_l]$ is the* type *(or, more precisely, the p^l-type) of C.*

With the help of the Smith normal form we directly see that for a $\mathbb{Z}_{p^l}$-code C with type $[k_0, \ldots, k_l]$ the type of the dual code $C^\perp$ is just the reversal $[k_l, \ldots, k_0]$, c.f. [48]. So, the notion of the type of a $\mathbb{Z}_{p^l}$-code generalizes the notion of the dimension of a $\mathbb{Z}_p$-code as an invariant. In our context we can e.g. use it to deduce that the incidence vectors of the affine 3-spaces in $\mathrm{PG}(3,2)$ span the $\mathbb{Z}_4$-kernel of the incidence matrix between planes and points in $\mathrm{PG}(3,2)$ by verifying that the $\mathbb{Z}_4$-code of the affine 3-spaces in $\mathrm{PG}(3,2)$ has type $[4,6,5]$.

	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$
$\begin{pmatrix}0 & 0 & 1\end{pmatrix}$	0	1	0	1	0	1	0
$\begin{pmatrix}0 & 1 & 0\end{pmatrix}$	1	0	0	1	1	0	0
$\begin{pmatrix}0 & 1 & 1\end{pmatrix}$	0	0	1	1	0	0	1
$\begin{pmatrix}1 & 0 & 0\end{pmatrix}$	1	1	1	0	0	0	0
$\begin{pmatrix}1 & 0 & 1\end{pmatrix}$	0	1	0	0	1	0	1
$\begin{pmatrix}1 & 1 & 0\end{pmatrix}$	1	0	0	0	0	1	1
$\begin{pmatrix}1 & 1 & 1\end{pmatrix}$	0	0	1	0	1	1	0

0	**1**	0	0	1	0	1
1	0	0	0	0	-1	3
0	0	**1**	0	1	1	0
0	0	0	0	0	0	**6**
0	0	0	**1**	-1	1	-1
0	0	0	0	**2**	0	-2
0	0	0	0	0	**2**	-2

0	**1**	0	0	1	0	1
1	0	0	0	0	1	1
0	0	**1**	0	1	1	0
0	0	0	0	0	0	0
0	0	0	**1**	1	1	1
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Table 3.1: The incidence matrix between points and lines in $\mathrm{PG}(2,2)$ and its kernel.

Example 3.21. *If we label the points by generating row vectors and the hyperplanes by orthogonal column vectors, then the incidence matrix A between points and hyperplanes in $\mathrm{PG}(2,2)$ is given on the left hand side of Table 3.1. We now apply the Gaussian elimination algorithm to the transposed matrix $A^\top$ without swapping rows or columns. In order to make results applicable for different domains, we perform all computations over $\mathbb{Z}$. More precisely, we only allow multiplications or divisions by the units $\{-1,1\}$ in $\mathbb{Z}$ and adding the λ-fold of a row to another row is only permitted for $\lambda \in \mathbb{Z}$. The result is displayed in the middle of Table 3.1. With this we can conclude that the $\mathbb{R}$-rank of $A^\top$ is 7 and the corresponding kernel of $A^\top$ has dimension zero. Reducing modulo 2 gives the result for the computations of $\mathbb{F}_2$, see the matrix on the right hand side of Table 3.1. I.e., the 2-rank of $A^\top$ is four and the corresponding kernel of $A^\top$ has dimension three.*

The associated $2^3 = 8$ *necessary conditions* $\mathbf{y}^\top \mathbf{z} = \mathbf{0}$ *are given by:*

$$\begin{array}{rcl}
\mathcal{M}(\langle\begin{pmatrix}0&0&1\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}0&1&0\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&0&0\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&1&1\end{pmatrix}\rangle) & \equiv & 0 \pmod 2\\
\mathcal{M}(\langle\begin{pmatrix}0&0&1\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}0&1&1\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&0&0\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&1&0\end{pmatrix}\rangle) & \equiv & 0 \pmod 2\\
\mathcal{M}(\langle\begin{pmatrix}0&1&0\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}0&1&1\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&1&0\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&1&1\end{pmatrix}\rangle) & \equiv & 0 \pmod 2\\
\mathcal{M}(\langle\begin{pmatrix}0&1&0\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}0&1&1\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&0&0\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&0&1\end{pmatrix}\rangle) & \equiv & 0 \pmod 2\\
\mathcal{M}(\langle\begin{pmatrix}0&0&1\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}0&1&1\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&0&1\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&1&1\end{pmatrix}\rangle) & \equiv & 0 \pmod 2\\
\mathcal{M}(\langle\begin{pmatrix}0&0&1\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}0&1&0\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&0&1\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&1&0\end{pmatrix}\rangle) & \equiv & 0 \pmod 2\\
\mathcal{M}(\langle\begin{pmatrix}1&0&0\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&0&1\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&1&0\end{pmatrix}\rangle)+\mathcal{M}(\langle\begin{pmatrix}1&1&1\end{pmatrix}\rangle) & \equiv & 0 \pmod 2\\
0 & \equiv & 0 \pmod 2
\end{array}$$

Note that the four points occurring in one of the first seven equations form an affine plane in each case, i.e., the complement is one of the $\begin{bmatrix}3\\2\end{bmatrix}_2 = 7$ *lines in* $\mathrm{PG}(2,2)$. *This is not a coincidence as we will see in the subsequent remark.*

Remark 3.22. *For a prime* p *the incidence matrix between the points and the* k*-spaces in* $\mathrm{PG}(v-1,p)$ *is the generator matrix of a projective generalized Reed–Muller code, see e.g. [3, Theorem 5.41]. A few simplified formulas for the Hamada* p*-rank formula for special cases and explicit bases can e.g. be found in [3, Section 5.9], see also the survey [50]. For a prime power* q *these so-called* geometric codes *admit a representation by polynomial functions, which is a rather natural description for generalized Reed–Muller codes, see [87] for the details.*[1]

For a direct entry observe that the intersection of each k*-space* K *with an arbitrary subspace* S *of codimension at most* $k-1$ *consists of* $[i]_q$ *points for some integer* $1 \le i \le k$ *and that these numbers all are congruent to* 1 *modulo* q. *Going over to complements we end up with numbers of points that are congruent to zero modulo* q. *If* $q = p^2$ *we can apply the same idea using Baer subspaces and in general we have to consider subfield subcodes, see e.g. [3, Section 5.8]. To compute the dimension of the resulting span and to compare it with the Hamada formula or one of its simplifications in special cases then is the, of course unavoidable, technical part if an exhaustive classification is desired.*

Example 3.23. *Consider the incidences between points and planes in* $\mathrm{PG}(3,2)$. *The incidence matrix* A *and the resulting matrix after applying the Gaussian elimination*

[1] A similar statement also applies to e.g. codes obtained from Hermitian varieties, see e.g. [138] or [12, Theorem 2.30].

algorithm of $\mathbb{Z}$, see Example 3.21 for the technical details, are given by

$$\begin{pmatrix}
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&0&0&1&1&0&0&1&1&0&0&1&1&0&0\\
0&0&1&1&0&0&1&1&0&0&1&1&0&0&1\\
1&1&1&0&0&0&0&1&1&1&1&0&0&0&0\\
0&1&0&0&1&0&1&1&0&1&0&0&1&0&1\\
1&0&0&0&0&1&1&1&1&0&0&0&0&1&1\\
0&0&1&0&1&1&0&1&0&0&1&0&1&1&0\\
1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
0&1&0&1&0&1&0&0&1&0&1&0&1&0&1\\
1&0&0&1&1&0&0&0&0&1&1&0&0&1&1\\
0&0&1&1&0&0&1&0&1&1&0&0&1&1&0\\
1&1&1&0&0&0&0&0&0&0&0&1&1&1&1\\
0&1&0&0&1&0&1&0&1&0&1&1&0&1&0\\
1&0&0&0&0&1&1&0&0&1&1&1&1&0&0\\
0&0&1&0&1&1&0&0&1&1&0&1&0&0&1
\end{pmatrix}$$

and

$$\begin{pmatrix}
0&\boldsymbol{1}&0&0&1&0&1&0&1&0&1&-1&2&-1&2\\
\boldsymbol{1}&0&0&0&0&1&1&0&0&1&1&-1&-1&2&2\\
0&0&\boldsymbol{1}&0&1&-1&0&0&-3&1&-2&1&-2&0&-3\\
0&0&0&0&0&0&0&0&0&0&16&0&0&0&12\\
0&0&0&\boldsymbol{1}&-1&1&-1&0&0&0&0&1&-1&1&-1\\
0&0&0&0&\boldsymbol{2}&0&0&0&0&0&-6&0&0&-2&-8\\
0&0&0&0&0&\boldsymbol{2}&0&0&0&0&-6&0&-2&0&-8\\
0&0&0&0&0&0&\boldsymbol{2}&0&0&0&-6&0&-2&-2&-6\\
0&0&0&0&0&0&0&\boldsymbol{1}&-1&1&-1&1&-1&1&-1\\
0&0&0&0&0&0&0&0&\boldsymbol{2}&0&-2&0&2&0&-2\\
0&0&0&0&0&0&0&0&0&\boldsymbol{2}&-2&0&0&2&-2\\
0&0&0&0&0&0&0&0&0&0&-12&\boldsymbol{2}&-2&-2&-14\\
0&0&0&0&0&0&0&0&0&0&12&0&0&\boldsymbol{4}&12\\
0&0&0&0&0&0&0&0&0&0&12&0&\boldsymbol{4}&0&12\\
0&0&0&0&0&0&0&0&0&0&12&0&0&0&16
\end{pmatrix}.$$

So, the 2-rank of A equals 5 and the corresponding kernel of $A^\top$ has dimension 10. The associated necessary conditions are sums over point multiplicities that are congruent to zero modulo 2. The conditions $\mathcal{M}(H) - \#\mathcal{M} \equiv 0 \pmod 4$ from Lemma 3.12 for each hyperplane H in $\mathrm{PG}(3,2)$ can be concluded from the kernel approach if we compute modulo 4. Here the rank of A equals 11 and the corresponding kernel of $A^\top$ has dimension 4. (As in Example 3.21 we again have the trivial constraint $0 \equiv 0 \pmod 4$, so that $2^4 = 1 + {4 \brack 3}_2$.)

Note that Exercise 3.11 can be used to deduce

$$\mathcal{M}(S) = \frac{1}{q^{v-s-1}} \cdot \left(\sum_{H \in \mathcal{H}\,:\, S \le H} \mathcal{M}(H) \;-\; [v-s-1]_q \cdot \#\mathcal{M} \right) \tag{3.6}$$

for a multiset of points $\mathcal{M}$ in $\mathrm{PG}(v-1,q)$, where S is an s-dimensional subspace with $1 \le s \le v-1$.

Exercise 3.24. *Show*

$$\chi_S = \frac{1}{q^{v-s-1}} \cdot \sum_{H\in\mathcal{H}\,:\,S\le H} \chi_H - \frac{[v-s-1]_q}{q^{v-s-1}} \cdot \chi_V \tag{3.7}$$

for an s-dimensional subspace S of $V = \operatorname{PG}(v-1,q)$ and deduce that χ_S is q^{s-1}-divisible from the q^{v-1}-divisibility of χ_V and the q^{v-2}-divisibility of χ_H for every $H \in \mathcal{H}$.

Since $\sum_{H\in\mathcal{H}} \mathcal{M}(H) = [v-1]_q \cdot \#\mathcal{M}$ the point multiplicities $\mathcal{M}(P)$ can be computed from the hyperplane multiplicities $\mathcal{M}(H)$ and vice versa. Interchanging the roles of the points $P \in \mathcal{P}$ and the hyperplanes $H \in \mathcal{H}$ yields the so-called *dual* multiset.

Exercise 3.25. *Show that each multiset of point $\mathcal{M}$ in $\operatorname{PG}(v-1,q)$ with $v \ge 2$ can be uniquely written as $\mathcal{M} = \sum_{H\in\mathcal{H}} \alpha_H \cdot \chi_H$ where*

$$\alpha_H = \frac{1}{q^{v-2}} \cdot \left(\mathcal{M}(H) - \frac{[v-2]_q}{[v-1]_q} \cdot \#\mathcal{M}\right) \in \mathbb{Q} \tag{3.8}$$

for every $H \in \mathcal{H}$.

Note that $\alpha_H \ge 0$ iff $[v-1]_q \cdot \mathcal{M}(H) \ge [v-2]_q \cdot \#\mathcal{M}$. If $\mathcal{M}$ is proper and $\alpha_H \ge 0$ for all $H \in \mathcal{H}$, then we can use Exercise 3.9 to deduce that there exists an integer x such that $\#\mathcal{M} = x[v-1]_q$ and $\min\{\mathcal{M}(H) : H \in \mathcal{H}\} = x[v-2]_q$, i.e. $\mathcal{M}$ is a $(x[v-1]_q, x[v-2]_q; v, q)$-minihyper, see Section 8.5. With this, Exercise 3.25 gives $q^{v-2} \cdot \alpha_H \in \mathbb{N}_0$ every hyperplane H, while we only have $q^{v-2}[v-1]_q \cdot \alpha_H \in \mathbb{Z}$ in general. See e.g. [165, Section 2] for more details.

4 Lengths of divisible codes

In this chapter we will consider the possible effective lengths of Δ-divisible linear codes over $\mathbb{F}_q$. Due to Theorem 3.2 it is sufficient to consider Δ-divisible codes where $\Delta = q^r$ with $r \in \mathbb{Q}$ such that $er \in \mathbb{N}$ for field sizes $q = p^e$. We will first consider the restricted case $r \in \mathbb{N}$, see [140], and then consider the general situation $r \in \mathbb{Q}$, see [153]. Since adding zero coordinates to codewords does not change the divisibility, see Exercise 2.3, we focus on the effective lengths and not the lengths of q^r-divisible linear codes over $\mathbb{F}_q$. We remark that we will mostly use the geometric language, i.e., consider the possible cardinalities of q^r-divisible multisets of points in $\mathrm{PG}(v-1,q)$.

There are a few very basic constructions for q^r-divisible multisets of points, see Example 2.19 and Exercise 2.27:

Lemma 4.1. *([140, Lemma 2])*

(i) Let U be a q-vector space of dimension $k \ge 1$. The set $\begin{bmatrix} U \\ 1 \end{bmatrix}$ of $[k]_q$ points contained in U is q^{k-1}-divisible.

(ii) For q^r-divisible multisets $\mathcal{M}$ and $\mathcal{M}'$ in V, the sum (or multiset union) $\mathcal{M} + \mathcal{M}'$ is q^r-divisible.

(iii) The q-fold repetition of a q^r-divisible multiset $\mathcal{M}$ is q^{r+1}-divisible.

Note that for a multiset of points $\mathcal{M}_1$ in V_1 and a multiset of points $\mathcal{M}_2$ in V_2 we can consider their embeddings $\mathcal{M}_1', \mathcal{M}_2'$ in $V_1 \times V_2$ and consider the sum $\mathcal{M}_1' + \mathcal{M}_2'$ in the ambient space $V_1 \times V_2$. By applying Lemma 4.1 we obtain:

Lemma 4.2. *The set of possible cardinalities of q^r-divisible multisets of points over $\mathbb{F}_q$ is closed under addition.*

For each integer r and each dimension $1 \le i \le r+1$ the q^{r+1-i}-fold repetition of an i-space in $\mathrm{PG}(v-1,q)$ is a q^r-divisible multiset of points of cardinality $q^{r+1-i} \cdot [i]_q$. So, for a fixed prime power q, a non-negative integer r, and $i \in \{0,\dots,r\}$, we define

$$s_q(r,i) := q^i \cdot [r-i+1]_q = \frac{q^{r+1}-q^i}{q-1} = \sum_{j=i}^{r} q^j = q^i + q^{i+1} + \ldots + q^r \tag{4.1}$$

and state:

Lemma 4.3. *For each $r \in \mathbb{N}_0$ and each $i \in \{0,\dots,r\}$ there is a q^r-divisible multiset of points of cardinality $s_q(r,i)$.*

As a consequence of Lemma 4.2 and Lemma 4.3 all integers $n = \sum_{i=0}^{r} a_i s_q(r,i)$ with $a_i \in \mathbb{N}_0$ are realizable cardinalities of q^r-divisible multisets of points. Later on we will prove in Theorem 4.6 that these integers are indeed the only possibilities. E.g. for $q = 2$ and $r = 2$ the possible cardinalities are given by $\{4, 6, 7, 8\} \cup \mathbb{N}_{\geq 10}$. The impossibility of cardinality 9 was shown in Example 3.10.

Frobenius coin problem

The *Frobenius coin problem* [37], named after the German mathematician Ferdinand Georg Frobenius (1849–1917), asks for the largest monetary amount $F(a_1, \dots, a_r)$ that cannot be obtained using only coins of specified denominations in $\{a_1, \dots, a_r\}$. If $\gcd(a_1, \dots, a_r) = 1$ the number $F(a_1, \dots, a_r)$ is always finite and we have $F(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$ in this case. For $r \geq 3$ no general formula is known.

In analogy to the Frobenius coin problem we define $\mathrm{F}_q(r)$ as the smallest integer such that a q^r-divisible multiset of cardinality n exists for all integers $n > \mathrm{F}_q(r)$ in $\mathrm{PG}(v-1, q)$ provided that the dimension v is sufficiently large. In other words, $\mathrm{F}_q(r)$ is the largest integer which is not realizable as the size of a q^r-divisible multiset of points over $\mathbb{F}_q$. If all non-negative integers are realizable then $\mathrm{F}_q(r) = -1$, which is the case for $r = 0$. We have $\mathrm{F}_2(2) = 9$ and will state a general formula for $\mathrm{F}_q(r)$ in Proposition 4.7. For the moment we just remark that for $r \geq 1$ the numbers $s_q(r,r) = q^r$ and $s_q(r,0) = 1 + q + q^2 + \ldots + q^r$ are coprime, so that $\mathrm{F}_q(r)$ is indeed finite and there is only a finite set of cardinalities which is not realizable as a q^r-divisible multiset for every choice of q and r. We remark that the classical Frobenius number is e.g. applied in [18] to the existence problem of vector space partitions.

Note that the number $s_q(r,i)$ is divisible by q^i, but not by q^{i+1}. This property allows us to create kind of a positional system upon the sequence of base numbers

$$S_q(r) := (s_q(r,0), s_q(r,1), \dots, s_q(r,r)).$$

Our next aim is to show that each integer n has a unique $S_q(r)$*-adic expansion*

$$n = \sum_{i=0}^{r} a_i s_q(r,i) \tag{4.2}$$

with $a_0, \dots, a_{r-1} \in \{0, \dots, q-1\}$ and *leading coefficient* $a_r \in \mathbb{Z}$. The idea is to consider Equation (4.2) modulo $q, q^2, \dots, q^r$ which gradually determines $a_0, a_1, \dots, a_{r-1} \in \{0, \dots, q-1\}$, using that $s_q(r,i)$ is divisible by q^i, but not by q^{i+1}. For the existence part, we give an algorithm that computes the $S_q(r)$-adic expansion:

Algorithm

Input: $n \in \mathbb{Z}$, field size q, exponent $r \in \mathbb{N}_0$

Output: representation $n = \sum\limits_{i=0}^{r} a_i s_q(r,i)$ with $a_0, \dots, a_{r-1} \in \{0, \dots, q-1\}$ and $a_r \in \mathbb{Z}$

$m \leftarrow n$

For $i \leftarrow 0$ To $r - 1$

  $a_i \leftarrow m \bmod q$

  $m \leftarrow \frac{m - a_i \cdot [r-i+1]_q}{q}$

$a_r \leftarrow m$

Here $m \bmod q$ denotes the remainder of the division of m by q.

Exercise 4.4. *Let $n \in \mathbb{Z}$ and $r \in \mathbb{N}_0$. Show that the above algorithm computes the unique $S_q(r)$-adic expansion of n.*

The $S_2(2)$-adic expansion of $n = 11$ is given by $11 = 1\cdot 7 + 0\cdot 6 + 1\cdot 4$ and the $S_2(2)$-adic expansion of $n = 9$ is given by $1\cdot 7 + 1\cdot 6 - 1\cdot 4$, i.e., the leading coefficient is -1.

Exercise 4.5. *Compute the $S_3(3)$-adic expansion of $n = 137$ and determine the leading coefficient.*

In Example 3.10 we have shown the non-existence of 4-divisible multisets of cardinality 9 over $\mathbb{F}_2$. Using the same tools, i.e., Lemma 3.4 and Lemma 3.8, we can show the following characterization on the lengths of q^r-divisible codes and multisets by induction:

Characterization of lengths of divisible codes

Theorem 4.6. *([140, Theorem 1]) For $n \in \mathbb{Z}$ and $r \in \mathbb{N}_0$ the following statements are equivalent:*

(i) There exists a q^r-divisible multiset of points of cardinality n over $\mathbb{F}_q$.

(ii) There exists a full-length q^r-divisible linear code of length n over $\mathbb{F}_q$.

(iii) The leading coefficient of the $S_q(r)$-adic expansion of n is non-negative.

So, the $S_q(r)$-adic expansion of n provides a certificate not only for the existence, but remarkably also for the non-existence of a q^r-divisible multiset of size n. As computed in Exercise 4.5, the leading coefficient of the $S_3(3)$-adic expansion of $n = 137$ is -2, so that there is no 27-divisible ternary linear code of effective length 137.

Theorem 4.6 allows us also to compute the Frobenius-coin-problem-like number $\mathrm{F}_q(r)$ as the largest integer n whose $S_q(r)$-adic expansion $n = \sum_{i=0}^{r-1} a_i s_q(r,i) + a_r q^r$ has leading coefficient $a_r < 0$. Clearly, this n is attained by choosing $a_0 = \ldots = a_{r-1} = q-1$ and $a_r = -1$.

Frobenius number for lengths of divisible codes

Proposition 4.7. *([140, Proposition 1]) For every prime power q and $r \in \mathbb{N}_0$ we have*

$$\mathrm{F}_q(r) = r \cdot q^{r+1} - [r+1]_q = rq^{r+1} - q^r - q^{r-1} - \ldots - 1.$$

Just for the ease of a direct usage, we spell out a few implications of Theorem 4.6 in the following.

Lemma 4.8. *Let n be the effective length of a non-trivial 2^1-divisible code over $\mathbb{F}_2$. Then, we have $n \geq 2$.*

Lemma 4.9. *Let n be the effective length of a non-trivial 2^2-divisible code over $\mathbb{F}_2$. Then, we have $n \in \{4, 6, 7, 8\}$ or $n \geq 10$.*

Lemma 4.10. *Let n be the effective length of a non-trivial 2^3-divisible code over $\mathbb{F}_2$. Then, we have $n \in \{8, 12, 14, 15, 16, 20, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32\}$ or $n \ge 34$.*

Lemma 4.11. *Let n be the effective length of a non-trivial 3^1-divisible code over $\mathbb{F}_3$. Then, we have $n \in \{3, 4\}$ or $n \ge 6$.*

Exercise 4.12. *Show that the effective length n of a non-trivial q^r-divisible code over $\mathbb{F}_q$ satisfies $n \ge q^r$ and describe the unique example where equality is attained.*

We remark that for the cases when the field size is a proper prime power $q = p^m$ Theorem 3.2 and Theorem 4.6 are not sufficient to determine the possible lengths of p^r-divisible codes of $\mathbb{F}_q$. Due to Theorem 3.2 it suffices to consider Δ-divisible codes over $\mathbb{F}_q$ where Δ is a power of p. More concretely, we will use the parameterization $\Delta = p^{am-b}$ where $a, b \in \mathbb{N}$ with $a \ge 1$ and $b \le m-1$. For non-negative integers a, b with $a \ge 1$, $b \le m-1$, and $i \in \{0, \dots, a\}$ we define

$$s_q(a, b, i) := [a+1]_q \tag{4.3}$$

if $i = 0$ and

$$s_q(a, b, i) := q^i \cdot [a-i+1]_q / p^b = p^{im-b} \cdot [a-i+1]_q = p^{m-b} \cdot \left(q^{i-1} + q^i + \cdots + q^{a-1}\right) \tag{4.4}$$

for $1 \le i \le a$. Note that for $i \ge 1$ the number $s_q(a, b, i)$ is divisible by p^{im-b} but not by p^{im-b+1}, where $im - b \ge 1$, and $s_q(a, b, 0)$ is coprime to p. This property allows us to create kind of a positional system upon the sequence of base numbers

$$S_q(a, b) := \left(s_q(a, b, 0), s_q(a, b, 1), \dots, s_q(a, b, a)\right).$$

As it can be easily shown, each integer n has a unique $S_q(a, b)$-*adic expansion*

$$n = \sum_{i=0}^{a} c_i \cdot s_q(a, b, i) \tag{4.5}$$

with $c_0 \in \left\{0, \dots, p^{m-b} - 1\right\}$, $c_1, \dots, c_a - 1 \in \{0 \dots, q-1\}$ and *leading coefficient* $c_a \in \mathbb{Z}$.

Theorem 4.13. *([153, Theorem 2]) Let $q = p^m$, $n \in \mathbb{Z}$, and $a, b \in \mathbb{N}$ with $a \ge 1$, $b \le m-1$. The following statements are equivalent:*

(i) There exists a p^{am-b}-divisible linear code of effective length n over $\mathbb{F}_q$.

(ii) The leading coefficient c_a of the $S_q(a, b)$-adic expansion of n is non-negative.

Example 4.14. *For $q = 4$ and $r = \frac{3}{2}$ the multisets of points of a 8-fold point, a 2-fold line, and a plane are 4^r-divisible of cardinalities 8, 10, and 21, respectively. The set of all positive integers that cannot be written as sums of 8s, 10s, and 21s is given by $E_1 \cup E_2$, where*

$$E_1 = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 23, 25, 27, 33, 35, 43\}$$

and

$$E_2 = \{2, 4, 6, 12, 14, 22\}.$$

Thus, $4^{3/2}$-divisible multisets of points of cardinality n over $\mathbb{F}_4$ exist for all $n \in \mathbb{N}_0 \backslash (E_1 \cup E_2)$.

The used constructions in Example 4.14 are rather straightforward generalizations of the situation of q^r-divisible multisets of points when r is an integer. More precisely, we consider i-spaces S_i with $1 \le i \le \lceil r \rceil + 1$ in order to construct the q^r-divisible multisets of points $q^{r-i+1} \cdot \chi_{S_i}$ having cardinality $q^{r-i+1} \cdot [i]_q$. In Theorem 4.13 it turns out that the possible cardinalities lengths of q^r-divisible multisets of points over $\mathbb{F}_q$ can always be attained by taking suitable unions of the basic constructions mentioned before.

Of course, similar questions also make sense for codes over rings instead over finite fields $\mathbb{F}_q$.

4.1 Applications

Now we are ready to treat the example from Subsection 1.1 from a more general point of view. First we need a notion of a complementary multiset of points.

Definition 4.15. *Let $\mathcal{M}$ be a multiset of points in $\operatorname{PG}(v-1, q)$ with maximum point multiplicity at most λ, i.e., $\mathcal{M}(P) \le \lambda$ for all points $P \in \mathcal{P}$. The* λ-complement $\mathcal{M}^{\mathsf{C}_\lambda}$ *of $\mathcal{M}$ is the multiset of points in $\operatorname{PG}(v-1, q)$ defined by $\mathcal{M}^{\mathsf{C}_\lambda}(P) = \lambda - \mathcal{M}(P)$ for all $P \in \mathcal{P}$.*

If $\mathcal{M}$ is the multiset of points in $\operatorname{PG}(9-1, 2)$ corresponding to the points of 20 solids and 30 planes with pairwise trivial intersection, then the maximum point multiplicity of $\mathcal{M}$ is 1. Here we have $\#\mathcal{M} = 510$ and the 1-complement $\mathcal{M}^{\mathsf{C}_1}$ has cardinality 1 and also a maximum point multiplicity of 1.

For a given ambient space $\operatorname{PG}(v-1, q)$ and a positive integer λ let $\mathcal{V}$ be the multiset $\lambda \cdot \mathcal{P}$ defined by $\mathcal{M}(P) = \lambda$ for all $P \in \mathcal{P}$. Since $\mathcal{V}$ is λq^{v-1}-divisible, the equation $\mathcal{M} + \mathcal{M}^{\mathsf{C}_\lambda} = \mathcal{V}$ implies:

Lemma 4.16. *Let $\lambda \in \mathbb{N}_0$ and $\mathcal{M}$ a multiset of points in $\operatorname{PG}(v-1, q)$ of maximum point multiplicity at most λ, $q = p^m$, and e the largest integer such that p^e divides λ. If $r \in \mathbb{Q}_{\ge 0}$ with $mr \in \mathbb{N}$ and $0 \le r \le \frac{e}{m} \cdot (v-1)$ exists, then, $\mathcal{M}$ is q^r-divisible iff its λ-complement $\mathcal{M}^{\mathsf{C}_\lambda}$ is.*

In the above example we have $v = 9$ and $\lambda = 1$, so that $\mathcal{M}^{\mathsf{C}_1}$ is 4-divisible since $\mathcal{M}$ is 4-divisible due to Lemma 3.12. Since there is no 4-divisible multiset of points of cardinality 1 over $\mathbb{F}_2$, no configuration of 20 solids and 30 planes with pairwise trivial intersection can exist in $\operatorname{PG}(9-1, 2)$.

Exercise 4.17. *Determine the maximum integer f such that there exists a non-empty p^f-divisible multiset of points in $\operatorname{PG}(v-1, q)$ with maximum point multiplicity λ, where $q = p^m$ and v, p, m, λ are arbitrary but fixed.*

For the case of multisets of subspaces of the same dimension we can state rather explicit results using sharpened rounding operators.

Sharpened rounding

Definition 4.18. *For $a \in \mathbb{Z}$ and $b \in \mathbb{Z}\setminus\{0\}$ let $\lfloor\!\lfloor a/b \rfloor\!\rfloor_{q^r}$ be the maximal $n \in \mathbb{Z}$ such that there exists a q^r-divisible $\mathbb{F}_q$-linear code of effective length $a - nb$. If no such code exists for any n, we set $\lfloor\!\lfloor a/b \rfloor\!\rfloor_{q^r} = -\infty$. Similarly, let $\lceil\!\lceil a/b \rceil\!\rceil_{q^r}$ denote the minimal $n \in \mathbb{Z}$ such that there exists a q^r-divisible $\mathbb{F}_q$-linear code of effective length $nb - a$. If no such code exists for any n, we set $\lceil\!\lceil a/b \rceil\!\rceil_{q^r} = \infty$.*

Note that the symbols $\lfloor\!\lfloor a/b \rfloor\!\rfloor_{q^r}$ and $\lceil\!\lceil a/b \rceil\!\rceil_{q^r}$ encode the four values a, b, q and r. Thus, the fraction a/b is a formal fraction and the power q^r is a formal power, i.e. we assume $1530/14 \neq 765/7$ and $2^2 \neq 4^1$ in this context.

Exercise 4.19. *Compute $\lfloor\!\lfloor 765/7 \rfloor\!\rfloor_{2^2}$ and $\lfloor\!\lfloor 1530/14 \rfloor\!\rfloor_{4^1}$. Verify*

$$\lfloor\!\lfloor 0/b \rfloor\!\rfloor_{q^r} = \lceil\!\lceil 0/b \rceil\!\rceil_{q^r} = 0$$

and

$$\begin{aligned} \ldots \le \lfloor\!\lfloor a/b \rfloor\!\rfloor_{q^2} \le \lfloor\!\lfloor a/b \rfloor\!\rfloor_{q^1} \le \lfloor\!\lfloor a/b \rfloor\!\rfloor_{q^0} = \left\lfloor \tfrac{a}{b} \right\rfloor \\ \le a/b \le \lceil a/b \rceil = \lceil\!\lceil a/b \rceil\!\rceil_{q^0} \le \lceil\!\lceil a/b \rceil\!\rceil_{q^1} \le \lceil\!\lceil a/b \rceil\!\rceil_{q^2} \le \ldots \end{aligned}$$

Exercise 4.20. *Develop an algorithm for the computation of $\lfloor\!\lfloor a/b \rfloor\!\rfloor_{q^r}$ and $\lceil\!\lceil a/b \rceil\!\rceil_{q^r}$. Minimize its necessary complexity.*

Having the notion of the sharpened rounding of Definition 4.18 at hand, we can state:

Lemma 4.21. *Let $k \in \mathbb{Z}_{\ge 1}$ and $\mathcal{U}$ be a multiset of k-spaces in $\operatorname{PG}(v-1,q)$.*

(i) If every point in $\mathcal{P}$ is covered by at most λ elements of $\mathcal{U}$, then

$$\#\mathcal{U} \le \lfloor\!\lfloor \lambda[v]_q/[k]_q \rfloor\!\rfloor_{q^{k-1}}.$$

(ii) If every point in $\mathcal{P}$ is covered by at least λ elements in $\mathcal{U}$, then

$$\#\mathcal{U} \ge \lceil\!\lceil \lambda[v]_q/[k]_q \rceil\!\rceil_{q^{k-1}}.$$

Exercise 4.22. *Prove Lemma 4.21 using the multisets of points $\mathcal{M}^{\complement_\lambda}$ and $\mathcal{M}' = \mathcal{M} - \lambda \cdot \operatorname{PG}(v-1,q)$, i.e., $\mathcal{M}'(P) = \mathcal{M}(P) - \lambda$ for all $P \in \mathcal{P}$.*

Example 4.23. *What is the maximum number of planes in $\operatorname{PG}(7,2)$ such that every point is covered at most three times? Counting points gives*

$$\left\lfloor \frac{3 \cdot [8]_2}{[3]_2} \right\rfloor = \left\lfloor 109 + \tfrac{2}{7} \right\rfloor = 109$$

as an upper bound, while Lemma 4.21 gives the upper bound

$$\left\lfloor\!\!\left\lfloor \frac{3 \cdot [8]_2}{[3]_2} \right\rfloor\!\!\right\rfloor_{2^2} = 107,$$

since no 2^2-divisible code of length 9 exists over $\mathbb{F}_2$. This bound is indeed tight, see e.g. [75, 76] where also more general packings of k-spaces are studied.

In some cases the sharpened rounding can even be computed when the input data is parametric. **Asymptotic maximum size of λ-fold partial spreads**

Example 4.24. *Let $v = tk + r$ with $r \in \{1, \dots, k-1\}$ and $\mathcal{U}$ be a multiset of k-spaces in $\mathrm{PG}(v-1, q)$ such that every point is covered at most $\lambda \in \mathbb{N}$ times. We will show*

$$\#\mathcal{U} \le \lambda \cdot \left(1 + \sum_{i=1}^{t-1} q^{ik+r}\right) = \lambda \cdot \left(\frac{q^v - q^{k+r}}{q^k - 1} + 1\right) < \lambda \frac{[v]_q}{[k]_q} \tag{4.6}$$

for $k > \lambda [r]_q$.

First we deduce

$$\begin{aligned} \lambda\left(1+\sum_{i=1}^{t-1} q^{ik+r}\right) &= \lambda q^{k+r} \cdot \frac{q^{k(t-1)}-1}{q^k-1} + \lambda = \lambda \cdot \frac{q^v - q^{k+r} + q^k - 1}{q^k - 1} \\ &= \lambda \frac{[v]_q - [k+r]_q + [k]_q}{[k]_q} < \lambda \frac{[v]_q}{[k]_q}, \end{aligned}$$

from the geometric series, so that we assume

$$\#\mathcal{U} = \lambda \cdot \left(1 + \sum_{i=1}^{t-1} q^{ik+r}\right) + 1 = \lambda \cdot \frac{[v]_q - [k+r]_q + [k]_q}{[k]_q} + 1$$

for a moment. From

$$\begin{aligned} (q-1)\sum_{i=0}^{k-2} s_q(k-1, i) &= (q-1)\sum_{i=0}^{k-2} q^i \cdot [k-i]_q = (q-1)\sum_{i=0}^{k-2} \frac{q^k - q^i}{q-1} \\ &= (k-1)q^k - [k-1]_q = kq^k - [k]_q \end{aligned}$$

we conclude the $S_q(k-1)$-adic expansion

$$\#\mathcal{M} = (\lambda[r]_q - k)\, s_q(k-1, k-1) + \sum_{i=0}^{k-2} (q-1) \cdot s_q(k-1, i)$$

of $\#\mathcal{M}$. Since $\mathcal{M}$ is q^{k-1}-divisible by Lemma 3.12 and Lemma 4.16, Theorem 4.6 yields that the leading coefficient $\lambda[r]_q - k$ is non-negative, which contradicts $k > \lambda[r]_q$.

We remark that one can easily give a matching construction, i.e., the stated upper bound in Inequality (4.6) is tight. The special case $\lambda = 1$ is the main theorem of [181]. While the proof is a bit technical, we have actually just applied Lemma 4.21 and evaluated the sharpened rounding analytically (for special parameters).

Exercise 4.25. *Let $\mathcal{U}$ be a multiset of k-spaces in $\mathrm{PG}(v-1, q)$ that covers each point at least once. Show*

$$\#\mathcal{U} \ge \left\lceil \frac{[v]_q}{[k]_q} \right\rceil$$

and determine for the case of equality the geometric structure of the (multi-)set of points that are covered more than once.

Exercise 4.26. *Let $\mathcal{U}$ be a multiset of* 4*-spaces in* PG(6, 2) *that cover every* 2*-space at least once. Show* $\#\mathcal{U} > 77$*.*
Hint: First show that a 2^3*-divisible multiset of points* $\mathcal{M}$ *of cardinality* 12 *over* $\mathbb{F}_2$ *is a* 4*-fold line, i.e.,* $\mathcal{M} = 4 \cdot \chi_L$ *for some line* L*.*

We remark that the best known published lower bound for the number of solids in PG(6, 2) that cover every line at least once is 77 and an example of 93 such solids is known, see [74]. Without proof we state that the lower bound can be improved to 86 and the upper bound to 91.

Research problem

Apply similar techniques to improve further lower bounds from [74].

5 Constructions for projective q^r-divisible codes or multisets of points with bounded maximum point multiplicity

In Section 4 we have completely characterized the possible cardinalities of q^r-divisible multisets of points over $\mathbb{F}_q$, where r is an arbitrary positive integer. As a refinement we now consider q^r-divisible multisets of points over $\mathbb{F}_q$ whose maximum point multiplicity is upper bounded by some positive integer λ, see e.g. [144]. In the extreme case $\lambda = 1$ the corresponding linear codes are projective. A first observation is that we can combine a q^r-divisible multiset $\mathcal{M}_1$ in an $\mathbb{F}_q$-vector space V_1 and another q^r-divisible multiset $\mathcal{M}_2$ in an $\mathbb{F}_q$-vector space V_2 to a q^r-divisible multiset $\mathcal{M}$ in $V_1 \times V_2$ by considering V_1 and V_2 as subspaces of $V_1 \times V_2$. Here we have $\#\mathcal{M} = \#\mathcal{M}_1 + \#\mathcal{M}_2$ and $\gamma_0(\mathcal{M}) = \max\{\gamma_0(\mathcal{M}_1), \gamma_0(\mathcal{M}_2)\}$, so that:

Lemma 5.1. *The set of possible cardinalities of q^r-divisible multisets of points over $\mathbb{F}_q$ with maximum point multiplicity at most λ is closed under addition.*

Let us start to consider constructions for multisets of points with maximum point multiplicity 1, i.e., sets of points.

Combinations of Simplex and first order Reed-Muller codes

In Example 2.19 and Example 2.25 we have seen the first two basic constructions of q^r-divisible sets.

Lemma 5.2. *Let u be an arbitrary positive integer and U be an arbitrary u-space in $\operatorname{PG}(v-1, q)$, where $v \geq u$. Then χ_U is a q^{u-1}-divisible set of cardinality $[u]_q$ and dimension u. If $u \geq 2$ and H is a hyperplane of U, i.e., a $(u-1)$-space that is contained in U, then $\chi_U - \chi_H$ is a q^{u-2}-divisible set of cardinality q^{u-1} and dimension u.*

The small cardinalities of q^r-divisible sets over $\mathbb{F}_q$ that cannot be attained by combinations of $(r+1)$-spaces and affine $(r+2)$-spaces can be determined easily:

Exercise 5.3. *Let $1 \leq n \leq rq^{r+1}$ such that no $u, v \in \mathbb{N}_0$ with $u \cdot [r+1]_q + v \cdot q^{r+1} = n$ exist. Then, there exist $a, b \in N_0$ with $a \leq r-1$, $b \leq q-2$, and*

$$(a(q-1)+b)[r+1]_q + a + 1 \leq n \leq (a(q-1)+b+1)[r+1]_q - 1.$$

Moreover, there are no $u, v \in \mathbb{N}_0$ with $u \cdot [r+1]_q + v \cdot q^{r+1} = rq^{r+1} + 1$.

Up to the bound rq^{r+1} the attainable cardinalities of q^r-divisible sets of points over $\mathbb{F}_q$ using the first two basic constructions only are given by

- $\{3, 4\}$ for $q = 2$ and $r = 1$;
- $\{7, 8, 14, 15, 16\}$ for $q = 2$ and $r = 2$;
- $\{15, 16, 30, 31, 32, 45, 46, 47, 48\}$ for $q = 2$ and $r = 3$;
- $\{4, 8, 9\}$ for $q = 3$ and $r = 1$;
- $\{13, 26, 27, 39, 40, 52, 53, 54\}$ for $q = 3$ and $r = 2$;
- $\{5, 10, 15, 16\}$ for $q = 4$ and $r = 1$;
- $\{21, 42, 63, 64, 84, 85, 105, 106, 126, 127, 128\}$ for $q = 4$ and $r = 2$.

Example 5.4. *An* ovoid *in* $\mathrm{PG}(3, q)$ *is a set* $\mathcal{M}$ *of* q^2+1 *points, no three collinear, such that every hyperplane contains* 1 *or* $q+1$ *points, i.e.,* $\mathcal{M}$ *is* q*-divisible. Ovoids exist for all* $q > 2$*, see e.g. [184].*

For the binary field a 2-divisible set of cardinality $2^2 + 1$ is contained in a different parametric family.

Definition 5.5. *A set of* $k+1$ *points in* $\mathrm{PG}(k-1, q)$*, where* $k \geq 2$*, such that any subset of* k *points span the full space is called a* k-dimensional projective base.

Exercise 5.6. *Show that the binary* k*-dimensional projective base is* 2*-divisible and has cardinality* $k+1$ *if* $k \geq 2$*. Moreover, show that a representation is given by the points* $\langle \mathbf{e}_1 \rangle, \ldots \langle \mathbf{e}_k \rangle$ *and* $\langle \mathbf{e}_1 + \cdots + \mathbf{e}_k \rangle$*, where* $\mathbf{e}_1, \ldots, \mathbf{e}_k$ *denote the unit vectors in* $\mathbb{F}_q^k$*.*

A cone construction

Definition 5.7. *Let* X*,* Y *be complementary subspaces of* $\mathrm{PG}(v-1, q)$ *and* $\mathcal{B}$ *be a set of points in* $\mathrm{PG}(Y)$*. The* cone with vertex X and base $\mathcal{B}$ *is the multiset of points* $\mathcal{M}$ *given by* $\mathcal{M} = \sum_{B \in \mathcal{B}} \chi_{\langle B, X \rangle}$*.*

If $\dim(X) = s$, then the set of points of $\langle P, X \rangle$ is q^s divisible for every point P. If $\mathcal{B}$ is q^r-divisible then we can easily check that the cone $\mathcal{M}$ with vertex X and base $\mathcal{B}$ is q^{r+s}-divisible and all points outside of X have multiplicity at most 1 while the points in X have multiplicity $\#\mathcal{B}$. Clearly we can subtract $(\#\mathcal{B} - 1) \cdot \chi_X$ or $\#\mathcal{B} \cdot \chi_X$ from $\mathcal{M}$ in order to obtain a set of points.

Exercise 5.8. *Let* X*,* Y *be complementary subspaces of* $\mathrm{PG}(v-1, q)$*,* $s = \dim(X)$*, and* $\mathcal{B}$ *be a* q^r*-divisible set of points in* $\mathrm{PG}(Y)$*. Show that*

$$\sum_{B \in \mathcal{B}} \chi_{\langle B, X \rangle \setminus X} \tag{5.1}$$

is q^{r+s}*-divisible of cardinality* $\#\mathcal{B} \cdot q^s$ *if* $\#\mathcal{B} \equiv 0 \pmod{q^{r+1}}$ *and*

$$\sum_{B \in \mathcal{B}} \chi_{\langle B, X \rangle \setminus X} + \chi_X \tag{5.2}$$

is q^{r+s}*-divisible of cardinality* $\#\mathcal{B} \cdot q^s + [s]_q$ *if* $\#\mathcal{B}(q-1) \equiv -1 \pmod{q^{r+1}}$*.*

Example 5.9. *For a 6-dimensional projective base $\mathcal{B}$ over $\mathbb{F}_2$ and an s-space X, where $s \ge 1$, (5.2) yields a 2^{s+1}-divisible set of $2^{s+3} - 1$ points over $\mathbb{F}_2$. Similarly, for a 7-dimensional projective base $\mathcal{B}$ over $\mathbb{F}_2$ and an s-space X, where $s \ge 1$, (5.1) yields a 2^{s+1}-divisible set of 2^{s+3} points over $\mathbb{F}_2$.*

Parity check bits

We remark that adding a so-called parity (check) bit to the codewords of a binary linear code yields a 2-divisible linear code whose length is increased by one. Since a binary 4-dimensional projective base gives a 2-divisible set of cardinality 5 over $\mathbb{F}_2$, there are 2-divisible sets of points of cardinality n over $\mathbb{F}_2$ for all $n \ge 3$.

In a certain sense we can generalize the idea of parity check bits to construct binary codes with higher divisibility. To this end, assume that we are given a 2^r-divisible $[n, k]_2$-code C that contains a 2^{r+1}-divisible $[n', k-1]_2$-code C'. Geometrically, C corresponds to a 2^r-divisible multiset of points $\mathcal{M}$ in $\mathrm{PG}(k-1, 2)$ and C' corresponds to a 2^{r+1}-divisible multiset of points $\mathcal{M}'$ in $\mathrm{PG}(k-2, 2)$. Moreover, there exists a point P in $\mathrm{PG}(k-1, 2)$ such that $\mathcal{M}'$ arises from $\mathcal{M}$ by projection trough P. Especially, we have $\mathcal{M}(P) = n - n'$ and for every hyperplane H of $\mathrm{PG}(k-1)$ we have $\mathcal{M}(H) \equiv \#\mathcal{M} \pmod{2^{r+1}}$ if $P \le H$ and $\mathcal{M}(H) \equiv \#\mathcal{M} + 2^r \pmod{2^{r+1}}$ otherwise. Defining the multiset of points $\widetilde{\mathcal{M}}$ in $\mathrm{PG}(k-1, 2)$ by $\widetilde{\mathcal{M}}(P) = \mathcal{M}(P) + 2^r$ and $\widetilde{\mathcal{M}}(Q) = \mathcal{M}(Q)$ for all other points $Q \ne P$, we obtain a 2^{r+1}-divisible multiset of points in $\mathrm{PG}(k-1, q)$. For the special case $r = 0$ we note that the set of codewords of even weights of an arbitrary $[n, k]_2$-code C forms a subcode of dimension $k-1$ or k. So, if C is not even itself, then the above geometric construction corresponds to adding a parity check bit.

Example 5.10. *Consider the matrix G consisting of the 16 column vectors in $\mathbb{F}_2^6$ that have Hamming weight 2 or 6 and let $\mathcal{M}$ be the corresponding set of 16 points in $\mathrm{PG}(5, 2)$.*[1] *By N we denote the unique point with Hamming weight 6. Let us describe the hyperplanes by the set of points being perpendicular to a vector $\mathbf{v} \in \mathbb{F}_2^6 \backslash \mathbf{0}$, i.e., we write $H(\mathbf{v})$. We can easily check that $\mathcal{M}(H(\mathbf{v})) = 10$ if $\mathrm{wt}(\mathbf{v}) \in \{1, 5\}$, $\mathcal{M}(H(\mathbf{v})) = 5$ if $\mathrm{wt}(\mathbf{v}) = 3$, $\mathcal{M}(H(\mathbf{v})) = 8$ if $\mathrm{wt}(\mathbf{v}) \in \{2, 4\}$, and $\mathcal{M}(H(\mathbf{v})) = 16$ if $\mathrm{wt}(\mathbf{v}) = 6$. So, G spans a $[16, 5]_2$-code C with non-zero weights in $\{6, 8, 10\}$. The codewords of weight 8 correspond to the hyperplanes containing N (and not being equal to $H(N)$). Increasing the multiplicity of N by 2 yields a 2^2-divisible multiset of cardinality 18 in $\mathrm{PG}(5, 2)$, with dimension 5 and N is the unique point with multiplicity larger than 1.*

By a little trick we can turn the above example into a 2^2-divisible set of 21 points in $\mathrm{PG}(6, 2)$. Instead of increasing the multiplicity of N by 2, we decrease it by 2, so that it becomes -1. Adding a suitable plane π containing N gives the desired multiset $\mathcal{M} + \chi_\pi - 2\chi_N$. Since $\dim(\mathcal{M}) = 5$ and $\dim(\chi_\pi) = 3$ an embedding in $\mathrm{PG}(6, 2)$ as a set of points is possible.

A switching construction

[1] This is a specific embedding of the complement of the parabolic quadric $Q(4, 2)$, see e.g. [123], in $\mathrm{PG}(5, 2)$. The subsequent point N is the nucleus of the quadric, i.e., every line trough N contains exactly one point of the quadric.

Lemma 5.11. *Let $\mathcal{M}$ be a q^r-divisible set of points in $\mathrm{PG}(v-1,q)$, where $k \in \mathbb{N}$, such that there exists an r-space S with $\mathcal{M}(P)=1$ for all points P in S, i.e., S is contained in the support $\operatorname{supp}(\mathcal{M})$ of $\mathcal{M}$. Then, there exists a q^r-divisible set of points $\mathcal{M}'$ with cardinality $\#\mathcal{M}' = \#\mathcal{M} + q^{r+1} - [r+1]_q$.*

Proof. Let $\widetilde{\mathcal{M}}$ be the embedding of $\mathcal{M}$ in $\mathrm{PG}(v'-1,q)$ for sufficiently large $v' \ge v$ (chosen later on) and $T_1, \cdots, T_{q-1}$ be $(r+1)$-spaces containing S. With this the multiset of points

$$\mathcal{M}' := \widetilde{\mathcal{M}} + \sum_{i=1}^{q-1} \chi_{T_i} - q \cdot \chi_S$$

is q^r-divisible and has cardinality

$$\#\mathcal{M}' = \#\mathcal{M} + (q-1)[r+1]_q - q[r]_q = \#\mathcal{M} + q^{r+1} - [r+1]_q.$$

If v' is sufficiently large then the T_i can clearly be chosen in such a way such that their pairwise intersection as well as their intersection with $\operatorname{supp}(\mathcal{M})$ equals S, so that $\gamma_0(\mathcal{M}') = 1$. □

The construction is called *switching construction* since an r-space is switched for $q-1$ affine $(r+1)$-spaces.

Starting from an $(r+1)$-space over $\mathbb{F}_q$ we can construct several non-isomorphic q^r-divisible sets of points over $\mathbb{F}_q$ with cardinality q^{r+1} if $q > 2$.

Exercise 5.12. *Show that for each $r \ge 1$ and each $r+2 \le k \le r+q$ there exists a q^r-divisible set of points over $\mathbb{F}_q$ with cardinality q^{r+1} and dimension k.*

The switching construction from Lemma 5.11 can be used to construct projective 2^r-divisible codes for an entire interval of effective lengths:

Corollary 5.13. *For each integer $r \ge 1$ and each $2^{2r} - 1 \le n \le 2^{2r} + 2^r$ there exists a 2^r-divisible set of points over $\mathbb{F}_2$ with cardinality n.*

Proof. Let $\mathcal{S}$ be an r-spread, i.e. a partition of $\mathrm{PG}(2r-1,2)$ into 2^r+1 pairwise disjoint r-spaces. The corresponding set of points is 2^{2r-1}-divisible, where $2r-1 \ge r$. For $0 \le j \le 2^r + 1$ of these r-spaces we can apply the switching construction from Lemma 5.11. □

For $r = 2$ we obtain 4-divisible sets of points over $\mathbb{F}_2$ with cardinalities between 15 and 20. Together with the examples for cardinalities 7, 8, and 14, we obtain examples for all cardinalities $n \ge 14$. Using the same construction for general field sizes q we obtain a sequence of possible cardinalities:

Corollary 5.14. *For each integer $r \ge 1$ and each $0 \le j \le q^r+1$ there exists a q^r-divisible set of points over $\mathbb{F}_q$ with cardinality $n = \begin{bmatrix} 2r \\ 1 \end{bmatrix}_q + j \cdot \left(q^{r+1} - [r+1]_q\right)$.*

For $q = 2$ we can use the switching construction to construct a 2^r-divisible set of points over $\mathbb{F}_2$ of cardinality $r \cdot 2^{r+1} + 1$ for all $r \ge 2$.

Exercise 5.15. *For an integer $r \ge 1$ consider the multisets of points $\mathcal{M}_i$ over $\mathbb{F}_2$ consisting of the 2^{r+1} points generated by the 2^{r+1} binary vectors in $e_{i+r+1} + \langle e_i, e_{i+1}, \ldots, e_{i+r}\rangle$ for all $1 \le i \le r$. Show that the $\mathcal{M}_i$ are affine $(r+2)$-spaces and that $\mathcal{M} = \sum_{i=1}^{r} \mathcal{M}_i$ is a 2^r-divisible set of points over $\mathbb{F}_2$ of cardinality $r \cdot 2^{r+1}$ and dimension $2r+1$ whose support contains the r-space $S = \langle e_{r+2}, \ldots, e_{2r+1}\rangle$. Use the switching construction to obtain a 2^r-divisible set of $r \cdot 2^{r+1} + 1$ points over $\mathbb{F}_2$.*

Using $r = 3$, $r = 4$, and $r = 5$ in the construction of Exercise 5.15 we obtain an 8-divisible set of 49, a 16-divisible set of 129, and a 32-divisible set of 321 points over $\mathbb{F}_2$.

It is also possible to extend the switching construction to a more general setting. To this end let Δ and Δ' be two integers such that $\rho := \frac{\Delta}{\Delta'} \in \mathbb{N}$. For a given prime power q let $\mathcal{M}$ be a Δ-divisible set of points over $\mathbb{F}_q$, $\mathcal{D}$ be a Δ'-divisible set of points over $\mathbb{F}_q$, $D = \langle \mathcal{D} \rangle$ be the subspace spanned by $\mathcal{D}$, and let $\mathcal{M}_1, \ldots, \mathcal{M}_{\rho-1}$ be Δ-divisible sets of points over $\mathbb{F}_q$ such that $\mathcal{M}_i(P) \ge \mathcal{D}(P)$ for all $1 \le i \le \rho - 1$ and $\mathcal{M}(P) \ge \mathcal{D}(P)$, where P ranges over all points in D. In other words, the set of points $\mathcal{M}_i$ and $\mathcal{M}$ all contain the set of points $\mathcal{D}$ as a subset. With this, the multiset of points given by

$$\mathcal{M} + \sum_{i=1}^{\rho-1} \mathcal{M}_i - \rho \cdot \mathcal{D} \tag{5.3}$$

is Δ-divisible over $\mathbb{F}_q$. If $\mathcal{M}|_D = \mathcal{D}|_D$ and $\mathcal{M}_i|_D = \mathcal{D}|_D$ for all $1 \le i \le \rho - 1$, then $\mathcal{M}$ and the $\mathcal{M}_i$ can clearly be embedded in suitable subspaces such that their pairwise intersection is given by the points in $\operatorname{supp}(\mathcal{D})$, so that the multiset of points given by Equation (5.3) is indeed a set of points.

Exercise 5.16. *Let $\Delta, \Delta' \in \mathbb{N}$ such that $\rho := \frac{\Delta}{\Delta'} \in \mathbb{N}$, $\mathcal{M}$ be a Δ-divisible set of points over $\mathbb{F}_q$, $\mathcal{D}$ be a Δ'-divisible set of points over $\mathbb{F}_q$, and let $\mathcal{M}_1, \ldots, \mathcal{M}_{\rho-1}$ be Δ-divisible sets of points over $\mathbb{F}_q$. Further assume that $\mathcal{M}|_D = \mathcal{D}|_D$ and $\mathcal{M}_i|_D = \mathcal{D}|_D$ for all $1 \le i \le \rho - 1$, where $D := \langle \mathcal{D} \rangle$. Show that there exists a Δ-divisible set of points over $\mathbb{F}_q$ of cardinality $\#\mathcal{M} + \sum_{i=1}^{\rho-1} \#\mathcal{M}_i - \rho \cdot \#\mathcal{D}$ and dimension $\dim(\mathcal{M}) + \sum_{i=1}^{\rho-1} \dim(\mathcal{M}_i) - \rho \cdot \dim(\mathcal{D}) \le k \le \dim(\mathcal{M}) + \sum_{i=1}^{\rho-1} \dim(\mathcal{M}_i) - (\rho - 1) \cdot \dim(\mathcal{D})$.*

For an application with a specific choice of the $\mathcal{M}_i$ and $\mathcal{D}$ we refer to Exercise 5.21.

5.1 Constructions using subfields

Considering $\mathbb{F}_{q^l}$ as an extension field of $\mathbb{F}_q$ we can assume $\mathbb{F}_q \subset \mathbb{F}_{q^l}$ for each integer $l \ge 2$. So, the field $\mathbb{F}_{q^l}$ can be also seen as an l-dimensional vector space over $\mathbb{F}_q$. If $\mathbf{v}$ is a k-dimensional vector over $\mathbb{F}_4 \simeq \mathbb{F}_2[x]/(x^2+x+1)\mathbb{F}_2[x]$, we can represent each entry $v_i \in F_4$ by $a_1 x + a_0$ with $a_0, a_1 \in \mathbb{F}_2$ and replace it by the vector $(a_0, a_1)^\top \in \mathbb{F}_2^2$. This yields a representation of $\mathbf{v}$ as an element in $\mathbb{F}_2^{2k}$ instead $\mathbb{F}_4^k$. So, starting from a multiset of points in $\operatorname{PG}(v-1, q^l)$ we can construct multiset of points in $\operatorname{PG}(vl-1, q)$. However, we have to be a bit careful when using the relation between vectors and points, i.e., 1-dimensional subspaces, for different field sizes. For a given vector $\mathbf{v} \in \mathbb{F}_{q^l}^k$ the point $\langle \mathbf{v} \rangle_{F_{q^l}}$ admits $q^l - 1$ representations $\langle \mathbf{v} \rangle_{\mathbb{F}_{q^l}} = \langle \alpha \cdot \mathbf{v} \rangle_{F_{q^l}}$, where $\alpha \in \mathbb{F}_{q^l} \backslash 0$. If $\mathbf{v}' \in \mathbb{F}_q^{kl}$ is a representation of $\mathbf{v}$ over $\mathbb{F}_q$, then the point $\langle \mathbf{v}' \rangle_{\mathbb{F}_q}$ admits only $q-1$ representations $\langle \mathbf{v}' \rangle_{\mathbb{F}_q} = \langle \alpha \cdot \mathbf{v}' \rangle_{F_q}$, where $\alpha \in \mathbb{F}_q \backslash 0$. So, if we want that all non-zero vectors are covered by the points, we have to replace a single point in $\mathrm{PG}(v-1, q^l)$ by $\frac{q^l-1}{q-1} = [l]_q$ points in $\mathrm{PG}(vl-1, q)$. In terms of linear codes this can be described by concatenation (with an l-dimensional simplex code). In the other direction, starting from a point $\langle \mathbf{v}' \rangle_{\mathbb{F}_q}$ in $\mathrm{PG}(v-1, q)$ we may also replace the point by the point $\langle \mathbf{v}' \rangle_{\mathbb{F}_{q^l}}$ in $\mathrm{PG}(v-1, q^l)$ using $\mathbb{F}_q \subset \mathbb{F}_{q^l}$. Note that $\alpha \mathbf{v}'$, where $\alpha \in \mathbb{F}_q \backslash 0$, leads to the same point in $\mathrm{PG}(v-1, q^l)$. The analog for linear codes is the interpretation of a given generator matrix of an $[n, k]_q$-code over $\mathbb{F}_{q^l}$. It remains to study how divisibility properties are transferred by these two constructions.

Concatenated codes

Concatenation was introduced by George David Forney Jr. in his PhD thesis [81]. Here we are given an *outer* $[N, K, D]_{q^l}$-code C_{out} and an *inner* $[n, l, d]_q$-code C_{in}, where we note that the dimension of the inner code C_{in} equals the degree $[\mathbb{F}_q^l : \mathbb{F}_q]$ of the field extension. Each vector in $\mathbb{F}_q^{lK}$ can be associated with a vector in $\mathbb{F}_{q^l}^K$ and then mapped via the outer code C_{out} to $\mathbb{F}_{q^l}^N$. Then each field element in $\mathbb{F}_{q^l}$ can be associated with an element in $\mathbb{F}_q^l$ and then mapped via the inner code C_{in} to $\mathbb{F}_q^n$. Putting everything together, the concatenation of C_{out} and C_{in} gives an $[nN, lK \geq dD]_q$-code C, where the minimum distance of C may also be strictly larger than dD, see e.g. [20, Theorem 5.9]. For more details, including an example of the computation of a generator matrix of the concatenated code C, we refer to [20, Section 5.2]. We remark that decomposing a given linear code over $\mathbb{F}_q$ as a concatenated code, if possible, is an interesting algorithmical problem, see e.g. [195]. While the determination of the weight distribution of a concatenated code often requires some extra work, see e.g. [213], the situation becomes much easier when C_{in} is an l-dimensional simplex code.

Exercise 5.17. *Let C be a projective Δ-divisible $[n, k]_{q^l}$-code. Show that the concatenation of C with an l-dimensional simplex code over $\mathbb{F}_q$ yields a projective Δq^{l-1}-divisible $[n \cdot [l]_q, kl]_q$-code.*

Example 5.18. *Let C be the projective 4-divisible $[17, 4]_4$-code corresponding to an ovoid in $\mathrm{PG}(3, 4)$, see Example 5.4. Concatenation with the projective 2-divisible $[3, 2]_2$-simplex-code yields a projective 8-divisible $[51, 8]_2$-code. Note that C as well as the concatenated code are two-weight codes. By construction, the corresponding set of points can be partitioned into 17 lines.*

We remark that, up to isomorphisms, there is a unique 8-divisible set of points of cardinality 51 over $\mathbb{F}_2$ [127, Lemma 24]. By puncturing the $[51, 8]_2$-code from Example 5.18 we obtain 8-divisible $[50, 7]_2$-codes, which however are not projective. Nevertheless 8-divisible sets of 50 points over $\mathbb{F}_2$ indeed exist. To this end we have enumerated all projective 8-divisible binary codes with length at most 51 using the software package

LinCode [34, 154]. Observe that a projective 8-divisible binary code with an effective length $49 \leq n_{\text{eff}} \leq 51$ does not contain codewords of weights 40 or 48 since the corresponding residual code would be a projective 4-divisible binary code with an effective length in $\{1, 2, 3, 9, 10, 11\}$, which does not exist as we will see in Lemma 7.2. We have tabulated the corresponding counts of projective 8-divisible binary codes in Table 5.1.

n / k	8	9	10	11	12	13	$\sum$
49	9	38	44	21	7	1	120
50	1	0	0	0	0	0	1
51	1	0	0	0	0	0	1

Table 5.1: Number of projective 8-divisible binary codes with $49 \leq n_{\text{eff}} \leq 51$ per dimension.

There is a unique projective 8-divisible binary linear code of length 50.

As shown by the above exhaustive enumeration, each 8-divisible binary code of length 50 has dimension 8 and is indeed unique up to isomorphism. A generator matrix is given by

$$\begin{pmatrix}
11111111111111111111111111111110000000000010000000\\
00000000000000011111111111111111111111000001000000\\
00000001111111100000000111111110001111111000100000\\
00011110000111100001111000011110110011011100010000\\
01100110011001100110011001100111010101101100001000\\
00111010101010101000111010101011100011011100000100\\
00101101010010010010001010010010100100001000000010\\
11111101000000010101101011111111000001101000000001
\end{pmatrix}.$$

The automorphism group of the code has order 3840 and the weight enumerator is given by $W_C(x) = 1 + 5x^{16} + 210x^{24} + 40x^{32}$.

We remark that the 8-divisible binary codes with length up to 48 have been enumerated in [15] and the counts of the corresponding subset of projective codes where stated in [113].

Research problem

- Find a parametric family of projective q^r-divisible linear codes containing the projective 8-divisible $[50, 8]_2$-code.
- Give a computer-free proof of the uniqueness of a projective 8-divisible binary linear code of length 50.

Generator matrices interpreted over extension fields

Let G be a generator matrix of an $[n, k]_q$-code C. Since $\mathbb{F}_q \subset \mathbb{F}_{q^l}$ for each integer $l \geq 2$ we can also interprete G as a generator matrix of an $[n, k]_{q^l}$-code C'. Let $\mathcal{M}$ and $\mathcal{M}'$ be the corresponding multisets of points in $\operatorname{PG}(k-1, q)$ and $\operatorname{PG}(k-1, q^l)$, respectively, and assume that C is q^r-divisible. Each hyperplane H in $\operatorname{PG}(k-1, q^l)$ corresponds to a subspace S in $\mathrm{PG}(k-1,q)$ with dimension $k - l\dim(S) \le k-1$, so that we can use Lemma 3.4 to conclude that C' is q^{r-l+1}-divisible.

Exercise 5.19. *Let G be a generator matrix of a q^r-divisible $[n,k]_q$-code C and C' be the row span over $\mathbb{F}_{q^l}$. Show that C' is a q^{r-l+1}-divisible $[n,k]_{q^l}$-code that has the same maximum point multiplicity as C.*

Example 5.20. *Let*

$$G = \begin{pmatrix} 1111000 \\ 0011110 \\ 0101011 \end{pmatrix} \in \mathbb{F}_2^{3\times 7} \quad \text{and} \quad G' = \begin{pmatrix} 1111000 \\ 0011110 \\ 0101011 \end{pmatrix} \in \mathbb{F}_4^{3\times 7}.$$

The code C spanned by G is a 3-dimensional simplex code over $\mathbb{F}_2$, i.e., a $[7,3,\{4\}]_2$-code, the code C' spanned by G' is a $[7,3]_4$-code, and let $\mathcal{M}'$ be the corresponding multiset of points in $\mathrm{PG}(2,4)$. Let us represent the hyperplanes in $\mathrm{PG}(2,4)$ by perpendicular points and the elements of $\mathbb{F}_4$ by linear polynomials over $\mathbb{F}_2$. If H is a hyperplane represented by a point $\langle a_1,a_2,a_3\rangle$ with $a_1,a_2,a_3 \in \mathbb{F}_2$ and $(a_1,a_2,a_3) \ne \mathbf{0}$, then $\mathcal{M}'(H) = 3$. In all other cases we have $\mathcal{M}'(H) = 1$, so that $\mathcal{M}'$ and C' are 2-divisible.

As another example we consider the projective 32-divisible binary linear code of length 321 obtained in Exercise 5.15. Over $\mathbb{F}_4$ we obtain a projective 16-divisible linear code of length 321.[2]

Baer subspaces

If G is the generator matrix of an l-dimensional simplex code over $\mathbb{F}_q$, cf. Example 5.20, and $\mathcal{M}'$ be the multiset of points in $\mathrm{PG}(v-1,q^2)$, where $v \ge l$, corresponding to the linear code over $\mathbb{F}_{q^2}$ spanned by G, then we call $\mathcal{M}'$ an l-dimensional *Baer subspace*. If $l=3$, then we speak of a *Baer plane*. Note that l-dimensional Baer subspaces are q^{l-2}-divisible, where $l \ge 2$. If S is an $(l+1)$-dimensional Baer subspace and T an l-dimensional Baer subspace that is contained in S, then $\chi_S - \chi_T$ is called $(l+1)$-*dimensional affine Baer subspace* and is q^{l-2}-divisible for $l \ge 2$.

We remark that Baer subspaces yield two-weight codes, cf. [47, Example RT1]. Affine Baer subspaces only give “few” weight codes.

As mentioned e.g. in [123], a partition of $\mathrm{PG}(v-1,q^l)$ into subgeometries $\mathrm{PG}(v-1,q)$ exists iff $\gcd(v,l)=1$. In particular, $\mathrm{PG}(v-1,q^2)$ can be partitioned into subgeometries $\mathrm{PG}(v-1,q)$, i.e. Baer subspaces, precisely when v is odd and is called *Baer subgeometry partition* (BSP) then. Using a Singer cycle, BSPs for $\mathrm{PG}(2,q^2)$ where constructed by Bruck [42]. While this this technique can be generalized to other parameters, also other constructions are known, see e.g. [4]. Since the set of all points $\mathrm{PG}(2,q^2)$ is q^4-divisible and can be partitioned into q^2-q+1 Baer planes, we can generalize the switching construction from Corollary 5.14 by switching Baer planes to affine Baer solids, i.e., we apply Exercise 5.16 choosing the $\mathcal{M}_i$ as Baer solids and $\mathcal{D}$ as a common Baer plane.

[2] This example does not occur in the proof of Lemma 7.12 since $321 = (85 + 4\cdot 43) + 64$ allows a different construction using the codes of Example 2.25 and Corollary 5.14.

Exercise 5.21. *Construct projective q^2-divisible codes of length*

$$n = \left(q^4 + q^2 + 1\right) + j \cdot \left(q^4 - [4]_q\right)$$

over $\mathbb{F}_{q^2}$ for all $0 \le j \le q^2 - q + 1$.

5.2 Computer searches

We have already reported that there is a unique projective 8-divisible binary linear code of length 50, see Table 5.1. This example was found using exhaustive generation of linear codes (with restrictions on the set of allowed weights). Suitable software packages are e.g. `QextNewEdition`, or its predecessor `Q-Extension` [33], and `LinCode`, see [34]. Further classifications for linear codes have e.g. been presented in [16] and [185], see also [136, Section 7.3].

The search problem for projective q^r-divisible codes can easily be formulated as an integer linear programming (ILP) problem using binary characteristic variables x_P for all points P of $\operatorname{PG}(v-1,q)$, i.e., x_P encodes the multiplicity of P. Prescribing the desired cardinality $n = \sum_{P\in\mathcal{P}} x_P$ and the dimension k, it remains to convert the restrictions induced by q^r-divisibility, see Equation (2.30), into linear constraints:

$$\sum_{P\le H} x_P = n - z_H \cdot q^r \tag{5.4}$$

for each hyperplane $H \in \mathcal{H}$, where $z_H \in \mathbb{N}_0$ and $z_H \le \lfloor n/q^r \rfloor$. Although, Lemma 3.4 allows to include *modulo-constraints* on the number of holes for subspaces other than hyperplanes, ILP solvers seem not to benefit from these extra constraints. If the desired divisible codes do not need to be projective, we can use integer variables $x_P \in \mathbb{N}_0$ (with an eventual upper bound $x_P \le \lambda$ in case of maximum point multiplicity λ). Of course, we may prescribe $x_{\langle \mathbf{e}_i \rangle} = 1$ (or $x_{\langle \mathbf{e}_i \rangle} \ge 1$) for all $1 \le i \le k$.

Since larger instances can not be successfully treated directly by customary ILP solvers, we have additionally prescribed some symmetry to find examples. This general approach is called the Kramer–Mesner method [147]. Giving a group G acting on the set of points $\mathcal{P}$ and the set of hyperplanes $\mathcal{H}$ we additionally assume $x_{P^g} = x_P$ and $z_{H^g} = z_H$ for each $P \in \mathcal{P}$, each $H \in \mathcal{H}$, and each $g \in G$, where $P^g \in \mathcal{P}$ and $H^g \in \mathcal{H}$ denote the image of the group operation of g applied to $P \in \mathcal{P}$ and $H \in \mathcal{H}$, respectively. This rather general method was e.g. applied to general linear codes [38] and two-weight codes [145]. For an exemplary application to the construction of constant-dimension codes we refer e.g. to [146]. Prescribing cyclic groups in our application, we found the following generator matrices:

- $q = 2$, $r = 3$, $n = 50$, $k = 8$, $\begin{pmatrix} 00000000000000000000000000111111111111111111111111 \\ 00000000000000111111111111000000000000111111111111 \\ 00000000111111000000111111000000111111000000111111 \\ 00000011000011000011000011001111001111001111001111 \\ 00001100000101001111011101000011010001110011010111 \\ 00110000001111000101101110000101110110010100100111 \\ 01010101011101011011100111011101011011110101111001 \\ 10010101011011011101010111011011110011011101111010 \end{pmatrix}$

- $q = 2$, $r = 3$, $n = 74$, $k = 12$, $W_C(x) = 1 + 3x^8 + 60x^{24} + 1423x^{32} + 2585x^{40} + 24x^{48}$,

$$\begin{pmatrix}
00001101111111111001000010001001110111111100101001100011011000011011000111 \\
11100001101111111100011011000111011000100111110010110001110010011001111000 0 \\
11001100101011011010100010001010100111000000000010000110000011011111110001 \\
00001110110001000010010100100111100011101010100110010000000011110001011101 \\
01000010111000110011111010110101010111110000101100000000101101111111110001 \\
00100111100011111010011100011111001000000000111111010011101111100010001101 11 \\
11001011110011111011010111100110101001001001001111001110000010101001101010 \\
01111000111001010010110000010111111111000101011110010000011101101101011100 \\
11110110101010101110110101110110101111000000011000111001011100010100101001 1010 \\
10101001110000000110011101101111001111100010101001011011001100101110101110 0 \\
00100000111111010110100111011110100011101001101111010100000100101100011011 1 \\
11111110000011011000101000101111100010011011011010001111000111001101001101 0
\end{pmatrix}$$

- $q = 2$, $r = 4$, $n = 161$, $k = 10$, $W_C(x) = 1 + 50x^{64} + 886x^{80} + 87x^{96}$,

$$\begin{pmatrix}
00000000000000000000000000000000000000000000000000000000000000000000000000000000 \\
00000000000000000000000000000000000000001111111111111111111111111111111111111111 \\
00000000000000000000000011111111111111110000000000000000000000001111111111111111 \\
00000000000000011111111111000000000011111100000000000111111111111111000000111111 1 \\
00000000011111100000011110000111111001111000000111100000000011111100111100001111 \\
00000111100011101111110011000100001111000100001100010001111100001101001111110001 \\
00011011101101100001111010011001101011111000100001101100011001101000111000100 10 \\
00101001100010100110101110101110101010010001011010010001101010111100111001101 00 \\
01000010100101101010101000010001011010010011101010100100100001100000000010111 10 \\
10001001101100000101101000111010000000110011010111010010101101110110101001011 10 \\
001111111111111111111111111111111111111111111111111111111111111111111111111111111 \\
110000000000000000000000000000000000000000111111111111111111111111111111111111111 \\
110000000000000000011111111111111111111111100000000000000000000111111111111111111 \\
110000000000111111000000000000000011111111110000001111111111000000000111111111111 1 \\
110000000111100001100000011111111000011111100001100000011110000111111000000111111 1 \\
110000110001000111000111000011110011011111100110100011111110001000010000110001111 1 \\
010011001110001000000100001110001010110001101010100101111110111001111000101001011 11 \\
010111111011101101100000100010110001100010110100000001000000100110101110111110011 0111 \\
010111001011101101011010101100111011101001111111000011001111000010010011110111101 1 \\
110001010011010001110010001001100000011101101101011101010111110001011100010011110 1
\end{pmatrix}$$

- $q = 2$, $r = 4$, $n = 162$, $k = 10$, $W_C(x) = 1 + x^{32} + 30x^{64} + 890x^{80} + 102x^{96}$,

$$\begin{pmatrix}
000000000000000000000000000000000000000000000000000000000000000000000000000000000 \\
000000000000000000000000000000000000000011111111111111111111111111111111111111111 \\
000000000000000000000011111111111111111111000000000000000000000011111111111111111 11 \\
000000000011111111111100000000011111111111110000000001111111111110000000001111111111 \\
000000111100000011111100001111000000111111000000011000000000111100000001100000000111 \\
00001100110001110001110000000000000010000010001110100001111001100011101000011110 01 \\
00010001010000000001000001100010001110001100010011000110001010101111111001100110 00 \\
00100100000110010110010101011001111101101001101100010101111011101011111101111101 1 \\
01000001100010101110110111101010101110111100100000000100010110111000100010101011 11 \\
10000100110111101011010100010011001000110101010011100001111111000010100100100110 1 \\
011111111111111111111111111111111111111111111111111111111111111111111111111111111 \\
100000000000000000000000000000000000000001111111111111111111111111111111111111111 \\
100000000000000000000011111111111111111110000000000000000000001111111111111111111 \\
100000000000011111111000000000000000111111110000000000000000111111110000000000000011111111 \\
100001111111100111111100001111111110011111110000000111111000011110000000111111100001111 \\
100110000111101000111001100001111010001110111111011111111111111110001110001110011001 1 \\
101110011001100000000101010111001110011011100111000111011100111110111111110101110 1 \\
100010000001000000010100010000101011100100110100100000110010101011001011001111101 10 \\
001001100010111011100100001111011111111011000010001010001101000010001010111010110 1 \\
100100101110110101111000000101101001011100100111011001101110101001000101000111010
\end{pmatrix}$$

- $q = 2$, $r = 4$, $n = 195$, $k = 10$, $W_C(x) = 1 + 33x^{80} + 855x^{96} + 135x^{112}$,

$$\begin{pmatrix}
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000 \\
00000000000000000000000000000000000000000000000000111111111111111111111111111111111111111111111111 \\
00000000000000000000000000011111111111111111111111110000000000000000000000000111111111111111111111 11 \\
00000000000000011111111111100000000000000111111111111100000000000000111111111111100000000000011111111111 \\
00000000011111100000011111100000011111100000011111100000011111100000011111100000011111100000011111 \\
00000000110001110001110000110000110001110001110000110000110001110000110000110001110001110000110001110000 1 \\
00000110101100000101100110000010101100100001100110101011101001011011011000111011100011011011001001 11 \\
00011010000100100000010001010110111010010111011101110111001110011101001101011000000101001101111001010 10 \\
01101000000010000111010110110011100110100010101101010101001101011010110110010000001001000111011100 0111 \\
10100000000110110110100001110101010111000010010110110001011101011010100011010010111010011000101101 1
\end{pmatrix}$$

$$\begin{pmatrix}
01111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111 \\
10000000000000000000000000000000000000000000000011111111111111111111111111111111111111111111111 \\
10000000000000000000000000111111111111111111111111100000000000000000000000001111111111111111111111 \\
10000000000000111111111111100000000000000111111111111100000000000001111111111111100000000000011111111 \\
10000001111110000000111111000000011111100000001111110000000111111000000011111100000001111111 \\
10011111000111000111001111100111110001110001110011111001111100011100011100111 \\
10000110010110010010100110100110010010110010100010100110110110110011100110101110010110110110 10011 \\
10011000111000111100101010111000101101110001100001100000110000011010000101011110111101010011011101 01 \\
10001010001011010101110111001110101000010111001100101010011100110001101001101000101010011101110 1100 \\
00101001100110101101010010101101100100001011100110100100111101101001101010100100110100101011101 0
\end{pmatrix}$$

- $q = 2$, $r = 4$, $n = 197$, $k = 10$, $W_C(x) = 1 + 10x^{80} + 837x^{96} + 176x^{112}$,

$$\left(\begin{array}{l}
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000\\
000000000000000000000000000000000000000000000000000000111111111111111111111111111111111111111111111\\
000000000000000000000000000001111111111111111111111111000000000000000000000000111111111111111111111\\
000000000000000001111111111110000000000001111111111111000000000000111111111111000000000000111111111\\
000000000011111110000000111110000000011111000000111111100000000111110000001111110000011111110000011\\
000000011100001110000011000110000111001110001100000110000011000110011100001110001100000110000011100\\
000001101100010010001101001000001001010010010000111110011101001010101101110110111100111010011001101\\
000110100100110000000110000110110001111110110101001010100101010001010100110010010101011100101011111\\
011010000000100010110111011000010010010110001101011110101010100101101010010110001101101001011001110\\
101000000010111010001101000101000110010011001101100111010100100110101101010110001011010101110110000011
\end{array}\right.$$

$$\left.\begin{array}{l}
00011111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111\\
11100000000000000000000000000000000000000000000000011111111111111111111111111111111111111111111111\\
11100000000000000000000000011111111111111111111111000000000000000000000000111111111111111111111111\\
11100000000000111111111111000000000000111111111111000000000000111111111111000000000000111111111111\\
11100000001111100000111111100000111111100000001111100000111111100000001111100000001111100000111111\\
11100011110001100111001111100011000111100111110011100111001111100011110001100111111001110001100011111\\
01100100010010101011010001100101011001101000110000101001110111100100110110101001110100111111001001\\
10100100100101010001100011101110001010111011010101100010111001100101011000010111110001101100111000001\\
10100101100010100111100110110100110010011001101011000001010011101001011111001110010010011101010110\\
01001001001011000011010111111001101100101011000110001010110110110010001010100010110011010110110110
\end{array}\right)$$

- $q = 5$, $r = 1$, $n = 41$, $k = 5$, $W_C(x) = 1 + 4x^{25} + 1360x^{30} + 1760x^{35}$,

$$\begin{pmatrix}
00000000000111111111111111111111111111111\\
00011111111000000001111111122233344444444\\
00100112344012223330011122403303300111224\\
01003013314040131332201203012211324033021\\
10003222331041124030121012410144041303440
\end{pmatrix}$$

- $q = 7$, $r = 1$, $n = 141$, $k = 4$, $W_C(x) = 1 + 30x^{112} + 1692x^{119} + 672x^{126} + 6x^{133}$,

$$\left(\begin{array}{l}
00000000000000000000001111111111111111111111111111111111111111111111111\\
00001111111111111111110000000000000000011111111111111111112222222222222\\
01110000011222234446660001122333444456666001112233344445666600022233344 55\\
10260135624025641250260460426056123400154645603245012611560120261455602
\end{array}\right.$$

$$\left.\begin{array}{l}
1111111111111111111111111111111111111111111111111111111111111111111111\\
2222233333333333333333444444444444444444455555555555666666666666666666\\
66666001223334445555666001112223445555666011122345666000111122335556666\\
0345624314234013124524512016012525035613561562424324045145606261350136
\end{array}\right)$$

In our ILP model the integer variable z_H may be replaced by several binary variables $y_{H,n'}$, which are equal to 1 iff hyperplane H contains exactly n' selected points, i.e., has multiplicity n'. This way, it is possible to exclude some specific multiplicities for hyperplanes or to count (and incorporate given bounds on) the number of hyperplanes with a given multiplicity. Restrictions for n and the n' are given by our exclusion results for q^r-divisible and q^{r-1}-divisible sets, respectively, see Section 4 or Section 7. Prescribing a specific solution of the MacWilliams identities directly translates to equations for the number of hyperplanes with a given multiplicity.

A Diophantine linear equation system, in the same vein as our ILP model, together with the prescription of a subgroup of the automorphism group of the code was used in [145] in order to construct two-weight codes with previously unknown parameters. In [34, Lemma 7] the ILP approach was adjusted to the situation where a given Δ-divisible $[n, k]_q$-code should be extended to a Δ-divisible $[n', k+1]_q$-code.

Research problem

Use the ILP approach and some carefully selected candidates for subgroups of the automorphism group of a potential projective q^r-divisible code over $\mathbb{F}_q$ whose length is currently unknown to exist, see Section 7.

5.3 Two-weight codes

A linear $[n, k]_q$ code C is called a *two-weight code* if the non-zero codewords of C attain just two possible weights, i.e., if it is an $[n, k, \{w_1, w_2\}]_q$ code for $w_1 \neq w_2 \in$

$\mathbb{N}$. An online-table for known two-weight codes is at `http://www.tec.hkr.se/~chen/research/2-weight-codes` and an exhaustive survey was given by Calderbank and Kantor [47]. As observed by Delsarte, a projective two-weight code typically has a large divisibility.

Projective two-weight codes are divisible

Lemma 5.22. *([60, Corollary 2])*
Let C be a projective two-weight code over $\mathbb{F}_q$, where $q = p^e$ for some prime p. Then there exist suitable integers u and t with $u \geq 1$, $t \geq 0$ such that the weights are given by $w_1 = up^t$ and $w_2 = (u+1)p^t$.

We remark that first order Reed–Muller codes or affine spaces, see Example 2.25, are examples of $\left[q^{r+1}, r+2, \left\{q^{r+1} - q^r, q^{r+1}\right\}\right]_q$ two-weight codes for all prime powers q and all $r \in \mathbb{N}_0$. (Repeated) simplex codes are the unique possibility for one-weight codes and Baer subspaces, see Subsection 5.1, yield two-weight codes. Solving Diophantine linear equation systems, similar to those discussed Section 5.2, leads to many examples of two-weight codes in [145]. Using Bose-Chaudhuri-Hocquenghem (BCH) codes a parametric family of two-weight codes was constructed in [21]:

Theorem 5.23. *(Cf. [21, Theorem 4]) For every prime-power q and every pair of natural numbers $m \leq n'$ there exists a projective $q^{n'+m-1}$-divisible $\left[q^m \cdot [n']_q \cdot \left(q^{n'} - q^{n'-m} + 1\right), 3n'\right]_q$-code.*

In some cases these codes can be obtained by concatenation with a suitable simplex code.

Due to their omnipresence a lot of research has been done on two-weight codes and many examples are available. Nevertheless the topic is studied for decades, new parametric families are still found, see e.g. [118]. In Table 5.2 we list those parameters that we will use in Section 7 as examples.

We remark that the example of a projective binary 32-divisible code of length 780 can be obtained by concatenation of the example of a projective quaternary 16-divisible code of length 260. For more results on field changes we refer to [47, Section 6] for two-weight codes and Subsection 5.1 for divisible codes.

n	k	$\{w_1, w_2\}$	Δ	q	description
51	8	$\{24, 32\}$	8	2	[47, Example CY1], Example 5.18
73	9	$\{32, 40\}$	8	2	computer search with prescribed automorphisms [145], optimal code
196	9	$\{96, 112\}$	16	2	BY construction in Theorem 5.23
198	10	$\{96, 112\}$	16	2	computer search with prescribed automorphisms [145], optimal code
231	10	$\{112, 128\}$	16	2	computer search with prescribed aut. [145], optimal code, [63]
234	12	$\{112, 128\}$	16	2	[47, Theorem 6.1] applied to Example FE3 over $\mathbb{F}_4$
273	12	$\{128, 144\}$	16	2	quasi-cyclic code [52]
276	11	$\{128, 144\}$	16	2	[47, Example RT5$^{\mathrm{d}}$]
455	12	$\{224, 256\}$	32	2	[47, Example CY1]
780	12	$\{384, 416\}$	32	2	BY construction in Theorem 5.23
845	12	$\{416, 448\}$	32	2	computer search with prescribed automorphisms [145]
975	12	$\{480, 512\}$	32	2	computer search with prescribed automorphisms [145]
1105	12	$\{544, 576\}$	32	2	computer search with prescribed automorphisms [145]
1170	12	$\{576, 608\}$	32	2	computer search with prescribed automorphisms [145]
10	4	$\{6, 9\}$	3	3	[47, Example CY1 and RT2], Example 5.4
11	5	$\{6, 9\}$	3	3	[47, Example RT6], ternary Golay code [9, 88, 204]
55	5	$\{36, 45\}$	9	3	optimal code [94], quasi-cyclic code [52]
56	6	$\{36, 45\}$	9	3	[47, Example FE2], Hill cap [121], optimal code
84	6	$\{54, 63\}$	9	3	BY construction, optimal code
98	6	$\{63, 72\}$	9	3	optimal code [94], quasi-cyclic code [52], [145]
260	6	$\{192, 208\}$	16	4	BY construction in Theorem 5.23
303	6	$\{224, 240\}$	16	4	[47, Example CY2]
304	6	$\{224, 240\}$	16	4	complement of [47, Example CY2]
39	4	$\{30, 35\}$	5	5	optimal code [63]
175	4	$\{147, 154\}$	7	7	[47, Example FE1]
205	4	$\{180, 189\}$	9	9	complement of [47, Example CY2]

Table 5.2: Parameters of a few selected two-weight codes.

6 Non-existence results for projective q^r-divisible codes

The aim of this section is to draw some parametric conclusions from the linear programming method for projective q^r-divisible codes. However, we will mainly use the geometric reformulation, i.e., the standard equations in Lemma 2.22. For parametric conclusions of the linear programming method for distance optimal linear codes we refer e.g. to [20, Section 15.3] and [19]. Our first example is an alternative version of Lemma 3.8. Given a multiset of points $\mathcal{M}$ in $\operatorname{PG}(v-1,q)$ let $\mathcal{T}(\mathcal{M}) := \{0 \le i \le \#\mathcal{M} \,:\, a_i > 0\}$ denote the set of attained hyperplane multiplicities, where a_i is the number of hyperplanes $H \in \mathcal{H}$ with $\mathcal{M}(H) = i$.

A "linear" condition

Lemma 6.1. *For integers $u \in \mathbb{Z}$, $m \ge 0$ and $\Delta \ge 1$ let $\mathcal{M}$ in be a Δ-divisible multiset of points in $\operatorname{PG}(v-1,q)$ of cardinality $n = u + m\Delta \ge 0$. Then, we have*

$$(q-1) \cdot \sum_{h \in \mathbb{Z}, h \le m} h a_{u+h\Delta} = (u + m\Delta - uq) \cdot \frac{q^{v-1}}{\Delta} - m, \tag{6.1}$$

where we set $a_{u+h\Delta} = 0$ if $u + h\Delta < 0$.

Exercise 6.2. *Use the standard equations from Lemma 2.22 to verify Equation (6.1).*

Corollary 6.3. *For integers $u, m \ge 0$ and $\Delta \ge 1$ let the multiset of points $\mathcal{M}$ in $\operatorname{PG}(v-1,q)$ satisfy $\#\mathcal{M} = u + m\Delta$ and $\mathcal{T}(\mathcal{M}) \subseteq \{u, u+\Delta, \dots, u+m\Delta\}$. Then, $u < \frac{m\Delta}{q-1}$ or $u = m = 0$.*

Example 6.4. *Applying Corollary 6.3 with $q = 2$, $\Delta = 2$, $u = 1$, and $m = 0$ yields that no 2-divisible multiset of points over $\mathbb{F}_2$ of cardinality 1 exists. With this we can choose $q = 2$, $\Delta = 4$, $u = 5$, and $m = 1$ in Corollary 6.3 to conclude that no 4-divisible multiset of points over $\mathbb{F}_2$ of cardinality 9 exists. Using this and the non-existence of a 4-divisible multiset of points over $\mathbb{F}_2$ of cardinality 1, we can choose $q = 2$, $\Delta = 8$, $u = 17$, and $m = 2$ in Corollary 6.3 to conclude that no 8-divisible multiset of points over $\mathbb{F}_2$ of cardinality 33 exists.*

Of course, the non-existence of an 8-divisible $[33, k]_2$ full-length code also follows from the methods presented in Section 4, which are essentially based on the averaging argument in Lemma 3.8 and a suitable induction. Arguably Lemma 3.8 has some advantages over Corollary 6.3 since we can directly start with an 8-divisible multiset $\mathcal{M}$ of points over $\mathbb{F}_2$ of cardinality 33 and conclude the existence of a hyperplane H with

$\mathcal{M}(H) \in \{1, 9\}$. The example of a potential 8-divisible $[33, k]_2$ full-length code is also interesting when using the linear programming method directly. First note that we will have to prescribe some suitable values for k. If we allow all weights in $\{8, 16, 24, 32\}$, then the MacWilliams equations admit a non-negative rational solution while the weights 32 and 24 might also be excluded with a separate linear programming computation. Thus, it definitely is useful to tabulated the possible lengths of (projective) q^r-divisible codes as we do in Section 7.

A "quadratic" condition

Lemma 6.5. *For integers $u \in \mathbb{Z}$, $m \geq 0$, and $\Delta \geq 1$ let $\mathcal{M}$ be a Δ-divisible set of points in $\operatorname{PG}(v-1, q)$ of cardinality $n = u + m\Delta \geq 0$. Then, we have*

$$(q-1) \cdot \sum_{h \in \mathbb{Z}, h \leq m} h(h-1) a_{u+h\Delta} = \tau_q(u, \Delta, m) \cdot \frac{q^{v-2}}{\Delta^2} - m(m-1), \tag{6.2}$$

where we set $\tau_q(u, \Delta, m) =$

$$m(m-q)\Delta^2 + \left(q^2u - 2mqu + mq + 2mu - qu - m\right)\Delta + (q-1)^2u^2 + (q-1)u \tag{6.3}$$

and $a_{u+h\Delta} = 0$ if $u + h\Delta < 0$.

Proof. Rewriting the standard equations from Lemma 2.22 yields

$$\begin{aligned}
(q-1) \cdot \sum_{h \in \mathbb{Z}, h \leq m} a_{u+h\Delta} &= q^2 \cdot q^{v-2} - 1, \\
(q-1) \cdot \sum_{h \in \mathbb{Z}, h \leq m} (u + h\Delta) a_{u+h\Delta} &= (u + m\Delta)(q \cdot q^{v-2} - 1), \\
(q-1) \cdot \sum_{h \in \mathbb{Z}, h \leq m} (u + h\Delta)(u + h\Delta - 1) a_{u+h\Delta} &= (u + m\Delta)(u + m\Delta - 1)(q^{v-2} - 1).
\end{aligned}$$

$u(u+\Delta)$ times the first equation minus $(2u+\Delta-1)$ times the second equation plus the third equation gives Δ^2 times the stated equation. □

The multipliers used in the proof of Lemma 6.5 can be directly read off from the following observation.

Lemma 6.6. *For pairwise different non-zero numbers a, b, c the inverse matrix of*

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 - a & b^2 - b & c^2 - c \end{pmatrix}$$

is given by

$$\begin{pmatrix} bc(c-b) & -(c+b-1)(c-b) & (c-b) \\ -ac(c-a) & (c+a-1)(c-a) & -(c-a) \\ ab(b-a) & -(b+a-1)(b-a) & (b-a) \end{pmatrix} \cdot \left((c-a)(c-b)(b-a)\right)^{-1}$$

Similar as for the "linear condition" we can conclude explicit non-existence criteria from Lemma 6.5:

Corollary 6.7. *For integers $u \in \mathbb{Z}$ and $\Delta, m \geq 1$ let $\mathcal{K}$ be a Δ-divisible arc of cardinality $n = u + m\Delta \geq 0$ in $\mathrm{PG}(v-1,q)$. If one of the following conditions hold, then $(q-1)\cdot\sum_{i=2}^{m} i(i-1)x_i \notin \mathbb{N}_0$, which is impossible.*

(a) $\tau_q(u,\Delta,m) < 0$*;*

(b) $\tau_q(u,\Delta,m)\cdot q^{v-2}$ *is not divisible by* Δ^2*;*

(c) $m \geq 2$ *and* $\tau_q(u,\Delta,m) = 0$*.*

We have the following special cases:

$$\begin{aligned}\tau_q(u,q^r,m) &= \big(m(m-q)q^r - 2mqu + q^2u + mq + 2mu - qu - m\big)\cdot q^r\\ &\quad + \big(q^2u^2 - 2qu^2 + qu + u^2 - u\big)\,,\\ \tau_2(u,2^r,m) &= (m(m-2)2^r - 2mu + m + 2u)\cdot 2^r + \big(u^2+u\big)\,.\end{aligned}$$

Exercise 6.8. *Conclude the non-existence of projective 4-divisible $[n,k]_2$-codes for all $n \in \{1,\dots,6\}\cup\{9,\dots,14\}$ and the non-existence of projective 8-divisible $[n,k]_2$-codes for all $n \in \{1,\dots,14\}\cup\{17,\dots,29\}\cup\{33,\dots,44\}$ from Corollary 6.3 and Corollary 6.7.*

Note that in order to apply Lemma 6.5, we have to choose a parameter $m \in \mathbb{N}_0$. Given m, we can easily analyze when $\tau_q(u,\Delta,m)$ is non-positive:

Lemma 6.9. *Given a positive integer m, we have $\tau_q(u,\Delta,m) \leq 0$ iff*

$$\begin{aligned}&(q-1)u - (m-q/2)\Delta + \frac{1}{2}\\ \in\ &\left[-\frac{1}{2}\sqrt{q^2\Delta^2 - 4qm\Delta + 2q\Delta + 1},\ \frac{1}{2}\sqrt{q^2\Delta^2 - 4qm\Delta + 2q\Delta + 1}\right]. \qquad (6.4)\end{aligned}$$

The last interval is non-empty, i.e., the radiant is non-negative, iff $1 \leq m \leq \lfloor (q\Delta+2)/4 \rfloor$. We have $\tau_q(u,\Delta,1) = 0$ iff $u = (\Delta-1)/(q-1)$.

Quadratic functions that are non-negative over the integers

We remark that [32, Theorem 1.B] is quite similar to Lemma 6.5 and its implications. Actually, their analysis grounds on [189] and is strongly related to the classical second-order Bonferroni Inequality [30, 84, 85] in Probability Theory. In simple words, the trick of Lemma 6.5 is that $h(h-1) = h^2 - h$ is non-negative for every integer h. Note that $f(x) = x^2 - x$ attains its minimum at $x = \frac{1}{2}$ with function value $-\frac{1}{4}$. So, in some sense we perform a (quadratic) integer rounding cut.

We can also use Corollary 6.3 and Corollary 6.7 to show that for all cardinalities $n \leq rq^{r+1}$ the attainable lengths of q^r-divisible sets of points over $\mathbb{F}_q$ are those that are attained by combinations of $(r+1)$-spaces and affine $(r+2)$-spaces, cf. Exercise 5.3.

Theorem 6.10. *[127, Theorem 11] Let $\mathcal{M}$ be a q^1-divisible set of points in $\operatorname{PG}(v-1,q)$ with cardinality n. If $2 \le n \le q^2$, then either $n = q^2$ or $q+1$ divides n. Additionally, the non-excluded cases can be realized.*

Theorem 6.11. *[127, Theorem 12] For the cardinality n of a q^r-divisible set of points in $\operatorname{PG}(v-1,q)$, where $r \in \mathbb{N}$, we have*

$$n \notin \left[(a(q-1)+b) \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q + a + 1, (a(q-1)+b+1) \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q - 1 \right],$$

where $a, b \in \mathbb{N}_0$ with $b \le q-2$ and $a \le r-1$. If $n \le rq^{r+1}$, then all other cases can be realized.

Similar as the conditions based on a linear and a quadratic polynomial in Lemma 6.1 and Lemma 6.5, we can also conclude a condition based on a cubic polynomial. To this end we consider an explicit example first.

Lemma 6.12. *No 2^3-divisible set of points in $\operatorname{PG}(v-1,2)$ of cardinality 52 exists.*

Proof. Using the abbreviation $y = 2^{v-3}$ the first four MacWilliams identities, see Equation (2.11), are given by

$$\begin{array}{rcl} A_0 + A_8 + A_{16} + A_{24} + A_{32} + A_{40} + A_{48} & = & 8y \\ \binom{52}{1} + \binom{44}{1}A_8 + \binom{36}{1}A_{16} + \binom{28}{1}A_{24} + \binom{20}{1}A_{32} + \binom{12}{1}A_{40} + \binom{4}{1}A_{48} & = & 4y \cdot 52 \\ \binom{52}{2} + \binom{44}{2}A_8 + \binom{36}{2}A_{16} + \binom{28}{2}A_{24} + \binom{20}{2}A_{32} + \binom{12}{2}A_{40} + \binom{4}{2}A_{48} & = & 2y \cdot \binom{52}{2} \\ \binom{52}{3} + \binom{44}{3}A_8 + \binom{36}{3}A_{16} + \binom{28}{3}A_{24} + \binom{20}{3}A_{32} + \binom{12}{3}A_{40} + \binom{4}{3}A_{48} & = & y \cdot \left(\binom{52}{3} + B_3 \right) \end{array}$$

Substituting $x = y \cdot B_3$ and rearranging yields

$$\begin{array}{rcl} A_8 & = & -4 + A_{40} + 4A_{48} + \frac{1}{512}x + \frac{7}{64}y \\ A_{16} & = & 6 - 4A_{40} - 15A_{48} - \frac{3}{512}x - \frac{17}{64}y \\ A_{24} & = & -4 + 6A_{40} + 20A_{48} + \frac{3}{512}x + \frac{397}{64}y \\ A_{32} & = & 1 - 4A_{40} - 10A_{48} - \frac{1}{512}x + \frac{125}{64}y. \end{array}$$

With this we compute

$$A_{16} + \frac{31}{20}A_8 = -\frac{1}{5} - \frac{49}{20}A_{40} - \frac{44}{5}A_{48} - \frac{123}{1280}y - \frac{29}{10240}x,$$

which contradicts $A_8, A_{16}, A_{40}, A_{48}, x, y \ge 0$. $\square$

We remark that Lemma 6.12 generalizes Example 2.8 and Example 2.14 dealing with all dimensions v, encoded in $y = 2^{v-3}$, simultaneously. To this end we have replaced the non-linear $y \cdot B_3$ by a new variable x, which relaxes the problem on the one hand but turns the problem into a linear one on the other hand.

Remark 6.13. *The non-existence of a 2^3-divisible set of cardinality $n = 52$ implies several upper bounds for partial spreads, see Section 9 and in particular Lemma 9.9. More precisely, we e.g. have $129 \le A_2(11, 8; 4) \le 132$, $2177 \le A_2(15, 8; 4) \le 2180$, and $34945 \le A_2(19, 8; 4) \le 34948$.*

The underlying idea of the proof of Lemma 6.12 can be generalized. Choosing a suitable basis for the first four MacWilliams equations, the multiplication with the inverse of a suitable 4×4-matrix, cf. Lemma 6.6 yields:

A "cubic" condition

Lemma 6.14. *Let $t \in \mathbb{Z}$ be an integer and $\mathcal{K}$ be Δ-divisible arc of cardinality $n > 0$ in $\operatorname{PG}(v-1, q)$. Then, we have*

$$\sum_{i \ge 1} \Delta^2 (i-t)(i-t-1) \cdot (g_1 \cdot i + g_0) \cdot A_{i\Delta} \; + qhx = n(q-1)(n-t\Delta)(n-(t+1)\Delta)g_2,$$

where $x \in \mathbb{R}_{\ge 0}$, $g_1 = \Delta q h$, $g_0 = -n(q-1)g_2$, $g_2 = h - (2\Delta q t + \Delta q - 2nq + 2n + q - 2)$ and

$$h = \Delta^2 q^2 t^2 + \Delta^2 q^2 t - 2\Delta n q^2 t - \Delta n q^2 + 2\Delta n q t + n^2 q^2 + \Delta n q - 2n^2 q + n^2 + nq - n.$$

Corollary 6.15. *Using the notation of Lemma 6.14, if $n/\Delta \notin [t, t+1]$, $h \ge 0$, and $g_2 < 0$, then there exists no Δ-divisible arc $\mathcal{K}$ of cardinality n in $\operatorname{PG}(v-1, q)$.*

Proof. First we observe $(i-t)(i-t-1) \ge 0$, $(n-t\Delta)(n-(t+1)\Delta) > 0$, and $g_1 \ge 0$. Since $g_2 < 0$, we have $g_0 \ge 0$ so that $g_1 i + g_0 \ge 0$. Thus, the entire left hand side is non-negative and the right hand side is negative – a contradiction. □

Applying Corollary 6.15 with $t = 3$ gives Lemma 6.12. Note that in Example 2.14 we have only used the first three MacWilliams equations. As a further example we consider the parameters $q = 2$, $\Delta = 2^4 = 16$, and $n = 235$. The condition $n/\Delta \notin [t, t+1]$ excludes $t = 14$. The condition $h \ge 0$ is satisfied for all integers t since the excluded interval $(6.700, 6.987)$ contains no integer. The condition $g_2 < 0$ just allows to choose $t = 7$, which also satisfies $qh \ge -g_0$.

We can perform a closer analysis in order to develop computational cheap checks. We have $g_2 < 0$ iff

$$n \in \left(\frac{\Delta q t + \frac{\Delta q}{2} - \frac{3}{2} - \frac{1}{2} \cdot \sqrt{\omega}}{q-1}, \frac{\Delta q t + \frac{\Delta q}{2} - \frac{3}{2} + \frac{1}{2} \cdot \sqrt{\omega}}{q-1} \right), \tag{6.5}$$

where $\omega = \Delta^2 q^2 - 4qt\Delta - 2\Delta q + 4q + 1$. Thus, $\omega > 0$, i.e., we have

$$t \le \left\lfloor \frac{q\Delta - 2}{4} + \frac{1}{\Delta} + \frac{1}{4q\Delta} \right\rfloor.$$

We have $h \geq 0$ iff

$$n \notin \left(\frac{\Delta qt + \frac{\Delta q}{2} - \frac{1}{2} - \frac{1}{2} \cdot \sqrt{\omega - 4q}}{q-1}, \frac{\Delta qt + \frac{\Delta q}{2} - \frac{1}{2} + \frac{1}{2} \cdot \sqrt{\omega - 4q}}{q-1} \right). \tag{6.6}$$

The most promising possibility, if not the only at all, seems to be

$$n \in \left(\frac{\Delta qt + \frac{\Delta q}{2} - \frac{3}{2} - \frac{1}{2} \cdot \sqrt{\omega}}{q-1}, \frac{\Delta qt + \frac{\Delta q}{2} - \frac{1}{2} - \frac{1}{2} \cdot \sqrt{\omega - 4q}}{q-1} \right], \tag{6.7}$$

which allows the choice of at most one integer n. In our example $q = 2$, $\Delta = 2^4 = 16$ the possible n for $t = 1, \dots, 7$ correspond to $33, 66, 99, 132, 166, 200, 235$, respectively. The two other conditions are automatically satisfied.

Exercise 6.16. *Show that no projective 2^5-divisible $[n, k]_2$-code with*

$$n \in \{325, 390, 456, 521, 587, 652, 718, 784, 850, 917, 985\}$$

exists.

Lemma 6.17. *No 3^2-divisible set of points in $\mathrm{PG}(k-1, 3)$ of cardinality 89 exists.*

Proof. We set $x = 3^{k-4}$, $y = 3^{k-4} \cdot B_3$, and $z = 3^{k-4} \cdot B_4$. Solving the first five MacWilliams equations for A_9, A_{54}, A_{63}, x, and y yields the equation

$$\begin{aligned} 99630A_9 + 121905A_{18} + 99873A_{27} + 60021A_{36} & \\ +22275A_{45} + 22518A_{72} + 61236A_{81} + z &= 0, \end{aligned}$$

so that $A_9 = A_{18} = A_{27} = A_{36} = A_{45} = A_{72} = A_{81} = z = 0$. With that, the equation system has the unique solution $x = 189$, $y = 33642$, $A_{54} = 6230$, and $A_{63} = 9078$. However, 189 is not a power of three, but $x = 3^{k-4}$. □

We remark that for the parameters of Lemma 6.17 the first four MacWilliams equations permit non-negative rational solutions for all dimensions $9 \leq k \leq 89$. When adding the fifth MacWilliams equation, the corresponding polyhedron gets empty.

Exercise 6.18. *Implement the non-existence criteria for lengths of projective q^r-divisible codes over $\mathbb{F}_q$ presented in this section, cf. Lemma 7.7, Lemma 7.10, and Lemma 7.14.*

Research problem

Conclude a general "quartic condition" from the linear programming method covering the parameters of Lemma 6.17.

7 Lengths of projective q^r-divisible codes

The aim of this section is to summarize the current knowledge on the possible lengths of projective q^r-divisible codes. Even for small parameters there are several lengths where the existence of a corresponding code still remains undecided. This leaves plenty of space for own research, i.e., new constructions, cf. Section 5, and more sophisticated techniques for non-existence proofs, cf. Section 6, are needed.

We will give brief proofs for our subsequent results. All of them are constructed in the same manner. On the constructive side we list some "base examples", i.e., examples for some small cardinalities/lengths. Specific parametric series are mentioned explicitly, for more details on the used two-weight codes we refer to Subsection 5.3 and Table 5.2, and for optimal linear codes we refer to Subsection 8.10 and Table 8.1. Explicit generator matrices obtained by computer searches are listed in Subsection 5.2. Without explicitly stating, we then invoke Lemma 5.1, i.e., we use the fact that the set of attainable lengths is closed under addition. For the non-existence results we list the utilized results from Section 6. In the statements we explicitly list those cardinalities/lengths where no non-existence results is mentioned and which are not implied by combinations of the base examples. Stating all details becomes a bit extensive when the parameters are not rather small. So, for a few medium sized parameters we only state the ranges of excluded cardinalities obtained via the methods outlined in Section 6, cf. Exercise 6.18. Here $[a,b]$ denotes the list of integers $a, a+1, \ldots, b$.

For the binary field the smallest open case is length 130 for projective 16-divisible codes. For $q = 3$ and $\Delta = 9$ the smallest open lengths are 70 and 77. If $q \ge 5$, then there are even open cases for projective q-divisible codes over $\mathbb{F}_q$, e.g., length 40 for $q = 5$.

Lemma 7.1. *Let $\mathcal{M}$ the a 2^1-divisible set of n points in* $\mathrm{PG}(v-1,2)$*, then $n \ge 3$ and all cases can be realized.*

Proof. The values $n \in \{1,2\}$ are excluded by Theorem 6.10. The *base examples* of cardinalities 3, 4, and 5 are given by Example 2.19, Example 2.25, and Exercise 5.6, respectively. □

Lemma 7.2. *Let $\mathcal{M}$ the a 2^2-divisible set of n points in* $\mathrm{PG}(v-1,2)$*, then $n \in \{7,8\}$ or $n \ge 14$ and all mentioned cases can be realized.*

Proof. The cases $1 \le n \le 6$ and $9 \le n \le 13$ are excluded by Theorem 6.11. *Base examples* for cardinalities 7 and 8 are given by Example 2.19 and Example 2.25. For the range $15 \le n \le 20$ we refer to Corollary 5.13. □

Lemma 7.3. *Let $\mathcal{M}$ the a 2^3-divisible set of n points in $\mathrm{PG}(v-1,2)$, then*

$$n \in \{15, 16, 30, 31, 32, 45, 46, 47, 48, 49, 50, 51\}$$

or $n \geq 60$ and all cases can be realized.

Proof. The cases $1 \leq n \leq 14$, $17 \leq n \leq 29$, and $33 \leq n \leq 44$ are excluded by Theorem 6.11. The case $n = 52$ is excluded by Corollary 6.15 with $t = 3$, see also Lemma 6.12. The cases $53 \leq n \leq 58$ are excluded by Lemma 6.5 using $m = 4$. The special case $n = 59$ is treated in [130].

Base examples for cardinalities 15, 16, and 49 are given by Example 2.19, Example 2.25, and Exercise 5.15, respectively. The range $63 \leq n \leq 72$ is covered by Corollary 5.13. There are two-weight codes for cardinalities $n \in \{51, 73\}$ and sporadic examples found by computer searches for cardinalities $n \in \{50, 74\}$. □

Lemma 7.4. *Let $\mathcal{M}$ the a 2^4-divisible set of n points in $\mathrm{PG}(v-1,2)$, then*

$$\begin{aligned} n \in\ & \{31, 32, 62, 63, 64, 93, \ldots, 96, 124, \ldots, 130, 155, \ldots, 165, 185, \ldots, 199, \\ & 215, \ldots, 234, 244, \ldots, 309\} \end{aligned}$$

or $n \geq 310$ and all cases, possibly except

$$\begin{aligned} n \in\ & \{130, 163, 164, 165, 185, 215, 216, 232, 233, 244, 245, 246, 247, \\ & 274, 275, 277, 278, 306, 309\}, \end{aligned}$$

can be realized.

Proof. The cases $1 \leq n \leq 30$, $33 \leq n \leq 61$, $65 \leq n \leq 92$, and $97 \leq n \leq 123$ are excluded by Theorem 6.11. The cases $133 \leq n \leq 154$, $167 \leq n \leq 184$, $201 \leq n \leq 214$, and $236 \leq n \leq 243$ are excluded by Lemma 6.5 using $m = 5$, $m = 6$, $m = 7$, and $m = 8$, respectively. The cases $n \in \{132, 166, 200, 235\}$ are excluded by Corollary 6.15 with $t = 4, \ldots, 7$, respectively. The special case $n = 131$ was treated in [155].

Base examples for cardinalities 31, 32, and 129 are given by Example 2.19, Example 2.25, and Exercise 5.15 respectively. The range $255 \leq n \leq 272$ is covered by Corollary 5.13. There are two-weight codes for cardinalities $n \in \{196, 198, 231, 234, 273, 276\}$. Additionally, we have a distance-optimal code for $n = 199$ and sporadic examples found by computer searches for cardinalities $n \in \{161, 162, 195, 197\}$. □

Research problem

Decide whether a projective 16-divisible binary linear code of length 130 exists.

Remark 7.5. *Due to Lemma 7.3 a projective 16-divisible binary linear code C of length 165 has codewords with weight at most 96. From the first three MacWilliams equations we conclude*

$$-5120A_{16} - 3072A_{32} - 1536A_{48} - 512A_{64} = 30 \cdot \left(256 - 2^{k-2}\right), \tag{7.1}$$

so that $k \geq 10$. For $k = 10$ we have $A_{16} = A_{32} = A_{48} = A_{64} = 0$, i.e., C is a projective two-weight code with weights 80 and 96. However, the residual code of a codeword of weight 96 is a $[69, 9, 32]_2$-code, which does not exist, see [36, Theorem 2].

Lemma 7.6. *Let $\mathcal{M}$ the a 2^5-divisible set of n points in* $\mathrm{PG}(v-1,2)$*, then*

$$n \in \{63,64,126,127,128,189,\dots,192,252,\dots,256,315,\dots,323,378,385,\dots,389,441,\dots,455,503,\dots,520,566,\dots,586,628,\dots,651,691,\dots,717,753,\dots,783,815,\dots,843,845,\dots,849,877,\dots,916,938,\dots,984\}$$

or $n \ge 998$ and all cases, possibly except

$$n \in \{322,323,385,\dots,389,449,\dots,454,503,513,\dots,517,520,566,577,\dots,580,584,\dots,586,628,629,641,642,648,\dots,651,691,692,705,712,\dots,717,753,\dots,755,776,\dots,779,781,\dots,783,815,\dots,818,840,841,842,846,\dots,849,877,\dots,881,904,905,911,\dots,916,938,\dots,944,968,976,\dots,984,998,\dots,1007,1057,\dots,1070,1121,\dots,1133,1185\},$$

can be realized.

Proof. The cases $1 \le n \le 62$, $65 \le n \le 125$, $129 \le n \le 188$, $193 \le n \le 251$, and $257 \le n \le 314$ are excluded by Theorem 6.11. The cases $326 \le n \le 377$, $391 \le n \le 440$, $457 \le n \le 502$, $522 \le n \le 565$, $588 \le n \le 627$, $653 \le n \le 690$, $719 \le n \le 752$, $785 \le n \le 814$, $851 \le n \le 876$, $918 \le n \le 937$, and $986 \le n \le 997$ are excluded by Lemma 6.5 using $m = 6,\dots,16$, respectively. The cases $n \in \{325,390,456,521,587,652,718,784,850,917,985\}$ are excluded by Corollary 6.15 with $t = 5,\dots,15$, respectively, cf. Exercise 6.16. The case $n = 324$ is excluded by Lemma 3.8 and Lemma 7.4.

Base examples for cardinalities 63, 64, and 321 are given by Example 2.19, Example 2.25, and Exercise 5.15, respectively. The range $1023 \le n \le 1056$ is covered by Corollary 5.13. There are two-weight codes for cardinalities $n \in \{455,780,845,975,1105,1170\}$. □

Lemma 7.7. *Let $\mathcal{M}$ the a 2^6-divisible set of n points in* $\mathrm{PG}(v-1,2)$*, then n is not contained in any of the intervals* $[1,126]$, $[129,253]$, $[257,380]$, $[385,507]$, $[513,634]$, $[641,761]$, $[772,888]$, $[902,1015]$, $[1032,1142]$, $[1161,1269]$, $[1291,1395]$, $[1420,1522]$, $[1549,1649]$, $[1678,1776]$, $[1808,1902]$, $[1937,2029]$, $[2066,2156]$, $[2196,2282]$, $[2325,2409]$, $[2455,2535]$, $[2585,2661]$, $[2714,2788]$, $[2844,2914]$, $[2974,3040]$, $[3104,3166]$, $[3234,3292]$, $[3364,3418]$, $[3495,3543]$, $[3626,3668]$, $[3757,3793]$, $[3889,3917]$*, and* $[4023,4039]$.

Lemma 7.8. *Let $\mathcal{M}$ the a 3^1-divisible set of n points in* $\mathrm{PG}(v-1,3)$*, then $n = 4$ or $n \ge 8$ and all cases can be realized.*

Proof. The values $1 \le n \le 3$ and $5 \le n \le 7$ are excluded by Theorem 6.10.

Base examples for cardinalities 4, 9, and 10 are given by Example 2.19, Example 2.25, and Example 5.4, respectively. Additionally, there exists a two-weight code of cardinality $n = 11$. □

Lemma 7.9. *Let $\mathcal{M}$ the a 3^2-divisible set of n points in* $\mathrm{PG}(v-1,3)$*, then*

$$n \in \{13,26,27,39,40,52,\dots,56,65,\dots,70,77,\dots,85,90,\dots,128\}$$

or $n \geq 129$ and all cases, possibly except

$$n \in \{70, 77, 99, 100, 101, 102, 113, 114, 115, 128\},$$

can be realized.

Proof. The cases $1 \leq n \leq 12$, $15 \leq n \leq 25$, $29 \leq n \leq 38$ and $43 \leq n \leq 51$ are excluded by Theorem 6.11. The case $57 \leq n \leq 64$, $72 \leq n \leq 76$, and $87 \leq n \leq 88$ are excluded by Lemma 6.5 using $m = 5, \dots, 7$, respectively. The cases $n \in \{71, 86\}$ are excluded by Corollary 6.15 with $t \in \{5, 6\}$, respectively. The case $n = 89$ is excluded in Lemma 6.17.

Base examples for cardinalities 13 and 27 are given by Example 2.19 and Example 2.25. There are two-weight codes for cardinalities $n \in \{55, 56, 84, 98\}$ and optimal codes for cardinalities $n \in \{85, 90, 127, 141\}$. $\square$

Lemma 7.10. *Let $\mathcal{M}$ the a 3^3-divisible set of n points in $\mathrm{PG}(v-1, 3)$, then n is not contained in any of the intervals* $[1, 39]$, $[41, 79]$, $[82, 119]$, $[122, 159]$, $[163, 199]$, $[203, 239]$, $[246, 279]$, $[287, 319]$, $[329, 359]$, $[370, 399]$, $[411, 439]$, $[452, 478]$, $[493, 518]$, $[535, 558]$, $[576, 597]$, $[618, 637]$, $[659, 676]$, $[701, 715]$, $[743, 754]$, *and* $[786, 793]$.

Lemma 7.11. *Let $\mathcal{M}$ the a 4^1-divisible set of n points in $\mathrm{PG}(v-1, 4)$, then*

$$n \in \{5, 10, 15, 16, 17\}$$

or $n \geq 20$ and all cases can be realized.

Proof. The values $1 \leq n \leq 4$, $6 \leq n \leq 9$, and $11 \leq n \leq 14$ are excluded by Theorem 6.10. The cases $n \in \{18, 19\}$ are excluded by Lemma 6.5 using $m = 4$.

Base examples for cardinalities 5, 16, and 17 are given by Example 2.19, Example 2.25, and Example 5.4, respectively. The cases $21 \leq n \leq 24$ are covered by Exercise 5.21. $\square$

Lemma 7.12. *Let $\mathcal{M}$ the a 4^2-divisible set of n points in $\mathrm{PG}(v-1, 4)$, then*

$$\begin{aligned} n \in \; & \{21, 42, 63, 64, 84, 85, 105, 106, 126, \dots, 129, 147, \dots 151, 168, \dots, 173, \\ & 189, \dots, 195, 210, \dots, 217, 231, \dots, 239, 251, \dots, 261, 272, \dots, 283, 293, \\ & \dots, 305, 313, \dots, 328\} \end{aligned}$$

or $n \geq 333$ and all cases, possibly except

$$\begin{aligned} n \in \; & \{129, 150, 151, 172, 173, 193, 194, 195, 215, 216, 217, 236, \dots, 239, 251, 258, \\ & 259, 261, 272, 279, 280, 282, 283, 293, 301, 305, 313, 314, 322, 326, 333, 334, \\ & 335\} \end{aligned}$$

can be realized.

Proof. The cases $1 \leq n \leq 20$, $22 \leq n \leq 41$, $44 \leq n \leq 62$, $66 \leq n \leq 83$, $87 \leq n \leq 104$, $109 \leq n \leq 125$, $131 \leq n \leq 146$, $153 \leq n \leq 167$, $174 \leq n \leq 188$, $196 \leq n \leq 209$, $218 \leq n \leq 230$, $240 \leq n \leq 250$, $262 \leq n \leq 271$, $284 \leq n \leq 292$, $306 \leq n \leq 312$, and $329 \leq$

$n \le 332$ are excluded by Lemma 6.5 using $m = 1, \ldots, 16$, respectively. The cases $n \in \{65, 130, 152\}$ are excluded by Corollary 6.15 with $t \in \{3, 6, 7\}$, respectively. Applying Corollary 6.3 with $m \in \{1, 4, 5\}$ gives the non-existence for $n \in \{43, 86, 107, 108\}$.

Base examples for cardinalities 21 and 64 are given by Example 2.19 and Example 2.25. Additionally, there exist two-weight codes with $n \in \{260, 303, 304\}$. For the sequence $n = 85 + 43 \cdot j$, where $0 \le j \le 17$, we refer to Corollary 5.14. □

Lemma 7.13. *Let $\mathcal{M}$ the a 5^1-divisible set of n points in $\mathrm{PG}(v-1, 5)$, then*

$$n \in \{6, 12, 18, 24, 25, 26, 30, 31, 32\}$$

or $n \ge 36$ and all cases, possibly except $n = 40$, can be realized.

Proof. The values $1 \le n \le 5$, $7 \le n \le 11$, $13 \le n \le 17$, and $19 \le n \le 23$ are excluded by Theorem 6.10. The cases $27 \le n \le 29$ and $34 \le n \le 35$ are excluded by Lemma 6.5 using $m = 5$ and $m = 6$, respectively. The case $n = 33$ is excluded by Corollary 6.15 with $t = 5$ respectively.

Base examples for cardinalities 6, 25, and 26 are given by Example 2.19, Example 2.25, and Example 5.4, respectively. Additionally, there exists a two-weight code of cardinality $n = 39$ and two sporadic examples found by computer searches for cardinalities $n \in \{41, 46\}$. □

Lemma 7.14. *Let $\mathcal{M}$ the a 5^2-divisible set of n points in $\mathrm{PG}(v-1, 5)$, then n is not contained in any of the intervals* $[1, 30]$, $[32, 61]$, $[63, 92]$, $[94, 123]$, $[126, 154]$, $[157, 185]$, $[188, 216]$, $[219, 247]$, $[252, 278]$, $[283, 309]$, $[316, 340]$, $[347, 371]$, $[379, 402]$, $[410, 433]$, $[442, 464]$, $[473, 495]$, $[505, 526]$, $[537, 557]$, $[568, 587]$, $[600, 618]$, $[632, 649]$, $[663, 680]$, $[695, 711]$, $[727, 742]$, $[758, 772]$, $[790, 803]$, $[822, 834]$, $[854, 864]$, $[886, 895]$, $[918, 925]$, *and* $[951, 955]$.

Research problem

Resolve one of the following open cardinalities of q-divisible sets in $\mathrm{PG}(v-1, q)$.

- $q = 7$: $\{75, 83, 91, 92, 95, 101, 102, 103, 109, 110, 111, 117, 118, 119, 125, 126, 127, 133, 134, 135, 142, 143, 151, 159, 167\}$;
- $q = 8$: $\{93, 102, 111, 120, 121, 134, 140, 143, 149, 150, 151, 152, 158, 159, 160, 161, 167, 168, 169, 170, 176, 177, 178, 179, 185, 186, 187, 188, 196, 197, 205, 206, 214, 215, 223, 224, 232, 233, 241, 242, 250, 251\}$;
- $q = 9$: $\{123, 133, 143, 153, 154, 175, 179, 185, 189, 195, 196, 199, 206, 207, 208, 209, 216, 217, 218, 219, 226, 227, 228, 229, 236, 237, 238, 239, 247, 248, 249, 257, 258, 259, 267, 268, 269, 277, 278, 279, 288, 289, 298, 299, 308, 309, 318, 319, 329, 339, 349, 359\}$.

8 Applications

In Theorem 3.2 we have seen that in order to study Δ-divisible codes it is sufficient to study q^r-divisible codes, where $r \in \mathbb{Q}$, $m \cdot r \in \mathbb{N}$, and $q = p^m$. The equivalence between q^r-divisible codes and q^r-divisible multisets of points in projective geometries have been discussed in Subsection 2.2. Besides that there are several relations to other combinatorial structures, which is the topic of this section. In the subsequent subsections we give brief descriptions and pointers to the literature, while we devote entire sections to the relations to partial spreads and vector space partitions, see sections 9 and 10. Our list is very far from being exhaustive. For applications in quantum computation we refer to [131]. The unique minimal[1] linear $[19, 5]_3$-code is 3-divisible [158]. The relation between minimality and divisibility is more extensively studied in [53].

8.1 Subspace codes

For two subspaces U and U' of $\operatorname{PG}(v-1, q)$ the *subspace distance* is given by $d_S(U, U') = \dim(U + U') - \dim(U \cap U')$. A set $\mathcal{C}$ of subspaces in $\operatorname{PG}(v-1, q)$, called codewords, with minimum subspace distance d is called a *subspace code*. Its maximal possible cardinality is denoted by $A_q(v, d)$, see e.g. [126]. If all codewords have the same dimension, say k, then we speak of a *constant dimension code* and denote the corresponding maximum possible cardinality by $A_q(v, d; k)$, see e.g. [77]. For known bounds, we refer to `http://subspacecodes.uni-bayreuth.de` [115] containing also the generalization to subspace codes of mixed dimension. For $2k \le v$ the cardinality $A_q(v, 2k; k)$ is the maximum size of a partial k-spread, see Section 9. For $d < 2k$ the recursive Johnson bound

$$A_q(v, d; k) \le \left\lfloor \begin{bmatrix} v \\ 1 \end{bmatrix}_q \cdot A_q(v-1, d; k-1) / \begin{bmatrix} k \\ 1 \end{bmatrix}_q \right\rfloor,$$

see [215], recurs on this situation. The involved rounding can be slightly sharpened using the non-existence of q^r-divisible multisets of a certain cardinality, see [140, Lemma 13] and Lemma 4.21:

$$A_q(v, d; k) \le \llbracket A_q(v-1, d; k-1) \cdot [v]_q / [k]_q \rrbracket_{q^{k-1}}. \tag{8.1}$$

For $d < 2k$ this gives the tightest known upper bound for $A_q(v, d; k)$ except $A_2(6, 4; 3) = 77 < 81$ [125] and $A_2(8, 6; 4) = 257 < 289$ [114]. For general subspace codes the underlying idea of the Johnson bound in combination with q^r-divisible multisets has been generalized in [129].

[1] A linear $[n, k]_q$-code C is called linear if there do not exist two non-zero codewords $c_1, c_2 \in C$ with $\operatorname{supp}(c_1) \subsetneq \operatorname{supp}(c_2)$. A major problem in this area is the determination of the minimum possible length $n = m_q(k)$ of a minimal $[n, k]_q$-code. We indeed have $m_3(5) = 19$.

8.2 Subspace packings and coverings

A constant-dimension code consisting of k-dimensional codewords in $\mathrm{PG}(v-1,q)$ has minimum subspace distance d iff each $(k-\frac{d}{2}+1)$-dimensional subspace is contained in at most one codeword. If we relax the condition a bit and require that for a multiset $\mathcal{U}$ of k-spaces each $(k-\frac{d}{2}+1)$-dimensional subspace is contained in at most λ codewords, then we have the definition of a *subspace packing*. Of course, similar to constant-dimension codes, q^r-divisible multisets can be used to obtain upper bounds on the cardinality of a subspace packing, see [75, 76]. Indeed, [140, Lemma 13] and Lemma 4.21 cover that case, i.e.,

$$A_q^{\lambda}(v,d;k) \le \llbracket A_q^{\lambda}(v-1,d;k-1)[v]_q/[k]_q \rrbracket_{q^{k-1}} \tag{8.2}$$

for $k \ge 2$, where $A_q^{\lambda}(v,d;k)$ denotes the maximum cardinality of a multiset $\mathcal{U}$ of k-spaces in $\mathrm{PG}(v-1,q)$ such that each $(k-\frac{d}{2}+1)$-dimensional subspace is covered at most λ times.

If we replace "contained in at most λ codewords" by "contained in at least λ codewords" we obtain so-called *subspace coverings*. For the special case of $\lambda = 1$ we refer e.g. to [74, 78]. Again, [140, Lemma 13] and Lemma 4.21 cover this situation and relate it to q^r-divisible multisets, i.e.,

$$B_q^{\lambda}(v,d;k) \ge \llbracket B_q^{\lambda}(v-1,d;k-1)[v]_q/[k]_q \rrbracket_{q^{k-1}} \tag{8.3}$$

for $k \ge 2$, where $B_q^{\lambda}(v,d;k)$ denotes the minimum cardinality of a multiset $\mathcal{U}$ of k-spaces in $\mathrm{PG}(v-1,q)$ such that each $(k-\frac{d}{2}+1)$-dimensional subspace is covered at least λ times.

8.3 Orthogonal arrays

A $t-(v,k,\lambda)$ orthogonal array, where $t \le k$, is a $\lambda v^t \times k$ array whose entries are chosen from a set X with v points such that in every subset of t columns of the array, every t-tuple of points of X appears in exactly λ rows. Here, t is called the strength of the orthogonal array. For a survey see e.g. [103]. A library of orthogonal arrays can be found at `http://neilsloane.com/oadir/`. A variant of the linear programming method for orthogonal arrays with mixed levels was presented in [199], see also [23]. Orthogonal arrays can be regarded as natural generalizations of orthogonal Latin squares[137], cf. [32]. Linear orthogonal arrays are ultimately linked to linear codes, see e.g. [103, Section 4.3], via:

Theorem 8.1. *Suppose that C is an $[n,k]_q$-code. Then $\mathrm{d}_H(C) \ge d$ iff $C^{\perp}$ is a linear $\mathrm{OA}_{\lambda}(d-1,n,q)$, where $\lambda = q^{n-k-d+1}$.*

8.4 (s,r,μ)-nets

Definition 8.2. *([67, Definition 2])*
Let J be an incidence structure. Define $B \parallel G$ for blocks B, G of J to mean that either

$B = G$ or $[B,G] = 0$. Then J is called an (s,r,μ)-net provided:

(i) $||$ is a parallelism;

(ii) $G \not\parallel H$ implies $[G,H] = \mu$;

(iii) there is at least one point, some parallel class has $s \geq 2$ blocks, and there are $r \geq 3$ parallel classes.

We note that the existence of an (s,r,μ)-net is equivalent to the existence of an orthogonal array of strength two, see Subsection 8.3. From partial spreads (s,r,μ)-nets can be constructed, see [67]. Additionally, there is a connection between 3-nets and Latin squares, see e.g. [137, Section 8.1].

Nets can be seen as a relaxation of a finite projective plane, see e.g. [186]. For the famous existence question of finite projective planes of small order we refer to [161, 187].

8.5 Minihypers

An $(f,m;v,q)$-*minihyper* is a pair (F,w), where F is a subset of the point set of $\mathrm{PG}(v-1,q)$ and w is a weight function $w\colon\ \mathrm{PG}(v-1,q) \to \mathbb{N}$, $x \mapsto w(x)$, satisfying

(1) $w(x) > 0 \Longrightarrow x \in F$,

(2) $\sum_{x\in F} w(x) = f$, and

(3) $\min\{\sum\limits_{x\in H} w(x) \mid H \in \mathcal{H}\} = m$, where $\mathcal{H}$ is the set of hyperplanes of $\mathrm{PG}(v-1,q)$.

We also say that a multiset $\mathcal{M}$ of points $\mathcal{M}$ in $\mathrm{PG}(v-1,q)$ is an (f,m)-*minihyper* if $\#\mathcal{M} = f$, $\mathcal{M}(H) \geq m$ for all $H \in \mathcal{H}$, and $\min\{\mathcal{M}(H) : H \in \mathcal{H}\} = m$. For a positive integer e (and field size q) write $f = \sum_{i=1}^{e} f_i[i]_q$ where $f_e = \lfloor f/[e]_q \rfloor$ and $f_j = \left\lfloor \left(f - \sum_{i=j+1}^{e} f_i[i]_q\right)/[j]_q \right\rfloor$ for $j = e-1,\ldots,1$. By $[f_e,\ldots,f_1]$ we denote the $[e]_q$-*expansion* of f. The expansion has the properties $f_e \geq 0$ and $0 \leq f_j \leq q$ for $1 \leq j \leq e-1$. Moreover, $f_j = q$ for some $1 \leq j < e$ implies $f_i = 0$ for all $1 \leq i < j$. With this, the mapping $f \mapsto [f_e,\ldots,f_1]$ is a bijection from $\mathbb{N}$ onto the set of e-element lists with the mentioned properties.

The Hamada bound

Theorem 8.3. *Let $\mathcal{M}$ be an (f,m)-minihyper in $\mathrm{PG}(v,q)$ and the $[v-1]_q$-expansion of f be $[f_{v-1},\ldots,f_1]$. Then,*

$$f \geq [f_{v-1},\ldots,f_1,0] = qf + \sum_{i=1}^{v-1} f_i. \tag{8.4}$$

For a proof we refer e.g. to [212, Theorem 4.1]. We remark that there exist several variants of this result where e.g. $\gamma_0(\mathcal{M}) = 1$ or $0 \leq f_i \leq q-1$ is assumed. The latter assumption also allows to drop the parameter e from the expansion. For a survey on minihypers we refer to [101].

An distinguished class of minihypers is given by $(x[t]_q, x[t-1]_q)$-minihypers in $\mathrm{PG}(t,q)$, see the discussion after Exercise 3.25. Here we have rather strong divisibility properties:

Theorem 8.4. *[165, Theorem 3.1] Let $\mathcal{M}$ be an $(x[t]_q, x[t-1]_q)$-minihyper in $\mathrm{PG}(t,q)$, where $x \le q - p^g$ for some non-negative integer g and the characteristic p of $\mathbb{F}_q$. Then, $\mathcal{M}$ is $p^{g+1}q^{t-2}$-divisible.*

For the other direction we consider $\mathcal{M} = \chi_E + 3\cdot\chi_L + 9\cdot\chi_P$ in $\mathrm{PG}(v-1,3)$ for a point P, a line L, and a plane E, where $v \ge 3$ and $P \le L \le E$. We can easily check that $\mathcal{M}$ is 9-divisible and has $[3]_3$-expansion $[2,2,0]$. However, $\mathcal{M}$ is not a $(34,10)$-minihyper, as one might hope in view of Theorem 8.3, but only a $(34,7)$-minihyper.

Exercise 8.5. *Let $\mathcal{M}$ be a q^t-divisible multiset of points of cardinality $x[t+1]_q$ in $\mathrm{PG}(v-1,q)$ with $1 \le t < v$. Show that $\mathcal{M}$ is an $(x[t+1]_q, x[t]_q)$-minihyper if $x \le q-1$. Assuming $\gamma_0(\mathcal{M}) = 1$, show that $\mathcal{M}$ is an $(x[t+1]_q, x[t]_q)$-minihyper if $x \le t(q-1)$. (Hint: Use Theorem 6.10.)*

Example 8.6. *Let K be a $(t+1)$-space in $\mathrm{PG}(v-1,q)$ with $v \ge t+1$, where $t \ge 2$. For a point $P \le K$ the set of points $\chi_K - \chi_P$ is an $(q[t]_q, q[t-1]_q)$-minihyper that is not q-divisible. If $S \le K$ is an s-space with $2 \le s \le t$ and $S' \le S$ is an $(s-1)$-space, then $\mathcal{M} := \chi_K - \chi_S + q\cdot\chi_{S'}$ is a $(q[t]_q, q[t-1]_q)$-minihyper that is not q^s-divisible.*

Exercise 8.7. *Show that each $(2[t]_2, 2[t-1]_2)$-minihyper in $\mathrm{PG}(v-1,2)$ is either a union of two t-spaces or covered by the construction in Exercise 8.6.*

Minihypers have e.g. been used to prove extendability results for partial spreads, see e.g. [80, 90, 91] and Section 12. If $\mathcal{P}$ is the set of holes of a partial k-spread, then the partial spread is extendible iff $\mathcal{P}$ contains all points of a k-dimensional subspace. As an example, in [128] the possible hole configurations of partial 3-spreads in $\mathrm{PG}(6,2)$ of cardinality 15 were classified. In four cases the partial spread is extensible and in one case it is not, cf. Example 5.9.

A close relation between divisible sets and minihypers can be found in [163]. To this end an (n,w)-arc in $\mathrm{PG}(k-1,q)$ is called t-quasidivisible iff every hyperplane has a multiplicity congruent to $n+i \pmod q$, where $i \in \{0,1,...,t\}$. With this, every t-quasidivisible arc associated with a linear code meeting the Griesmer bound, and satisfying an additional numerical condition, is t times extendable. For more papers using minihypers to study codes meeting the Griesmer bound see e.g. [101, 122].

Research problem

Can some results obtained using minihypers be improved by using the properties of divisible codes?

The use of classification results for projective q^{k-1}-divisible codes, see e.g. Section 11, for extendability results for partial k-spreads can be generalized to extendability results for constant-dimension codes, see [179] and Section 12.

8.6 Few-weight codes

A linear $[n,k]_q$ code C is called an *s-weight code* if the non-zero codewords of C attain (at most) s possible weights. For $s=1$ repetitions of simplex codes give the only examples. The case $s=2$ is discussed in Subsection 5.3. For projective two-weight codes there is a strong relation to q^r-divisible codes, see Lemma 5.22. While we do not have such a strong relation for $s\ge 3$, it turns out that many examples of codes with relatively few weights are q^r-divisible, where r is relatively large. For some literature on three-weight codes, see e.g. [46, 62, 119, 141, 205, 216, 218]. Few-weight codes, i.e., s-weight codes with $s\ge 4$ but s still being relatively small, are e.g. treated in [102, 166, 214].

8.7 k-dimensional dual hyperovals

A set $\mathcal{K}$ of k-spaces in $\mathrm{PG}(v-1,q)$ with $\#\mathcal{K}\ge 3$ such that

- $\dim(X\cap Y)=1$ for any distinct $X,Y\in\mathcal{K}$;
- $\dim(X\cap Y\cap Z)=0$ for any distinct $X,Y,Z\in\mathcal{K}$; and
- the points in the elements of $\mathcal{K}$ generate $\mathrm{PG}(v-1,q)$

is called *k-dimensional dual arc* (in $\mathrm{PG}(v-1,q)$). The associated multiset $\mathcal{M}$ of points is q^{k-1}-divisible. Each point of an arbitrary element $X\in\mathcal{K}$ is contained in at most one further element of $\mathcal{K}$ so that $\#\mathcal{K}\le [k]_q+1$, see e.g. [217, Lemma 2.2]. In the case of equality $\mathcal{K}$ is called *(k-dimensional) dual hyperoval.* Here we have $\mathcal{M}(P)\in\{0,2\}$ for all points $P\in\mathcal{P}$, so that $\frac{1}{2}\mathcal{M}$ is a q^{k-1}-divisible set of $\frac{1}{2}[k]_q([k]_q+1)$ points if q is odd, which we assume in the following. Note that the number of elements of $\mathcal{K}$ that are contained in a given hyperplane H has to be even, so that we can conclude $v\ge 2k$. For $v=2k$ it was shown in [58, Proposition 2.11] that each hyperplane contains either 0 or $2k-2$ elements from $\mathcal{K}$, i.e., $\frac{1}{2}\mathcal{M}$ corresponds to a two-weight code, cf. Example SU2 in [47]. If $\frac{1}{2}\mathcal{M}$ can be the set of double points of a k-dimensional dual hyperoval seems to be a rather hard question. A few more necessary conditions are known, see e.g. [58, 217].

8.8 q-analogs of group divisible designs

Let K and G be sets of positive integers. A *q-analog of a group divisible design* of index λ and order v is a triple $(\mathcal{V},\mathcal{G},\mathcal{B})$, where

- $\mathcal{V}$ is a vector space over $\mathbb{F}_q$ of dimension v,
- $\mathcal{G}$ is a vector space partition whose dimensions lie in G, and
- $\mathcal{B}$ is a family of subspaces (blocks) of $\mathcal{V}$,

that satisfies

1. $\#\mathcal{G}>1$,

2. if $B \in \mathcal{B}$ then $\dim B \in K$,

3. every 2-dimensional subspace of $\mathcal{V}$ occurs in exactly λ blocks or one group, but not both.

This notion was introduced in [44] and generalizes the classical definition of a group divisible design in the set case, see e.g. [41]. If $K = \{k\}$ and $G = \{g\}$, then we speak of a $(v, g, k, \lambda)_q$-GDD. All necessary existence conditions of the set case can be easily transferred to the q-analog case. Moreover, there is an additional necessary existence condition whose proof is based on q^r-divisible multisets:

Lemma 8.8. *([44, Lemma 5]) Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a $(v, g, k, \lambda)_q$-GDD and $2 \le g \le k$, then q^{k-g} divides λ.*

Note that in the set case the divisibility by 1^{k-g} is trivially satisfied.

8.9 Codes of nodal surfaces

In algebraic geometry, a nodal surface is a surface in a (usually complex) projective space whose only singularities are nodes, i.e., a very simple type of a singularity. A major problem about them is to find the maximum number of nodes of a nodal surface of given degree. In [14] to each such nodal surface is assigned a linear code with a certain divisibility and the problem was solved for quintic surfaces. Using the link to linear codes it was shown in [134] that a sextic surface can have at most 65 nodes. In [188, Theorem 5.5.9] a unique irreducible 3-parameter family of 65-nodal sextics, containing the famous Barth sextic [10], was determined. The uniqueness of the associated 8-divisible binary linear code was established in [150]. In general, the binary codes associated to nodal surfaces are either doubly-even or triply even, depending on whether the degree of the surface is odd or even, see [49].

For another type of singularities, so-called cusps, we end up with 3-divisible codes over $\mathbb{F}_3$, see e.g. [11].

8.10 Distance-optimal codes

Given a field size q the possible parameters n, k, and d of an $[n, k, d]_q$-code allow different optimizations, i.e., we can fix two parameters and optimize the third. The codes attaining the maximum possible value for the minimum distance d, given length n and dimension k, are called *distance-optimal* codes. Among the distance-optimal codes, there are quite some q^r-divisible codes with a relatively large value of r. E.g. all ten "base examples" used in the proof of Lemma 7.9 are distance-optimal. This phenomenon can partially be explained by our search technique screening the lists of available optimal linear codes and checking them for divisibility. Our sources were `http://www.codetables.de` maintained by Markus Grassl, `http://mint.sbg.ac.at` maintained at the university of Salzburg, and the database of *best known linear codes* implemented in `Magma`. In Table 8.1 we list the parameters and references of those cases that appear as "base examples" in the proofs of Section 7, but are not two-weight codes, see Subsection 5.3, or have an explicit construction In Section 5. We remark that there are way more possible lengths

n	k	d	Δ	q	reference
199	11	96	8	2	BCH code extended with a parity check bit [68]
85	7	54	9	3	[22]
90	8	54	9	3	[176]
127	7	81	9	3	[92]
141	7	90	9	3	[92]

Table 8.1: Parameters of a few selected distance-optimal codes.

of distance-optimal projective q^r-divisible linear codes. However, in many cases the corresponding lengths can be obtained as the sum of smaller base examples. Note that it is unknown whether $[90, 8, 55]_3$- or $[90, 8, 56]_3$-codes exist.

For some cases it can be shown that distance-optimal codes have to admit a certain divisibility. To this end we have to mention the *Griesmer bound*, see [93], stating that each $[n, k, \geq d]_q$-code C satisfies

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil . \tag{8.5}$$

Code attaining Inequality (8.5) with equality are called *Griesmer codes* or codes meeting the Griesmer bound. Those codes have a high divisibility, at least if the field size is a prime:

Theorem 8.9. *([209, Theorem 1]) Let C be an $[n, k, d]_p$-code, where p is a prime, meeting the Griesmer bound. If p^e divides d, where $e \in \mathbb{N}$, then C is p^e-divisible.*

Similar results also hold for distance-optimal non-Griesmer codes, see e.g. [6]. An interesting example is given by the $[46, 9, 20]_2$-code found in [157]. It is optimal, unique, and does not have any non-trivial automorphism. So, heuristic searches prescribing automorphisms had to be unsuccessful for this example. Like prescribing automorphisms, prescribing Δ-divisibility might help to reduce search spaces to a more manageable size while still permitting solutions.

Research problem

Try to improve the best known lower bounds for distance-optimal codes for a few parameters by assuming q^r-divisibility for the largest possible r so that the minimum distance d is divisible by q^r.

9 Partial spreads

A *partial t-spread* $\mathcal{T}$ in $\operatorname{PG}(v-1,q)$ is a set of t-dimensional subspaces such that the points of $\operatorname{PG}(v-1,q)$ are covered at most once.[1] In other words, the non-zero vectors in $\mathbb{F}_q^v$ are covered at most once by the non-zero vectors of the t-dimensional subspaces, i.e., the elements of the partial t-spread. Using the notion of vector space partitions, see Section 10, a partial t-spread is a vector space partition of type $t^{m_t}1^{m_1}$. The m_1 uncovered points are also called *holes*. By $A_q(v,2t;t)$ we denote the maximum value of m_t.[2]

If we replace the elements of a partial t-spread by their $[t]_q$ points, we obtain a set of points in $\operatorname{PG}(v-1,q)$ with cardinality at most $[v]_q$, so that

$$\#\mathcal{T} \le A_q(v,2t;t) \le \left\lfloor \frac{[v]_q}{[t]_q} \right\rfloor . \tag{9.1}$$

Observe that $[v]_q$ is divisible by $[t]_q$ iff v is divisible by t. If $\mathcal{T}$ is a partial t-spread in $\operatorname{PG}(v-1,q)$ attaining Inequality (9.1) with equality, then we speak of a *t-spread*. Those perfect packings of the points indeed exist for all positive integers t and v where t divides v. To this end we can consider the set of all points in $\operatorname{PG}\big(v/t-1,q^t\big)$ and concatenate the corresponding linear codes with a t-dimensional simplex code over $\mathbb{F}_q$, see Subsection 5.1 for more details and e.g. [127, Example 1] for a concrete example. The $[v/t]_{q^t} = [v]_q/[t]_q$ points in $\operatorname{PG}(v/t-1,q^t)$ and the corresponding t-dimensional simplex codes form the spread elements. Spreads arising by the sketched construction are also called *Desarguesian spreads*.

In order to construct large partial t-spreads we need:

Lemma 9.1. *([17],[89, Lemma 1.3] If π_a is an a-space in $\operatorname{PG}(a+b-1,q)$, where $a \ge b \ge 1$, then it is possible to partition the points of $\operatorname{PG}(a+b-1,q)\backslash\pi_a$ by a set of q^a b-spaces.*

Proof. Embed $\operatorname{PG}(a+b-1,q)$ in $\operatorname{PG}(2a-1,q)$ and take an a-spread $\mathcal{S}$ in $\operatorname{PG}(2a-1,q)$ containing π_a. The elements of $\mathcal{S}\backslash\{\pi_a\}$ intersect $\operatorname{PG}(a+b-1,q)$ in a b-spread of $\operatorname{PG}(a+b-1,q)\backslash\pi_a$. □

If $v \ge 2t$, then by choosing $a = v-t$ and $b = t$ we can recursively construct partial t-spreads using Lemma 9.1. If $t \le v < 2t$, then we can choose an arbitrary t-space. Note that we end up with t-spreads if v is divisible by t. Otherwise we have:

[1] Note that we use the algebraic dimension, while authors in papers with a geometric background speak of partial $(t-1)$-spreads.

[2] The more general notation $A_q(v,2t-2w;t)$ denotes the maximum cardinality of a collection of t-dimensional subspaces, whose pairwise intersections have a dimension of at most w, see e.g. Subsection 8.1.

Proposition 9.2. *([17]) If $v = tk + s$, where $t \ge 2$ and $1 \le s \le t-1$, then we have*

$$A_q(v, 2t; t) \ge 1 + \sum_{i=1}^{k-1} q^{v-it} = 1 + \sum_{i=1}^{k-1} q^{it+s} = \frac{q^v - q^{t+s}}{q^t - 1} + 1. \tag{9.2}$$

The same lower bound can be also obtained from the Echelon–Ferrers construction.

In [17, Theorem 4.1], see also [124] for $q = 2$, it was shown that the lower bound in Inequality (9.2) is attained with equality if $s = 1$. In his original proof Beutelspacher considered the set of holes N and the average number of holes per hyperplane, which is less than the total number of holes divided by q. An important insight was the relation $\#N \equiv \#(H \cap N) \pmod{q^{k-1}}$ for each hyperplane $H \in \mathcal{H}$. In [152, Corollary 2.6] the case $s = 2$ was completely resolved for $q = 2$. The original proof is based on a case analysis on possible vector space partitions in subspaces of codimension 2. In [156] it was observed that it suffices to study the number of holes in subspaces of codimension 2. A major breakthrough was obtained by Năstase and Sissokho:

Theorem 9.3. *([181, Theorem 5]) If $v = kt + s$ with $0 < s < t$ and $t > [s]_q$, then $A_q(v, 2t; t) = \frac{q^n - q^{t+r}}{q^t - 1} + 1$, i.e., Inequality (9.2) is tight.*

Ignoring the technical details one might say that a main idea was the study of the number of holes in subspaces of larger codimenson by a clever inductive approach. All these results where obtained without using the notion of q^r-divisible (multi-)sets of points in $\mathrm{PG}(v-1, q)$. In retro perspective there is now an easy explanation. The set of holes of a partial t-spread in $\mathrm{PG}(v-1, q)$ is q^{t-1}-divisible, see Lemma 3.12 and Lemma 4.16. As shown in Section 4 the easy averaging argument used by Beutelspacher and the inheritance of divisibility properties to subspaces is sufficient to completely characterize the possible cardinalities of q^{t-1}-divisible multisets over $\mathbb{F}_q$, see Theorem 4.6. The property that the set of holes actually is a set and not just a multiset is not necessary and indeed Example 4.24 slightly generalizes Theorem 9.3 to a wider context and gives a proof that is reduced to a single technical computation.

Additionally using the set property, i.e., considering possible cardinalities of q^{t-1}-divisible sets of points over $\mathbb{F}_q$, allows to explain another classical result from a different point of view. For a long time the best upper bound for partial spreads was given by Drake and Freeman:

Theorem 9.4. *([67, Corollary 8]) If $v = kt + s$ with $0 < s < t$, then*

$$A_q(v, 2t; t) \le \sum_{i=0}^{k-1} q^{it+s} - \lfloor \theta \rfloor - 1 = q^s \cdot \frac{q^{kt} - 1}{q^t - 1} - \lfloor \theta \rfloor - 1 = \frac{q^v - q^s}{q^t - 1} - \lfloor \theta \rfloor - 1,$$

where $2\theta = \sqrt{1 + 4q^t(q^t - q^s)} - (2q^t - 2q^s + 1)$.

Example 9.5. *If we apply Theorem 9.4 with $q = 5$, $v = 16$, $t = 6$, and $s = 4$, we obtain $\theta \approx 308.81090$ and $A_5(16, 12; 6) \le 9765941$.*

The proof of Theorem 9.4 is based on the work of Bose and Bush [32] and uses nets, see Subsection 8.4, as crucial objects. A quadratic polynomial plays an important role. Starting from the "quadratic condition" in Lemma 6.5 and its implication in Corollary 6.7.(a) we consider the condition $\tau_q(u, \Delta, m) < 0$, where $\Delta = q^{t-1}$ is the divisibility, u depends linearly on the cardinality of the set of holes, and m is a free parameter. Noting that $\tau_q(u, \Delta, m)$ is a quadratic polynomial in m, we can minimize $\tau_q(u, \Delta, m)$ in order to obtain a suitable choice for m. Instead of the total number of holes we can also consider the number of holes in a suitable subspace, which gives us a parameter y as a degree of freedom. Referring to [127] or [156] for details and explanations, we state:

Theorem 9.6. *([127, Theorem 10],[156, Theorem 2.10]) For integers $s \ge 1$, $k \ge 2$, $y \ge \max\{s, 2\}$, $z \ge 0$ with $\lambda = q^y$, $y \le t$, $t = [s]_q + 1 - z > s$, $v = kt + s$, and $l = \frac{q^{v-t}-q^s}{q^t-1}$, we have*

$$A_q(v, 2t; t) \le lq^t + \left\lceil \lambda - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\lambda\left(\lambda - (z + y - 1)(q - 1) - 1\right)} \right\rceil. \tag{9.3}$$

Using Theorem 9.6 with $q = 5$, $t = 6$, $v = 15$, $s = 3$, $z = 17$, and $y = 5$ gives $A_5(15, 12; 6) \le 1953186$. Choosing $y = t$ we obtain Theorem 9.4. Theorem 9.6 also covers [180, Theorems 6,7] and yields improvements in a few instances, e.g. $A_3(15, 12; 6) \le 19695$. Compared to Theorem 9.4 we have e.g. improvements from $A_2(15, 12; 6) \le 516$, $A_2(17, 14; 7) \le 1028$, and $A_9(18, 16; 8) \le 3486784442$ to $A_2(15, 12; 6) \le 515$, $A_2(17, 14; 7) \le 1026$, and $A_9(18, 16; 8) \le 3486784420$, respectively.

Complementing Theorem 9.3 for smaller values of t there is another parametric upper bound:

Theorem 9.7. *([127, Corollary 7]) For integers $s \ge 1$, $k \ge 2$, and $u, z \ge 0$ with $t = [s]_q + 1 - z + u > s$ we have $A_q(v, 2t; t) \le lq^t + 1 + z(q - 1)$, where $l = \frac{q^{v-t}-q^s}{q^t-1}$ and $v = kt + s$.*

Choosing $z = 0$ implies Theorem 9.3.

While explicit parametric upper bounds like Theorem 9.6 and Theorem 9.7 are certainly useful, they are in principle implied by the following observation:

Projective q^{t-1}-divisible codes of length

$$n = [v]_q - A_q(v, 2t; t) \cdot [t]_q$$

and dimension at most v do exist over $\mathbb{F}_q$.

As a refinement of the sharpened rounding from Definition 4.18 we introduce:

Definition 9.8. *For $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$ let $\llbracket a/b \rrbracket_{q^r,\lambda}$ be the maximal $n \in \mathbb{Z}$ such that there exists a q^r-divisible multisets of points in $\mathrm{PG}(v-1, q)$ for suitably large v with maximum point multiplicity at most λ and cardinality $a - nb$. If no such multiset exists for any n, we set $\llbracket a/b \rrbracket_{q^r,\lambda} = -\infty$.*

With this our observation can be reformulated as:

Lemma 9.9. *Let $\mathcal{U}$ be a set of k-spaces in* $\mathrm{PG}(v-1,q)$*, where* $1 \le k \le v$*, with pairwise trivial intersection. Then, we have*

$$\#\mathcal{U} \le \llbracket [v]_q/[k]_q \rrbracket_{q^{k-1};1}. \tag{9.4}$$

Δ	n	bounds	reference
2^3	52	$129 \le A_2(11,8;4) \le 132$	Corollary 6.15 with $t=3$
2^4	131	$257 \le A_2(13,10;5) \le 259$	[155]
2^4	200	$1025 \le A_2(16,12;6) \le 1032$	Corollary 6.15 with $t=6$
2^5	850	$2049 \le A_2(17,12;6) \le 2066$	Corollary 6.15 with $t=13$
3^3	493	$2188 \le A_3(11,8;4) \le 2201$	Corollary 6.15 with $t=12$
3^4	1586	$6562 \le A_3(13,10;5) \le 6574$	Corollary 6.15 with $t=13$
3^4	4396	$19684 \le A_3(14,10;5) \le 19727$	Corollary 6.15 with $t=36$
3^5	14236	$59050 \le A_3(16,12;6) \le 59090$	Corollary 6.15 with $t=39$
3^5	39797	$177148 \le A_3(17,12;6) \le 177280$	Corollary 6.15 with $t=109$
3^6	43760	$177148 \le A_3(18,14;7) \le 177187$	Corollary 6.15 with $t=40$
4^4	10592	$65537 \le A_4(13,10;5) \le 65568$	Corollary 6.15 with $t=31$
4^4	10250	$262145 \le A_4(15,12;6) \le 262174$	Corollary 6.15 with $t=30$
4^5	648716	$4194305 \le A_4(17,12;6) \le 4194852$	Corollary 6.15 with $t=475$
4^6	693632	$4194305 \le A_4(18,14;7) \le 4194432$	Corollary 6.15 with $t=127$
5^1	33	$78126 \le A_5(12,10;5) \le 78132$	Corollary 6.15 with $t=5$
5^4	230551	$1953126 \le A_5(14,10;5) \le 1953454$	Corollary 6.15 with $t=295$
7^4	3232754	$40353608 \le A_7(14,10;5) \le 40354853$	Corollary 6.15 with $t=1154$
8^3	144568	$2097153 \le A_8(11,8;4) \le 2097416$	Corollary 6.15 with $t=247$
8^2	1759	$2097153 \le A_8(12,10;5) \le 2097177$	Corollary 6.15 with $t=24$
8^2	1539	$16777217 \le A_8(14,12;6) \le 16777237$	Corollary 6.15 with $t=21$
9^2	3559	$59050 \le A_9(8,6;3) \le 59090$	Corollary 6.15 with $t=39$
9^4	2679394	$43046722 \le A_9(13,10;5) \le 43047086$	Corollary 6.15 with $t=363$

Table 9.1: Sporadic upper bounds for partial spreads

The construction from Proposition 9.2 and the non-existence of a 8-divisible set of 52 points over $\mathbb{F}_2$, see e.g. Lemma 6.12, imply

$$2^4 \cdot \frac{2^{4k-1}-2^3}{2^4-1} + 1 \le A_2(4k+3,8;4) \le 2^4 \cdot \frac{2^{4k-1}-2^3}{2^4-1} + 4 \tag{9.5}$$

for all $k \ge 2$. In general lower and upper bounds, if obtained by non-existence results of projective q^{t-1}-divisible codes, for $A_q(v,2t;t)$ come in parametric series for $v = kt+s$ with $k \in \mathbb{N}_{\ge 2}$, see [127] for details or [156] for more examples. In Table 9.1 we list the known upper (and corresponding lower) bounds that do not follow from Theorem 9.6 or Theorem 9.7 directly. Here Δ is the divisibility constant and n the cardinality of the non-existent set of points that leads to the stated upper bound for a partial spread. When we mention the application of Corollary 6.15 as reference, then typically also the "linear" and the "quadratic" condition introduced in Section 6 are involved, see Example 9.10 for exemplarily details. As a measurement for our state of knowledge we also state the corresponding lower bound for the partial spread obtained via Proposition 9.2.

Example 9.10. *In order to show the upper bound* $A_8(12, 10; 5) \le 2097177$*, we actually have to show the non-existence of a* 8^4*-divisible set of* 177887 *points over* $\mathbb{F}_8$*. Assuming its existence, Lemma 3.8 yields the existence of a* 8^3*-divisible set of points over* $\mathbb{F}_8$ *with a cardinality contained in* $\{18143 - i \cdot 8^4 \,:\, i \in \mathbb{N}_0\} \cap \mathbb{N}_0 = \{18143, 14047, 9951, 5855, 1759\}$*. Since* 8^3*-divisible sets of* 8^4 *points over* $\mathbb{F}_8$ *exist, it suffices to exclude the existence of a* 8^3*-divisible set of* 18143 *points over* $\mathbb{F}_8$*. Assuming its existence, Lemma 3.8 yields the existence of a* 8^2*-divisible set of points over* $\mathbb{F}_8$ *with a cardinality contained in* $\{1759 - i \cdot 8^3 \,:\, i \in \mathbb{N}_0\} \cap \mathbb{N}_0 = \{1759, 1247, 735, 223\}$*. Again it is sufficient to exclude cardinality* 1759*. As mentioned in Table 9.1 we can apply Corollary 6.15 with* $t = 24$ *in order to conclude the non-existence of a* 8^2*-divisible set of* 1759 *points over* $\mathbb{F}_8$*.*

Exercise 9.11. *Verify all details leading to the upper bounds* $A_8(14, 12; 6) \le 16777237$ *and* $A_5(12, 10; 5) \le 78132$*.*

A few more remarks on Lemma 9.9 and Table 9.1 are in order. So far it occurs that upper bounds for partial spreads that are based on non-existence results of projective q^{t-1}-divisible codes certified by the linear programming method using the first three MacWilliams equations only, can be obtained via Theorem 9.6 or Theorem 9.7 directly. However, this is not a proven fact at all. The explicit parametric conditions introduced in Section 6 are quite handy for automatic computations cf. Exercise 6.18. Since the numbers grow very quickly and the linear programming method reveals its full power only if applied recursively, efficient algorithms are indeed an issue. For Table 9.1 we remark that we have applied the mentioned tools for $v \le 19$ if $q \le 4$, for $v \le 16$ if $q = 5$, and for $v \le 14$ if $7 \le q \le 9$ only.

q	Δ	n	bounds
2	2^4	232, 263	$513 \le A_2(14, 10; 5) \le 521$
2	2^5	322, 385	$513 \le A_2(15, 12; 6) \le 515$
2	2^5	913, 976	$2049 \le A_2(17, 13; 6) \le 2066$
3	3^3	244	$730 \le A_3(10, 8; 4) \le 732$

Table 9.2: Open cases with implications for partial spread bounds

It should be mentioned that the existence of a projective q^{t-1}-divisible code C over $\mathbb{F}_q$ (of suitable length) does not imply the existence of a partial spread with matching parameters. In other words, Lemma 9.9 is just a necessary condition. Since all currently known upper bounds for partial spreads are implied by Lemma 9.9, we list the open case of relevant lengths of projective q^{t-1}-divisible codes, i.e., where their existence is currently undecided, in Table 9.2.

We remark that the non-existence of a binary projective 2^4-divisible code of length 130 would imply the non-existence of a binary projective 2^5-divisible code of length 322, so that $A_2(15,12;6) \le 514$ would follow. In the other direction we remark that binary projective 2^r-divisible codes of length n with $(r,n) \in \{(3,67),(4,162),(5,519)\}$ indeed exist, so that the upper bounds $A_2(11,8;4) \le 132$, $A_2(13,10;5) \le 259$, and $A_2(16,12;6) \le 1032$ might be attained with equality.

Research problem

Improve the lower bound $129 \le A_2(11,8;4) \le 132$.

9.1 Realizations of q^r-divisible sets of points as partial spreads

As discussed, the non-existence of q^r-divisible sets of cardinality n over $\mathbb{F}_q$ sometimes has implications to upper bounds for partial spreads. However, not all cardinalities are directly relevant to that extend. E.g. the non-existence of a binary projective triply–even code of length 59, shown in [130], has no such implication. More precisely, there is no set of pairwise disjoint 4-spaces in $\operatorname{PG}(v-1,2)$ such that there are exactly 59 holes. To this end, we observe that $[v]_2 \mod [4]_2 \in \{0,1,3,7\}$ while $59 \mod [4]_2 = 14$. An implication for the existence of vector space partitions of a certain type is stated in Section 10.

Definition 9.12. *A q^r-divisible set $\mathcal{M}$ of n points over $\mathbb{F}_q$ is said to be* realizable as a partial spread *if a partial $(r+1)$-spread over $\mathbb{F}_q$ exists whose set of holes is equivalent to $\mathcal{M}$ (eventually embedded in an ambient space of dimension larger than $\dim(\mathcal{M})$). We use the same notation for codes using their correspondence to multisets of points.*

With this, we e.g. have $A_2(11,8;4) < 132$ if none of the projective binary triply–even codes of length 67 are realizable as a partial spread.

The only partial spread better than Beutelspacher's construction

So far, the only known cases in which the construction of Proposition 9.2 has been surpassed are derived from the sporadic example of a partial 3-spread of cardinality 34 in $\operatorname{PG}(7,2)$ [70], which has 17 holes and can be used to show $A_2(3m+2,6;3) \ge (2^{3m+2}-18)/7$, which exceeds the lower bound of Proposition 9.2 by one. A first step towards the understanding of the sporadic example is the classification of all 2^2-divisible sets of points with cardinality 17 in $\operatorname{PG}(k-1,2)$. It turns out that there are exactly 3 isomorphism types, one configuration $\mathcal{H}_k$ for each dimension $k \in \{6,7,8\}$. Generating matrices for the corresponding doubly-even codes are given by

$$\begin{pmatrix} 10000110010101110 \\ 01000010111011100 \\ 00100100000011000 \\ 00010111001110100 \\ 00001001100111110 \\ 00000011100111011 \end{pmatrix}, \begin{pmatrix} 10000011110100110 \\ 01000001111111000 \\ 00100010000110000 \\ 00010010000101000 \\ 00001001001000100 \\ 00000101001000010 \\ 00000010101011111 \end{pmatrix}, \begin{pmatrix} 10000000111011110 \\ 01000000010110000 \\ 00100000011100000 \\ 00010000001110000 \\ 00001001100000010 \\ 00000101000001010 \\ 00000011000000110 \\ 00000001111011101 \end{pmatrix}. \tag{9.6}$$

While the classification, so far, is based on computer calculations, see e.g. [66] and `http://www.rlmiller.org/de_codes`, one can easily see that there are exactly three solutions of the MacWilliams equations.

Exercise 9.13. *Let C be a projective 2^2-divisible $[17,k]_2$-code. Conclude $k \in \{6,7,8\}$ from the MacWilliams equations and determine the unique weight enumerator in each case.*

The set of holes of the partial 3-spread in [70] corresponds to $\mathcal{H}_7$. A geometric description using coordinates, of this configuration is given in [162, p. 84]. We have computationally checked that indeed all three hole configurations can be realized by a partial 3-spread of cardinality 34 in $\mathrm{PG}(7,2)$.[3] All three $\mathcal{H}_i$ are special instances of parametric geometrical constructions, see [127, Subsection 6.1] for the details.

The Hill cap

While not all automorphisms of the set of holes may extend to the entire partial spread, q^{t-1}-divisible sets of points with a large automorphism group may have some chances to be realized a partial spread with at least some automorphisms. To this end, we want to mention another interesting and rather small example. Over the ternary field the smallest open case for partial spreads is given by $244 \le A_3(8,6;3) \le 248$. A putative plane spread in $\mathrm{PG}(7,3)$ of size 248 would have a 3^2-divisible set $\mathcal{H}$ of holes of cardinality 56. Such a point set is unique up to isomorphism and has dimension $\dim(\mathcal{H}) = 6$. It corresponds to an distance-optimal two-weight code with weight enumerator $W_C(x) = x^0 + 616x^{36} + 112x^{45}$. The set $\mathcal{H}$ was first described by Raymond Hill [121] and is known as the *Hill cap*. A generator matrix is e.g. given by

$$\begin{pmatrix} 10000022110100110202111100101201021211111220012002012211 \\ 01000011101210101121120010211222111210000212022200222010 \\ 00100022220221020011200101120020202002111221211222001112 \\ 00010010112222022102002210010101002222100222112122221200 \\ 00001020121022112112001021102211121000021202220212201001 \\ 00000112202002201012122002011020121222221200210020211222 \end{pmatrix}.$$

The automorphism group of $\mathcal{H}$ has order 40320.

Research problem

Improve the lower bound $244 \le A_3(8,6;3) \le 248$.

If we do not restrict ourselves to partial spreads of the maximum possible size, we can mention another example. In [128] the 14445 isomorphism types of partial 3-spreads in $\mathrm{PG}(6,2)$ of size 16, i.e. one less than the maximum, were classified. In this context, the five non-isomorphic 2^2-divisible sets of 15 points over $\mathbb{F}_2$ were classified without the help of computer enumerations. Four of them consist of the union of plane and an affine solid, while the fifth example is obtained by applying the cone construction to a 6-dimensional projective base, see Example 5.9. The interesting point is that, again, all divisible sets can be realized.

[3] 624 non-isomorphic examples can be downloaded at `http://subspacecodes.uni-bayreuth.de`. Several thousand non-isomorphic examples have been found so far, some of them admitting an automorphism group of order 8.

A maximal partial t-spread in $\mathrm{PG}(n-1,q)$ is a set of t-dimensional subspaces that are pairwise disjoint which cannot be extended by a another t-dimensional subspace without violating the property of pairwise disjointness. The corresponding complement $\mathcal{H}$ is a set of holes/points that is q^{t-1}-divisible, does not contain any t-space in its support, and satisfies $\#\mathcal{H} \equiv [n]_q \pmod{[t]_q}$. As an example let us consider maximal partial 4-spreads in $\mathrm{PG}(7,2)$. Up to equivalence there are exactly 9316 point sets that can be covered by 8 pairwise disjoint solids in $\mathrm{PG}(7,2)$. In 181 cases those point sets cannot be extended by an additional solid, i.e., a corresponding set of 8 solids forms a maximal partial 4-spread in $\mathrm{PG}(7,2)$. Note that there may be several configurations of t-spaces resulting the same set of covered points. We have computationally checked that the minimum size of a maximal partial solid spread in $\mathrm{PG}(7,2)$ is indeed 8, see Table 9.3.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
# point sets	1	1	1	3	22	341	3726	9316	5442	1336	303	42	6	1	1	1	1
# maximal point sets	0	0	0	0	0	0	0	181	1343	317	58	5	3	0	0	0	1

Table 9.3: Non-isomorphic point sets covered by r solids in $\mathrm{PG}(7,2)$.

The three different hole configurations of maximal partial 4-spreads of size 13 in $\mathrm{PG}(7,2)$ are given by:

$$\begin{pmatrix}
000000000000000000000000000011111111111111111111111111111111\\
000000000000111111111111111100000000000000001111111111111111\\
000011111111000000001111111100000000111111110000000011111111\\
111100001111000011110000111100001111000011110000111100001111\\
001100110011000000110011001111110011001100111111001100110011\\
010101010101111101010101010111110101010101010000010101010101\\
111100001111001111000110101000111010001110010011100101011100\\
000011111111010101010111001001010101011011000101001110011010
\end{pmatrix}$$

$$\begin{pmatrix}
000000000000000000000000000011111111111111111111111111111111\\
000000000000111111111111111100000000000000001111111111111111\\
000011111111000000001111111100000000111111110000000011111111\\
111100001111000011110000111100001111000011110000111100001111\\
001100010111001100110111011111110011011100010011001100010001\\
010001110011111101110001001100110100000010110110001110110110\\
110100101111010111010010110011011001001110000100100100111110\\
001011111101011101101000100101000101111010101001000110011011
\end{pmatrix}$$

$$\begin{pmatrix}
000000000000000000000000000011111111111111111111111111111111\\
000000000000111111111111111100000000000000001111111111111111\\
000011111111000000001111111100000000111111110000000011111111\\
111100001111000011110000111100001111000011110000111100001111\\
001100110011111100110011001111110011001100110000001100110011\\
010101010101111101010101010100000101010101011111010101010101\\
111100001111001111000110101000111010001110010011100101011100\\
011111111000010100101100111001010001101010110101010010011101
\end{pmatrix}$$

The corresponding automorphism groups of those projective $[60,8,\{24,32\}]_2$-codes have orders 576, 14400, and 96, respectively. We remark that there are 12 non-isomorphic projective $[60,8,\{24,32\}]_2$-codes. Those three codes with automorphism groups of orders 120, 288, and 4320 are the disjoint union of four solids, i.e., realizing partial spreads can be completed to solid spreads. In one further case, with an automorphism group of order 720, the set of holes contains a solid but not two disjoint solids in its support. The remaining 8 sets of holes might in principle be realized by maximal partial 4-spreads. However, as already mentioned, only the stated three hole configurations can be realized by maximal partial 4-spreads in $\mathrm{PG}(7,2)$. There are two non-isomorphic projective $[45, 8, \{16, 24\}]_2$-codes: a disjoint union of three solids and the concatenation of a Baer solid with a line over $\mathbb{F}_2$, see Example 9.14 for a generalization. The latter case cannot be realized in $\mathrm{PG}(7,2)$ as a maximal partial 4-spread. Indeed both cases can be obtained by concatenating a $[15, 4, \{8, 12\}]_4$-code with a two-dimensional simplex code over $\mathbb{F}_2$. From the 12 $[60, 8, \{24, 32\}]_2$-codes only 7 can be obtained via the concatenation of a $[20, 4, \{12, 16\}]_4$-code with a two-dimensional simplex code over $\mathbb{F}_2$.

Example 9.14. *For an arbitrary prime power q and an integer $t \ge 3$ let $\mathcal{B}$ be the set of points in $\mathrm{PG}(t-1, q^2)$ whose points can be written with coordinates in $[GF](q) < [GF](q^2)$, i.e., a Baer-type construction. So, we have $\#\mathcal{B} = [t]_q$. By concatenating with a two-dimensional simplex code over $\mathbb{F}_q$ we obtain a q^{t-1}-divisible projective two-weight code over $\mathbb{F}_q$ with dimension $k = 2t$ and length $n = (q+1)[t]_q$, i.e., having the same parameters as the disjoint union of $q+1$ t-spaces but not containing a t-space in its support.*

In Table 9.4 we have listed the possible sizes of maximal partial t-spreads in $\mathrm{PG}(n-1, q)$ for small parameters. For $(t, n) = (2, 4)$ and $q \in \{7, 8, 9\}$ the listed sizes may be incomplete below the smallest listed size, c.f. [56, Section 7.3], [57, Section 4], and possibly in the interval 26–29 for $q = 8$.

t	ambient space	cardinalities	references
2	$\mathrm{PG}(3,2)$	5	[202]
2	$\mathrm{PG}(4,2)$	5,7,9	[196, 202]
2	$\mathrm{PG}(5,2)$	13,15–19,21	[89, 132]
3	$\mathrm{PG}(5,2)$	5,9	
3	$\mathrm{PG}(6,2)$	9–17	[128]
4	$\mathrm{PG}(7,2)$	8–13,17	
2	$\mathrm{PG}(3,3)$	7,10	[202]
2	$\mathrm{PG}(3,4)$	11–14,17	[135, 202]
2	$\mathrm{PG}(3,5)$	13–22,26	[105]
2	$\mathrm{PG}(3,7)$	*23–45, 50*	[27, 104, 106, 133]
2	$\mathrm{PG}(3,8)$	*25,30–58,65*	[8, 172, 203]
2	$\mathrm{PG}(3,9)$	*36–74,82*	[79, 111, 172]

Table 9.4: Possible cardinalities of maximal partial t-spreads in $\mathrm{PG}(n-1, q)$.

In [1, Lemma 4.15] it was shown that the minimum size of a maximal partial t-spread in $\mathrm{PG}(2t-1, 2)$ is at least 5, which is met with equality for $t \in \{2, 3\}$. In general, the minimum size of a maximal partial t-spread in $\mathrm{PG}(2t-1, q)$ is at least $2q-1$, see [57, Theorem 3.6], and at least $2q$ if $t = 2$, see [86].

In Remark 3.22 we have observed that for non-prime field sizes q the kernel of the incidence matrix between points and k-spaces yields further conditions on the multiset of points associated to a multiset of k-spaces that are not captured by the q^{k-1}-divisibility.

Example 9.15. *In [110] two non-isomorphic* 9*-divisible sets of* 60 *points in* $\mathrm{PG}(3,9)$ *were stated and characterized. None of these two point sets contains a full line. Using a result of [28] the authors showed that both point sets cannot be realized as the set of holes of a partial spread, see [110, Theorem 1] and [110, Theorem 2].*

Exercise 9.16. *Compare [28, Lemma 2.1] with the implications of the kernel approach, cf. Remark 3.22, for the two* 9*-divisible point sets of cardinality* 60 *in* $\mathrm{PG}(3,9)$ *from [110].*

Research problem

Find an example of a p^{t-1} divisible set of points over $\mathbb{F}_p$ that cannot be realized as a partial t-spread and does not admit a rather trivial justification.

10 Vector space partitions

A *vector space partition* $\mathcal{V}$ of $\mathrm{PG}(v-1,q)$ is a set of subspaces with the property that every point P of $\mathrm{PG}(v-1,q)$, or every non-zero vector in $\mathbb{F}_q^v$, is contained in a unique member of $\mathcal{V}$. If $\mathcal{V}$ contains m_d subspaces of dimension d, then $\mathcal{V}$ is of type $k^{m_k}\dots 1^{m_1}$, where we may leave out some of the cases with $m_d = 0$. If there is at least one dimension $d > 1$ with $m_d > 0$ and $m_v = 0$, then $\mathcal{V}$ is called non-trivial. By $\#\mathcal{V}$ we denote the number $\sum_{i=1}^{v} m_i$ of elements of the vector space partition.

The relation between vector space partitions and divisible sets can be directly read of from Lemma 3.12 (noting that the 1-spaces indeed form a set):

Lemma 10.1. *Let $\mathcal{V}$ be a vector space partition of type $t^{m_t}\dots s^{m_s}1^{m_1}$ of $\mathrm{PG}(v-1,q)$, where $v > t \ge s \ge 2$ Then, the 1-dimensional elements of $\mathcal{V}$ form a q^{s-1}-divisible set of cardinality m_1 in $\mathrm{PG}(v-1,q)$.*

Since there is no 2^1-divisible set of 2-points over $\mathbb{F}_2$ there is e.g. no vector space partition of type $4^{16}3^1 2^2 1^2$ of $\mathrm{PG}(7,2)$. For a potential vector space partition of type $4^{17}3^{35}2^2 1^5$ of $\mathrm{PG}(8,2)$ we cannot apply the argument directly since a 2^1-divisible set of 5 points over $\mathbb{F}_2$ indeed exists. However, if we replace the two lines by their three points each, we would end up with a 2^2-divisible set of 11 points over $\mathbb{F}_2$ which does not exist.

Lemma 10.2. *Let $\mathcal{V}$ be a vector space partition of type $t_k^{m_k}\dots t_1^{m_1}$ of $\mathrm{PG}(v-1,q)$, where $v > t_k > \dots > t_1 > 0$. Then, for each index $1 \le s < k$ the $\sum_{i=1}^{s} m_i\cdot[t_i]_q$ points contained in the elements of dimension at most t_s in $\mathcal{V}$ elements of $\mathcal{V}$ form a $q^{t_{s+1}-1}$-divisible set in $\mathrm{PG}(v-1,q)$.*

Note that the values of the m_i, for $i > s$, and the t_i, for $i > s+1$, as well as the dimension v of the ambient space are irrelevant.

Exercise 10.3. *Show that no vector space partition of type $4^a 3^b 2^c$ of $\mathrm{PG}(7,2)$ exists if*

$$\begin{aligned}(a,b,c) \in \big\{&(1,33,3),(4,27,2),(5,24,4),(7,21,1),(8,18,3),\\ &(11,12,2),(12,9,4),(14,6,1),(15,3,3)\big\}.\end{aligned}$$

As an example for a construction we remark that each partial t-spread of size n in $\mathrm{PG}(v-1,q)$ gives to a vector space partition of $\mathrm{PG}(v-1,q)$ of type $t^n 1^{m_1}$, where $m_1 = [v]_q - n[t]_q$, by complementing the set of t-spaces of the partial spread with its set of holes. If t divides v, then t-spreads in $\mathrm{PG}(v-1,q)$ directly give a vector space partition of $\mathrm{PG}(v-1,q)$ of type t^{m_t}, where $m_t = [v]_q/[t]_q$. Also Lemma 9.1 gives a vector space partition.

The packing and the dimension condition

Counting points gives the necessary condition

$$\sum_{1\le i\le v} m_i \cdot [i]_q = [v]_q \tag{10.1}$$

for the existence of a vector space partition of type $v^{m_v}\dots 1^{m_1}$ in $\operatorname{PG}(v-1,q)$. Since an a-space and a disjoint b-space span an $(a+b)$-space, we also have

$$m_i \cdot m_j = 0 \tag{10.2}$$

for all $1 \le i \le j \le v$ with $i+j > v$ and $m_i \le 1$ for all $i > v/2$.

Definition 10.4. *If a vector space partition $\mathcal{V}$ of $\operatorname{PG}(v-1,q)$ arises from a vector space partition $\mathcal{V}_1$ of $\operatorname{PG}(v-1,q)$ where one a-space of $\mathcal{V}_1$ is replaced by a vector space partition $\mathcal{V}_2$ of $\operatorname{PG}(a-1,q)$ with $\#\mathcal{V}_2 \neq 1$, then we say that $\mathcal{V}$ is* reducible *and* composed *of $\mathcal{V}_1$ and $\mathcal{V}_2$.*

Exercise 10.5.

(a) Show the existence of a vector space partition of type $4^{m_4}2^{m_2}$ of $\operatorname{PG}(7,2)$, where $m_4 = 17-i$ and $m_2 = 5i$, for all $0 \le i \le 17$.

(b) Show the existence of a vector space partition of type $3^{33}2^8$ of $\operatorname{PG}(7,2)$. Hint: Construct a vector space partition of type $5^1 3^{32}$ of $\operatorname{PG}(7,2)$ first.

If we only focus on the occurring dimensions in a type of a vector space partition, then the following general existence result was shown using the Frobenius number:

Theorem 10.6. *([18, Theorem 2]) Let $T = \{t_1 < t_2 < \dots < t_k\}$ be a set of positive integers with $d := \gcd(T)$. If v is an integer with*

$$v > 2t_1 \left\lceil \frac{t_k}{dk} \right\rceil + t_2 + \dots + t_k, \tag{10.3}$$

then a vector space partition of $\operatorname{PG}(v-1,q)$ of a type satisfying $\{i : m_i > 0\} = T$ exists iff $\gcd(T)$ divides v.

Types of vector space partitions in $\operatorname{PG}(v-1,q)$ for $v \le 5$

Exercise 10.7. *Show that for $v \le 4$ conditions (10.1) and (10.2) are sufficient to characterize all possible types of vector a space partition in $\operatorname{PG}(v-1,q)$. More precisely:*

- *the possible vector space partitions of $\operatorname{PG}(1,q)$ are given by 1^{q+1};*
- *the possible vector space partitions of $\operatorname{PG}(2,q)$ are given by $2^1 1^{q^2}$ and 1^{q^2+q+1};*
- *the possible vector space partitions of $\operatorname{PG}(3,q)$ are given by $2^{q^2+1-j}1^{(q+1)j}$, where $0 \le j \le q^2+1$, and $3^1 1^{q^3}$.*

For vector space partitions of type $2^{m_2}1^{m_1}$ in $\mathrm{PG}(4,q)$ conditions (10.1) and (10.2) only imply $m_2 = q^3+q-j$ and $m_1 = 1+(q+1)j$ for $0 \le j \le q^3+q$. Lemma 10.1 (or $A_q(5,4;2) = q^3+1$)) yields $j \ge q-1$.

Exercise 10.8. *Show that the conditions (10.1), (10.2) and Lemma 10.1 are sufficient to characterize all possible types of vector a space partition in* $\mathrm{PG}(4,q)$. *More precisely, the possible vector space partitions of* $\mathrm{PG}(4,q)$ *are given by* $4^1 1^{q^4}$, $3^1 2^{q^3-j} 1^{(q+1)j}$ *for* $0 \le j \le q^3$, *and* $2^{q^3+1-j} 1^{q^2+(q+1)j}$, *where* $0 \le j \le q^3+1$.

There is little hope to classify all feasible types of vector space partitions of $\mathrm{PG}(v-1,q)$ unless the parameters are relatively small. Already the determination of the minimum possible m_1 such that a vector space partition of type $t^{m_t}1^{m_1}$ of $\mathrm{PG}(v-1,q)$ exists, i.e., the determination of $A_q(v,2t;t)$, is a really hard problem if v and t get large. More precisely, already the exact value of $A_q(8,6;3)$ is unknown if $q>2$. Nevertheless, the mentioned classification is an ongoing, very hard, major project, see e.g. [72, 107, 108, 109, 164, 194]. Currently all feasible types of vector space partitions of $\mathrm{PG}(v-1,2)$ with $v \le 7$ are characterized [72]. The feasible types of vector space partitions of $\mathrm{PG}(7,2)$ that do not contain elements of dimension 1 are classified in [69]. For the characterization of all feasible vector space partitions of $\mathrm{PG}(8-1,2)$ see [159]. Here we want to focus on non-existence results.

Using classification results for divisible codes

In $\mathrm{PG}(5,2)$ the only infeasible type of a vector space partition that is not excluded by conditions (10.1), (10.2) or Lemma 10.2 is $3^7 2^3 1^5$, see e.g. [72] for constructions in the other cases. Here, 2^1 divisible sets of 5 or 2^2-divisible sets of 14 points indeed exist over $\mathbb{F}_2$. However, in the latter case the 14 points always form two disjoint planes, see Lemma 11.6 in Section 11. Since two lines contained in the same plane have to intersect non-trivially, type $3^7 2^3 1^5$ is infeasible. The same argument also excludes the existence of a vector space partition of type $4^1 3^{14} 2^3 1^5$ of $\mathrm{PG}(7,2)$, cf. [72, Proposition 6.4], and can easily be generalized to:

Lemma 10.9. *Let* $\mathcal{V}$ *be a vector space partition of type* $t_k^{m_k} \dots t_1^{m_1}$ *of* $\mathrm{PG}(v-1,q)$, *where* $v > t_k > \dots > t_1 > 0$. *If* $1 \le s < k$ *is an index with* $l := \sum_{i=1}^{s} m_i \cdot [t_i]_q / [t_{s+1}]_q \in \mathbb{N}$ *and every* $q^{t_{s+1}-1}$*-divisible set of* $l[t_{s+1}]_q$ *is the disjoint union of* l t_{s+1}*-spaces, then we have*

$$\sum_{1 \le i \le s \,:\, 2t_i > t_{s+1}} m_i \le l. \tag{10.4}$$

Another example is the exclusion of a vector space partition of type $3^{26} 2^4 1^{10}$ of $\mathrm{PG}(5,3)$ since 3^2-divisible sets of 26 points over $\mathbb{F}_3$ can be partitioned into two disjoint planes, see Lemma 11.6 in Section 11.

Also other classification or characterization results for q^r-divisible sets of points can be used to exclude the existence of certain types of vector space partitions. In Exercise 11.15 we see that for each q^r-divisible set $\mathcal{M}$ of q^{r+1} points over $\mathbb{F}_q$, where $r \in \mathbb{N}$, there exists an empty hyperplane $H \in \mathcal{H}$, i.e., $\mathcal{M}(H) = 0$. So, in particular $\mathrm{supp}(\mathcal{M})$ does not contain a line. As a consequence, there is no vector space partition of type $3^{17} 2^1 1^5$ of $\mathrm{PG}(7,2)$, cf. [72, Proposition 6.5] . By replacing an arbitrary line in a vector space partition of type $4^1 3^{13} 2^7 1^0$ of $\mathrm{PG}(7,2)$ by its three points we obtain a vector space partition of type $4^1 3^{13} 2^6 1^3$, cf. Definition 10.4. Since each q^r-divisible set of $[r+1]_q$ points is the characteristic function of an $(r+1)$-space, see Lemma 11.2, here the mapping also works in the other direction, i.e., a vector space partition of type $4^1 3^{13} 2^6 1^3$ implies the existence of a vector space partition of type $4^1 3^{13} 2^7 1^0$.

Exercise 10.10. *Show that no vector space partition of type* $4^{16} 3^1 2^1 1^5$ *of* $\mathrm{PG}(8,2)$ *exists. Hint: e.g. Use Lemma 11.2 or [109, Lemma 2].*

Research problem

Do vector space partitions of type $4^4 3^{135} 1^{18}$ or $4^3 3^{137} 1^{19}$ exist in $\mathrm{PG}(9,2)$?

The previously mentioned non-existence results, except the non-existence of type $4^1 3^{13} 2^7$ that we will prove later on, and suitable constructions give the full characterization of vector space partitions of $\mathrm{PG}(v-1,2)$ for all $v \le 7$, see [72] for the details. For the vector space partitions of $\mathrm{PG}(7,2)$ without 1-dimensional elements conditions (10.1), (10.2) and Lemma 10.2 are sufficient except for the type $4^{13} 3^6 2^6$, cf. Exercise 10.3, Example 10.19, and [69].

The "tail condition"

Another, very explicit, necessary criterion for the existence of vector space partitions is the so-called *tail condition*:

Theorem 10.11. *([107, Theorem 1]) Let* $\mathcal{V}$ *be a vector space partition of type* $t^{m_t} \ldots d_2{}^{m_{d_2}} d_1{}^{m_{d_1}}$ *of* $\mathrm{PG}(v-1,q)$*, where* $m_{d_2}, m_{d_1} > 0$ *and* $n_1 = m_{d_1}$*,* $n_2 = m_{d_2}$*.*

(i) if $q^{d_2-d_1}$ *does not divide* n_1 *and if* $d_2 < 2d_1$*, then* $n_1 \ge q^{d_1} + 1$*;*

(ii) if $q^{d_2-d_1}$ *does not divide* n_1 *and if* $d_2 \ge 2d_1$*, then* $n_1 > 2q^{d_2-d_1}$ *or* d_1 *divides* d_2 *and* $n_1 = \left(q^{d_2}-1\right)/\left(q^{d_1}-1\right)$*;*

(iii) if $q^{d_2-d_1}$ *divides* n_1 *and* $d_2 < 2d_1$*, then* $n_1 \ge q^{d_2} - q^{d_1} + q^{d_2-d_1}$*;*

(iv) if $q^{d_2-d_1}$ *divides* n_1 *and* $d_2 \ge 2d_1$*, then* $n_1 \ge q^{d_2}$*.*

We remark that the proof is based on so-called *mixed perfect codes*, see e.g. [107, 120] for details. In this context we would like to mention [26], which translates a similar results obtained via mixed perfect codes into geometry.

The tail of a vector space partition consists of the elements of the smallest occurring dimension. This notion was generalized to the so-called *supertail* containing all elements of the vector space partition with a dimension below a certain bound. For details and results on e.g. the minimum possible cardinality or the minimum possible number of covered points by the supertail we refer to [109, 182, 183]. In some cases the structure of the minimum tails or supertails can be completely characterized, see e.g. [107, 183] and cf. Exercise 10.10. For a few observations from the point of view of divisible codes we refer to Subsection 10.1.

Theorem 10.11 was slightly improved and reformulated in [151].

Definition 10.12. *([151, Definition 4]) Let $\mathcal{N}$ be a set of pairwise disjoint k-subspaces in $\mathrm{PG}(v-1,q)$. If there exists a positive integer r such that*

$$\#\{N \in \mathcal{N} : N \le H\} \equiv \#\mathcal{N} \pmod{q^r} \tag{10.5}$$

*for every hyperplane $H \in \mathcal{H}$, then we call $\mathcal{N}$ q^r-*divisible.

Exercise 10.13. *Let $\mathcal{V}$ be a vector space partition of type $t^{m_t} \ldots {d_2}^{m_{d_2}} {d_1}^{m_{d_1}}$ of $\mathrm{PG}(v-1,q)$, where $m_{d_2}, m_{d_1} > 0$ and $n_1 = m_{d_1}$, $n_2 = m_{d_2}$. Show that the set $\mathcal{N}$ of the d_1-dimensional elements of $\mathcal{V}$ is $q^{d_2-d_1}$-divisible.*

Theorem 10.14. *([151, Theorem 12]) For a non-empty q^r-divisible set $\mathcal{N}$ of k-subspaces in $\mathrm{PG}(v-1,q)$ the following bounds on $n = \#\mathcal{N}$ are tight.*

- *(i) We have $n \ge q^k + 1$ and if $r \ge k$ then either k divides r and $n \ge \frac{q^{k+r}-1}{q^k-1}$ or $n \ge \frac{q^{(a+2)k}-1}{q^k-1}$, where $r = ak + b$ with $0 < b < k$ and $a, b \in \mathbb{N}$.*
- *(ii) Let q^r divide n. If $r < k$ then $n \ge q^{k+r} - q^k + q^r$ and $n \ge q^{k+r}$ otherwise.*

For (i) the lower bounds are attained by k-spreads. For (ii) the second lower bound is attained by the construction of Lemma 9.1. In the other case the two-weight codes constructed in [21, Theorem 4] attain the lower bound.

Corollary 10.15. *Let $\mathcal{V}$ be a vector space partition of type ${d_l}^{u_l} \ldots {d_2}^{u_2} {d_1}^{u_1}$ of $\mathrm{PG}(v-1,q)$, where $u_1, u_2 > 0$ and $d_l > \cdots > d_2 > d_1 \ge 1$.*

- *(i) We have $u_1 \ge q^{d_1} + 1$ and if $d_2 \ge 2d_1$ then either d_1 divides d_2 and $u_1 \ge \frac{q^{d_2}-1}{q^{d_1}-1}$ or $u_1 \ge \frac{q^{(a+1)d_1}-1}{q^{d_1}-1}$, where $d_2 = ad_1 + b$ with $0 < b < d_1$ and $a, b \in \mathbb{N}$.*
- *(ii) Let $q^{d_2-d_1}$ divide u_1. If $d_2 < 2d_1$ then $u_1 \ge q^{d_2} - q^{d_1} + q^{d_2-d_1}$ and $u_1 \ge q^{d_2}$ otherwise.*

Example 10.16. *Let $\mathcal{N}$ be a non-empty 2^1-divisible set of lines in $\mathrm{PG}(v-1,2)$. From Theorem 10.14.(i) we conclude $\#\mathcal{N} \ge 5$ and Theorem 10.14.(ii) gives $\#\mathcal{N} \ge 6$ is $\#\mathcal{N} \equiv 0 \pmod 2$. A 2-spread of $\mathrm{PG}(3,2)$ and a vector space partition of type $3^1 2^8$ of $\mathrm{PG}(4,2)$ give examples for $\#\mathcal{N} \in \{5,8\}$. For $n \in \{6,7,9\}$ there exist projective $4^{1/2}$-divisible codes of length n over $\mathbb{F}_4$. Concatenation with a two-dimensional simplex code gives examples with $\#\mathcal{N} \in \{6,7,9\}$. By combining these examples we can attain all $\#\mathcal{N} \ge 5$.*

Exercise 10.17. *Show that non-empty 2^1-divisible sets of lines over $\mathbb{F}_2$ exist iff*

$$\#\mathcal{N} \in \{5, 10, 15, 16, 17\} \cup \mathbb{N}_{\ge 20}.$$

Hint: For the constructive direction consider projective 4^1-divisible codes of suitable lengths over $\mathbb{F}_4$. For the non-existence results consider the possible lengths of projective 2^3-divisible binary codes.

A generalization of q^r-divisible sets of points

Choosing $k = 1$ in Definition 10.12 we end up with q^r-divisible sets of points, so that we have some kind of a generalization for $k > 1$. So, we can again ask for the sets of possible cardinalities depending of k, q and r. Since k-spreads and the construction of Lemma 9.1 give examples with coprime cardinalities a finite Frobenius-type number and only finitely many non-feasible cardinalities exist.

Research problem

Characterize the possible cardinalities of q^r-divisible sets of k-spaces over $\mathbb{F}_q$ for some small parameters q, r, and k, i.e. continue Example 10.16 and Exercise 10.17

Now we show the non-existence of a vector space partition of type $4^1 3^{13} 2^7$ of $\operatorname{PG}(7,2)$. Note that a 2-divisible set of seven lines $\mathcal{N}$ over $\mathbb{F}_2$ indeed exists. However, we can deduce some information on those sets. To this end let $\mathcal{M}$ be a corresponding spanning set of 21 points in $\operatorname{PG}(k-1,2)$. Since $\#\mathcal{N} > \#\{N \in \mathcal{N} : N \le H\} \equiv \#\mathcal{N} \equiv 1 \pmod 2$ for every hyperplane $H \in \mathcal{H}$, we have $\mathcal{M}(H) \in \{9, 13, 17\}$, i.e., $\mathcal{M}$ is 4-divisible. However $\mathcal{M}(H) = 17$ is impossible, since removing five lines would give a 2-divisible set of 2 points over $\mathbb{F}_2$, which does not exist. With this, the corresponding standard equations are given by

$$\begin{aligned} a_9 + a_{13} &= [k]_2 \\ 9a_9 + 13a_{13} &= 21 \cdot [k-1]_2 \\ 36a_9 + 78a_{13} &= 210 \cdot [k-2]_2 \end{aligned}$$

and have the unique solution $k = 6$, $a_9 = 42$, $a_{13} = 12$.[1] Now consider the hyperplane H that contains all seven lines. Since H intersects solids in at least $[3]_2$ and planes in at least $[2]_2$ points, the intersection of H with a vector space partition of type $4^1 3^{13} 2^7$ consists of at least

$$1 \cdot [3]_2 + 13 \cdot [2]_2 + 7 \cdot [2]_2 = 67 > 63 = [6]_1 \qquad (10.6)$$

points, which is a contradiction. We remark that a vector space partition of type $3^{70} 2^7$ of $\operatorname{PG}(8,2)$ indeed exists, see [71]. A similar proof can be found in [72, Proposition 6.2]. We can also write down equations similar to the standard equations directly for the elements of $\mathcal{N}$ or work with the counts of different types of vector space partitions in hyperplanes, see e.g. [108, 164] and the subsequent example.

Exercise 10.18. *Show that a 2-divisible set $\mathcal{N}$ of six lines over $\mathbb{F}_2$ has dimension* $\dim(\mathcal{N}) = 6$.

Example 10.19. *Assume that $\mathcal{V}$ is a vector space partition of type $4^{13} 3^6 2^6$ of $\operatorname{PG}(7,2)$. Let $4^r 3^s 2^t 1^u$ be the type of the intersection of $\mathcal{V}$ with a hyperplane $H \in \mathcal{H}$, so that*

[1] According to [35] there are exactly 2 two-weight codes with these parameters having automorphism groups of order 336 and 1008. The latter two-weight code is given as Example SU2 in [47], i.e., the union of three non-intersecting planes in $\mathbb{F}_2^6$, with automorphism group $\operatorname{GL} \times S_3$. The other example has automorphism group $\operatorname{GL} \ltimes \mathbb{Z}_2$ and arises by concatenating the $[7, 3, \{4, 6\}]_4$ code, see [47, Example RT1], with a two-dimensional binary simplex code. In both cases there are $B_3 = 28$ lines and the 21 points admit several possibilities as a partition of 7 lines.

$r+s+t+u=13+6+6=25$ and $r[4]_2+s[3]_2+t[2]_2+u[1]_2=[7]_2$. Since two solids in $H\cong \mathrm{PG}(6,2)$ have to intersect non-trivially, we have $r\in\{0,1\}$. Since there is no 2-divisible set of $n\le 2$ points over $\mathbb{F}_2$, we have $u=0$ or $u\ge 3$. This gives the following possible types for H:

(a) $4^1 3^{13} 2^5 1^6$; *(b)* $4^1 3^{12} 2^8 1^4$; *(c)* $4^0 3^{16} 2^3 1^6$; *(d)* $4^0 3^{15} 2^6 1^4$; *(e)* $4^0 3^{13} 2^{12} 1^0$.

Let us denote their corresponding counts by a, b, c, d, and e, respectively. Counting the number of hyperplanes gives $a+b+c+d+e=[8]_2=255$. Counting the number of solid-hyperplane incidences gives $a+b=13\cdot[4]_2=195$, so that $c+d+e=60$. From Exercise 10.18 we know that the six lines to form a 6-dimensional subspace, so that $e=[2]_1=3$,[2] i.e., $c+d=57$. Counting pairs of planes gives $\binom{3}{2}c+d=\binom{6}{2}\cdot[2]_2$, i.e., $3c+d=45$, so that c has to be negative.

Generalizations of vector space partitions

The notion of a vector space partition can be generalized in several directions. A λ-fold vector space partition of $\mathrm{PG}(v-1,q)$ is a (multi-) set of subspaces such that every point $P\in\mathcal{P}$ is covered exactly λ times, see e.g. [73]. Here non-existence results for q^r-divisible multisets of points over $\mathbb{F}_q$ with point multiplicity at most λ can be utilized, cf. Subsection 8.2. Another variant considers set of subspaces such that every t-subspace is covered exactly once, see [112]. Also here divisible codes can be used for non-existence results for those vector space t-partitions. We remark that the upper bound $A_2(8,6;4)<289$ for constant-dimension codes is also implied by a non-existence result of certain vector space 2-partitions [112]. Vector space partitions of affine spaces have been considered in [7]. Another variant are multispreads [148].

10.1 Partitions of q^r-divisible sets of points

Following up the idea of the tail, see e.g. Theorem 10.11, and the supertail of a vector space partition in the context of divisible codes, we say that a set of points $\mathcal{M}$ over $\mathbb{F}_q$ *admits a partition*, or is *partitionable*, *of type* $k^{m_k}\dots 1^{m_1}$ if there exists a set $\mathcal{S}$ of m_i i-subspaces for $1\le i\le k$, such that $\mathcal{M}=\sum_{S\in\mathcal{S}}\chi_S$, i.e., the set of points of the elements of $\mathcal{S}$ coincides with $\mathcal{M}$. We are mainly interested in q^r-divisible partionable sets of points where $r\ge k$. In this context, the non-existence of a vector space partition of type $4^1 3^{14} 2^3 1^5$ of $\mathrm{PG}(7,2)$ follows from the non-existence of a 2^2-divisible set of points with partition type $2^3 1^5$, i.e., in general no vector space partition over $\mathbb{F}_2$ can end with $2^3 1^5$. The classification of q^r-divisible partition types of the form 1^{m_1} over $\mathbb{F}_q$ corresponds to the classification of the possible lengths of q^r-divisible sets of points over $\mathbb{F}_q$, see Section 7.

Let us consider 2^2-divisible sets of points of partition type $2^{m_2}1^{m_1}$ over $\mathbb{F}_2$ for a moment. In Example 10.16 we have shown that type $2^{m_2}1^0$ is feasible iff $m_2\ge 5$ (or the trivial case $m_2=0$). Since there are no 2^1-divisible sets of cardinality 1 or 2 over $\mathbb{F}_2$, the types $2^{m_2}1^1$ and $2^{m_2}1^2$ are infeasible in general.

[2] In the original proof of [69, Theorem 7] the estimation $e\le 7$ was used.

Exercise 10.20. *Let $\mathcal{M}$ be a q^r-divisible multiset of points over $\mathbb{F}_q$. Show that if a k-space S is completely contained in* $\operatorname{supp}(\mathcal{M})$*, then $\mathcal{M} - \chi_S$ is $q^{\min\{r,k-1\}}$-divisible.*

Exercise 10.21. *Let $0 \le j \le 5$. Show that 2^2-divisible set of points over $\mathbb{F}_2$ of partition type $2^{m_2}1^{3j}$ exist iff $m_2 \ge 5 - j$.*

Using Lemma 7.2 we can easily conclude that type $2^{m_2}1^4$ is impossible for $m_2 \in \{0, 2, 3\}$ while type $2^1 1^4$ is e.g. attained by a vector space partition of type $2^1 1^4$ of $\operatorname{PG}(2, 2)$, so that we have constructions for all $m_2 \ge 6$. For $m_2 \in \{4, 5\}$ it remains to be checked if the 2^2-divisible sets of 16 or 19 points can contain sufficiently many disjoint lines. Of course this amounts to a finite computation.

Exercise 10.22. *Show that a 2^2-divisible set of points over $\mathbb{F}_2$ of partition type $2^{m_2}1^{m_1}$ exist for all $m_2 \in \mathbb{N}_0$, $m_1 \in \mathbb{N}_{\ge 29}$. Hint: Use Lemma 7.2 and Example 10.16.*

Research problem

Complete the classification of the possible parameters (m_2, m_1) of a 2^2-divisible set of points over $\mathbb{F}_2$ of partition type $2^{m_2}1^{m_1}$.

Of course, also other parameters are of interest and the general classification problem is widely open. Also the question of the representation of such results arises. Taking Lemma 7.2 as given, we may summarize the presented knowledge on non-existence results of 2^2-divisible sets of points over $\mathbb{F}_2$ of partition type $2^{m_2}1^{m_1}$ by the forbidden types $2^1 1^5$ and $2^3 1^5$. For 3^2-divisible sets of points over $\mathbb{F}_3$ of partition type $2^{m_2}1^{m_1}$ we mention that the forbidden pattern $2^4 1^{10}$ is implied by the forbidden pattern $2^3 1^{14}$.

11 Classification results for q^r-divisible codes

Sets of points $\mathcal{M}$ where each hyperplane has the same multiplicity can be easily classified using the standard equations:

Exercise 11.1. *Let $\mathcal{M}$ be a spanning set of points in $\mathrm{PG}(k-1,q)$, where $k \ge 2$, such that $\mathcal{M}(H) = c \in \mathbb{N}$ for every hyperplane $H \in \mathcal{H}$. Show that $c = [k-1]_q$, $\#\mathcal{M} = [k]_q$, $\mathcal{M}$ is q^{k-1}-divisible, and $\mathcal{M} = \chi_{\mathcal{P}}$, i.e., $\mathcal{M}$ is the full k-space.*

As a direct implication we obtain:

Lemma 11.2. *Let $\mathcal{M}$ be a q^r-divisible set of $[r+1]_q$ points, where $r \in \mathbb{N}$. Then $\mathcal{M} = \chi_S$ for some $(r+1)$-space S, i.e., the corresponding points form an $(r+1)$-space.*

If we consider multisets of points in Exercise 11.1 instead sets of points, then we end up with λ-fold k-spaces, i.e., $\mathcal{M} = \lambda \cdot \chi_S$, see [31]. So Lemma 11.2 also applies to multisets of points. Point sets with two different hyperplane multiplicities have a very rich diversity, see Subsection 5.3. However, we can generalize Lemma 11.2 in a different direction.

Exercise 11.3. *Let $\mathcal{M}$ be a q^r-divisible set of $2[r+1]_q$ points over $\mathbb{F}_q$, where $r \in \mathbb{N}$ and $(q,r) \neq (2,1)$. Show that the standard equations have a unique solution corresponding to the disjoint union of two $(r+1)$-spaces, so that especially $\dim(\mathcal{M}) = 2r+2$ and there are $a_{2[r]_q} = \left(q^{r+1}-1\right) \cdot [r+1]_q$ hyperplanes of multiplicity $2[r]_q$ and $a_{[r]_q+[r+1]_q} = 2[r+1]_q$ hyperplanes of multiplicity $[r]_q + [r+1]_q$.*

We remark that over $\mathbb{F}_2$ a 5-dimensional projective base gives a spanning 2-divisible set of 6 points in $\mathrm{PG}(4,2)$. Given a set of points $\mathcal{M}$ as in Exercise 11.3, we observe $\mathcal{M}(H) \in \{2[r]_q, [r]_q + [r+1]_q\}$, i.e., there are just two different hyperplane multiplicities. If $\mathcal{M}(S) > 2[r]_q$ for an $(r+1)$-space S, then Equation (3.6) yields

$$\begin{aligned}
\mathcal{M}(S) &= \frac{1}{q^{v-s-1}} \cdot \left(\sum_{H \in \mathcal{H} \,:\, S \le H} \mathcal{M}(H) \;-\; [v-s-1]_q \cdot \#\mathcal{M} \right) \\
&= \frac{1}{q^r} \cdot \big([r+1]_q \cdot ([r]_q + [r+1]_q) - [r]_q \cdot 2[r+1]_q \big) = [r+1]_q,
\end{aligned}$$

i.e., $S \subseteq \operatorname{supp}(\mathcal{M})$ so that applying Lemma 11.2 to $\mathcal{M} - \chi_S$ gives that $\mathcal{M}$ is the disjoint union of two $(r+1)$-spaces. For $r = 1$ the existence of a line L with $\mathcal{M}(L) > 2$ can be deduced from $B_3 > 0$, which is satisfied for a projective q-divisible $[2q+2,k]_q$-code with $(q,k) \neq (2,5)$, so that:

Lemma 11.4. *For $q \ge 3$ every q-divisible set of $2q+2$ points over $\mathbb{F}_q$ is the disjoint union of two lines.*

The large number of hyperplanes of multiplicity $2[r]_q$ concluded in Exercise 11.3 can be used for an induction argument:

Exercise 11.5. *Let $\mathcal{M}$ be q^r-divisible set of $2[r+1]_q$ points over $\mathbb{F}_q$, where $r \in \mathbb{N}_{\ge 2}$, such that for each hyperplane $H \in \mathcal{H}$ with $\mathcal{M}(H) = 2[r]_q$ the restricted point set $\mathcal{M}|_H$ is the disjoint union of two r-spaces.*

(a) Show that each s-space S with $1 \le s \le r-1$ and $S \subseteq \operatorname{supp}(\mathcal{M})$ is contained in an $(s+1)$-space S' with $S' \subseteq \operatorname{supp}(\mathcal{M})$.

(b) Show that each $(r-1)$-space F with $F \subseteq \operatorname{supp}(\mathcal{M})$ is contained in two different r-spaces R_1 and R_2 with $R_1, R_2 \subseteq \operatorname{supp}(\mathcal{M})$.

(c) Show that the $(r+1)$-dimensional space $X := \langle R_1, R_2\rangle$ satisfies $\mathcal{M}(H) \ne 2[r]_q$ for each hyperplane $H \in \mathcal{H}$ containing X.

(d) Show that $\mathcal{M}(X) = [r+1]_q$ and that $\mathcal{M}$ is the disjoint union of two $(r+1)$-spaces.

A quick computer enumeration reveals that each 2^2-divisible set of 14 points over $\mathbb{F}_2$ is indeed the disjoint union of two planes,[1] so that we obtain:

Lemma 11.6. *Let $\mathcal{M}$ be q^r-divisible set of $2[r+1]_q$ points over $\mathbb{F}_q$, where $r \in \mathbb{N}$ and $(q, r) \ne (2, 1)$. Then, $\mathcal{M}$ is the disjoint union of two $(r+1)$-spaces.*

Using results on blocking sets in $\mathrm{PG}(2, q)$, actually a much stronger classification result for minihypers of a certain type was proven in [91].

Theorem 11.7. *([91, Theorem 13]) Let $\mathcal{M}$ be a q^r-divisible multiset of cardinality $\delta[r+1]_q$ over $\mathbb{F}_q$. If $q > 2$ and $1 \le \delta < \varepsilon$, where $q+\varepsilon$ is the size of the smallest non-trivial blocking sets in $\mathrm{PG}(2, q)$, then there exists $(r+1)$-spaces $S_1, \dots, S_\delta$ such that*

$$\mathcal{M} = \sum_{i=1}^{\delta} \chi_{S_i},$$

i.e., $\mathcal{M}$ is the sum of $(r+1)$-spaces.

Theorem 11.8. *If $q+\varepsilon$ is the size of the smallest non-trivial blocking sets in $\mathrm{PG}(2, q)$, then*

(a) $\varepsilon = (q+3)/2$ if $q > 2$ is prime [25];

[1] A computer-free proof can roughly run as follows. First show that a 2-divisible set of 6 points over $\mathbb{F}_2$ is either the disjoint union of two lines or a 5-dimensional projective base that does not contain a full line. Let $\mathcal{M}$ be a 2^2-divisible set of 14 points over $\mathbb{F}_2$. From the MacWilliams equations for the corresponding code we conclude $B_3 > 0$ so that there exists a line L with $L \subseteq \operatorname{supp}(\mathcal{M})$. For this line L we can proceed as in Exercise 11.5 since every hyperplane H containing L with multiplicity $\mathcal{M}(H) = 6$ is the disjoint union of two lines.

(b) $\varepsilon = \sqrt{q} + 1$ *if q is square [43];*

(c) $\varepsilon \geq c_p q^{2/3} + 1$, *where* $c_2 = c_3 = 2^{-1/3}$ *and* $c_p = 1$ *for* $p > 3$, *if* $q = p^h$ *with* $h > 2$ *and* $h \equiv 1 \pmod 2$ *[29].*

Note that Theorem 11.7 does not apply to $q = 2$ and Lemma 11.6 applies to $q = 2$ for $r \geq 2$ only. Moreover, Lemma 11.6 is tight in the sense that 2^2-divisible sets of 21 points over $\mathbb{F}_2$ that are not the union of three planes indeed exist. From e.g. [113] we know that the number of non-isomorphic such sets is given by 2, 7, 9, and 6 for dimensions 6, 7, 8, and 9, respectively. So, there is even a projective $[21, 6, \{8, 12\}]_2$ two-weight code which is not given by [47, Example SU2], as there is just one isomorphism type, see:

Exercise 11.9. *Let $\mathcal{M}$ be the set of points of three pairwise disjoint r-spaces. Sow that $2r \leq \dim(\mathcal{M}) \leq 3r$ and that there is a unique isomorphism type for each possible dimension.*

The second two-weight code has a nice geometric description. By the Klein correspondence there exist two disjoint planes π_1, π_2 in the Klein quadric $\mathsf{Q}^+(5, q)$. If $\mathcal{K}$ is the set of points of the Klein quadric in $\mathrm{PG}(5, 2)$, then $\mathcal{K} - \chi_{\pi_1} - \chi_{\pi_2}$ is a 2^2-divisible set of 21 points.

Exercise 11.10. *Show that the points of the Klein quadric form a 2^2-divisible set $\mathcal{K}$ of 35 points over $\mathbb{F}_2$. If π_1 and π_2 are two disjoint planes contained in the support of $\mathcal{K}$, then $\mathcal{K}' := \mathcal{K} - \chi_{\pi_1} - \chi_{\pi_2}$ is a 2^2-divisible set of 21 points that can be partitioned into 7 lines and only attains two different hyperplane multiplicities.*

From e.g. [113] we also know that the number of non-isomorphic 2^3-divisible sets of 45 points over $\mathbb{F}_2$ is given by 2, 1, 1, 1, and 1 for dimensions $8 \leq k \leq 12$. Thus, beside the examples that arise as the disjoint union of three solids, there is a unique other projective $[45, 8, \{16, 24\}]_2$ two-weight code, which is e.g. described in [99, Theorem 4.1].[2] So, Lemma 11.6 is also tight for $q = 2$, $r = 3$. However, the number of cases, which are not given as the union of three disjoint $(r + 1)$-spaces, seem to decrease. And indeed, enumerating all projective 2^4-divisible codes of length 93 over $\mathbb{F}_2$, with `LinCode` [34], yields that all examples arise as the disjoint union of three 5-spaces.

Exercise 11.11. *Show that the weights of a projective q^r-divisible code of length $\delta[r+1]_q$ over $\mathbb{F}_q$ are contained in $\{iq^r : 1 \leq i \leq \delta\}$ for all $r, \delta \in \mathbb{N}$.*

Exercise 11.12. *Show by induction that every 2^r-divisible set of $3[r + 1]_2$ points over $\mathbb{F}_2$ is the disjoint union of three $(r + 1)$-spaces for all $r \geq 4$.*

Conjecture 11.13. *There exists a function $f \colon \mathbb{N} \to \mathbb{N}$ such that every 2^r-divisible set of $f(r) \cdot [r + 1]_2$ points over $\mathbb{F}_2$ is the disjoint union of $f(r)$ $(r + 1)$-spaces and $\lim_{r\to\infty} f(r) = \infty$.*

[2] The residual $[21, 7]_2$-codes correspond to the construction directly following Example 5.10.

A few remarks on the case $q = 8$ are also contained in Section 12.

Let n be the cardinality of a q^r-divisible set of points over $\mathbb{F}_q$, where $r, n \in \mathbb{N}$. So far we have studied the isomorphism types for $n = \delta[r+1]_q$ over $\mathbb{F}_q$, where r and δ are positive integers, which includes the smallest possible cardinality attained at $\delta = 1$. From Theorem 6.11 we known that for $n \le rq^{r+1}$ all possible values of n can be written as $a[r+1]_q + bq^{r+1}$ with $a, b \in \mathbb{N}_0$. So, the next interesting case is cardinality $n = q^{r+1}$, which will be treated in the subsequent subsection. For $b \ge 1 \wedge a + b \ge 2$ the situation seems to be more complicated. For $q = 2$ the cases $(a, b) = (1, 1)$ and $(0, 2)$ correspond to 2^r-divisible sets of $2^{r+2} - 1$ or 2^{r+2} points over $\mathbb{F}_2$. Examples that are not the union of subspaces and affine subspaces are obtained in Example 5.9 via the cone construction.

Research problem

Classify all 2^r-divisible sets of $2^{r+2} - 1$ or 2^{r+2} points over $\mathbb{F}_2$.

The case of $r = 2$ and cardinality 15 is solved in [128].

Exercise 11.14. *Consider the* $\left[2^{k-1} + l\left(2^k - 1\right), k, (2l+1)2^{k-2}\right]_2$ *code* C*, where* $k \ge 1$ *and* $l \ge 0$ *are integers. Let* $\mathcal{M}$ *be the corresponding multiset of points. Show that* $\mathcal{M}(P) \in \{l, l+1\}$ *for all* $P \in \mathcal{P}$ *and that the* $2^{k-1} - 1$ *points with multiplicity* l *form a hyperplane in* $\mathbb{F}_2^k$*.*

11.1 The (generalized) cylinder conjecture

Applying the cone construction with a base $\mathcal{B}$ of arbitrary q points and an r-space as center X gives a q^r-divisible set of q^{r+1} points over $\mathbb{F}_q$, see (5.1). For $r = 1$ these sets consist of q affine lines meeting in a common point, which is not part of the point set, so that one can speak of a *cylinder*. For general $r \ge 1$ we also speak of cylinders, or more precisely r-*cylinders* in these cases. As an abbreviation, we say that the cylinder conjecture is true for (v, r, q) if each q^r-divisible set $\mathcal{M}$ of q^{r+1} points in $\mathrm{PG}(v-1, q)$ with $\dim(\mathcal{M}) = v$ is a cylinder. The origin of the cylinder conjecture was the idea of classifying all sets of p^2 points in $\mathrm{AG}(3, p)$ determining few directions, see [5], and is a continuation of similar results in $\mathrm{AG}(2, p)$ starting in [171, 193]. The assumption on the number of directions was weakened to the property that every hyperplane contains $0 \pmod p$ of the points in [55]. There the authors proved the cylinder conjecture for $(4, 1, 2)$ and $(4, 1, 3)$. A relaxed version of the cylinder conjecture for $(4, 1, p)$ was proven for all primes $p \le 13$, see [55] for the details.

Our first observation is that the standard equations can be used to deduce the existence of a hyperplane with multiplicity zero.

Exercise 11.15. *Let* $\mathcal{M}$ *be a* q^r*-divisible set of* q^{r+1} *points in* $\mathrm{PG}(v-1, q)$ *with* $\dim(\mathcal{M}) = v$ *and* $r \in \mathbb{N}$*. Use the standard equations to show* $a_0 \ge \frac{q^{v-r-1}-1}{q-1} \ge 1$*.*

In other words, it makes no difference if we consider sets of points in $\mathrm{AG}(v-1, q)$ or $\mathrm{PG}(v-1, q)$. However, the (or at least a) assumption on the maximum point multiplicity is essential, since q arbitrary points of multiplicity q each also form a q-divisible multiset of cardinality q^2 that is not a cylinder unless the q points form an affine line. If $v \le r+1$

then no set of q^{r+1} points does exist in $\mathrm{PG}(v-1,q)$ at all. For dimension $v = r+2$ the existence of the empty plane leaves the affine $(r+2)$-space as the unique possibility, so that the cylinder conjecture is true for $(r+2,r,q)$.

In [160, Corollary 20] it was shown that the cylinder conjecture is true for (v,r,q) iff it is true for $(v-r+1,1,q)$, i.e., it suffices to study the case $r=1$. Dimension $v=4$ is indeed the smallest case where things start to get non-trivial. In [160] the cylinder conjecture was shown to be true for $(4,1,q)$ for all $q \le 7$ and some partial results for $q=8$ were obtained. If the field size is not a prime and $v > r+3$ is chosen suitably, then cylinders over subfields certify that the cylinder conjecture is wrong for (v,r,q), see [160] for the details.

To sum up, the classification of q^r-divisible sets of q^{r+1} points is quite a challenge, while there is a precise conjecture for field sizes that are prime.

Conjecture 11.16. *The cylinder conjecture is true for all (v,r,p), where p is a prime.*

We remark that e.g. the cylinder conjecture is wrong for $(4,1,8)$. Abbreviating the elements $c_0 + c_1x + c_2x^2 \in \mathbb{F}_2[x]/\left(x^3+x+1\right)$ as $c_0 + 2c_1 + 4c_2$, a generator matrix is given by:

$$\begin{pmatrix}
1111111111111111111111111111111111111111111111111111111000001000\\
0000000000000111111222222333333444444555555666666777777111110100\\
1233345556777123457045666012345123555015557024567012456444570010\\
1001261467356476012037146574610666012023561651000101163134000001
\end{pmatrix}.$$

Research problem

Can this specific counter example be explained from a geometric point of view and generalized to other field sizes?

Exercise 11.17. *Let $\mathcal{M}$ be an 8-divisible set of 64 points in $\mathrm{PG}(3,8)$ that is not a cylinder. Show $a_0 = 29$, $a_8 = 528$, and $a_{16} = 28$ for the spectrum. Moreover, the total number b_i of i-lines is given by $b_0 = 1753$, $b_1 = 1536$, $b_2 = 1344$, $b_4 = 112$ and in a 16-plane the distribution has to be $b_0 = 13$, $b_2 = 48$, $b_4 = 12$.*

12 Extendability results

t-spreads in $\mathrm{PG}(st-1,q)$ exists for all $s \in \mathbb{N}_{\ge 2}$, $t \in \mathbb{N}_{\ge 1}$, see Section 9. If a partial t-spread in $\mathrm{PG}(st-1,q)$ has cardinality $[st]_q/[t]_q-\delta$ then we say that it has *deficiency* δ. The corresponding set of holes, i.e., the set of $\delta[t]_q$ uncovered points, is q^{t-1}-divisible, so that the results stated in Section 11 can be used to show the extendability to a t-spread. More precisely, if δ is small enough such that every q^{t-1}-divisible set of $\delta[t]_q$ points is the disjoint union of δ t-spaces, then each partial t-spread in $\mathrm{PG}(st-1,q)$ with deficiency δ can be extended to a t-spread. For the other direction, e.g. the existence of a maximal partial line spread of size 45 in $\mathrm{PG}(3,7)$, see [106], shows the existence of a 7-divisible set of 40 points over $\mathbb{F}_7$ that is not the disjoint union of five lines.

The non-existence of maximal partial t-spreads does not necessarily imply classification results for $\mathbf{q^{t-1}}$-divisible sets of points.
Note that in general the non-existence of a maximal partial t-spread in $\mathrm{PG}(st-1,q)$ of deficiency δ does not imply that every q^{t-1}-divisible set of $\delta[t]_q$ points over $\mathbb{F}_q$ contains a t-space in its support, see Example 9.15.

Exercise 12.1. *Consider the non-existence proof of maximal partial line spreads of deficiency* 5 *and* 6 *in* $\mathrm{PG}(3,8)$ *given in [59]. Do the details of the proof imply that each* 8*-divisible set of* 9δ *points over* $\mathbb{F}_8$ *is the disjoint union of* δ *lines for all* $\delta \le 6$*? (A maximal partial line spread of deficiency* 7 *is indeed known.)*

In principal we can ask the question of extendability also for partial t-spreads in $\mathrm{PG}(v-1,q)$ where the dimension v of the ambient space is not divisible by t. More generally, we consider a vector space partition $\mathcal{V}$ of type $t^{m_t}\dots s^{m_s}1^{m_1}$ of $\mathrm{PG}(v-1,q)$, see Section 10. Due to Lemma 10.1 the set of holes $\mathcal{H}$, i.e., the set of 1-dimensional elements, is q^{s-1}-divisible. We call $\mathcal{V}$ *k-extendable* if the support of $\mathcal{H}$ contains a full k-space and *extendable* if it is k-extendable for some $k \ge 2$. As an example we refer to a hypothetical vector space partition $\mathcal{V}$ of type $4^1 3^{13} 2^6 1^3$ in $\mathrm{PG}(6,2)$ discussed in Section 10. It would be 2-extendable. However, the non-existence of a vector space partition of type $4^1 3^{13} 2^7$ implies the non-existence of $\mathcal{V}$. So, the question arises for which cardinalities n every q^r-divisible set of n points over $\mathbb{F}_q$ contains a k-space in its support. Certainly, the most restricted and interesting case is $k=r+1$. In this context we mention Example 5.9 and the construction of 2^r-divisible sets of $2^{r+2}-1$ points over $\mathbb{F}_2$ not containing an $(r+1)$-space in its support. So in principal, maximal partial $(r+1)$-spreads in $\mathrm{PG}(2r,2)$ with size one less than the maximum possible cardinality $A_2(2r+1,2r+2;r+1)$, see [17, Theorem 4.1] and Proposition 9.2, may exist. They do indeed exist for $r=2$ as shown in [128]. What about $r>2$ or general field sizes $q>2$?

As partial spreads are just a special case of constant-dimension codes, see Subsection 8.1, one may wonder whether results on the structure of divisible multisets of points can be used to deduce extendability results for constant-dimension codes. To our knowledge, the first extenability results for constant-dimension codes, that are not partial spreads, was shown in [179].

Theorem 12.2. *([179, Theorem 4.2]) Let $\mathcal{C}$ be a set of $\begin{bmatrix} v \\ t \end{bmatrix}_q / \begin{bmatrix} k \\ t \end{bmatrix}_q - \delta$ k-spaces in $\operatorname{PG}(v-1,q)$ such that every t-space is contained in at least one element of $\mathcal{C}$, where $1 < t < k < v$. If $v - i \equiv 0 \pmod{k-i}$ for $i = 0, 1, \dots, t-1$ and $\delta \le (q+1)/2$, then $\mathcal{C}$ can be extended by δ k-spaces without destroying the property on the covering of the t-spaces.*

Corollary 12.3. *([179, Corollary 4.3]) Let $1 < t < k < v$ be integers with $v - i \equiv 0 \pmod{k-i}$ for $i = 0, 1, \dots, t-1$. Then, either $A_q(v, 2k-2t+2; k) = \begin{bmatrix} v \\ t \end{bmatrix}_q / \begin{bmatrix} k \\ t \end{bmatrix}_q$ or $A_q(v, 2k-2t+2; k) < \begin{bmatrix} v \\ t \end{bmatrix}_q / \begin{bmatrix} k \\ t \end{bmatrix}_q - (q+1)/2$.*

A 2-analog of the Fano plane

Let $\mathcal{C}$ be a set of planes in $\operatorname{PG}(6,2)$ such that every line is covered at most once. What is the maximum size $A_2(7,4;3)$ of $\mathcal{C}$? If every line would be covered exactly once, then we would have $\#\mathcal{C} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}_2 / \begin{bmatrix} 3 \\ 2 \end{bmatrix}_2 = 381$ and $\mathcal{C}$ would be called a 2-*analog of the Fano plane.*[1] Corollary 12.3 gives $A_2(7,4;4) = 381$ or $A_2(7,4;4) \le 379$. So, assume $\#\mathcal{C} = 380$ for a moment. Double-counting lines yields that exactly seven lines $L_1, \dots, L_7$ of $\operatorname{PG}(6,2)$ are uncovered by the elements of $\mathcal{C}$. From Lemma 3.12 we know that the multiset of points $\mathcal{M}$ of all points of the elements of $\mathcal{C}$ is 2^2-divisible. Let $\mathcal{C}_P := \{C \in \mathcal{C} : P \le C\}$ denote the elements of $\mathcal{C}$ that contain an arbitrary but fixed point $P \in \mathcal{P}$. Moding out P from $\mathcal{C}_P$ yields a (partial) line spread in $\operatorname{PG}(5,2)$, so that $\#\mathcal{C}_P \le [6]_2/[2]_2 = 21$. Thus, the 21-complement $\overline{\mathcal{M}} := \mathcal{M}^{\complement_{21}}$ of $\mathcal{M}$ is a 2^2-divisible multiset of points with cardinality 7 in $\operatorname{PG}(6,2)$ by Lemma 4.16. Moreover, $\overline{\mathcal{M}} = \chi_\pi$ for a plane π, see e.g. [31], so that $\mathcal{C} \cup \{\pi\}$ covers each point of $\operatorname{PG}(6,2)$ exactly 21 times. In principal, an element $C \in \mathcal{C}$ with $\dim(\pi \cap C) \ge 2$ might exist. However, the seven uncovered lines partition $3 \cdot \overline{\mathcal{M}}$, i.e. $3 \cdot \overline{\mathcal{M}} = \sum_{i=1}^{7} \chi_{L_i}$, so that $\dim(\pi \cap C) \le 1$ for all $C \in \mathcal{C}$ and $A_2(7,4;3) = 381$. We will slightly tighten the “gap” result in a moment.

The currently best lower bound is $A_2(7,4;3) \ge 333$ [116] and if $A_2(7,4;3) = 381$ then a matching code can have an automorphism group of order at most 2 [143].

In the following paragraph we want to generalize the idea of using classification results for divisible multisets of points to show that either $A_2(7,4;3) = 381$ or $A_2(7,4;3) \le 378$. After that example we give a general problem statement in Definition 12.6. Note that we actually have not used the information on the line covering of $3 \cdot \overline{\mathcal{M}}$ for its classification.

Let $\mathcal{M}$ be a k-dimensional 2^2-divisible multiset of cardinality 14 in $\operatorname{PG}(k-1,2)$ such that 14 lines $L_1, \dots, L_{14}$ exist with $3\mathcal{M} = \sum_{i=1}^{14} \chi_{L_i}$. Here the latter condition will be essential, since e.g. $2 \cdot \mathcal{B}$ for a 6-dimensional projective base $\mathcal{B}$ over $\mathbb{F}_2$ is a 2^2-divisible multiset of points in $\operatorname{PG}(6,2)$ with cardinality 14 not fitting our subsequent classification result, see Lemma 12.4. Counting points gives that each hyperplane $H \in \mathcal{H}$ contains exactly $(3 \cdot \mathcal{M}(H) - 14)/2$ out of the 14 lines, so that $\mathcal{M}(H) \in \{6, 10\}$. With this, the standard equations give $a_6 = 3 \cdot 2^{k-2} + 1$, $a_{10} = 2^{k-2} - 2$, and $2^{6-k} - 1 = \sum_{i \ge 2} \binom{i}{2} \lambda_i$.

[1] A Fano plane is a configuration of seven 3-element subsets $\mathcal{B}$ of a 7-set V such that every 2-subset of V is contained in exactly one element $B \in \mathcal{B}$.

Now let H_6 be an arbitrary hyperplane with multiplicity 6. Since $\mathcal{M}|_H$ is 2-divisible and 2 out of the 14 lines are contained in H, there exists a line L in the support of $\mathcal{M}|_H$. Since $\mathcal{M}|_H - \chi_L$ is a 2-divisible multiset of cardinality 3 over $\mathbb{F}_2$, $\mathcal{M}|_H$ is the sum of two lines L, L', i.e. $\mathcal{M}|_H = \chi_L + \chi_{L'}$. Now let P be an arbitrary point with positive multiplicity. Since P is contained in $2^{k-1} - 1$ hyperplanes, P is contained in a hyperplane H of multiplicity 6, so that a line L with $P \le L$ and $L \subseteq \operatorname{supp}(\mathcal{M})$ exists. Since L is contained in $2^{k-2} - 1$ hyperplanes there are at least

$$2^{k-1} - 1 - \left(2^{k-2} - 1\right) - a_{10} = 2$$

hyperplanes of multiplicity 6 that contain P but not L, so that there exists another line $L' \neq L$ with $P \le L'$ and $L' \subseteq \operatorname{supp}(\mathcal{M})$. Now let $E = E(P)$ be the plane spanned by L and L'. If $\mathcal{M}(P) = 1$, then E cannot be contained in a hyperplane of multiplicity 6 due to their classification as the sum of two lines. Thus, every hyperplane H through E has multiplicity $\mathcal{M}(H) = 10$, so that counting points gives $\mathcal{M}(E) = 6 + 2^{6-k}$. Since $2^{k-2} - 2 = a_{10} \ge 0$ and $2^{6-k} - 1 = \sum_{i \ge 2} \binom{i}{2} \lambda_i \ge 0$, we have $3 \le k \le 6$, so that $\lambda_2 + 3\lambda_3 + 6\lambda_4 = 2^{6-k} - 1 \le 7$ and $\lambda_i = 0$ for $i \ge 5$. If $\lambda_1 = 0$, then $k = 3$ and $\lambda_2 = 7$, i.e., $\mathcal{M} = 2 \cdot \chi_\pi$ for some plane π. If $k = 6$, then $\lambda_1 = 14$, and we can apply Lemma 11.6 to deduce that $\mathcal{M}$ is the sum of two planes. If $\lambda_1 > 0$, then we can choose a point P with multiplicity $\mathcal{M}(P) = 1$ and construct the plane $E(P)$ as described above. For $k = 5$ we conclude $\mathcal{M}(E(P)) = 8$, $\lambda_2 = 1$, and $\lambda_i = 0$ for $i \ge 3$, so that $E(P) \subseteq \operatorname{supp}(\mathcal{M})$. For $k = 4$ we conclude $\mathcal{M}(E(P)) = 10$ and $\lambda_2 + 3\lambda_3 + 6\lambda_4 = 6$, so that $\lambda_2 = 3$, $\lambda_3 = \lambda_4 = 0$, and $E(P) \subseteq \operatorname{supp}(\mathcal{M})$. So, in both remaining cases $k \in \{4, 5\}$ the plane $E(P)$ is contained in the support of $\mathcal{M}$ and $\mathcal{M} - \chi_{E(P)}$ is a 2^2-divisible multiset of points of cardinality 7 over $\mathbb{F}_2$. Thus, we have:

Lemma 12.4. *Let $\mathcal{M}$ be a 2^2-divisible multiset of points of cardinality 14 over $\mathbb{F}_2$. If there exist 14 lines $L_1, \dots, L_{14}$ such that $3 \cdot \mathcal{M} = \sum_{i=1}^{14} \chi_{L_i}$, then $\mathcal{M}$ is the sum of two planes.*

Exercise 12.5. *Use Lemma 12.4 to show that either $A_2(7,4;3) = 381$ or $A_2(7,4;3) \le 378$.*

Definition 12.6. *For integers $1 \le t \le r$ we denote by $m_q(r,t)$ the smallest number δ such that there exists a q^r-divisible multiset of points $\mathcal{M}$ over $\mathbb{F}_q$ with cardinality $\delta[r+1]_q$ that is not the union of δ $(r+1)$-spaces but where for each $1 \le j \le t$ there exist $\delta \begin{bmatrix} r+1 \\ j \end{bmatrix}_q$ j-spaces S_1^j, S_2^j, … such that*

$$\begin{bmatrix} r+2-j \\ j-1 \end{bmatrix}_q \cdot \mathcal{M} = \sum_{i=1}^{\delta \left[\begin{smallmatrix} r+1 \\ j \end{smallmatrix}\right]_q} \chi_{S_i^j}. \qquad (12.1)$$

In Lemma 12.4 we have shown $m_2(2,2) \ge 3$ and the two-weight code in Example 11.10, which can be partitioned into seven lines, yields $m_2(2,2) \le 3$, so that $m_2(2,2) = 3$.

Exercise 12.7. *Show that $m_2(3,2) \ge 3$.*

Exercise 12.8. *Show that the* $[45, 8, \{21, 29\}]_2$ *two-weight code described in [99, Theorem 4.1] can be partitioned into* 15 *lines.*

So, we have $m_2(3, 2) = 3$.

An application for partial MRD codes

For two $m \times n$-matrices A and B the *rank distance* is given by the rank $\mathrm{rk}(A - B)$ of their difference. A set $\mathcal{M}$ of such matrices over $\mathbb{F}_q$ with minimum rank distance d is called a *maximum rank distance* (MRD) code if it has the maximum possible size $q^{\max\{m,n\}\cdot(\min\{m,n\}-d+1)}$ and those codes indeed exist for all parameters. Considering the row spans of the matrices $(I|M)$ for all $M \in \mathcal{M}$, where a $m \times m$ identity matrix was put in front of M, gives a constant-dimension code $\mathcal{C}$, called *lifted MRD code*, of m-spaces such that there exists an n-space S which is disjoint to the elements of $\mathcal{C}$. In the remaining part we choose the specific parameters $q = 2$, $m = n = 4$, and $d = 3$. Here we have $\#\mathcal{C} = 256$, every line with trivial intersection with the special solid S is covered exactly once by the elements of S, and each point P not contained in S is contained in exactly 16 elements from $\mathcal{C}$. Now assume that $\mathcal{C}$ satisfies the same conditions as before but does not have the maximum possible cardinality, so that we speak of a lifted partial MRD code. If $\mathcal{M}$ is the multisets of points of the elements of $\mathcal{C}$, then we can apply our result $m_2(3, 2) \ge 3$ to the 16-complement of $\mathcal{M} + 16 \cdot \chi_S$ in order to conclude that for $\#\mathcal{C} \in \{256 - 1, 256 - 2\}$ an extension to a lifted MRD code exists, cf. [117, Proposition 6]. Clearly, also the analog statement for the partial MRD code holds.

Exercise 12.9. *Let* $\mathcal{M}$ *be an* $m \times n$ *rank-metric code, where* $m \le n$*, over* $\mathbb{F}_q$ *with minimum rank distance* d *whose cardinality is* δ *less than that of an MRD code with the same parameters. Show that if* $\delta < m_q(m - 1, m + 1 - d)$*, then* $\mathcal{M}$ *is extendable to an MRD code.*

Research problem

Does there exist a set of $256-3 = 253$ 4×4-matrices over $\mathbb{F}_2$ with minimum rank distance 3 that is maximal, i.e., where no further matrix can be added without decreasing the minimum rank distance.

Covering radius

The covering radius $\rho(C)$ of a code C in V is the smallest integer r such that every element of V has a distance of at most r to an element of C. If the covering radius is larger or equal to the minimum distance of C, then C is extendable. For rank-metric codes results on the covering radius can be found in [45].

Another application of $m_2(2, 2) = 3$ is that each set of $93\cdot3-2$ planes in $\mathrm{PG}(5, 2)$ such that very line is covered at most thrice can be completed to a corresponding 2-(subspace) design with $\lambda = 3$ over $\mathbb{F}_2$.

13 Dimensions of divisible codes

The dimension k of a linear code of length n over $\mathbb{F}_q$ can be at most n, which is attained by a identity matrix as generator matrix. An even binary linear code of length n can have dimension at most $n-1$, which is attained by the codes consisting of all even weight codewords, i.e., projective bases. Higher divisibility implies tighter bounds. A double-even binary linear code with effective length n has dimension at most

$$\begin{array}{lcl} 4\lfloor n/8 \rfloor & : & \operatorname{rem}(n,8) \in \{0,1,2,3\}, \\ 4\lfloor n/8 \rfloor + 1 & : & \operatorname{rem}(n,8) \in \{4,5\}, \\ 4\lfloor n/8 \rfloor + \operatorname{rem}(n,8) - 4 & : & \operatorname{rem}(n,8) \in \{6,7\}, \end{array} \tag{13.1}$$

where $\operatorname{rem}(n,a)$ denotes the remainder of n divided by a. Equality can indeed be attained, see e.g. [83, Section VIII]. In general the dimension is upper bounded by $n/2$ and equality is attained for self-dual codes only, where especially n is divisible by 8. For 2^3-divisible binary linear codes with effective length n the dimension is at most $5n/16$, see [170] for the details.

When using the linear programming method to exclude the existence of certain lengths of divisible codes, upper bounds on the dimension are certainly useful. With respect to lower bounds we remark that a single codeword of weight Δ always generates a 1-dimensional code. If the maximum point (or column) multiplicity is at most γ then k clearly has to at least as large so that $\gamma[k]_q \geq n$, where n is the effective length. In [130] a general tool was used to compute an upper bound on the minimal dimension of a projective binary linear code C of length n. To this end let $\mathcal{M}$ be the corresponding set of points in $V := \mathrm{PG}(k-1,2)$. Let Q be a point not contained in $\mathcal{M}$, i.e., $\mathcal{M}(Q) = 0$. Consider the projection of $\mathcal{M}$ modulo Q, that is the multiset image of $\mathcal{M}$ under the map $V \to V/Q$, $\mathbf{v} \mapsto (\mathbf{v}+Q)/Q$. The resulting multiset $\mathcal{M}'$ in $\mathrm{PG}(V/Q) \cong \mathrm{PG}(k-2,2)$ arises by identifying points of $\mathcal{M}$ on the same line through Q. The corresponding linear code C' is a subcode of C of effective length n and dimension $k-1$. If C is Δ-divisible, so is C'. The assumed minimality of k implies that C' is not projective. Equivalently, there is a secant through Q, that is a line whose remaining two points are contained in $\mathcal{M}$. So each of the $[k]_2 - n$ points of V not contained in $\mathcal{M}$ lies on a secant. Since $\mathcal{M}$ admits at most $\binom{\#\mathcal{M}}{2} = \binom{n}{2}$ secants, covering at most $\binom{n}{2}$ different points not in $\mathcal{M}$, we get

$$[k]_2 - n \leq \binom{n}{2} \quad\iff\quad 2^k \leq \frac{n^2+n+2}{2}. \tag{13.2}$$

In [130] this inequality was used to conclude the non-existence of a projective 2^3-divisible binary linear code of length 59 from the non-existence of projective 2^3-divisible $[59, \leq 10]_2$-codes.

The divisible code bound

Let $q = p^f$ and v_p be the *p-adic valuation on* $\mathbb{Z}$, i.e., $v_p(x)$ is the exponent of the highest power of p dividing x, with $v_p(0) = \infty$.

Theorem 13.1. *([208], [210, Theorem 7], [211, Theorem 6]) Let C be an $[n,k]_q$-code whose non-zero word weights lie in the sequence $(b-m+1)\Delta, \ldots, b\Delta$ of m consecutive multiples of Δ. Then*

$$kv_p(q) \le m\big(v_p(\Delta) + v_p(q)\big) + v_p\left(\binom{b}{m}\right). \tag{13.3}$$

Example 13.2. *Let C be a projective 2^3-divisible binary linear code of length 59. Since the residual code of each codeword $\mathbf{c}$ is a projective 2^2-divisible binary linear code of length $59 - \mathrm{wt}(\mathbf{c})$, the non-zero weights of C are contained in $\{8, 16, 24, 32, 40\}$. Theorem 13.1 with $b = 5$, $m = 5$, and $\Delta = 8$ gives $k \le 20$.*

Improvements of the divisible code bound can be found in [168, 169].

Exercise 13.3. *Show that each 3^r-divisible $\left[3^{r+1}, k\right]_3$-code satisfies $k \le 3r + 3$ for all $r \in \mathbb{N}$. Additionally assume that there is no codeword of weight 3^r to deduce $k \le 2r + 3$.*

In [160] it was shown that each projective 3^r-divisible $\left[3^{r+1}, k\right]_3$-code satisfies $k \le r+3$ for all $r \in \mathbb{N}$.

Research problem

Improve the divisible code bound for projective q^r-divisible codes over $\mathbb{F}_q$.

Exercise 13.4. *Let C be a binary q^r-divisible code effective length n that is spanned by codewords of weight 2^r, where $r \in \mathbb{N}_{\ge 2}$. Show that the dimension k of C satisfies $k \le n \cdot (r+2)/2^{r+1}$ and that equality can be attained if n is divisible by 2^{r+1}.*
Hint: Use the classification in Section 14.

Conjecture 13.5. *Let r be an integer and $\eta_q(r,k)$ be the minimum possible length n of a q^r-divisible $[n,k]_q$-code. For $r \ge 2$ we have $\eta_2(r,k) = 2^{r-k+1} \cdot [k]_2$ for $k \le r+1$ and $\eta_2(r,k) = \eta_2(r, k-r-2) + 2^{r+1}$ for $k \ge r+2$.*

14 Enhancing the linear programming method with additional constraints

Here we want to continue our discussion of the linear programming method from Subsection 2.1 and discuss a few additional conditions. First we note that the number of even-weight codewords of an $[n,k]_2$-code can just take one of two possible values, i.e.,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} A_{2i} \in \left\{2^{k-1}, 2^k\right\}. \tag{14.1}$$

Exercise 14.1. *Let C be an $[n,k]_2$-code. Show that the set of codewords of even weight forms a subcode of dimension at least $k-1$.*

Example 14.2. *We can use Equation (14.1) in order to e.g. show that each $[\le 16, 4, 7]_2$ code contains at least one codeword of weight 8, cf. [157, Lemma 3.1]. Assume that C is an $[n,4,7]_2$ code with $n \le 16$ and $A_8 = 0$. From the first two MacWilliams equations we conclude*

$$A_7 + A_9 + \sum_{i\ge 10} A_i = 2^4 - 1 = 15 \quad \textit{and} \quad 7A_7 + 9A_9 + \sum_{i \ge 10} iA_i = 2^3 n = 8n,$$

so that

$$2A_9 + 3A_{10} + \sum_{i\ge 11} (i-7)A_i = 8n - 105.$$

Thus, the number of even weight codewords is at most $8n/3 - 34$. Since at least half of the codewords have to be of even weight, we obtain $n \ge \lceil 15.75 \rceil = 16$. In the remaining case $n = 16$ we use the linear programming method with the first four MacWilliams identities, $A_8 = 0$, $B_1 = 0$, and the fact that there are exactly 8 even weight codewords to conclude $A_{11} + \sum_{i\ge 13} A_i < 1$, i.e., $A_{11} = 0$ and $A_i = 0$ for all $i \ge 13$. With this and rounding to integers we obtain the bounds $5 \le B_2 \le 6$, which then gives the unique solution $A_7 = 7$, $A_9 = 0$, $A_{10} = 6$, and $A_{12} = 1$. Computing the full dual weight distribution unveils $B_{15} = -2$, which is negative.

The subcode in Exercise 14.1 is also called *even weight subcode* and its dimension equals k iff C is even itself. We have the following generalization, see [40, Section IV]:

Proposition 14.3. *Let C be an even $[n,k]_2$-code and t be the maximum dimension of a doubly-even subcode. Then, for the set D of codewords of C whose Hamming weight is divisible by 4 we have*

$$|D| = \sum_{i=0}^{\lfloor n/4 \rfloor} A_{4i} \in \left\{2^{k-1} - 2^t, 2^{k-1}, 2^{k-1} + 2^{t-1}, 2^k\right\}. \tag{14.2}$$

In the context of linear codes with maximum possible minimum distance it suffices to consider even codes, so that Proposition 14.3 gives an extra condition for the linear programming method. In the context of (binary) divisible codes we commonly have even higher divisibility constants and [40, Theorem 2] states that the number of codewords with weight divisible by 2^a of a 2^{a-1} binary linear code C is at least $|C|/2^a$. This bound was e.g. used in [40] in order to show the non-existence of $[124, 9, 60]_2$-code. We have the following refinement and generalization of Proposition 14.3:

Proposition 14.4. *([64, Proposition 5], see also [197]) Let C be an $[n, k, d]_2$-code with all weights divisible by $\Delta := 2^a$ and let $(A_i)_{i=0,1,\dots,n}$ be the weight distribution of C. Put*

$$\begin{aligned} \alpha &:= \min\{k-a-1, a+1\}, \\ \beta &:= \lfloor (k-a+1)/2 \rfloor, \text{ and} \\ \delta &:= \min\{2\Delta i \mid A_{2\Delta i} \neq 0 \wedge i > 0\}. \end{aligned}$$

Then the integer

$$T := \sum_{i=0}^{\lfloor n/(2\Delta) \rfloor} A_{2\Delta i}$$

satisfies the following conditions.

(i) T is divisible by $2^{\lfloor (k-1)/(a+1) \rfloor}$.

(ii) If $T < 2^{k-a}$, then

$$T = 2^{k-a} - 2^{k-a-t}$$

for some integer t satisfying $1 \le t \le \max\{\alpha, \beta\}$. Moreover, if $t > \beta$, then C has an $[n, k-a-2, \delta]_2$-subcode and if $t \le \beta$, it has an $[n, k-a-t, \delta]_2$-subcode.

(iii) If $T > 2^k - 2^{k-a}$, then

$$T = 2^k - 2^{k-a} + 2^{k-a-t}$$

for some integer t satisfying $0 \le t \le \max\{\alpha, \beta\}$. Moreover, if $a = 1$, then C has an $[n, k-t, \delta]_2$-subcode. If $a > 1$, then C has an $[n, k-1, \delta]_2$-subcode unless $t = a+1 \le k-a-1$, in which case it has an $[n, k-2, \delta]_2$-subcode.

Example 14.5. *An implication of Proposition 14.4 is that no projective $[32, 10, \{8, 16, 24\}]_2$-code exists, see [142] for the context and an application. From the first three Mac Williams equations we compute $A_8 = 61$, $A_{16} = 899$, and $A_{24} = 63$. Applying Proposition 14.4 with $a = 3$ gives $\Delta = 8$, $\alpha = 4$, $\beta = 4$, $\delta = 16$, and $T = 900$. As required by Part (i), T is divisible by 4. However, Part (iii) gives $t = 5$, which contradicts $0 \le t \le \max\{\alpha, \beta\} = 4$, so that such a code cannot exist.*

The general idea behind Proposition 14.3 is to consider $\sum_{i \in I} A_i$, for some subset $I \subseteq \{1, \dots, n\}$, as weights of codewords in (generalized) Reed-Muller codes, see [95, 197] for the details. It is well known that the occurring weights of generalized Reed-Muller codes have some gaps, e.g.:

Proposition 14.6. *([177], see also [167]) Let C be a second order q-ary generalized Reed-Muller code of length q^k. Then, all non-zero weights of C are of the form*

$$q^k - q^{k-1} - \nu q^{k-1-j}, \tag{14.3}$$

where $\nu \in \{0, \pm 1, \pm(q-1)\}$ and $0 \le j \le \lfloor k/2 \rfloor$.

For more such "gap" results we refer to e.g. [95]. Results similar to Proposition 14.4 for field sizes $q \in \{3, 4\}$ were used in e.g. [95, 96, 97, 98].

Δ-divisible codes spanned by codewords of weight Δ

The characterization of indecomposable self-orthogonal binary codes which are spanned by codewords of weight 4 from [191, Theorem 6.5] was generalized in [139, Theorem 1]:

Theorem 14.7. *Let Δ be a positive integer and let a be the largest integer such that q^a divides Δ. Let C be a q-ary Δ-divisible linear code that is spanned by codewords of weight Δ. Then C is isomorphic to the direct sum of codes of the following form, possibly extended by zero positions:*

(i) The $\frac{\Delta}{q^{k-1}}$-fold repetition of the q-ary simplex code of dimension $k \in \{1, \ldots, a+1\}$.

In the binary case $q = 2$ additionally:

(ii) The $\frac{\Delta}{2^{k-2}}$-fold repetition of the binary first order Reed-Muller code of dimension $k \in \{3, \ldots, a+2\}$.

(iii) For $a \ge 1$: The $\frac{\Delta}{2}$-fold repetition of the binary parity check code of dimension $k \ge 4$.

Up to the order, the choice of the codes is uniquely determined by C.

We remark that if $C = C_1 \oplus C_2 \oplus \cdots \oplus C_l$ is the direct sum of l linear codes C_i, then we have

$$W_C(x) = W_{C_1}(x) W_{C_2}(x) \ldots W_{C_l}(x)$$

for the weight enumerator. We have $A(C_i)_\Delta = [k]_q$, $A(C_i)_\Delta = [k]_2 - 1$, and $A(C_i)_\Delta = \binom{k+1}{2}$, in cases (i), (ii), and (iii) of Theorem 14.7, respectively. This can of course be used to compute $A(C)_\Delta$.

Exercise 14.8. *Let C be a binary linear code with non-zero weights in $\{8, 16, 24\}$ that is spanned by codewords of weight 8. Then, we have*

$$\begin{aligned} A_8 \in \; & \{0,1,2,3,4,6,7,8,9,10,11,13,14,15, \\ & 16,17,18,21,22,25,29,30,31,33,37,45\}. \end{aligned}$$

Note that the non-existence result in Example 14.5 is a direct implication.

Exercise 14.9. *Let $a \in \mathbb{N}_{\ge 3}$, $\Delta = 2^a$, and C be a (projective) Δ-divisible $[4\Delta, k]_2$-code. Show $k \le 2a + 4$, cf. [170, Theorem 4].*

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