Generalized Triangular Numbers and Combinatorial Explanations

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Abstract

The formula for the sums of the first integers, which are known as triangular numbers, is well known and there are many proofs for it: by induction, graphical, by combinatorics, etc. The sum of the first triangular numbers is known as tetrahedral numbers. In this article[1], we discuss a generalization of triangular and tetrahedral numbers where the number of summation symbols is variable. We repeat results from the literature that state that these so-called generalized triangular numbers can be represented via multicombinations, i.e. combinations with repetitions, and give an illustrative explanation for this formula, which is based on combinatorics. Via high-dimensional illustrations, we show that these generalized triangular numbers are figurate numbers, namely hyper-tetrahedral numbers, see Figure 1. Additionally, we demonstrate that there is a relation between the height and the dimension of these hypertetrahedra, i.e. a series of generalized triangular numbers with fixed dimension and varying height can be represented as such a series with fixed height and varying dimension, and vice versa.

1 Motivation

Long before the time of Carl Friedrich Gauss[2] the formula

$$\sum_{n_1=1}^{n_2} n_1 = \frac{(n_2+1)n_2}{2} = \binom{n_2+1}{2}$$

for the triangular numbers was known, e.g., in the time of the medieval Irish monk Dicuil[3] [2], probably also in the time of the ancient Greeks, i.e. in the time of Pythagoras or the

[1]This article is building upon the paper [1] from the same author.

[2]a.k.a. Carl Friedrich Gauß

[3]a.k.a. Dikuil and Dicuilus

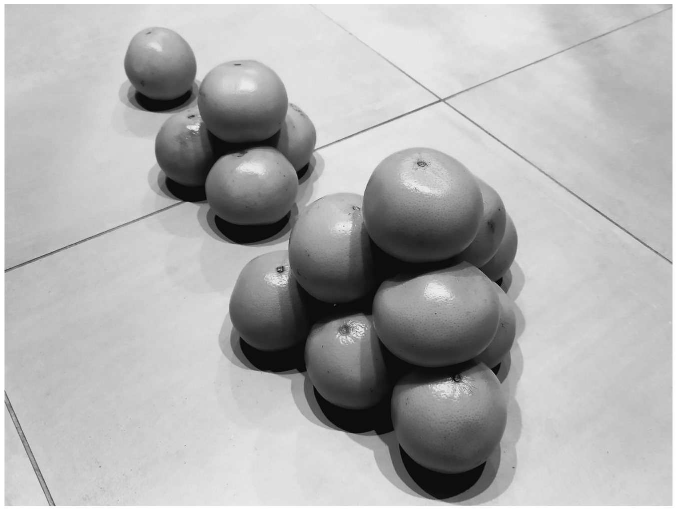

Figure 1: A photography of a four-dimensional hypertetrahedron of height three consisting of 15 grapefruits. A four-dimensional hypertetrahedron is a line of three-dimensional pyramids. Note that it is not a coincidence that the number of grapefruits is a triangular number $\left(\sum_{n_1=1}^{5} n_1 = 15\right)$, cf. Equation (5).

Pythagoreans [3], and maybe even before them (see also [4, 5]). There is an overwhelming number of visual, illustrative, formal, and/or combinatorial proofs for it, some of them are so-called "proofs without words." Known is a story about the young Gauss discovering the formula (i.a. [6, 7]), which is why the formula is sometimes called "Gaussian Summation Formula" or briefly "Little Gauss." Note that we intentionally have used here the terms $n_2, n_1 \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and not, say, n, i. The reason is the following: in this article we are going to have a look at generalizations of the triangular numbers (and of $n_1 = \binom{n_1}{1}$), such as the tetrahedral numbers[4]

$$\sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1 = \frac{(n_3+2)(n_3+1)n_3}{6} = \binom{n_3+2}{3}$$

In [1], a generalization of the formula for triangular and tetrahedral numbers, namely

$$\sum_{n_{k-1}=1}^{n_k} \sum_{n_{k-2}=1}^{n_{k-1}} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1 = \binom{n_k+k-1}{k} \tag{1}$$

[4]A famous example of tetrahedral numbers is hidden in the English Christmas carol "The Twelve Days of Christmas," whose oldest known printed version dates back to the 18th century (see [8, 9]). In it, on day 1, 1 present of type 1 is received; on day 2, 1 present of type 1 and 2 presents of type 2 are received; on day 3, 1 present of type 1, 2 presents of type 2, and 3 presents of type 3 are received; and so forth. In total, there are $\sum_{n_2=1}^{12} \sum_{n_1=1}^{n_2} n_1 = \sum_{n_2=1}^{12} \frac{n_2(n_2+1)}{2} = 364$ presents received, which is the twelfth tetrahedral number, i.e. the sum of the first twelve triangular numbers. Another way to count the total number of presents is to sum over the different types, leading to $\sum_{i=1}^{12} i(13-i) = 364$. Confer [10].

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with $n_k \in \mathbb{N}$ and $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, is given and proven. The cases $k = 0$ and $k = 1$ need some further explanation when having a look at the left-hand side of Equation (1). In the case $k = 1$, there is no summation sign, i.e., the left-hand side equals n_1. The case $k = 0$ is more tricky: when having a look at Figure 2, we observe that for $k = 3$ the numbers can be illustrated as tetrahedra (3d), for $k = 2$ this is true for triangles (2d), for $k = 1$ they are lines (1d), thus, for $k = 0$ it is convenient to illustrate the "numbers" as a dot (0d), leading to the definition of the left-hand side as 1, *constant*. We call $\sum_{n_{k-1}=1}^{n_k} \sum_{n_{k-2}=1}^{n_{k-1}} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1$ the n_k-th generalized triangular number of dimension k. For example, the series of the three-dimensional generalized triangular numbers is the series of the tetrahedral numbers. The generalized triangular numbers equal one whenever $k = 0$ or $n_k = 1$. Here we note that $k - 1$ for $k = 2, 3, 4, \ldots$ is the number of summation signs and $n_k \in \mathbb{N}$ is the number on top of the first summation sign (i.e. on the far left). The number n_k is called the height. It does not depend on k (i.e., it is not $n(k)$), but the index k is only used to distinguish the k different variables $n_1, n_2, \ldots, n_k$. The variables $n_1, \ldots, n_{k-1}$ are bound while n_k and k are free.

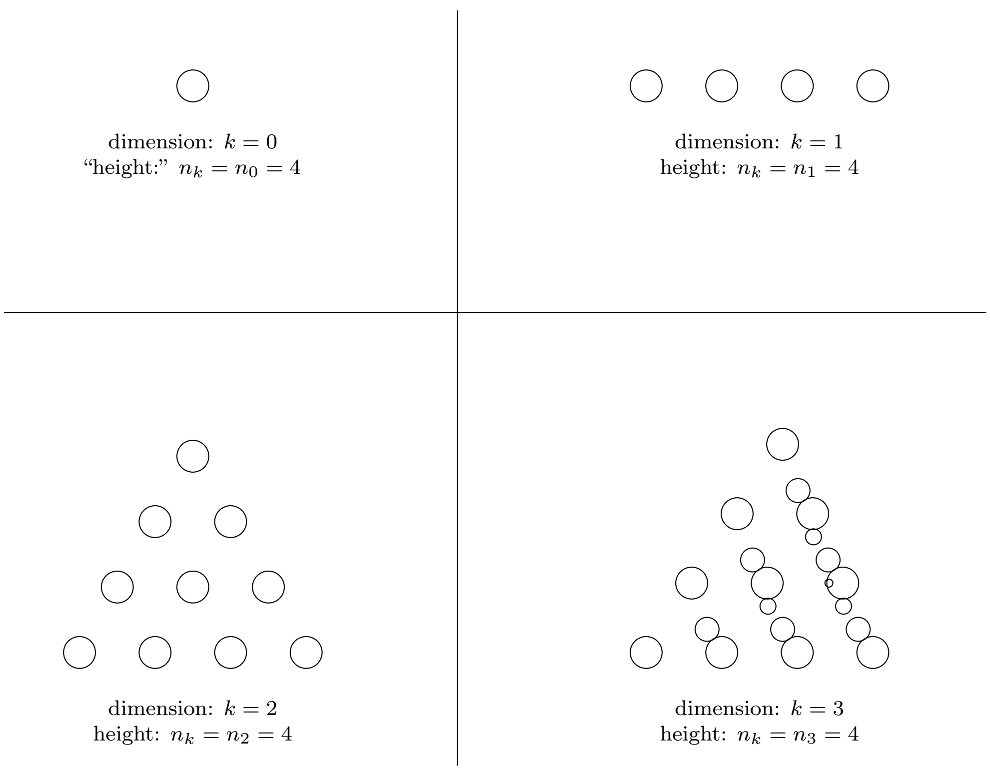

Figure 2: Hypertetrahedra of height four in dimensions zero (dot), one (line), two (triangle), and three (tetrahedron). The triangle in the lower left corner, i.e. the hypertetrahedron of dimension two and height four, consists of ten dots; it was of special interest to the Pythagoreans and is called "Tetractys."

In [1], Equation (1) is proven by induction and an illustrative derivation is presented.

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This illustrative derivation goes as follows: It is well known that if in a group of $n_2 + 1$ people everyone shakes hands with everyone, there will be a total of $1 + 2 + 3 + \ldots + n_2$ handshakes (the first one greets n_2 people, the second one another $n_2 - 1$, etc.), but there will also be $\binom{n_2+1}{2}$ greetings, since there are just as many pairs. So, according to the principle of double counting,[5] these two numbers are equal. The same reasoning, with a group of $n_k + k - 1$ people where all subgroups of exactly k people perform some "handshake" together, leads to the given formula.

The purpose of this article is twofold: First, we shortly rephrase the formal proof from [1]—which is not written in English, but in German—in order to ensure self-containedness of the work at hand. Second, we will give an illustrative combinatorial explanation of it. This will use multicombinations (i.e. combinations with repetitions).[6] Further, to show in which sense generalized triangular numbers are figurate numbers, geometric illustrations are given, cf. [12, 13]. Via these geometric illustrations further insights, esp. on heights two and three, are given. Additionally, it is explained how dimensions and heights are related.

2 Proof of Equation (1)

The proof is very briefly rephrased from [1] and is included here for completeness. First, we show that for all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, it holds:

$$\sum_{i=1}^{n} \frac{(i+k-1)!}{(i-1)!} = \frac{(n+k)!}{(k+1)(n-1)!} \tag{2}$$

Let $n = 1$ and $k \in \mathbb{N}_0$ arbitrary, then on both sides $k!$ remains. Now, suppose the equation is true for some $n \in \mathbb{N}$ and all $k \in \mathbb{N}_0$. Let $k \in \mathbb{N}_0$ be arbitrary. We compute:

$$\sum_{i=1}^{n+1} \frac{(i+k-1)!}{(i-1)!} = \frac{(n+k)!}{n!} + \frac{(n+k)!}{(k+1)(n-1)!} = \frac{(n+k)!(n+k+1)}{n!(k+1)}$$

This completes the proof of Equation (2). Now, we turn to the formal proof of Equation (1). For $k = 0$, both sides equal 1. For $k = 1$ both sides equal n_1. Now, we assume that Equation (1) holds for some $k \in \mathbb{N}_0$ and all $n_k \in \mathbb{N}$. Let $n_{k+1} \in \mathbb{N}$ be arbitrary. We

[5] The principle of double counting states that when there is a finite set, which might depend on a parameter (or also on some parameters), and we find two (or more) ways to count its cardinality and to express these numbers, i.e. the cardinalities via one term each, these terms have to be equal.

[6] The author is grateful to Jochen Ziegenbalg (Pädagogische Hochschule Karlsruhe, Karlsruhe, Germany; i.R.) for suggesting to include a real world photography (Figure 1) for the purpose of illustration and to think about the relationship between generalized triangular numbers and multicombinations after he read the article [1] by the author, which is related to [11].

calculate:

$$\sum_{n_k=1}^{n_{k+1}} \sum_{n_{k-1}=1}^{n_k} \sum_{n_{k-2}=1}^{n_{k-1}} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1 = \sum_{n_k=1}^{n_{k+1}} \binom{n_k+k-1}{k}$$
$$= \frac{1}{k!} \sum_{n_k=1}^{n_{k+1}} \frac{(n_k+k-1)!}{(n_k-1)!}$$
$$\overset{(2)}{=} \frac{(n_{k+1}+k)!}{(k+1)!(n_{k+1}-1)!}$$

Quod erat demonstrandum.

3 Multicombinations

A well-known problem from combinatorics is the following: *How many ways are there to distribute k indistinguishable balls into n numbered (sufficiently large) drawers?* This number is known as multicombinations[7] and denoted by $\left(\!\binom{n}{k}\!\right)$, see [14]. One can calculate this number of possibilities by considering that it is literally the same

- to distribute k indistinguishable balls into n numbered (sufficiently large) drawers

and

- to distribute the $n-1$ separators, which represent the boundaries between the n drawers, and the k balls into $n+k-1$ places (where the places are numbered, but both the balls and the separators are indistinguishable in each case).

Hence, it follows by the principle of double counting that

$$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$$

Not only due to the definition of binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, but also by the principle of double counting it is clear that $\binom{n+k-1}{k} = \binom{n+k-1}{n-1} = \left(\!\binom{n}{k}\!\right)$ holds since it does not matter whether to distribute the $n-1$ separators or the k balls. Furthermore, it holds $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{n-1} = \binom{(k+1)+(n-1)-1}{(n-1)} = \left(\!\binom{k+1}{n-1}\!\right)$. When reconsidering Equation (1), we observe that

$$\sum_{n_{k-1}=1}^{n_k} \sum_{n_{k-2}=1}^{n_{k-1}} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1 = \left(\!\binom{n_k}{k}\!\right) \tag{3}$$

holds. In the next paragraph, we will explain this equality in a combinatorial way.

[7]Multicombinations are also named combinations with repetitions, multiset coefficients, or multiset numbers.

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4 The combinatorial explanation

First of all, we mention that the value of the n_k-th k-dimensional generalized triangular number $\sum_{n_{k-1}=1}^{n_k} \sum_{n_{k-2}=1}^{n_{k-1}} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1$ can be illustrated as a k-dimensional hypertetrahedron with height n_k, see Figure 2 for dimensions zero to three and Figure 1 for a photography illustrating a four-dimensional hypertetrahedron. Since the imagination of figures in dimensions four and higher is not that easy, in Figures 3 and 4 hypertetrahedra of height three are depicted in different dimensions (dimensions zero to eight).[8] Note that in all dimensions hypertetrahedra of height one are just zero-dimensional single dots.

With that, we may rephrase Equation (1) a little bit informally as:

$$\begin{aligned}
&\#\text{“Dots in a } k\text{-dimensional hypertetrahedron of height } n_k\text{''} = \\
&= \sum_{n_{k-1}=1}^{\text{height}} \sum_{n_{k-2}=1}^{n_{k-1}} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1 \\
&= \binom{\text{height plus dimension minus one}}{\text{dimension}}
\end{aligned}$$

Why higher-dimensional hypertetrahedra can be illustrated as we have done, we now explain briefly. A 3d tetrahedron can be drawn by iteratively drawing 2d triangles (see Figure 3, top right and middle left) in the following way. One draws a triangle in the size the tetrahedron shall have, then a triangle that is one row smaller, a triangle that is again one row smaller and so on until finally a single dot is drawn. In the same way a 2d triangle can be drawn by iteratively drawing 1d lines (see Figure 3, top center and top right). One draws a line, then a line that is one dot smaller, a line that is again one dot smaller, and so on until a single dot is drawn in the end. Thus, a 4d hypertetrahedron can be drawn by iteratively drawing 3d tetrahedra (see Figure 3, middle left and middle right). One draws one tetrahedron in the desired size, one tetrahedron that is one plane smaller, one that is again one plane smaller, etc., up to single dot. And that way it works for all other dimensions, too.

Now, we are going to deal with the question how to count the number of dots in a k-dimensional hypertetrahedron of height n_k. We find two different ways to do this and use the principle of double counting to deduce Equation (1). It is obvious that the number of dots in a k-dimensional hypertetrahedron of height n_k is equal to the sum on the left-hand side of Equation (1). Thus, let us now attempt to find a combinatorial explanation for why it also corresponds to the right-hand side.

Before we turn to the k-dimensional case, we first look at Figure 6. On the left-hand side we see a triangle of height four. The upper right side of this triangle is marked in gray. It is a (one-dimensional) line of length (or height) four. We can represent this line as a hypertetrahedron with $k = 1$ and $n_k = n_1 = 4$. The triangle has $k + 1 = 2$ dimensions.

[8] Additionally, in Figure 5 hypertetrahedra of height two are depicted in different dimensions (zero to nine). However, due to the shallowness of the hypertetrahedra of height two, Figure 5 might not be as insightful as Figures 3 and 4.

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Clearly, the line consists of $n_1 = 4$ dots. From another point of view we see that there is a unique black dot opposite the fixed gray line. Further, we observe that if we always jump in the direction from the black dot to the gray line from one dot to a neighboring dot, we can get from this unique black dot to the gray line within $n_k - 1 = 3$ steps, each in two (i.e., $k + 1$) possible directions. In which dot we end up is determined by how many times we go in which of the $k + 1$ directions; but not by the order in which we do so. We always arrive at the line after $n_k - 1 = 3$ steps, and each dot can be reached within three such steps. That is, the number of dots in the line corresponds to the number of ways to distribute $n_k - 1 = 3$ steps arbitrarily among $k + 1 = 2$ directions. *How many ways are there to distribute $n_k - 1 = 3$ steps arbitrarily to $k + 1 = 2$ directions? Put another way, how many ways are there to arbitrarily distribute $n_k - 1 = 3$ balls among $k+1 = 2$ drawers?* The answer is $\binom{(n_k-1)+(k+1)-1}{(n_k-1)} = \binom{n_k-1+k}{k} = \left(\!\binom{n_k}{k}\!\right) = \binom{3+2-1}{3} = 4 = n_1 = n_k$.

On the right-hand side of Figure 6 a tetrahedron of height four is depicted. The right side of this tetrahedron is marked in gray and it is a (two-dimensional) triangle of height four. This triangle can be represented as a (low-dimensional) hypertetrahedron with $k = 2$ and $n_k = n_2 = 4$. Hence, the 'higher-dimensional' hypertetrahedron, which is a tetrahedron, has $k + 1 = 3$ dimensions. The triangle consists of $\sum_{n_1=1}^{n_2} n_1$ dots. Again, we see that there is a unique black dot opposite the fixed gray side. When jumping in the direction from the black dot to the gray side from one dot to a neighboring dot, one can get from the unique black dot to the fixed side within $n_k - 1 = 3$ steps, in each of which we can go into three (i.e., $k + 1$) possible directions. Which dot one reaches is determined by how many times one goes in which of the $k + 1$ directions; but not by the order in which this is done. The fixed side is always reached after $n_k - 1 = 3$ such steps, and each dot can be reached within three of these steps. That is, the number of dots in the triangle equals the number of ways to distribute $n_k - 1 = 3$ steps arbitrarily among $k + 1 = 3$ directions. Again, we ask: *How many ways are there to distribute $n_k - 1 = 3$ steps arbitrarily to $k+1 = 3$ directions? How many ways are there to arbitrarily distribute $n_k - 1 = 3$ balls among $k+1 = 3$ drawers?* We calculate $\binom{(n_k-1)+(k+1)-1}{(n_k-1)} = \binom{n_k-1+k}{k} = \left(\!\binom{n_k}{k}\!\right) = \binom{3+3-1}{3} = 10 = \sum_{n_1=1}^{n_2} n_1 = \sum_{n_1=1}^{n_k} n_1$. That means, instead of counting and adding numbers of dots, we can use combinatorics to calculate the number of dots. This idea is generalized next.

Each side of a $(k + 1)$-dimensional ($(k+1)$d) hypertetrahedron is a k-dimensional hypertetrahedron, see Figure 7. When fixing one side of a $(k + 1)$-dimensional hypertetrahedron, there is one unique corner opposite of this side. When having a look at Figure 8, it turns out that if the dimension of a fixed side is one, two, or three, there are two, three, or four possible directions, resp., for moving one step from a dot to a neighboring dot towards the fixed side. This pattern is also true for higher dimensions, though this is more difficult to picture.

Whenever starting in the unique corner, see Figure 9, and moving $n_k - 1$ steps towards the fixed side, one of the dots of this side is reached. Analogously, it is easy to see that all dots of this side can be reached by moving $n_k - 1$ such steps. For reaching one specific dot in the fixed k-dimensional side, it is not important in which order of directions the steps are moved. Rather, only the numbers of steps in each direction matters. For example, in Figure 9 in the subfigure with the two-dimensional side, when moving three steps from the unique corner towards the fixed side, this side is reached. All paths $(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)$ lead to the same dot, but no other path does.

Thus, the number of dots in the side is, on the one hand, the number of dots in a line, a triangle, a tetrahedron, or, in general, a hypertetrahedron, i.e., n_1, $\sum_{n_1}^{n_2} n_1$, $\sum_{n_2=1}^{n_3}\sum_{n_1}^{n_2} n_1$, or, in general, $\sum_{n_{k-1}=1}^{n_k}\sum_{n_{k-2}=1}^{n_{k-1}}\dots\sum_{n_2=1}^{n_3}\sum_{n_1=1}^{n_2} n_1$. On the other hand, the number of dots equals the number of possibilities to move n_k-1 steps in some of the $k+1$ directions, where the order is not important and it can be moved more than once into the same direction. *How many ways are there to distribute n_k-1 balls—the number of steps, which is the height minus one—into $k+1$ drawers—labeled with the possible directions?* The answer is $\binom{(n_k-1)+k}{(n_k-1)} = \binom{n_k+k-1}{n_k-1} = \left(\!\!\binom{n_k}{k}\!\!\right)$ since there are $k=(k+1)-1$ separators between the $k+1$ drawers. Hence, by the principle of double counting Equations (3) and, thus, (1) follow.

5 Further insights

It is straightforward to compute with Equation (1) that for $n_k = 2$ and arbitrary $k \in \mathbb{N}_0$

$$\sum_{n_{k-1}=1}^{2}\sum_{n_{k-2}=1}^{n_{k-1}}\dots\sum_{n_2=1}^{n_3}\sum_{n_1=1}^{n_2} n_1 = k+1 \tag{4}$$

holds. That means, the series of the number of dots for all hypertetrahedra with height two with increasing dimension equals the series of natural numbers $(n)_{n=1,2,3,\dots}$.

This relationship becomes obvious when having a look at Figure 5 or when calculating:

$$\begin{aligned}
\sum_{n_{k-1}=1}^{2}\sum_{n_{k-2}=1}^{n_{k-1}}\dots\sum_{n_2=1}^{n_3}\sum_{n_1=1}^{n_2} n_1 &= 1 + \sum_{n_{k-2}=1}^{2}\dots\sum_{n_2=1}^{n_3}\sum_{n_1=1}^{n_2} n_1\\
&= 2 + \sum_{n_{k-3}=1}^{2}\dots\sum_{n_2=1}^{n_3}\sum_{n_1=1}^{n_2} n_1\\
&\vdots\\
&= (k-3) + \sum_{n_2=1}^{2}\sum_{n_1=1}^{n_2} n_1\\
&= (k-2) + \sum_{n_1=1}^{2} n_1 = (k-2) + (1+2) = k+1
\end{aligned}$$

Somehow more interesting is the following equation, which uses $\binom{n}{k} = \binom{n}{n-k}$:

$$\sum_{n_{k-1}=1}^{3} \sum_{n_{k-2}=1}^{n_{k-1}} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1 = \binom{k+2}{k} = \binom{(k+1)+2-1}{2} = \sum_{k_1=1}^{k+1} k_1 \qquad (5)$$

That means, the series of the number of dots for all hypertetrahedra with height three with increasing dimension equals the series of triangular numbers $\left(\frac{n(n+1)}{2}\right)_{n=1,2,3,\ldots}$.

To understand the meaning of Equation (5) the reader might have a look at Figures 3 and 4, however, the following calculations might be more helpful.

$$\begin{aligned}
\sum_{n_{k-1}=1}^{3} \sum_{n_{k-2}=1}^{n_{k-1}} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1 &= 1 + \sum_{n_{k-2}=1}^{2} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1 \\
&\quad + \sum_{n_{k-2}=1}^{3} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1 \\
&= 2 + k + \sum_{n_{k-3}=1}^{2} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1 \\
&\quad + \sum_{n_{k-3}=1}^{3} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1 \\
&= 3 + (k + (k-1)) + \sum_{n_{k-4}=1}^{2} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1 \\
&\quad + \sum_{n_{k-4}=1}^{3} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{n_2} n_1 \\
&\quad \vdots \\
&= (k-3) + \sum_{k_1=5}^{k} k_1 \\
&\quad + \sum_{n_2=1}^{2} \sum_{n_1=1}^{n_2} n_1 + \sum_{n_2=1}^{3} \sum_{n_1=1}^{n_2} n_1 \\
&= (k-2) + \sum_{k_1=4}^{k} k_1 + \sum_{n_1=1}^{2} n_1 + \sum_{n_1=1}^{3} n_1 \\
&= (k-1) + \sum_{k_1=3}^{k} k_1 + (2+3) = \sum_{k_1=1}^{k+1} k_1
\end{aligned}$$

This pattern can be generalized, too, to

$$\begin{aligned}\sum_{n_{k-1}=1}^{n_k}\sum_{n_{k-2}=1}^{n_{k-1}}\cdots\sum_{n_2=1}^{n_3}\sum_{n_1=1}^{n_2} n_1 &= \left(\!\!\binom{n_k}{k}\!\!\right) = \binom{n_k+k-1}{k} \\ &= \binom{(k+1)+(n_k-1)-1}{(n_k-1)} = \left(\!\!\binom{k+1}{n_k-1}\!\!\right) \\ &= \sum_{k_{n_k-2}=1}^{k+1}\sum_{k_{n_k-3}=1}^{k_{n_k-2}}\cdots\sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2} k_1 \end{aligned} \tag{6}$$

which equals exemplarily for $n_k = 4$

$$\sum_{n_{k-1}=1}^{4}\sum_{n_{k-2}=1}^{n_{k-1}}\cdots\sum_{n_2=1}^{n_3}\sum_{n_1=1}^{n_2} n_1 = \binom{k+3}{k} = \binom{k+2}{3} = \sum_{k_2=1}^{k+1}\sum_{k_1=1}^{k_2} k_1$$

i.e., the series of the number of dots for all hypertetrahedra with height four with increasing dimension equals the series of tetrahedral numbers$\left(\frac{n(n+1)(n+2)}{3!}\right)_{n=1,2,3,\ldots}$. Note that in Equation (6) there are $k-1$ many summation signs on the left-hand side with the number n_k on top of the first (i.e. the most left) one and $n_k - 2$ many summation signs on the right-hand side with the number $k+1$ on top of the first one.

6 Conclusion

In this article, we presented the results

$$\begin{aligned}\sum_{n_{k-1}=1}^{n_k}\sum_{n_{k-2}=1}^{n_{k-1}}\cdots\sum_{n_2=1}^{n_3}\sum_{n_1=1}^{n_2} n_1 &= \binom{n_k+k-1}{k} \\ &= \left(\!\!\binom{n_k}{k}\!\!\right) = \sum_{k_{n_k-2}=1}^{k+1}\sum_{k_{n_k-3}=1}^{k_{n_k-2}}\cdots\sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2} k_1 \end{aligned} \tag{7}$$

where the first line is known from [1]. We rephrased the formal proof from that source and briefly explained an illustrative derivation of this formula from the same source, using the principle of double counting and the numbers of possibilities of "handshakes." In the main part of the article at hand, we illustrated these generalized triangular numbers as hypertetrahedra and gave illustrative insights why it is quite natural that these numbers are multicombinations. Additionally, we showed that there is a relation between the dimension and the height of hypertetrahedra as can be seen in Equation (7): on the left-hand side there is the n_k-th generalized triangular number of dimension k (i.e., $k-1$ summation signs and the number n_k on top of the first one) and on the right-hand side there is the $(k+1)$-th generalized triangular number of dimension $n_k - 1$ (i.e., $n_k - 2$ summation signs and the number $k+1$ on top of the first one).

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Dedication

This work is dedicated to Luzia.

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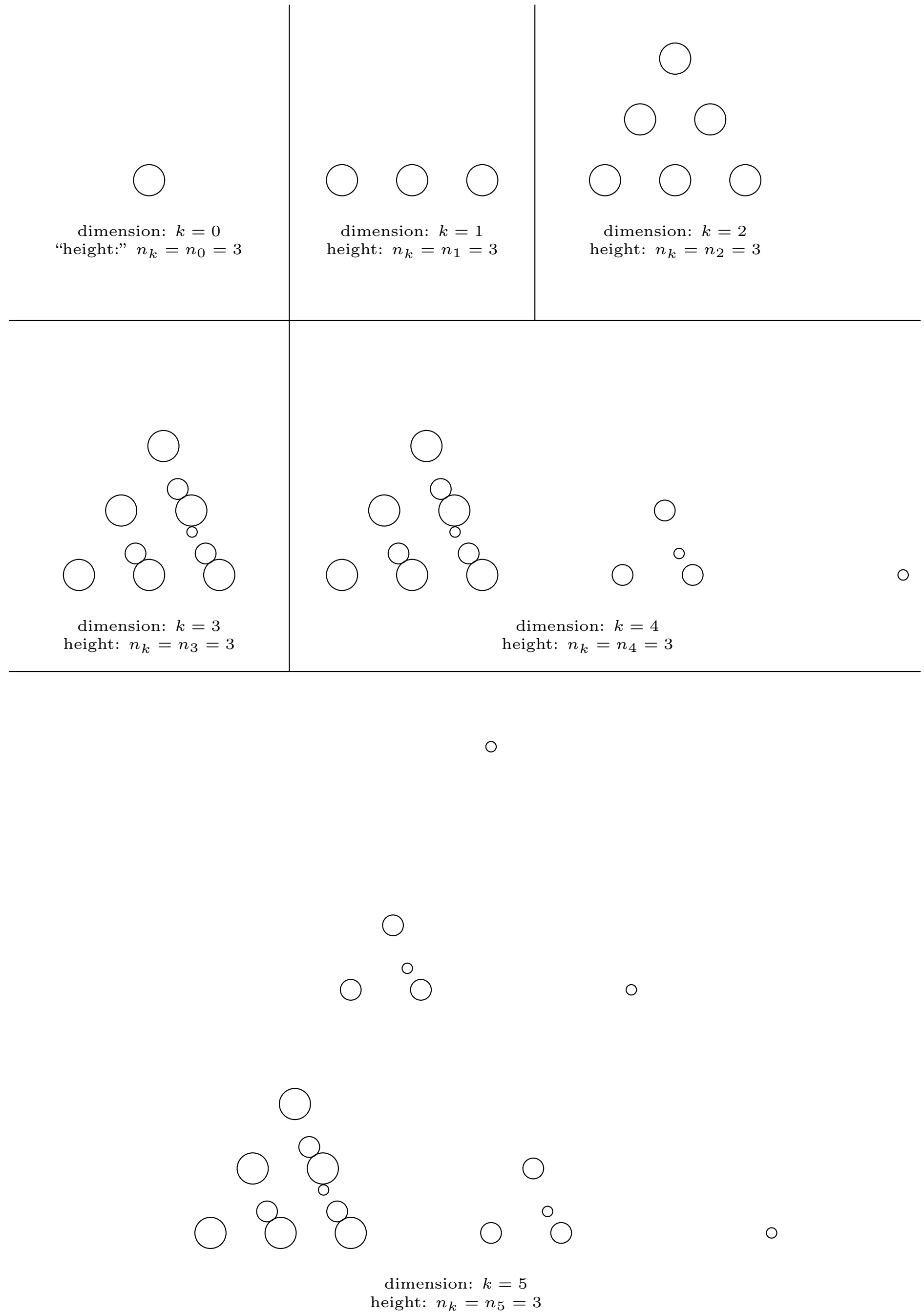

Figure 3: Hypertetrahedra of height three in dimensions zero, one, two, three, four, and five. High-dimensional hypertetrahedra can be represented as low-dimensional hypertetrahedra consisting of other hypertetrahedra.

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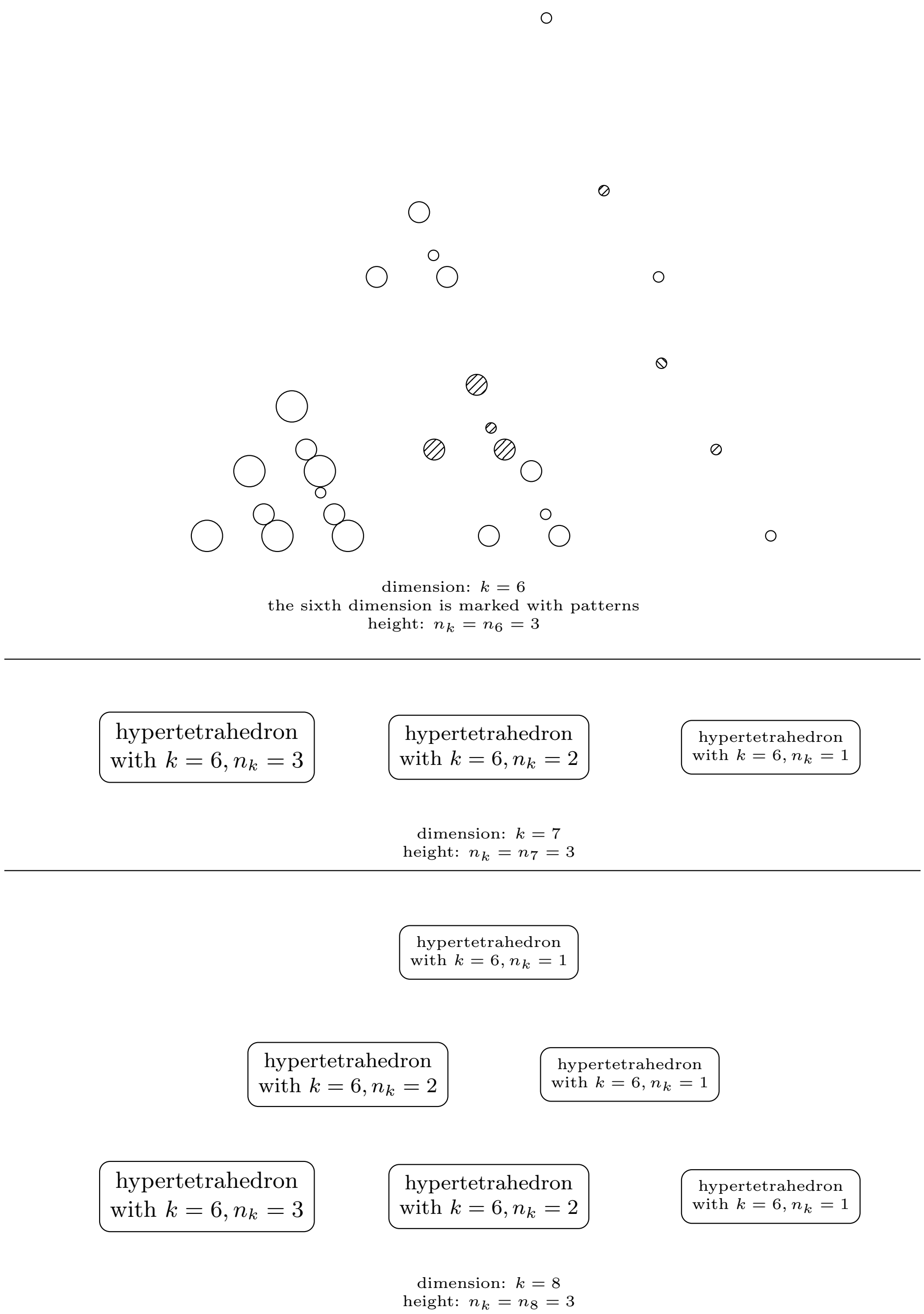

Figure 4: Hypertetrahedra of height three in dimensions six, seven, and eight. High-dimensional hypertetrahedra can be represented as low-dimensional hypertetrahedra consisting of other hypertetrahedra. The depicted pattern from this figure and Figure 3 can be continued straightforwardly.

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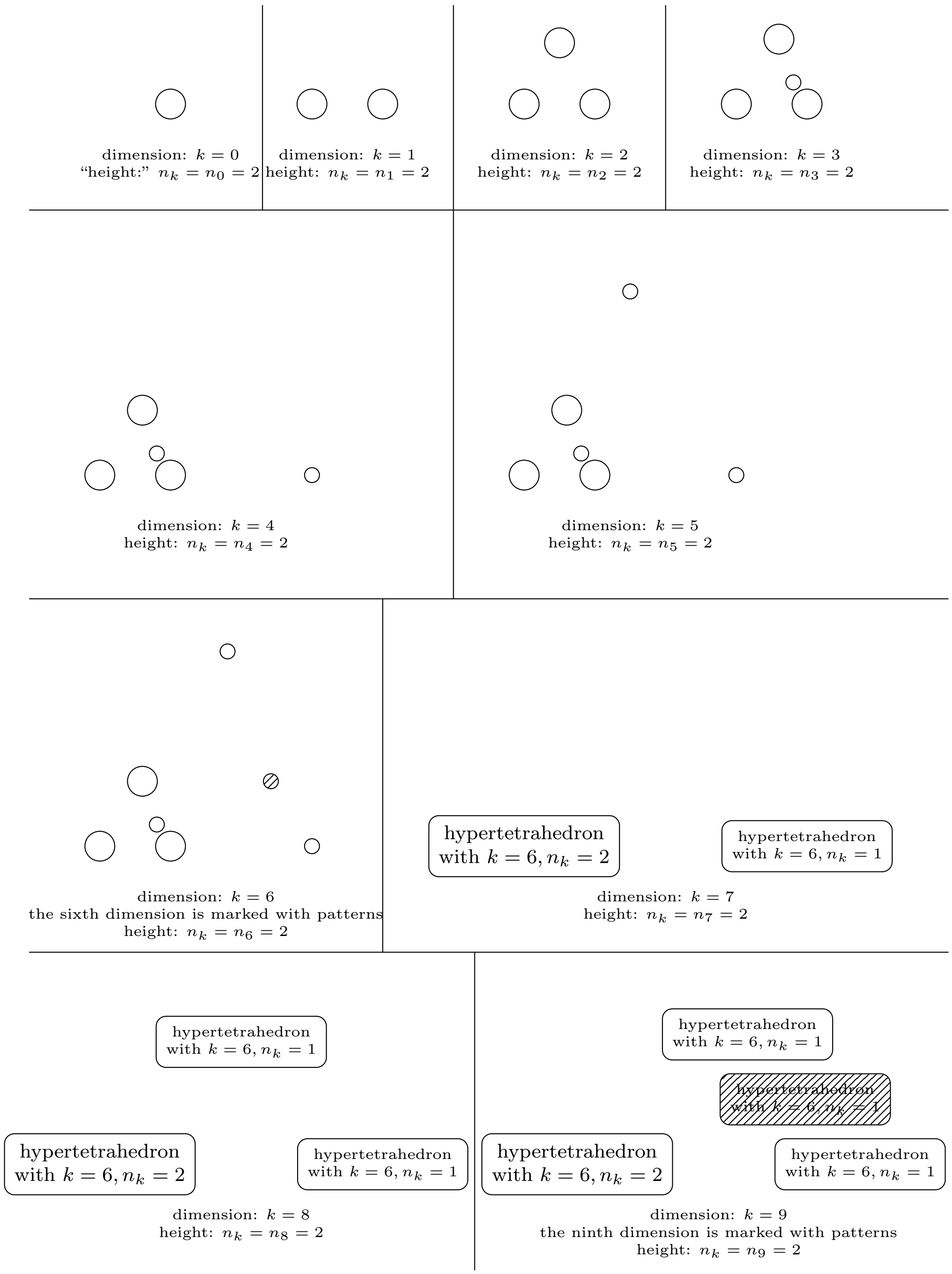

Figure 5: Hypertetrahedra of height two in dimensions zero to nine. High-dimensional hypertetrahedra can be represented as low-dimensional hypertetrahedra consisting of other hypertetrahedra. This pattern can be continued straightforwardly.

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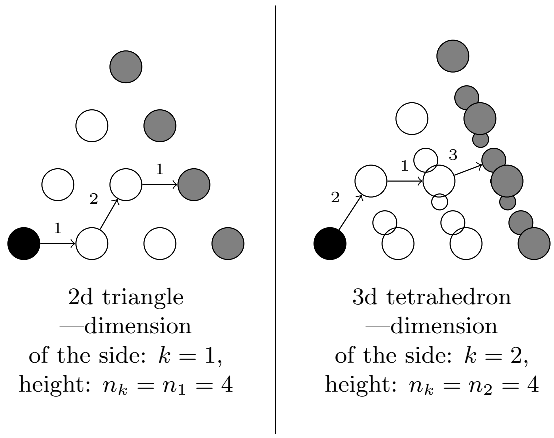

Figure 6: A triangle and a tetrahedron with one side marked gray (each), which are a line resp. a triangle. Additionally, paths from the respective opposite dots to the marked sides are depicted (cf. Figure 8).

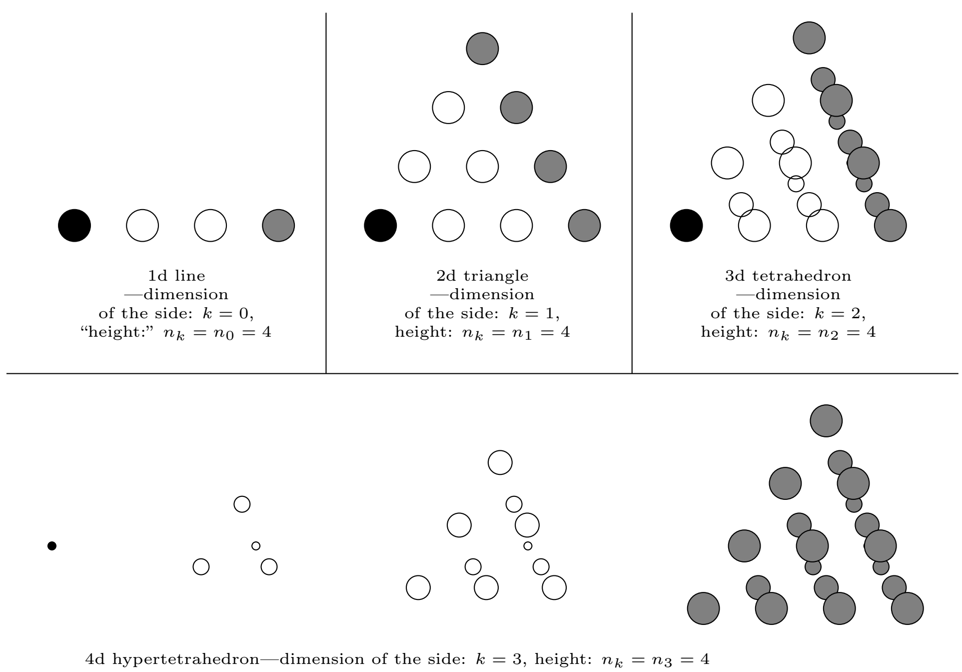

Figure 7: Hypertetrahedra of height four in dimensions one, two, three, and four ($k+1 = 1, 2, 3, 4$). In each case, one side, which is a hypertetrahedron of dimension zero, one, two, or three, resp., ($k = 0, 1, 2, 3$) and the unique opposite corner are marked.

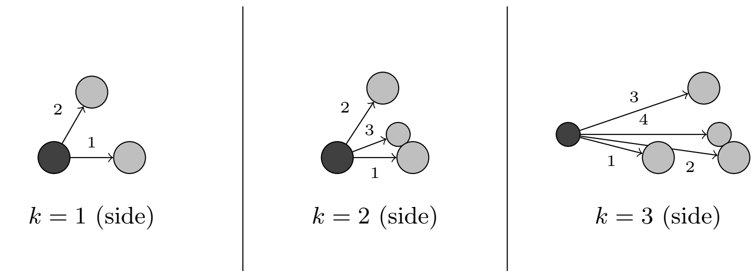

Figure 8: Possibilities to move one step to a neighboring dot from the unique corner in the direction of the fixed side.

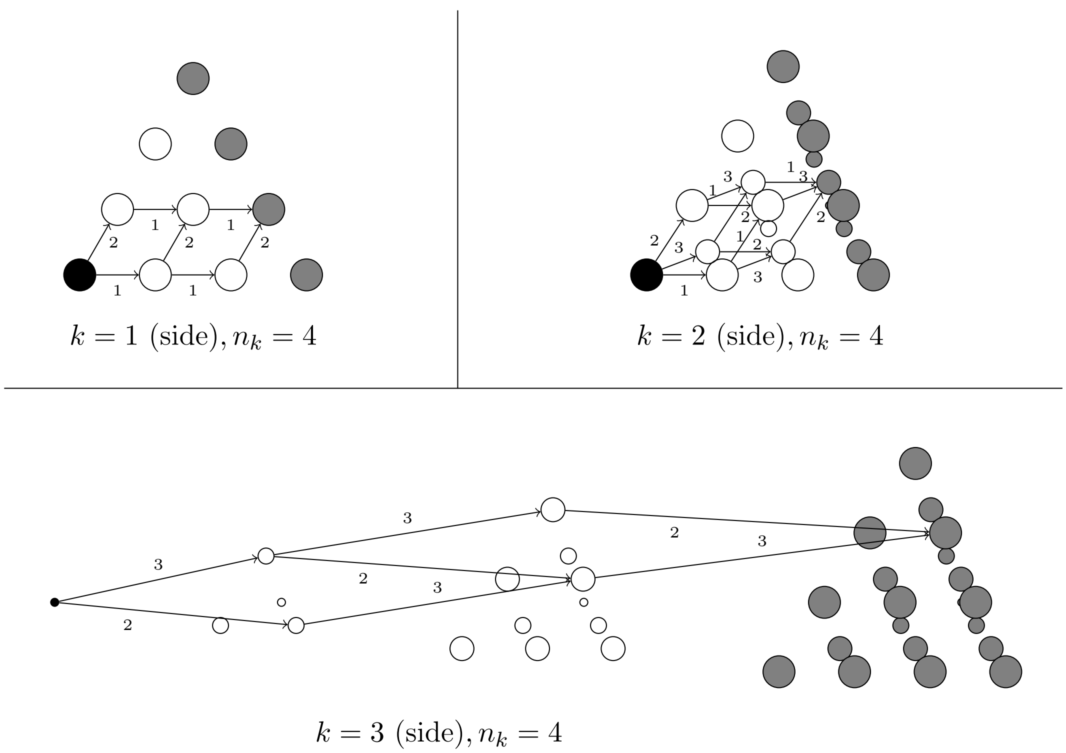

Figure 9: The dot that is reached by the arcs is not determined by the order of the directions but only by their numbers (cf. Figure 8). All chains of length $n_k - 1$ from the unique corner towards the side reach the marked side. All dots in the side can be reached by such a chain of arcs.

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