Cost design for predictive controllers: theoretical considerations and application to the start-up of a combined cycle power plant

Lars Grüne[1,*]

Chair of Applied Mathematics, University of Bayreuth, 95440 Bayreuth, Germany

Matthias Höger, Kilian Link

Siemens Energy, Siemenspromenade 9, 91058 Erlangen, Germany

Abstract

We explain two aspects of the design of stage cost in MPC schemes. In the theoretical part of this paper, we summarize recent result for the analysis of MPC stability and performance for general cost functions based on strict dissipativity and the turnpike property. In the application part, we explain how to design an MPC controller for the challenging task of starting up a combined cycle power plant. The paper thus describes the topic from two sides and shows how the theoretical insights can inform decisions in applications.

Keywords:
Model Predictive Control; stage cost; dissipativity; turnpike property; control of power systems; optimal start-up

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1. Introduction

Predictive or Model Predictive Control (henceforth MPC) is one of the most successful advanced control techniques. Its popularity stems on the one

*Corresponding Author
Email address: `lars.gruene@uni-bayreuth.de` (Lars Grüne)

[1]This paper elaborates results that the first author presented in his plenary lecture at the Control Conference Africa 2024

hand from the fact that, while the underlying idea is simple, its mathematical analysis is challenging and its efficient implementation requires state-of-the art techniques from optimization and numerics. This has led to a huge number of academic papers, survey articles and monographs, of which we only mention Camacho and Bordons (2004), Rawlings et al. (2017), and Grüne and Pannek (2017) as a small selection. On the other hand, MPC is also very popular in industry, due to its ability to handle constraints, optimization objectives, and nonlinear dynamics, see, e.g., Qin and Badgwell (2003), Forbes et al. (2015), or Ferreau et al. (2016).

While often the constraint handling is the main argument for using MPC in practice, because this is a feature that most classical controller designs do not share, there is a growing interest in additionally guaranteeing the (approximate) minimization of certain performance criteria. The corresponding branch of MPC research was originally termed "economic MPC" (Amrit et al., 2011; Angeli et al., 2012; Faulwasser et al., 2018), but should probably rather be called "general MPC", as it incorporates general cost objectives that should be minimized. A prime example of control problems in which this is of high relevance are problems in which a certain objective should be reached with minimal energy consumption. In Section 4 of this paper we present the energy-optimal startup of a combined cycle power plant as industrial use case from this field, which is published here for the first time.

The key idea of MPC is to optimize a certain objective over a finite horizon, the prediction horizon, using an available model for computing the future system trajectory inside the optimization algorithm, starting from the estimated current state of the system. Then a first piece, in discrete time usually the first value, of the resulting optimal control is applied. This process is successively repeated on shifted horizons, resulting in an online algorithm for computing a feedback controller and a corresponding closed-loop solution (for a formalization see Algorithm 2.1, below). Clearly, for general optimal control problems there is no guarantee that this procedure leads to a closed-loop solution that has any (approximate) optimality properties. To this end, certain properties are needed, and this is what we discuss in Section 3 of this paper. We present a concise explanation why two classical properties from mathematical systems theory and optimal control, i.e., strict dissipativity and the turnpike property, enables near optimality of the MPC closed-loop. While the results presented in this section are known, the presentation and in particular the illustrating example in this section is new in this context. The presentation borrows ideas from Grüne (2018) but focuses on those aspects that are most important for the use case and adds proof parts that are not treated in this reference. In the use case, we then show which difficulties can arise when applying these results and how to circumvent them.

The remainder of this paper is arranged as follows. In Section 2 we explain the problem setting and present preliminary results. Section 3 introduces the turnpike property and strict dissipativity and shows rigorously and by means of a continuing example how these properties ensure near optimality of the MPC closed-loop solution. Section 3 presents the use case of the optimal startup of a combined cycle power plant and in particular discusses the choice of the cost function for which the turnpike property can be ensured with reasonable computational effort. Finally, Section 5 concludes the paper.

2. Setting and preliminaries

2.1. Optimal and model predictive control

In this article we consider nonlinear model predictive control (MPC) schemes, where in each step of the scheme an optimal control problem of the form

$$\text{Minimize } J_N(x_0, u) \text{ with respect to the control sequence } u \tag{2.1}$$

is solved. Here the objective is

$$J_N(x_0, u) = \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

with $N \in \mathbb{N}$ and $x(\cdot)$ satisfies the dynamics and the initial condition

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0 \tag{2.2}$$

as well as the combined state and input constraints

$$(x(k), u(k)) \in \mathbb{Y} \;\; \forall k = 0, \dots, N-1 \quad \text{and} \quad x(N) \in \mathbb{X} \tag{2.3}$$

for all $k \in \mathbb{N}$ for which the respective values are defined. Here $\mathbb{Y} \subset X \times U$ is the constraint set, X and U are the the state and input value set, respectively, and $\mathbb{X} := \{x \in X \,|\, \exists u \in U \text{ with } (x, u) \in \mathbb{Y}\}$ is the state constraint set. The sets X and U are metric spaces with metrics $d_X(\cdot, \cdot)$ and $d_U(\cdot, \cdot)$. Because there is no danger of confusion we usually omit the indices X and U in the metrics. We denote the solution of (2.2) by $x_u(k, x_0)$. Moreover, for the distance of a point $x \in X$ to another point $y \in X$ we use the short notation $|x|_y := d(x, y)$.

The basic MPC algorithm that builds upon this optimal control problem then reads as follows. For the formal definition of an optimal control sequence see Subsection 2.2, below.

Algorithm 2.1. (Basic Model Predictive Control Algorithm)

(Step 0) *Fix an optimization horizon* $N \in \mathbb{N}$ *and set* $k := 0$*;* *let an initial value* $x_{MPC}(0)$ *be given*

(Step 1) *Compute an optimal control sequence* $u_N^\star$ *of Problem* (2.1) *for* $x_0 = x_{MPC}(k)$

(Step 2) *Define the MPC feedback law value* $\mu_N(x_{MPC}(k)) := u_N^\star(0)$

(Step 3) *Apply the control value* $\mu_N(x_{MPC}(k))$ *to the plant, obtain* $x_{MPC}(k+1)$*, set* $k := k+1$ *and go to (Step 1)*

This algorithm generates a feedback law μ_N mapping $\mathbb{X}$ to U that is derived from the first element $u_N^\star(0)$ of an open loop optimal control sequence $u_N^\star$. Controls in feedback form are indispensable to be able to react to deviations between the theoretically planned solution and the behavior of the real process. In MPC this feedback mechanism is realized by computing a new optimal control based on the most current state information $x_{MPC}(k)$ (or an estimate thereof). Typically[2], one does not compute an explicit formula for μ_N. Instead, the control value $\mu_N(x_{MPC}(k))$ is obtained by numerically solving the optimal control problem in Step 1 of Algorithm 2.1 online once $x_{MPC}(k)$ becomes available.

The nominal closed-loop system resulting from this algorithm is given by

$$x_{MPC}(k+1) := f(x_{MPC}(k+1), \mu_N(x_{MPC}(k))).$$

Nominal means that this would be the solution from Algorithm 2.1 if f was an exact model for the plant. While this is hardly ever the case in real-world applications, for studying theoretical properties of MPC is is a justified simplifying assumption, because if the methods does not produce good results in the absence of modeling errors, then we cannot expect it to perform well in a real-world situation.

[2]An exception are so-called explicit MPC schemes (Alessio and Bemporad, 2009), which, however, are only applicable for relatively low-dimensional systems.

In our theoretical considerations we will also consider Problem 2.1 with infinite time horizon $N = \infty$ and we write $N \in \mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$.

For $x_0 \in \mathbb{X}$ and $N \in \mathbb{N}$ we define the set of admissible control sequences as

$$\mathbb{U}^N(x_0) := \left\{ u \in U^N \;\middle|\; \begin{array}{l} (x_u(k, x_0), u(k)) \in \mathbb{Y} \;\; \forall\, k = 0, \dots, N-1 \\ \text{and } x_u(N, x_0) \in \mathbb{X} \end{array} \right\}$$

and

$$\mathbb{U}^\infty(x_0) := \{u \in U^\infty \,|\, (x_u(k, x_0), u(k)) \in \mathbb{Y} \;\; \forall\, k \in \mathbb{N}\}.$$

The focus of this article is on stability and performance. To avoid difficulties with the MPC scheme not being recursively feasible, during out theoretical considerations we assume that $\mathbb{U}^N(x_0) \neq \emptyset$ for all $x_0 \in \mathbb{X}$ and all $N \in \mathbb{N}_\infty$. If desired, this assumption can be avoided using techniques from, e.g., Faulwasser and Bonvin (2015), (Grüne and Pannek, 2017, Chapter 7), (Kerrigan, 2000, Chapter 5), or Primbs and Nevistić (2000).

2.2. Dynamic programming

Dynamic programming is a classical technique to characterize optimal controls and solutions. It can be used as the basis for computational schemes, but these are very inefficient in high space dimensions. This is why here we only use dynamic programming as a theoretical framework for analyzing MPC schemes. The center piece of dynamic programming considerations is the optimal value function

$$V_N(x_0) := \inf_{u \in \mathbb{U}^N(x_0)} J(x_0, u)$$

and we say that a control sequence $u_N^\star \in \mathbb{U}^N(x_0)$ is optimal for initial value $x_0 \in \mathbb{X}$ if $J(x_0, u_N^\star) = V_N(x_0)$ holds.

For the *finite horizon problem* the following equations and statements hold for all $N \in \mathbb{N}$ and all $K \in \mathbb{N}$ with $K \leq N$ (using $V_0(x) \equiv 0$ in case $K = N$). They all describe different ways to express the so-called dynamic programming principle.

$$V_N(x) = \inf_{u \in \mathbb{U}^K(x)} \{J_K(x, u) + V_{N-K}(x_u(K, x))\}. \tag{2.4}$$

If $u_N^\star \in \mathbb{U}^N(x)$ is an optimal control for initial value x and horizon N, then

$$V_N(x) = J_K(x, u_N^\star) + V_{N-K}(x_{u_N^\star}(K, x)) \tag{2.5}$$

and

$$\begin{aligned}&\text{the sequence } u_K := (u_N^\star(K), \ldots, u_N^\star(N-1)) \in \mathbb{U}^{N-K}(x_{u_N^\star}(K,x)) \\ &\text{is an optimal control for initial value } x_{u_N^\star}(K,x) \text{ and horizon } N-K.\end{aligned} \tag{2.6}$$

Moreover, for all $x \in \mathbb{X}$ the MPC feedback law μ_N satisfies

$$V_N(x) = \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x))). \tag{2.7}$$

These relations hold in a similar way *infinite horizon problem.* More precisely, for all $K \in \mathbb{N}$ it holds that

$$V_\infty(x) = \inf_{u \in \mathbb{U}^K(x)} \{J_K(x,u) + V_\infty(x_u(K,x))\}. \tag{2.8}$$

If $u_\infty^\star$ is an optimal control for initial value x and horizon N, then

$$V_\infty(x) = J_K(x, u_\infty^\star) + V_\infty(x_{u_\infty^\star}(K,x)) \tag{2.9}$$

and

$$\begin{aligned}&\text{the sequence } u_K := (u_\infty^\star(K), u_\infty^\star(K+1), \ldots) \in \mathbb{U}^\infty(x_{u_\infty^\star}(K,x)) \\ &\text{is an optimal control for initial value } x_{u_\infty^\star}(K,x).\end{aligned} \tag{2.10}$$

The goal of the ensuing Section 3 is to present fundamental results from the last years, together with (as we hope) intuitive proofs for the core results, which show that MPC yields approximately optimal infinite horizon solutions. In particular, this means that the stage cost used Problem (2.1) in Step 1 of Algorithm 2.1 is not merely an auxiliary object, but indeed defines in which sense the MPC solution is approximately optimal.

A priori, it is not clear, at all, whether the trajectory x_{MPC} generated by the MPC algorithm enjoys approximate optimality properties or qualitative properties like stability. An important goal of the next section is thus to give suitable conditions on the stage cost and the dynamics under which such properties can be guaranteed. In order to measure the optimality of the closed loop trajectory, we introduce its closed loop finite horizon value

$$J_K^{cl}(x, \mu_N) := \sum_{k=0}^{K-1} \ell(x_{MPC}(k), \mu_N(x_{MPC}(k)))$$

and the infinite horizon averaged value

$$\overline{J}_\infty^{cl}(x, \mu_N) := \limsup_{K \to \infty} \frac{1}{K} J_K^{cl}(x, \mu_N),$$

where in both cases the initial value is $x_{MPC}(0) = x$. In practical applications, one then has to find a tradeoff between the quantity that should be minimized in the application, the conditions a cost function should satisfy in order to guarantee near-optimal behavior of the MPC closed loop and the necessity of being able to solve the resulting optimal control problem in real time. We illustrate some of the challenges that come along with these partially conflicting goals in an industrial use case in Section 4.

3. Turnpikes, Dissipativity, and MPC

3.1. The turnpike property and MPC

For MPC to produce approximately optimal infinite-horizon solutions, we need a property of the optimal control problem that guarantees that finite-horizon and infinite-horizon solutions are close to each other. Clearly, if these solutions exhibit a very different behavior, then it is unlikely that the concatenation of pieces of finite-horizon optimal trajectories can ever approximate an infinite horizon trajectory. In this context, an important question is in which sense they should be close. As we will see below, it is sufficient that they are close in a weak sense, i.e., that their values (or, rather, the values of pieces of such trajectories) does not differ much. This is intuitively reasonable, since in the end we would like to produce a trajectory whose value is close to the optimal one.

The next important question is whether there is any hope to identify a class of optimal control problem that has this closeness property. Here, fortunately, a classical property of optimal control problems comes to our help, the so-called turnpike property. Informally, it describes the property that there is a particular trajectory, the *turnpike*, which has the property that all optimal trajectories stay near the turnpike most of the time. In many examples the turnpike is an optimal equilibrium (von Neumann, 1945; Faulwasser and Grüne, 2022), but it may also be an optimal growth path (Ramsey, 1928; von Neumann, 1937), a periodic trajectory (Müller and Grüne, 2016; Zanon et al., 2017), a general time-varying trajectory (Grüne and Pirkelmann, 2019), or — in stochastic optimal control problems — even a stochastic process (Schießl et al., 2025). The optimal control literature lists a huge number of examples in which the turnpike property occurs and Google Scholar yields more than 50 000 citations when searching for *turnpike property*. The name turnpike property was coined in Dorfman et al. (1958) and alludes to the use of the name turnpike for toll roads or, more generally, highways: When driving on a network of roads, the turnpike is the one that brings the driver to her or his destination in the fastest way. As an example for an optimal control problem exhibiting the turnpike property, we consider the following simple optimal investment problem.

Example 3.1. *The following is a simplified version of a classical 1d macroeconomic model going back to Brock and Mirman (1972). The goal is to maximize the sum over a utility function that depends on the consumption in each time step. The capital that can be consumed stems from an investment in a firm, which yields a return depending on the amount of re-invested capital in each time step. In mathematical notation, the invested capital at time* k *is denoted by* $x(k)$ *and the control* $u(k)$ *specifies the invested capital at time* $k+1$*, leading to the (very simple) dynamics* $x(k+1) = u(k)$*. In one time step, the capital increases to* $Ax(k)^\alpha$*, hence the amount of capital that is available for consumption amounts to* $C(k) = Ax(k)^\alpha - u(k)$*. In economics, the utility function is a measure of the satisfaction the consumer gets from the consumption. It is typically chosen to be concave (twice as much consumption yields less than twice as much satisfaction) and the simplest choice is the logarithm, i.e.,* $\ln(C(k))$*. Since the utility should be maximized but in this article we make the convention that Problem* (2.1) *is a minimization problem, we minimize the negative utility.*

In formulas, the problem thus reads: Minimize the objective

$$\sum_{k=0}^{N-1} \ell(x_u(k), u(k)) \quad \text{with} \quad \ell(x,u) = -\ln(Ax^\alpha - u)$$

for the dynamics $x(k+1) = u(k)$*. For the numerical simulation in Figure 1 we set* $A = 5$ *and* $\alpha = 0.34$ *and impose the state and input constraints* $\mathbb{X} = \mathbb{U} = [0, 10]$*. The figure shows that regardless of the length of the optimization horizon* N*, most of the time the solutions are close to a value of* ≈ 2.25*, indicated by the blue dashed line. This is a typical instance of the turnpike property. Clearly, as one only gets utility from spending capital and utility shall be maximized, at the end of the horizon the capital will always be* 0*. However, in between for increasing* N *it stays longer and longer at a point that can be seen as the optimal compromise between spending and re-investing.*

Figure 1 already suggests that the turnpike property implies that initial pieces of optimal trajectories with different horizons have almost identical

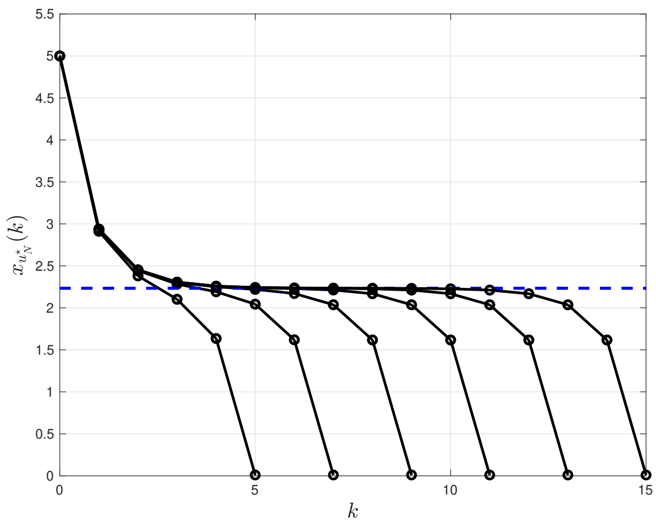

Figure 1: Optimal trajectories for Example 3.1 for horizons $N = 5, 7, \ldots, 15$

values and we will now prove formally that this is the case. When formally defining the turnpike property, many different options are possible:

- Which trajectories exhibit the turnpike property? Choices are optimal trajectories, near optimal trajectories, or trajectories with values below a certain threshold.
- Which quantities exhibit the turnpike property? Possible quantities are, e.g., states, outputs, controls, or adjoint states.
- How do we measure that trajectories stay near the turnpike? This could be based on, e.g., measures, integrals, or exponentially decaying bounds.

A selection of possible combinations of these options can be found in Faulwasser and Grüne (2022). Here we limit ourselves to one particular choice that has turned out to be suitable for analyzing MPC schemes. For simplicity of exposition, we only consider the case when the turnpike is an equilibrium.

Definition 3.2 (Turnpike property). *Let* (x^e, u^e) *be an equilibrium of* (2.2). *We say that the optimal control problem* (2.1) *has the* turnpike property *at*

x^e, if for each $\delta > 0$ there exists $\sigma_\delta \in \mathcal{L}$ such that[3] *for all $N, P \in \mathbb{N}$, $x \in \mathbb{X}$ and $u \in \mathbb{U}^N(x)$ with $J_N(x,u) \leq N\ell(x^e, u^e) + \delta$, the set*

$$\mathcal{Q}(x,u,P,N) := \{k \in \{0,\dots,N-1\} \mid |x_u(k,x)|_{x^e} \geq \sigma_\delta(P)\}$$

has at most P elements.

This formalization of the turnpike property specifies the number of exceptional points P at which a considered trajectory is not close to the turnpike x^e. The function $\sigma_\delta(P)$ then defines the size of the neighborhood of the turnpike, in which all but at most P points on the considered trajectory lie. As σ_δ is an $\mathcal{L}$-function, if P becomes larger and larger, the size of this neighborhood shrinks to 0. Hence, σ_δ describes the tradeoff between how close the trajectory it to the turnpike x^e and for how many points (at most) it is not that close. Here we demand that the turnpike property holds for all trajectories whose value is smaller than the value in the equilibrium plus some offset δ, where δ influences the size of the turnpike neighborhood, since σ_δ depends on δ. Typically, a larger δ will lead to a larger σ_δ, i.e., to a larger neighborhood. Under suitable bounds on the optimal value function all optimal and near optimal trajectories satisfy the imposed inequality for J_N, cf. Remark 3.4, below.

The turnpike property implies that any two trajectories $x_{u_j}(\cdot, x_j)$, $j = 1, 2$ satisfying the bound on J_{N_j} are $\sigma_\delta(P)$-close to the turnpike and thus also to each other for any $M \in \{0, 1, \dots, \min\{N_1, N_2\}\}$ with $M \notin \mathcal{Q}(x_1, u_1, P, N_1) \cup \mathcal{Q}(x_2, u_2, P, N_2)$. Whenever $N_j > 2P$ for $j = 1$ and 2, such an M exists and we thus find a time M at which $x_{u_1}(M, x_1) \approx x_{u_2}(M, x_2)$.

Now, if both trajectories are optimal, we can use this to conclude that their value summed up to time K only differs slightly. To this end, we need one more ingredient. Namely, we need that the cost of the remaining optimal trajectories starting from $x_{u_1}(K, x_1)$ and $x_{u_2}(K, x_2)$, respectively, are also close to each other when $x_{u_1}(K, x_1) \approx x_{u_2}(K, x_2)$ holds. The following assumption implies this property. It only requires closeness of the optimal values near x^e, which holds if and only if the values of the optimal trajectories starting in $x \approx x^e$ and x^e, respectively, are close.

Assumption 3.3. *(i) The optimal control problem has the turnpike property from Definition 3.2.*

[3] The space $\mathcal{L}$ contains all functions $\sigma : [0, \infty) \to [0, \infty)$ which are continuous and strictly decreasing with $\lim_{t\to\infty} \sigma(t) = 0$.

(ii) There exist functions $\gamma_V \in \mathcal{K}_\infty$ and $\omega_V \in \mathcal{L}$ such that the inequality

$$|V_N(x) - V_N(x^e)| \leq \gamma_V(|x|_{x^e}) + \omega_V(N)$$

holds for all $x \in \mathbb{X}$ and all $N \in \mathbb{N}_\infty$.

Remark 3.4. *We note that Assumption 3.3 implies that any optimal trajectory satisfies the condition $J_N(x,u) \leq N\ell(x^e,u^e)+\delta$ from Definition 3.2. This is because for initial condition x^e, staying in the equilibrium for all $k = 1,\dots,N$ is always an admissible trajectory, hence $V_N(x^e) \leq N\ell(x^e,u^e)$ follows. Then Assumption 3.3 yields $J_N(x,u_N^\star) = V_N(x) \leq N\ell(x^e,u^e)+\delta$ with $\delta = \gamma_V(|x|_{x^e})+\omega_V(N)$. Note that δ here depends on the distance of x to the turnpike equilibrium x^e. Generally, the larger δ is, the larger $\sigma_\delta(P)$ is for a fixed P. Hence, for a fixed number of exceptional points P, the closeness of the optimal trajectory to the turnpike depends on its initial distance to the turnpike.*

With Assumption 3.3 we can prove the following lemma. In this lemma we use the following notation for (in)equalities that hold up to an error term: for a sequence of functions $a_J : \mathbb{X} \to \mathbb{R}$, $J \in \mathbb{N}$, and another function $b : \mathbb{X} \to \mathbb{R}$ we write $a_J(x) \approx_J b(x)$ if $\lim_{J\to\infty} \sup_{x\in\mathbb{X}} |a_J(x) - b(x)| = 0$ and we write $a_J(x) \lesssim_J b(x)$ if $\limsup_{J\to\infty} \sup_{x\in\mathbb{X}} a_J(x) - b(x) \leq 0$. In words, $\approx_J$ means "$=$ up to terms which are independent of x and vanish as $J \to \infty$", and $\lesssim_J$ means the same for $\leq$. We note that precise quantitative statements can be made for the error terms "hiding" in the $\approx_J$-notation. Essentially, these terms depend on the distance between the optimal trajectories to the turnpike equilibrium x^e, as measured by the function σ_δ in Remark 3.11(ii), and by the functions from Assumption 3.3. For details we refer to (Grüne and Pannek, 2017, Chapter 8).

Lemma 3.5. *Let $\mathbb{X}$ be bounded, let Assumption 3.3 hold and assume $\ell(x^e,u^e) = 0$. Then the following approximate equalities hold.*

$$\begin{array}{llll} (i) & V_\infty(x) \approx_P J_M(x,u_\infty^\star) + V_\infty(x^e) & \text{for all } M \notin \mathcal{Q}(x,u_\infty^\star,P,\infty) \\ (ii) & J_M(x,u_\infty^\star) \approx_S J_M(x,u_N^\star) & \text{for all } M \in \{1,\dots,N\} \text{ with} \\ & & M \notin \mathcal{Q}(x,u_N^\star,P,N) \cup \mathcal{Q}(x,u_\infty^\star,P,\infty). \end{array}$$

Here $P \in \mathbb{N}$ is an arbitrary number, $S := \min\{P, N-M\}$ and $u_\infty^\star$ and $u_N^\star$ are the controls minimizing $J_\infty(x,u)$ and $J_N(x,u)$, respectively.

Proof: (i) The infinite horizon dynamic programming equation (2.9) yields

$$V_\infty(x) = J_M(x, u_\infty^\star) + V_\infty(x_{u_\infty^\star}(M, x)).$$

Hence, we obtain

$$V_\infty(x) = J_M(x, u_\infty^\star) + V_\infty(x^e) + \Big[V_\infty(x_{u_\infty^\star}(M, x)) - V_\infty(x^e)\Big].$$

From the turnpike property and Assumption 3.3 for $N = \infty$ we obtain that the term in square brackets is $\approx_P 0$ for all $M \notin \mathcal{Q}(x, u_\infty^\star, P, \infty)$, which shows (i).

(ii) The finite horizon dynamic programming equations (2.4) and (2.5) imply that $u = u_N^\star$ minimizes the expression $J_M(x, u) + V_{N-M}(x_u(M, x))$. Using the turnpike property and Assumption 3.3 this yields

$$\begin{aligned} J_M(x, u_N^\star) + V_{N-M}(x^e) &\approx_S J_M(x, u_N^\star) + V_{N-M}(x_{u_N^\star}(M, x)) \\ &\leq\ J_M(x, u_\infty^\star) + V_{N-M}(x_{u_\infty^\star}(M, x)) \ \approx_S\ J_M(x, u_\infty^\star) + V_{N-M}(x^e). \end{aligned}$$

for all $M \notin \mathcal{Q}(x, u_N^\star, P, N)$ and $S = \min\{P, N - M\}$.

Conversely, the infinite horizon dynamic programming equations (2.8) and (2.9) imply that $u_\infty^\star$ minimizes the expression $J_M(x, u_\infty^\star) + V_\infty(x_{u_\infty^\star}(M, x))$. Using the turnpike property and Assumption 3.3 for V_∞ this yields

$$\begin{aligned} J_M(x, u_\infty^\star) + V_\infty(x^e) &\approx_P\ J_M(x, u_\infty^\star) + V_\infty(x_{u_\infty^\star}(M, x)) \\ &\leq\ J_M(x, u_N^\star) + V_\infty(x_{u_N^\star}(M, x)) \ \approx_P\ J_M(x, u_N^\star) + V_\infty(x^e) \end{aligned}$$

for all $M \notin \mathcal{Q}(x, u_\infty^\star, P, \infty)$. Combining these two approximate inequalities then implies (ii). □

Figure 2 sketches the idea of the proof of (ii). The black solid line with solid circles visualizes the finite horizon optimal trajectory $x_{u_N^\star}$ and the blue dashed line with blue solid dots visualizes the infinite horizon optimal trajectory $x_{u_\infty^\star}$. Since at time K the black and the blue trajectory are very close to x^e, Assumption 3.3 implies that the finite horizon trajectory of length $N - K$ starting in $x_{u_\infty^\star}(K, x)$ (black dotted line and black and white circles) has a value that is very close to the value of the black solid one. Likewise, the infinite horizon trajectory starting in $x_{u_N^\star}(K, x)$ (blue dotted line and blue and white circles) has a very similar value as the tail of the dashed blue infinite-horizon optimal trajectory starting at time K. As a consequence, if the summed cost of the dashed blue trajectory up to time K would be much smaller than the cost of the solid black trajectory up to time K, the cost of the dashed blue trajectory up to time K and the dotted black trajectory from time K onward would be smaller than the cost of the dashed black trajectory, which contradicts optimality of this trajectory. Likewise, up to time K the cost of the solid black trajectory cannot be much smaller than the cost of the dashed blue one.

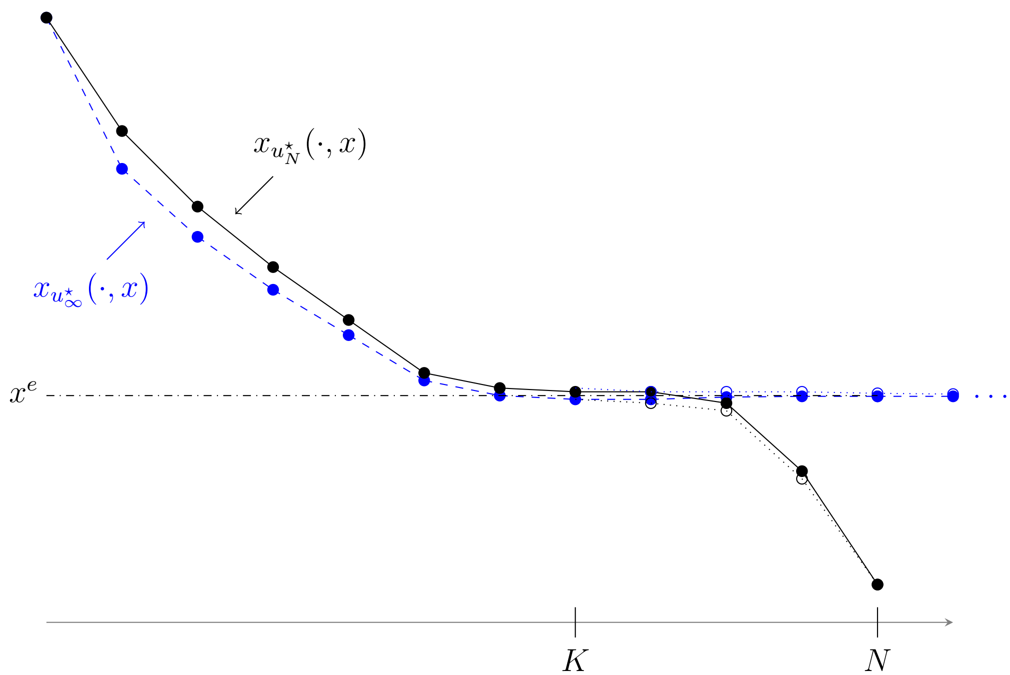

Figure 2: Sketch of the idea of the proof of Lemma 3.5(ii)

Using this lemma we can prove the first theorem on the performance of MPC.

Theorem 3.6. *Consider the MPC scheme from Algorithm 2.1 with an optimal control problem* (2.1) *for which Assumption 3.3 holds, let* $\ell(x^e, u^e) = 0$ *and* $\mathbb{X}$ *be bounded. Then there is* $\delta_1 \in \mathcal{L}$ *such that for all* $x \in \mathbb{X}$ *the closed loop solution* $x_{MPC}(k)$ *generated by this scheme with* $x_{MPC}(0) = x$ *satisfies the inequality*

$$J_K^{cl}(x, \mu_N) + V_\infty(x_{MPC}(K)) \leq V_\infty(x) + K\delta_1(N) \qquad (3.1)$$

for all $K, N \in \mathbb{N}$.

Proof: We pick $x \in \mathbb{X}$ and abbreviate $x^+ := f(x, \mu_N(x))$. For the corresponding optimal control $u_N^\star$, the relation (2.6) yields that $u_N^\star(\cdot + 1)$ is an optimal control for initial value x^+ and horizon $N-1$. Hence, for each $M \in \{1, \ldots, N\}$ we obtain

$$\begin{aligned} \ell(x, \mu_N(x)) \;=\; V_N(x) - V_{N-1}(x^+) \;&=\; J_N(x, u_N^\star) - J_{N-1}(x^+, u_N^\star(\cdot + 1)) \\ &=\; J_M(x, u_N^\star) - J_{M-1}(x^+, u_N^\star(\cdot + 1)), \end{aligned}$$

where the last equality follows from the fact that the omitted terms in the sums defining $J_M(x, u_N^\star)$ and $J_{M-1}(x^+, u_N^\star(\cdot + 1))$ coincide. Using Lemma 3.5(i) for N, x and M and for $N-1$, x^+ and $M-1$, respectively, yields

$$\begin{aligned} V_\infty(x) - V_\infty(x^+) \;&\approx_P\; J_M(x, u_\infty^\star) + V_\infty(x^e) - J_{M-1}(x^+, u_\infty^\star) - V_\infty(x^e) \\ &\approx_P\; J_M(x, u_\infty^\star) - J_{M-1}(x^+, u_\infty^\star). \end{aligned}$$

Putting the two (approximate) equations together and using Lemma 3.5(ii) yields

$$\ell(x, \mu_N(x)) \approx_S V_\infty(x) - V_\infty(x^+). \tag{3.2}$$

for all $M \in \{1, \ldots, N\}$ satisfying $M \notin \mathcal{Q}(x, u_N^\star, P, N) \cup \mathcal{Q}(x, u_\infty^\star, P, \infty)$ and $M - 1 \notin \mathcal{Q}(x^+, u_N^\star(\cdot + 1), P, N-1) \cup \mathcal{Q}(x^+, u_\infty^\star(\cdot + 1), P, \infty)$. Since each of the four $\mathcal{Q}$ sets contains at most P elements, their union contains at most $4P$ elements and hence if $N > 8P$ then there is at least one such M with $M \leq N/2$.

Thus, choosing $P = \lfloor (N-1)/8 \rfloor$ yields the existence of $M \leq N/2$ such that (3.2) holds with $S = \lfloor (N-1)/8 \rfloor$, implying that $\approx_S$ in (3.2) can be replaced by $\approx_N$. Hence, the error in (3.2) can be bounded by $\delta_1(N)$ for a function $\delta_1 \in \mathcal{L}$, yielding

$$\ell(x, \mu_N(x)) \leq V_\infty(x) - V_\infty(x^+) + \delta_1(N). \tag{3.3}$$

Applying (3.3) for $x = x_{MPC}(k)$, $k = 0, \ldots, K-1$, we can then conclude

$$\begin{aligned} J_K^{cl}(x, \mu_N) &= \sum_{k=0}^{K-1} \ell(x_{MPC}(k), \mu_N(x_{MPC}(k))) \\ &\leq \sum_{k=0}^{K-1} \Big(V_\infty(x_{MPC}(k)) - V_\infty(x_{MPC}(k+1)) + \delta_1(N) \Big) \\ &\leq V_\infty(x) - V_\infty(x_{MPC}(K)) + K\delta_1(N). \end{aligned}$$

This proves the claim. □

Inequality (3.1) is not self-explaining. What is states is that if we concatenate the closed loop trajectory $(x_{MPC}(0), \dots, x_{MPC}(K))$ and the infinite horizon optimal trajectory emanating from $x_{MPC}(K)$, then the overall cost $J_K^{cl}(x, \mu_N) + V_\infty(x_{MPC}(K))$ is less than the optimal cost $V_\infty(x)$ plus the error term $K\delta_1(N)$. In other words, for large N the initial piece of the MPC closed loop trajectory is an initial piece of an approximately optimal infinite horizon trajectory.

Remark 3.7. *If we assume that V_∞ is bounded from below on $\mathbb{X}$ (which is a mild assumption), then* (3.1) *implies*

$$\overline{J}_\infty^{cl}(x, \mu_N) \le \limsup_{K\to\infty} \frac{1}{K}\Big(V_\infty(x) - V_\infty(x_{MPC}(K)) + K\delta_1(N)\Big) = \delta_1(N). \quad (3.4)$$

This is an estimate for the long term average cost $\overline{J}_\infty^{cl}$ of x_{MPC}. Particularly, since the turnpike property implies that $\ell(x_{u_N^\star}(k), u_N^\star) \approx \ell(x^e, u^e)$ most of the time, we obtain that $\frac{1}{N}V_N(x) \ge \ell(x^e, u^e) - C/N$ for some constant $C > 0$. As we have assumed that $\ell(x^e, u^e) = 0$, this implies $\overline{J}_\infty^{cl}(x, u) = 0$ for all admissible control sequences u. Thus, (3.4) *says that the infinite horizon average cost $\overline{J}_\infty^{cl}(x, \mu_N)$ of the MPC closed loop is near optimal, up to the term $\delta_1(N)$ which tends to* 0*. Hence,* (3.4) *is intuitively easier to understand than* (3.1)*. Moreover, it has the advantage that it can be extended to the case $\ell(x^e, u^e) = 0$, because a constant that is added to ℓ simply appears additively in $\overline{J}_\infty^{cl}(x, \mu_N)$, i.e., we obtain $\overline{J}_\infty^{cl}(x, \mu_N) \le \ell(x^e, u^e) + \delta_1(N)$.*

Yet, (3.4) *is also weaker than* (3.1)*. This is because a closed loop solution that produces very large cost up to some time $\widehat{K}$ but then constantly produces stage costs $\ell(x_{MPC}(k), \mu_N(x_{MPC}(k))) \le \delta_1(N)$ will satisfy* (3.4) *but not* (3.1)*, since without averaging the large cost on the interval $\{0, \dots, \widehat{K}\}$ cannot in general be compensated for at times $k > \widehat{K}$. Thus, while* (3.4) *makes transparent that MPC produces near optimal cost in the long run,* (3.1) *also implies that MPC yields good performance on finite time horizons.*

3.2. Strict dissipativity and MPC

While (3.1) and (3.4) already yield valuable information on the closed-loop performance of MPC, more structured behavior can be observed in simulations. In particular, one sees that the closed loop solutions converge to a neighborhood of the turnpike equilibrium x^e and it would be desirable if we could prove this and estimate the size of this neighborhood. Moreover, it would be desirable to give an estimate for J_K^{cl} without having to use the infinite-horizon optimal value function, which would then be interpretable also in the case that $\ell(x^e, u^e) \neq 0$.

To this end, we make use of the following systems theoretic property.

Definition 3.8 (Strict Dissipativity and Dissipativity). *We say that an optimal control problem with stage cost ℓ is* strictly dissipative *at an equilibrium $(x^e, u^e) \in \mathbb{Y}$ if there exists a* storage function $\lambda : \mathbb{X} \to \mathbb{R}$ *bounded from below and satisfying $\lambda(x^e) = 0$, and a function $\rho \in \mathcal{K}_\infty$ such that for all $(x, u) \in \mathbb{Y}$ the inequality*

$$\ell(x, u) - \ell(x^e, u^e) + \lambda(x) - \lambda(f(x, u)) \geq \rho(|x|_{x^e}) \tag{3.5}$$

holds.

We note that the assumption $\lambda(x^e) = 0$ can be made without loss of generality because adding a constant to λ does not invalidate (3.5).

Example 3.9. *One checks with elementary computations that the optimal control problem from Example 3.1 is strictly dissipative at the equilibrium $(x^e, u^e) \approx (2.2344, 2.2344)$ with $\lambda(x) \approx 0.2306x$. This example falls into the class of optimal control problems with linear dynamics, strictly convex cost (in x) and convex constraints. For this kind of problems, a general procedure for computing x^e, u^e and λ is known, see Damm et al. (2014, Proposition 4.3), which was applied here.*

The observation that strict dissipativity is the "right" property in order to analyze economic MPC schemes was first made by Diehl et al. (2011), where strict duality, i.e., strict dissipativity with a linear storage function, was used. The extension to the nonlinear notion of strict dissipativity was then made by in Angeli and Rawlings (2010).

Remark 3.10. *While at a first glance this is not obvious, at all, it turns out that strict dissipativity is essentially equivalent to the existence of the turnpike property from Definition 3.2. More precisely, it was shown in Grüne and Müller (2016, Corollary 4.2) that, under mild technical conditions, for* *a system that is locally controllable in a neighborhood of x^e, strict dissipativity holds if and only if the turnpike property from Definition 3.2 holds. Example 4.3 in Grüne and Müller (2016) shows that the implication "turnpike property $\Rightarrow$ strict dissipativity" fails to hold when removing the local controllability, although it is an open question whether we could replace it by a weaker property. The converse implication "strict dissipativity $\Rightarrow$ turnpike property" holds regardless of whether the system is locally controllable or not.*

Remark 3.11. *Besides the turnpike property, strict dissipativity implies*

(i) The equilibrium $(x^e, u^e) \in \mathbb{Y}$ from Definition 3.8 is a strict optimal equilibrium in the sense that $\ell(x^e, u^e) < \ell(x, u)$ for all other admissible equilibria of f, i.e., all other $(x, u) \in \mathbb{Y}$ with $f(x, u) = x$. This follows immediately from (3.5)*.*

(ii) Under mild technical conditions, nonlinear detectability implies strict dissipativity, see Höger and Grüne (2019). This means that if we design the cost ℓ such that a trajectory not converging to x^e (eventually) yields a positive cost, i.e., that the distance $|x|_{x^e}$ is detectable through ℓ, then the optimal control problem is strictly dissipative.

(iii) If we define the modified or rotated cost $\tilde{\ell}(x, u) := \ell(x, u) - \ell(x^e, u^e) + \lambda(x) - \lambda(f(x, u))$, then this modified cost satisfies $\tilde{\ell}(x^e, u^e) = 0$ and $\tilde{\ell}(x, u) \geq \rho(|x|_{x^e})$, i.e., it is a so-called positive definite *cost w.r.t. x^e.*

The last property enables us to use the optimal control problem with modified cost $\tilde{\ell}$ as an auxiliary problem in our analysis. For this purpose we define the functional and optimal value function for $\tilde{\ell}$ as

$$\widetilde{J}_N(x_0, u) = \sum_{k=0}^{N-1} \tilde{\ell}(x(k), u(k)), \qquad \widetilde{V}_N(x_0) := \inf_{u \in \mathbb{U}^N(x_0)} \widetilde{J}(x_0, u).$$

The relation between $\widetilde{J}_N$ and J_N is then given by

$$\widetilde{J}_N(x, u) = J_N(x, u) + \lambda(x) - \lambda(x_u(N, x)) - N\ell(x^e, u^e). \tag{3.6}$$

In addition to the rotated cost, we consider the shifted cost $\hat{\ell}(x, u) := \ell(x, u) - \ell(x^e, u^e)$ and denote the corresponding objective and optimal value function by $\widehat{J}_N$ and $\widehat{V}_N$, respectively.

In what follows, we combine arguments from Grüne (2013); Grüne (2016); Grüne and Stieler (2014). As in the previous section, we have to assume that near x^e the values of the optimal value function do not change too much. Now we need it both for the original cost $\tilde{\ell}$ and

To this end we make the following assumptions.

Assumption 3.12. *(i) The optimal control problem is strictly dissipative in the sense of Definition 3.8.*

(ii) There exist functions $\gamma_{\widehat{V}}$, $\gamma_{\widetilde{V}}$ and $\gamma_\lambda \in \mathcal{K}_\infty$ as well as $\omega, \tilde{\omega} \in \mathcal{L}$ such that the following inequalities hold for all $x \in \mathbb{X}$ and all $N \in \mathbb{N}_\infty$:

$$\begin{array}{llll} (a) & |\widehat{V}_N(x) - \widehat{V}_N(x^e)| & \leq & \gamma_{\widehat{V}}(|x|_{x^e}) + \omega(N) \\ (b) & |\widetilde{V}_N(x) - \widetilde{V}_N(x^e)| & \leq & \gamma_{\widetilde{V}}(|x|_{x^e}) + \tilde{\omega}(N) \\ (c) & |\lambda(x) - \lambda(x^e)| & \leq & \gamma_\lambda(|x|_{x^e}) \end{array}$$

Part (ii) of this assumption is a uniform continuity assumption in x^e. For the optimal value functions V_N and $\widetilde{V}_N$ it can, e.g., be guaranteed by local controllability around x^e, see (Grüne, 2013, Theorem 6.4). We note that this assumption together with the obvious inequality $V_N(x^e) \leq N\ell(x^e, u^e)$ and boundedness of $\mathbb{X}$ implies $V_N(x) \leq N\ell(x^e, u^e) + \delta$ with $\delta = \sup_{x\in\mathbb{X}} \gamma_{\widehat{V}}(|x|_{x^e}) + \omega(0)$. Hence, the optimal trajectories have the turnpike property according to Remark 3.11(ii).

With these assumptions and notation we can now prove the following relations. For simplicity of exposition in what follows we limit ourselves to a bounded state space $\mathbb{X}$. If this is not satisfied, the following considerations can be made for bounded subsets of $\mathbb{X}$. As we will see, dynamic programming arguments are ubiquitous in the following considerations.

Lemma 3.13. *Let $\mathbb{X}$ be bounded. Then under Assumptions 3.12 the following approximate equalities hold.*

$$\begin{array}{llll} (i) & V_N(x) \approx_S J_M(x, u_N^\star) + V_{N-M}(x^e) & \text{for all } M \notin \mathcal{Q}(x, u_N^\star, P, N) \\ (ii) & V_N(x^e) \approx_S M\ell(x^e, u^e) + V_{N-M}(x^e) & \text{for all } M \notin \mathcal{Q}(x^e, u_N^{\star e}, P, N) \\ (iii) & \widetilde{V}_N(x) \approx_N V_N(x) + \lambda(x) - V_N(x^e) & \\ (iv) & V_N(x) \approx_N V_{N-1}(x) + \ell(x^e, u^e) & \end{array}$$

Here $P \in \mathbb{N}$ is an arbitrary number, $S := \min\{P, N - M\}$, $u_N^\star$ is the control minimizing $J_N(x, u)$, $u_N^{\star e}$ is the control minimizing $J_N(x^e, u)$, and $\mathcal{Q}$ is the *set from Definition 3.2 (which exists because of Remark 3.10). Moreover, (i) and (ii) also apply to the optimal control problem with stage cost $\widetilde{\ell}$.*

Proof: We first observe that it is sufficient to prove all statements under the assumption $\ell(x^e, u^e) = 0$. This is because the addition of a constant c to ℓ cancels out in all four approximations. This is because in (i), (ii), and (iv) it appears as Nc on both sides, while in (iii) it appeary as Nc and $-Nc$ on the right hand side. We can thus assume without loss of generality that $V_N = \widehat{V}_N$ and $J_N = \widehat{J}_N$.

(i) Using the constant control $u \equiv u^e$ we can estimate $V_N(x^e) \leq J_N(x^e, u) = N\ell(x^e, u^e)$. Thus, using Assumption 3.12 we get $J_N(x, u_N^\star) \leq N\ell(x^e, u^e) + \gamma_{\widehat{V}}(|x|_{x^e}) + \omega(N)$, hence the turnpike property from Remark 3.11(ii) applies to the optimal trajectory with $\delta = \gamma_{\widehat{V}}(|x|_{x^e}) + \omega(N)$. This in particular ensures $|x_{u_N^\star}(M, x)|_{x^e} \leq \sigma_\delta(P)$ for all $M \notin \mathcal{Q}(x, u_N^\star, P, N)$.

Now the dynamic programming equation (2.5) yields

$$V_N(x) = J_M(x, u_N^\star) + V_{N-M}(x_{u_N^\star}(M, x)).$$

Hence, (i) holds with remainder terms $R_1(x, M, N) = V_{N-M}(x_{u_N^\star}(M, x)) - V_{N-M}(x^e)$. For any $P \in \mathbb{N}$ and any $M \notin \mathcal{Q}(x, u_N^\star, P, N)$ the inequality $|R_1(x, M, N)| \leq \gamma_{\widehat{V}}(|x_{u_N^\star}(M, x)|_{x^e}) + \omega(N - M) \leq \gamma_{\widehat{V}}(\sigma_\delta(P)) + \omega(N - M)$ holds, and thus (i).

(ii) From the dynamic programming equation (2.4) and $u \equiv u^e$ we obtain

$$V_N(x^e) \leq M\ell(x^e, u^e) + V_{N-M}(x^e).$$

On the other hand, from (2.5) we have

$$\begin{aligned} V_N(x^e) &= J_M(x, u_N^{\star e}) + V_{N-M}(x_{u_N^{\star e}}(M, x^e)) \\ &= \underbrace{\widetilde{J}_M(x, u_N^{\star e})}_{\geq 0} - \lambda(x^e) + \lambda(x_{u_N^{\star e}}(M, x^e)) + M\ell(x^e, u^e) + V_{N-M}(x_{u_N^{\star e}}(M, x^e)) \\ &\geq V_{N-M}(x^e) + M\ell(x^e, u^e) + \left[V_{N-M}(x_{u_N^{\star e}}(M, x^e)) - V_{N-M}(x^e)\right] \\ &\qquad + \left[\lambda(x_{u_N^{\star e}}(M, x^e)) - \lambda(x^e)\right] \end{aligned}$$

Now since V_{N-M} and λ satisfy Assumption 3.12(ii) and $x_{u_N^{\star e}}(M, x^e) \approx_P x^e$ for all $M \notin \mathcal{Q}(x^e, u_N^{\star e}, P, N)$, we can conclude that the differences in the squared brackets have values $\approx_S 0$ which shows the assertion.

(iii) Fix $x \in \mathbb{X}$ and let $u_N^\star$ and $\tilde{u}_N^\star \in \mathbb{U}^N(x)$ denote the optimal control minimizing $J_N(x, u)$ and $\widetilde{J}_N(x, u)$, respectively. We note that if the optimal control problem with cost ℓ is strictly dissipative then the problem with cost $\tilde{\ell}$ is strictly dissipative, too, with bounded storage function $\lambda \equiv 0$ and same $\rho \in \mathcal{K}_\infty$. Moreover, $V_N(x) \le N\ell(x^e,u^e) + \gamma_{\widehat{V}}(|x|_{x^e}) + \omega(N)$ and $\widetilde{V}_N(x) \le N\tilde{\ell}(x^e,u^e) + \gamma_{\widehat{V}}(|x|_{x^e})$, since $V_N(x^e) \le N\ell(x^e,u^e)$ and $\widetilde{V}_N(x^e) = 0$. Hence, the turnpike property from Remark 3.11(ii) applies to the optimal trajectories for both problems, yielding $\sigma_\delta \in \mathcal{L}$ and $\mathcal{Q}(x,u_N^\star,P,N)$ for $x_{u_N^\star}$ and $\tilde{\sigma}_{\tilde{\delta}}$ and $\widetilde{\mathcal{Q}}(x,\tilde{u}_N^\star,P,N)$ for $x_{\tilde{u}_N^\star}$. For all $M \notin \widetilde{\mathcal{Q}}(x,\tilde{u}_N^\star,P,N) \cup \mathcal{Q}(x^e,u_N^{\star e},P,N)$ we can estimate

$$\begin{aligned} V_N(x) &\le J_M(x,\tilde{u}_N^\star) + V_{N-M}(x_{\tilde{u}_N^\star}(M)) \\ &\le J_M(x,\tilde{u}_N^\star) + V_{N-M}(x^e) + \gamma_{\widehat{V}}(\tilde{\sigma}_{\tilde{\delta}}(P)) + \omega(N-M) \\ &\le \widetilde{J}_M(x,\tilde{u}_N^\star) - \lambda(x) + \lambda(x^e) + M\ell(x^e,u^e) + V_{N-M}(x^e) \\ &\qquad + \gamma_{\widehat{V}}(\tilde{\sigma}_{\tilde{\delta}}(P)) + \gamma_\lambda(\tilde{\sigma}_{\tilde{\delta}}(P)) + \omega(N-M) \\ &\lesssim_S \widetilde{V}_N(x) - \lambda(x) + V_N(x^e) \end{aligned}$$

for $S = \min\{P, N-M\}$, where we have applied the dynamic programming equation (2.4) in the first inequality, the turnpike property for $x_{\tilde{u}_N^\star}$ and Assumption 3.12 and (3.6) in the second and third inequality and (i) applied to $\widetilde{V}_N$, and (ii) applied to ℓ in the last step. Moreover, $\lambda(x^e) = 0$ and $\widetilde{V}_N(x^e) = 0$ were used.

By exchanging the two optimal control problems and using the same inequalities as above, we get

$$\widetilde{V}_N(x) \lesssim_S V_N(x) + \lambda(x) - V_N(x^e)$$

for all $M \notin \mathcal{Q}(x,u_N^\star,P,N) \cup \widetilde{\mathcal{Q}}(x^e,\tilde{u}_N^{\star e},P,N)$. Together this implies

$$\widetilde{V}_N(x) \approx_S V_N(x) + \lambda(x) - V_N(x^e)$$

for all $M \notin \mathcal{Q}(x,u_N^\star,P,N) \cup \widetilde{\mathcal{Q}}(x,u_N^\star,P,N) \cup \mathcal{Q}(x,u_N^{\star e},P,N) \cup \widetilde{\mathcal{Q}}(x^e,\tilde{u}_N^{\star e},P,N)$ and $S = \min\{P, N-M\}$.

Now, choosing $P = \lfloor N/5 \rfloor$, the union of the four $\mathcal{Q}$-sets has at most $4N/5$ elements, hence there exists $M \le N/5$ for which this approximate inequality holds. This yields $S = \lfloor N/5 \rfloor$ and thus $\approx_S$ implies $\approx_N$, which shows (iii).

(iv) Let $u_{N-1}^\star$ denote the optimal control for initial value $x \in \mathbb{X}$ and horizon length $N-1$. We apply (i) with $N-1$ in place of N, choosing

$P = \lfloor N/2 \rfloor$ and $M \le N/2 - 1$, which implies that $S = O(N)$ and thus $\approx_S$ implies $\approx_N$. This yields

$$V_{N-1}(x) \approx_N J_M(x, u^\star_{N-1}) + V_{N-M-1}(x^e).$$

The proof of (i) moreover shows that $|x_{u^\star_{N-1}}(M,x)|_{x^e} \le \sigma_\delta(P)$. Defining a control u by setting $u(k) := u^\star_{N-1}(k)$ for $k = 0, \dots, M-1$ and $u(M + k) := u^\star_{N-M}(k)$, where $u^\star_{N-M}$ denotes the optimal control for initial value $x_{u^\star_{N-1}}(M,x)$ and horizon $N - M$. Since $P = O(N)$, we obtain

$$J_N(x,u) = J_M(x, u^\star_{N-1}) + V_{N-M}(x_{u^\star_{N-1}}(M,x)) \approx_N J_M(x, u^\star_{N-1}) + V_{N-M}(x^e).$$

Now a feasible trajectory of length $N - M$ for initial value x^e is to use u^e in the first step and to use the optimal control for x^e horizon $N - M - 1$ for the remaining steps, leading to the cost $\ell(x^e, u^e) + V_{N-M-1}(x^e)$. With this we can estimate

$$\begin{aligned} V_N(x) &\le J_N(x,u) \approx_N J_M(x, u_{N-1}) + V_{N-M}(x^e) \\ &\le J_M(x, u_{N-1}) + \ell(x^e, u^e) + V_{N-M-1}(x^e) = V_{N-1}(x) + \ell(x^e, u^e). \end{aligned}$$

□

Using Lemma 3.13(iii) and (iv) and the dynamic programming equation (2.7), for $x^+ = f(x, \mu_N(x))$ we obtain

$$\begin{aligned} \widetilde{V}_N(x^+) &\approx_N V_N(x^+) + \lambda(x^+) - V_N(x^e) \\ &\approx_N V_{N-1}(x^+) + \ell(x^e, u^e) + \lambda(x^+) - V_N(x^e) \\ &= V_N(x) - \ell(x, \mu_N(x)) + \ell(x^e, u^e) + \lambda(x^+) - V_N(x^e) \\ &\approx_N \widetilde{V}_N(x) \underbrace{-\ell(x, \mu_N(x)) + \ell(x^e, u^e) + \lambda(x^+) - \lambda(x)}_{= -\tilde{\ell}(x, \mu_N(x))}. \end{aligned} \tag{3.7}$$

This implies that the modified optimal value function decays in each step, except for an error term which vanishes as $N \to \infty$. Since $\widetilde{V}_N(x) \ge \rho(|x|_{x^e})$ and $\tilde{\ell}(x, \mu_N(x)) \ge \rho(|x|_{x^e})$, from this we can conclude that as $k \to \infty$ the closed loop solution $x_{MPC}(k)$ converges to a neighbourhood of x^e, which shrinks down to x^e for $N \to \infty$ (for a rigorous application of this argument see (Grüne and Pannek, 2017, Section 8.6)). In fact, due to the upper bound on $\widetilde{V}_N$ induced by Assumption 3.12(ii), we can even conclude the existence of $\beta \in \mathcal{KL}$ and $\eta \in \mathcal{L}$ such that for all $x \in \mathbb{X}$ the MPC closed loop solution $x_{MPC}(k)$ with $x_{MPC}(0) = x$ satisfies

$$|x_{MPC}(k)|_{x^e} \leq \max\{\beta(|x|_{x^e}, k), \, \eta(N)\} \tag{3.8}$$

for all $N, k \in \mathbb{N}$, cf. (Grüne and Pannek, 2017, Theorem 8.33). This means that the optimal equilibrium x^e is practically asymptotically stable for the MPC closed loop.

Example 3.14. *Figure 3 shows the MPC closed-loop trajectories (red solid) and corresponding open-loop trajectories (black dashed) for Example 3.1 with* $N = 3$ *(left) and* $N = 5$ *(right). In addition the blue dashed line indicates the optimal equilibrium* x^e*, i.e., the turnpike. One clearly sees that the closed-loop solutions converge to a neighborhood of* x^e *that shrinks with increasing* N*, but not to* x^e *itself.*

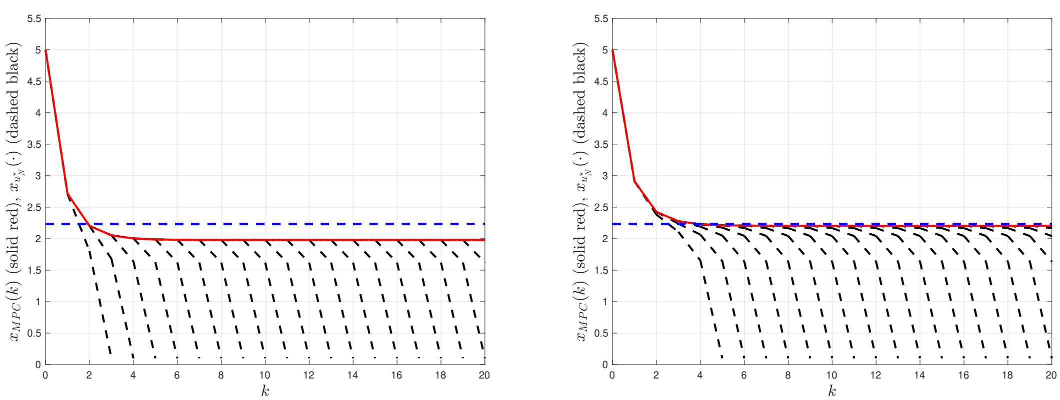

Figure 3: MPC closed-loop and open-loop trajectories for Example 3.1 for horizons $N = 3$ (left) and $N = 5$ (right)

Remark 3.15. *As discussed in and after Remark 3.4, the constants that affect the error of the "*$\approx_N$*" approximation depend on the distance of* x *to* x^e*. This means that in order to obtain a decay of* $\widetilde{V}_N$ *via (3.7), one may need a larger prediction horizon* N *if* $\mathbb{X}$ *is large. Yet, the error only needs to be small relative to* $|\tilde{\ell}(x, \mu_N(x))|$*, which because of Remark 3.11(iii) also becomes large as* $|x|_{x^e}$ *becomes large. This is why no or only a moderate increase of* N *is* *needed when the initial condition x is far away from the turnpike equilibrium x^e. Intuitively speaking, the cost may "guide" the optimal solutions towards the turnpike if it increases with the distance of the state from the turnpike.*

As Example 3.14 shows, already in very simple examples exact convergence to the optimal equilibrium x^e will not hold for the MPC closed loop. Hence, unless additional techniques are introduced (cf. Remark 3.19, below) practical asymptotic stability of x^e is in general the best one can obtain. This also explains the factor K before $\delta_1(N)$ in the estimate from Theorem 3.6 (and likewise in Theorem 3.16, below). Since the trajectory always has a little distance to the optimal equilibrium, in each step we collect a small error and these errors sum up from 0 to $K-1$, resulting in the factor K in front of the error term. However, as we will discuss in Remark 3.17(ii) and Example 3.18, below, the relative error usually stays bounded independent of K.

Due to the fact that the closed loop solution converges to a neighbourhood of x^e, we can now give a characterization of the optimality of $J_K^{cl}(x, \mu_M)$ that does not rely on the optimal value functions of the infinite horizon problem.

Theorem 3.16. *Consider the MPC scheme from Algorithm 2.1 with an optimal control problem* (2.1) *satisfying Assumption 3.12 and let $\mathbb{X}$ be bounded. Then there are $\delta_1, \delta_2, \delta_3 \in \mathcal{L}$ such that for all $x \in \mathbb{X}$ the closed loop solution $x_{MPC}(k)$ generated by this scheme with $x_{MPC}(0) = x$ satisfies the inequality*

$$J_K^{cl}(x, \mu_N) \leq \inf_{u \in \mathbb{U}^K_{\delta(K)}(x)} J_K(x, u) + \delta_1(N) + K\delta_2(N) + \delta_3(K) \tag{3.9}$$

for all $K, N \in \mathbb{N}$, where

$$\mathbb{U}^K_{\delta(K)}(x) := \{u \in \mathbb{U}^K(x) \,|\, |x_u(K, x)|_{x^e} \leq \delta(K)\}$$

with $\delta(K) := |x_{MPC}(K)|_{x^e}$.

Proof: From (3.7) we obtain

$$\tilde{\ell}(x, \mu_N(x)) \approx_N \widetilde{V}_N(x) - \widetilde{V}_N(f(x, \mu_N(x))).$$

We denote the error in this approximate equation by $\delta_2(N)$. Summing $\tilde{\ell}(x, \mu_N(x))$ along the closed-loop trajectory then yields

$$\sum_{k=0}^{K-1} \tilde{\ell}(x_{MPC}(k), \mu_N(x_{MPC}(k))) \leq \widetilde{V}_N(x) - \widetilde{V}_N(x_{MPC}(K)) + K\delta_2(N). \tag{3.10}$$

Now the dynamic programming equation (2.4) and Assumption 3.12(ii) yield for all $K \in \{1, \dots, N\}$ and all $u \in \mathbb{U}^K_{\delta(K)}(x)$

$$\begin{aligned} \widetilde{J}_K(x,u) &= \underbrace{\widetilde{J}_K(x,u) + \widetilde{V}_{N-K}(x_u(K,x))}_{\geq \widetilde{V}_N(x)} - \underbrace{\widetilde{V}_{N-K}(x_u(K,x))}_{\leq \gamma_{\widetilde{V}}(\delta(K))} \\ &\geq \widetilde{V}_N(x) - \gamma_{\widetilde{V}}(\delta(K)). \end{aligned} \tag{3.11}$$

Due to the non-negativity of $\tilde{\ell}$, for $K \geq N$ we get $\widetilde{J}_K(x,u) \geq \widetilde{V}_N(x)$ for all $u \in \mathbb{U}^K(x)$. Hence (3.11) holds for all $K \in \mathbb{N}$. Moreover, we have $\widetilde{V}_N(x) \geq 0$. Using (3.10), (3.11), (3.6) and the definition of δ_2, for all $u \in \mathbb{U}^K_{\delta(K)}(x)$ we obtain

$$\begin{aligned} J^{cl}_K(x,\mu_N(x)) &= \sum_{k=0}^{K-1} \tilde{\ell}(x_{MPC}(k), \mu_N(x_{MPC}(k))) - \lambda(x) + \lambda(x_{MPC}(K)) \\ &\leq \widetilde{V}_N(x) - \widetilde{V}_N(x_{MPC}(K)) + K\delta_2(N) - \lambda(x) + \lambda(x_{MPC}(K)) \\ &\leq \widetilde{J}_K(x,u) + \gamma_{\widetilde{V}}(\delta(K)) - \widetilde{V}_N(x_{MPC}(K)) + K\delta_2(N) - \lambda(x) + \lambda(x_{MPC}(K)) \\ &= J_K(x,u) + \gamma_{\widetilde{V}}(\delta(K)) - \widetilde{V}_N(x_{MPC}(K)) + K\delta_2(N) - \lambda(x_u(K,x)) \\ &\qquad + \lambda(x_{MPC}(K)) \\ &\leq J_K(x,u) + \gamma_{\widetilde{V}}(\delta(K)) + K\delta_2(N) + 2\gamma_\lambda(\delta(K)). \end{aligned}$$

Now from (3.8) we obtain

$$\begin{aligned} \gamma_{\widetilde{V}}(\delta(K)) + 2\gamma_\lambda(\delta(K)) &\leq \underbrace{\sup_{x \in \mathbb{X}} \gamma_{\widetilde{V}}(\beta(|x|_{x^e}, K)) + 2\gamma_\lambda(\beta(|x|_{x^e}, K))}_{=:\delta_3(K)} \\ &\quad + \underbrace{\gamma_{\widetilde{V}}(\kappa(N)) + 2\gamma_\lambda(\kappa(N))}_{=:\delta_1(N)} \end{aligned}$$

which finishes the proof. □

The interpretation of this result is as follows: among all solutions steering x into the $\delta(K)$-neighbourhood of the optimal equilibrium x^e, MPC yields the cheapest one up to error terms vanishing for large K and larger N.

Remark 3.17. *(i) In Theorem 3.6 we had to make the assumption $\ell(x^e,u^e) = 0$ in order to obtain a well defined optimal value function V_∞ for the inequality (3.1). For the result in Theorem 3.16 we did not need to make this assumption, because in the final result V_∞ does not appear, in the assumption we* *could replace it by $\widehat{V}_\infty$, which uses the cost $\hat{\ell}$ that satisfies $\hat{\ell}(x^e, u^e) = 0$ by construction, and in the proof of the underlying Lemma 3.13 we could assume that $\widehat{V}_N = V_N$ without loss of generality since its assertions are invariant under additions of constants to ℓ.*

However, there is no straightforward way to interpret (3.1) if $\ell(x^e, u^e) \neq 0$. If this is desired, the concept of overtaking optimality can be employed, as it was done in the context of time-varying problems in Grüne and Pirkelmann (2019).

(ii) In fact, $\ell(x^e, u^e) \neq 0$ is the typical situation, since there will usually be a certain minimal non-zero cost that cannot be avoided (or a non-zero yield that can be achieved in case ℓ is a negative yield) in the long run. In this case, the optimal cost J_K grows linearly in K, just as the error term $K\delta_2(N)$. This implies that the relative error behaves like $\sim \delta_2(N)$ and is thus independent of N.

Example 3.18. *For Example 3.1 we have evaluated $J_K^{cl}(x, \mu_N)$ for $x = 5$ and $N = 2$ and $N = 5$ numerically. The values are plotted in Figure 4. The effect of the $K\delta_2(N)$ term is visible in the steeper slope of the red dashed curve, leading to visibly smaller values for sufficiently large K. The asymptotic slope of the infinite horizon optimal solution is $\ell(x^e, u^e) \approx -1.4673$ and differs from the slope of $J_K^{cl}(x, \mu_5)$ by less than 10^{-3}. This difference would not be discernible in the plot and is thus not shown.*

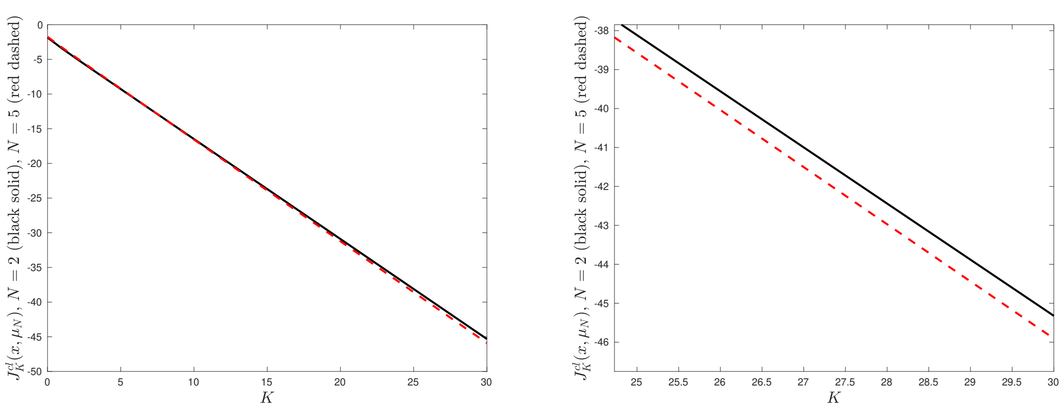

Figure 4: MPC closed-loop values $J_K^{cl}(x, \mu_N)$ for $N = 2$ (solid black) and $N = 5$ (red dashed). The right figure is a zoom into the lower right corner of the left one.

Remark 3.19. *By using so-called terminal ingredients in* (2.1)*, i.e., terminal costs*

$$J_N(x_0, u) = \sum_{k=0}^{N-1} \ell(x(k), u(k)) + F(x(N))$$

with $F : \mathbb{X}_f \to \mathbb{R}$ *and terminal constraints* $x(N) \in \mathbb{X}_f$*, one can improve the above results. For the dissipativity-based analysis of MPC, such terminal ingredients in which* F *is a local control Lyapunov function were introduced in Amrit et al. (2011). An alternative linear terminal cost was suggested in Zanon and Faulwasser (2018).*

If $\mathbb{X}_f$ *and* F *are suitably designed, then* δ_1 *in Theorem 3.6,* δ_2 *in Theorem 3.16, and* η *in* (3.8) *can be set to* 0*, i.e., removed from the respective inequalities. In particular, we can obtain "true" instead of only practical asymptotic stability. However, the design of* $\mathbb{X}_f$ *and* F *with these properties may not always be feasible, particularly for complex and high-dimensional problems such as in the use case we will present in Section 4. Hence, in practical applications often no terminal ingredients are used.*

We would like to note that the results from this section have been extended in various ways. For instance, in many examples it can be observed that the error terms $\delta_j(N)$ converge to 0 exponentially fast as $N \to \infty$, i.e., that they are of the form $\delta_j(N) = C\Theta^N$ for $C > 0$ and $\Theta \in (0, 1)$. Conditions under which this can be rigorously proved can be found in Grüne and Stieler (2014). Another extension concerns replacing the optimal equilibrium x^e by a periodic orbit. Corresponding results can be found, e.g., in Angeli et al. (2012); Müller and Grüne (2016); Zanon et al. (2017). An extension to general time-varying problems can be found in Grüne and Pirkelmann (2019) and a first result for stochastic MPC schemes is given in Schießl et al. (2024), based on stochastic dissipativity and turnpike results from Schießl et al. (2025).

4. Use case: Startup of a combined cycle power plant

CCPPs (Combined cycle power plants) generate electrical energy via a combination of gas and steam turbines. Their superior efficiency stems from the fact that the exhaust gas from the gas turbine produces steam for a steam turbine. Thermodynamically, it is a combination of the Brayton cycle of the gas turbine (GT) with the Rankine water steam cycle including the

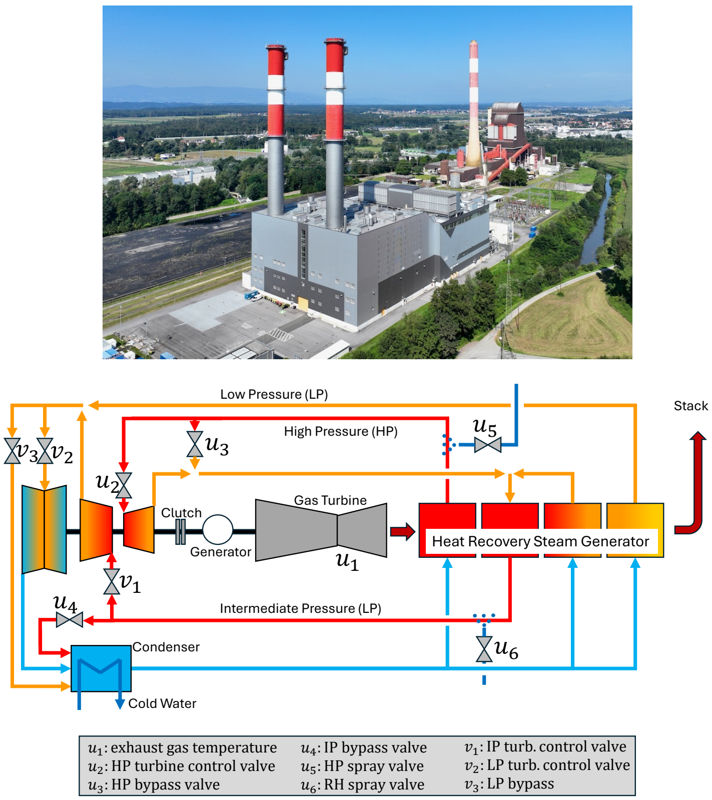

Figure 5: Top: Combined cycle power plant in Mellach, Austria (C. Stadler/Bwag, wikipedia). Bottom: Schematic sketch of the plant architecture

steam turbine (ST). This way, up to 64% efficiency can be reached, and even more if the remaining heat is used for district heating. Historically CCPPs had the purpose to server as medium and peak load units, that delivered the residual load for the less flexible nuclear and coal fired power plants. Today, they fulfill the same purpose and supply the residual load for renewables, however, with much higher flexibility requirements in a much more dynamic market environment.

4.1. Control objective

The goal of the control is to develop control strategies for flexible, fast, and economic start-up. The corresponding cost function for power plants in general is

$$\min_{u \in U} \Phi = CF_{fatigue} + \int_{t_s}^{t_f} \left(c_{fuel} - r_{production}\right) dt \tag{4.1}$$

This cost function covers all all economically relevant factors from t_s the begin of the startup until t_f, the end of the startup. Although t_s is typically easy to identify, since it is the time at which the GT is switched on by the operators, t_f is more difficult to identify since it requires the CCPP to reach a specified load level and several more conditions must be met. In the integral part, all costs related to the operation are lumped into c_{fuel}, which includes, e.g., the cost of CO_2, maintenance cost, and so on. Revenue-generating earnings are lumped into $r_{production}$. It should be mentioned that, especially $r_{production}$ is a time-dependent function, since electricity rates usually change significantly during CCPP startup. In contrast to the integral and time-dependent part, $CF_{fatigue}$ represents the financial loss caused by the degradation of components during the period of transient operation. The special thing about this term is that a statement can only be made if the entire horizon of the process is considered.

Modifications to (4.1) are often needed to take into account the processes of customer organizations or the design of the market. A strongly simplified variant of (4.1) is

$$\min_{u \in U} \Phi = \int_{t_s}^{t_f} \dot{m}_{fuel} dt, \tag{4.2}$$

which was recently implemented in several power plants. Here, at t_f the CCPP must have reached a specified load level with the ST in operation to allow a central dispatcher to operate the plant according to the market boundary conditions from that point in time. In this simplified cost the revenue-generated earnings are neglected, fuel consumption is used as a proxy for the costs of the operations, and the term measuring degradation of components is replaced by constraints on the thermal stress, see below. Thus, the task is to reach a given set with minimal energy consumption, which is a typical "economic" MPC problem. Suitable implementations of this cost function will be discussed later.

Especially during start-up, the operation of complex systems is subject to many restrictions. To name a few limitations that need to be considered:

- Maximal pressures and temperatures.
- Maximal and minimal mass flow rates.

- Maximal or minimal change rates.
- Maximal and minimal thermal stress induced by temperature changes.

While it might look beneficial for the optimization goal to ramp up the GT load as fast as possible, the optimal solution is much different in reality. On one hand due the limitations above, on the other due the fact several criteria need to be met before the conditions of the produced steam allow safe ST operation and all produced steam is dumped to the condenser i.e. wasted until then. Furthermore the GT, the heat recovery steam generator, the main steam piping and the ST need to be heated up to their operation temperature. Since these components cool down to lower temperatures the longer the shutdown period is, startups of cold plants after longer shutdown periods take significantly longer compared to hotter plants after shorter shutdown periods.

Technically, the MPC solution described below is implemented by calculating set points via MPC and sending them to the DCS (distributed control system = the conventional controller) of the power plant. Underlying controllers inside the DCS try to act according to the MPC set points. Thus, the hierarchical control structure "MPC $\rightarrow$ DCS $\rightarrow$ actuators" arises. For instance, the MPC controller may produce a temperature set point and DCS adjusts injection valve position such that measurement is driven towards this set point. Additional constraints that need to be taken into account are:

- Safety critical limitations, like too high pressures, which may lead to plant trips triggered by the fail-safe system. These are already maintained by the DCS, i.e., before the plant reaches a critical state, countermeasures are taken. Nevertheless, to avoid unplanned unavailability of the plant, which causes high cost for the plant owner, these constraints must be modeled as well.
- The control behavior of relevant DCS logics, such as ramp-rate limitations on set points or start criteria for the steam turbine. In order not to generate set points that violate these logics, they must be integrated as constraints in the MPC scheme.
- Model scope constraints, i.e., the model must be evaluated only in regions in which it is physically meaningful. For instance, we assume there is no backward flow through the components.

4.2. Model

The plant model consists of (sub-)models each representing relevant real plant components, for instance pipes, valves, heat-exchangers, gas-/steam-turbine. The basic principle for all those components are mass rate and energy rate balances, which lead to the following first principle equations. We note that the control inputs of the system are the quantities denoted u_1 to u_6 and v_1 to v_3 in the scheme in Figure 5.

- **Conservation of mass**, described by the equations

$$\dot{m} = \sum_{i \in I_m} \dot{m}_i \qquad [kg \cdot s^{-1}]$$

  with the variables

  m: system mass $[kg]$
  m_i: mass of medium entering (> 0) / leaving (< 0) the system $[kg]$
  I_m: set of indices for medium entering/leaving the system.

- **Conservation of energy**, leading to the equations

$$\frac{d}{dt}(mu) = \sum_{i \in I_Q} \dot{Q}_i - \sum_{i \in I_W} \dot{W}_i + \sum_{i \in I_m} h_{m_i} \dot{m}_i \qquad [W]$$

  where the variables are:

  m: system mass $[kg]$
  u: specific internal energy of system $[J \cdot kg^{-1}]$
  Q_i: heat transfer into (> 0) / out (< 0) of the system $[J]$
  W_i: work performed by (> 0) / on (< 0) the system $[J]$
  m_i: mass of medium entering (> 0) / leaving (< 0) the system $[kg]$
  h_i: specific enthalpy of medium entering / leaving the system $[J{\cdot}kg^{-1}]$
  I_Q: set of indices for heat transferred into/out of the system
  I_W: set of indices for work performed by/on the system
  I_m: set of indices for medium entering/leaving the system.

- **Steady state momentum balance**, leading to pressure loss equations of the type

$$\Delta p = f(v_{fluid})$$

  Δp: pressure difference between inlet and outlet of a component
  v: velocity of flow

- **Convective heat transfer**, e.g., between steam and wall, described by

$$\dot{Q} = \alpha \cdot A \cdot (T - T_f) \qquad [W]$$

with

Q: heat transfer $[J]$
α: heat transfer coefficient $[W \cdot m^{-2} \cdot K^{-1}]$
A: contact area between fluid and material $[m^2]$
T: material temperature $[K]$
T_f: fluid temperature $[K]$.

- **Conductive heat transfer** inside the material:

$$\dot{Q} = \lambda \cdot A \cdot \frac{dT}{dx} \qquad [W]$$

Q: heat transfer $[J]$
λ: conductivity $[W \cdot m^{-1} \cdot K^{-1}]$
A: cross-sectional area $[m^2]$
T: material temperature $[K]$
x: location $[m]$.

Moreover, the **kinetic energy of the rotating shaft** and the **work performed by turbines** are taken into account in the model.

After coupling the equations, the overall model consists of coupled ordinary and partial differential equations as well as algebraic equations. The model equations, the path and box constraints and the cost functions are implemented in Modelica®,[4] and its optimization extension Optimica, see Åkesson (2008). After spatial discretization, the model consists of about 1200 equations with 600 variables and 4500 parameters and constants. By means of model reduction techniques (for instance, by eliminating alias variables and parameters), the model is reduced to a size of about 30 differential states, 300 algebraic states, and 430 constraints. CasADi is used to apply a collocation method on the optimal control problem in order to obtain a nonlinear program (NLP) that can be solved with IPOPT. This NLP consists of about $(30 + 300) \times N$ variables and approximately $430 \times N$[5] (in-)equality constraints, where N denotes the number of collocation points.

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[4] https://modelica.org/

[5] The exact number differs since not each constraint is implemented in all collocation points

4.3. Designing the cost and the turnpike

A direct formulation of the optimization objective as an infinite horizon functional is

$$\int_{t_0}^{\infty} \chi(x(t)) m_{fuel}(t) dt, \tag{4.3}$$

with $\chi(x) = 1$ if the startup finalized conditions are not yet met at state x and $\chi(x) = 0$ otherwise. Given the complexity of the model, it is difficult to check rigorously whether the resulting optimal control problem is strictly dissipative. Numerical simulations suggest that the turnpike property is present, hence by Remark 3.10 it is likely that also strict dissipativity holds. However, the optimization horizon needed to realize the turnpike property is relatively large. This is because due to the choice of χ the above cost grows quickly (even discontinuously) near the turnpike, but it does not increase further when the state moves further away from the turnpike. Hence, there is no significant guiding effect towards the turnpike as described in Remark 3.15 and thus a long prediction horizon is needed to obtain the turnpike property, which in turn is needed to ensure good performance of the MPC controller. A consequence of this long prediction horizon is that solving the resulting optimization problem needs a long time, which leads to the problem that this solution is very difficult to implement in real time. Changing the optimization objective by including a component that penalizes the distance to the desired load level might change this, but this would also change the optimal solutions, which would not be fuel-optimal anymore, which is undesirable. Hence, we have to find another way to circumvent this problem.

The chosen remedy for this problem is to compute the optimal solution for the functional (4.3) offline and use it as a reference $x_{ref}(t)$ for an MPC scheme with tracking-type cost. To this end, two selections of variables $z \in Z$ and $y \in Y$ are tracked by minimizing the functional

$$\int_{t_0}^{t_f} \sum_{z \in Z} \lambda_z (z(t) - z_{ref}(t))^2 + \sum_{y \in Y} \lambda_y \max\{0, y(t) - y_{ref}\} dt$$

with weights $\lambda_z > 0$ and $\lambda_y > 0$. The aim of this functional is to drive and keep the quantities z close to z_{ref}, which denotes quantities obtained from the optimal trajectory of the offline optimal control problem. These are the quantities we want to track as good as possible. The quantities y represent the temperature differences in the thick walls modeling the thermal stress. These are special in the case that we do not necessarily want to track the corresponding y_{ref}, but only want to avoid by penalization that the y values become larger than the corresponding reference values y_{ref}. Examples for such variables are temperature differences in thick-walled components. Large temperature difference (for instance, the modulus of surface temperature minus wall middle temperature) means high thermal stress. If during the reference tracking MPC start-up the temperature differences get larger than in the reference, we want to penalize this. On the other hand, if the temperature difference is lower compared to the reference, then we do not want to track the corresponding reference, as this would mean that we increase the thermal stress just for the sake of tracking.

The initial state x_0 of the MPC problem results from the cool down behavior of the power plant during shutdown. Depending on the shutdown period and other boundary conditions it strongly differs and is not known in the offline phase. Since a wide range of cool down states of the power plant needs to be covered, a whole dictionary of optimal start-up trajectories for different x_0 should be available, from which the most suitable should be chosen at run time. The ability of MPC to control the system back to the reference when it is not exactly at the reference implies that it is sufficient to have references that start near the true x_0 but not necessarily exactly at this point. With this tracking approach and the above functional, the optimal reference trajectory defines the (now time-varying) turnpike of the MPC problem, which is not known exactly but is expected to be close to the reference trajectory. As a consequence, the initial distance to the turnpike becomes smaller, hence the optimization horizon can be reduced to a couple of minutes, which in turn reduces the computation times for the optimization, leading to a real-time capable scheme.

4.4. Findings of the on-site implementation

The following figures show experimental results that were obtained during implementation of the Siemens Energy product “Low Loss Start (NMPC)” at several CCPP units. In all units the predictive optimal start-up control as discussed above has been successfully implemented and performs as expected. Therefore it is used daily by the customers to their satisfaction. In all graphs the following color code is used:

- *solid black*: the optimal reference used for the tracking MPC
- *solid red, green, purple, and orange*: results from MPC controlled optimal start-up

- *dashed light blue, red, green, and pink:* results from conventional start-up

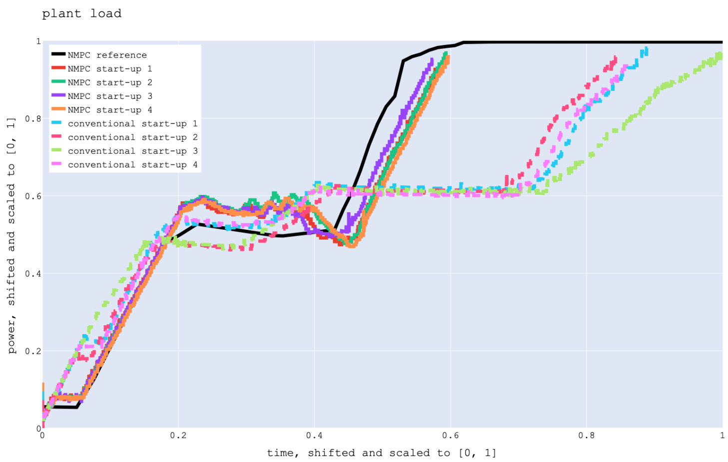

Figure 6: Plant load over time

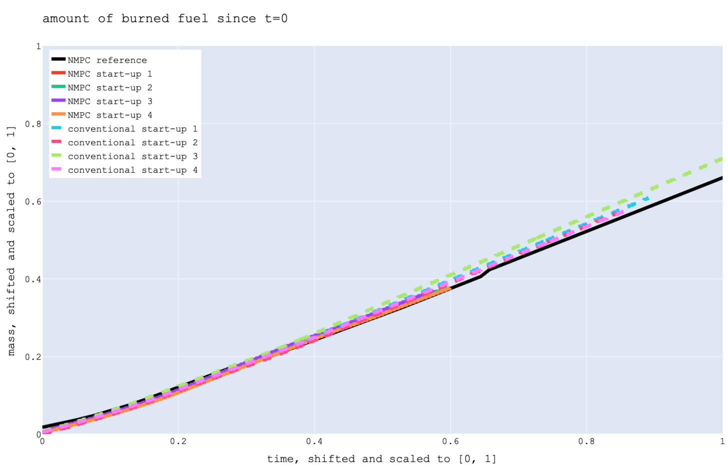

Figure 7: Amount of burnt fuel over time

Figures 6 and 7 show that the start-up controlled with MPC reaches the desired load level about 40% faster and needs about 40% less fuel for this.

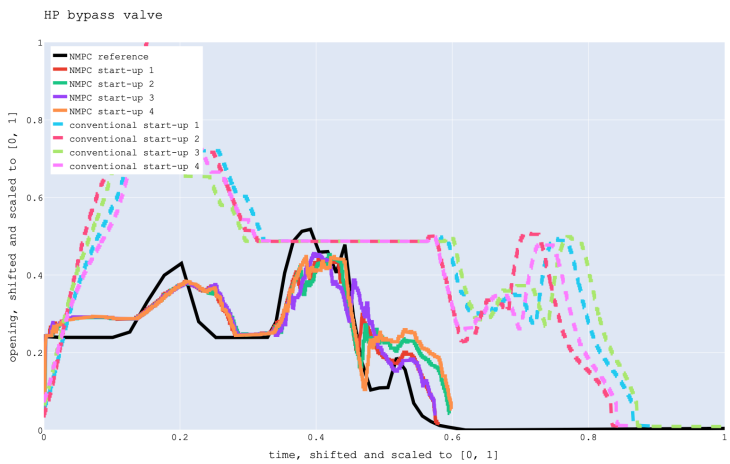

Figure 8: Opening of high-pressure bypass valve over time

Figure 8 reveals one of the fundamental differences between the conventional start-up and the one controlled by MPC: With MPC, the high-pressure bypass valve is closed much earlier and thus directs much more steam to the steam turbine at earlier times then in the conventional way. As a consequence, the steam turbine can be ramped up much faster, allowing to reach the desired load level significantly earlier, as Figure 9 shows.

Finally, Figure 10 shows that by directing more steam earlier to the steam turbine the MPC controller not only leads to a much faster and energy efficient start-up, but also achieves a more gentle warming up of the thicked-walled components, in which the temperature gradient is significantly smaller than in the conventional start-up.

5. Conclusions

In this paper we have shown how MPC with approximately optimal performance can be achieved for a challenging industrial use case. In this use case the direct optimization of the desired cost functional leads to prohibitively large computation times. This is because for this cost long prediction horizons are needed in order to obtain the turnpike property that ensures approximately optimal performance of the MPC closed-loop. A remedy is the computation of optimal reference trajectories in an offline phase and the use

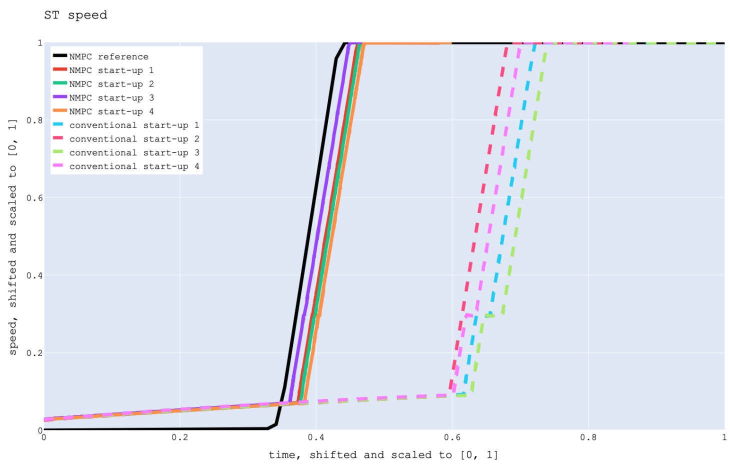

Figure 9: Speed of steam turbine over time

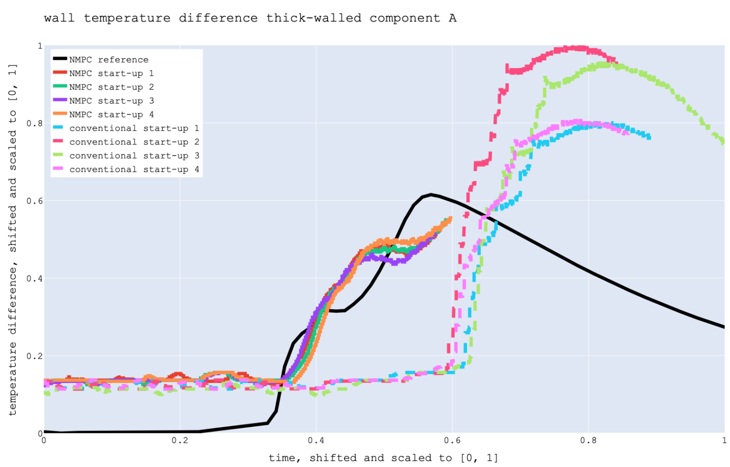

Figure 10: Temperature difference in thick walled component

of an auxiliary cost penalizing deviations of selected quantities from the reference quantities in the online scheme. For this cost the turnpike property is obtained for significantly shorter prediction horizons, allowing for a real-time implementation of the MPC controller.

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