



Towards New Teaching in Mathematics

Peter Baptist **Experiencing Mathematics**

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Experiencing Mathematics

Suggestions for Exploring Individual Paths towards Learning

Mathematics – a Strange World for Many

Mathematics appears to be a particularly difficult subject for people in Germany. Some years ago the German magazine *Psychologie heute* discussed the issue of "Why so many people are lost when it comes to numbers and facts". I quote the article's first few lines:

"In many circles, deficiencies in mathematics appear to be regarded as excusable peccadilloes. Party guests come out with statements like 'In school, mathematics was always my worst subject', and no one raises an eyebrow. Even educated people are often at odds with numbers. Why is it that many people cannot handle mathematical and statistical information?"

About the same time, the German weekly DIE ZEIT had the following to say:

"The shock was enormous. The way German students fared in the international PISA study was regarded as scandalous, disastrous, a catastrophic failure. Since then, the nation as a whole has been denouncing the failings of today's teachers and students. But among those deploring the results there are many who should be glad that they themselves were not put to the test of solving ninth-grade problems. ... In terms of arithmetic and abstract thinking, the knowledge society in Germany presents a rather gloomy picture, even among university graduates."

This statement is supported by two examples. According to a statistician at the University of Hamburg, many medical students (not to mention practising physicians) encounter great difficulties when presented with test questions along the following lines:

Of 3,000 people 1.4% have cancer. How many people are affected?

Even after a few minutes' reflection, some people cannot solve this problem. Among business school students the situation is no better. A professor at the University of Leipzig presented his students with two problems from the PISA test. The outcome was "a disaster" (sic!). Yet only very few feel such a result to be embarrassing.

In autumn 1999, the Hamburg English Studies professor Dietrich SCHWANITZ (1940–2004) hit the headlines with his book about education. The subtitle was "All one needs to know". But there is nothing in it about the natural sciences and mathematics. Are they subjects we do not need to know anything about? Asked about this in an interview, Schwanitz answered: "In the German tradition, natural sciences and mathematics have not been perceived under the term [general] education". But if so, what is [general] education all about? Schwanitz gave a rather flippant answer: "Among educated people you will make a complete fool of yourself if you don't know who Molière was. But you can easily get by if you have no

idea about the second law of thermodynamics. Traditionally, the natural sciences are not considered part of general knowledge."

In other words, someone who does not know who Shakespeare, Goethe or Mozart was is uneducated and ignorant. But if you've never heard of Euler or Gauss, people are more forgiving, because those great figures are familiar only to specialists at best. Knowledge of outstanding mathematicians or even of elementary mathematics is obviously not essential to a good educational background.

No wonder, then, that in an EMNID survey one third of the German citizens questioned proved unable to correctly interpret the meaning of "40 percent". Some of them thought it was equivalent to one-fourth, while the rest hazarded the guess that it might mean "one in forty"! Our press review continues. The Süddeutsche Zeitung quoted Hans-Olaf HENKEL, former BDI president and chairman of the Leibniz-Gesellschaft (non-mathematician, by the way!), as saying: "Mathematics is not everything. But without mathematics, everything is nothing". What does that mean? We are living in a society that is crucially dependent on mathe-matics. Without mathematics there would be no such thing as technical or medical progress. The situation is paradoxical. The more important mathematics has become for society, the more it recedes into the background. If we do not look closely, we fail to realize that mathematics plays a significant role in our lives. Most of us don't know and do not want to know. Traffic-light control circuits and intelligent transportation systems are inconceivable without mathematics. Without mathematics we would not have notebooks, mobile phones, DVD and MP3 players, navigation systems, CAT scans, cash cards or scanner checkout counters. We could extend this list almost indefinitely because ultimately mathematics is at the root of all the electronic components embedded in devices and vehicles. Even if math is mostly invisible in the finished product, it had to be developed in the first place and had to be applied to the respective problem. It is certainly no exaggeration to assert that mathematics is the prime mover of the age of information we live in.

Student Perceptions of Mathematics

Everyone who attends school encounters mathematics. Of course, teachers need to convey the technicalities of the subject. But apart from that, they also assume a highly responsible task. The way they present mathematics will define how mathematics is perceived throughout most of society.

What perceptions of mathematics are entertained by students as a result of the classroom instruction they have received? In an American survey in 1992, Alan SCHOENFELD found the following typical student perceptions of the essential features of mathematics:

- Mathematical problems have one and only one correct solution.
- For any mathematical problem there is only one valid approach usually the one most recently demonstrated by the teacher.
- "Ordinary" students do not expect to achieve a true grasp of mathematics; they assume it is sufficient to learn things by heart and apply mechanically what they have learned.

- Students who have grasped mathematical subject matter are in a position to solve any problem in a matter of a few minutes.
- Mathematics as taught in school has little or nothing to do with the real world.
- Engaging with mathematics is a solitary activity indulged in by "loners".
- > Mathematics is a formal system. It has nothing to do with intuition and creativity.

Reasons why Our Society Has an Impaired View of Mathematics

Why do students display such a "skewed" perception of mathematics that does no justice at all to the significance and pertinence of the subject matter? One cause is certainly to be seen in the type of instruction they get. In comprehensive studies conducted under the direction of Franz E. WEINERT (1930–2001), Max-Planck-Institut für psychologische Forschung, instruction and learning in Germany do not come off very well at all:

- Generally, classroom lessons are much too content-oriented and not sufficiently talentoriented.
- Paradoxically, instruction is geared too much to achievement and not enough to learning.
- On average, classroom lessons are geared to getting through the workload rather than discerning talent.
- Generally, instruction is not variable enough in terms of teaching methods.

How is mathematical instruction assessed in general? Here we find a highly questionable attitude:

When a sufficiently large number of students can correctly solve a sufficient number of problems of a specific known type, the instruction is considered to be successful.

This cannot and must not be the purpose and aim of instruction! Mathematical instruction must not be equated with the students reproducing what the teacher has demonstrated. We must regularly check

- how well students have really grasped and processed newly acquired knowledge;
- how well students can really handle the methods, terms, and rules they have just learnt;
- how flexible students are in applying what they have learnt to problems that are as yet unfamiliar to them.

This approach is also the one used in the PISA survey. The idea is not to test the content of a curriculum. PISA assumes that fifteen-year old boys or girls who have received several years of mathematics education should have acquired a certain degree of mathematical literacy that they can put to active use. The test is designed to find out whether students have grasped basic mathematical concepts in such a way that with the help of these tools they can deal with problems from different contexts. Unfortunately, German students fall down on that score.

Why? Martin WAGENSCHEIN (1896–1988) aptly described the teaching of mathematics as a tragedy:

Mathematics, the pattern for clarity and free thinking, is perceived by many students as a pattern for lack of clarity and meaninglessness.

When they leave school most students embrace this attitude. Ultimately, they are glad to have cleared that hurdle and will do their best to avoid any further contact with mathematics. In an article in 1996, the *Zürcher Tagesanzeiger* printed a passage that gives us food for thought:

In our civilization, mathematics is considered to be an arcane science, an inaccessible terrain barricaded by thousands of formulas. That is a prejudice nurtured by many factors: unimaginative and dry-as-dust teaching approaches, the intellectual laziness of the public, and the self-isolating attitude of many mathematicians.

This statement was made by Hans Magnus ENZENSBERGER, one of the greatest German writers of our age. But he is not content just to accuse and lament. Enzensberger reacted by writing a remarkable book titled *The Number Devil (Der Zahlenteufel)*. This "pillow book for all those who suffer from math anxiety" introduces the reader to diverse, interesting, and exciting subsections of mathematics. The book sets out to persuade children and math-impaired adults to explore their own paths to mathematics.

Defining General Mathematics Education

Hans Magnus Enzensberger wrote his mathematics book (not only) for children. The reverse situation is an English mathematician from the Victorian era who wrote famous children's books still read today by children and adults alike. At least one title is known all over the world. What is probably less well-known is that Lewis CARROLL was the pen name of the mathematician Charles Lutwidge DODGESON (1832–1898), who taught at Christ Church College, Oxford in the second half of the 19th century.

When we leaf through *Alice's Adventures in Wonderland*, we meet a rather clueless Alice who does not know which turn to take at the fork of a road in the woods. Rather haltingly, she asks the Cheshire cat sitting up in a tree: "Would you tell me, please, which way I ought to go from here?" "That depends a good deal on where you want to get to," said the cat.

Applying this situation to the classroom, we need to ask:

- Which qualifications should mathematics instruction aim at?
- Which form should the lessons take?

Which qualifications should be aimed at in the framework of general mathematics instruction? They are

- the ability to cope with open-ended tasks, since, as a rule, realistic problems and tasks are not defined precisely,
- the ability to recognize that mathematical concepts and models can be applied to everyday problems and to problems of a complex nature,
- ▶ the ability to translate problems into suitable mathematical representations,
- adequate knowledge and mastery of problem-solving routines.

Classroom instruction that appropriates these aims is not characterized by a monoculture of dreary tasks such as the following:

4. a)
$$\frac{4x}{5y}: \frac{5y}{4x}$$
 b) $\frac{18m}{35n}: \frac{7m}{9n}$ c) $\frac{108}{25pq}: \frac{135p}{q^2}$ d) $\frac{72ab^2}{27c}: \frac{16a^2b}{45c^2}$
• 5. a) $\frac{147(uv^2w)^2}{1024ab^3}: \frac{196u^3w}{640(ab)^3}$ b) $(\frac{12m^3n}{55xyz^2})^2: \frac{27m^4x^2}{242ny^2z}$
6. a) $\frac{5(2a-b)}{9}: \frac{4a-2b}{15}$ b) $\frac{5m}{3n-5}: \frac{m^2}{5-3n}$
7. a) $\frac{rs}{24r-32s+20t}: \frac{st}{30r-40s+25t}$ b) $\frac{8x-28y}{5yz-3y}: \frac{21y-6x}{20z-12}$
8. a) $(a^2-9b^2): \frac{a+3b}{a-3b}$ b) $\frac{6x-6y}{5x+5y}: (3x^2-3y^2)$
9. a) $(2m^2+20m+50): \frac{m-5}{15+3m}$ b) $\frac{x-3}{3x+15}: (x^2+2x-15)$
10. a) $\frac{25a^2-20ab+4b^2}{4c^2+56c+196}: \frac{5a-2b}{3c+21}$ b) $\frac{8u-20v}{9+3w}: \frac{4u^2-25v^2}{w^2+6w+9}$
11. a) $\frac{9p^2-36q^2}{3-r}: \frac{2p-4q}{r^2-9}$ b) $\frac{18s-2}{81s^2+18s+1}: \frac{2}{1-81s^2}$
•12. a) $\frac{x^2-1}{1-y^2}: (\frac{x-1}{1-y})^2$ b) $\frac{(x-1)^2+4x}{x^2+4y^2}: \frac{3x^2+6x+3}{4xy-(x+2y)^2}$
13. a) $(\frac{1}{a}:\frac{1}{b}):\frac{1}{c}$ b) $\frac{1}{a}: (\frac{1}{b}:\frac{1}{c})$ c) $\frac{1}{b}: (\frac{1}{c}:\frac{1}{a})$
14. a) $(\frac{w}{v}:\frac{w}{w}):\frac{w}{x}$ b) $\frac{w}{v}: (\frac{w}{w}:\frac{w}{x})$ c) $(\frac{w}{w}:\frac{w}{v}):\frac{x}{w}$
15. a) $(\frac{1}{y}:y^2):\frac{1}{y^3}$ b) $(\frac{12m^3n}{12}:\frac{1}{2a}+\frac{1}{12}$
•16. a) $\frac{x^2-9}{6y}: (z:\frac{xyz}{x+3})$ b) $\frac{54(a^2-b^2)}{13a}: (51(ab+b^2)]$
(c) $\frac{5ab(x-y)}{6xy(x+y)}:\frac{10ax(x-y)}{3by(x+y)}$ d) $\frac{38a(a-b)}{51b(x-y)}:\frac{57b(a^2-b^2)}{68a(x^2-y^2)}$

Unfortunately, such textbook problems often still determine the course of math lessons and are characteristic of the way this subject is taught. Instead, teaching should focus on real-life requirements, as postulated by the outstanding German-Dutch mathematician Hans FREUDENTHAL (1905–1990). He developed a comprehensive idea of general education in mathematics. His central statements are:

- Mathematical concepts, structures, and ideas were invented to explore and structure the phenomena of the physical, social, and intellectual world.
- In teaching and learning mathematics, reality must be the starting point rather than "ready-made" mathematics.
- The aim of homing in on phenomena is to create sustainable mental models for mathematical terms.

Future Teaching

What requirements need to be satisfied for classroom instruction to provide the qualifications listed above? What is an appropriate teaching process? Numerous suggestions are made in the literature. We select one which emphasizes a crucial aspect:

"The kind of instruction we need is one that teaches students in a way that does not make them suffer but equips them with active enjoyment in acquiring knowledge or skills. Teaching, presenting, transmitting are features of a teaching approach that belongs to the past, and they are of little value for the present. ... Of course, in future students will also need to knowledge and skills – and we hope they will do so even better than in the past. But we do not want to ram them down their gullets, we want them to acquire them for themselves. ... This changes the teacher's tasks in all fields. Instead of merely presenting material, he will have to encourage the abilities of the students. ... And the student's job is not merely to be on the receiving end, but to actively participate in the process. ..."

These remarks are a clarion call for independent learning. This topic is a highly relevant one today – although Johannes KÜHNEL (1869–1920) voiced his considerations regarding the teaching process of the future as early as 1916.

This shows clearly that we have an implementation problem. We know precisely how learning processes should operate, i.e. students learn best by doing, not by listening. But in spite of this insight, teachers still do the talking for 60 to 80% of the time spent in class. Wilhelm BUSCH (1832–1908) put it very trenchantly:

"What memory of lessons do we keep? One voice drones on, the others sleep."

What is the upshot of such lecturing in the classroom? It is quite simple. All the parties involved are dissatisfied with everyday life at school.

Teaching Must Change!

Teaching must change because the attitude of children and parents to school has changed. That does not mean that schools have to adjust. They must not choose the line of least resistance by continually reducing standards. When planning their teaching, instructors must consider the changes taking place in society. We cannot go on teaching mathematics as we did in the 1950s or the 19th century.

Heinz KLIPPERT, a proponent of classroom reform, clearly points out that the situation has changed:

- ► Today's students are not less competent than in the past, they are just different.
- Many students are extremely spoiled and overprotected. The result is chronic helplessness and laziness!
- Many students are highly unreceptive toward subject matter dealt with in school. This results from the influence of everyday media consumption.

Klippert therefore urges educationalists to think and think again. I quote:

- The traditional methods of leading and presenting are becoming increasingly questionable. The teacher as "entertainer" is having a much harder time of it.
- ► This is also due to the fact that 90% of our middle-school students prefer a lively hands-on learning environment. In order to learn, they need first-hand experience.

Of course there are many dedicated teachers who have recognized these problems and who try to explore new paths. Conventional instruction has changed considerably at many places in the past few years.

- The subject matter is often presented in problem form. Mere knowledge of facts is no longer a prime concern.
- Textbooks have more illustrations. Instruction material has been edited in a more student-friendly way.

Yet most students are "brilliant at forgetting", as Berlin headmaster Wolfgang HARNISCH-MACHER said so appositely in a newspaper article. Heinz Klippert, referred to earlier, underscores the weaknesses of everyday instruction even more outspokenly:

"Knowledge is served up, ingurgitated, forgotten."

If this method of instruction does not change ("Knowledge is served up"), no reduction of the syllabus will do anything to improve matters. Instructors must rethink their role, radically redefine it. In the course of lessons, instructors should no longer set out to convey as much knowledge as possible or to be the walking encyclopedia at the heart of the proceedings. Teachers must learn to restrain themselves and also to desist from teaching pre-fabricated lessons. They must discard the notion that it is their job to show off everything they know. Instead, they should make an effort to establish learning situations that enable students to work independently and to instill in them a sense of responsibility for their own progress. The guiding principle is not what teachers say and do in class, but what they manage to "get across" to the students.

Classroom teaching of this kind ties in ideally with the ideas of Johannes Kühnel quoted earlier. The vital thing is to ensure that school develops from a place of instruction to a place of learning. Countries with a good showing in comparative student-performance tests have shown that this is possible.

Working Independently – Learning from Success

There is certainly no panacea for the effective implementation of independent working. But a look at nations with high scores in international student assessment studies is certainly worthwhile. So let us home in on the front runner in the TIMS-study, i.e. Japan. How does education there differ from ours?

Video studies and live observations suggest that the following basic pattern is typical of Japanese classroom lessons:

- 1. Present a problem and ensure that the question is understood.
- 2. Independent work and solution-finding on the part of the students (individually or in groups).
- 3. Collect different solutions and exchange thoughts on them

At first sight, we might perhaps not detect any difference from German problem-solving classes. This was indeed the reaction displayed by many of our teachers and students after they had watched Japanese video streams. But there is, in my view, one crucial difference that contributes substantially to the discrepancies in the scores achieved by German and Japanese students. The crucial point is the transition from phase 1 to phase 2. The Japanese teacher checks whether the students have understood the problem, and after that the students work independently. No potential solutions are discussed with the class, nor does the teacher ever hint at any such solution. The students work on the problem without outside help. This approach prompts students to work autonomously, which almost automatically leads to differing methods of problem-solving. Discussion of the problem is only undertaken after the students have had a chance to engage with it themselves. True, the discussion is still led by the teacher, but the students have meanwhile become more competent partners with regard to the problem posed and are thus in a better position to grasp the explanations offered to them.

Let us look at a comparable classroom situation in our country. The transition from phase 1 to phase 2 is introduced with a question like this: "What is a possible approach to solving this problem?" Suggestions made by students are taken up and discussed. Then the teacher will use a question-and-answer method to outline the approach to be taken – I emphasize the approach to be taken. "Independent" problem-solving by students then takes the form of using the approach the teacher has outlined. With the best intentions in the world, this is tantamount to disempowering our students. No wonder we hear screams of horror whenever we present a slightly unfamiliar problem: "We've never done that!" is the universal lament. This standardised instruction based on asking the "right" questions and getting the "right"

answers precludes the proposal of a variety of possible solutions. And the principle of independent learning so stoutly upheld by educationists also falls by the wayside. With this approach, we divide challenging and complex problems into manageable portions, i.e. clearly defined partial problems, thus excluding individual approaches from the outset.

What conclusions can we draw from this observation? What is the formula for teaching success? Leave students to their own devices? Certainly not! Hardly anyone will learn to play the piano just by being plonked down in front of the instrument and left to figure out the chords without any assistance. Five-finger exercises stand at the beginning of any pianist's career. Transposing this to school, this means teacher-controlled instruction, which, however, is not the equivalent of small-scale, question-and-answer instruction, as the Japan example shows.

George Polya – the Classic Problem-Solver

What are the five-finger exercises needed to become a successful problem-solver? Let us see what the grand master of successful problem-solving, George POLYA (1887–1985), has to say on the subject. In his book *How to Solve It* (German translation: *Schule des Denkens*), he propagates the following approach:

How to Solve It

- 1. Understanding the problem.
- 2. Devising a plan.
- 3. Carrying out the plan.
- 4. Looking back.

In a Japanese classroom lesson, phase 1 takes the form of discourse led by the teacher. The crucial step then follows in phase 2, i.e. when the Japanese students are left to their own devices. Our major concern here is to find out how a teacher can prod students towards discovery without hinting at possible solutions. Polya advises teachers to provide help only from "within" (intrinsically), i.e. only to offer suggestions the student could also have found on his own, had he thought of it. The teacher should avoid "extrinsic" help, which means he is not to reveal any parts of a solution that go beyond the student's current knowledge. "Help-ing them to help themselves" is very important, although providing such help is anything but easy. The teacher needs excellent knowledge both of his students and of the problem they are faced with.

The schema set out above does not furnish a watertight guarantee for successful problemsolving, but it does help students to marshal their thoughts and to learn independence. Useful suggestions are also provided by the various individual, phase-related questions Polya proposes in his book. Phase 2 is obviously the most crucial. What can you do if you are unable to find a promising approach to solving the problem? Here, even Polya does not have an all-purpose formula up his sleeve. All he says is: "The first rule of discovery is to have brains and good luck", and "The second rule of discovery is to sit tight till you get a bright idea". The fourth phase (reviewing the whole process) is also highly significant for instruction. It encourages thorough understanding and serves to deepen and expand the problem.

Calculation Is Less Important than Understanding

Polya's proposals represent a radically new approach to dealing with problems. Merely solving a problem is replaced by getting deeply involved in the context of the task at hand. Transmitting content is important, but equally vital are the underlying learning processes. Therefore, we expand and modify Polya's scheme. Instruction comprises the following strongly interactive phases:

Orientation phase:

Understanding the problem. The aim is to grasp the problem and identify all necessary data.

- Creative processing phase: Retrieve missing data, if any. Look at related problems. Recognize patterns and devise strategies. Develop a plan for solving the problem. Actual problem-solving phase: Implement the plan. Find the solution.
- Evaluation phase:

Examine the solution process and the solution obtained. Which new insights has the problem generated? Was the strategy familiar? Can we derive the solution differently?

Expansion and link-up phase:
 Find connections with knowledge and problems already familiar, generalize, suggest variations of the problem.

The schema outlined above is designed to provide guidance in working with problems. It facilitates access to systematic problem-solving approaches. But Polya also has very clear ideas about the type of problems to be presented. He prefers problems that are about one-third geared to the students' mathematical literacy and about two-thirds geared to their common sense. These requirements are certainly met by the well-known balloon problem.

How much air is needed to fill the balloon?



Foto: Kropsoq, commons.wikimedia.org

The grasp of standard subject matter can also be tested intelligently. The following problem requires students to activate and use elementary geometry:

In a triangle one angle is 38°. Determine the other two angles to obtain

a) an acute-angled triangle,
b) a right-angled triangle,
c) an obtuse-angled triangle,
d) an isosceles triangle,
e) an equilateral triangle.



The tasks a) to e) are characterized by a variety of potential solutions (several, one, none). In addition, there is productive repetition of knowledge about the forms triangles can take and the sum of their angles.

Training arithmetical skills does not necessarily mean slogging away at cripplingly boring problems, as is demonstrated by the next example:

The European Central Bank plans to issue three-euro and five-euro coins only. Can every round-figure amount higher than seven euros be paid for exactly just by using these two coin denominations?

Here students can be made aware of content-related goals revolving around the divisibility of numbers, plus various heuristic approaches and strategies (trial and error, systematic trial, looking at a special case). What matters is not only to achieve results but to figure out the specific mental processes leading to those results.

The Journey Is the Goal

In working out the subject matter for math lessons, learning methods and problem-solving should also be issues of prime concern. You cannot learn to solve problems just by obeying problem-solving rules. You can only learn by dealing with practical problems and analyzing their solution. It is crucial for success that such endeavours should not be sporadic. This type of instruction must be undertaken systematically.

Potential strategies of this kind are important not only for mathematics but also for other subjects, as well as for coping with problems in everyday situations. Here are some examples:

- testing and systematic re-testing by trial and error,
- working with special cases,
- generalizing,
- finding analogies,
- restructuring,
- working forward or backward,
- going back to a known case,
- recognizing patterns.

Alexander WITTENBERG (1926–1965) gives us an apt summary of these considerations. His book *Bildung und Mathematik* (*Mathematics and Education*) is definitely worth reading. He writes:

> "The real essence of teaching is not found in the subject-related material but in the process of working on it."

Independent Learning – Implementation in the Classroom

If we want students to develop a living relationship to mathematics, they should not experience mathematics instruction as an artificial world that has nothing to do with their personal lives and feelings. This requirement grows from Freudenthal's ideas about general mathematics education. Students must be given greater independence vis-à-vis the teachers. Students should learn to structure, process, and present their own ideas. Cognitive psychology stresses the enormous significance of autonomous learning. A successful learning process is active, constructive, cumulative and goal-oriented. For school education, this means, among other things, that the teacher is not an entertainer, and the student is not a mere consumer. How can we implement all these concepts and ideas in the classroom? We can start by adopting successful countries as role models:

- ► Teachers present students with challenging and realistic problems.
- Problems are presented and discussed clearly, however without anticipating any solutions.
- Very often the problems presented can be solved in a variety of ways or may have several solutions.
- Approaches taken are discussed in detail. Frequently, the approaches are more important than the solutions themselves.
- Lessons are not dominated by test situations but by learning situations.
- Approaches to solving the problem, including wrong approaches and the wrong solutions that go with them, are accepted, discussed, and exploited for the learning potential they have.

The structure of a teaching unit geared to developing student autonomy is illustrated below. We also discuss the individual phases.



- 1. As a rule, the *problem* is presented by the instructor. Of course, this task can also be assumed by students, usually by varying a problem that has already been treated.
- 2. Independent work. Students work independently over relatively long periods of time (as individuals and/or in small groups). The teacher monitors the students. At most, he acts as an advisor (helping them to help themselves). The teacher notes the approaches and/or solutions that are taking shape and can then select students for presentations.
- 3. Selected students present their solutions and/or approaches to the problem. The *presentations* are compared and discussed. The teacher should exercise restraint and only intervene when necessary.
- 4. Following the students' active working phases, the teacher *summarizes* the results. This teacher-centered phase can also be used to add further remarks and introduce new concepts and formalisms.

This approach to classroom teaching is aimed at the interaction between the students' active, independent working phases and teacher-centered instruction phases. Independent know-ledge formation does not rule out instructional support or the systematic transmission of knowledge. In the end, it is the interaction that makes for effective and sustainable learning processes.

Comparison with Traditional Classes

This approach has been used successfully in the schools enrolled in the BLK pilot study SINUS. Comparing it with traditional methods of instruction, we found

- that learning subject matter is not limited to one single method but involves several processes handled by every student for him- or herself;
- that students are prompted to work actively and independently;
- that presentations of different solutions demonstrate different approaches and that mathematical communication among students is promoted;
- ▶ that initially, the teacher is a mere observer and, at most, an advisor;
- that the teacher does not offer explanations until the students have attempted to come up with a solution;
- that in the students' activity phase, weak and unmotivated students cannot duck for cover as effectively as they can in lecture-style teaching environments;
- that initially, it is more time-consuming;
- that at first it is unfamiliar, more complex, and more demanding both for students and teachers;
- that this demanding type of instruction is also enjoyed by students.

Writing, Sketching, Writing

In our mathematics classes, students rarely write their own texts. One crucial aim of education, and also a crucial element for achieving this aim, must be to enable students to formulate the results and/or the insights arrived at in clear language, in their own words, and not in standard phrases they have learnt. This means:

Learning outcomes and learning processes should be put down in writing as frequently as possible.

Assignments, e.g. in the form of worksheets, should be designed in such a way that they support this aim. The worksheet designed by our Swiss colleague Peter GALLIN for solving quadratic equations is a good example.

Solve the quadratic equation $ax^2 + bx + c = 0$

Solve for every single equation. As a rule, every equation has two (!) solutions. Write down your lines of thought. In particular, for every problem note down what is new or different compared to the preceding one. Be careful with the transition from 5. to 6.

1.
$$x^2 = 4$$

2. $x^2 - 3 = 0$
3. $2x^2 - 1 = 0$
4. $x^2 = 6$
5. $(x + 2)^2 = 6$
6. $x^2 - 6x + 9 = \frac{25}{4}$
7. $x^2 - 6x = 31$
8. $x^2 + 4x = -\frac{7}{4}$
9. $x^2 - \frac{2}{3}x = -\frac{1}{9}$
10. $x^2 - 3x = -\frac{25}{4}$
11. $2x^2 + 4x - 7 = 0$
12. $\frac{1}{6}x^2 - \frac{1}{4}x - \frac{1}{6} = 0$
13. $x^2 + 2px + q = 0$
14. $ax^2 + bx + c = 0$

In addition to the carefully considered sequence of equations, the crucial element here is requesting the students to note down their lines of thought. In addition, they are also asked to define how every single equation differs from the preceding one. And they are asked to put down their ideas in writing.

In many joint publications, Peter Gallin and Urs RUF have demonstrated that such procedures lead to independent, constructive work. After all, in order to be able to write about something, we must have reflected on it more thoroughly beforehand.

Students are also asked to keep so-called study journals in lieu of the standard exercise books. They are required to note down everything essential to them in just one exercise book: homework, exercises, teachers' comments, personal ideas and perceptions, summaries. Teachers looking through such study journals can then establish quite easily what makes their students "tick" mathematically, and whether they have understood the syllabus or not.

In traditional classroom environments, teachers mostly offer their own or generally accepted ideas about the topic being dealt with. However, this does not lead to sustainable understanding. Students must be given the opportunity to develop their own ideas, associations, and images. This requires students to consider tasks on their own, discuss them with others, draw figures in their study journals, and write down their personal notes. This is the only way to foster a personal relationship with mathematics. It makes the subject matter more familiar and more transparent. Personal notes supplement textbooks and official worksheets.

Working with study journals can start very gently, in small steps. At the end of every teaching unit, students are asked to write down their answers to the following standard questions:

- What new insights have I gained today?
- What questions have remained unsolved for me personally?

Working with Open-Ended Problems and Learning Situations

Key definition: A problem is open-ended when students are empowered, expected, and allowed to work on a problem independently.

Questions arising from open-ended problems:

- What is the given situation?
- How am I going to proceed, and what am I going to do?
- How do I interpret the result?

Open-ended problems are characterized by questions that result in different, perhaps equally valid answers. These differences are the subject of discussion in working on the problem and in class. We can define open-ended problems in terms of their opposite, so-called closed-ended or closed tasks. This, in turn, confronts us with the new problem of finding out how to describe closed or narrowly defined tasks. Closed-ended tasks contain all the necessary data. The solution process is either obvious or clearly prescribed. For teachers, error correction is extraordinarily easy, and this is the reason why many teachers favor this kind of task. But if such tasks prevail, students will gain a skewed perception of mathematics. Everyone can think of typical examples from his/her own school days. Here is one example:

A rectangle with sides a and b has a perimeter of 25cm. One side is 3 cm longer than the other. How long is it? Start with an equation.

A hard-working student can hardly go wrong, unless he or she miscalculates. Students are not required to embark on considerations of their own. Individual paths toward a solution are not possible. The aim of the task is to practice solving equations. The request to "calculate the side length of a rectangle" is a minor matter. The situation changes completely when we focus on the content of the task. We are concerned with a rectangle for which the side lengths have to be determined. This piece of information is important for persons asked to draw a sketch of the rectangle. Here is the task opened up, now it is a problem:

A rectangle with a perimeter of 25 cm has one side 3cm longer than the other. Sketch the rectangle.

But it is not the problem alone that determines an open- or closed-ended approach. The much more important factor is the decision on the ways and means we use in the classroom to deal with the task and what we make of it.

"The real essence of teaching is not in the subject-related material but in the process of working on it."

Alexander Wittenberg (1926–1965)

The example above illustrates that closely defined tasks can generally be reformulated to obtain open-ended problems. Classroom lessons are given a more realistic touch, and every student can work on the problem in a way suited to his individual skills and knowledge. There are various aspects involved in generating open-ended approaches:

- extend problems (= woolly formulations), no narrowly conceived questions, permit variations of the question and different approaches to a solution,
- make students get involved with the subject matter underlying the problem presented,
- encourage students to develop a keener interest in different mathematical concepts.

How Do We Define Open-Ended Approaches in Mathematics Teaching?

Students are confronted with a situation or a challenge. General examples are: newspaper item, image, experience, real-life situation, geometrical configuration, number material.

Specific examples:

Adenauer monument: Determine the height of a statue that fits to the head of the first German chancellor Konrad Adenauer.





Balcony railing: Investigate the quadratic pattern between two posts of the balcony railing.

Traffic sign: How steep is the hill when the sign shows 100%?





Number sequences: Explain the adjacent number sequences and find out further elements. In open-ended situations we pursue the following aims:

- students are meant to gain their own individual overview of the situation presented to them by selecting, procuring, and evaluating information;
- developing situative questions or conjectures;
- exploring mathematical processes and paths in order to answer the questions posed or to validate conjectures;
- exchanging insights and knowledge with other students (the rest of the class).

How to Proceed in Introducing Open-Ended Problems

- Traditional procedure: The teacher asks questions about a given situation. Students answer the question or work on the respective problem.
- Students devise further questions in connection with the initial situation. This catalog of questions is discussed with all students in the class and may be arranged in a certain order, or the teacher adds new questions of his own. Here are some examples:
 - What is interesting about this question?
 - Which paths toward a solution are meaningful?
 - > What additional information or materials are required to answer the question?
- Students work on some of the questions or problems.

Once the students have gradually familiarized themselves with open-ended approaches, the teacher may change this procedure depending on the achievement and capability level attained. Questions from the teacher will be progressively reduced. Finally, they can be done without altogether. The teacher should, or rather must get used to "letting go". The teacher's task is to instill curiosity in his students. This also means that students must learn to ask appropriate questions. Teachers using this method consistently and on a regular basis will observe that their students gain increasing independence in studying such problems.

The specific importance of proposing questions in mathematics was emphasized by the German mathematician Georg CANTOR (1845 - 1918), best known as the inventor of set theory. He even talks about "the art of asking questions." In his oral doctoral examination in 1867, one of the theses he defended was: "In re mathematica ars proponendi questiones pluris facienda est quam solvendi". (In mathematics the art of asking questions is more commonly applied than that of solving problems.)

In class, the teacher needs to keep very much in the background and should always be prepared to act as an observer and advisor (an ideal worth aiming at!) However, in preparing such lessons, teachers must thoroughly explore the subject matter to be conveyed via such open-ended approaches. The teacher must be capable of indicating to his students that there are certain expectations. In addition, he should be prepared to offer additional questions and further suggestions and materials.

Examples:

1. Speeding Drivers, source: Norderneyer Badezeitung (newspaper), 1991:

Some years ago, every tenth driver exceeded the speed limit at some time or other. Nowadays it is only every fifth. But even five percent of drivers are too many. So speed checks are still with us, and drivers exceeding the speed limit will have to pay up.

Comments:

- ► Feasible instruction: Discuss the information given in this piece of news.
- Students familiar with such tasks have no need for such instructions.
- Teachers must make sure that students do not indulge a bias toward non-mathematical content.
- Diagnosis function: The teacher can deduce from the answers the understanding of percentages that the students have developed.
- Evaluating open-ended problems: An analysis of student solutions submitted in writing can serve as a basis for developing transparent assessment criteria that must be communicated to the students. (cf. Assessing Essays in German Instruction). Unfortunately, the following assertion is only too true: Most students do not take open-ended problems seriously unless they are declared to be test-relevant.
- 2. "Graphs Tell Stories":

A graph may tell a story. It could be a story about a lake, a bathtube, or whatever you imagine.

Invent a story to go with the adjacent graph:

water level in m

time in min '

3. Calculating with the hours of the day

The face of a clock shows the numbers 1 through 12. Take all twelve numbers and use addition and subtraction to create a term resulting in zero. a) Try to find several ways of doing it. b) What do your results have in

common?



Foto: High Contrast, commons.wikimedia.org

Classroom objectives:

- Implementing the task and producing a term,
- training addition and subtraction,
- discovering strategies leading to solutions.

Expected solutions:

Ad a) trial and error;

- systematic trial and error: determine sum total (78), divide sum by 2; find variations for half the sum.
- digression: How do we determine the sum of natural integers in arithmetical progression?

Ad b) All plus terms always solve to give 39. There is a minimum of 4 plus terms and a maximum of 8 plus terms.

Varying the task

- ► Take the six even numbers 2, 4, 6, 8, 10, 12 and use addition and subtraction to produce a term resulting in zero. Is this possible?
- Can you do the same with the six uneven numbers?
- Can you do the same with the numbers 1 through 11?
- Can you do the same with the numbers 1 through 10?
- Suppose the clock falls down and the face splits into three parts. Is it possible for the sum of the numbers on each part to be the same?

Working with the open-ended approaches described here is important for the training of basic skills, and it promotes the following important activities:

- working on problems independently,
- working out independent strategies to solve problems,
- presenting solutions to fellow students,
- comparing and evaluating different solutions.

At the end of the teaching unit, the teacher should sum up, evaluate the results achieved, and perhaps give further examples of problems with related mathematical content.

Sustainable results are achieved only when we have a balance between in-struction (provided by the teacher) and independent construction. Proceeding in the way described above is an attempt to achieve this kind of balance. The task posed induces students to work actively and independently. Teacher instruction can and must wait until after the students' work phase.

Characteristic Features of Open-Ended Situations

Open-ended situations

- are generally highly motivating,
- can be very true-to-life,
- enable students to devise questions and problems independently,
- ensure that students express themselves understandably,
- promote cross-linking what students know about other school subjects,
- enable students to solve problems with their own strategies via paths they have chosen independently,
- have only limited "product orientation" (the journey is the reward).

Advantages of Working with Open-Ended Approaches in Teaching Mathematics

- Students dealing with open-ended situations make more of an effort to gain insights into the subject matter of a problem than they do when solving conventional problems.
- In working on open-ended questions, students develop a variety of different questions. They design and tackle different problems depending on pre-acquired skills, interests, and capabilities (competencies). The students discover different approaches and possibilities for arriving at solutions. This makes for natural differentiation.
- When students tackle traditional tasks, we can mostly expect their answers to be either "100% correct" or "100% wrong". With open-ended problems, teachers must also expect "partly correct" solutions.
- Solving traditional math problems often gives rise to the so-called "captain's syndrome". Counter-productive ways of thinking are much rarer when students work on open-ended problems.
- What is the captain's syndrome? In 1989 the book *How old is the captain*? by Stella BARUK caused a stir. 76 out of 97 second and third graders questioned, "solved" the task "On a ship there are 26 sheep and 10 goats. How old is the captain?" Typically their answer was: 36 years! An extension of this (French) investigation to more tasks and pupils also of other grades confirmed this very upsetting result.
- Working on open-ended tasks does not result in any gender-specific differences. Female students benefit enormously from open-ended approaches, while males are not disadvantaged.

One remark needs to be made here. Our aim is not to replace all traditional problems by open-ended ones or to work exclusively with open-ended situations in mathematics classes. We merely want to extend the range of methods at our disposal. Routine tasks still retain their importance for practicing or inculcating certain procedures, patterns, or skills. Open-ended problems are more true-to-life. Here, students can show whether they are in a position to apply the knowledge they have acquired without assistance from the teacher.

Conditions for Successful Changes in the Approach to Teaching

We must attempt to make schools more efficient by changing teaching from within. This must be done in small steps that can be implemented at relatively low expense. Here we list some of the factors required:

Cooperation among teaching staff

The 'lone wolf' is merely wasting his energy. It makes little or no sense for an individual teacher to attempt to implement thoroughgoing changes to conventional teaching approaches. If students have been accustomed to conventional, lecture-type teaching for years, such an attempt would be doomed to failure because students will be reluctant to relinquish their "conveniently" passive role. Instruction can be changed effectively only by cooperation. A "critical mass" of like-minded teachers consistently working toward independent learning must "band together". This requires teachers to reconsider their own teaching style. A change of role is required. Primarily, the teacher

does not pass on his own knowledge to his students. Instead, he enables them to gain access to knowledge.

More autonomy for schools and teachers

In Germany, classroom instruction and the number and type of performance evaluations are subject to a rather rigid regime of curricula, syllabuses, school and other regulations. In other countries, e.g. Switzerland or The Netherlands, teachers have more latitude to act on their own responsibility. This generates greater motivation, dedication, and professionalism. No one should be allowed to hide behind regulations just because they are too lazy to do otherwise. Let us finally create more latitude! We already have positive approaches.

To prompt students to explore their own learning paths, to establish teaching methods such as opening up tasks, varying tasks, or acting on the principle of I-you-we, teachers need an appropriate teaching environment. This encompasses

- class sizes small enough to enable instructors to devote time to individual students or to take up their individual ideas
- departing from the staccato 45-minute rhythm; working on interesting problems only makes sense if we have ample time to do so.
- Separating learning environments from test situations

Pressure to achieve good grades must not be allowed to prevail in environments where new subject matter is being explored and practised. Students should be told unequivocally that content is at the focus of teaching activities, not the assessment of their scholastic achievement. Learning and testing environments must be separated clearly from each other because they are characterized by very different aims.

Learning situations are characterized by such things as

- learning something new
- closing knowledge gaps
- improving understanding of issues requiring clarification
- discovering connections.

Here, the teacher acts as advisor and helper and fellow students act as partners in learning.

A test or exam situation is characterized by

- avoiding mistakes
- not disclosing lack of knowledge
- projecting a positive image of oneself
- scoring as well as possible.

Here the teacher is the evaluator. Fellow students are mostly competitors.

Testing achievement

Changed forms of instruction cannot be sustainable or successful unless scholastic achievement tests are changed as well. Test questions that can be answered successfully by memorization and schematic responses endanger an approach to instruction geared to understanding and problem-solving skills.

Changing ingrained habits is a long and arduous process that can only be implemented in small steps. One step in the right direction would be to consider the following questions when compiling a class test (a suggestion from a colleague in Baden-Württemberg):

- > Do tasks always have to be presented in the form of arithmetical problems?
- Can we not set problems in a verbal form?
- Can we include explanations and arguments?
- Can we present problems that allow for different solutions?

These are simple and moderate suggestions for change, but their effect is potentially enormous if they are implemented consistently and developed further.

Do We Learn for School or for Life?

There have been a lot of controversial discussions about the knowledge acquired at school. The question "Why do we need this?" betrays over-emphasis on immediate application. General schools have an educational function that goes far beyond the transmission of directly applicable everyday know-how. Of course, such know-how must also have its place in an educational environment, but this cannot be the whole story. School is a fact of life. Therefore we do not learn for school, nor do we learn for life. We learn for life at school.

We usually acquire knowledge within a certain context. This is why we have difficulties in transferring knowledge to other situations and contexts. No wonder that knowledge acquired in school is often not readily available in extracurricular environments. This is described as so-called inert knowledge. To apply such knowledge, we must first adapt it to the individual situation. We must modify and/or expand it. To do this is much easier when the knowledge acquired has already been applied to different contexts. This means that we need both situational and systematic learning. They should complement each other. In spite of the restrictions placed on immediate applicability in school, knowledge acquired by such methods is never useless.

The quality of school learning can hardly be determined by immediate practical application. School learning is always quality learning when it instills a capability for effective and independent learning both in and out of school.

"School" is an abstract term. The concrete factors are "students" and "teachers" (...). Important is not so much what is taught as how it is taught. In my view classical education can be a good preparation for modern life – as long as the teaching is up to scratch!

The BLK pilot studies SINUS and SINUS-Transfer provide a number of suggestions for instruction of the kind called for by Heinrich MANN (1871–1950) in his thoughts about schools quoted above. If teachers and students expect to embark on experiencing mathematics, this initially strange world will rapidly change into a familiar one, centering both on the importance of mathematics for technical, medical, and economic progress and on the formation of innovative and precise thinking.

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