On the Classification of Rigid Three-Dimensional Torus Quotients with Canonical Singularities

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Julia Kotonski

Contents

Abstract

Since the beginning of the 20th century, quotients of complex tori by groups acting freely, also known as generalized hyperelliptic manifolds, have been studied extensively (e.g. [BF08], [ES10], [UY76], [Lan01], [CD20b], [Dem22], [DG22]). In [DG22], the authors observed that actions on complex tori that are rigid and free are only possible in dimension at least 4. However, by allowing isolated canonical singularities, rigid quotients in dimension 3 do also occur. A notable example is Beauville's construction $X_{3,3} = E^3/\langle \zeta_3 \cdot \mathrm{id} \rangle$, where $E = \mathbb{C}/\mathbb{Z}[\zeta_3]$ is Fermat's elliptic curve (cf. [Bea83]). Despite some previous work on this topic (cf. [Ogu96c], [OS01], [BG21]), a complete classification has not yet been established.

This thesis focuses on studying rigid quotients of complex tori with isolated canonical singularities in dimension 3. More precisely, we provide a classification of all finite groups G that admit a holomorphic, faithful, and translation-free action with isolated fixed points on a three-dimensional complex torus T such that the quotient $X = T/G$ is rigid and has canonical singularities. Moreover, we classify the corresponding quotients up to biholomorphism and homeomorphism, and we construct crepant terminalizations and resolutions of the singular quotients that preserve the rigidity, yielding rigid smooth manifolds.

In joint work with Christian Gleissner ([GK22], [GK24]), we show that any admissible group G is isomorphic to one of the following:

$$\mathbb{Z}_3, \quad \mathbb{Z}_7, \quad \mathbb{Z}_9, \quad \mathbb{Z}_{14}, \quad \mathbb{Z}_3^2, \quad \mathbb{Z}_3^3, \quad \mathrm{He}(3) = \mathbb{Z}_3^2 \rtimes \mathbb{Z}_3, \quad \text{or} \quad \mathbb{Z}_9 \rtimes \mathbb{Z}_3.$$

Furthermore, we provide a fine classification of the quotients: They form 21 biholomorphism classes and 15 homeomorphism classes. For each class, we give an explicit description of the torus and the action. Using methods from toric geometry, we construct crepant terminalizations and resolutions of singularities as required above.

For the classification of the groups, we first determine the possible singularities and then apply the orbifold Riemann-Roch formula and methods from group and representation theory. To achieve a fine classification of the quotients, we rely extensively on the observation that the orbifold fundamental group of a torus quotient is a crystallographic group, allowing us to use Bieberbach's structure theorems and their geometric consequences. We adapt the classification framework from [DG22] and [HL21] to the singular case.

During the investigation of possible linear parts of affine linear homeomorphisms between quotients, we encountered a homomorphism from a finite group to the group of semi-projective transformations of a finite-dimensional vector space. Such a map is referred to as a *semi-projective representation*. We study them in the last chapter of the thesis and extend Schur's concept of a representation group for projective representations to the semi-projective case, assuming the

field to be algebraically closed. This work was carried out in collaboration with Massimiliano Alessandro and Christian Gleissner in [AGK23].

Zusammenfassung

Seit Beginn des 20. Jahrhunderts werden Quotienten komplexer Tori nach freien Gruppenwirkungen, welche auch als verallgemeinerte hyperelliptische Mannigfaltigkeiten bekannt sind, ausführlich untersucht (z.B. [BF08], [ES10], [UY76], [Lan01], [CD20b], [Dem22], [DG22]). In [DG22] bewiesen die Autoren, dass Wirkungen auf komplexen Tori, die starr und frei sind, erst ab Dimension 4 möglich sind. Erlaubt man jedoch isolierte kanonische Singularitäten, so treten starre Quotienten bereits in Dimension 3 auf. Ein Beispiel dafür ist Beauvilles Konstruktion $X_{3,3} = E^3/\langle \zeta_3 \cdot \mathrm{id}\rangle$, wobei $E = \mathbb{C}/\mathbb{Z}[\zeta_3]$ Fermats elliptische Kurve bezeichnet (s. [Bea83]). Auch wenn es einige frühere Arbeiten zu diesem Thema gibt (s. [Ogu96c], [OS01], [BG21]), liegt bislang keine vollständige Klassifikation vor.

Diese Arbeit konzentriert sich auf die Untersuchung starrer Quotienten von komplexen Tori mit isolierten kanonischen Singularitäten in Dimension 3. Genauer gesagt klassifizieren wir alle endlichen Gruppen G, die eine holomorphe, treue und translationsfreie Wirkung mit isolierten Fixpunkten auf einem dreidimensionalen komplexen Torus T erlauben mit der Eigenschaft, dass der Quotient $X = T/G$ starr ist und kanonische Singularitäten hat. Darüber hinaus klassifizieren wir die Quotienten bis auf Biholomorphie und Homöomorphie und konstruieren krepante Terminalisierungen sowie Auflösungen der singulären Quotienten, die die Starrheit erhalten und zu starren glatten Mannigfaltigkeiten führen.

In Zusammenarbeit mit Christian Gleissner ([GK22], [GK24]) zeigen wir, dass eine solche Gruppe G isomorph zu einer der folgenden Gruppen ist:

$$\mathbb{Z}_3, \quad \mathbb{Z}_7, \quad \mathbb{Z}_9, \quad \mathbb{Z}_{14}, \quad \mathbb{Z}_3^2, \quad \mathbb{Z}_3^3, \quad \mathrm{He}(3) = \mathbb{Z}_3^2 \rtimes \mathbb{Z}_3 \quad \text{oder} \quad \mathbb{Z}_9 \rtimes \mathbb{Z}_3.$$

Außerdem klassifizieren wir die zugehörigen Quotienten vollständig: Sie bilden 21 Biholomorphie- und 15 Homöomorphieklassen. Für jede Klasse geben wir eine konkrete Beschreibung des Torus und der Wirkung an. Mithilfe von Methoden der torischen Geometrie konstruieren wir krepante Terminalisierungen sowie Auflösungen von Singularitäten mit den gewünschten Eigenschaften.

Für die Klassifikation der Gruppen bestimmen wir zunächst die möglichen Singularitäten und wenden dann die Orbifold-Riemann-Roch-Formel sowie Methoden aus der Gruppen- und Darstellungstheorie an. Um die feine Klassifikation der Quotienten zu erreichen, nutzen wir die Tatsache, dass die Orbifold-Fundamentalgruppe eine kristallographische Gruppe ist, was uns erlaubt, die Struktursätze von Bieberbach und deren geometrische Konsequenzen anzuwenden. Dabei adaptieren wir das in [DG22] und [HL21] entwickelte Konzept zur Klassifikation auf den singulären Fall.

Im Zuge der Bestimmung möglicher Lineararteile von affin-linearen Homöomorphismen zwischen Quotienten stießen wir auf einen Homomorphismus von einer endlichen Gruppe in die Gruppe

semi-projektiver Transformationen eines endlich-dimensionalen Vektorraums. Solche Abbildungen nennt man *semi-projektive Darstellungen*. Diese werden im letzten Kapitel der Arbeit behandelt. Dort erweitern wir Schurs Konzept der Darstellungsgruppe von projektiven Darstellungen auf den semi-projektiven Fall, unter der Voraussetzung, dass der zugrunde liegende Körper algebraisch abgeschlossen ist. Dieser Teil der Arbeit entstand in Zusammenarbeit mit Massimiliano Alessandro und Christian Gleissner (s. [AGK23]).

1. Introduction

A *generalized hyperelliptic manifold* is defined as a quotient of a complex torus by a free action of a non-trivial finite group which does not contain translations. Note that in dimension one, no hyperelliptic manifolds exist, since finite group actions on elliptic curves always have fixed points. The study of torus quotients has been a classic subject of research, dating back to the early 20th century. Bagnera and de Franchis [BF08] as well as Enriques and Severi [ES10] provided a full classification in the case of surfaces. They identified seven families of quotients and established that all occurring groups are cyclic. Their work earned them the prestigious Bordin prize.

Later on, in the 1970s, Uchida and Yoshihara expanded previous work by producing a complete list of all finite groups that give rise to hyperelliptic threefolds [UY76]. This finite list consists of 16 abelian groups and the non-abelian dihedral group $\mathcal{D}_4$ of order 8. The classification of the resulting quotients for the abelian groups was settled by Lange [Lan01] using results of Fujiki [Fuj88], while Catanese and Demleitner [CD20b] completed the classification by addressing the case involving the group $\mathcal{D}_4$. It is worth noting that quotients by the dihedral group were also discussed in [DHS09] and [Joh19] as examples of Ricci-flat Riemannian manifolds with complex structure. Independently of the work just mentioned, Oguiso and Sakurai [OS01] discovered hyperelliptic threefolds with groups $\mathbb{Z}_2^2$ and $\mathcal{D}_4$ in their investigation of Calabi-Yau threefolds that arise as free torus quotients.

Moving to dimension 4, Demleitner provided a complete list of the groups associated to hyperelliptic fourfolds in his PhD-thesis ([Dem20]). This list includes 79 groups, making a full classification of all possible quotients quite extensive. Rather than attempting such an exhaustive classification, Demleitner and Gleissner focused on classifying those quotients that are rigid (cf. [DG22]). Under this assumption, it turned out that the group must be either isomorphic to $\mathbb{Z}_3^2$ or to the Heisenberg group He(3) of order 27. They observed furthermore that *rigid* hyperelliptic manifolds do not exist in dimensions less than 4 (cf. [DG22, Theorem 1.1]).

However, when allowing the quotient to have isolated canonical singularities, rigid examples do occur in dimension 3 (cf. [Bea83], [BG21]). It is important to note that in dimension 2, no such examples exist, as the Kodaira dimension of rigid surfaces is either $-\infty$ or 2 (cf. [BC18]). This naturally gives rise to three key questions:

(1) Is it possible to provide an entire list of all finite groups allowing a holomorphic, faithful, and translation-free action on a three-dimensional torus such that the quotient is rigid and has isolated canonical singularities?

(2) Can one classify the corresponding quotients up to biholomorphism and homeomorphism?

(3) Do there exist resolutions (crepant terminalizations) of the singular quotients that preserve the rigidity?

The primary focus of this thesis is to address these questions in full. In fact, the problems posed in these questions have been solved entirely.

However, before presenting the new results, we review the partial findings that were previously known. If the group action preserves the volume form of the torus, the singularities of the quotients are Gorenstein, and the quotients X admit crepant resolutions $\psi\colon \hat{X} \to X$ resulting in smooth rigid Calabi-Yau three-folds. Oguiso studied smooth Calabi-Yau three-folds Z by analyzing possible contractions $f\colon Z \to W$ and subdividing the pairs (Z, f) into six classes based on certain numerical invariants (cf. [Ogu93], [Ogu96a]). Further details are provided in Section 4.1. The pairs $(\hat{X}, \psi)$ mentioned above are categorized as "fibered Calabi-Yau three-folds of type III_0". In the simply connected case, Oguiso offered a complete classification in [Ogu96c]: The group is cyclic of order 3 or 7, and for each group, there exists one and only one quotient up to biholomorphism. Together with Sakurai, Oguiso also explored the case where the quotients have non-trivial fundamental group. Here, the authors showed that the group is isomorphic to either $\mathbb{Z}_3^2$ or $\mathrm{He}(3)$, and they described the linear part of the actions (cf. [OS01]). However, different choices for the translation part of the action may result in quotients that are not biholomorphic or even not homeomorphic, leaving the finer classification an open problem.

The complementary case where the geometric genus of the quotient is zero, was even less well understood. Bauer and Gleissner investigated specific cases under additional restrictions on the group actions and the torus, using product-quotient techniques (cf. [BG21]).

Now, let us address the questions raised earlier. The first step is to analyze the relevant groups, which leads to one of the key results presented in the joint paper with Christian Gleissner [GK24, Theorem 3.6]. It forms one of the main theorems of the thesis:

Main Theorem 1. *Let G be a finite group acting holomorphically, without translations, and with isolated fixed points on a complex torus T of dimension 3, such that $X = T/G$ is rigid with canonical singularities. Then G is isomorphic to one of the following groups:*

$$\mathbb{Z}_3, \quad \mathbb{Z}_7, \quad \mathbb{Z}_9, \quad \mathbb{Z}_{14}, \quad \mathbb{Z}_3^2, \quad \mathbb{Z}_3^3, \quad \mathrm{He}(3), \quad \textit{or} \quad \mathbb{Z}_9 \rtimes \mathbb{Z}_3.$$

Let us briefly outline the strategy to derive this list. First, we determine the possible orders of automorphisms of three-dimensional complex tori, which we find to be bounded above by 18. We then show that all stabilizer groups are cyclic, calculate their possible orders, and characterize the types of singularities that can arise. With this information, we apply the orbifold Riemann-Roch formula to deduce the possible baskets of singularities. By counting fixed points of elements, analyzing p-Sylow subgroups, and using techniques from representation theory, we ultimately reach the desired classification. Note that the rigidity-condition imposes very strong constraints concerning the representation of the group which describes the linear parts of the action. Some of the proofs are supported by computer-aided calculations using the computer algebra system MAGMA ([BCP97]), specifically drawing on its database of small groups and precomputed character tables.

Note that Birkenhake, González, and Lange previously provided a complete list of all possible finite automorphism groups of three-dimensional complex tori ([BGL99]). In theory, one could follow their list and determine which of these groups (and their subgroups) allow rigid actions with isolated fixed points that lead to canonical singularities. However, due to the length of their list and the restrictive nature of the rigidity condition, we decided to derive the classification from scratch.

Knowing the possible groups, we can proceed to answer the second question, which is to classify the corresponding quotients up to biholomorphism and homeomorphism. The classification is divided into two cases: first, the quotients with $p_g = 1$, and second, those with $p_g = 0$. In both cases, the work was carried out in collaboration with Christian Gleissner, and the results are presented in [GK22, Theorem 1.1] and [GK24, Theorems 1.1 and 1.2], respectively.

Main Theorem 2. *Let G be a finite group admitting a rigid, holomorphic, and translation-free action on a three-dimensional complex torus T with finite fixed locus and such that the quotient $X = T/G$ has canonical singularities.*

(1) If $p_g(X) = 1$, then there are precisely 8 biholomorphism classes of quotients, which are pairwise topologically distinct.

(2) If $p_g(X) = 0$, then there are precisely 13 biholomorphism classes of quotients, which form 11 homeomorphism classes.

In total, the quotients $X = T/G$ form 21 biholomorphism classes and 15 homeomorphism classes.[1] Moreover, the diffeomorphism and homeomorphism classes coincide.

In Table 4.1 and Table 4.2 of Section 4.3, we give explicit descriptions for the tori and the actions for one representative of each biholomorphism class. We want to point out that for the presentation of the classification in this thesis, we decided to distinguish two cases depending on whether the group is cyclic or not instead of treating the cases of the different geometric genera of the quotients separately.

Since holomorphic maps between complex tori are affine linear, any action $\Phi\colon G \hookrightarrow \mathrm{Bihol}(T)$ can be expressed as

$$\Phi(g)(z) = \rho(g) \cdot z + \tau(g),$$

where $\rho(g) \in \mathrm{GL}(3, \mathbb{C})$ maps the lattice of T to itself and represents the linear part and $\tau(g) \in T$ the translation part of $\Phi(g)$. The map $\rho\colon G \to \mathrm{GL}(3, \mathbb{C})$ is a group homomorphism, known as the *analytic representation*, while the map $\tau\colon G \to T$ is not a homomorphism, but a 1-cocycle, hence defines a class in $H^1(G, T)$, where the G-module structure of T is induced by ρ. Conversely, any class in $H^1(G, T)$ together with ρ yields a well-defined action on T, up to conjugation by a translation. Therefore, the general strategy for classification can be outlined as follows:

Strategy for the classification.

(1) For each group G in Theorem 1, determine all possible representations $\rho\colon G \hookrightarrow \mathrm{GL}(3, \mathbb{C})$ fulfilling a “rigidity” and an “integrality” condition, up to equivalence of representations and automorphisms of G.

(2) For each group G and each representation ρ, do the following:

[1] Note that it may happen that quotients with different geometric genus are homeomorphic.

(a) Determine all lattices Λ that have a G-module structure via ρ.

(b) For each $T = \mathbb{C}^3/\Lambda$, determine all cohomology classes in $H^1(G, T)$ that lead to an action with finite fixed locus and fix a representative τ for each class.

(c) Decide which quotients of T by the actions given by ρ and τ are biholomorphic or homeomorphic, respectively.

(3) If there are groups G admitting more than one representation ρ, analyze which biholomorphism and homeomorphism classes coming from different representations coincide.

We now provide more details on point (2c): How can we determine whether two classes in $H^1(G, T)$ yield biholomorphic or homeomorphic quotients? To address this, we adapt the classification framework for free torus quotients, as described in [DG22] and [HL21], to the singular case. More precisely, we assume that the action of G on $T = \mathbb{C}^n/\Lambda$ is free in codimension 1. Each class $[\tau] \in H^1(G, T) \simeq H^2(G, \Lambda)$ corresponds to an extension

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

of G by the lattice Λ. The group Γ has a geometrical interpretation: It represents the group of all lifts of all elements in G to $\mathbb{C}^n$, also known as the *orbifold fundamental group*, and coincides with the fundamental group of the regular locus of $X = T/G$. Two quotients X and X' corresponding to classes $[\tau]$ and $[\tau']$ are homeomorphic if and only if the orbifold fundamental groups Γ and Γ' are isomorphic. An important observation is that these groups embed into the Euclidean group $\mathbb{E}(2n) = \mathbb{R}^{2n} \rtimes \mathrm{O}(2n)$ and that they are discrete and cocompact, hence they are *crystallographic* groups. According to Bieberbach's structure theorems ([Bie11], [Bie12]), isomorphisms between crystallographic groups are given by conjugation with affine linear transformations. By analyzing the implications for the tori and the quotients in greater detail, we obtain an action of a certain group (consisting of the linear parts of potential affine linear homeomorphisms) on $H^1(G, T)$. Two quotients are homeomorphic if and only if they belong to the same orbit under this action. For the biholomorphic classification, we restrict to those affine transformations whose linear parts are $\mathbb{C}$-linear. Once the possible linear parts are determined, the division into the orbits can be carried out with the help of computer algebra systems (we use MAGMA).

While determining the possible linear parts in the case where $G = \mathrm{He}(3)$, we had to determine a "lift" of a homomorphism

$$f\colon \operatorname{Stab}(\chi) \longrightarrow \mathrm{PGL}(n, \mathbb{C}) \rtimes \operatorname{Aut}(\mathbb{C})$$

to $\mathrm{GL}(n, \mathbb{C}) \rtimes \operatorname{Aut}(\mathbb{C})$. This problem led to a broader study of such homomorphisms, namely semi-projective representations, and lifting problems. In collaboration with Massimiliano Alessandro and Christian Gleissner, we explored these topics in depth and presented the results in the article [AGK23]. Chapter 6 of this thesis covers the contents of this paper.

Finally, using methods from toric geometry, we can give a positive answer to the last question (see [GK24, Proposition 6.1]):

Main Theorem 3. *All quotients X in Theorem 2 admit a rigid crepant terminalization and a rigid resolution.*

In addition to this introduction, this thesis comprises five chapters and an appendix. We now roughly explain the content of the individual parts and indicate which results were achieved in joint work.

- In Chapter 2, we introduce the theoretical foundations about complex tori, holomorphic maps between them, and abelian varieties of CM-type, and present the necessary notions and tools from representation theory and group cohomology. Moreover, we discuss crystallographic groups and Bieberbach's structure theorems. Finally, we give an overview of rigid manifolds as well as (canonical) singularities and their properties.
- Chapter 3 covers the classification of all finite groups allowing a rigid, holomorphic, faithful, and translation-free action on a three-dimensional complex torus with isolated fixed points such that the quotient has canonical singularities. Here, we give a proof of Main Theorem 1. The classification was achieved in collaboration with Christian Gleissner (cf. [GK24]).
- In Chapter 4, we deal with the classification of the corresponding quotients up to biholomorphism and homeomorphism. After recalling known results and the connection to Calabi-Yau three-folds and their classification, we explain the classification strategy, and finally provide the fine classification, which proves Main Theorem 2. This chapter is again based on joint work with Christian Gleissner. Section 4.2 and the classification of the quotients with geometric genus $p_g(X) = 1$ is presented in [GK22], the classification in the case $p_g = 0$ in [GK24].
- In Chapter 5, we first introduce some tools from toric geometry and then construct crepant terminalizations and resolutions of singularities that preserve the rigidity. This proves Main Theorem 3, and is part of the afore mentioned paper [GK24].
- Chapter 6 is about the joint work with Massimiliano Alessandro and Christian Gleissner [AGK23], where we study semi-projective representations and extend Schur's concept of a representation group for projective representations to the semi-projective case in the case that the underlying field is algebraically closed.
- In Appendix A, we provide our MAGMA-codes, which we use throughout the thesis.

Notation 1.0.1. In this thesis, all varieties are defined over the field of complex numbers $\mathbb{C}$, and we use standard notation from complex algebraic geometry, see for example [GH78] or [Har77]. The notions concerning complex tori basically follow those in [BL04]. Moreover, we use the following notation and conventions:

- For a complex torus T, $\mathrm{Bihol}(T)$ denotes the group of biholomorphic self-maps of T, whereas $\mathrm{Aut}(T)$ stands for the group of *linear* biholomorphisms of T.
- For an endomorphism α of a finite dimensional vector space, we denote its set of eigenvalues by $\mathrm{Eig}(\alpha)$.
- Let $p\colon X \to Y$ be a covering; then $\mathrm{Deck}(p)$ denotes its group of covering isomorphisms.
- If G acts on a set X, then for $x \in X$, $\mathrm{Stab}(x) = \{g \in G \mid g \cdot x = x\}$ denotes the stabilizer (isotropy) group of x.
- By $\mathbb{Z}_d$, we denote the cyclic group of order d.

- $\mathrm{GL}(n, \mathbb{K})$ and $\mathrm{SL}(n, \mathbb{K})$ are the general linear and special linear groups of $n \times n$ matrices over a field $\mathbb{K}$, respectively, where "special" means that determinants of the matrices are equal to 1.
- $\mathrm{O}(n)$ and $\mathrm{U}(n)$ denote the groups of orthogonal and unitary $n \times n$ matrices, respectively.
- By $\mathrm{AGL}(n, \mathbb{K})$, we denote the group of affine linear transformations of $\mathbb{K}^n$, where $\mathbb{K}$ is any field.
- $\mathcal{D}_n = \langle s, t \mid s^2 = t^n = 1,\ sts = t^{-1} \rangle$ is the dihedral group of order $2n$.
- $\mathcal{S}_n$ and $\mathcal{A}_n$ denote the symmetric and alternating group on the set $\{1, \ldots, n\}$, respectively.
- $\mathrm{He}(3) = \langle g, h, k \mid g^3 = h^3 = k^3 = [g, k] = [h, k] = 1,\ [g, h] = k \rangle$ is the Heisenberg group of order 27.
- Given a positive integer d, we fix the d-th primitive root of unity $\zeta_d := \exp(\frac{2\pi i}{d})$, and Φ_d denotes the d-th cyclotomic polynomial.

The notions of toric geometry used in Chapter 5 are explained in that chapter, and the conventions needed in Chapter 6 are introduced within that part of the thesis.

2. Preliminaries

In this chapter, we establish the theoretical foundations necessary for this thesis. We begin with introducing the main objects of study, namely complex tori, holomorphic maps between them, and important properties of finite automorphisms and their fixed points. Special projective complex tori with many automorphisms, known as "abelian varieties of CM-tpye", are introduced in the subsequent section.
Since holomorphic maps between complex tori are affine linear, group actions on them have a particular shape: They can be described as a combination of a representation and a class in the first cohomology group $H^1(T, G)$. Therefore, it is essential to develop a basic understanding of representation and character theory of finite groups and group cohomology.
Next, we introduce crystallographic groups and explain Bieberbach's structure theorems as their geometrical consequences will be crucial for the classification of torus quotients.
Since the focus of this thesis is on rigid quotients, we will next provide an overview of rigid manifolds and actions. Finally, we will discuss certain notions and properties of singularities, as the quotients involved in our classification will be singular.

Most of the results will be presented without proof. The interested reader can find them in the respective literature.

2.1. Complex tori and biholomorphisms

We start with the introduction and collection of basic notions and properties of complex tori and holomorphic maps between them. For more details, we refer the reader to the textbook [BL04].

Definition 2.1.1. Let V be a finite dimensional complex vector space.

(1) A *lattice* in V is a discrete subgroup of maximal rank in V.

(2) A *complex torus* is a quotient $T = V/\Lambda$ of V by a lattice Λ that acts by addition on V.

(3) A complex torus T is called an *abelian variety* if it is a projective variety, meaning that it admits an embedding into some projective space $\mathbb{P}^N$.

Remark 2.1.2.

(1) A complex torus $T = V/\Lambda$ is a connected compact complex manifold of the same dimension as V. The addition on V induces a group structure on V so that T is a compact complex Lie group.

(2) Unless otherwise stated, we will assume that $V = \mathbb{C}^n$. Note furthermore that Λ is isomorphic to $\mathbb{Z}^{2n}$ as $\mathbb{Z}$-module.

(3) In the following, we will mostly drop the square brackets to denote residue classes modulo Λ and just write $z \in T$.

Proposition 2.1.3. *Let $f\colon T \to T'$ be a holomorphic map that is induced by a linear map from V to V'.*

(1) The image $\mathrm{im}(f)$ *is a subtorus of T'.*

(2) The kernel $\ker(f)$ *is a closed subgroup of T, and the connected component* $(\ker(f))^0$ *containing 0 is a subtorus of T of finite index in* $\ker(f)$*.*

Holomorphic maps between complex tori have a particular simple shape: They are all affine linear.

Proposition 2.1.4. *Let $T = V/\Lambda$ and $T' = V'/\Lambda'$ be two complex tori and $F\colon T \to T'$ a holomorphic map. Then F is induced by an affine linear map*

$$f\colon V \longrightarrow V', \quad z \mapsto Cz + d,$$

where $C \in \mathrm{Hom}_{\mathbb{C}}(V, V')$ with $C(\Lambda) \subset \Lambda'$. The linear map C and the class of d in T' are uniquely determined by F.
We call C the linear part *of F and d (viewed as element in T') the* translation part *of F.*

Note that the linear maps $T \to T'$ are precisely the holomorphic maps that are compatible with the group structures of the tori. Clearly, F is biholomorphic if and only if C is an isomorphism.

Notation 2.1.5. For a complex torus T, we introduce the following notions:

- $\mathrm{Hol}(T) \coloneqq \{f\colon T \to T \mid f \text{ is holomorphic}\}$
- $\mathrm{End}(T) \coloneqq \{f\colon T \to T \mid f \text{ is holomorphic and linear}\}$
- $\mathrm{Bihol}(T) \coloneqq \{f\colon T \to T \mid f \text{ is biholomorphic}\}$
- $\mathrm{Aut}(T) \coloneqq \{f\colon T \to T \mid f \text{ is biholomorphic and linear}\}$

For each of these $\mathbb{Z}$-algebras, adding the index $\mathbb{Q}$ means tensoring the algebra with $\mathbb{Q}$ over $\mathbb{Z}$.

By sending a holomorphic map $T \to T$ to its linear part, we obtain two homomorphisms of $\mathbb{Z}$-algebras

$$\rho\colon \mathrm{Hol}(T) \longrightarrow \mathrm{End}(V) \qquad \text{and} \qquad \rho_\Lambda\colon \mathrm{Hol}(T) \longrightarrow \mathrm{End}(\Lambda),$$

which are called the *analytic* and *rational representation*, respectively. We can extend them both to $\mathrm{Hol}_{\mathbb{Q}}(T) = \mathrm{Hol}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. They are related as follows:

Proposition 2.1.6. *The extended rational representation*

$$\rho_\Lambda \otimes \mathrm{id}_{\mathbb{C}}\colon \mathrm{Hol}_{\mathbb{Q}}(T) \otimes \mathbb{C} \longrightarrow \mathrm{End}_{\mathbb{C}}(\Lambda \otimes \mathbb{C}) \simeq \mathrm{End}_{\mathbb{C}}(V \times V)$$

is equivalent to the direct sum $\rho \oplus \overline{\rho}$.

Corollary 2.1.7. *The characteristic polynomial of* $\rho(f) \oplus \overline{\rho(f)}$ *has integer coefficients for all* $f \in \mathrm{Hol}(T)$.

In this thesis, we are mainly interested in actions of finite groups on complex tori, so all occurring biholomorphisms $f \in \mathrm{Bihol}(T)$ have finite order. Let d be the order of such a biholomorphism f. Then all eigenvalues of $\rho(f)$ are d-th roots of unity. Since the characteristic polynomial of $\rho(f) \oplus \overline{\rho(f)}$ has integer coefficients, this implies that it is a product of cyclotomic polynomials. Analyzing this in more details leads to very strong constraints for the possible orders of the automorphisms, which will be very helpful for the classification of the possible groups. More precisely, we will use the following properties of cyclotomic polynomials and roots of unity:

Remark 2.1.8.

(1) The *Euler totient function* is given by

$$\varphi\colon \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}_{>0}, \quad d \mapsto \#\{m \mid 1 \leq m \leq d-1,\ \gcd(m,d)=1\}$$

and counts the number of units in $\mathbb{Z}_d$. It has the following properties:

(a) It is a multiplicative function in the following sense: If $d_1, d_2 \in \mathbb{Z}_{>0}$ are coprime, then $\varphi(d_1 d_2) = \varphi(d_1)\varphi(d_2)$.

(b) For any prime number p and $k \in \mathbb{Z}_{>0}$, it holds that $\varphi(p^k) = p^{k-1}(p-1)$.

(2) The minimal polynomial of a primitive d-th root of unity over $\mathbb{Q}$ is given by the d-th cyclotomic polynomial

$$\Phi_d := \prod_{\zeta \in \mu_d^*} (X - \zeta) \in \mathbb{Z}[X],$$

where μ_d^* is the set of primitive d-th roots of unity. Its degree equals $\varphi(d)$.

Lemma 2.1.9 ([Dem22], Lemma 3.1.6)**.** *Let* $\alpha \in \mathrm{Aut}(T)$ *be a linear automorphism of order* d. *For each positive divisor* k *of* d*, we define*

$$\nu_k := \max\{\nu \geq 0 \mid \Phi_k^\nu \textit{ divides the characteristic polynomial of } \alpha \oplus \overline{\alpha}\}.$$

Furthermore, we denote by $\mathrm{mult}(\zeta)$ *the (algebraic) multiplicity of a* d*-th root of unity* ζ *as eigenvalue of* α*. Then the following holds:*

(1) The function

$$\mathrm{mult}_k\colon \mu_k^* \longrightarrow \mathbb{Z}, \quad \zeta \mapsto \mathrm{mult}(\zeta) + \mathrm{mult}(\overline{\zeta}),$$

is constant and equal to ν_k *for any positive divisor* k *of* d.

(2) $2\dim(T) = \sum_{k|d} \varphi(k)\nu_k$.

Since rigid actions on three-dimensional complex tori are never free (cf. [DG22, Theorem 1.1]), we have to deal with fixed points. Next, we clarify the terminology and quote a useful lemma for counting the number of fixed points of automorphisms.

Notation 2.1.10. Let T be a complex torus and $f \in \mathrm{Bihol}(T)$ a biholomorphism. Then we denote by $\mathrm{Fix}(f)$ the set of fixed points of f. If $G \subset \mathrm{Bihol}(T)$ is a finite group, then $\mathrm{Fix}(G)$ denotes the set of all points in T with non-trivial stabilizer group and we will refer to it as the *set of fixed points* or the *fixed locus.*

Remark 2.1.11.

(1) A biholomorphism $f(z) = \rho(f)z + d$ of T acts with fixed points if and only if the equation

$$(\rho(f) - \mathrm{id}) \cdot z = -d$$

has a solution in T. In particular, if f acts freely, then 1 is an eigenvalue of $\rho(f)$.

(2) If f acts with fixed points and 1 is an eigenvalue of $\rho(f)$, then its fixed locus has positive dimension. In particular, if f has order d and acts non-freely, then the fixed locus of the group generated by f is finite if and only if all the eigenvalues of $\rho(f)$ are primitive d-th roots of unity.

Lemma 2.1.12 ([BL04], Corollary 13.2.4, Proposition 13.2.5)**.** *Let T be a complex torus of dimension n and $\alpha \in \mathrm{Aut}(T)$ an automorphism of order d whose eigenvalues are all primitive d-th roots of unity. Then the following holds:*

$$\# \mathrm{Fix}(\alpha) = \begin{cases} p^{2n/\varphi(d)}, & \textit{if } d = p^s \textit{ for some prime } p, \\ 1, & \textit{else.} \end{cases}$$

Finally, we discuss the concept of isogenies of complex tori. These induce an equivalence relation on the set of complex tori, which is somewhat coarser than isomorphism, but provides sufficient information in many cases.

Definition 2.1.13. An *isogeny* of two complex tori T and T' is a surjective linear map $T \to T'$ with finite kernel.

Remark 2.1.14. A linear map $f \colon T \to T'$ is an isogeny if and only if it is surjective and the dimensions of T and T' coincides.

Remark 2.1.15. Let $K \subset T$ be a finite subgroup. The quotient T/K is a complex torus, and the quotient map $p \colon T \to T/K$ is an isogeny. Conversely, up to isomorphism, every isogeny is of this type.

Proposition 2.1.16.

(1) Isogenies define an equivalence relation on the set of complex tori.

(2) An element in $\mathrm{End}(T)$ is an isogeny if and only if it is a unit in $\mathrm{End}_{\mathbb{Q}}(T)$.

Definition 2.1.17. For $m \in \mathbb{Z}_{>0}$, the kernel of the map given by multiplication by m is called the *group of m-torsion points* of T and is denoted by $T[m]$.

Proposition 2.1.18. *For any positive integer $m \in \mathbb{Z}_{>0}$, the group of m-torsion points fulfills:*

$$T[m] \simeq (\mathbb{Z}_m)^{2n},$$

where n denotes the dimension of T. In particular, the multiplication map is an isogeny.

Example 2.1.19. Let $E_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be an elliptic curve. Then $E[m]$ is spanned by $\frac{1}{m}$ and $\frac{\tau}{m}$.

2.2. Abelian varieties of CM-type

Given a complex torus T, obviously $\pm \mathrm{id}_X$ are automorphisms of T. It is well-known that there are no further automorphisms for a general torus, so tori with automorphism groups different from $\{\pm \mathrm{id}_T\}$ are quite special.
For example, in the one-dimensional case when $T = E$ is an elliptic curve, the only examples of complex tori that have an automorphism of order at least 3 are Fermat's elliptic curve $E = \mathbb{C}/\mathbb{Z}[\zeta_3]$, whose lattice is given by the ring of Eisenstein integers $\mathbb{Z}[\zeta_3]$, and the harmonic elliptic curve $E = \mathbb{C}/\mathbb{Z}[i]$, where $\mathbb{Z}[i]$ denotes the ring of Gaussian integers (cf. [BL04, Corollary 13.3.4]). Looking closer at these two lattices, we observe that they both are the ring of integers of a number field, namely $\mathbb{Q}(\zeta_3)$ and $\mathbb{Q}(i)$, respectively. This in fact is not a coincidence. Starting with special number fields, one can construct complex tori (in any dimension) that have automorphisms of higher order, so-called "abelian varieties of CM-type" ("CM" stands for "complex multiplication"), which will be important for the classification of the quotients in this thesis.

Definition 2.2.1. Let K be a number field, i.e., a finite extension $\mathbb{Q} \subset K$.

(1) An embedding $\sigma\colon K \hookrightarrow \mathbb{C}$ is called *real* if its image is contained in $\mathbb{R}$, and *complex* otherwise.

(2) The *complex conjugate* of an embedding $\sigma\colon K \hookrightarrow \mathbb{C}$ is defined as

$$\overline{\sigma}\colon K \hookrightarrow \mathbb{C}, \quad a \mapsto \overline{\sigma(a)}.$$

(3) K is called *totally real (complex)* if every embedding of K into $\mathbb{C}$ is real (complex).

(4) K is called a *CM-field* if it is a totally complex quadratic extension of a totally real number field over $\mathbb{Q}$.

Example 2.2.2.

(1) Let $d < 0$ be a square-free integer. Then the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ is a CM-field. The totally real subfield is just the field of rationals.

(2) Any cyclotomic field $\mathbb{Q}(\zeta_n)$ with $n \geq 3$ is a CM-field: It is a totally imaginary quadratic extension of the totally real field $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$.

Since any finite extension of $\mathbb{Q}$ is separable, the total number of embeddings of a number field K into $\mathbb{C}$ equals the degree of the extension $\mathbb{Q} \subset K$.

Definition 2.2.3. Let K be a CM-field of degree $2n$ over $\mathbb{Q}$. A *CM-type of* K is a set of n pairwise different and not complex conjugate embeddings $K \hookrightarrow \mathbb{C}$.

Definition 2.2.4. Let K be a CM-field and $\Phi = \{\sigma_1, \ldots, \sigma_n\}$ a CM-type of K. An abelian variety $T = \mathbb{C}^n/\Lambda$ is said to be of *CM-type* (K, Φ) if there is an embedding $\iota\colon K \hookrightarrow \mathrm{End}_{\mathbb{Q}}(T)$ such that $\rho \circ \iota$ is equivalent to

$$\mathrm{diag}(\sigma_1, \ldots, \sigma_n)\colon K \longrightarrow \mathrm{Mat}(n \times n, \mathbb{C}), \quad a \mapsto \mathrm{diag}(\sigma_1(a), \ldots, \sigma_n(a)),$$

over $\mathbb{C}$, where ρ denotes the analytic representation of T.

Remark 2.2.5. Let T be an abelian variety of CM-type (K, Φ). Then the embedding $\iota\colon K \hookrightarrow \mathrm{End}_{\mathbb{Q}}(T)$ induces an embedding of the ring of integers $\mathcal{O}_K$ of K into the endomorphism algebra $\mathrm{End}(T)$ of T. Thus, we obtain an embedding

$$\mathcal{O}_K^* \longhookrightarrow \mathrm{Aut}(T)$$

of the group of units of $\mathcal{O}_K$ into the automorphism group of T. In particular, if $\mathcal{O}_K$ has more units than ± 1, the abelian variety T has many non-trivial automorphisms. Consider for example the cyclotomic field $K = \mathbb{Q}(\zeta_n)$ with $n \geq 3$.

Next, we explain that for every CM-field K and CM-type Φ, there is in fact an abelian variety of CM-type (K, Φ). For this, let $[K : \mathbb{Q}] = 2n$. Then $K \otimes_{\mathbb{Q}} \mathbb{R}$ is a $2n$-dimensional $\mathbb{R}$-vector space, and the CM-type $\Phi = \{\sigma_1, \ldots, \sigma_n\}$ induces a complex structure on it via

$$(\sigma_1, \ldots, \sigma_n) \otimes \mathrm{id}_{\mathbb{R}}\colon K \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \mathbb{C}^n.$$

Proposition 2.2.6 ([Neu99], Proposition 5.2)**.** *Let $\Phi = \{\sigma_1, \ldots, \sigma_n\}$ be the CM-type of a CM-field K. Then the ring of integers $\mathcal{O}_K$ is a lattice of rank $2n$ in $\mathbb{C}^n$ via the embedding*

$$\mathcal{O}_K \longhookrightarrow \mathbb{C}^n, \quad a \mapsto (\sigma_1(a), \ldots, \sigma_n(a)).$$

Definition 2.2.7. Let K be a CM-field with CM-type Φ. Then we define

$$T(K, \Phi) := {}^{K \otimes_{\mathbb{Q}} \mathbb{R}}/_{\mathcal{O}_K}.$$

If $[K : \mathbb{Q}] = 2n$, then $T(K, \Phi)$ is a complex torus of dimension n.

Proposition 2.2.8 ([BL04], Proposition 13.3.1)**.** *The complex torus $T(K, \Phi)$ is an abelian variety of CM-type (K, Φ).*

Let T be any complex torus and $f \in \mathrm{Aut}(T)$ a non-trivial automorphism of order d. Recall that the set of fixed points of the action of $\langle f \rangle$ on T is finite if and only if all eigenvalues of f are primitive d-th roots of unity. In this case, the characteristic polynomial of $\rho(f) \oplus \overline{\rho(f)}$ equals Φ_d^m, where $\varphi(d) \cdot m = 2n$. In particular,

- $\varphi(d)$ divides $2n$, and
- the set of eigenvalues of $\rho(f)$ contains $\varphi(d)/2$ pairwise not complex conjugate primitive d-th roots of unity.

Theorem 2.2.9 ([BL04], Theorem 13.3.2)**.** *Let T be a complex torus of dimension n and f an automorphism of T of order $d \geq 3$ such that the action of the group $\langle f \rangle$ has finite fixed locus. Assume that the set of eigenvalues Φ_f of $\rho(f)$ has precisely $\varphi(d)/2$ elements. Then Φ_f is a CM-type of $\mathbb{Q}(\zeta_d)$, and there are $k = 2n/\varphi(d)$ abelian varieties $T_1, \ldots, T_k$ of CM-type $(\mathbb{Q}(\zeta_d), \Phi_f)$ such that*

$$T \simeq T_1 \times \ldots \times T_k,$$

and f decomposes into a product of automorphisms of the T_ν of order d.

Corollary 2.2.10. *Let T be a complex torus and $f \in \mathrm{Aut}(T)$ an automorphism of order $d \geq 3$ such that $\varphi(d) = 2 \cdot \dim(T)$ and such that all eigenvalues of $\rho(f)$ are primitive d-th roots of unity. Then the set of eigenvalues Φ_f of $\rho(f)$ is a CM-type of $\mathbb{Q}(\zeta_d)$, and T is an abelian variety of CM-type $(\mathbb{Q}(\zeta_d), \Phi_f)$.*

A natural question is whether the number of isomorphism classes of abelian varieties of a given CM-type is finite, and if so, how to compute it. In fact, the answer is yes, and the number of isomorphism classes can be computed by an invariant of the number field K, which we will introduce next.

Remark 2.2.11. Let K be any number field and $\mathcal{O}_K$ its ring of integers. A *fractional ideal* of K is a finitely generated $\mathcal{O}_K$-submodule of K. Together with the multiplication

$$\mathfrak{a} \cdot \mathfrak{b} := \left\{ \sum_{\text{finite}} a_i b_i \mid a_i \in \mathfrak{a},\ b_i \in \mathfrak{b} \right\},$$

the set of fractional ideals defines an abelian group, the *ideal group* I_K of K. Denote by P_K the subgroup of principal fractional ideals of K. Then the quotient

$$\mathrm{Cl}_K := {}^{I_K}/_{P_K}$$

is called the *ideal class group* of K. This group is finite (cf. [Has63, III., §29, p. 542]), and its order is called the *class number* of K.
Note that the class group measures how "close" the ring $\mathcal{O}_K$ is to be a unique factorization domain. More precisely, the ideal class group is trivial if and only if all ideals of $\mathcal{O}_K$ are principal. Observe that the ring of integers of a number field is a Dedekind domain, which is a unique factorization domain if and only if it is a principal ideal domain.

Theorem 2.2.12 ([ST61], p. 60, Proposition 17)**.** *Let (K, Φ) be a CM-type. Then the number of isomorphism classes of abelian varieties of CM-type (K, Φ) equals the number of ideal-classes of K. In particular, it is finite.*

If K is a cyclotomic field $\mathbb{Q}(\zeta_d)$ and d is "small", then the class number equals often 1, so there is one and only one abelian variety for a given CM-type. More precisely, the following holds:

Theorem 2.2.13 ([Was97], Theorem 11.1)**.** *Let d be a positive integer such that $d \not\equiv 2 \mod 4$. Then $\mathbb{Q}(\zeta_d)$ has class number 1 if and only if d is one in the following list:*

1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 24, 25, 27, 28, 32, 33, 35, 36, 40, 44, 45, 48, 60, 84.

Remark 2.2.14. Note that the numbers $d \equiv 2 \mod 4$ are redundant since $\mathbb{Q}(\zeta_{2n}) = \mathbb{Q}(\zeta_n)$ if n is odd.

2.3. Group actions on complex tori

Let G be a finite group that acts holomorphically on a complex torus T. Write the action as

$$\Phi(g)(z) = \rho(g) \cdot z + \tau(g),$$

where $\rho\colon G \to \mathrm{GL}(n, \mathbb{C})$ is the restriction of the analytic representation of T to G (we will call this the analytic representation of the group action), and $\tau\colon G \to T$ maps g to the translation part of $\Phi(g)$. Since Φ is a group action, the map $g \mapsto \rho(g)$ is a homomorphism, so a linear representation of a finite group. In contrast, the translation part $\tau\colon G \to T$ is not a homomorphism but we have for all $g, h \in G$ the relation

$$\tau(gh) = \rho(g)\tau(h) + \tau(g).$$

Such maps are so-called *1-cocycles* in the theory of group cohomology.

Therefore, we need some basic knowledge about representation theory and group cohomology of finite groups in order to understand actions of finite groups on complex tori better.

2.3.1. Representations and characters of finite groups

In this subsection, V always denotes a finite dimensional $\mathbb{K}$-vector space, $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, and G is a finite group. As a reference, we refer to [Isa76].

Definition 2.3.1. A *representation* of G on V is a homomorphism

$$\rho\colon G \longrightarrow \mathrm{GL}(V).$$

The *character* of ρ is the function

$$\chi_\rho\colon G \longrightarrow \mathbb{K}, \quad g \mapsto \mathrm{tr}(\rho(g)).$$

The *degree* of ρ or χ_ρ is the dimension of V, and we call ρ or χ_ρ *faithful* if ρ is injective.

Remark 2.3.2. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation with character χ_ρ. Since G is finite, $\rho(g)$ is diagonalizable (over $\mathbb{C}$) for all $g \in G$ and one can deduce:

$$g \in \ker(\rho) \iff \chi_\rho(g) = \chi_\rho(1) = \deg(\chi_\rho).$$

Hence, the faithfulness of ρ can be read off its character.

Remark 2.3.3. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. Then the following holds:

(1) The degree of ρ fulfills
$$\deg(\rho) = \deg(\chi_\rho) = \chi_\rho(1).$$

(2) The character $\chi_\rho\colon G \to \mathbb{K}$ is not a homomorphism in general, except if $\deg(\chi_\rho) = 1$.

(3) The character χ_ρ belongs to the $\mathbb{K}$-vector space of class-functions:
$$\mathrm{CF}_{\mathbb{K}}(G) := \{f\colon G \longrightarrow \mathbb{K} \mid f \text{ is constant on the conjugacy classes of } G\}.$$

Example 2.3.4.

(1) The *trivial* representation is given by
$$\rho_{\mathrm{triv}}\colon G \longrightarrow \mathbb{C}^*, \quad g \mapsto 1.$$
We denote its character by χ_{triv}.

(2) Another important representation is the *regular representation*: Let V be the $\mathbb{K}$-vector space of maps from G to $\mathbb{K}$. It has the natural basis $\{e_g \mid g \in G\}$, where e_g denotes the map that sends g to 1 and all other elements of G to 0. The regular representation is defined as
$$\rho_{\mathrm{reg}}\colon G \longrightarrow \mathrm{GL}(V), \quad g \mapsto \left[\sum_{h\in G} \lambda_h e_h \mapsto \sum_{h\in G} \lambda_h e_{gh}\right],$$
and its character, which is called the *regular character* of G, is given by
$$\chi_{\mathrm{reg}}(g) = \begin{cases} |G|, & \text{if } g = 1_G, \\ 0, & \text{else.} \end{cases}$$

(3) Let $\rho_1\colon G \to \mathrm{GL}(V_1)$ and $\rho_2\colon G \to \mathrm{GL}(V_2)$ be two representations. The *direct sum* of ρ_1 and ρ_2 is defined as
$$\rho_1 \oplus \rho_2\colon G \longrightarrow \mathrm{GL}(V_1 \oplus V_2), \quad g \mapsto [v_1 \oplus v_2 \mapsto \rho_1(g)(v_1) \oplus \rho_2(g)(v_2)],$$
and its character is $\chi_{\rho_1\oplus\rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$.

(4) Let $\rho_1\colon G_1 \to \mathrm{GL}(V_1)$ and $\rho_2\colon G_2 \to \mathrm{GL}(V_2)$ be two representations of two finite groups G_1 and G_2. The *tensor product* of ρ_1 and ρ_2 is defined as
$$\rho_1 \otimes \rho_2\colon G_1 \times G_2 \longrightarrow \mathrm{GL}(V_1 \otimes V_2), \quad (g_1, g_2) \mapsto [v_1 \otimes v_2 \mapsto \rho_1(g_1)(v_1) \otimes \rho_2(g_2)(v_2)],$$
and its character is $\chi_{\rho_1\otimes\rho_2} = \chi_{\rho_1} \cdot \chi_{\rho_2}$, where $(\chi_{\rho_1} \cdot \chi_{\rho_2})(g_1, g_2) = \chi_{\rho_1}(g_1) \cdot \chi_{\rho_2}(g_2)$.
In particular, by choosing $G_1 = G_2 =: G$ and identifying G with the diagonal in $G \times G$, it follows that the product of two characters is again a character.

(5) Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. Then we obtain in a natural way representations

$$\mathrm{Sym}^q(\rho)\colon G \longrightarrow \mathrm{GL}(\mathrm{Sym}^q(V)) \qquad \text{and} \qquad \Lambda^q(\rho)\colon G \longrightarrow \mathrm{GL}(\Lambda^q V).$$

For $q = 2$, their characters $\mathrm{Sym}^2(\chi)$ and $\Lambda^2(\chi)$ fulfill:

$$\mathrm{Sym}^2(\chi)(g) = \frac{\chi(g)^2 + \chi(g^2)}{2} \quad \text{and} \quad \Lambda^2(\chi)(g) = \frac{\chi(g)^2 - \chi(g^2)}{2}.$$

In particular, $\chi^2 = \mathrm{Sym}^2(\chi) + \Lambda^2(\chi)$.

(6) The *dual representation* of a representation $\rho\colon G \to \mathrm{GL}(V)$ is defined as

$$\rho^*\colon G \to \mathrm{GL}(V^*), \quad g \mapsto \rho(g^{-1})^*,$$

and has as character $\chi_{\rho*} = \overline{\chi_\rho}$.

Definition 2.3.5. A *homomorphism of representations* $\rho_1\colon G \to \mathrm{GL}(V_1)$ and $\rho_2\colon G \to \mathrm{GL}(V_2)$ is a linear map $f\colon V_1 \to V_2$ such that for all $g \in G$, the diagram

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "f"] & V_1 \arrow[d, "f"] \\ V_2 \arrow[r, "\rho_2(g)"] & V_2 \end{tikzcd}$$

is commutative. We define $\mathrm{Hom}_G(V_1, V_2)$ to be the space of all homomorphisms of the representations ρ_1 and ρ_2. The representations ρ_1 and ρ_2 are *equivalent* if there is an isomorphism of representations between them.

Clearly, equivalent representations have the same character, and a natural question is to ask whether the converse holds. To answer this question, we need the following notions.

Definition 2.3.6. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation.

(1) A linear subspace $W \subset V$ is called *G-invariant* if for all $g \in G$, it holds that $\rho(g)(W) \subset W$.

(2) The representation ρ is called *irreducible* if $\{0\}$ and V are the only G-invariant subspaces of V.

(3) The character χ_ρ is called *irreducible* if ρ is irreducible. We define

$$\mathrm{Irr}(G) \coloneqq \{\text{irreducible characters of } G\}.$$

Remark 2.3.7. Let $\mathbb{K} = \mathbb{C}$, G be abelian, and $\rho\colon G \to \mathrm{GL}(V)$ be a representation. Since G is abelian, the automorphisms $\rho(g)$ are simultaneously diagonalizable for all $g \in G$, and the subspaces generated by the eigenvectors are G-invariant. Hence, ρ is irreducible if and only if $\deg(\rho) = 1$.

Theorem 2.3.8 (Maschke)**.** *Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. Then any G-invariant subspace has a G-invariant complement in V. In particular, ρ can be decomposed into a direct sum of irreducible representations.*

Lemma 2.3.9 (Schur). *Let* $\rho_1 \colon G \to \mathrm{GL}(V_1)$ *and* $\rho_2 \colon G \to \mathrm{GL}(V_2)$ *be two irreducible representations.*

(1) If $f \in \mathrm{Hom}_G(V_1, V_2)$ *and* $f \neq 0$*, then* f *is an isomorphism.*

(2) $\mathrm{End}_G(V) = \mathrm{Hom}_G(V, V)$ *is a skew-field.*

Moreover, if $\mathbb{K} = \mathbb{C}$*, then* $\mathrm{End}_G(V) = \mathbb{C} \cdot \mathrm{id}_V$*, and* $\mathrm{Hom}_G(V_1, V_2)$ *is either trivial or one-dimensional.*

To determine whether two representations are equivalent, we must compute the dimension of the space $\mathrm{Hom}_G(V_1, V_2)$ of homomorphisms between them and establish that this dimension is at least 1. Due to a theorem of Schur, we need to evaluate for this a certain inner product on the space of class-functions $\mathrm{CF}_{\mathbb{K}}(G)$ in the characters of the representations. The inner product is defined as follows:

Definition 2.3.10. The *inner product* on $\mathrm{CF}_{\mathbb{K}}(G)$ is defined as

$$\langle -, - \rangle \colon \mathrm{CF}_{\mathbb{K}}(G) \times \mathrm{CF}_{\mathbb{K}}(G) \longrightarrow \mathbb{K}, \quad (f_1, f_2) \mapsto \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

The inner product on $\mathrm{CF}_{\mathbb{K}}(G)$ is a Hermitian product if $\mathbb{K} = \mathbb{C}$ and an Euclidean product if $\mathbb{K} = \mathbb{R}$.

Proposition 2.3.11. *Let* $\rho \colon G \to \mathrm{GL}(V)$ *be a representation with character* χ_ρ*. Then*

$$\dim(V^G) = \langle \chi_\rho, \chi_{\mathrm{triv}} \rangle,$$

where $V^G = \{v \in V \mid \rho(g)(v) = v \text{ for all } g \in G\}$*.*

Theorem 2.3.12 (Schur).

(1) Let $\rho_1 \colon G \to \mathrm{GL}(V_1)$ *and* $\rho_2 \colon G \to \mathrm{GL}(V_2)$ *be two representations. Then*

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = \dim_{\mathbb{K}}(\mathrm{Hom}_G(V_1, V_2)).$$

In particular, representations with the same character are equivalent.

(2) Let $\mathbb{K} = \mathbb{C}$*, and let* d *be the number of conjugacy classes of* G*. Then* G *has precisely* d *irreducible characters, which form an orthonormal basis of the vector space* $\mathrm{CF}_{\mathbb{C}}(G)$*.*
In particular, a character χ *is irreducible if and only if* $\langle \chi, \chi \rangle = 1$*.*

Remark 2.3.13.

(1) If $\mathbb{K} = \mathbb{R}$, the set $\mathrm{Irr}(G)$ of irreducible characters of G is no longer a basis of the vector space of class functions $\mathrm{CF}_{\mathbb{R}}(G)$ in general. Nevertheless, it forms an orthogonal system. In particular, any character χ can be written as

$$\chi = \sum_{\eta \in \mathrm{Irr}(G)} \frac{\langle \chi, \eta \rangle}{\langle \eta, \eta \rangle} \cdot \eta.$$

(2) Consider the regular representation $\rho_{\mathrm{reg}}\colon G \to \mathrm{GL}(V)$. Then

$$\chi_{\mathrm{reg}} = \sum_{\eta \in \mathrm{Irr}(G)} \frac{\eta(1)}{\langle \eta, \eta \rangle} \cdot \eta.$$

In particular, the regular character contains any irreducible character, and

$$|G| = \chi_{\mathrm{reg}}(1) = \sum_{\eta \in \mathrm{Irr}(G)} \frac{\eta(1)^2}{\langle \eta, \eta \rangle}.$$

If we want to determine all irreducible characters of a finite group, the following properties are useful in addition to the previous remark.

Proposition 2.3.14. *If $\mathbb{K} = \mathbb{C}$ and χ is an irreducible character of G, then* $\deg(\chi) = \chi(1)$ *divides the group order.*

Lemma 2.3.15. *Let $\mathbb{K} = \mathbb{C}$, and let G_1 and G_2 be two finite groups; then*

$$\mathrm{Irr}(G_1 \times G_2) = \{\chi_1 \cdot \chi_2 \mid \chi_j \in \mathrm{Irr}(G_j)\}.$$

Example 2.3.16. Let G be a finite abelian group. We want to compute the irreducible characters of G in the case $\mathbb{K} = \mathbb{C}$. Since any finite abelian group is the product of cyclic groups, it is enough to consider cyclic groups by Lemma 2.3.15. Let $\mathbb{Z}_n$ be the cyclic group of order n with generator g and ζ_n a primitive n-th root of unity. Then the irreducible characters (which all have degree 1 by Remark 2.3.7) are given by $\chi_j(g) \coloneqq \zeta_n^j$.

If $\mathbb{K} = \mathbb{C}$, the irreducible characters of a particular group G are usually presented in a *character table*: This is a square $d \times d$-matrix, where d is the number of conjugacy classes of G, and whose rows correspond to the irreducible characters and the columns to the conjugacy classes. The (i, j)-th entry of the matrix is the value of the i-th character on the j-th conjugacy class.

Example 2.3.17. We want to compute the (complex) character table of the Heisenberg group of order 27:

$$\mathrm{He}(3) = \mathbb{Z}_3^2 \rtimes \mathbb{Z}_3 = \langle g,\ h,\ k \ \mid\ g^3 = h^3 = k^3 = [g,k] = [h,k] = 1,\ [g,h] = k \rangle.$$

It has eleven conjugacy classes, which are represented by

$$1,\ k,\ k^2,\ g,\ g^2,\ h,\ h^2,\ gh,\ g^2h,\ gh^2,\ \text{and}\ g^2h^2.$$

Hence, He(3) admits eleven irreducible complex representations. Nine of them have degree 1 and are obtained by composing the irreducible representations of $\mathbb{Z}_3^2$ with the quotient map

$$\mathrm{He}(3) \longrightarrow \mathrm{He}(3)/\langle k \rangle \simeq \mathbb{Z}_3^2.$$

Furthermore, the Heisenberg group has two irreducible representations of degree 3, namely

$\rho\colon G \to \mathrm{GL}(3,\mathbb{C})$ given by

$$\rho(g) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(h) = \begin{pmatrix} 1 & & \\ & \zeta_3^2 & \\ & & \zeta_3 \end{pmatrix}, \quad \rho(k) = \zeta_3 \cdot \mathrm{id},$$

and its complex conjugate $\overline{\rho}$. They are not equivalent, since their characters take different values on k. Since

$$\sum_{\chi\in\mathrm{Irr}(G)} \chi(1)^2 = |G| = 27 = 9\cdot 1 + 2\cdot 3^2,$$

He(3) has no further irreducible representations (up to equivalence). The character table of the Heisenberg group is displayed in Table 2.1.

	1	k	k^2	g	g^2	h	h^2	gh	g^2h	gh^2	g^2h^2
χ_{triv}	1	1	1	1	1	1	1	1	1	1	1
χ_1	1	1	1	1	1	ζ_3	ζ_3^2	ζ_3	ζ_3	ζ_3^2	ζ_3^2
χ_2	1	1	1	1	1	ζ_3^2	ζ_3	ζ_3^2	ζ_3^2	ζ_3	ζ_3
χ_3	1	1	1	ζ_3	ζ_3^2	1	1	ζ_3	ζ_3^2	ζ_3	ζ_3^2
χ_4	1	1	1	ζ_3	ζ_3^2	ζ_3	ζ_3^2	ζ_3^2	1	1	ζ_3
χ_5	1	1	1	ζ_3	ζ_3^2	ζ_3^2	ζ_3	1	ζ_3	ζ_3^2	1
χ_6	1	1	1	ζ_3^2	ζ_3	1	1	ζ_3^2	ζ_3	ζ_3^2	ζ_3
χ_7	1	1	1	ζ_3^2	ζ_3	ζ_3	ζ_3^2	1	ζ_3^2	ζ_3	1
χ_8	1	1	1	ζ_3^2	ζ_3	ζ_3^2	ζ_3	ζ_3	1	1	ζ_3^2
χ_9	3	$3\zeta_3$	$3\zeta_3^2$	0	0	0	0	0	0	0	0
χ_{10}	3	$3\zeta_3^2$	$3\zeta_3$	0	0	0	0	0	0	0	0

Table 2.1.: Character table of He(3).

Above, we gave a criterion for complex characters to be irreducible: A complex character χ is irreducible if and only if $\langle\chi,\chi\rangle = 1$. For characters of real representations, this is not true in general, but one can adapt the condition to obtain a criterion for real characters as well.

Definition 2.3.18. Let χ be a (complex or real) character of G. Then

$$\nu(\chi) \coloneqq \frac{1}{|G|}\sum_{g\in G}\chi(g^2)$$

is called the *Frobenius-Schur indicator* of χ.

The next proposition is well-known but we could not find a suitable reference, so we give a proof here.

Proposition 2.3.19. *Let $\mathbb{K} = \mathbb{R}$ and $\rho\colon G \to \mathrm{GL}(V)$ be a representation with character χ_ρ. Then ρ is irreducible if and only if $\langle\chi_\rho,\chi_\rho\rangle + \nu(\chi_\rho) = 2$.*

Proof. By Example 2.3.4, it holds that

$$\langle\chi_\rho,\chi_\rho\rangle + \nu(\chi_\rho) = 2\cdot\langle\mathrm{Sym}^2(\chi_\rho),\chi_{\mathrm{triv}}\rangle = 2\cdot\dim(\mathrm{Sym}^2(V))^G.$$

Thus, we have to show that ρ is irreducible if and only if $\dim(\mathrm{Sym}^2(V))^G = 1$, which means that, up to a scalar, there is one and only one non-zero G-invariant symmetric bilinear form or equivalently, one non-zero G-invariant quadratic form.
Assume first that ρ is reducible and choose a non-trivial, G-invariant decomposition $V = U_1 \oplus U_2$ and positive definite quadratic forms $Q_i \colon U_i \to \mathbb{R}$. We can assume without loss of generality that Q_1 and Q_2 are G-invariant since otherwise, we can replace them by

$$\widetilde{Q}_i \colon U_i \longrightarrow \mathbb{R}, \quad u \mapsto \frac{1}{|G|} \sum_{g \in G} (Q_i \circ \rho(g))(u).$$

Clearly, the compositions of the Q_i with the projection maps $p_i \colon V \to U_i$ define linearly independent G-invariant quadratic forms on V.
Conversely, let $Q_1, Q_2 \colon V \to \mathbb{R}$ be two linearly independent G-invariant quadratic forms, and assume without loss of generality that Q_1 is positive definite. Choose a basis of V such that the matrix of Q_1 is the identity. Let S be the matrix of Q_2 with respect to this basis, and let λ be an eigenvalue of S, which is real since S is symmetric. Then the symmetric bilinear form B corresponding to $\lambda Q_1 - Q_2 \neq 0$ is G-invariant and degenerated. Hence,

$$\{v \in V \mid B(v,u) = 0 \quad \text{for all} \quad u \in V\}$$

is a non-trivial G-invariant subspace of V, so ρ is reducible. $\square$

Example 2.3.20. Consider the group $G = \mathbb{Z}_3^2$. Let B be the rotation matrix by the angle $\alpha = 120°$:

$$B = -\frac{1}{2} \cdot \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

We claim that the non-trivial irreducible real representations of $\mathbb{Z}_3^2$ (up to equivalence) are given by

$$\rho_1(a,b) = B^a, \quad \rho_2(a,b) = B^b, \quad \rho_3(a,b) = B^{a+b}, \quad \rho_4(a,b) = B^{2a+b}.$$

Since B is not diagonalizable over $\mathbb{R}$, the representations $\rho_1, \dots, \rho_4$ are irreducible (alternatively, one can prove that $\langle \chi_{\rho_j}, \chi_{\rho_j} \rangle + \nu(\chi_{\rho_j}) = 2$). Furthermore, the representations are pairwise not equivalent because they have different characters. Finally, since

$$1 + \sum_{j=1}^{4} \frac{\chi_{\rho_j}(1)^2}{\langle \chi_{\rho_j}, \chi_{\rho_j} \rangle} = 1 + 4 \cdot \frac{4}{2} = 9 = |G|,$$

there are no further irreducible representations.

Let $\rho \colon G \to \mathrm{GL}(V)$ be a real irreducible representation with character χ. By Schur's Lemma, the endomorphism algebra of ρ is a skew field, whose dimension as $\mathbb{R}$-vector space equals $\langle \chi, \chi \rangle$ and is in particular finite. Thus, by Frobenius' Theorem [Fro78], the endomorphism algebra of ρ is either $\mathbb{R}$, $\mathbb{C}$, or the quaternions $\mathbb{H}$, and we call χ of *real*, *complex* or *quaternionic type*, respectively. Since χ is irreducible, it holds that $\nu(\chi) = 2 - \dim_{\mathbb{R}}(\mathrm{End}_G(V))$, thus the type can be directly read off the Frobenius-Schur indicator:

- $\nu(\chi) = 1$: real type
- $\nu(\chi) = 0$: complex type
- $\nu(\chi) = -2$: quaternionic type.

Given a complex irreducible representation $\rho\colon G \to \mathrm{GL}(V)$ with character χ, we can consider the decomplexification $\rho_{\mathbb{R}}\colon G \to \mathrm{GL}(V_{\mathbb{R}})$, whose character $\chi_{\mathbb{R}}$ fulfills

$$\chi_{\mathbb{R}} \otimes \mathrm{id}_{\mathbb{C}} = \chi + \overline{\chi},$$

and ask whether $\chi_{\mathbb{R}}$ is irreducible as real character and if so, how the type of $\chi_{\mathbb{R}}$ is related to properties of χ. The main ingredient for the answer is the following theorem:

Theorem 2.3.21 (Frobenius-Schur, [Isa76], Theorem 4.5)**.** *Let χ be a complex irreducible character. Then $\nu(\chi) \in \{-1, 0, 1\}$, and $\nu(\chi) \neq 0$ if and only if χ takes only real values. More precisely, the following holds:*

(1) $\nu(\chi) = 0$ if and only if χ is not real.

(2) $\nu(\chi) = 1$ if and only if χ can be realized as the character of a real representation.

(3) $\nu(\chi) = -1$ if and only if χ takes only real values but cannot be realized as the character of a real representation.

Corollary 2.3.22. *Let $\rho\colon G \to \mathrm{GL}(V)$ be a complex irreducible representation with character χ, and denote by $\rho_{\mathbb{R}}$ its decomplexification with character $\chi_{\mathbb{R}}$. Then $\rho_{\mathbb{R}}$ is an irreducible real representation if and only if χ cannot be realized as the character of a real representation, and the following holds:*

(1) χ is not real if and only if $\chi_{\mathbb{R}}$ is of complex type.

(2) χ takes only real values but is not realizable as the character of a real representation if and only if $\chi_{\mathbb{R}}$ is of quaternionic type.

2.3.2. Group cohomology and group extensions

In the textbook [Bro82], group cohomology is introduced in an abstract, homological way. We will use instead a more “ad-hoc” approach like in [Cha86, Section I.5] or [Szc12, Section 2.2.1], which works for finite groups and gives a very concrete description of the objects and maps so that computations can be done explicitly.

Definition 2.3.23. Let G be a group. A G*-module* is an abelian group M together with a G-action on M such that $g \cdot (m + m') = g \cdot m + g \cdot m'$ for any $g \in G$ and $m, m' \in M$.

Definition 2.3.24. Let G be a finite group, M a G-module and $n \in \mathbb{Z}$. The set

$$C^n(G, M) := \begin{cases} \{f\colon G^n \to M\}, & \text{if } n \geq 1, \\ M, & \text{if } n = 0, \\ 0, & \text{else,} \end{cases}$$

together with the addition of functions is called the *group of n-cochains*.
The *coboundary-operator*

$$\partial^n \colon C^n(G, M) \longrightarrow C^{n+1}(G, M)$$

is defined as follows: if $n \geq 1$, then

$$(\partial^n f)(g_0, \ldots, g_n) \coloneqq g_0 \cdot f(g_1, \ldots, g_n) + \sum_{j=1}^{n} (-1)^j f(g_0, \ldots, g_{j-2}, g_{j-1}g_j, g_{j+1}, \ldots, g_n) + \\ + (-1)^{n+1} f(g_0, \ldots, g_{n-1}),$$

if $n = 0$, then

$$(\partial^0 m)(g_0) \coloneqq g_0 \cdot m - m,$$

and $\partial^n \coloneqq 0$ for $n < 0$.

Remark 2.3.25. The coboundary-operator is a group homomorphism and fulfills $\partial^{n+1} \circ \partial^n = 0$.

Definition 2.3.26. The subgroups $Z^n(G, M) \coloneqq \ker(\partial^n)$ and $B^n(G, M) \coloneqq \operatorname{im}(\partial^{n-1})$ are called the group of *n-cocycles* and *n-coboundaries*, respectively. The quotient

$$H^n(G, M) \coloneqq {Z^n(G, M)}/{B^n(G, M)}$$

is the *n-th cohomology group* of G with values in M.

Example 2.3.27. Let M be a G-module.

(1) Since $B^0(G, M) = \{0\}$, it holds that

$$H^0(G, M) = Z^0(G, M) = \ker(\partial^0) = M^G,$$

so the 0-th cohomology group coincides with the G-invariant part of M.

(2) Let G act trivially on M; then $B^1(G, M) = \{0\}$ and hence,

$$\begin{aligned} H^1(G, M) &= Z^1(G, M) \\ &= \{f \colon G \to M \mid 0 = (\partial^1 f)(g_0, g_1) = f(g_1) - f(g_0 g_1) + f(g_0) \text{ for all } g_0, g_1 \in G\} \\ &= \operatorname{Hom}(G, M). \end{aligned}$$

The next proposition shows that the order of each element in $H^n(G, M)$ divides the group order of G.

Proposition 2.3.28. *Let G be a finite group, $|G| = k$, M a G-module, and $n \geq 1$. Then*

$$k \cdot H^n(G, M) = 0.$$

In particular, if multiplication by k is an isomorphism of M, then $H^n(G, M) = 0$.

If M is a finitely generated G-module, then the higher cohomology groups $H^n(G, M)$ are finitely generated as well. Since the elements in the cohomology group are all torsion by the previous proposition, we obtain:

Corollary 2.3.29. *Let G be a finite group and M a finitely generated G-module. Then $H^n(G, M)$ is finite for any $n \geq 1$.*

A short exact sequence $0 \to M \to N \to P \to 0$ of G-modules yields an induced exact sequence:

$$0 \longrightarrow M^G \longrightarrow N^G \longrightarrow P^G.$$

In general, the last map is not surjective, but as usual in cohomology theory, one can construct a long exact cohomology sequence:

Proposition 2.3.30. *Let $0 \to M \xrightarrow{i} N \xrightarrow{\pi} P \to 0$ be an exact sequence of G-modules. Then the maps i and π induce homomorphisms $H^n(i)$ and $H^n(\pi)$ between the cohomology groups, and there exist natural maps $\sigma^n \colon H^n(G, P) \to H^{n+1}(G, N)$ for any n such that the sequence*

$$0 \longrightarrow M^G \xrightarrow{i} N^G \xrightarrow{\pi} P^G \xrightarrow{\sigma^0} H^1(G, M) \xrightarrow{H^1(i)} H^1(G, N) \xrightarrow{H^1(\pi)} H^1(G, P) \longrightarrow$$
$$\xrightarrow{\sigma^1} H^2(G, M) \longrightarrow \dots$$

is exact.

Remark 2.3.31. We do not prove the previous proposition but we sketch the construction of the connecting homomorphism σ^1, since it will be relevant later on.
For this, we view M as submodule of N via i. Let $\tau \colon G \to P$ be a 1-cocycle. For each $g \in G$, choose a preimage $n_g \in N$ of $\tau(g)$ under π, and set $\beta(g) := n_g$. This yields a 1-cochain $\beta \colon G \to N$ fulfilling $\pi \circ \partial^1_N(\beta) = 0$, where ∂^1_N is the first coboundary operator of the G-module N. Hence, $\partial^1_N(\beta)$ takes only values in M, and we can set

$$\sigma^1([\tau]) := [\partial^1_N(\beta)] \in H^2(G, M).$$

Example 2.3.32. Let $T = \mathbb{C}^n/\Lambda$ be a complex torus and G a finite group together with a representation $\rho \colon G \to \mathrm{GL}(n, \mathbb{C})$ such that for all $g \in G$, the matrix $\rho(g)$ maps the lattice Λ to itself. Then

$$0 \longrightarrow \Lambda \longrightarrow \mathbb{C}^n \longrightarrow T \longrightarrow 0$$

is a short exact sequence of G-modules (via ρ). By Proposition 2.3.28, the higher cohomology groups with values in $\mathbb{C}^n$ vanish. Hence, the connecting homomorphisms of the long exact cohomology sequence define isomorphisms

$$H^n(G, T) \xrightarrow{\simeq} H^{n+1}(G, \Lambda), \qquad \text{for} \qquad n \geq 1.$$

Since $\Lambda \simeq \mathbb{Z}^{2n}$ is finitely generated, the higher cohomology groups $H^n(G, \Lambda)$ are finite by Corollary 2.3.29. In particular, the group $H^1(G, T)$ is finite.

The translation part of an action on T with analytic representation ρ defines a class in $H^1(G,T)$, and conversely, every cohomology class yields an action on T, which is well-defined up to conjugation with a translation. Thus, we can conclude that for a given linear representation ρ, there are only finitely many corresponding actions on T, up to conjugation with a translation.

Moreover, analyzing the actions of G on T with fixed analytic representation is the same as studying $H^1(G,T)$, which is isomorphic to $H^2(G,\Lambda)$. This second cohomology group parametrizes the group extensions of G by Λ. The interplay between the second cohomology group and group extensions will be explained next, as it will be important for the classification of the quotients in this thesis on the one hand, and for the study of so-called "semi-projective representations" in Chapter 6 on the other hand.

Definition 2.3.33. Let A and G be two groups where A is abelian. A *group extension* of G by A is a short exact sequence of groups

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1.$$

Identifying A with its image $i(A)$, we will view A as subgroup of E. Therefore, we will interpret A as multiplicative group, although it is abelian, and write a "1" at the beginning of the sequence.

Definition 2.3.34. Two extensions $1 \to A \to E \to G \to 1$ and $1 \to A \to E' \to G \to 1$ of G by A are said to be *equivalent* if there exists a group homomorphism $F\colon E \to E'$ such that the diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"] & E \arrow[r] \arrow[d, "F"] & G \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & A \arrow[r] & E' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

is commutative.

Remark 2.3.35. The homomorphism F is in fact an isomorphism, so, equivalent extensions induce isomorphic groups. Note that the converse is not true in general.

In order to obtain a bijection between group extensions $1 \to A \to E \to G \to 1$ and classes in $H^2(G,A)$, we have to introduce a G-module structure on A: Choose any set-theoretic section with $s(1) = 1$ and set

$$g * a := s(g) \cdot a \cdot s(g)^{-1}, \qquad \text{for} \qquad g \in G,\ a \in A.$$

Since A is abelian, this action is independent of the choice of the section. Moreover, equivalent extensions induce the same action since the corresponding isomorphism between the extensions is the identity on A.

Theorem 2.3.36 ([Bro82], Chapter IV, Theorem 3.12)**.** *Let G be a finite group and A a G-module. Then there is a bijection between the set of equivalence classes of those group extensions*

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1$$

of G by A that induce the fixed G-module structure on A and $H^2(G,A)$.

Since we will need it later on, let us explain the construction of the bijection:

(1) Let $1 \to A \to E \to G \to 1$ be an extension. Let $s\colon G \to E$ be a set-theoretic section with $s(1) = 1$. In general, s is not a homomorphism but for $g, h \in G$, there exists $\beta(g, h) \in A$ such that
$$\beta(g, h) \cdot s(gh) = s(g) \cdot s(h).$$
In this way, we obtain a map $\beta\colon G \times G \to A$, which is in fact a 2-cocycle since it fulfills
$$(g * \beta(h, k)) \cdot \beta(gh, k)^{-1} \cdot \beta(g, hk) \cdot \beta(g, h)^{-1} = 1$$
for all $g, h, k \in G$, and hence defines a class $[\beta] \in H^2(G, A)$.

(2) Conversely, let $[\beta] \in H^2(G, A)$, and assume without loss of generality that β is *normalized*, which means that $\beta(g, 1) = \beta(1, g) = 1$ for all $g \in G$. Then the set $A \times G$ together with the multiplication
$$(a_1, g_1) \cdot (a_2, g_2) := (a_1 \cdot (g_1 * a_2) \cdot \beta(g_1, g_2), g_1g_2)$$
is a group with neutral element $(1, 1)$. We will denote this by $\Gamma := A \times_\beta G$. The obvious sequence
$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$
is exact with corresponding cohomology class $[\beta] \in H^2(G, A)$.

2.4. Crystallographic groups and Bieberbach's structure theorems

In this section, we collect the tools from crystallographic group theory that we will need to derive the classification of the quotients in Chapter 4. Crystallographic groups are subgroups of the Euclidean group $\mathbb{E}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$ with special properties. In the course of torus quotients, they appear as so-called "orbifold-fundamental groups", which we introduce first. Our references for the theory of crystallographic groups are the textbooks [Cha86] and [Szc12].

Definition 2.4.1. Let $T = \mathbb{C}^n/\Lambda$ be a complex torus and G a finite group of biholomorphisms acting on T without translations. Let $\pi\colon \mathbb{C}^n \to T$ be the universal cover. Then we define the *orbifold fundamental group* as
$$\pi_1^{\mathrm{orb}}(T, G) := \{\gamma\colon \mathbb{C}^n \to \mathbb{C}^n \mid \exists\, g \in G \text{ s.t. } \pi \circ \gamma = g \circ \pi\}.$$

As a reference for the orbifold fundamental group, we refer to [Cat15, Section 6.1].

Remark 2.4.2.

(1) The orbifold-fundamental group $\Gamma := \pi_1^{\mathrm{orb}}(T, G)$ fits into the exact sequence
$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

since $\Lambda = \pi_1(T) = \mathrm{Deck}(\pi)$. Hence, $G \simeq \Gamma/\Lambda$.

(2) Let $p\colon T \to T/G = X$ be the quotient map. Then $\pi \circ p\colon \mathbb{C}^n \to X$ is Galois with group Γ. In particular,

$$X = {}^{T}\!/_{G} = {}^{\mathbb{C}^n}\!/_{\Gamma}.$$

(3) If the action of G on T is free, then $\pi \circ p$ is the universal cover of X, hence $\pi_1(X) = \Gamma$. If the action is not free, this is no longer true; but if the action is at least free in codimension 1, it holds that

$$\Gamma = \pi_1^{\mathrm{orb}}(T, G) = \pi_1(X^\circ),$$

where $X^\circ := X \setminus \mathrm{Sing}(X)$ is the regular locus of X. The reason behind this is the following: It holds that $\mathrm{Sing}(X) = p(F)$, where

$$F := \{x \in T \mid \exists\, g \in G \setminus \{1\} \text{ s.t. } g(x) = x\}$$

is the set of points with non-trivial stabilizer group. By assumption, the analytic set F has codimension at least two. Since the restriction $p\colon T \setminus F \to X \setminus \mathrm{Sing}(X)$ is finite and unramified, the composition

$$\mathbb{C}^n \setminus \pi^{-1}(F) \to T \setminus F \to X \setminus \mathrm{Sing}(X)$$

is an unramified cover. It is universal because $\mathbb{C}^n \setminus \pi^{-1}(F)$ is simply connected (cf. [Pri67, p. 378]). Finally, the Galois group of this cover, which is the fundamental group of the regular locus $X^\circ = X \setminus \mathrm{Sing}(X)$, equals the orbifold fundamental group $\Gamma = \pi_1^{\mathrm{orb}}(T, G)$.

(4) Since G is finite, the group Γ acts properly discontinuously on $\mathbb{C}^n$, and hence, it is discrete, meaning that the orbits are discrete. By (2), it is furthermore cocompact because the quotient space $\mathbb{C}^n/\Gamma = T/G$ is compact. Moreover, since G is finite, we may assume that the analytic representation ρ is unitary. Via the identification

$$\mathbb{C}^n \to \mathbb{R}^{2n}, \quad (z_1, \dots, z_n) \mapsto (x_1, y_1, \dots, x_n, y_n), \qquad \text{where} \quad z_j = x_j + \sqrt{-1}y_j,$$

we can view ρ as a real representation $\rho_{\mathbb{R}}\colon G \to \mathrm{O}(2n)$ and consider $\Gamma = \pi_1^{\mathrm{orb}}(T, G)$ as a subgroup of the Euclidean group $\mathbb{E}(2n) = \mathbb{R}^{2n} \rtimes \mathrm{O}(2n)$.

Definition 2.4.3. A discrete cocompact subgroup Γ of the Euclidean group $\mathbb{E}(n)$ is called a *crystallographic group*. If Γ acts furthermore freely on $\mathbb{R}^n$, then it is called a *Bieberbach group*.

Such groups are called "crystallographic" since they are symmetry groups of repeating patterns like in crystals. Note that the above remark guarantees that the orbifold fundamental group $\pi_1^{\mathrm{orb}}(T, G)$ is crystallographic.

Remark 2.4.4. Let $\Gamma \subset \mathbb{E}(n)$ be a crystallographic group. Then the discreteness of the group ensures that the action of Γ on $\mathbb{R}^n$ is properly discontinuous in the following sense: For all $x \in \mathbb{R}^n$, there exists a neighborhood $U = U(x)$ such that the following set is finite:

$$\{\gamma \in \Gamma \mid \gamma U \cap U \neq \emptyset\}.$$

Moreover, the quotient space $\mathbb{R}^n/\Gamma$ is a Hausdorff space.
Note furthermore that the action of Γ is free, that is Γ is a Bieberbach group, if and only if it does not contain any torsion elements different from the identity.

Example 2.4.5. Clearly, each lattice Λ in $\mathbb{R}^n$ defines a Bieberbach group whose quotient $\mathbb{R}^n/\Lambda$ is a real torus. Another example is given as follows: Let $\Gamma \subset \mathbb{E}(2)$ be the subgroup generated by the elements

$$\gamma_1(x) \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot x + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \quad \text{and} \quad \gamma_2(x) \coloneqq x + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.$$

Then Γ is a Bieberbach group, and the quotient $\mathbb{R}^2/\Gamma$ is homeomorphic to the Klein bottle.

The fist part of Hilbert's 18th problem ([Hil02]) asks whether there are only finitely many crystallographic groups in each dimension up to isomorphism. Bieberbach gave a positive answer to this question in 1911 and furthermore described the structure of crystallographic groups and isomorphisms between them. In Chapter 4, we will use geometrical consequences of his theorems in order to classify the quotients up to biholomorphism and homeomorphism. Because the statements are so central for the classification in this thesis, we state them now and give a proof, at least in outline form. The given proof is based on the argumentation in [Szc12].

Theorem 2.4.6 ([Bie11], [Bie12])**.**

(1) The translation subgroup $\Lambda \coloneqq \Gamma \cap \mathbb{R}^n$ of a crystallographic group $\Gamma \subset \mathbb{E}(n)$ is a lattice of rank n, and Γ/Λ is finite. All other normal abelian subgroups of Γ are contained in Λ.

(2) Let $\Gamma_1, \Gamma_2 \subset \mathbb{E}(n)$ be two crystallographic groups and $f \colon \Gamma_1 \to \Gamma_2$ be an isomorphism. Then there exists an affine transformation $\alpha \in \mathrm{AGL}(n, \mathbb{R}) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$ such that $f(g) = \alpha \circ g \circ \alpha^{-1}$ for all $g \in \Gamma_1$.

(3) In each dimension, there are only finitely many isomorphism classes of crystallographic groups.

Proof. (1) Let $\Gamma \subset \mathbb{E}(n)$ be a crystallographic group. Since in the setup of this thesis, the translation subgroup Λ of Γ coincides with the lattice of the torus because the action on T is translation-free, and $\Gamma/\Lambda = G$ is finite, we do not give a prove for the first statement. But we explain why the translation subgroup is the unique maximal normal abelian subgroup of Γ. For this, let $N \trianglelefteq \Gamma$ be an abelian normal subgroup and $\gamma(x) = Ax + b$ an element in N. We set $U \coloneqq \ker(A - \mathrm{id})$ and show that $U = \mathbb{R}^n$.
Let $v \in \Lambda \subset \mathbb{R}^n$ be an element in the lattice, and identify it with the translation $t_v(x) = x+v \in \Gamma$. By the normality of N, the element

$$\hat{\gamma}(x) \coloneqq (t_v \circ \gamma \circ t_v^{-1})(x) = Ax - Av + b + v$$

belongs to N as well, and since N is abelian, it commutes with γ. This implies that v belongs to the kernel of $(A - \mathrm{id})^2$, so $(A - \mathrm{id})v$ belongs to U. Now, write $v = u + w$ with $u \in U$ and $w \in U^\perp$. Then the equation $(A - \mathrm{id})v = (A - \mathrm{id})w$ is true, and moreover, since $A \in \mathrm{O}(n)$, it holds for every $u' \in U$ that

$$\langle (A - \mathrm{id})w, u' \rangle = \langle Aw, u' \rangle = \langle Aw, Au' \rangle = \langle w, u' \rangle = 0.$$

Hence, $(A - \mathrm{id})v$ is an element in $U^\perp \cap U = \{0\}$, which implies that $Av = v$ for all $v \in \Lambda$. Since Λ spans the whole $\mathbb{R}^n$, the claim follows.

(2) For this part of the proof, we write an element $\gamma(x) = Bx + d \in \mathbb{E}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$ as tuple (B, d) and denote by I the identity matrix. Let $f\colon \Gamma_1 \to \Gamma_2$ be an isomorphism between two crystallographic groups. By (1), the restriction $f_{|\Lambda_1}$ is given by multiplication with a matrix $A \in \mathrm{GL}(n, \mathbb{R})$, and f induces an isomorphism of the quotients

$$f_1\colon \Gamma_1/\Lambda_1 \longrightarrow \Gamma_2/\Lambda_2.$$

By identifying the quotients with the images of Γ_i under the projection $p_1\colon \mathbb{E}(n) \to \mathrm{O}(n)$, we can write f as

$$f(B, d) = (f_1(B), f_2(B, d)).$$

First, we show that f_1 is given by conjugation with A. Let $(I, e) \in \Lambda$. Applying f to the equation

$$(B, d) \cdot (I, e) \cdot (B, d)^{-1} = (I, Be),$$

yields

$$(f_1(B), f_2(B, d)) \cdot (I, Ae) \cdot (f_1(B^{-1}), f_2(B^{-1}, -B^{-1}d)) = (I, ABe),$$

which implies that $f_1(B) = ABA^{-1}$. Next, we consider the isomorphism

$$F\colon \Gamma_1 \longrightarrow F(\Gamma_1), \quad \gamma \mapsto (A, 0) \cdot \gamma \cdot (A, 0)^{-1},$$

and define $G := f \circ F^{-1}\colon F(\Gamma_1) \to \Gamma_2$. Since $f_1(B) = ABA^{-1}$, the isomorphism G is of the form

$$G(B, d) = (B, G_2(B, d)).$$

We claim that G is given by conjugation with a translation $(I, b) \in \mathbb{E}(n)$. Assume for the moment that this is proven, then

$$f(\gamma) = (G \circ F)(\gamma) = (A, b) \cdot \gamma \cdot (A, b)^{-1}.$$

The isomorphism G is given by conjugation with (I, b) if and only if $G_2(B, d) = (I - B)b + d$ holds for all (B, d) in $F(\Gamma_1)$. Equivalently, all $(B, d) \in F(\Gamma_1)$ satisfy the equation

$$(I, b) \cdot (B, d - G_2(B, d)) \cdot (I, -b) = (B, 0). \tag{2.4.1}$$

Thus, we have to show the existence of an element $b \in \mathbb{R}^n$ with this property. By construction, G is the identity on $F(\Gamma_1) \cap \mathbb{R}^n$. Consider the subgroup

$$H := \{(B, e) \in \mathbb{E}(n) \mid \exists\ (B, d) \in F(\Gamma_1)\colon e = d - G_2(B, d)\}$$

of $\mathbb{E}(n)$, which has the property that $p_1(H) = p_1(F(\Gamma_1))$. In particular, $p_1(H)$ is finite. Since for any translation $(I, e) \in H$, there exists an element $(I, d) \in F(\Gamma_1)$ such that $e = d - G_2(I, d) = 0$, it holds that $H \simeq p_1(H)$, so H is finite as well. For

$$b := -\frac{1}{|H|} \sum_{(B,e)\in H} e,$$

the equation 2.4.1 is fulfilled for all $(B, d) \in F(\Gamma_1)$.
(3) By (1), every crystallographic group $\Gamma \subset \mathbb{E}(n)$ fits into an exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

where Λ is the subgroup of translations of Γ, and $G \simeq \Gamma/\Lambda$ is finite and can be identified with $p_1(\Gamma)$. Since G acts effectively on Λ via conjugation, $G \subset \mathrm{O}(n)$ is isomorphic to a finite subgroup of $\mathrm{GL}(n,\mathbb{Z})$. For such groups, there are only finitely many possibilities (cf. [Aus65]), which can be shown using a theorem of Jordan (cf. [Jor78]). Fix such a finite group G, and assume without loss of generality that $\Lambda = \mathbb{Z}^n$. We explain now that it is enough to consider finitely many G-module structures on $\mathbb{Z}^n$: Each G-module structure on $\mathbb{Z}^n$ corresponds to an embedding of G into $\mathrm{GL}(n,\mathbb{Z})$, and it is enough to consider these embeddings up to conjugation in $\mathrm{GL}(n,\mathbb{Z})$ since conjugate embeddings yield the same set of extensions. By the Jordan-Zassenhaus-Theorem (cf. [Zas37]), there are only finitely many conjugacy classes of finite subgroups of $\mathrm{GL}(n,\mathbb{Z})$. This finishes the proof because for a fixed G-module structure on $\mathbb{Z}^n$, the extensions of $\mathbb{Z}^n$ by G are parametrized by the finite group $H^2(G,\mathbb{Z}^n)$ (cf. Theorem 2.3.36). □

Since the third part of Bieberbach's Theorem ensures that in each dimension, there are only finitely many isomorphism classes of crystallographic and Bieberbach groups, a natural problem is to ask for a classification of them. Such a classification is known up to dimension 6, but was associated with considerable effort from dimension four on and could only be realized with computer support as the number of isomorphism classes very quickly becomes very large (cf. Table 2.2). The crystallographic groups in dimension 2, also known as *wallpaper groups*, have been known for a long time; a rigorous proof has been given by Fedorov in 1890. Several people were involved in the classification in the three-dimensional case, e.g. Hessel, Bravais, Jordan, Sohncke, Fedorov, and Schoenflies, and it was completed in 1891. For more details, we refer to [Bro+78]. There, Brown, Bülow, Neubüser, and Zassenhaus also worked out the classification in dimension 4. Plesken and Schulz achieved the classification of crystallographic groups in dimension 5 and 6 in 2000 ([PS00]), and Cid together with Schulz determined the Bieberbach groups among them one year later ([CS01]).

dimension	1	2	3	4	5	6
# classes of crystallographic groups	1	17	219	4783	222 018	28 927 922
# classes of Bieberbach groups	1	2	10	74	1060	38 746

Table 2.2.: Number of isomorphism classes of crystallographic and Bieberbach groups

Bieberbach groups are also of particular interest in the field of differential geometry: Quotients of $\mathbb{R}^n$ by Bieberbach groups are so-called *flat Riemannian manifolds*, and each such manifold is of this form. In the rest of thesis section, we will give a brief overview. For more details about Riemannian geometry, we refer the reader to [Car92].

Definition 2.4.7. Let M be a differentiable real manifold. A *Riemannian metric* g is a collection of inner products $(g(-,-)_p)_{p\in M})$ on the tangent spaces T_pM varying smoothly in p. The pair (M, g) is called *Riemannian manifold.*

Example 2.4.8. Consider $M = \mathbb{R}^n$. The derivations $\partial_1, \dots, \partial_n$ form a basis of the tangent space at each point, and a Riemannian metric on $\mathbb{R}^n$ is given by $g(\partial_i, \partial_j)_p := \delta_{ij}$. This metric is called the *flat metric*.

Definition 2.4.9.

(1) A diffeomorphism $f\colon (N, g_N) \to (M, g_M)$ between Riemannian manifolds is called an *isometry* if the differential $J_pf\colon T_pN \to T_{f(p)}M$ is an isometry in each point $p \in N$, that is, the equation

$$g_N(v_1, v_2)_p = g_M(J_pf(v_1), J_pf(v_2))_{f(p)}$$

holds for all $v_1, v_2 \in T_pN$.

(2) A Riemannian manifold is *flat* if it is locally isometric to $\mathbb{R}^n$ equipped with the flat metric.

If $\Gamma \subset \mathbb{E}(n)$ is a Bieberbach group, then the flat metric on $\mathbb{R}^n$ naturally induces a Riemannian metric on the manifold $M = \mathbb{R}^n/\Gamma$, such that the projection map $\pi\colon \mathbb{R}^n \to M$ becomes a local isometry. Thus, $\mathbb{R}^n/\Gamma$ is a flat Riemannian manifold. Moreover, even the converse is true: All compact flat Riemannian manifolds are of this form:

Theorem 2.4.10. *[Car92], Proposition 4.3; [Cha86], Corollary 5.1] A compact Riemannian manifold (M, g) is flat if and only if there exists a Bieberbach group $\Gamma \subset \mathbb{E}(n)$ such that M is isometric to $\mathbb{R}^n/\Gamma$.*

We do not give an entire proof of this theorem but sketch the main ideas: First, by the Hopf-Rinow theorem ([HR31]), a compact flat manifold is geodesically complete, which means that every maximal geodesic is defined on $(-\infty, \infty)$ (a geodesic is roughly speaking locally the shortest path between points). Let $\pi\colon \widetilde{M} \to M$ be the universal cover, which is locally an isometry. Hence, $\widetilde{M}$ is flat and geodesically complete as well, and of course simply connected. The Clifford-Klein theorem, which was first proven by Hopf ([Hop26]), states that such Riemannian manifolds are globally isometric to $\mathbb{R}^n$ (equipped with the flat metric). Thus, M is isometric to $\mathbb{R}^n/\operatorname{Deck}(\pi)$. Every covering isomorphism of π is an isometry, and the isometries of $\mathbb{R}^n$ in the sense of Definition 2.4.9 are precisely the elements in $\mathbb{E}(n)$, which can be seen as follows: Isometries in the sense of Definition 2.4.9 map geodesics to geodesics, and the geodesics on $\mathbb{R}^n$ are precisely the lines. Hence, by the fundamental theorem of affine geometry, each covering isomorphism is affine linear, and since it preserves the inner product on $\mathbb{R}^n$, the matrix is orthogonal. Thus, $\operatorname{Deck}(\pi)$ is a Bieberbach group.

2.5. Rigid manifolds

In this thesis, we focus on *rigid* quotients of complex tori. Therefore, we need some notions from deformation theory, which we introduce now. Moreover, we will explain, how we can directly see from the analytic representation of a group G acting on T whether the quotient T/G is rigid.

Definition 2.5.1. Let X be a reduced compact complex space.

(1) A *deformation* of X consists of the following data:
   - a flat and proper holomorphic map $\pi\colon \mathfrak{X} \to B$ of connected complex spaces,
   - a point $b_0 \in B$,
   - an isomorphism $\pi^{-1}(\{b_0\}) \simeq X$.

(2) An *infinitesimal deformation* of X is a germ $\pi\colon (\mathfrak{X}, X) \to (B, b_0)$ of a deformation.

(3) We call X *(locally) rigid* if for every deformation $\pi\colon \mathfrak{X} \to B$ of X, there is an open neighborhood $U \subset B$ of b_0 such that $X \simeq \pi^{-1}(\{t\})$ for all $t \in U$.

(4) We call X *infinitesimally rigid* if $\operatorname{Ext}^1(\Omega^1_X, \mathcal{O}_X) = 0$.

For compact complex *manifolds*, Kuranishi [Kur62] showed that they always admit a uniquely determined *semi-universal* deformation $\pi\colon (\mathfrak{X}, X) \to (\operatorname{Def}(X), 0)$, which means that every infinitesimal deformation of X is obtained via-pullback of this particular deformation, which is called the *Kuranishi-familiy*. Moreover, by the theory of Kodaira, Kuranishi, and Spencer, the Zariski tangent space of the base $\operatorname{Def}(X)$ in the origin is given by $H^1(X, \Theta_X)$, where $\Theta_X = \mathscr{H}om\,(\Omega^1_X, \mathcal{O}_X)$ denotes the holomorphic tangent sheaf (cf. [Cat88, Remark 5.2]).
Grauert [Gra74] generalized Kuranishi's result to non-smooth reduced compact complex spaces. In the general situation, the Zariski tangent space of the base $\operatorname{Def}(X)$ in the origin is given by $\operatorname{Ext}^1(\Omega^1_X, \mathcal{O}_X)$. The space $H^1(X, \Theta_X)$ has still a geometric meaning: It describes the tangent space of the subspace of $\operatorname{Def}(X)$ that contains the equisingular infinitesimal deformations of X, which are those preserving the singularities (cf. [Cat13, Section 2]). It is connected with $\operatorname{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ via the short-term exact sequence of the local-to-global Ext spectral sequence:

$$0 \longrightarrow H^1(X, \Theta_X) \longrightarrow \operatorname{Ext}^1(\Omega^1_X, \mathcal{O}_X) \longrightarrow H^0(X, \mathscr{E}xt^{\,1}(\Omega^1_X, \mathcal{O}_X)).$$

The space $H^0(X, \mathscr{E}xt^{\,1}(\Omega^1_X, \mathcal{O}_X))$ is trivial if and only if every infinitesimal deformation of X is equisingular. If this is the case, we say that the singularities of X are *infinitesimally rigid* and obtain an isomorphism

$$H^1(X, \Theta_X) \simeq \operatorname{Ext}^1(\Omega^1_X, \mathcal{O}_X).$$

In particular, we obtain:

Corollary 2.5.2. *If X has infinitesimally rigid singularities or is smooth, then X is infinitesimally rigid if and only if $H^1(X, \Theta_X) = 0$.*

In this thesis, only isolated quotient singularities in dimension 3 occur. These are always infinitesimally rigid by a theorem of Schlessinger (cf. [Sch71]).

Next, we want to compare the notions “infinitesimal rigid” and “rigid”. For this, let X be any reduced compact complex space. Since $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ is the Zariski tangent space of the base of the semi-universal Kuranishi-family, we deduce:

Lemma 2.5.3. *Every infinitesimally rigid variety is rigid.*

The question whether the converse is true or not was posed by Morrow and Kodaira in 1971 [MK71]. It turned out that the answer is “No”, even if we restrict to complex manifolds. In this setup, Bauer and Catanese proved:

Theorem 2.5.4 ([BC18], Theorem 2.3)**.** *A compact complex manifold X is rigid if and only if the Kuranishi space* $\mathrm{Def}(X)$ *(base of the Kuranishi family of deformations) is 0-dimensional.*

Even if $\mathrm{Def}(X)$ is of dimension 0, it may happen that it consists of a non-reduced point, so its tangent space $H^1(X, \Theta_X)$ is non-trivial, and hence, X is not infinitesimally rigid. The first counter example was given by Bauer and Pignatelli in [BP21]. For further examples, see also [BG20] and [BBP22]. However, in the situation of torus quotients, these two notions coincide. Before we quote the statement, we introduce the notion of an infinitesimally rigid group action.

Definition 2.5.5. A holomorphic action of a finite group G on a compact complex manifold Y is called *infinitesimally rigid* if

$$H^1(Y, \Theta_Y)^G = 0.$$

If the action of G on Y is free in codimension 1, then $H^1(Y, \Theta_Y)^G = H^1(Y/G, \Theta_{Y/G})$. Thus, if furthermore the singularities of X are infinitesimally rigid, we obtain that the quotient $X = Y/G$ is infinitesimally rigid if and only if the action of G on Y is so.

Proposition 2.5.6 ([DG22], Proposition 2.5)**.** *Let $X = T/G$ be a torus quotient of dimension at least* 3 *by an action with at most isolated fixed points. Then X is rigid if and only if it is infinitesimally rigid.*

Remark 2.5.7. Due to [CD20a], any complex torus quotient has an algebraic approximation. In particular, rigid torus quotients are projective.

In this thesis, we want to classify the groups admitting rigid actions on complex three-dimensional tori. Therefore, it will be crucial to have a criterion whether an action is rigid or not that can easily be checked. Fortunately, the situation for actions on complex tori is quite simple:

Proposition 2.5.8 ([DG22], Corollary 2.6)**.** *Let G be a finite group acting holomorphically on a complex torus T. Then the action is infinitesimally rigid if and only if the analytic representation ρ and its complex conjugate $\overline{\rho}$ do not have any common subrepresentations, that is, $\langle \chi, \overline{\chi} \rangle = 0$, where χ denotes the character of ρ.*

Since the proposition is of particular relevance for this thesis, we give a proof here.

Proof. By definition, the action is rigid if and only if $H^1(T,\Theta_T)^G$ is trivial. Using Dolbeault's interpretation of cohomology, we have

$$H^1(T,\Theta_T) \simeq H^{0,1}_{\overline{\partial}}(\Theta_T) = \left\langle d\overline{z_i} \otimes \tfrac{\partial}{\partial z_j} \mid 1 \leq i,j, \leq n \right\rangle.$$

Via this isomorphism, G acts on $H^1(T,\Theta_T)$ with character χ^2, and the G-invariant part is trivial if and only if χ^2 does not contain the trivial character (cf. Proposition 2.3.11). Now, the claim follows since $\langle \chi^2, \chi_{\mathrm{triv}} \rangle = \langle \chi, \overline{\chi} \rangle$. □

The criterion in the above proposition is not only useful to decide for a given action whether it is rigid or not, but also yields theoretical implications like the following:

Corollary 2.5.9 ([DG22], Remark 2.7)**.** *Let $X = T/G$ be a rigid torus quotient; then the irregularities $q_i(X) = h^i(X,\mathcal{O}_X)$ vanish for $i = 1$ and 2.*

Proof. Let $\psi\colon \hat{X} \to X$ be a resolution of $X = T/G$. Since quotient singularities are rational[1], the irregularities of X and $\hat{X}$ coincide. Let χ be the character of the analytic representation ρ. As $H^0(\hat{X},\Omega^i_{\hat{X}}) \simeq H^0(T,\Omega_T)^G$, we can compute the irregularities as follows:

$$q_i(X) = q_i(\hat{X}) = \dim_{\mathbb{C}}(H^0(T,\Omega^i_T)^G) = \langle \Lambda^i(\overline{\chi}), \chi_{\mathrm{triv}} \rangle.$$

Since ρ and $\overline{\rho}$ have no common subrepresentations by Proposition 2.5.8, it follows that $q_1 = 0$. In order to show that $q_2 = 0$, we use the formula $\chi^2 = \Lambda^2(\chi) + \mathrm{Sym}^2(\chi)$ given in Example 2.3.4(5) and obtain:

$$0 = \langle \overline{\chi}, \chi \rangle = \langle \overline{\chi}^2, \chi_{\mathrm{triv}} \rangle = \langle \Lambda^2(\overline{\chi}), \chi_{\mathrm{triv}} \rangle + \langle \mathrm{Sym}^2(\overline{\chi}), \chi_{\mathrm{triv}} \rangle = q_2(X) + \langle \mathrm{Sym}^2(\overline{\chi}), \chi_{\mathrm{triv}} \rangle.$$

As both, q_2 and $\langle \mathrm{Sym}^2(\overline{\chi}), \chi_{\mathrm{triv}} \rangle$ are non-negative, the claim follows. □

Theorem 2.5.10 ([DG22], Theorem 1.1)**.** *Let T be a complex torus of dimension 3, and let $G \subset \mathrm{Bihol}(T)$ a finite group such that the action on T is rigid. Then the action has fixed points and the quotient X is singular.*

Since we are in particular interested in rigid *manifolds*, we will construct resolutions of our singular quotients that preserve the rigidity. A sufficient and computable condition on the resolution to not change the first cohomology group with values in the tangent sheaf can be derived from the short-term exact sequence of Leray's spectral sequence and has been used several times in the literature (e.g. [BG20], [BG21], [BGK]). Let $\psi\colon \hat{X} \to X$ be a resolution of singularities. Then the mentioned sequence reads

$$0 \longrightarrow H^1(X,\psi_*\Theta_{\hat{X}}) \longrightarrow H^1(\hat{X},\Theta_{\hat{X}}) \longrightarrow H^0(X,R^1\psi_*\Theta_{\hat{X}}).$$

Thus, we obtain:

[1]See Section 2.6 for a definition and more information.

Proposition 2.5.11. *Let X be a complex variety and $\psi\colon \hat{X} \to X$ a (partial) resolution of singularities with the properties*

(1) $\psi_* \Theta_{\hat{X}} \simeq \Theta_X$ *and*

(2) $R^1\psi_*\Theta_{\hat{X}} = 0$.

Then it holds that $H^1(\hat{X}, \Theta_{\hat{X}}) \simeq H^1(X, \Theta_X)$.

Remark 2.5.12. Note that not all resolutions of singularities satisfy the conditions of Proposition 2.5.11 (cf. [BG20, Remark 5.4]).

2.6. Singularities

As discussed in the previous section, the torus quotients studied in this thesis are not smooth but exhibit singularities. Therefore, we now collect the relevant notions and properties, focusing in particular on why we restrict our attention to *canonical* singularities. Moreover, we give a criterion how to decide for cyclic quotient singularities whether they are canonical or not.

In the following, we mean by a *variety* an irreducible reduced complex space.

Definition 2.6.1. Let X be a normal variety. A *resolution of singularities* is a proper biholomorphic map

$$\psi\colon \hat{X} \to X,$$

where $\hat{X}$ is smooth, such that ψ induces an isomorphism $\hat{X} \setminus \psi^{-1}(X_{\mathrm{Sing}}) \simeq X \setminus X_{\mathrm{Sing}}$, where X_{Sing} denotes the singular locus of X.

Let us recall the standard notions for singularities, which can be found in [KM98].

Definition 2.6.2. Let X be a normal variety.

(1) X has *rational* singularities if for all resolutions $\psi\colon \hat{X} \to X$, the higher direct images vanishes, that is:

$$R^q\psi_*\mathcal{O}_{\hat{X}} = 0 \qquad \text{for all} \qquad q \geq 1.$$

(2) X has *canonical (terminal) singularities* if the canonical divisor K_X is $\mathbb{Q}$-Cartier and if for a resolution $\psi\colon \hat{X} \to X$ with exceptional prime divisors E_i, the rational numbers a_i determined by

$$K_{\hat{X}} = \psi^*(K_X) + \sum a_i E_i$$

fulfill $a_i \geq 0$ ($a_i > 0$).

(3) X is *Gorenstein* or has *Gorenstein singularities* if it is Cohen-Macauly[2] and its canonical Weil divisor is a Cartier divisor.

One important property of varieties with only canonical singularities is that they have the same plurigenera as any resolution of their singularities. In particular, the following holds:

[2]For a definition of *Cohen-Macauly*, we refer for example to [KM98, Section 5.1].

Proposition 2.6.3. *Let X be a normal variety with canonical singularities, and let $\psi\colon \hat{X} \to X$ be a resolution of singularities. Then $\hat{X}$ and X have the same Kodaira dimension, $\kappa(\hat{X}) = \kappa(X)$.*

Moreover quasi-étale morphisms, i.e., morphisms with ramification of codimension at least 2, between varieties with canonical singularities preserve the Kodaira dimension.

Proposition 2.6.4 ([Cat07], Section 3)**.** *Let $\pi\colon Y \to X$ be a quasi-étale morphism between normal varieties with canonical singularities. Then X and Y have the same Kodaira dimension, $\kappa(X) = \kappa(Y)$.*

Overall, we conclude from the last two propositions that the quotients and their resolutions in our setup, where we only allow isolated canonical singularities, all have Kodaira dimension 0. This is the primary reason for restricting our attention to canonical singularities. In order to apply Proposition 2.6.4, we could allow actions that are free only in codimension 1, but the isolatedness of singularities is essential for certain rigidity conditions, as explained in Section 2.5.

In general, the canonical divisor of a variety with canonical singularities changes under resolution. However, if we do not fully resolve the singularities but allow terminal singularities to remain, there are always partial resolutions that preserve the canonical divisor, at least in dimension 3.

Theorem 2.6.5 ([Rei87], Theorem 3.2 b)**.** *Let X be a three-dimensional variety with canonical singularities. Then there exists a crepant terminalization $\psi\colon \hat{X} \to X$, that is, $\hat{X}$ has only terminal singularities and $\psi^*(K_X) = K_{\hat{X}}$.*

For Gorenstein quotient singularities in dimension 3, even crepant resolutions exists. Before we will state the result of Roan, we will explain the notion of quotient singularities.

Definition 2.6.6. Let X be a variety. A point $p \in X$ is a *(finite) quotient singularity* if there is a (finite) subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ such that $p \in X$ is locally analytically equivalent to $0 \in \mathbb{C}^n/G$. If G is abelian (cyclic), then p is called *abelian (cyclic) quotient singularity.*

Remark 2.6.7. Note that a quotient singularity not necessarily need to be a point in the singular locus of X: Smooth points are always quotient singularities; choose G as the trivial group.
Note moreover that finite quotient singularities are always rational (cf. [KM98, Proposition 5.15]).

Remark 2.6.8. According to [Wat74], a finite quotient singularity $\mathbb{C}^n/G$ is Gorenstein if and only if $G \subset \mathrm{SL}(n, \mathbb{C})$.

Theorem 2.6.9 ([Roa96])**.** *If X is a three-dimensional variety with only Gorenstein quotient singularities, then X admits a crepant resolution.*

Remark 2.6.10. Let G be a finite group acting holomorphically on a complex torus T with analytic representation ρ. Consider a point $p \in T$, and denote by x its class in $X = T/G$. Then locally near x, the quotient X is given by $\mathbb{C}^n/H$, where $H = \{\rho(g) \mid g \in \mathrm{Stab}(p)\}$. Thus, the quotient $X = T/G$ has only quotient singularities.

Let $G = \langle g \rangle \subset \mathrm{GL}(n, \mathbb{C})$ be a cyclic group. Then we can assume without loss of generality that g is diagonal, $g = \mathrm{diag}(\zeta_d^{a_1}, \ldots, \zeta_d^{a_n})$, where $d = \mathrm{ord}(g)$, $a_1, \ldots, a_n \in \{0, \ldots, d-1\}$, and $\gcd(a_1, \ldots, a_n, d) = 1$. We say that the cyclic quotient singularity $0 \in U = \mathbb{C}^n/G$ is of *type*

$$\tfrac{1}{d}(a_1, \ldots, a_n).$$

Note that the singularity is isolated if and only if $\gcd(a_i, d) = 1$ for all $i = 1, \ldots, n$.

For cyclic quotient singularities[3], there is the famous criterion of Reid, Shepherd-Baron and Tai to decide whether the singularity is canonical (terminal) or not:

Proposition 2.6.11 ([Rei80], Theorem 3.1)**.** *Let (U, x_0) be a cyclic quotient singularity of type $\frac{1}{d}(a_1, \ldots, a_n)$. The singularity is canonical (or terminal) if and only if for all $k = 1, \ldots, d-1$, it holds that*

$$\sum_{j=1}^{n} [k \cdot a_j] \geq d,$$

(respectively $> d$). Here, $[\ \cdot\ ]$ denotes the residue modulo d.

In Chapter 3, we will show that actions of finite groups on three-dimensional tori with isolated fixed points – precisely the cases considered in this thesis – lead to quotients with only isolated cyclic quotient singularities. They have been classified by Morrison:

Theorem 2.6.12 ([MS84], [Mor85])**.** *Every isolated, canonical, cyclic quotient singularity (U, x_0) has precisely one of the following types:*

(1) $\frac{1}{d}(1, a, d-a)$, *where* $\gcd(d, a) = 1$ *(terminal),*

(2) $\frac{1}{d}(1, a, d-a-1)$, *where* $\gcd(d, a) = \gcd(d, a+1) = 1$ *(Gorenstein),*

(3) $\frac{1}{9}(1, 4, 7)$ *or* $\frac{1}{14}(1, 9, 11)$.

[3] The criterion even holds true for abelian quotient singularities if one checks every element of G, but we only need it in the cyclic case.

3. Classification of the groups

This chapter is devoted to the classification of all finite groups G that admit a holomorphic action with isolated fixed points on a complex torus T of dimension 3 such that $X = T/G$ is rigid and has canonical singularities. We assume that the action is translation-free, which is equivalent to requiring that it has a faithful analytic representation

$$\rho\colon G \hookrightarrow \mathrm{GL}(3, \mathbb{C}).$$

Some proofs will be performed computer-based with the computer algebra system MAGMA. The code can be found in the Appendix A.1.

During the classification process, we will frequently make use of some basic observations:

Remark 3.0.1. Given an action of a finite group G as above, then the following holds:

(1) The restriction to every subgroup U shares the same properties apart from the rigidity of the quotient T/U.

(2) An element $g \in G$ acts with fixed points if and only if the equation

$$(\rho(g) - \mathrm{id}) \cdot z = -\tau(g)$$

has a solution in T. In particular, if g acts freely, then 1 is an eigenvalue of $\rho(g)$.

(3) If $g \in G$ has fixed points and order d, then all the eigenvalues of $\rho(g)$ must be primitive d-th roots of unity since otherwise, the fixed locus of some power of g has positive dimension.

From now on, we fix a finite group G and assume that it admits an action with the above properties. First, we determine the possible orders of the elements in such a group.

Lemma 3.0.2. *Let $g \in G$ be a non-trivial element.*

(1) If g acts freely on T, then $\mathrm{ord}(g) \in \{2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12\}$.

(2) If g acts with fixed points, then $\mathrm{ord}(g) \in \{2,\ 3,\ 4,\ 6,\ 7,\ 9,\ 14,\ 18\}$.

In particular, $\mathrm{ord}(g) \in \{2, \ldots,\ 10,\ 12,\ 14,\ 18\}$*, elements of order* $7,\ 9,\ 14,\ 18$ *always have fixed points, and elements of order* $5,\ 8,\ 10,\ 12$ *always act freely.*

Proof. Let $d := \mathrm{ord}(g)$. Assume first that g acts freely. Then $\rho(g)$ has eigenvalue 1. If the other two eigenvalues have the same order, then $\varphi(d) \leq 4$, where φ denotes the Euler totient function. Otherwise, the orders d_1 and d_2 of the other two eigenvalues fulfill $\varphi(d_1) + \varphi(d_2) \leq 4$ (cf. Lemma 2.1.9(2)) and $d = \mathrm{lcm}(d_1, d_2)$. It is now easy to determine all possible values for d. If g acts with fixed point, then all its eigenvalues are primitive d-th roots of unity. In this case, $\varphi(d)$ divides 6 by the same lemma. This implies $d \in \{2,\ 3,\ 4,\ 6,\ 7,\ 9,\ 14,\ 18\}$. $\square$

Lemma 3.0.3. *Assume that G contains an abelian subgroup U such that every element in U acts non-freely. Then U is cyclic.*

Proof. Since U is abelian, we can assume that ρ restricted to U is the direct sum of three one-dimensional representations. Each of them must be faithful because the identity is the only element that has eigenvalue 1 since every element acts with fixed points. Hence, U is cyclic. $\square$

Corollary 3.0.4. *If G has a 7-Sylow subgroup S_7, then S_7 is cyclic of order 7.*

Proof. By Sylow's Theorem, it suffices to exclude that G has a subgroup U of order 7^2. Since U does not contain elements of order 49 by Lemma 3.0.2, the subgroup U is not cyclic, and every element acts with fixed points. Thus, $U \simeq \mathbb{Z}_7^2$, which contradicts Lemma 3.0.3. $\square$

Now, we are ready to determine the possible non-trivial stabilizer groups and the corresponding singularities of the quotient. It turns out that all of them are cyclic.

Theorem 3.0.5. *Let G be a finite group acting holomorphically and with isolated fixed points on a three-dimensional complex torus T. Then for all $p \in T$, the stabilizer group $\mathrm{Stab}(p)$ is cyclic of order $1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 9,\ 14,$ or 18.*
In particular, the quotient X has only isolated cyclic quotient singularities. The possible types of the canonical ones are $\frac{1}{d}(1,1,d-1)$, where $d = 2,3,4,6$, and $\frac{1}{3}(1,1,1)$, $\frac{1}{7}(1,2,4)$, $\frac{1}{9}(1,4,7)$, and $\frac{1}{14}(1,9,11)$.

Proof. Let $p \in T$ be a point with non-trivial stabilizer $H := \mathrm{Stab}(p)$. Moving the origin of T, we may assume that H acts linearly. In particular, every element of H has $0 \in T$ as fixed point. First, we prove that H is cyclic. By Lemma 3.0.3, it is enough to show that H is abelian.
We start with summing some relevant properties of H and its elements:

(1) For every non-trivial element $g \in H$, the matrix $\rho(g)$ has 1 not as eigenvalue, since the fixed points are assumed to be isolated.

(2) By Lemma 3.0.2, for all $g \in G$, it holds that $\mathrm{ord}(g) \in \{1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 9,\ 14,\ 18\}$. In particular, $|H| = 2^a \cdot 3^b \cdot 7^c$.

(3) Let $g \in H$ be an element of order 2. Then $\rho(g) = -\mathrm{id}$. In particular, H contains at most one element of order 2 since ρ is faithful.

(4) By Corollary 3.0.4, we have $c \in \{0,1\}$.

Next, we analyze the 2- and 3-Sylow subgroups of H. We claim that they are cyclic of order 2 or 4, and 3 or 9, respectively (if existent). By Sylow's Theorem and Lemma 3.0.3, it is enough to show that H has no subgroups of order p^3 for $p = 2,3$. Such a subgroup U cannot be cyclic by item (2) and hence not abelian by Lemma 3.0.3. If $p = 2$, then the three-dimensional representation ρ restricted to this subgroup has a one-dimensional subrepresentation, which has to be faithful by item (1) – a contradiction. If $p = 3$, then U is either $\mathrm{He}(3)$ or $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$. Both of them contain $\mathbb{Z}_3^2$ as a subgroup contradicting Lemma 3.0.3. In particular, $a, b \in \{0,1,2\}$.
Finally, we show that there is no non-abelian group fulfilling all these conditions. Note that if H is not abelian, the representation ρ needs to be irreducible. In particular, $3 = \chi_\rho(1)$ has to divide the group order, so $b \neq 0$. We now analyze the two possible cases for c separately:

- $\underline{c=0}$: If $a=0$, then H is abelian.
  If $a=1$, then the 3-Sylow subgroup is normal due to Sylow's Theorems. By item (3), H has only one 2-Sylow subgroup $\mathbb{Z}_2$, which is normal (its generator acts with $-\mathrm{id}$), too. Hence, H is abelian.
  If $a=2$, then the only groups admitting an irreducible representation of dimension 3 are $\mathcal{A}_4$, $\mathcal{A}_4 \times \mathbb{Z}_3$ and $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_9$, all of which have more than one element of order 2.

- $\underline{c=1}$: The only groups that have at most one element of order 2 and admit an irreducible representation of dimension 3 contain $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ or $\mathbb{Z}_7 \rtimes \mathbb{Z}_9$ as a subgroup. Thus, we only have to exclude these groups. Up to complex conjugation and equivalence of representations, the only irreducible three-dimensional representation of

$$\mathbb{Z}_7 \rtimes \mathbb{Z}_3 = \langle t,\ s \mid t^7 = s^3 = 1,\ sts^{-1} = t^4 \rangle$$

  is given by
$$s \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} \zeta_7^4 & 0 & 0 \\ 0 & \zeta_7^2 & 0 \\ 0 & 0 & \zeta_7 \end{pmatrix}.$$

  But then the matrix of s has eigenvalue 1. The group $\mathbb{Z}_7 \rtimes \mathbb{Z}_9$ has an element of order 21.

Thus, H is cyclic, and by Lemma 3.0.2, its order d belongs to $\{2,\ 3,\ 4,\ 6,\ 7,\ 9,\ 14,\ 18\}$. By Morrison's classification (cf. Theorem 2.6.12), each isolated cyclic canonical quotient singularity is isomorphic to precisely one of the following:

- $\frac{1}{d}(1, a, d-a)$, where $\gcd(d, a) = 1$ (terminal)
- $\frac{1}{d}(1, a, d-a-1)$, where $\gcd(d, a) = \gcd(d, a+1) = 1$ (Gorenstein)
- $\frac{1}{9}(1, 4, 7)$ or $\frac{1}{14}(1, 9, 11)$.

The condition $\gcd(d, a) = \gcd(d, a+1) = 1$ in the Gorenstein case implies that d is odd. Recall that for a linear automorphism $\alpha \in \mathrm{Aut}(T)$ of order d with only primitive d-th roots of unity as eigenvalues, the function

$$\mu_d^* \longrightarrow \mathbb{Z}, \quad \zeta \mapsto \mathrm{mult}(\zeta) + \mathrm{mult}(\overline{\zeta}),$$

is constant, where $\mathrm{mult}(\zeta)$ denotes the multiplicity of ζ as eigenvalue of α and μ_d^* denotes the set of primitive d-th roots of unity (cf. Lemma 2.1.9(1)).
In the terminal case, each generator of the stabilizer has two eigenvalues that are complex conjugate to each other. Thus, $\varphi(d) \leq 2$ or equivalently $d \in \{2, 3, 4, 6\}$.
Analyzing the remaining cases yields the singularities in the theorem. Note that there is no singularity of order 18, since 18 is even and $\varphi(18) = 6$. □

The main result of this chapter is the following:

Theorem 3.0.6. *Let G be a finite group acting holomorphically, without translations, and with isolated fixed points on a complex torus T of dimension 3 such that $X = T/G$ is rigid with canonical singularities.*

(1) If $p_g(X) = 1$*, then* $G \simeq \mathbb{Z}_7,\ \mathbb{Z}_3,\ \mathbb{Z}_3^2$*, or* $\mathrm{He}(3)$.

(2) If $p_g(X) = 0$*, then* $G \simeq \mathbb{Z}_9,\ \mathbb{Z}_{14},\ \mathbb{Z}_3^2,\ \mathbb{Z}_3^3$*, or* $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$.

Remark 3.0.7. We point out that the classification of the groups and the analytic representations in the case $p_g(X) = 1$ was already achieved by Oguiso and Sakurai ([OS01, Theorem 3.4]). Instead of the rigidity, they assumed that the action has non-empty (isolated) fixed locus and that the quotients have vanishing irregularity q_1. From their description of the analytic representations (see also Theorem 4.1.5 in the next chapter), the rigidity follows immediately from Proposition 2.5.8. Conversely, any rigid action on a three-dimensional torus has fixed points by [DG22, Theorem 1.1]. In the rest of the section, we therefore only need to consider the case $p_g(X) = 0$.

A first step towards the classification of the groups in the case $p_g(X) = 0$ is to analyze the possible baskets of singularities. For this purpose, we make use of a relative version of the orbifold Riemann-Roch formula (cf. [Rei87, Corollary 10.3]), adapted to our setup (cf. [Gle16, Section 4.3] for a similar situation).

Proposition 3.0.8. *If* $p_g(X) = 0$*, then*

$$1 = \tfrac{1}{16}N_2 + \tfrac{1}{9}N_3 + \tfrac{5}{32}N_4 + \tfrac{35}{144}N_6 + \tfrac{1}{3}N_9 + \tfrac{7}{16}N_{14},$$

where N_d *denotes the number of singularities of type* $\frac{1}{d}(1,1,d-1)$ *for* $d = 2,\ 3,\ 4,\ 6$*, and* N_9 *and* N_{14} *the number of singularities of type* $\frac{1}{9}(1,4,7)$ *and* $\frac{1}{14}(1,9,11)$*, respectively.*

Proof. The orbifold Riemann-Roch formula (cf. [Rei87, Corollary 10.3]) reads:

$$\chi(\mathcal{O}_X) = \frac{1}{24}\left(-K_X \cdot c_2(X) + \sum_{x \text{ terminal}} \frac{m_x^2 - 1}{m_x}\right),$$

where the sum runs over all terminal singularities $\frac{1}{m_x}(1, a_x, m_x - a_x)$ of a crepant terminalization of X, which we obtain by looking locally at each isolated singular point. The Gorenstein singularities have a crepant resolution, so they do not contribute. The crepant terminalization of $\frac{1}{9}(1,4,7)$ consists of three copies of $\frac{1}{3}(1,1,2)$, and the one of $\frac{1}{14}(1,9,11)$ consists of seven nodes $\frac{1}{2}(1,1,1)$ (cf. Chapter 5). Since the remaining singularities are all terminal (cf. Theorem 3.0.5), we do not have to modify them.

By the rigidity of the action, we have $q_1(X) = q_2(X) = 0$, which implies that $\chi(\mathcal{O}_X) = 1$. Moreover, the intersection product $K_X \cdot c_2(X)$ is 0 since $|G| \cdot K_X \sim_{lin} 0$. Hence, the claim follows. ☐

Corollary 3.0.9. *The candidates for the values of* $[N_2, N_3, N_4, N_6, N_9, N_{14}]$ *are*

k	$[N_2, N_3, N_4, N_6, N_9, N_{14}]$	k	$[N_2, N_3, N_4, N_6, N_9, N_{14}]$	k	$[N_2, N_3, N_4, N_6, N_9, N_{14}]$
1	[0, 1, 2, 1, 1, 0]	6	[5, 1, 0, 1, 1, 0]	11	[16, 0, 0, 0, 0, 0]
2	[0, 4, 2, 1, 0, 0]	7	[5, 4, 0, 1, 0, 0]	12	[0, 0, 0, 0, 3, 0]
3	[1, 0, 6, 0, 0, 0]	8	[6, 0, 4, 0, 0, 0]	13	[0, 3, 0, 0, 2, 0]
4	[2, 0, 0, 0, 0, 2]	9	[9, 0, 0, 0, 0, 1]	14	[0, 6, 0, 0, 1, 0]
5	[4, 0, 2, 0, 0, 1]	10	[11, 0, 2, 0, 0, 0]	15	[0, 9, 0, 0, 0, 0]

Next, we want to derive a formula which allows us to compute the order of the group G in terms of the N_i. If the image of the analytic representation ρ contains non-trivial scalar matrices, we can derive such a formula from the Lefschetz fixed-point formula, which has a particularly simple shape on complex tori (cf. Lemma 2.1.12).

Lemma 3.0.10.

(1) If $-\mathrm{id} \in \mathrm{im}(\rho)$*, then*

$$2^6 = |G| \cdot (\tfrac{1}{2}N_2 + \tfrac{1}{4}N_4 + \tfrac{1}{6}N_6 + \tfrac{1}{14}N_{14}).$$

(2) If $\zeta_3 \cdot \mathrm{id} \in \mathrm{im}(\rho)$*, then*

$$3^3 = |G| \cdot (\tfrac{1}{3}N_{3,gor} + \tfrac{1}{9}N_9),$$

where $N_{3,gor}$ *denotes the number of singularities of type* $\frac{1}{3}(1,1,1)$*.*

Proof. We only give a proof for the first statement. The reasoning for the second one is similar. Let $g \in G$ be the unique element with $\rho(g) = -\mathrm{id}$. The fixed points of g are precisely the elements in T with stabilizer of even order. Thus:

$$\mathrm{Fix}(g) = \bigsqcup_{j \in \{2,4,6,14\}} \{y \in T \mid \mathrm{Stab}(y) \simeq \mathbb{Z}_j\}.$$

By Lemma 2.1.12, g has 2^6 fixed points. If x is a singularity of order j, then the fiber of x under the projection map $\pi\colon T \to X$ contains $|G|/j$ elements, all having a stabilizer group isomorphic to $\mathbb{Z}_j$. □

Remark 3.0.11. If there exists an even j such that $N_j \neq 0$, then G has an element g of order 2 with fixed points. Thus, $\rho(g) = -\mathrm{id}$.
Similarly, if $N_9 \neq 0$, there is an element $h \in G$ of order 9 such that $\rho(h)$ is similar to $\mathrm{diag}(\zeta_9, \zeta_9^4, \zeta_9^7)$. Hence, $\rho(h^3) = \zeta_3 \cdot \mathrm{id}$.
In these cases, we can deduce the possible orders of G from the baskets of singularities given in Corollary 3.0.9 and the above lemma, which results in finitely many possible groups. Not all of them allow a rigid action on T with the properties of Theorem 3.0.6, which we recall below:

Notation 3.0.12. In the following, we shall say that a group G enjoys the *standard conditions*, if and only if, for all $g \in G$, it holds that

$$\mathrm{ord}(g) \in \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 12,\ 14\},$$

and there is a three-dimensional representation $\rho\colon G \to \mathrm{GL}(3,\mathbb{C})$ such that:

- ρ is faithful (the action contains no translations),
- its character χ contains no complex conjugated irreducible characters (the action is rigid),
- for each $g \in G$, the characteristic polynomial of $\rho(g) \oplus \overline{\rho}(g)$ has integer coefficients ($\rho(g)$ maps the lattice of the torus to itself),

- if $\mathrm{ord}(g) \in \{5,\ 8,\ 10,\ 12\}$, then $1 \in \mathrm{Eig}(\rho(g))$ (these elements have to act freely), and
- if $\mathrm{ord}(g) \in \{7,\ 9,\ 14\}$, then $1 \notin \mathrm{Eig}(\rho(g))$ (the fixed points are isolated).

In the following, we will frequently use the following version of Burnside's Lemma counting singularities in two different ways.

Lemma 3.0.13. *Let T be a complex torus, G a finite group acting holomorphically on T such that all stabilizer groups are cyclic. Let m be a divisor of $|G|$. Assume that the order of each non-trivial element of G which has fixed points is not a proper multiple of m. Let s_m be the number of elements of G of order m acting with fixed points, and let ℓ be the number of fixed points of such an element. Then*

$$\#\{[x] \in T/G \mid \mathrm{Stab}(x) \simeq \mathbb{Z}_m\} \cdot \frac{|G|}{m} = \ell \cdot \frac{s_m}{\varphi(m)}$$

Proof. The left hand side of the equation counts the number of points in T with stabilizer isomorphic to $\mathbb{Z}_m$. Each stabilizer contains $\varphi(m)$ elements of order m, all of which have the same fixed points. Moreover, generators of different stabilizer groups have disjoint sets of fixed points by the "maximality" of m. Thus, the claim follows. □

Proposition 3.0.14. *If $-\mathrm{id} \in \mathrm{im}(\rho)$, then $k = 9$ in Corollary 3.0.9 and $G \simeq \mathbb{Z}_{14}$. In particular, the cases $k = 1, \ldots, 8, 10, 11$ cannot occur.*

Proof. By Remark 3.0.11, $-\mathrm{id} \in \mathrm{im}(\rho)$ holds if and only if $k \in \{1, \ldots, 11\}$. By Lemma 3.0.10, the number of singularities of even order determine the group order uniquely. They are displayed in the following table:

k	1	2	3	4	5	6	7	8	9	10	11
$\lvert G\rvert$	96	96	32	56	$\frac{224}{9}$	24	24	16	14	$\frac{32}{3}$	8

Obviously, the cases $k = 5$ and $k = 10$ are not possible.

If $k = 1$ or $k = 6$, then the group order is not divisible by 9. Hence, the group G doesn't contain any element of order 9 – a contradiction to $N_9 \neq 0$.

If $k = 3,\ 8,\ 11$, then G is a 2-group of order $8, 16$ or 32. None of these groups fulfills the standard conditions and the following constraints: If $|G| = 8$, then we are in case $k = 11$ and $N_4 = 0$, and thus, also the elements of order 4 act freely. In the other two cases, all elements of order 4 whose linear parts do not have eigenvalue 1 have as set of eigenvalues $\{i, -i\}$ (the fixed points of elements of order 4 lead to singularities of type $\frac{1}{4}(1, 1, 3)$).

If $k = 2$ or $k = 7$, then $N_6 = 1$. By Lemma 3.0.13, G has precisely $2 \cdot |G|/6$ elements of order 6 whose set of eigenvalues is $\{\zeta_6, \zeta_6^5\}$. Note that all other elements of order 6 have the eigenvalue 1. For the case $k = 7$, we observe furthermore that all elements of order 4 act freely, as $N_4 = 0$ and stabilizer groups of order $4s$ do not occur for $s \geq 2$ (cf. Theorem 3.0.5). No group of order 96 or 24 enjoys both, the standard and these additional conditions.

If $k = 4$, then $|G| = 56$ and $N_{14} = 2$. By Lemma 3.0.13, G has $2 \cdot 56 \cdot 6/14 = 48$ elements of order 14. Note that G has at least six elements of order 7 and a 2-Sylow subgroup of order 8, whose elements have an order that divides 8. Therefore, G can have at most $56 - 6 - 8 = 42$ elements of order 14 – a contradiction.

If $k = 9$, then $|G| = 14$ and since $N_{14} \neq 0$, the group G has an element of order 14. Thus, G is cyclic of order 14, and the proposition is proven. □

Proposition 3.0.15. *If $\zeta_3 \cdot \mathrm{id} \in \mathrm{im}(\rho)$, then $k = 12$ and $G \simeq \mathbb{Z}_9$ or $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$, or $k = 15$ and $G \simeq \mathbb{Z}_3^2$ or $\mathbb{Z}_3^3$. In particular, the cases $k = 13$ and 14 cannot occur.*

Proof. By Proposition 3.0.14 and Remark 3.0.11, $-\zeta_3 \cdot \mathrm{id}$ belongs to the image of ρ if and only if $k \in \{12, \ldots, 15\}$. As a consequence of Lemma 3.0.10, N_9 cannot be 2. Hence, the case $k = 13$ can be excluded.

If $k = 14$, then $N_9 = 1$ and so, $N_{3,gor} = 0$ and $|G| = 3^5$. By Lemma 3.0.13, G has $3^5/9 \cdot 6/3 = 54$ elements of order 9 and no elements of order greater than 9. The only groups of order 3^5 with these properties are the ones with MAGMA-ID $\langle 243, 53\rangle$ and $\langle 243, 58\rangle$. Both of these groups do not fulfill the standard conditions – hence, $k = 14$ is also not realizable.

If $k = 12$, we have $N_9 = 3$, and thus, $|G| = 3^a$ for some $a \in \{2, 3, 4\}$. If $a = 4$, then $N_{3,gor} = 0$, and following the same argument as in the case $k = 14$, we obtain a contradiction. If $a = 3$, then G is either isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_9$ or to $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ (we need an element of order 9), but the first group does not enjoy the standard conditions. If $a = 2$, then $G \simeq \mathbb{Z}_9$ since G contains an element of order 9.

If $k = 15$ then

$$3^3 = N_{3,gor} \cdot \tfrac{|G|}{3}$$

because $N_9 = 0$. Since $N_3 \neq 0$, the order of G is strictly greater than 3. Note furthermore that G doesn't contain any element of order 9 because $N_9 = 0$, i.e., the group has exponent 3.
Clearly, if $|G| = 9$, then $G \simeq \mathbb{Z}_3^2$ is the only possibility, and if $|G| = 27$, then $G \simeq \mathbb{Z}_3^3$ because the group $\mathrm{He}(3)$ is not possible, since $N_3 \neq 0$ and the images of the three-dimensional irreducible representations of $\mathrm{He}(3)$ belong to $\mathrm{SL}(3, \mathbb{C})$.
The only groups of order 81 with exponent 3 are $\mathbb{Z}_3^4$ and $\mathbb{Z}_3 \times \mathrm{He}(3)$. Both groups do not admit a faithful three-dimensional representation. In case of the first group $\mathbb{Z}_3^4$, the representation is the sum of three one-dimensional characters. Since each of them takes values in $\langle\zeta_3\rangle$, the image of the representation has at most 3^3 elements. The second 3-group, $\mathbb{Z}_3 \times \mathrm{He}(3)$, is not abelian, hence, the representation is irreducible. By Schur's Lemma, its center $\mathbb{Z}_3^2$ acts by scalar multiples of the identity matrix of order 3. So, the kernel of the representation is non-trivial. □

In the rest of the section, we consider the remaining case, where the image of ρ does not contain any scalar matrices. Thus we cannot apply Lemma 3.0.13 to control the group order. Instead, we apply Sylow's Theorems to bound the orders of the p-subgroups of G. The goal is to prove the following:

Proposition 3.0.16. *If $-\mathrm{id},\ \zeta_3 \cdot \mathrm{id} \notin \mathrm{im}(\rho)$, then $k = 15$ and $G \simeq \mathbb{Z}_3^2$.*

Remark 3.0.17. By Remark 3.0.11, it is clear that $k = 15$ is the only case, where $-\mathrm{id},\ \zeta_3 \cdot \mathrm{id} \notin \mathrm{im}(\rho)$ holds. In this situation, we have the following basket of singularities:

$$9 \times \tfrac{1}{3}(1,1,2), \quad N_7 \times \tfrac{1}{7}(1,2,4).$$

In particular, G has no elements of order 9, since such an element would act with fixed points but $N_9 = 0$.
Recall furthermore, that $|G| = 2^a \cdot 3^b \cdot 5^c \cdot 7^d$ with $d \in \{0,1\}$, and $b \geq 1$ because $N_3 \neq 0$.

Lemma 3.0.18. *If G has a 5-Sylow subgroup S_5, then S_5 is cyclic of order 5, thus, $c \in \{0,1\}$.*

Proof. By Lemma 3.0.2 and Sylow's Theorem, it suffices to exclude that G contains a copy of $\mathbb{Z}_5^2$. Assuming the existence of such a subgroup, the restriction of ρ to $\mathbb{Z}_5^2$ would be of the form

$$\rho \colon \mathbb{Z}_5^2 \longrightarrow \mathrm{GL}(3,\mathbb{C}), \qquad (a,b) \mapsto \mathrm{diag}(\zeta_5^a, \zeta_5^b, \zeta_5^{\lambda a + \mu b}),$$

up to equivalence of representations and automorphisms of $\mathbb{Z}_5^2$. All of the representation matrices must have 1 as an eigenvalue because the elements of $\mathbb{Z}_5^2$ cannot have fixed points (cf. Lemma 3.0.2). This implies $\lambda = \mu = 0$. Since the characteristic polynomial of $\rho(1,0) \oplus \overline{\rho}(1,0)$ does not have integer coefficients, we observe that an action with such a linear part cannot exist. □

Furthermore, G has no 7-Sylow subgroups:

Lemma 3.0.19. *The group G has no elements of order 7; hence, $N_7 = d = 0$.*

Proof. Let n_7 denote the number of 7-Sylow subgroups of G, and assume that $n_7 \geq 1$, so $d = 1$. First, we show that $|G| = 7 \cdot n_7$. By Lemma 3.0.13 and since $N_{14} = 0$, it holds that

$$N_7 \cdot \frac{|G|}{7} = 7 \cdot n_7.$$

Since 7^2 does not divide the group order, this implies that N_7 is divisible by 7. By Sylow's Theorems, there exists an integer m such that $n_7 \cdot m = |G|/7$. Hence, $N_7 \cdot m = 7$, which implies $N_7 = 7$ and $|G| = 7 \cdot n_7$.
Note that the group has $n_7 \cdot 6$ elements of order 7 and at least $2/9 \cdot |G|$ elements of order 3 (cf. Lemma 3.0.13, recall that $N_{3,gor} = N_9 = 0$). These are in total more elements than G has:

$$(n_7 \cdot 6 + \tfrac{2}{9} \cdot |G|) - |G| = n_7 \cdot (6 + \tfrac{2}{9} \cdot 7 - 7) = n_7 \cdot \tfrac{5}{9} > 0. \qquad \square$$

Lemma 3.0.20. *The 2-Sylow subgroups contain at most 2^5 elements, and the 3-Sylow subgroups are all isomorphic to $\mathbb{Z}_3^2$. In particular, $a \leq 5$ and $b = 2$.*

Proof. Let S_p be a p-Sylow group of G. Then S_p and all its subgroups fulfill the standard conditions except possibly for the rigidity. By Sylow's Theorems, S_p has subgroups of order p^m for all m such that $p^m \leq |S_p|$. So, we determine successively by increasing order all possible p-groups for $p = 2, 3$.

If $p = 3$, then by assumption, $\zeta_3 \cdot \mathrm{id}$ is not contained in the image of the representation and the group does not contain elements of order 9. This excludes all groups of order 3^3, and the entire list of possible 3-groups contains only $\mathbb{Z}_3$ and $\mathbb{Z}_3^2$. By Lemma 3.0.13, the number of elements of order 3 with fixed points equals $2/9 \cdot |G|$. In particular, 9 divides the group order, so the only possibility for a 3-Sylow subgroup is $\mathbb{Z}_3^2$.
If $p = 2$, then for each element h in S_2, the matrix $\rho(h)$ has to have eigenvalue 1 since all elements of even order have to act freely. Together with the described strategy, a MAGMA-computation shows that the 2-subgroups of G have order at most 2^5. □

Proof of Proposition 3.0.16. First, we assume $|G| = 2^a \cdot 3^2 \cdot 5$ with $a \in \{0, \dots, 5\}$. The only group that has at least $2/9 \cdot |G|$ elements of order 3 and fulfills that the order of each element of G belongs to $\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12\}$ has MAGMA-ID $\langle 360, 118\rangle$. But this group is not abelian and has no irreducible character of degree 2 or 3 – a contradiction. Hence, 5 does not divide the group order and $|G| = 2^a \cdot 3^2$.
If moreover $a = 0$, then G is isomorphic to $\mathbb{Z}_3^2$.
In order to exclude the case $a \geq 1$, we check with MAGMA that there is no group G of order $2^a \cdot 3^2$ with $a \in \{1, \dots, 5\}$ with the following properties:

- G enjoys the standard conditions,
- for all $g \in G$, it holds that $\mathrm{ord}(g) \in \{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12\}$,
- G has at least $2/9 \cdot |G|$ elements of order 3,
- $\#\{g \in G \mid \mathrm{ord}(g) = 3,\ 1 \notin \mathrm{Eig}(\rho(g))\} = \frac{2}{9} \cdot |G|$, and
- if $\mathrm{ord}(g) = 3$ and $1 \notin \mathrm{Eig}(\rho(g))$, then $\mathrm{Eig}(\rho(g)) = \{\zeta_3, \zeta_3^2\}$. □

Proof of Theorem 3.0.6. Let G be a finite group as in the theorem. Since T has geometric genus $p_g(T) = 1$, the geometric genus of the quotient $X = T/G$ is either 1 or 0 depending whether the volume form $dz_1 \wedge dz_2 \wedge dz_3$ is G-invariant or not. The classification of the groups in the case that $p_g(X) = 1$ was achieved in [OS01] as explained in Remark 3.0.7. If $p_g(X) = 0$, then Corollary 3.0.9 describes the possible baskets of singularities. In Proposition 3.0.14, it is proven that if $-\mathrm{id} \in \mathrm{im}(\rho)$, then G is isomorphic to $\mathbb{Z}_{14}$. Proposition 3.0.15 settles the classification if $\zeta_3 \cdot \mathrm{id}$ belongs to the image of ρ: G is isomorphic to $\mathbb{Z}_9$, $\mathbb{Z}_3^2$, $\mathbb{Z}_3^3$, or $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$. If $-\mathrm{id}$ as well as $\zeta_3 \cdot \mathrm{id}$ are both not contained in $\mathrm{im}(\rho)$, then Proposition 3.0.16 ensures that G is isomorphic to $\mathbb{Z}_3^2$. □

4. Classification of the quotients

4.1. Known results and Calabi-Yau threefolds

In this section, we collect the results about rigid quotients $X = T/G$ of complex tori in dimension 3 which have already been established. In particular, we explain the connection to Calabi-Yau threefolds and their contractions.

Let us start with the case $p_g(X) = 1$. Since $p_g(X) = \dim(H^0(T, \Omega_T^3))^G$, this means precisely that the top form $dz_1 \wedge dz_2 \wedge dz_3$ of the torus is invariant under the action of G, and thus descends to a nowhere vanishing top-form on the quotient $X = T/G$. In particular, the canonical divisor of X is trivial. Due to the rigidity, the irregularity q_1 vanishes. Furthermore, the invariance of the volume form implies that the image of the analytic representation belongs to the special linear group $\mathrm{SL}(3, \mathbb{C})$, so all singularities of X are Gorenstein and X admits a crepant resolution $f\colon \hat{X} \to X$, which means $K_{\hat{X}}$ is again trivial.

Definition 4.1.1 ([OS01])**.** A *Calabi-Yau threefold* is a $\mathbb{Q}$-factorial terminal projective three-dimensional variety X (defined over $\mathbb{C}$) such that $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$ and $q_1(X) = h^1(\mathcal{O}_X) = 0$.

Looking at this definition, we see that the resolution $\hat{X}$ is a Calabi-Yau threefold together with a *contraction* $f\colon \hat{X} \to X$, i.e., a surjective holomorphic map with connected fibers onto a normal, projective variety of positive dimension. A Calabi-Yau threefold X together with a contraction $f\colon X \to W$ is called *fibered Calabi-Yau threefold*. Such threefolds were extensively studied by Oguiso at the end of the 20th century (cf. [Ogu93], [Ogu96a], [Ogu96b], [Ogu96c]), and together with Sakurai in [OS01]. Each such contraction $f\colon X \to W$ is defined by the complete linear system of a nef divisor D, which has non-negative intersection $c_2(X) \cdot D \geq 0$ due to a result of Miyaoka-Yau [Miy87]. For the analysis, Oguiso divided the fibered Calabi-Yau threefolds into six classes, according to their values of the Iitaka dimension of D, which is equal to the dimension of W, and the property whether the intersection number $c_2(X) \cdot D$ is 0 or strictly positive. If the intersection number is 0, then Oguiso and Sakurai call the contraction a *c_2-contraction*. In our setup, where f is the crepant resolution of a (rigid) torus quotient with $p_g = 1$, f is a c_2-contraction and D is big, so the Iitaka dimension equals 3. This class is labeled by Oguiso as type III_0. In fact, every fibered Calabi-Yau threefold of type III_0 arises via this construction: Oguiso showed that the three-dimensional base W of the contraction $f\colon X \to W$ has trivial canonical divisor K_W and vanishing second Chern class. A famous result of Shepherd-Barron and Wilson [SBW94] ensures that W is biholomorphic to a quotient of a projective complex torus T by a finite translation-free group G which acts with finite and non-empty fixed locus and preserves the volume form of T. Based on this result, Oguiso and Sakurai proved:

Theorem 4.1.2 ([Ogu96c], [OS01]). *Let (X, f) be a fibered Calabi-Yau threefold of type III_0. Then the pair is isomorphic to the crepant resolution $f\colon X' \to T/G$ of a quotient of a projective complex torus T by a finite translation-free group G which acts with finite and non-empty fixed locus and preserves the volume form of T.*

In particular, the pair (X, f) is uniquely determined by T/G, so classifying fibered Calabi-Yau threefolds of type III_0 is the same as classifying the quotients T/G with the required properties.

Let us point out the following: In our setup, we do not require the torus to be projective and the action to have fixed points, but this follows from the rigidity of the action (cf. [DG22, Theorem 1.1]). On the other hand, rigidity is not required in the above theorem. However, it turns out that the quotients with these properties are all automatically rigid, as we will explain later on.

If the quotient T/G is simply connected, then a fine classification is already established (cf. [RY87], [Roa89], [Ogu96c], [Roa03]):

Theorem 4.1.3. *Let G be a finite group acting holomorphically, without translations and with finite and non-empty fixed locus on an abelian variety T of dimension 3 such that the action preserves the volume form of T. If the quotient $X = T/G$ is simply connected, then G is isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_7$ and the quotient X is biholomorphic to*

$$Z_1 := \mathrm{Jac}(Q)/\langle \mathrm{diag}(\zeta_7,\ \zeta_7^2,\ \zeta_7^4)\rangle \qquad \text{or} \qquad Z_2 := E^3/\langle \zeta_3 \cdot \mathrm{id}\rangle,$$

where $\mathrm{Jac}(Q) = H^0(\Omega^1_Q)^*/H_1(Q)$ denotes the Jacobian of Klein's plane quartic curve

$$Q = \{x_0x_1^3 + x_1x_2^3 + x_2x_0^3 = 0\} \subset \mathbb{P}^2_{\mathbb{C}},$$

and $E = \mathbb{C}/\mathbb{Z}[\zeta_3]$ is Fermat's elliptic curve.

Note that the torus $\mathrm{Jac}(Q)$ has many other descriptions: For example, it is biholomorphic to $E_{u_7}^3$, where $E_{u_7} = \mathbb{C}/\mathbb{Z}[u_7]$ with $u_7 := \zeta_7 + \zeta_7^2 + \zeta_7^4$, or to the torus $\mathbb{C}^3/\Lambda(\zeta_7, \zeta_7^2, \zeta_7^4)$, where $\Lambda(\zeta_7, \zeta_7^2, \zeta_7^4)$ is the lattice with basis $\{(\zeta_7^k, \zeta_7^{2k}, \zeta_7^{4k}) \mid k = 0, \dots, 5\}$. Indeed, by Theorems 2.2.9, 2.2.12 and 2.2.13, any complex torus with automorphism $\mathrm{diag}(\zeta_7,\ \zeta_7^2,\ \zeta_7^4)$ is biholomorphic to $\mathrm{Jac}(Q)$.

Remark 4.1.4. Recently, Gachet [Gac22] generalized Theorem 4.1.3 to higher dimensions: Let T be an abelian variety of dimension $n \geq 3$ and G a finite group acting freely in codimension two on T. If the quotient T/G has a simply connected Calabi-Yau manifold as a crepant resolution, then T is isogenous to E^n or $E_{u_7}^n$, and the group G is generated by those elements that act with fixed points.

In the case that the quotient $X = T/G$ has non-trivial fundamental group, only the isomorphism classes of the groups and the analytic representations are known. In fact, Oguiso and Sakurai proved in [OS01, Thoerem 3.4]:

Theorem 4.1.5. *Let G be a finite group acting holomorphically, without translations, and with finite and non-empty fixed locus on an abelian variety T of dimension 3 such that the action preserves the volume form of T. If the quotient $X = T/G$ is not simply connected, then G is isomorphic to $\mathbb{Z}_3^2$ or $\mathrm{He}(3)$. More precisely, the following holds:*

(1) If $G = \mathbb{Z}_3^2 = \langle h, k\rangle$, then the analytic representation is, up to automorphisms of G, equivalent to

$$\rho(h) = \mathrm{diag}(1,\ \zeta_3^2,\ \zeta_3), \quad \rho(k) = \mathrm{diag}(\zeta_3,\ \zeta_3,\ \zeta_3).$$

(2) If $G = \mathrm{He}(3) = \langle g, h, k \mid g^3 = h^3 = k^3 = [g,k] = [h,k] = 1,\ [g,h] = k\rangle$, then the analytic representation is, up to automorphisms of G, equivalent to

$$\rho(g) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \rho(h) = \begin{pmatrix} 1 & & \\ & \zeta_3^2 & \\ & & \zeta_3 \end{pmatrix}, \qquad \rho(k) = \begin{pmatrix} \zeta_3 & & \\ & \zeta_3 & \\ & & \zeta_3 \end{pmatrix}.$$

Remark 4.1.6. From the description of the analytic representation, it follows directly from Proposition 2.5.8 that all the corresponding quotients are rigid.

Nevertheless, different translations parts may lead to different biholomorphism or even homeomorphism classes of quotients T/G. The fine classification will be settled in this chapter.

If the quotient $X = T/G$ has geometric genus $p_g = 0$, then much less is known. The classification of the groups was the content of Chapter 3. In [BG21], the authors obtained some partial classification results, but they had more restrictive assumptions: They required that the torus in question is a product of three elliptic curves and the action of the group on the product is diagonal and faithful on each factor, which allowed them to use product quotient techniques.

Although we will see later on that all our quotients are abstractly isomorphic to a product of three elliptic curves, namely three copies of either E or E_{u_7}, conjugating the actions with these isomorphisms may lead to non-diagonal actions, which do not fit in the setup of [BG21]. Thus, we need to apply different methods to reach the classification, which we will explain next.

4.2. Classification machinery

In this section, we explain how to use Bieberbach's structure theorems about crystallographic groups (cf. Section 2.4) to decide if two torus quotients, which we view as normal complex spaces in the sense of Cartan (cf. [Car57]), are biholomorphic or homeomorphic. Since this works in arbitrary dimension, we discuss it in the general setting of a complex torus $T = \mathbb{C}^n/\Lambda$ of dimension $n \geq 2$. This generalizes the classification strategy outlined in [DG22], which is similar to [HL21] and where the authors only discussed quotients of tori by free actions.

Remark 4.2.1. Let G be a finite group of biholomorphisms acting on a complex torus $T = \mathbb{C}^n/\Lambda$ without translations, and denote by $\pi\colon \mathbb{C}^n \to T$ the universal cover. Recall that the orbifold fundamental group is defined as

$$\pi_1^{\mathrm{orb}}(T, G) := \{\gamma\colon \mathbb{C}^n \to \mathbb{C}^n \mid \exists\, g \in G \text{ s.t. } \pi \circ \gamma = g \circ \pi\}.$$

In Section 2.4, we collected several properties of $\Gamma := \pi_1^{\mathrm{orb}}(T, G)$. In particular, the orbifold fundamental group is a crystallographic group, that is a discrete and cocompact subgroup of

$\mathbb{E}(2n)$, and fits into the exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

where the lattice Λ of the torus coincides with the subgroup of translations of Γ, and for the quotient $X = T/G$, it holds that

$$X = {}^{T}\!/_{G} = {}^{\mathbb{C}^n}\!/_{\Gamma}.$$

Moreover, if the action of G on T is free in codimension 1, then $\Gamma = \pi_1^{\mathrm{orb}}(T, G)$ is the fundamental group of the regular locus X° of X.

Furthermore, recall the most important properties of crystallographic groups that are summarized in Bieberbach's structure theorems:

Theorem 4.2.2 ([Bie11], [Bie12])**.**

(1) The translation subgroup $\Lambda := \Gamma \cap \mathbb{R}^n$ of a crystallographic group $\Gamma \subset \mathbb{E}(n)$ is a lattice of rank n, and Γ/Λ is finite. All other normal abelian subgroups of Γ are contained in Λ.

(2) Let $\Gamma_1, \Gamma_2 \subset \mathbb{E}(n)$ be two crystallographic groups and $f \colon \Gamma_1 \to \Gamma_2$ be an isomorphism. Then there exists an affine transformation $\alpha \in \mathrm{AGL}(n, \mathbb{R}) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$ such that $f(g) = \alpha \circ g \circ \alpha^{-1}$ for all $g \in \Gamma_1$.

(3) In each dimension, there are only finitely many isomorphism classes of crystallographic groups.

In the following, we explain the geometric consequences of this theorem and how to apply them to classify torus quotients. For this, let $T = \mathbb{C}^n/\Lambda$ and $T' = \mathbb{C}^n/\Lambda'$ be two complex tori of dimension $n \geq 2$ and G and G' finite groups acting without translations and freely in codimension 1 on T and T', respectively. Denote the quotients by X and X' and the orbifold-fundamental groups by Γ and Γ', respectively.

The next lemma gives a link between homeomorphisms of torus quotients and their orbifold fundamental groups. This will allow us to apply Bieberbach's structure theorems to distinguish between the quotients T/G.

Lemma 4.2.3. *Every homeomorphism $f \colon X \to X'$ restricts to a homeomorphism of the smooth loci of X and X'. In particular, it induces an isomorphism of the orbifold fundamental groups $\Gamma \simeq \Gamma'$.*

Proof. The homeomorphism f induces an isomorphism of the local fundamental groups $\pi_1^{\mathrm{loc}}(X, p)$ and $\pi_1^{\mathrm{loc}}(X', f(p))$ for every $p \in X$. Since the actions are free in codimension one, these groups are trivial if and only if the points p and $f(p)$, respectively, are smooth (cf. [Pri67]). Therefore, f maps the smooth locus of X homeomorphically to the smooth locus of X'. □

The structure theorems of Bieberbach ensure that even the converse is true:

Corollary 4.2.4. *The quotients X and X' are homeomorphic if and only if the orbifold fundamental groups Γ and Γ' are isomorphic. In particular, there are only finitely many homeomorphism classes of quotients.*

Proposition 4.2.5. *Let $\Phi\colon G \to \mathrm{Bihol}(T)$ and $\Phi'\colon G' \to \mathrm{Bihol}(T')$ be translation-free holomorphic actions of finite groups G and G' which are free in codimension 1. Assume that the quotients $X = T/G$ and $X' = T'/G'$ are homeomorphic. Then the following holds:*

(1) The groups G and G' are isomorphic.

(2) There exists an affine transformation $\alpha \in \mathrm{AGL}(2n, \mathbb{R})$ inducing diffeomorphisms $\widehat{\alpha}$ and $\widetilde{\alpha}$, such that the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\widetilde{\alpha}} & T' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\widehat{\alpha}} & X'. \end{array}$$

In particular, the quotients X and X' are homeomorphic if and only if they are diffeomorphic.

Furthermore, any biholomorphism $f\colon X \to X'$ lifts to a biholomorphism of the tori, i.e., it is induced by an affine transformation $\alpha \in \mathrm{AGL}(n, \mathbb{C})$.

Proof. Any homeomorphism $f\colon X \to X'$ induces an isomorphism between the orbifold fundamental groups $f_*\colon \Gamma \to \Gamma'$ by Lemma 4.2.3, which is given by conjugation with an affinity α due to Theorem 4.2.2. Since Λ and Λ' are the unique maximal normal abelian subgroups of Γ and Γ', respectively, the isomorphism f_*, and thus α, maps Λ to Λ'. This proves the first and the second assertion of the proposition.
Now, let $f\colon X \to X'$ be a biholomorphic map. Clearly, this map restricts to a biholomorphism between the smooth loci of the quotients, $f\colon X^\circ \to (X')^\circ$, and lifts to the universal covers:

$$\begin{array}{ccc} \mathbb{C}^n \setminus \pi^{-1}(F) & \xrightarrow{\widetilde{f}} & \mathbb{C}^n \setminus (\pi')^{-1}(F') \\ \downarrow & & \downarrow \\ X^\circ & \xrightarrow{\quad f \quad} & (X')^\circ. \end{array}$$

Since $\pi^{-1}(F) \subset \mathbb{C}^n$ is analytic and of codimension at least two, there exists a unique biholomorphic extension $\widetilde{F}\colon \mathbb{C}^n \to \mathbb{C}^n$ of $\widetilde{f}$ by Riemann's second extension theorem (cf. [FG02, Theorem 6.12]). Because of the commutativity of the diagram, we can find for any $\gamma \in \Gamma$ an element $\gamma' \in \Gamma'$ such that $\widetilde{F} \circ \gamma = \gamma' \circ \widetilde{F}$ holds for all $z \in \mathbb{C}^n \setminus \pi^{-1}(F)$, hence, for all $z \in \mathbb{C}^n$ by the identity theorem. Therefore, the biholomorphism $\widetilde{F}$ induces a biholomorphic map $F\colon X \to X'$, that coincides with f. Since conjugation with $\widetilde{F}$ gives an isomorphism between the crystallographic groups Γ and Γ', the biholomorphism $\widetilde{F}$ maps the lattice Λ to Λ' and induces a well-defined map between the tori T and T'. In particular, $\alpha := \widetilde{F}$ is affine linear. $\square$

Corollary 4.2.6. *The quotients X and X' are homeomorphic if and only if they are diffeomorphic.*

As in [DG22, Remarks 4.6 and 4.7], we make use of the following observations and notations.

Remark 4.2.7. Let $f\colon X \to X'$ be a homeomorphism induced by an affine transformation $\alpha(x) = Cx+d$. Then the commutativity of the diagram in Proposition 4.2.5 is equivalent to the existence of an isomorphism $\varphi\colon G \to G'$ such that

$$\text{(a) } C\rho_{\mathbb{R}}(g)C^{-1} = \rho'_{\mathbb{R}}(\varphi(g)) \qquad \text{and} \qquad \text{(b) } (\rho'_{\mathbb{R}}(\varphi(g)) - \mathrm{id})d = C\tau(g) - \tau'(\varphi(g))$$

hold for all $g \in G$, where the second item is an equation holding on T'. Note that $\varphi = \varphi_C$ is uniquely determined by C due to the faithfulness of the analytic representations.
Item (a) means that $\rho_{\mathbb{R}}$ and $\rho'_{\mathbb{R}} \circ \varphi$ are equivalent as real representations (or as complex representations if f is holomorphic). If we consider T and T' as G and G'-modules, then this item implies that the matrix C induces a φ-twisted equivariant module isomorphism $C\colon T \to T'$.
Item (b) tells us that the cocycles τ' and

$$C * \tau := C \cdot (\tau \circ \varphi_C^{-1})$$

differ by a coboundary.

Notation 4.2.8. We define

$$\mathcal{N}_{\mathbb{R}}(\Lambda, \Lambda') := \{C \in \mathrm{GL}(2n, \mathbb{R}) \mid C\Lambda = \Lambda',\ C \cdot \mathrm{im}(\rho_{\mathbb{R}}) = \mathrm{im}(\rho'_{\mathbb{R}}) \cdot C\}$$

and

$$\mathcal{N}_{\mathbb{C}}(\Lambda, \Lambda') := \mathcal{N}_{\mathbb{R}}(\Lambda, \Lambda') \cap \mathrm{GL}(n, \mathbb{C}).$$

In summary, the following holds:

Proposition 4.2.9.

(1) Two quotients $X = T/G$ and $X' = T'/G$ are homeomorphic (biholomorphic) if and only if there exists a matrix $C \in \mathcal{N}_{\mathbb{R}}(\Lambda, \Lambda')$ ($C \in \mathcal{N}_{\mathbb{C}}(\Lambda, \Lambda')$) such that $C * \tau$ and τ' belong to the same cohomology class in $H^1(G, T')$.

(2) If $X = T/G$ and $X' = T'/G$ are homeomorphic, then T and T' are isomorphic as G-modules up to an automorphism of G.

In the special case where $\rho = \rho'$ and $T = T'$, the sets $\mathcal{N}_{\mathbb{R}}(\Lambda, \Lambda)$ and $\mathcal{N}_{\mathbb{C}}(\Lambda, \Lambda)$ are the normalizers of $\mathrm{im}(\rho_{\mathbb{R}})$ in the group of linear diffeomorphisms or biholomorphisms of T. For simplicity, we denote them by $\mathcal{N}_{\mathbb{R}}(\Lambda)$ and $\mathcal{N}_{\mathbb{C}}(\Lambda)$. They act on $H^1(G, T)$ by $C * \tau$. The quotients corresponding to τ and τ' are homeomorphic (or biholomorphic) if and only if they belong to the same orbit under this action.

Using all these results, a classification of the quotients can be achieved performing the following steps:

Strategy for the classification 4.2.10.

(1) For each group G in Theorem 3.0.6, determine all possible representations $\rho\colon G \hookrightarrow \mathrm{GL}(3, \mathbb{C})$ fulfilling the rigidity and integrality condition, up to equivalence of representations and automorphisms of G.

(2) For each group G and each representation ρ, do the following:

   (a) Determine all lattices Λ that have a G-module structure via ρ.

   (b) For each $T = \mathbb{C}^3/\Lambda$, determine all cohomology classes in $H^1(G,T)$ that lead to an action with finite fixed locus and fix a representative τ for each class. We will refer to such classes as *good* cohomology classes.

   (c) Decide which quotients of T by the actions given by ρ and τ are biholomorphic or homeomorphic, respectively, using Proposition 4.2.9.

(3) If there are groups G admitting more than one representation ρ, analyze which biholomorphism and homeomorphism classes coming from different representations coincide.

Remark 4.2.11. The cocycles $C * \tau$ and τ' belong to the same cohomology class in $H^1(G,T')$ if and only if there there exists an element $d \in T'$ such that $(C * \tau - \tau')(g) = \rho'(g)d - d$ for all $g \in G$. Conjugation by the affinity $\alpha(x) = Cx + d$ induces isomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "C"] & \pi_1^{\mathrm{orb}}(T,G) \arrow[r] \arrow[d, "\mathrm{conj}_\alpha"] & G \arrow[r] \arrow[d, "\varphi_C"] & 1 \\
0 \arrow[r] & \Lambda' \arrow[r] & \pi_1^{\mathrm{orb}}(T',G) \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

Conversely, every isomorphism of the orbifold fundamental groups is given by conjugation with an affinity yielding C and d as above.

Remark 4.2.12. By Example 2.3.32, the short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \mathbb{C}^3 \longrightarrow T \longrightarrow 0$$

of G-modules, where G acts via the analytic representation ρ, induces an isomorphism

$$\sigma^1 \colon H^1(T,G) \longrightarrow H^2(T,\Lambda).$$

Thus, the choice of 1-cocycle representing the translation part of an action Φ of G on T yields a class $[\beta] \in H^2(G,\Lambda)$, which defines a group extension

$$0 \longrightarrow \Lambda \longrightarrow \Gamma = (\Lambda \times_\beta G) \longrightarrow G \longrightarrow 1$$

as explained in Theorem 2.3.36. On the other hand, we have the extension

$$0 \longrightarrow \Lambda \longrightarrow \pi_1^{\mathrm{orb}}(T,G) \longrightarrow G \longrightarrow 1, \tag{4.2.1}$$

induced by the action of G on T defined by τ. In fact, these two extensions agree (up to isomorphism): The 2-cocycle β' induced by the section $s(g) \coloneqq \Phi(g)$ of the extension (4.2.1) is the same as $\beta = \sigma^1(\tau)$. To see this, let $g, h \in G$; then

$$\begin{aligned}\beta'(g,h)(z) &= \left(s(g)s(h)s(gh)^{-1}\right)(z)\\ &= \rho(g)\cdot\left(\rho(h)\rho(gh)^{-1}\cdot z - \rho(h)\rho(gh)^{-1}\tau(gh) + \tau(h)\right)\\ &= z - \tau(gh) + \rho(g)\tau(h) + \tau(g)\\ &= z + \sigma^1(\tau)(g,h).\end{aligned}$$

If T' is another torus admitting and action of G with linear part ρ and translation part τ', then by Proposition 4.2.9, the corresponding quotients X and X' are homeomorphic if and only if there exists a matrix $C \in \mathcal{N}_{\mathbb{R}}(\Lambda, \Lambda')$ such that $C * \tau$ and τ' belong to the same cohomology class in $H^1(T', G)$. Analogous to $C * \tau$, we can define $C * \beta$ via

$$C * \beta(g,h) \coloneqq C \cdot \beta(\varphi_C^{-1}(g), \varphi_C^{-1}(h))$$

for $\beta \in Z^2(G, \Lambda)$. Since the connecting homomorphism $\sigma^1 \colon H^1(G,T) \to H^2(G,\Lambda)$ is compatible with this action of C, we conclude:
The quotients X and X' are homeomorphic if and only if the corresponding short exact sequences

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \quad \text{and} \quad 0 \longrightarrow \Lambda' \longrightarrow \Gamma' \longrightarrow G \longrightarrow 1$$

are isomorphic.
The same is true if we replace "homeomorphic" by "biholomorphic", but then the matrix C has to be $\mathbb{C}$-linear and not only $\mathbb{R}$-linear.
In view of this discussion, our strategy for the classification 4.2.10 is similar to the scheme for classifying Bieberbach groups outlined in [Cha86, Chapter III, Section 2]. The differences are that in our setup, actions with fixed points are allowed, so we have to replace "Bieberbach groups" by "crystallographic groups", and for the biholomorphic classification of the quotients, we have to refine the equivalence relation for crystallographic groups to conjugation by holomorphic affinities, i.e., affinities with $\mathbb{C}$-linear matrices.

4.3. Fine Classification

In this section, we finally classify all rigid quotients of three-dimensional complex tori with isolated canonical singularities. By Theorem 3.0.6, the possible Galois groups are:

$$\mathbb{Z}_3, \quad \mathbb{Z}_7, \quad \mathbb{Z}_9, \quad \mathbb{Z}_{14}, \quad \mathbb{Z}_3^2, \quad \mathbb{Z}_3^3, \quad \mathrm{He}(3), \quad \text{and} \quad \mathbb{Z}_9 \rtimes \mathbb{Z}_3.$$

The goal of the section is to prove the main theorem:

Theorem 4.3.1. *Let G be a finite group admitting a rigid, holomorphic, and translation-free action on a three-dimensional complex torus T with finite fixed locus and such that the quotient $X = T/G$ has canonical singularities.*

(1) If $p_g(X) = 1$, then there are precisely 8 biholomorphism classes of quotients, which are pairwise topologically distinct. Table 4.1 contains precisely one representative Z_i for each class.

(2) If $p_g(X) = 0$, then there are precisely 13 biholomorphism classes of quotients. Table 4.2 contains precisely one representative Y_i of each class. These 13 threefolds form 11 homeomorphism classes, $Y_4 \simeq_{\text{homeo}} Y_{4'}$ and $Y_{10} \simeq_{\text{homeo}} Y_{10'}$. Explicit explicit homeomorphisms are given by

$$\begin{aligned} Y_4 &\longrightarrow Y_{4'}, \quad (z_1,\ z_2,\ z_3) \mapsto (-z_1,\ \overline{z_3},\ \overline{z_2}), \\ Y_{10} &\longrightarrow Y_{10'}, \quad (z_1,\ z_2,\ z_3) \mapsto (\overline{z_1},\ \overline{z_2},\ \overline{z_3}). \end{aligned}$$

In total, the quotients $X = T/G$ form 21 biholomorphism classes and 15 homeomorphism classes

$$\begin{gathered} Z_1, \quad Z_2, \quad Z_7, \quad Z_8, \quad Y_1, \quad Y_2, \quad Y_7, \quad Y_8, \quad Y_9, \quad Y_{11} \\ Z_3 \simeq_{\text{homeo}} Y_3, \quad Z_4 \simeq_{\text{homeo}} Y_4 \simeq_{\text{homeo}} Y_{4'}, \quad Z_5 \simeq_{\text{homeo}} Y_5, \quad Z_6 \simeq_{\text{homeo}} Y_6, \quad Y_{10} \simeq_{\text{homeo}} Y_{10'}. \end{gathered}$$

Explicit homeomorphisms $Z_k \to Y_k$ for $k = 3, \ldots, 6$ are given by $(z_1,\ z_2,\ z_3) \mapsto (z_1,\ z_2,\ -\overline{z_3})$. Moreover, the diffeomorphism and homeomorphism classes coincide.

i	G	Λ	action	singularities	$\pi_1(Z_i)$
1	$\mathbb{Z}_7$	$\Lambda(\zeta_7, \zeta_7^2, \zeta_7^4)$	$\Phi(1)(z) = \operatorname{diag}(\zeta_7, \zeta_7^2, \zeta_7^4) \cdot z$	$7 \times \frac{1}{7}(1,2,4)$	$\{1\}$
2	$\mathbb{Z}_3$	$\mathbb{Z}[\zeta_3]^3$	$\Phi(1)(z) = \operatorname{diag}(\zeta_3, \zeta_3, \zeta_3) \cdot z$	$27 \times \frac{1}{3}(1,1,1)$	$\{1\}$
3	$\mathbb{Z}_3^2$	$\mathbb{Z}[\zeta_3]^3$	$\Phi(h)(z) = \operatorname{diag}(1, \zeta_3^2, \zeta_3) \cdot z + (t,t,t)$ $\Phi(k)(z) = \operatorname{diag}(\zeta_3, \zeta_3, \zeta_3) \cdot z$	$9 \times \frac{1}{3}(1,1,1)$	$\mathbb{Z}_3$
4	$\mathbb{Z}_3^2$	$\mathbb{Z}[\zeta_3]^3 + \mathbb{Z}(t,t,0)$	$\Phi(h)(z) = \operatorname{diag}(1, \zeta_3^2, \zeta_3) \cdot z + \frac{1}{3}(1,1,3t)$ $\Phi(k)(z) = \operatorname{diag}(\zeta_3, \zeta_3, \zeta_3) \cdot z$	$9 \times \frac{1}{3}(1,1,1)$	$\mathbb{Z}_3$
5	$\mathbb{Z}_3^2$	$\mathbb{Z}[\zeta_3]^3 + \mathbb{Z}(t,t,t)$	$\Phi(h)(z) = \operatorname{diag}(1, \zeta_3^2, \zeta_3) \cdot z + \frac{1}{3}(1,1,1)$ $\Phi(k)(z) = \operatorname{diag}(\zeta_3, \zeta_3, \zeta_3) \cdot z$	$9 \times \frac{1}{3}(1,1,1)$	$\mathbb{Z}_3$
6	$\mathbb{Z}_3^2$	$\mathbb{Z}[\zeta_3]^3 + \mathbb{Z}(t,t,t) + \mathbb{Z}(t,-t,0)$	$\Phi(h)(z) = \operatorname{diag}(1, \zeta_3^2, \zeta_3) \cdot z + \frac{1}{3}(1,1,1)$ $\Phi(k)(z) = \operatorname{diag}(\zeta_3, \zeta_3, \zeta_3) \cdot z$	$9 \times \frac{1}{3}(1,1,1)$	$\mathbb{Z}_3$
7	He(3)	$\mathbb{Z}[\zeta_3]^3 + \mathbb{Z}(t,t,t)$	$\Phi(g)(z) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot z + (t,0,0)$ $\Phi(h)(z) = \operatorname{diag}(1, \zeta_3^2, \zeta_3) \cdot z + \frac{2}{3}(1,1,1)$	$3 \times \frac{1}{3}(1,1,1)$	$\mathbb{Z}_3^2$
8	He(3)	$\mathbb{Z}[\zeta_3]^3 + \mathbb{Z}(t,t,t) + \mathbb{Z}(t,-t,0)$	$\Phi(g)(z) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot z + (t,0,0)$ $\Phi(h)(z) = \operatorname{diag}(1, \zeta_3^2, \zeta_3) \cdot z + \frac{2}{3}(1,1,1)$	$3 \times \frac{1}{3}(1,1,1)$	$\mathbb{Z}_3^2$

Table 4.1.: Calabi-Yau quotients. In the table, $t := (1 + 2\zeta_3)/3$ and $\Lambda(\zeta_7, \zeta_7^2, \zeta_7^4)$ has the basis $\{(\zeta_7^k, \zeta_7^{2k}, \zeta_7^{4k}) \mid k = 0, \ldots, 5\}$.

We want to point out that the quotients Z_1 and Z_2 are precisely the ones described in Theorem 4.1.3. Furthermore, the quotients $Y_3,\ Y_5,\ Y_7$, and Y_8 are those already found in [BG21] using product quotient techniques.

i	G	Λ	action	singularities	$\pi_1(Y_i)$
1	$\mathbb{Z}_9$	$\Lambda(\zeta_9, \zeta_9^4, \zeta_9^7)$	$\Phi(1)(z) = \mathrm{diag}(\zeta_9, \zeta_9^4, \zeta_9^7) \cdot z$	$8 \times \frac{1}{3}(1,1,1)$ $3 \times \frac{1}{9}(1,4,7)$	$\{1\}$
2	$\mathbb{Z}_{14}$	$\Lambda(\zeta_{14}, \zeta_{14}^9, \zeta_{14}^{11})$	$\Phi(1)(z) = \mathrm{diag}(\zeta_{14}, \zeta_{14}^9, \zeta_{14}^{11}) \cdot z$	$9 \times \frac{1}{2}(1,1,1)$ $3 \times \frac{1}{7}(1,2,4)$ $1 \times \frac{1}{14}(1,9,11)$	$\{1\}$
3	$\mathbb{Z}_3^2$	$\mathbb{Z}[\zeta_3]^3$	$\Phi(h)(z) = \mathrm{diag}(1, \zeta_3^2, \zeta_3^2) \cdot z + (t,t,t)$ $\Phi(k)(z) = \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3^2) \cdot z$	$9 \times \frac{1}{3}(1,1,2)$	$\mathbb{Z}_3$
4	$\mathbb{Z}_3^2$	$\mathbb{Z}[\zeta_3]^3 + \mathbb{Z}(t,t,0)$	$\Phi(h)(z) = \mathrm{diag}(1, \zeta_3^2, \zeta_3^2) \cdot z + \frac{1}{3}(1,1,3t)$ $\Phi(k)(z) = \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3^2) \cdot z$	$9 \times \frac{1}{3}(1,1,2)$	$\mathbb{Z}_3$
4'	$\mathbb{Z}_3^2$	$\mathbb{Z}[\zeta_3]^3 + \mathbb{Z}(t,0,t)$	$\Phi(h)(z) = \mathrm{diag}(1, \zeta_3^2, \zeta_3^2) \cdot z + \frac{1}{3}(1,3t,2\zeta_3^2)$ $\Phi(k)(z) = \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3^2) \cdot z$	$9 \times \frac{1}{3}(1,1,2)$	$\mathbb{Z}_3$
5	$\mathbb{Z}_3^2$	$\mathbb{Z}[\zeta_3]^3 + \mathbb{Z}(t,t,t)$	$\Phi(h)(z) = \mathrm{diag}(1, \zeta_3^2, \zeta_3^2) \cdot z + \frac{1}{3}(1,1,2)$ $\Phi(k)(z) = \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3^2) \cdot z$	$9 \times \frac{1}{3}(1,1,2)$	$\mathbb{Z}_3$
6	$\mathbb{Z}_3^2$	$\mathbb{Z}[\zeta_3]^3 + \mathbb{Z}(t,t,t) + \mathbb{Z}(t,-t,0)$	$\Phi(h)(z) = \mathrm{diag}(1, \zeta_3^2, \zeta_3^2) \cdot z + \frac{1}{3}(1,1,2)$ $\Phi(k)(z) = \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3^2) \cdot z$	$9 \times \frac{1}{3}(1,1,2)$	$\mathbb{Z}_3$
7	$\mathbb{Z}_3^2$	$\mathbb{Z}[\zeta_3]^3$	$\Phi(h)(z) = \mathrm{diag}(\zeta_3, \zeta_3, 1) \cdot z + (t,t,t)$ $\Phi(k)(z) = \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3^2) \cdot z$	$9 \times \frac{1}{3}(1,1,1)$ $9 \times \frac{1}{3}(1,1,2)$	$\{1\}$
8	$\mathbb{Z}_3^2$	$\mathbb{Z}[\zeta_3]^3 + \mathbb{Z}(t,t,t)$	$\Phi(h)(z) = \mathrm{diag}(\zeta_3, \zeta_3, 1) \cdot z + \frac{1}{3}(1,1,1)$ $\Phi(k)(z) = \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3^2) \cdot z$	$9 \times \frac{1}{3}(1,1,1)$ $9 \times \frac{1}{3}(1,1,2)$	$\{1\}$
9	$\mathbb{Z}_3^3$	$\mathbb{Z}[\zeta_3]^3$	$\Phi(h)(z) = \mathrm{diag}(1, \zeta_3^2, \zeta_3) \cdot z + (-t,-t,t)$ $\Phi(g)(z) = \mathrm{diag}(\zeta_3, 1, 1) \cdot z + (-t,0,-t)$ $\Phi(k)(z) = \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3) \cdot z$	$3 \times \frac{1}{3}(1,1,1)$ $9 \times \frac{1}{3}(1,1,2)$	$\{1\}$
10	$\mathbb{Z}_3^3$	$\mathbb{Z}[\zeta_3]^3 + \mathbb{Z}(t,t,0)$	$\Phi(h)(z) = \mathrm{diag}(1, \zeta_3^2, \zeta_3) \cdot z + \frac{1}{3}(-\zeta_3^2, 2, 3t)$ $\Phi(g)(z) = \mathrm{diag}(\zeta_3, 1, 1) \cdot z + \frac{1}{3}(-\zeta_3^2, 2\zeta_3, 0)$ $\Phi(k)(z) = \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3) \cdot z$	$3 \times \frac{1}{3}(1,1,1)$ $9 \times \frac{1}{3}(1,1,2)$	$\{1\}$
10'	$\mathbb{Z}_3^3$	$\mathbb{Z}[\zeta_3]^3 + \mathbb{Z}(t,t,0)$	$\Phi(h)(z) = \mathrm{diag}(1, \zeta_3^2, \zeta_3) \cdot z + \frac{1}{3}(\zeta_3, \zeta_3^2, 3t)$ $\Phi(g)(z) = \mathrm{diag}(\zeta_3, 1, 1) \cdot z + \frac{1}{3}(\zeta_3^2, \zeta_3^2, 0)$ $\Phi(k)(z) = \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3) \cdot z$	$3 \times \frac{1}{3}(1,1,1)$ $9 \times \frac{1}{3}(1,1,2)$	$\{1\}$
11	$\mathbb{Z}_9 \rtimes \mathbb{Z}_3$	$\mathbb{Z}[\zeta_3]^3$	$\Phi(h)(z) = \mathrm{diag}(1, \zeta_3^2, \zeta_3) \cdot z + (t,t,t)$ $\Phi(g)(z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta_3 & 0 & 0 \end{pmatrix} \cdot z$	$2 \times \frac{1}{3}(1,1,1)$ $3 \times \frac{1}{9}(1,4,7)$	$\{1\}$

Table 4.2.: Quotients with $p_g = 0$. In the table, $t := (1+2\zeta_3)/3$, $\Lambda(\zeta_9, \zeta_9^4, \zeta_9^7)$ has the basis $\{(\zeta_9^k, \zeta_9^{4k}, \zeta_9^{7k}) \mid \gcd(k,9) = 1\}$ and $\Lambda(\zeta_{14}, \zeta_{14}^9, \zeta_{14}^{11})$ has the basis $\{(\zeta_{14}^k, \zeta_{14}^{9k}, \zeta_{14}^{11k}) \mid \gcd(k,14) = 1\}$.

4.3.1. Quotients with respect to cyclic groups

We start with the classification of the quotients by cyclic groups. Here, the situation is quite clear:

Proposition 4.3.2. *For each of the groups $G = \mathbb{Z}_3,\ \mathbb{Z}_7,\ \mathbb{Z}_9$, and $\mathbb{Z}_{14}$, there exists up to biholomorphism one and only one quotient $X = T/G$. They are represented by Z_1 and Z_2 of Table 4.1, and Y_1 and Y_2 of Table 4.2, respectively.*

Proof. Moving the origin, we can assume that G acts linearly with generators

$$\mathrm{diag}(\zeta_3,\ \zeta_3,\ \zeta_3),\quad \mathrm{diag}(\zeta_7,\ \zeta_7^2,\ \zeta_7^4),\quad \mathrm{diag}(\zeta_9,\ \zeta_9^4,\ \zeta_9^7),\quad \text{or}\quad \mathrm{diag}(\zeta_{14},\ \zeta_{14}^9,\ \zeta_{14}^{11}),$$

respectively (cf. Theorem 3.0.5). This implies that T is of CM-type in each case (cf. Theorem 2.2.9) and uniquely determined due to Theorems 2.2.12 and 2.2.13. □

Note that the classification for the groups $\mathbb{Z}_3$ and $\mathbb{Z}_7$ was already discussed by several authors, e.g., [RY87], [Roa89], [Ogu96c].

4.3.2. Quotients with respect to non-cyclic groups

For the non-cyclic groups, the situation is more involved because it is not possible to assume that the action is linear. Here, we use the "classification machinery" as explained in Section 4.2. We start with step (1) of the classification strategy 4.2.10 and determine the possible analytic representations of the groups.

Proposition 4.3.3. *The analytic representation $\rho\colon G \hookrightarrow \mathrm{GL}(3,\mathbb{C})$ of a rigid, faithful, and translation-free action on a three-dimensional complex torus by a non-cyclic group is, up to an automorphism of G, equivalent to the following representations:*

(1) If $G = \mathbb{Z}_3^2$, then ρ is equivalent to

$$\rho_1(a,b) = \mathrm{diag}(\zeta_3^a,\ \zeta_3^b,\ \zeta_3^{2a+2b}),\ \rho_2(a,b) = \mathrm{diag}(\zeta_3^a,\ \zeta_3^b,\ \zeta_3^{a+b}),\ \textit{or}\ \rho_3(a,b) = \mathrm{diag}(\zeta_3^a,\ \zeta_3^a,\ \zeta_3^b).$$

(2) If $G = \mathbb{Z}_3^3$, then

$$\rho(a,b,c) = \mathrm{diag}(\zeta_3^a,\ \zeta_3^b,\ \zeta_3^c).$$

(3) If $G = \mathrm{He}(3) = \langle g,h,k \mid g^3 = h^3 = k^3 = [g,k] = [h,k] = 1,\ [g,h] = k\rangle$, then

$$\rho(g) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\quad \rho(h) = \begin{pmatrix} 1 & & \\ & \zeta_3^2 & \\ & & \zeta_3 \end{pmatrix},\quad \textit{and}\quad \rho(k) = \begin{pmatrix} \zeta_3 & & \\ & \zeta_3 & \\ & & \zeta_3 \end{pmatrix}.$$

(4) If $G = \mathbb{Z}_9 \rtimes \mathbb{Z}_3 = \langle g,h \mid h^3 = g^9 = 1,\ hgh^{-1} = g^4\rangle$, then

$$\rho(g) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta_3 & 0 & 0 \end{pmatrix} \qquad \textit{and} \qquad \rho(h) = \begin{pmatrix} 1 & & \\ & \zeta_3^2 & \\ & & \zeta_3 \end{pmatrix}.$$

Proof. We start with the group $G = \mathbb{Z}_3^2$. Let $\rho\colon \mathbb{Z}_3^2 \hookrightarrow \mathrm{GL}(3,\mathbb{C})$ be a representation with the properties in the proposition. Write $\rho = \mathrm{diag}(\chi_1,\chi_2,\chi_3)$ with characters χ_j of degree 1, which is possible since $\mathbb{Z}_3^2$ is abelian. Since ρ is faithful and the action is rigid, we can assume that χ_1 and χ_2 are linearly independent in the group of characters. In other words

$$\rho(a,b) = \mathrm{diag}(\zeta_3^a,\ \zeta_3^b,\ \zeta_3^{\lambda a+\mu b}) \qquad \text{for some } \lambda,\mu \in \{0,1,2\}.$$

Due to the rigidity, it holds that ρ equals either $\rho_1,\ \rho_2,\ \rho_3$, or $\rho_4(a,b) = \mathrm{diag}(\zeta_3^a,\ \zeta_3^b,\ \zeta_3^{a+2b})$, up to a permutation of coordinates. Twisting ρ_2 by the automorphism $(a,b) \mapsto (a, 2a+b)$ of $\mathbb{Z}_3^2$ gives a representation equivalent to ρ_4.
If $G = \mathbb{Z}_3^3$, then the three characters have to be linearly independent due to the faithfulness of ρ. Thus, the claim follows.
In the case $G = \mathrm{He}(3)$, the geometric genus of the quotient is always equal to 1 and in this situation, the shape of the analytic representation was already proven in [OS01, Theorem 3.4], see also Theorem 4.1.5.
Finally, if $G = \mathbb{Z}_9 \rtimes \mathbb{Z}_3$, then the analytic representation has to be irreducible of degree 3 since the group is not abelian and does not admit irreducible characters of degree 2. Up to equivalence of representations, the only irreducible representations of $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ are given by the one in the proposition and its complex conjugate, which coincide by twisting $\overline{\rho}$ with the automorphism φ given by $\varphi(g) = g^2$ and $\varphi(h) = h$. □

Notation 4.3.4. From now on we fix the analytic representations given in the above proposition and will work with the following generators of the abelian groups:

G	$\mathbb{Z}_3^2,\ \rho = \rho_1$ or ρ_2	$\mathbb{Z}_3^2,\ \rho = \rho_3$	$\mathbb{Z}_3^3$
generators	$h := (0,2)$	$h := (1,0)$	$h := (0,2,1)$
	$k := (1,1)$	$k := (1,1)$	$k := (1,1,1)$
			$g := (1,0,0)$

Furthermore, in the case $G = \mathbb{Z}_9 \rtimes \mathbb{Z}_3$, we set $k := g^3$, so $\rho(k) = \zeta_3 \cdot \mathrm{id}$.

We observe the following:

- If $G = \mathbb{Z}_3^2$, then ρ_1 and ρ_2 are the sum of three pairwise different characters, whereas ρ_3 contains the same character twice.
- The representations of $\mathbb{Z}_3^3$ and $\mathrm{He}(3)$ restricted to the subgroup $\langle h,\ k\rangle$ coincide with the representation ρ_1 of $\mathbb{Z}_3^2$.
- Quotients by actions of the group $G = \mathbb{Z}_3^2$ can have either geometric genus 1 or 0, depending on whether the top form $dz_1 \wedge dz_2 \wedge dz_3$ of T is G-invariant. The latter is the case if and only if $\rho = \rho_1$. For the other two representations, the quotients have $p_g = 0$.

The next step (2a) in the classification strategy is to determine for all groups G all lattices Λ admitting a G-module structure with respect to the analytic representations. For this, the next lemma is crucial:

Lemma 4.3.5. *Let T be a three-dimensional torus admitting an action of $G = \mathbb{Z}_3^2,\ \mathbb{Z}_3^3,\ \mathrm{He}(3)$, or $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ with analytic representation as in Proposition 4.3.3. Then:*

(1) If $G = \mathbb{Z}_3^2$ and $\rho = \rho_3$, then T admits subtori $T',\ E_3 \subset T$ such that the addition map

$$\mu \colon T' \times E_3 \longrightarrow T$$

defines an equivariant isogeny, $E_3 \simeq E = \mathbb{C}/\mathbb{Z}[\zeta_3]$, and T' is equivariantly isomorphic to E^2.

(2) *In the other situations, T admits subtori $E_1,\ E_2,\ E_3 \subset T$, all isomorphic to E, such that the addition map*

$$\mu \colon E_1 \times E_2 \times E_3 \longrightarrow T$$

defines an equivariant isogeny.

In particular, T is equivariantly isomorphic to E^3/K, where K is the finite kernel of the addition map.

Proof. (1) Consider the subtori $T' := \ker(\rho_3(h^2k) - \mathrm{id}_T)^0$ and $E_3 := \ker(\rho_3(h) - \mathrm{id}_T)^0$, where the superscript 0 denotes the connected component of the identity. Then the addition map

$$\mu \colon T' \times E_3 \longrightarrow T$$

defines an equivariant isogeny, where the action of $\mathbb{Z}_3^2$ on $T' \times E$ is the natural one induced by the action of $\mathbb{Z}_3^2$ on T. Since ζ_3 acts on E_3, this curve is isomorphic to $E = \mathbb{C}/\mathbb{Z}[\zeta_3]$. The action of ρ restricted to T' is given by $\zeta_3^a \cdot \mathrm{id}_{T'}$; hence, T' is equivariantly isomorphic to E^2 (cf. [BL04, Corollary 13.3.5]).
(2) Consider the subtori

$$E_1 := \ker(\rho(h) - \mathrm{id}_T)^0, \quad E_2 := \ker(\rho(hk) - \mathrm{id}_T)^0, \quad E_3 := \ker(\rho(hk^2) - \mathrm{id}_T)^0,$$

which are all isomorphic to E by construction. It can be shown similarly to the first part of the proof that the addition map $\mu \colon E_1 \times E_2 \times E_3 \to T$ is an equivariant isogeny. □

Remark 4.3.6. We will assume from now on that T is of the form E^3/K, where K is finite, thus, $T = \mathbb{C}^3/\Lambda_K$, where

$$\Lambda_K = \mathbb{Z}[\zeta_3]^3 + K.$$

Since the maps

$$E_i \hookrightarrow T = (E_1 \times E_2 \times E_3)/K = E^3/K,$$

for $i = 1, 2, 3$, and

$$E_3 \hookrightarrow T = (T' \times E_3)/K = E^3/K, \quad T' \hookrightarrow T = (T' \times E_3)/K = E^3/K,$$

are injective, the kernel K does not contain non-zero multiples of unit vectors e_j. In the case that $G = \mathbb{Z}_3^2$ and $\rho = \rho_3$, the kernel K does not even contain non-zero elements of the form $\lambda e_1 + \mu e_2$. Furthermore, K is fixed by $\rho(u)$ for all $u \in G$.

Lemma 4.3.7. *In the second case of Lemma 4.3.5, it holds that $E[3]$ is not contained in $p_i(K)$ for all $i = 1, 2, 3$, where $p_i \colon K \to E_i$ denotes the projection onto the i-th factor.*

Proof. Since the action Φ is a group homomorphism, it has to satisfy $\Phi(hk^n) = \mathrm{id}_T$ for $n = 0, 1, 2$, which is equivalent to requiring $\tau(hk^n) = 0$. This is fulfilled if and only if the coordinates of $\tau(h) = (a_1,\ a_2,\ a_3)$ are all 3-torsion points of E. Assume now that $E[3] \subset p_i(K)$ for some i. Then we can find an element in K whose i-th coordinate equals a_i. Thus, the element h (for

$i = 1$), hk (for $i = 2$), or hk^2 (for $i = 3$) has fixed points, respectively: For example, $\Phi(h)$ has $z = (z_1, z_2, z_3) \in T$ as fixed point if and only if

$$(a_1,\ (\zeta_3^2 - 1)z_2 + a_2,\ (\zeta_3 - 1)z_3 + a_3) \in K,$$

which is equivalent to requiring that K contains an element of the form $(a_1, *, *)$.
But this is a contradiction, since these fixed points are not isolated because the linear parts of the actions of these elements have the eigenvalue 1. □

Lemma 4.3.8. *The kernel K is contained in* $\mathrm{Fix}_{\zeta_3}(E)^3$.

Proof. For $u \in G$, we view $\rho(u)$ as an automorphism of E^3 mapping K to itself. Let (t_1, t_2, t_3) be an element of K.
We first consider the case that $G = \mathbb{Z}_3^2$ and $\rho = \rho_3$. Then, for $u = h$ and h^2k, we obtain that the elements

$$\begin{aligned}(\rho(h) - \mathrm{id}_T)(t) &= ((\zeta_3 - 1)t_1,\ (\zeta_3 - 1)t_2,\ 0) \qquad \text{and}\\ (\rho(h^2k) - \mathrm{id}_T)(t) &= (0,\ 0,\ (\zeta_3 - 1)t_3)\end{aligned}$$

belong to K. This implies that $((\zeta_3 - 1)t_1, (\zeta_3 - 1)t_2) = 0$ in $T' = E^2$, and $(\zeta_3 - 1)t_3 = 0$ in $E_3 = E$. Thus, $(t_1, t_2, t_3) \in \mathrm{Fix}_{\zeta_3}(E)^3$ by Remark 4.3.6.
In all other cases, we first show that each coordinate t_i is a 3-torsion point of $E_i = E$: This follows from

$$\begin{aligned}(\rho(hk) - \mathrm{id}_{E^3}) \circ (\rho(hk^2) - \mathrm{id}_{E^3})(t_1,\ t_2,\ t_3) &= 3t_1 \cdot (1,\ 0,\ 0),\\ (\rho(h) - \mathrm{id}_{E^3}) \circ (\rho(hk^2) - \mathrm{id}_{E^3})(t_1,\ t_2,\ t_3) &= 3t_2 \cdot (0,\ 1,\ 0),\\ (\rho(h) - \mathrm{id}_{E^3}) \circ (\rho(hk) - \mathrm{id}_{E^3})(t_1,\ t_2,\ t_3) &= 3t_3 \cdot (0,\ 0,\ 1)\end{aligned}$$

since K contains no non-trivial multiples of unit vectors. Assume now that there is an element in K that has one coordinate t_i that is not fixed by multiplication with ζ_3. Hence, $t_i \neq \zeta_3 t_i$ are two linearly independent elements in $E[3] \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$, and so, they span the whole group of 3-torsion points. This would imply that $p_i(K) = E[3]$ – a contradiction to Lemma 4.3.7. □

Remark 4.3.9. The fixed locus of multiplication with ζ_3 on E is given by

$$\mathrm{Fix}_{\zeta_3}(E) = \{0, t, -t\} \simeq \mathbb{Z}_3, \qquad \text{where} \qquad t := \tfrac{1}{3}(1 + 2\zeta_3).$$

Corollary 4.3.10. *If $G = \mathrm{He}(3)$ or $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$, then the kernel K is one of the following:*

$$K_0 := \{0\}, \quad K_1 := \langle (t,\ t,\ t) \rangle, \quad K_2 := \langle (t,\ t,\ t),\ (t,\ -t,\ 0) \rangle.$$

Proof. These three subspaces of $\mathrm{Fix}_{\zeta_3}(E)^3$ are the only ones that are fixed under the action of $\rho(g)$ and do not contain non-zero multiples of unit-vectors. □

If G is abelian, then all subspaces of $\mathrm{Fix}_{\zeta_3}(E)^3$ are preserved under ρ, so the list of possible kernels is quite long. We show next that the normalizer $\mathcal{N} := N_{\mathrm{Aut}(E^3)}(\rho(G))$ of $\rho(G)$ in the automorphism group of E^3 acts on the set of kernels and that it is enough to consider one representative of each orbit.

Proposition 4.3.11. *Let G be either $\mathbb{Z}_3^2$ or $\mathbb{Z}_3^3$, $\rho\colon G \hookrightarrow \mathrm{GL}(3,\mathbb{C})$ one of the corresponding representations, K and K' two kernels, Φ and Φ' two actions with linear part ρ, and X and X' the corresponding quotients, respectively. Then any biholomorphism $f\colon X \to X'$ is induced by a biholomorphic map*

$$\hat{f}\colon E^3 \longrightarrow E^3, \quad z \mapsto Cz + d,$$

such that $CK = K'$. This means that C is contained in the normalizer $\mathcal{N} := N_{\mathrm{Aut}(E^3)}(\rho(G))$ of $\rho(G)$ in the automorphism group of E^3.

Proof. By Proposition 4.2.5, the map f is induced by an affine linear map $\hat{f}(z) = Cz + d$, where $C \in \mathrm{GL}(3,\mathbb{C})$ solves the equation $C \cdot \rho \cdot C^{-1} = \rho \circ \varphi$ for some $\varphi \in \mathrm{Aut}(G)$ and $C \cdot \Lambda_K = \Lambda_{K'}$. Assume for the moment that we are not in the case $\rho = \rho_3$ for $G = \mathbb{Z}_3^2$, then ρ is the sum of three distinct one-dimensional representations. Hence, Schur's Lemma implies that C is the product of a permutation matrix P_C and a diagonal matrix D_C. The condition $C\Lambda_K = \Lambda_{K'}$ then ensures that the diagonal entries of D_C belong to $\mathbb{Z}[\zeta_3]^* = \langle -\zeta_3 \rangle$: The element $Ce_i \in \Lambda_{K'}$ is a non-zero multiple of a unit vector that cannot belong to K', hence its non-zero element belongs to $\mathbb{Z}[\zeta_3]$, so $C \in \mathrm{Mat}(3 \times 3, \mathbb{Z}[\zeta_3])$. Analogously, $C^{-1} \in \mathrm{Mat}(3 \times 3, \mathbb{Z}[\zeta_3])$. Thus, C is well-defined as an automorphism of E^3 and belongs to $N_{\mathrm{Aut}(E^3)}(\rho(G))$.
If $G = \mathbb{Z}_3^2$ and $\rho = \rho_3$, then the first two one-dimensional characters are the same, whereas the third one differs. Hence, again by Schur's Lemma, C is of block diagonal form

$$C = \begin{pmatrix} C' & 0 \\ 0 & c \end{pmatrix}$$

with $C' \in \mathrm{GL}(2,\mathbb{C})$ and $c \in \mathbb{C}^*$. Since the kernels K and K' do not even contain non-zero elements of the form $\lambda e_1 + \mu e_2$, the claim follows as before. □

Remark 4.3.12. For $G = \mathbb{Z}_3^2$ or $\mathbb{Z}_3^3$, the normalizer $\mathcal{N} := N_{\mathrm{Aut}(E^3)}(\rho(G))$ acts on the set of possible kernels $\mathcal{K}$ by Proposition 4.3.11. The proposition tells us that quotients corresponding to kernels of different orbits cannot be biholomorphic. On the other hand, it suffices to consider one representative of each orbit since the sets of possible actions on two abelian varieties defined by kernels in the same orbit are conjugate and therefore lead to the same biholomorphism classes of quotients. Finally, to determine one representative of each orbit, we have to compute the normalizer $\mathcal{N} = N_{\mathrm{Aut}(E^3)}(\rho(G))$, which we will do separately for each group and representation in the following.

The previous proposition immediately leads to the following description of the sets $\mathcal{N}_{\mathbb{C}}(\Lambda_K, \Lambda_{K'})$ of possible linear parts of biholomorphisms between quotients for the groups $G = \mathbb{Z}_3^2$ and $\mathbb{Z}_3^3$:

Corollary 4.3.13. *If $G = \mathbb{Z}_3^2$ or $\mathbb{Z}_3^3$, then*

$$\mathcal{N}_{\mathbb{C}}(\Lambda_K, \Lambda_{K'}) = \{C \in \mathcal{N} = N_{\mathrm{Aut}(E^3)}(\rho(G)) \mid CK = K'\}.$$

Notation 4.3.14. After possibly conjugating the action with a translation (which does not change the linear part), the translation part $\tau\colon G \to T$ can be written as follows:

G	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^3$ or $\mathrm{He}(3)$	$\mathbb{Z}_9 \rtimes \mathbb{Z}_3$
τ	$\tau(h) = (a_1,\ a_2,\ a_3)$ $\tau(k) = (0,\ 0,\ 0)$	$\tau(h) = (a_1,\ a_2,\ a_3)$ $\tau(k) = (0,\ 0,\ 0)$ $\tau(g) = (b_1,\ b_2,\ b_3)$	$\tau(h) = (a_1,\ a_2,\ a_3)$ $\tau(g) = (0,\ 0,\ 0)$

We will refer to such a translation part as a *cocycle in standard form.*
Furthermore, we call a cocycle *good* if the corresponding action has only isolated fixed points.

With these conventions, we can determine a finite list of all potential translation parts of biholomorphisms for all groups.

Lemma 4.3.15. *Let X and X' be two quotients of $T = E^3/K$ corresponding to good cocycles of G in standard form. Then the translation part of a biholomorphism $f\colon X \to X'$ is one of the 27 fixed points of $\rho(k)$ on T.*

Proof. Let f be induced by $z \mapsto Cz + d$. Then, by Remark 4.2.7 evaluated in $k \in G$, it holds that

$$\text{(a) } C\rho(k)C^{-1} = \rho(\varphi(k)) \qquad \text{and} \qquad \text{(b) } (\rho(\varphi(k)) - \mathrm{id})d = C\tau(k) - \tau'(\varphi(k)).$$

By item (a), the matrices $\rho(k)$ and $\rho(\varphi(k))$ are similar, hence $\varphi(k) = k$ in all cases. Since all cocycles in standard form vanish in k, the claim follows. □

For the implementation in MAGMA, we need a finite list of candidates for the fixed points of $\rho(k)$. For this, the following is helpful:

Lemma 4.3.16. *Let $T = E^3/K$, where $K \subset \mathrm{Fix}_{\zeta_3}(E)^3$ does not contain non-zero multiples of unit vectors, and let A be a linear automorphism of T of the form $A = P \cdot \mathrm{diag}(\zeta_3^{n_1}, \zeta_3^{n_2}, \zeta_3^{n_3})$, where P is a permutation matrix and $n_i \in \{1, 2\}$. Then every fixed point of A belongs to $E[3]^3$.*

Proof. Let $z = (z_1, z_2, z_3) \in T$ be a fixed point of A, so $(A - \mathrm{id}_{E^3})(z) \in K$. In particular, $(\zeta_3^{n_j} - 1) \cdot z_i$ is fixed by ζ_3 for a suitable $j \in \{1, 2, 3\}$ depending on the permutation matrix P. Thus, the equation

$$0 = (\zeta_3^{n_j} - 1)^2 \cdot z_i = -3\zeta_3^{n_j} z_i$$

holds in E. Since multiplication with $-\zeta_3$ is an automorphism of E, this implies that z_i is a 3-torsion point of E. □

In the rest of the section, we will treat the different groups and analytic representations separately in order to derive a proof of Theorem 4.3.1.

The case $G = \mathbb{Z}_3^2$, $\rho = \rho_1$

We start with the classification of the quotients of T by actions of $\mathbb{Z}_3^2 = \langle h, k\rangle$ with the analytic representation $\rho = \rho_1$ of Proposition 4.3.3, which is given by

$$\rho(h) = \operatorname{diag}(1,\ \zeta_3^2,\ \zeta_3) \qquad \text{and} \qquad \operatorname{diag}(\zeta_3,\ \zeta_3,\ \zeta_3).$$

Recall that a cocycle in standard form is of the form

$$\tau(h) = (a_1,\ a_2,\ a_3) \qquad \text{and} \qquad \tau(k) = (0,\ 0,\ 0).$$

Lemma 4.3.17. *Let τ be a cocycle in standard form; then $\tau(h) \in \operatorname{Fix}_{\zeta_3}(T)$. Conversely, any $(a_1,\ a_2,\ a_3) \in \operatorname{Fix}_{\zeta_3}(T)$ yields a well-defined cocycle in standard form.*

Proof. The corresponding action Φ has to be a group homomorphism, so it has to preserve the relations of the generators of $\mathbb{Z}_3^2$. Since $\tau(k^3) = 0$ holds anyway, this leads to the following conditions:

- $\tau(h^3) = 0 \iff a_1 \in E[3]$,
- $\tau(hk) = \tau(kh) \iff \rho_1(k)\tau(h) = \tau(h) \iff \tau(h) \in \operatorname{Fix}_{\zeta_3}(T)$.

Since $\operatorname{Fix}_{\zeta_3}(T)$ is contained in $E[3]^3$ by Lemma 4.3.16, the second condition implies the first one. □

Lemma 4.3.18. *A cocylce $\tau\colon \mathbb{Z}_3^2 \to T$ in standard form is good if and only if the elements a_i are never the i-th coordinate of an element in K for all $i = 1, 2, 3$.*

Proof. The cocycle is good if the corresponding action has only isolated fixed points. This is precisely the case if all elements whose linear part has eigenvalue 1 act freely, i.e., the elements $h,\ hk,\ hk^2,\ h^2,\ h^2k$, and h^2k^2 have no fixed points. Since these elements have order 3, they act freely if and only if their squares act freely, so we have to ensure that the elements $h,\ hk,\ hk^2$ have no fixed points. With the argumentation presented in the proof of Lemma 4.3.7, we see that this the case if and only if the elements a_i are never the i-th coordinate of an element in K for all $i = 1, 2, 3$. □

Lemma 4.3.19. *The normalizer group $\mathcal{N} = N_{\operatorname{Aut}(E^3)}(\rho_1(\mathbb{Z}_3^2))$ of $\rho_1(\mathbb{Z}_3^2)$ in $\operatorname{Aut}(E^3)$ is finite of order 1296 and generated by the matrices*

$$\begin{pmatrix} -\zeta_3 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \textit{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The action of $\mathcal{N}$ on the set of possible kernels $\mathcal{K}$ has four orbits with representatives

$$K_0 := \{0\}, \quad K_1 := \langle (t,\ t,\ 0)\rangle, \quad K_2 := \langle (t,\ t,\ t)\rangle, \quad K_3 := \langle (t,\ t,\ t),\ (t,\ -t,\ 0)\rangle.$$

Proof. Let $C \in \mathcal{N}$. By the proof of Proposition 4.3.11, C is the product of a permutation matrix P_C and a diagonal matrix D_C, whose non-zero entries belong to $\mathbb{Z}[\zeta_3]^* = \langle -\zeta_3 \rangle$. To determine the possible permutations, we use that the condition $C \cdot \rho_1 \cdot C^{-1} = \rho_1 \circ \varphi$ for some automorphism $\varphi \in \mathrm{Aut}(\mathbb{Z}_3^2)$ implies that the matrices $\rho_1(u)$ and $\rho_1(\varphi(u))$ are similar for all u in $\mathbb{Z}_3^2$. Thus, $\varphi(k) = k$ and $\varphi(h) \in \{h,\ hk,\ hk^2,\ h^2,\ h^2k,\ h^2k^2\}$, which implies that the group generated by the possible permutation matrices P_C is isomorphic to $\mathfrak{S}_3$, and the claim follows. $\square$

Our MAGMA code determines all possible actions, i.e., all good cocycles, and in particular a representative for each good cohomology class in $H^1(\mathbb{Z}_3^2, E^3/K_i)$. The number of possible actions and good classes is displayed in the table below.

i	kernel K_i	# of actions	# of good classes
0	$\{0\}$	8	8
1	$\langle (t,\ t,\ 0) \rangle$	12	4
2	$\langle (t,\ t,\ t) \rangle$	18	6
3	$\langle (t,\ t,\ t), (t,\ -t,\ 0) \rangle$	6	2

By Remark 4.3.12, quotients of tori with different kernels in the above list cannot be biholomorphic. Next, we prove that they cannot even be homeomorphic. For this, we use the structure of the decomplexification $(\rho_1)_\mathbb{R}$:

Remark 4.3.20. The decomplexification $(\rho_1)_\mathbb{R}$ of the analytic representation ρ_1 of $\mathbb{Z}_3^2$ is given by

$$(\rho_1)_\mathbb{R} \colon \mathbb{Z}_3^2 \longrightarrow \mathrm{GL}(6,\mathbb{R}), \quad h^a k^b \mapsto \begin{pmatrix} B^b & & \\ & B^{2a+b} & \\ & & B^{a+b} \end{pmatrix}, \quad \text{where} \quad B = -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

The two-dimensional real representations B^b, B^{2a+b}, and B^{a+b} are irreducible and pairwise not equivalent. Furthermore, the following holds:

(1) The $\mathbb{R}$-algebra of matrices commuting with B is

$$\left\{ \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mid \lambda, \mu \in \mathbb{R} \right\} \simeq \mathbb{C}.$$

(2) The $\mathbb{R}$-vector space of matrices H with $HB = B^2H$ is

$$\left\{ \begin{pmatrix} \lambda & \mu \\ \mu & -\lambda \end{pmatrix} \mid \lambda, \mu \in \mathbb{R} \right\} \simeq \mathbb{R}^2.$$

The matrices in (1) define $\mathbb{C}$-linear maps and the matrices in (2) $\mathbb{C}$-antilinear maps. In complex coordinates $z = x + iy$, set $w := \lambda + i\mu$, and identify the maps with

$$m_w \colon \mathbb{C} \longrightarrow \mathbb{C}, \quad z \mapsto wz, \qquad \text{and} \qquad \overline{m}_w \colon \mathbb{C} \longrightarrow \mathbb{C}, \quad z \mapsto w\overline{z}.$$

Proposition 4.3.21. *Let Λ_K and $\Lambda_{K'}$ be two different lattices in the list in Lemma 4.3.19. Then the set $\mathcal{N}_{\mathbb{R}}(\Lambda_K, \Lambda_{K'})$ is empty, hence corresponding quotients are never homeomorphic.*

Proof. Assume the converse, let $C \in \mathcal{N}_{\mathbb{R}}(\Lambda_K, \Lambda_{K'})$, and view it as a map $C\colon \mathbb{C}^3 \to \mathbb{C}^3$. Using Schur's Lemma and the above remark, and arguing similarly to the proof of Proposition 4.3.11, we see that C is, up to a permutation of the coordinates, a sum of $\mathbb{C}$-linear and $\mathbb{C}$-antilinear maps:

$$m_{w_j}(z) = w_j z \quad \text{or} \quad \overline{m}_{w_j}(z) = w_j \bar{z}, \quad \text{where} \quad w_j \in \mathbb{Z}[\zeta_3]^*.$$

In particular, C descends to a group isomorphism

$$C\colon \Lambda_K/\mathbb{Z}[\zeta_3]^3 \longrightarrow \Lambda_{K'}/\mathbb{Z}[\zeta_3]^3.$$

By this, the sublattice $\mathbb{Z}[\zeta_3]^3$ has to have the same index in Λ_K as in $\Lambda_{K'}$. Therefore, only the case where $\Lambda_K = \Lambda_{K_1}$ and $\Lambda_{K'} = \Lambda_{K_2}$ remains. But then, C has to map $(t,\ t,\ 0)$ to a generator of $\langle (t,\ t,\ t)\rangle$, which is impossible. □

Proposition 4.3.22. *There are precisely 4 biholomorphism classes of rigid quotients of three-dimensional tori by a rigid action of $\mathbb{Z}_3^2$ with analytic representation ρ_1 and isolated fixed points. For each torus $T_0 = E^3/K_0, \ldots, T_3 = E^3/K_3$, there is one class. They are represented by $Z_3,\ Z_4,\ Z_5$, and Z_6 of Theorem 4.3.1 and are all topologically distinct.*

Proof. We use MAGMA to verify that

$$\mathcal{N}_{\mathbb{C}}(\Lambda_{K_i}) = \{C \in \mathcal{N} = N_{\mathrm{Aut}(E^3)}(\rho_1(\mathbb{Z}_3^2)) \mid CK_i = K_i\}$$

acts transitively on the good cohomology classes in $H^1(\mathbb{Z}_3^2, E^3/K_i)$ for each kernel K_i. Proposition 4.3.21 completes the proof. □

The case $G = \mathbb{Z}_3^2$, $\rho = \rho_2$

Next, we classify the quotients of T by actions of $\mathbb{Z}_3^2 = \langle h, k\rangle$ with the analytic representation $\rho = \rho_2$ of Proposition 4.3.3, which is given by

$$\rho(h) = \mathrm{diag}(1,\ \zeta_3^2,\ \zeta_3^2) \qquad \text{and} \qquad \mathrm{diag}(\zeta_3,\ \zeta_3,\ \zeta_3^2).$$

Recall that a cocycle in standard form is of the form

$$\tau(h) = (a_1,\ a_2,\ a_3) \qquad \text{and} \qquad \tau(k) = (0,\ 0,\ 0).$$

Similar to the case $\rho = \rho_1$, the following holds:

Lemma 4.3.23. *Let τ be a cocycle in standard form; then $\tau(h) \in T$ is fixed by $\rho_2(k)$. Conversely, any $(a_1,\ a_2,\ a_3) \in \mathrm{Fix}_{\rho_2(k)}(T)$ yields a well-defined cocycle in standard form.*
Moreover, a cocycle in standard form is good if and only if the elements a_i are never the i-th coordinate of an element in K for all $i = 1, 2, 3$.

Lemma 4.3.24. *The normalizer group $\mathcal{N} = N_{\mathrm{Aut}(E^3)}(\rho_2(\mathbb{Z}_3^2))$ of $\rho_2(\mathbb{Z}_3^2)$ in $\mathrm{Aut}(E^3)$ is finite of order 432 and generated by the matrices*

$$\begin{pmatrix} -\zeta_3 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & \\ & 1 & \\ & & -\zeta_3 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The action of $\mathcal{N}$ on the set of possible kernels $\mathcal{K}$ has five orbits with representatives

$$K_0 := \{0\}, \quad K_1 := \langle (t,\ t,\ 0) \rangle, \quad K_{1'} := \langle (t,\ 0,\ t) \rangle, \quad K_2 := \langle (t,\ t,\ t) \rangle,$$
$$K_3 := \langle (t,\ t,\ t),\ (t,\ -t,\ 0) \rangle.$$

Proof. We follow the proof of Lemma 4.3.19. This time, $\varphi(h) \in \{h,\ h^2k^2\}$, so the permutation matrix P_C is either the identity or the one permuting the first two coordinates. □

Our MAGMA code determines all possible actions, i.e., all good cocycles, and in particular a representative for each good cohomology class in $H^1(\mathbb{Z}_3^2, E^3/K_i)$. The number of actions and good classes is displayed in the table below.

i	kernel K_i	# of actions	# of good classes
0	$\{0\}$	8	8
1	$\langle (t,\ t,\ 0) \rangle$	12	4
$1'$	$\langle (t,\ 0,\ t) \rangle$	12	4
2	$\langle (t,\ t,\ t) \rangle$	18	6
3	$\langle (t,\ t,\ t), (t,\ -t,\ 0) \rangle$	6	2

Proposition 4.3.25. *There are precisely 5 biholomorphism classes of rigid quotients of three-dimensional tori by a rigid action of $\mathbb{Z}_3^2$ with analytic representation ρ_2 and isolated fixed points. For each torus $T_0 = E^3/K_0, \ldots, T_3 = E^3/K_4$, there is one class, they are represented by Y_3, Y_4, $Y_{4'}$, Y_5, and Y_6 of Theorem 4.3.1. The quotients Y_4 and $Y_{4'}$ are homeomorphic, but all other quotients are pairwise not homeomorphic.*

Proof. We use MAGMA to verify that

$$\mathcal{N}_{\mathbb{C}}(\Lambda_{K_i}) = \{C \in \mathcal{N} = N_{\mathrm{Aut}(E^3)}(\rho_2(\mathbb{Z}_3^2)) \mid CK_i = K_i\}$$

acts transitively on the good cohomology classes in $H^1(\mathbb{Z}_3^2, E^3/K_i)$ for each kernel K_i. Similarly to the proof of Proposition 4.3.21, we see that $\mathcal{N}_{\mathbb{R}}(\Lambda_K, \Lambda_{K'})$ is empty for different kernels K and K' except for $\{K, K'\} = \{K_1, K_{1'}\}$. A homeomorphism $Y_4 \to Y_{4'}$ is induced by the map

$$C\colon \mathbb{C}^3 \longrightarrow \mathbb{C}^3, \quad (z_1,\ z_2,\ z_3) \mapsto (-z_1,\ \overline{z_3},\ \overline{z_2}).$$ □

The case $G = \mathbb{Z}_3^2$, $\rho = \rho_3$

Next, we classify the quotients of T by actions of $\mathbb{Z}_3^2 = \langle h, k\rangle$ with the analytic representation $\rho = \rho_3$ of Proposition 4.3.3, which is given by

$$\rho(h) = \operatorname{diag}(\zeta_3,\ \zeta_3,\ 1) \qquad \text{and} \qquad \operatorname{diag}(\zeta_3,\ \zeta_3,\ \zeta_3).$$

Recall that a cocycle in standard form is of the form

$$\tau(h) = (a_1,\ a_2,\ a_3) \qquad \text{and} \qquad \tau(k) = (0,\ 0,\ 0).$$

Similar to the previous cases, the following holds:

Lemma 4.3.26. *Let τ be a cocycle in standard form; then $\tau(h) \in \operatorname{Fix}_{\zeta_3}(T)$. Conversely, any $(a_1,\ a_2,\ a_3) \in \operatorname{Fix}_{\zeta_3}(T)$ yields a well-defined cocycle in standard form.*
Moreover, a cocycle in standard form is good if and only if K contains no elements of the form $(*, *, a_3)$ or $(a_1, a_2, *)$.

Lemma 4.3.27. *The normalizer group $\mathcal{N} = N_{\operatorname{Aut}(E^3)}(\rho_3(\mathbb{Z}_3^2))$ of $\rho_3(\mathbb{Z}_3^2)$ in $\operatorname{Aut}(E^3)$ is given by*

$$\mathcal{N} = \left\{ \begin{pmatrix} C' & 0 \\ 0 & c \end{pmatrix} \,\middle|\, c \in \langle -\zeta_3\rangle,\ C' \in \operatorname{GL}(2, \mathbb{Z}[\zeta_3]) \right\}.$$

The action of $\mathcal{N}$ on the set of possible kernels $\mathcal{K}$ has two orbits with representatives

$$K_0 := \{0\} \quad \textit{and} \quad K_1 := \langle (t,\ t,\ t)\rangle.$$

Proof. The proof of the description of the normalizer is analogous to the other two cases for $G = \mathbb{Z}_3^2$, but one needs to be careful that the representation ρ_3 contains the same character twice. Since the normalizer is now infinite, we cannot compute the orbits of its action on the set of possible kernels $\mathcal{K}$ with MAGMA, so we determine them by hand: Each 2-dimensional subspace of $\operatorname{Fix}_{\zeta_3}(E)^3$ contains a non-trivial element of the form $\mu e_1 + \tau e_2$ and can therefore be excluded as a kernel. The only 1-dimensional subspaces without such elements and without non-zero multiples of e_3 are $\langle (t,\ 0,\ t)\rangle$, $\langle (0,\ t,\ t)\rangle$, and $\langle (t,\ t,\ t)\rangle$. The first and the second are mapped to the third by

$$\begin{pmatrix} 1 & 0 & \\ 1 & 1 & \\ & & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 & \\ 0 & 1 & \\ & & 1 \end{pmatrix},$$

respectively. Thus, they all belong to the $\mathcal{N}$-orbit of $\langle (t, t, t)\rangle$. □

Proposition 4.3.28. *There are precisely 2 biholomorphism classes of rigid quotients of three-dimensional tori by rigid actions of $\mathbb{Z}_3^2$ with analytic representation ρ_3. For each torus $T_0 = E^3/K_0$ and $T_1 = E^3/K_1$, there is one class. They are represented by Y_7 and Y_8 of Theorem 4.3.1 and are not homeomorphic.*

Proof. Since $H^0(\mathbb{Z}_3^2, T_0) \simeq \mathbb{Z}_3^3$ and $H^0(\mathbb{Z}_3^2, T_1) \simeq \mathbb{Z}_3^2$, a quotient of T_0 cannot be homeomorphic to a quotient of T_1 by Proposition 4.2.9.
Next, we prove that for both tori, the normalizer $\mathcal{N}_{\mathbb{C}}(\Lambda_{K_i})$ acts transitively on the set of good cocycles. Since all matrices $C \in N_{\mathrm{Aut}(E^3)}(\rho(\mathbb{Z}_3^2))$ are in block-form $C = \mathrm{diag}(C', c)$, where $C' \in \mathrm{GL}(2, \mathbb{Z}[\zeta_3])$ and $c \in \langle -\zeta_3 \rangle$ by Lemma 4.3.27, they commute with ρ. Hence, it suffices to show that for any two good cocycles τ and τ' in standard form, there exists a matrix $C \in \mathcal{N}_{\mathbb{C}}(\Lambda_{K_i})$ such that $C \cdot \tau = \tau'$. Evaluated in k, this equation automatically holds since $\tau(k) = \tau'(k) = 0$. So, it suffices to check that $C \cdot \tau(h) = \tau'(h)$.
We start with $T_0 = E^3$. Let $\tau(h) = (a_1,\ a_2,\ a_3)$. Then by Lemma 4.3.26, the cocycle τ is good if and only if $a_i \in \mathrm{Fix}_{\zeta_3}(E)$, $a_3 \neq 0$, and $(a_1,\ a_2) \neq (0,0)$. Let $\tau(h) := (t,\ 0,\ t)$, and let $\tau'(h) = (a_1',\ a_2',\ a_3')$ be an arbitrary cocycle. Choose $c \in \{\pm 1\}$ such that $c \cdot t = a_3'$. Then a suitable matrix

$$C' \in \left\{ \begin{pmatrix} \pm 1 & 0 \\ \pm 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

yields $C := \mathrm{diag}(C', c) \in \mathcal{N}_{\mathbb{C}}(\Lambda_{K_i})$ with $C \cdot (t,\ 0,\ t) = \tau'(h)$.
Finally, we consider the quotients of $T_1 = E^3/K_1$. There are six good cohomology classes in $H^1(\mathbb{Z}_3^2, T_1)$, represented by

i	1	2	3	4	5	6
$\tau_i(h)$	$\frac{1}{3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\frac{1}{3} \cdot \begin{pmatrix} \zeta_3 \\ 1 \\ 1 \end{pmatrix}$	$\frac{1}{3} \cdot \begin{pmatrix} 1 \\ \zeta_3 \\ 1 \end{pmatrix}$	$\frac{2}{3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\frac{2}{3} \cdot \begin{pmatrix} \zeta_3 \\ 1 \\ 1 \end{pmatrix}$	$\frac{2}{3} \cdot \begin{pmatrix} 1 \\ \zeta_3 \\ 1 \end{pmatrix}$

The following matrices $C_{ij} \in \mathcal{N}_{\mathbb{C}}(\Lambda_{K_1})$ fulfill $0 = C_{ij} \cdot \tau_i(h) - \tau_j(h)$ in T_1:

$$C_{ij} := \begin{cases} \mathrm{diag}(\zeta_3, 1, 1), & \text{if } (i,j) \in \{(1,2), (4,5)\}, \\ \mathrm{diag}(1, \zeta_3, 1), & \text{if } (i,j) \in \{(1,3), (4,6)\}, \\ -\mathrm{id}, & \text{if } (i,j) = (1,4). \end{cases}$$

Hence, all classes belong to the same orbit. □

The case $G = \mathbb{Z}_3^3$

We continue with the classification of the quotients of T by actions of $\mathbb{Z}_3^3 = \langle h, k, g \rangle$ with the analytic representation ρ of Proposition 4.3.3, which is given by

$$\rho(h) = \mathrm{diag}(1,\ \zeta_3^2,\ \zeta_3), \qquad \rho(k) = \mathrm{diag}(\zeta_3,\ \zeta_3,\ \zeta_3), \qquad \text{and} \qquad \rho(g) = \mathrm{diag}(\zeta_3,\ 1,\ 1).$$

Recall that a cocycle in standard form is of the form

$$\tau(h) = (a_1,\ a_2,\ a_3), \qquad \tau(k) = (0,\ 0,\ 0), \qquad \text{and} \qquad \tau(g) = (b_1,\ b_2,\ b_3).$$

Lemma 4.3.29. *Let τ be a cocycle in standard form; then $\tau(h)$ and $\tau(g)$ belong to $\mathrm{Fix}_{\zeta_3}(T)$ and the following holds:*

- $a_1 \in E[3]$,
- $(0, 3b_2, 3b_3) \in K$,
- $v := ((\zeta_3 - 1)a_1, (1 - \zeta_3^2)b_2, (1 - \zeta_3)b_3) \in K$.

Conversely, two elements $(a_1, a_2, a_3), (b_1, b_2, b_3)$ *in* $\mathrm{Fix}_{\zeta_3}(T)$ *that fulfill these conditions yield a well-defined cocycle in standard form.*

Lemma 4.3.30. *A cocycle in standard form is good if and only if the following conditions are satisfied:*

(1) For all $i = 1, 2, 3$, *the element* a_i *is never the* i*-th coordinate of an element in* K.

(2) There are no elements in K *of the forms*

$$(*, b_2, b_3), \quad (\zeta_3 a_1 + b_1, *, a_3 + b_3), \quad (2\zeta_3 a_1 + b_1, -\zeta_3 a_2 + b_2, *).$$

(3) b_1 *is never the first coordinate of an element in* K.

(4) $a_2 + b_2$ *is never the second coordinate of an element in* K.

(5) $-\zeta_3^2 a_3 + b_3$ *is never the third coordinate of an element in* K.

Proof. The action has only isolated fixed points if all elements whose linear parts of the action have 1 as eigenvalue act freely. Since all non-trivial elements in $\mathbb{Z}_3^3$ have order 3, the elements u and u^2 have the same fixed points. Thus, the action has isolated fixed points if and only if the elements h, hk, hk^2, g, ghk^2, gh^2k^2, gk^2, ghk, gh^2k act freely. This translates to the given conditions. □

As in the case where $G = \mathbb{Z}_3^2$ and $\rho = \rho_1$, the following holds:

Lemma 4.3.31. *The normalizer group* $\mathcal{N} = N_{\mathrm{Aut}(E^3)}(\rho(\mathbb{Z}_3^3))$ *of* $\rho(\mathbb{Z}_3^3)$ *in* $\mathrm{Aut}(E^3)$ *is finite of order* 1296 *and generated by the matrices*

$$\begin{pmatrix} -\zeta_3 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \textit{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The action of $\mathcal{N}$ *on the set of possible kernels* $\mathcal{K}$ *has four orbits with representatives*

$$K_0 := \{0\}, \quad K_1 := \langle (t,\ t,\ 0) \rangle, \quad K_2 := \langle (t,\ t,\ t) \rangle, \quad K_3 := \langle (t,\ t,\ t),\ (t,\ -t,\ 0) \rangle.$$

Our MAGMA code determines all good cocycles and a representative for each good cohomology class in $H^1(\mathbb{Z}_3^3, E^3/K_i)$. The number of actions and good classes is displayed in the table below.

i	kernel K_i	# of actions	# of good classes
0	$\{0\}$	16	16
1	$\langle (t,\ t,\ 0) \rangle$	48	16
2	$\langle (t,\ t,\ t) \rangle$	0	0
3	$\langle (t,\ t,\ t),\ (t,\ -t,\ 0) \rangle$	0	0

In particular, there are no actions with isolated fixed points on the tori E^3/K_2 and E^3/K_3.

Proposition 4.3.32. *There are precisely 3 biholomorphism classes of rigid quotients of three-dimensional tori by rigid actions of $\mathbb{Z}_3^3$ with isolated fixed points. More precisely, the following holds:*

- *One class is realized as a quotient of $T_0 = E^3/K_0 = E^3$ and corresponds to Y_9 of Theorem 4.3.1. The other two are realized as quotients of $T_1 = E^3/K_1$ and correspond to Y_{10} and $Y_{10'}$.*
- *The quotients Y_{10} and $Y_{10'}$ are diffeomorphic to each other but not diffeomorphic to Y_9.*

Proof. We use MAGMA to verify that the action of

$$\mathcal{N}_{\mathbb{C}}(\Lambda_{K_i}) = \{C \in \mathcal{N} = N_{\mathrm{Aut}(E^3)}(\rho(\mathbb{Z}_3^3)) \mid CK_i = K_i\}$$

on the good cohomology classes in $H^1(\mathbb{Z}_3^3, E^3/K_i)$ has one orbit if $i = 0$ and two orbits if $i = 1$. Since $H^0(\mathbb{Z}_3^3, T_1) \simeq \mathbb{Z}_3^3$ and $H^0(\mathbb{Z}_3^3, T_2) \simeq \mathbb{Z}_3^2$, quotients of T_0 cannot be homeomorphic to quotients of T_1 by Proposition 4.2.9. The map

$$C \colon \mathbb{C}^3 \longrightarrow \mathbb{C}^3, \quad (z_1, z_2, z_3) \mapsto (\overline{z_1}, \overline{z_2}, \overline{z_3}),$$

induces a homeomorphism between the named quotients of T_1. $\square$

The case $G = \mathrm{He}(3)$

We continue with the classification of the quotients of T by actions of

$$G = \mathrm{He}(3) = \langle g, h, k \mid g^3 = h^3 = k^3 = [g, k] = [h, k] = 1, \ [g, h] = k\rangle$$

with the analytic representation ρ of Proposition 4.3.3, which is given by

$$\rho(h) = \begin{pmatrix} 1 & & \\ & \zeta_3^2 & \\ & & \zeta_3 \end{pmatrix}, \qquad \rho(k) = \begin{pmatrix} \zeta_3 & & \\ & \zeta_3 & \\ & & \zeta_3 \end{pmatrix}, \qquad \text{and} \qquad \rho(g) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Recall that a cocycle in standard form is of the form

$$\tau(h) = (a_1, \ a_2, \ a_3), \qquad \tau(k) = (0, \ 0, \ 0), \qquad \text{and} \qquad \tau(g) = (b_1, \ b_2, \ b_3),$$

and the torus is E^3/K, where K is one of the following:

$$K_0 := \{0\}, \qquad K_1 := \langle (t, \ t, \ t)\rangle, \qquad K_2 := \langle (t, \ t, \ t), \ (t, \ -t, \ 0)\rangle.$$

Lemma 4.3.33. *Let τ be a cocycle in standard form; then $\tau(h)$ and $\tau(g)$ belong to $\mathrm{Fix}_{\zeta_3}(T)$ and the following holds:*

- $v_1 := (b_1 + b_2 + b_3,\ b_1 + b_2 + b_3,\ b_1 + b_2 + b_3) \in K$,
- $v_2 := \big(\zeta_3 a_1 - a_3 + (\zeta_3 - 1)b_1,\ \zeta_3 a_2 - a_1,\ \zeta_3 a_3 - a_2 + (\zeta_3^2 - 1)b_3\big) \in K$.

Conversely, two elements $(a_1, a_2, a_3), (b_1, b_2, b_3)$ in $\mathrm{Fix}_{\zeta_3}(T)$ that fulfill these conditions yield a well-defined cocycle in standard form.

Note that each element $(t_1,\ t_2,\ t_3) \in K$ has the property that $t_1 + t_2 + t_3 = 0$ in E. This observation is useful to prove a simple criterion for a cocycle $\tau\colon \mathrm{He}(3) \to T = E^3/K$ to be good.

Lemma 4.3.34. *A coycle in standard form is good if and only if the following conditions are satisfied:*

(1) $b_1 + b_2 + b_3 \neq 0$ *in* E.

(2) a_1 *is never the first coordinate of an element in* K.

(3) $\zeta_3^2(b_1 + b_2) + b_3 + \zeta_3^2(a_1 + a_3) + a_2 \neq 0$ *in* E.

(4) $\zeta_3(b_1 + b_2) + b_3 - \zeta_3(a_1 + a_2) - a_3 \neq 0$ *in* E.

Proof. The cocycle is good if and only if k and k^2 are the only elements acting with fixed points. Since u and u^2 as well as all conjugates of u have the same fixed points, the latter is equivalent to requiring that g, h, gh, and gh^2 act freely. Suppose that $\Phi(gh)$ has a fixed point $z = (z_1, z_2, z_3) \in T$. This means that

$$(\zeta_3 z_3 - z_1 + a_3 + b_1,\ z_1 - z_2 + a_1 + b_2,\ \zeta_3^2 z_2 - z_3 + a_2 + b_3) \in K,$$

or, equivalently,

$$(z_3 - \zeta_3^2 z_1 + \zeta_3^2(a_3 + b_1),\ \zeta_3^2 z_1 - \zeta_3^2 z_2 + \zeta_3^2(a_1 + b_2),\ \zeta_3^2 z_2 - z_3 + a_2 + b_3) \in K$$

because the coordinates of elements in K are fixed by ζ_3. Hence, the sum of the coordinates

$$\zeta_3^2(b_1 + b_2) + b_3 + \zeta_3^2(a_1 + a_3) + a_2$$

is zero in E by the above observation. Conversely, if this term is zero, then

$$z = (0,\ a_1 + b_2,\ -\zeta_3^2(a_3 + b_1))$$

is a fixed point of $\Phi(gh)$. The freeness of the action of g, h and gh^2 gives the other three conditions in a similar way. □

The trivial kernel $K = \{0\}$ can be excluded since we cannot find a good cocycle on the corresponding torus:

Lemma 4.3.35. *In the case $K = \{0\}$, there is no good cohomology class in standard form.*

Proof. Let us assume the contrary. Then, as the vector v_1 belongs to K by Lemma 4.3.33, this implies that $b_1 + b_2 + b_3 = 0$ in E – a contradiction to Lemma 4.3.34. □

Our MAGMA code determines all good cocycles and a representative for each good cohomology class in $H^1(\mathrm{He}(3), E^3/K_i)$. The number of actions and good classes is displayed in the table below.

i	kernel K_i	# of actions	# of good classes
1	$\langle (t,\ t,\ t) \rangle$	54	6
2	$\langle (t,\ t,\ t),\ (t,\ -t,\ 0) \rangle$	18	6

Lemma 4.3.36. *For both lattices Λ_{K_1} and Λ_{K_2}, the normalizer groups $\mathcal{N}_{\mathbb{C}}(\Lambda_{K_i})$ are finite of order 1296 and generated by the matrices*

$$C_1 := \begin{pmatrix} \zeta_3 & & \\ & \zeta_3^2 & \\ & & 1 \end{pmatrix}, \quad C_2 := -t \cdot \begin{pmatrix} 1 & \zeta_3^2 & \zeta_3^2 \\ \zeta_3^2 & 1 & \zeta_3^2 \\ \zeta_3^2 & \zeta_3^2 & 1 \end{pmatrix}, \quad \text{and} \quad C_3 := t \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 \end{pmatrix}.$$

Proof. Let $C \in \mathcal{N}_{\mathbb{C}}(\Lambda_{K_i})$. Then

$$C \cdot \rho \cdot C^{-1} = \rho \circ \varphi \tag{4.3.1}$$

for some $\varphi \in \mathrm{Aut}(\mathrm{He}(3))$, uniquely determined by C, and $C \cdot \Lambda_{K_i} = \Lambda_{K_i}$. In particular, ρ and $\rho \circ \varphi$ are equivalent irreducible representations, and hence have the same character χ. So, $\varphi \in \mathrm{Stab}(\chi)$. As the Heisenberg group has precisely two irreducible representations of degree 3, which are complex conjugates of each other, the stabilizer group $\mathrm{Stab}(\chi)$ is a subgroup of index 2 of $\mathrm{Aut}(\mathrm{He}(3)) \simeq \mathrm{AGL}(2, \mathbb{F}_3)$. Therefore, it has 216 elements.
Conversely, for every $\varphi \in \mathrm{Stab}(\chi)$, there exists a matrix $C_\varphi \in \mathrm{GL}(3, \mathbb{C})$ fulfilling (4.3.1). By Schur's Lemma, the matrix C_φ is unique up to a non-zero scalar in $\mathbb{C}$. Thus, we obtain a faithful projective representation

$$\Xi \colon \mathrm{Stab}(\chi) \longrightarrow \mathrm{PGL}(3, \mathbb{C}), \quad \varphi \mapsto [C_\varphi].$$

Hence,

$$\mathcal{N}_{\mathbb{C}}(\Lambda_{K_i}) = \{C \in \mathrm{GL}(3, \mathbb{C}) \mid [C] \in \mathrm{im}(\Xi),\ C \cdot \Lambda_{K_i} = \Lambda_{K_i}\}.$$

Let C_φ be a representative of the class of $\Xi(\varphi)$ fixing the lattice $\Lambda_{K_i} =: \Lambda$, and let $\mu \in \mathbb{C}^*$ such that μC_φ shares the same property. Then, $\mu\Lambda = \mu C_\varphi \Lambda = \Lambda$. Since e_1 belongs to Λ, we conclude $\mu \in \mathbb{Z}[\zeta_3]$. Furthermore, it holds that $\mu^{-1}\Lambda = \Lambda$, so μ is one of the six units of $\mathbb{Z}[\zeta_3]$.
Thus, the normalizer group $\mathcal{N}_{\mathbb{C}}(\Lambda_{K_i})$ has at most $6 \cdot 216 = 1296$ elements. Since the matrices C_1, C_2, and C_3 are contained in $\mathcal{N}_{\mathbb{C}}(\Lambda_{K_i})$ for both i and generate a group of order 1296, the claim follows. □

Proposition 4.3.37. *There are precisely 2 biholomorphism classes of rigid quotients of three-dimensional tori by a rigid action of $\mathrm{He}(3)$ with isolated fixed points. For each torus $T_1 = E^3/K_1$ and $T_2 = E^3/K_2$, there is one class. They are represented by Z_7 and Z_8 of Theorem 4.3.1 and are not homeomorphic.*

Proof. We use MAGMA to verify that $\mathcal{N}_{\mathbb{C}}(\Lambda_{K_i})$ acts transitively on the good cohomology classes in $H^1(\mathrm{He}(3), E^3/K_i)$ for each kernel K_i.
Since $H^0(\mathrm{He}(3), E^3/K_1) \simeq \mathbb{Z}_3$, whereas $H^0(\mathrm{He}(3), E^3/K_2) \simeq \mathbb{Z}_3^2$, quotients of $T_1 = E^3/K_1$ can neither be biholomorphic nor homeomorphic to quotients of $T_2 = E^3/K_2$. □

The case $G = \mathbb{Z}_9 \rtimes \mathbb{Z}_3$

Finally, we classify the quotients of T by actions of

$$G = \mathbb{Z}_9 \rtimes \mathbb{Z}_3 = \langle g, h \mid h^3 = g^9 = 1,\ hgh^{-1} = g^4 \rangle$$

with the analytic representation ρ of Proposition 4.3.3, which is given by

$$\rho(g) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta_3 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(h) = \begin{pmatrix} 1 & & \\ & \zeta_3^2 & \\ & & \zeta_3 \end{pmatrix}.$$

Recall that a cocycle in standard form is of the form

$$\tau(g) = (0,\ 0,\ 0) \qquad \text{and} \qquad \tau(h) = (a_1,\ a_2,\ a_3).$$

Lemma 4.3.38. *Let τ be a cocycle in standard form; then $\tau(h) \in T$ is fixed by $\rho(g)$. Conversely, any $(a_1,\ a_2,\ a_3) \in \mathrm{Fix}_{\rho(g)}(T)$ yields a well-defined cocycle in standard form.*
Moreover, a cocycle in standard form is good if and only if the elements a_i are never the i-th coordinate of an element in K for all $i = 1, 2, 3$.

Proposition 4.3.39. *There is one and only one biholomorphism class of rigid quotients of three-dimensional tori by rigid actions of $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ with isolated fixed points, which can be realized as a quotient of $T = E^3$ and corresponds to Y_{11} of Theorem 4.3.1.*

Proof. First recall that $T = E^3/K$, where K is one of the following:

$$K_0 := \{0\}, \quad K_1 := \langle (t,\ t,\ t) \rangle, \quad K_2 := \langle (t,\ t,\ t),\ (t,\ -t,\ 0) \rangle.$$

A MAGMA computation shows that there are no actions with isolated fixed points if $K = K_1$ or $K = K_2$ and that the only two possible actions for $K = K_0$, thus $T = E^3$, are given by

$$\tau_1(h) = t \cdot (1, 1, 1) \qquad \text{and} \qquad \tau_2(h) = -t \cdot (1, 1, 1).$$

Since multiplication by -1 induces a biholomorphism between the corresponding quotients, the claim follows. □

Proof of Theorem 4.3.1. Let G be a finite group admitting a rigid, holomorphic, and translation-free action on a three-dimensional complex torus T with finite fixed locus and such that the quotient $X = T/G$ has canonical singularities. By Theorem 3.0.6, G is isomorphic to one of the following groups:

$$\mathbb{Z}_3, \quad \mathbb{Z}_7, \quad \mathbb{Z}_9, \quad \mathbb{Z}_{14}, \quad \mathbb{Z}_3^2, \quad \mathbb{Z}_3^3, \quad \mathrm{He}(3), \quad \text{or} \quad \mathbb{Z}_9 \rtimes \mathbb{Z}_3.$$

More precisely, if $p_g(X) = 1$, then G is either $\mathbb{Z}_3,\ \mathbb{Z}_7,\ \mathbb{Z}_3^2$, or He(3), and if $p_g(X) = 0$, then G is either $\mathbb{Z}_9,\ \mathbb{Z}_{14},\ \mathbb{Z}_3^2,\ \mathbb{Z}_3^3$, or $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$. The geometric genus of the quotient can be seen directly by looking at the analytic representation: If its image is contained in $\mathrm{SL}(3, \mathbb{C})$, then $p_g(X) = 1$, otherwise $p_g(X) = 0$.
As stated in Proposition 4.2.5, quotients obtained by different groups cannot be homeomorphic, so in particular not biholomorphic.

For the cyclic groups, the classification is easy: There is one and only one biholomorphism class for each group (cf. Proposition 4.3.2). The classes are represented by the quotients Z_1 and Z_2 ($p_g = 1$) in Table 4.1, and Y_1 and Y_2 ($p_g = 0$) in Table 4.2.

In the other cases, the situation is more involved. Proposition 4.3.3 shows that there are three possibilities for the analytic representation of $\mathbb{Z}_3^2$, where precisely one, namely ρ_1, leads to quotients with geometric genus 1, whereas the representations of $\mathbb{Z}_3^3$, He(3) and $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ are unique up to equivalence of representations and automorphisms of the groups. In any case, we can deduce the structure of the torus from the description of the linear part of the action: It is the quotient of three copies of Fermat's elliptic curve E by a subgroup K of $\mathrm{Fix}_{\zeta_3}(E)^3$, which is the kernel of an isogeny given by addition (cf. Lemma 4.3.5 and Lemma 4.3.8). If G is abelian, then Remark 4.3.12 explains that the normalizer group $N_{\mathrm{Aut}(E^3)}(\rho(G))$ acts on the set of possible kernels and it is enough to consider one kernel for each orbit.

If $G = \mathbb{Z}_3^2$, we have three subcases according to the three choices $\rho_1,\ \rho_2$, and ρ_3 of the analytic representation. They lead to distinct biholomorphism classes of quotients because, even twisting with automorphisms of $\mathbb{Z}_3^2$, the three representations are pairwise not equivalent. Moreover, if we consider ρ_1 and ρ_3 as real representations, they do not belong to the $\mathrm{Aut}(\mathbb{Z}_3^2)$-orbit of $\mathbb{Z}_3^2$, so the corresponding quotients are even not homeomorphic, whereas ρ_1 and ρ_2 are equivalent as real representations (ρ_2 is obtained by complex conjugation of the third coordinate of ρ_1), so here, quotients can be homeomorphic. The classification of the quotients where $\rho = \rho_1$ is summarized in Proposition 4.3.22: There are four biholomorphism classes of quotients that can be represented by $Z_3,\ Z_4,\ Z_5$, and Z_6 of Table 4.1 and are topologically distinct. If the analytic representation equals ρ_2, then by Proposition 4.3.25, there are five biholomorphism classes, which can be represented by $Y_3,\ Y_4,\ Y_{4'},\ Y_5$, and Y_6 of Table 4.2, but here Y_4 and $Y_{4'}$ are homeomorphic. The quotients Z_i and Y_i for $i = 3, \ldots, 6$ are homeomorphic via

$$Z_i \longrightarrow Y_i, \quad (z_1,\ z_2,\ z_3) \mapsto (z_1,\ z_2, -\overline{z_3}).$$

For the last analytic representation ρ_3 of $\mathbb{Z}_3^2$, two kernels are possible and the fine classification is explained in Proposition 4.3.28.

If $G = \mathbb{Z}_3^3$, then there are four candidates for the kernels but only two of them allow actions with isolated fixed points. The classification is settled in Proposition 4.3.32.

In the case $G = \mathrm{He}(3)$, only three subspaces of $\mathrm{Fix}_{\zeta_3}(E)^3$ are preserved under ρ and do not contain non-zero multiple of unit-vectors, so there are three candidates for the kernels but only two of them allow actions with isolated fixed points (cf. Lemma 4.3.35). For each kernel, there is exactly one biholomorphism class and they are topologically distinct (cf. Proposition 4.3.37).

Finally, if $G = \mathbb{Z}_9 \rtimes \mathbb{Z}_3$, then there is one and only one biholomorphism class. The proof is given in Proposition 4.3.39.

Next, we explain how to determine the baskets of singularities in each case. In Theorem 3.0.5, we prove that all stabilizer groups are cyclic and determine the possible types of canonical singularities. If $p_g(X) = 1$, then it is well-known that the quotient Z_1 has seven singularities of type $\frac{1}{7}(1, 2, 4)$. The other quotients have only singularities of type $\frac{1}{3}(1, 1, 1)$ given by the images of the 27 fixed points of the automorphism $\zeta_3 \cdot \mathrm{id}$ of T under the quotient map. So, the number of these singularities can be computed with the formula in Lemma 3.0.10.
If $p_g(X) = 0$, then the orbifold Riemann-Roch formula (Proposition 3.0.8) allows us to compute the possible baskets of all non-Gorenstein singularities. The Propositions 3.0.14, 3.0.15 and 3.0.16 ensure that only the cases $k = 9$ ($G = \mathbb{Z}_{14}$), $k = 12$ ($G = \mathbb{Z}_9$ or $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$), and $k = 15$ ($G = \mathbb{Z}_3^2$ or $\mathbb{Z}_3^3$) of Corollary 3.0.9 can occur. To count the singularities of type $\frac{1}{3}(1, 1, 1)$, we use the Lefschetz fixed-point formula (Lemma 2.1.12 and Lemma 3.0.10). In the case $G = \mathbb{Z}_{14}$, we use the formula

$$7 = |G| \cdot (\tfrac{1}{7}N_7 + \tfrac{1}{14}N_{14}),$$

which can be proven similarly to Lemma 3.0.10, to deduce the number N_7 of singularities of type $\frac{1}{7}(1, 2, 4)$.

The last column of the two tables 4.1 and 4.2 contains the fundamental groups of the quotients. They will be analyzed in the next section. □

4.4. Fundamental groups and covering spaces of quotients of complex tori

Let G be a finite group acting holomorphically on a complex torus T of arbitrary dimension by

$$\Phi(g)(z) = \rho(g) \cdot z + \tau(g).$$

The goal of this section is to study the structure of the fundamental group and the covering space of the quotient $X = T/G = \mathbb{C}^n/\Gamma$, where Γ is the orbifold fundamental group

$$\Gamma := \pi_1^{orb}(T, G) = \{\gamma \colon \mathbb{C}^n \to \mathbb{C}^n \mid \exists\, g \in G \colon g \circ p = p \circ \gamma\}$$

sitting inside the exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1$$

as explained in Section 4.2. Applying the results to the quotients in our classification will allow us to compute their fundamental groups and covering spaces explicitly.

Notation 4.4.1. We denote by G_{fix} and Γ_{fix} the subgroups of G and Γ generated by the elements acting with fixed points on T and $\mathbb{C}^n$, respectively. These subgroups are normal.

By a theorem of Armstrong (cf. [Arm68]), it holds that

$$\Gamma/\Gamma_{\text{fix}} \simeq \pi_1\left(T/G\right).$$

Since the quotient map $\pi\colon \Gamma \to G$ restricts to a surjection $\pi\colon \Gamma_{\text{fix}} \to G_{\text{fix}}$ with kernel $\Lambda \cap \Gamma_{\text{fix}}$, we obtain the following 9-diagram with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
0 \arrow[r] & \Lambda \cap \Gamma_{\text{fix}} \arrow[r] \arrow[d] & \Gamma_{\text{fix}} \arrow[r] \arrow[d] & G_{\text{fix}} \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \Lambda \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \Lambda/(\Lambda \cap \Gamma_{\text{fix}}) \arrow[r] \arrow[d] & \Gamma/\Gamma_{\text{fix}} \arrow[r] \arrow[d] & G/G_{\text{fix}} \arrow[r] \arrow[d] & 1 \\
 & 0 & 1 & 1 &
\end{tikzcd} \tag{4.4.1}$$

Corollary 4.4.2. *The universal cover of T/G is*

$$\mathbb{C}^n/\Gamma_{\text{fix}} \simeq (\mathbb{C}^n/\Lambda')/(\Gamma_{\text{fix}}/\Lambda') \simeq T'/G_{\text{fix}},$$

where $\Lambda' := \Lambda \cap \Gamma_{\text{fix}}$ and $T' = \mathbb{C}^n/\Lambda'$ is a possibly non-compact torus.
In particular, the universal cover of T/G is compact if and only if Γ_{fix} is crystallographic, or equivalently if and only if T' is compact.

Corollary 4.4.3. *If there is an element $g \in G_{\text{fix}}$ such that all lifts of g to $\mathbb{C}^n$ belong to Γ_{fix}, then $\Lambda \subset \Gamma_{\text{fix}}$, and, in particular,*

$$\pi_1\left(T/G\right) \simeq G/G_{\text{fix}},$$

$T' = T$, and T/G_{fix} is the universal cover of the quotient $X = T/G$.

Remark 4.4.4. If G contains an element g such that 1 is not an eigenvalue of $\rho(g)$, then $g \in G_{\text{fix}}$ and all lifts of g have a fixed point. Hence, the condition $\Lambda \subset \Gamma_{\text{fix}}$ of Corollary 4.4.3 is satisfied. In all our examples, there is such an element, which allows us to compute the fundamental group and the universal cover immediately. In particular, Z_2 of Table 4.1 is the universal cover of $Z_3, \dots, Z_8$, and the universal cover of $Y_3, \dots, Y_6$ of Table 4.2 is given by

$$E^3/\langle \operatorname{diag}(\zeta_3, \zeta_3, \zeta_3^2)\rangle,$$

which is not rigid but homeomorphic to the rigid threefold $Z_2 = E^3/\langle \zeta_3 \cdot \mathrm{id}\rangle$. This finishes the proof of Theorem 4.3.1.

Remark 4.4.5. In the general case, where $\Lambda \neq (\Lambda \cap \Gamma_{\text{fix}})$, we still have a description of the fundamental group in terms of the G-action as an extension:

$$\pi_1\left({}^{T}/_{G}\right) \simeq {}^{\Lambda}/_{(\Lambda \cap \Gamma_{\text{fix}})} \times_{\overline{\beta}} {}^{G}/_{G_{\text{fix}}},$$

where the 2-cocyle $\overline{\beta}$ can be derived from the translation part τ of the action on the torus. For completeness, we sketch the construction:

As discussed in Remark 4.2.12, the connecting homomorphism

$$\sigma^1 \colon H^1(G,T) \longrightarrow H^2(G,\Lambda)$$

corresponding to the short exact sequence $0 \to \Lambda \to \mathbb{C}^n \to T \to 0$ of G-modules is an isomorphism, and the image of the class of τ under this isomorphism yields the crystallographic group Γ as an extension of G by Λ.

We describe how to compute a representative $\beta \in Z^2(G,\Lambda)$ of the image of τ which descends to a 2-cocyle $\overline{\beta}$ of G/G_{fix} with values in $\Lambda/(\Lambda \cap \Gamma_{\text{fix}})$ that give the extension

$$\pi_1\left({}^{T}/_{G}\right) \simeq {}^{\Lambda}/_{(\Lambda \cap \Gamma_{\text{fix}})} \times_{\overline{\beta}} {}^{G}/_{G_{\text{fix}}}.$$

Compare also the explanation after Theorem 2.3.36.

(1) Let $H := G/G_{\text{fix}}$ and $s \colon H \to G$ be a section. Then every $g \in G$ can be written uniquely as $g = g_{\text{fix}} \cdot s(h)$.

(2) For g_{fix}, choose a lift $\hat{\tau}(g_{\text{fix}}) \in \mathbb{C}^n$ of $\tau(g_{\text{fix}}) \in T$ such that the corresponding affine transformation

$$\gamma_{\text{fix}}(z) := \rho(g_{\text{fix}})z + \hat{\tau}(g_{\text{fix}})$$

belongs to Γ_{fix}. For $s(h)$, choose an arbitrary lift $\hat{\tau}(s(h)) \in \mathbb{C}^n$ of $\tau(s(h))$. Then

$$\hat{\tau} \colon G \longrightarrow \mathbb{C}^n, \quad g_{\text{fix}} \cdot s(h) \mapsto \rho(g_{\text{fix}})\hat{\tau}(s(h)) + \hat{\tau}(g_{\text{fix}})$$

is a lift of $\tau \colon G \to T$.

(3) The image of $[\tau]$ under σ^1 is represented by

$$\beta(g_1,g_2) := \rho(g_1)\hat{\tau}(g_2) - \hat{\tau}(g_1g_2) + \hat{\tau}(g_1) \in \Lambda.$$

Lemma 4.4.6. *The cocycle β descends to a cocycle $\overline{\beta} \in H^2(G/G_{\text{fix}}, \Lambda/(\Lambda \cap \Gamma_{\text{fix}}))$ describing the extension*

$$\pi_1\left({}^{T}/_{G}\right) \simeq {}^{\Lambda}/_{(\Lambda \cap \Gamma_{\text{fix}})} \times_{\overline{\beta}} {}^{G}/_{G_{\text{fix}}}.$$

Proof. The proof follows the reasoning in [Cha86, Chapter V, Section 3], adapted to our specific situation. First, we notice that the conjugation action of G on Λ fixes $\Lambda \cap \Gamma_{\text{fix}}$ since Γ_{fix} is normal in Γ. Moreover, the induced action of G_{fix} on the quotient $\Lambda/(\Lambda \cap \Gamma_{\text{fix}})$ is trivial. Thus, the action of G on Λ induces an action of G/G_{fix} on $\Lambda/(\Lambda \cap \Gamma_{\text{fix}})$ which coincides with the conjugation action. Next, we observe that the lift $\hat{\tau}$ of τ defines a section s_Γ of the extension $0 \to \Lambda \to \Gamma \to G \to 1$ via $s_\Gamma(g)(z) = \rho(g)z + \hat{\tau}(g)$. The section s_Γ has the following properties:

- For all $g_{\text{fix}} \in G_{\text{fix}}$ and $h \in G/G_{\text{fix}}$, it holds that $s_\Gamma(g_{\text{fix}} \cdot s(h)) = s_\Gamma(g_{\text{fix}}) \cdot s_\Gamma(s(h))$,
- For all $g_{\text{fix}} \in G_{\text{fix}}$, it holds that $s_\Gamma(g_{\text{fix}}) \in \Gamma_{\text{fix}}$, and
- For all $g \in G,\ g_{\text{fix}} \in G_{\text{fix}}$, it holds that $s_\Gamma(g_{\text{fix}} \cdot g) \cdot \Gamma_{\text{fix}} = s_\Gamma(g) \cdot \Gamma_{\text{fix}}$,

where the first property results directly from the construction and implies the other two properties. Hence, s_Γ descends to a section $\overline{s_\Gamma}$ of the bottom extension of Diagram (4.4.1). Thus, the 2-cocycle $\widetilde{\beta}$ defined by s_Γ, which describes the middle extension, descends to a cocycle

$$\overline{\beta} \in H^2(G/G_{\text{fix}}, \Lambda/(\Lambda \cap \Gamma_{\text{fix}}))$$

describing the bottom row. Now, the 2-cocycle $\widetilde{\beta}$ coincides with the cocycle β:

$$\begin{aligned}
\widetilde{\beta}(g_1, g_2)(z) &= s_\Gamma(g_1) s_\Gamma(g_2) s_\Gamma(g_1 g_2)^{-1}(z) = \\
&= (\rho(g_1)z + \hat{\tau}(g_1)) \circ (\rho(g_2)z + \hat{\tau}(g_2)) \circ (\rho(g_1g_2)^{-1}z - \rho(g_1g_2)^{-1}\hat{\tau}(g_1g_2))(z) = \\
&= z - \hat{\tau}(g_1, g_2) + \rho(g_1)\hat{\tau}(g_2) + \hat{\tau}(g_1) = \beta(g_1, g_2)(z).
\end{aligned}$$

Thus, the claim follows. □

Above, we described the universal covers of torus quotients. They are again torus quotients, although it may happen that the torus is not compact. Conversely, we show that any complex variety uniformized by a compact torus quotient is again a torus quotient. In the proof, the main ingredient is the lifting property that follows from Bieberbach's theorems.

Proposition 4.4.7. *Let $X = T/G$ be a simply connected n-dimensional compact torus quotient, $n \geq 2$, where G is a finite group acting holomorphically, without translations, and freely in codimension 1 on T, and let X' be any complex variety uniformized by X. Then X' is a compact torus quotient. More precisely, there is a torus T' and a finite group G' acting holomorphically, without translations, and freely in codimension 1 such that*

$$X' \simeq T'/G'.$$

Furthermore, if X is rigid, then X' is rigid as well.

Remark 4.4.8. In particular, the previous proposition shows that any complex variety uniformized by one of the quotients in our lists 4.1 and 4.2, is contained in the list as well.

Proof of Proposition 4.4.7. Let $p\colon X = T/G \to X'$ be the universal cover, which is finite by the compactness of X, and Galois, and let $\mathrm{pr}\colon T \to X = T/G$ be the quotient map, which is finite and Galois as well, but possibly ramified. Consider the possibly ramified cover

$$h \coloneqq p \circ \mathrm{pr}\colon T \longrightarrow T/G \longrightarrow X'.$$

Since any $f \in \mathrm{Deck}(p)$ can be lifted to T by Proposition 4.2.5, h is Galois. Let $H \trianglelefteq \mathrm{Deck}(h)$ be the subgroup of translations, $G' \coloneqq \mathrm{Deck}(h)/H$, and $T' \coloneqq T/H$. Then G' acts translation-free on the compact complex torus T' and $X' \simeq T'/G'$. Since the universal cover p is unramified

and the ramification locus of the quotient map pr has codimension at least 2 by assumption, the action of G' on T' is free in codimension 1.
Assume now that X is rigid. Since the action of G' is free in codimension 1, it follows:

$$H^1(T', \Theta_{T'})^{G'} = H^1(X', \Theta_{X'}) = H^1(X, \Theta_X)^{\mathrm{Deck}(p)} = 0.$$

Thus, X' is rigid, too. $\square$

5. Terminalizations and resolutions of singularities

In this chapter, we will construct crepant terminalizations and resolutions of our singular quotients with the property that the obtained manifolds are still (infinitesimally) rigid. The occurring singularities, isolated cyclic quotient singularities, can be described as germs of so-called *toric varieties*. Hence, we can and will use tools from toric geometry to construct the maps and verify the required properties of the resolutions and terminalizations.

5.1. Toric geometry and singularities

In this section, we introduce affine and abstract toric varieties as well as important properties and constructions. For further information and proofs, we refer to the textbooks [Ful93] and [CLS11].

Definition 5.1.1.

(1) An *algebraic torus* is an affine algebraic group $\mathbb{T}$ that is isomorphic to $(\mathbb{C}^*)^n$ as algebraic group.

(2) A *toric variety* is an irreducible algebraic variety X that contains an algebraic torus $\mathbb{T}$ as Zariski dense subset such that the action of $\mathbb{T}$ on itself extends to an algebraic action $\mathbb{T} \times X \to X$.

The basic idea of toric geometry is the following: Given combinatorial data like lattices, cones, and fans, associate toric varieties to them, and then, derive geometrical properties from combinatorial ones that can be computed in an easier way.

For the following, fix a lattice $N \simeq \mathbb{Z}^n$ and the corresponding $\mathbb{R}$-vector space $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 5.1.2. A subset $\sigma \subset N_{\mathbb{R}}$ is called a *convex polyhedral cone* if there are vectors $v_1, \ldots, v_r \in N$ such that

$$\sigma = \operatorname{cone}(v_1, \ldots, v_r) := \left\{ \sum \lambda_i v_i \mid \lambda_i \in \mathbb{R}_{\geq 0} \right\}.$$

The *dual lattice* is defined as $M := N^\vee = \operatorname{Hom}(N, \mathbb{Z})$, and

$$\sigma^\vee := \{ u \in M \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle u, v \rangle \geq 0 \quad \text{for all} \quad v \in \sigma \}$$

is called the *dual cone*, where $\langle -, - \rangle$ denotes the dual pairing.

By Gordan's Lemma (cf. [CLS11, Proposition 1.2.17]), the semigroup $\sigma^\vee \cap M$ is finitely generated and hence, the same holds for the associated $\mathbb{C}$-algebra $\mathbb{C}[\sigma^\vee \cap M]$. Therefore, we can define the *affine toric variety* U_σ as

$$U_\sigma \coloneqq \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M]).$$

Note that U_σ is irreducible and furthermore normal. If σ is moreover *strongly convex*, that is $\sigma \cap (-\sigma) = \{0\}$, then $\dim(U_\sigma) = n$, and U_σ contains the algebraic torus

$$\mathbb{T} = \operatorname{Spec}(\mathbb{C}[M]) \simeq (\mathbb{C}^*)^n$$

as Zariski-dense subset. The $\mathbb{C}$-algebra homomorphism

$$\mathbb{C}[\sigma^\vee \cap M] \longrightarrow \mathbb{C}[M] \otimes \mathbb{C}[\sigma^\vee \cap M], \quad \chi^a \mapsto \chi^a \otimes \chi^a,$$

induces an algebraic action of $\mathbb{T}$ on U_σ, which extends the action of $\mathbb{T}$ on itself. Hence, U_σ is in fact a toric variety in the sense of Definition 5.1.1.

From now on, we will always assume that all cones are strongly convex polyhedral cones.

Proposition 5.1.3 ([CLS11], Theorem 1.3.12)**.** *The affine toric variety U_σ is smooth if and only if σ is generated by a part of a $\mathbb{Z}$-basis of the lattice N, in which case*

$$U_\sigma \simeq \mathbb{C}^k \times (\mathbb{C}^*)^{n-k},$$

where $k \coloneqq \dim(\operatorname{Span}(\sigma))$.

Example 5.1.4. Isolated cyclic quotient singularities are toric[1].

Proof. Let (U, p) be an isolated cyclic quotient singularity of type

$$\tfrac{1}{d}(a_1, \ldots, a_n).$$

Recall that $(U, p) \simeq (\mathbb{C}^n/G, 0)$, where $G \coloneqq \langle \operatorname{diag}(\zeta_d^{a_1}, \ldots, \zeta_d^{a_n}) \rangle$. Here, ζ_d is a primitive d-th root of unity and $\gcd(a_i, d) = 1$ for all $i = 1, \ldots, n$. Consider the lattice

$$N \coloneqq \mathbb{Z}^n + \mathbb{Z} \cdot \tfrac{1}{d}(a_1, \ldots, a_n)$$

and the cone $\sigma \coloneqq \operatorname{cone}(e_1, \ldots, e_n)$. We claim that $U \simeq U_\sigma$. To see this, we first observe that $\sigma^\vee = \sigma$ and

$$M = \{(m_1, \ldots, m_n) \in \mathbb{Z}^n \mid a_1 m_1 + \ldots + a_n m_n \equiv 0 \mod d\}.$$

Hence, $\mathbb{C}[\sigma^\vee \cap M]$ is precisely the ring of invariants $\mathbb{C}[x_1, \ldots, x_n]^G$, which is the coordinate ring of $\mathbb{C}^n/G$. Thus,

$$U = \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_n]^G) = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M]) = U_\sigma. \qquad \square$$

In order to obtain toric varieties that are generally not affine, we glue suitable collections of affine toric varieties.

[1] In fact, all abelian quotient singularities are toric.

Definition 5.1.5.

(1) A *fan* in $N_{\mathbb{R}}$ is a finite collection Σ of strongly convex polyhedral cones in $N_{\mathbb{R}}$ such that
   - each face of a cone in Σ is a cone in Σ, and
   - the intersection of two cones in Σ is a face of each of them.

(2) By $\Sigma(k)$ we denote the subset of all cones in Σ of dimension k.

(3) The *support* of a fan Σ is defined as

$$|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma \quad \subset N_{\mathbb{R}}.$$

(4) The toric variety X_Σ associated to the fan Σ is the variety obtained by gluing the affine varieties $\{U_\sigma \mid \sigma \in \Sigma\}$ along the open sets $U_{\sigma \cap \sigma'}$ of U_σ and $U_{\sigma'}$ for all cones σ and σ' of Σ.

(5) A *ray* ρ of Σ is a one-dimensional cone. Its *primitive* element u_ρ is the unique generator of $N \cap \rho$.

Since smoothness is a local property, the variety X_Σ is smooth if and only if every cone $\sigma \in \Sigma$ is smooth.

Example 5.1.6. We want to realize the complex projective plane as toric variety. For this, let $N = \mathbb{Z}^2$, and let $\Sigma \subset N_{\mathbb{R}} = \mathbb{R}^2$ be the fan with maximal cones

$$\sigma_0 = \operatorname{cone}(e_1, e_2), \quad \sigma_1 = \operatorname{cone}(e_2, -e_1 - e_2), \quad \operatorname{cone}(e_1, -e_1 - e_2),$$

as shown on the left hand side of Figure 5.1. Since the three maximal cones are smooth, the corresponding affine toric varieties U_{σ_i} are all isomorphic to $\mathbb{C}^2$, and one can verify that the toric variety associated to Σ is in fact $\mathbb{P}^2$.

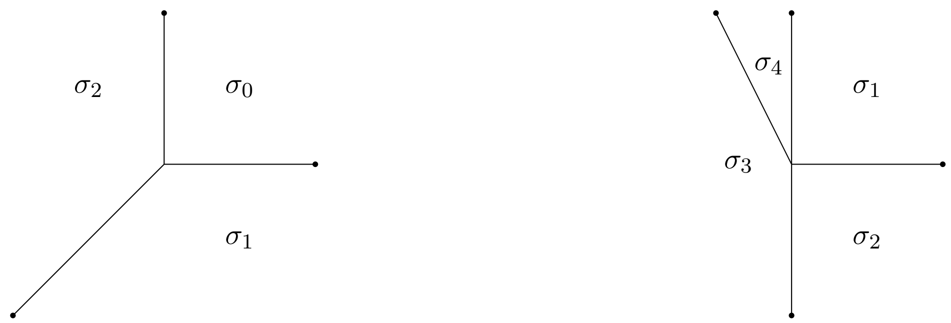


Figure 5.1.: The fans of $\mathbb{P}^2$ and $\mathcal{H}_2$.

Example 5.1.7. The Hirzebruch surface $\mathcal{H}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ has the following toric description (see Figure 5.1 on the right): The lattice N is just $\mathbb{Z}^2$, and the maximal cones of the fan Σ are given by

$$\sigma_1 = \operatorname{cone}(e_1, e_2), \quad \sigma_2 = \operatorname{cone}(e_1, -e_2), \quad \sigma_3 = \operatorname{cone}(-e_2, -e_1 + 2e_2),$$
$$\sigma_4 = \operatorname{cone}(e_2, -e_1 + 2e_2).$$

Remark 5.1.8. Let N' and N be two lattices, and let $\Sigma' \subset N'_{\mathbb{R}}$ and $\Sigma \subset N_{\mathbb{R}}$ be two fans. Let $\varphi\colon N' \to N$ be a $\mathbb{Z}$-linear map such that for all $\sigma' \in \Sigma'$, there is a cone $\sigma \in \Sigma$ such that the $\mathbb{R}$-linear extension of φ maps σ' to σ; we say that φ is *compatible* with the fans. Then φ induces in a natural way a toric morphism $\phi\colon X_{\Sigma'} \to X_{\Sigma}$, i.e., a morphism of algebraic varieties mapping the torus of $X_{\Sigma'}$ to the torus of X_{Σ} such that $\phi_{|\mathbb{T}_{N'}}$ is a homomorphism of groups (cf. [CLS11, Theorem 3.3.4]).

Definition 5.1.9. Let $\Sigma \subset N_{\mathbb{R}}$ be a fan.

(1) A fan $\Sigma' \subset N_{\mathbb{R}}$ is called a *refinement* of Σ if for all $\sigma' \in \Sigma'$, there is a cone $\sigma \in \Sigma$ such that $\sigma' \subset \sigma$, or equivalently, if the identity $\mathrm{id}\colon N \to N$ is compatible with the fans Σ' and Σ.

(2) Let $0 \neq v \in |\Sigma| \cap N$. The *star subdivision* of Σ along v is the fan $\Sigma^*(v)$ that consists of the following cones:

  (a) all cones $\sigma \in \Sigma$ with $v \notin \sigma$,

  (b) all cones $\mathrm{cone}(\tau, v)$, where $v \notin \tau \in \Sigma$ such that $\{v\} \cup \tau$ is contained in a cone in Σ.

In vivid terms, we obtain the star subdivision $\Sigma^*(v)$ of Σ along v by subdividing all cones of Σ that contain v along the ray generated by v. Clearly, it is in particular a refinement of Σ.

Theorem 5.1.10 ([CLS11], Theorem 11.1.9)**.**

(1) Let Σ' be a refinement of a fan $\Sigma \subset N_{\mathbb{R}}$. Then the induced toric morphism

$$\phi\colon X_{\Sigma'} \longrightarrow X_{\Sigma}$$

is proper and birational.

(2) Iterating finitely many star subdivisions along appropriate elements in $N \cap |\Sigma|$ leads to a resolution of singularities of X_{Σ}.

Remark 5.1.11. Let U_σ be an isolated cyclic quotient singularity as in Example 5.1.4, and assume that Σ is obtained from $\sigma = \mathrm{cone}(e_1, \ldots, e_n)$ by finitely many star subdivisions along $v_1, \ldots, v_s$. Then the partial resolution $\phi\colon X_{\Sigma} \to U_\sigma$ is crepant if and only if all the elements $v_1, \ldots, v_s$ belong to the hyperplane

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \ldots + x_n = 1\}.$$

This follows from [CLS11, Lemma 11.4.10].

Remark 5.1.12. Let $\Sigma \subset N_{\mathbb{R}}$ be a fan and $\rho = \mathbb{R}_{\geq 0} u_\rho$ a ray in Σ with primitive generator u_ρ. We can associate to ρ a prime divisor D_ρ as follows: Define the quotient lattice

$$N(\rho) := {}^{N}\!/_{N \cap \mathbb{R} u_\rho}.$$

Then

$$\mathrm{Star}(\rho) := \{\overline{\sigma} \subset N(\rho)_{\mathbb{R}} \mid \sigma \in \Sigma \text{ that have } \rho \text{ as facet}\}$$

is a fan, and its associated toric variety $D_\rho \coloneqq X_{\mathrm{Star}(\rho)}$ is an irreducible divisor in X_Σ.
Note that these divisors are invariant under the action of the torus and their classes generate the class group $\mathrm{Cl}(X_\Sigma)$ (cf. [CLS11, Theorem 4.1.3]).

Remark 5.1.13 ([CLS11], Proposition 11.1.10). The exceptional divisor of a partial toric resolution $\phi\colon X_{\Sigma'} \to X_\Sigma$ that is given by finitely many star subdivisions is the union of the divisors associated to the added rays.

Definition 5.1.14. Let $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ be a divisor of the toric variety X_Σ. The *polyhedron associated to* D is defined as

$$P_D \coloneqq \{m \in M_\mathbb{R} \mid \langle m, u_\rho\rangle \geq -a_\rho \quad \text{for all} \quad \rho \in \Sigma(1)\}.$$

Proposition 5.1.15 ([BG20], Proposition 5.8)**.** *Let $\rho\colon X_\Sigma \to U_\sigma$ be a toric partial resolution of an abelian quotient singularity $U_\sigma = \mathbb{C}^n/G$. Let $D_i \subset X_\Sigma$ and $D_i' \subset U_\sigma$ be the divisors corresponding to the rays $\mathbb{R}_{\geq 0}e_i$. Then the inclusion $\rho_*\Theta_{X_\Sigma} \subset \Theta_{U_\sigma}$ is an isomorphism if and only if the polyhedra P_{D_i} and $P_{D_i'}$ contain the same integral points, that is, for all $1 \leq i \leq n$, it holds that*

$$P_{D_i} \cap M = P_{D_i'} \cap M.$$

Proposition 5.1.16. *Let $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ be a Cartier divisor on a toric variety X_Σ. Then for each $\sigma \in \Sigma$, there exists an element $m_\sigma \in M$ such that for all rays $\rho \in \sigma(1)$, it holds that*

$$\langle m_\sigma, u_\rho\rangle = -a_\rho.$$

The collection $\{m_\sigma\}_{\sigma\in\Sigma}$ is called the Cartier data *of D.*

Knowing the Cartier-data of a divisor, it is easy to decide whether the divisor has base points or not:

Proposition 5.1.17. *Let $\Sigma \subset N_\mathbb{R} \simeq \mathbb{R}^n$ be an n-dimensional fan and $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ a Cartier divisor on X_Σ with Cartier data $\{m_\sigma\}_{\sigma\in\Sigma}$. Then the following are equivalent:*

(1) The divisor D is basepoint free.

(2) For all maximal cones $\sigma \in \Sigma(n)$, the element m_σ belongs to P_D.

If a divisor is basepoint free, then it is also nef. Under mild additional conditions, the higher cohomology groups of nef divisors on toric varieties vanish:

Theorem 5.1.18 (Demazure-Vanishig, [CLS11], Theorem 9.2.3)**.** *Let D be a $\mathbb{Q}$-Cartier divisor on X_Σ. If $|\Sigma|$ is convex and D is nef, then*

$$H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = 0 \qquad \text{for all} \qquad p > 0.$$

If a divisor is not nef, we cannot use Demazure-vanishing. However, the following criterion helps to compute at least the first cohomology since it coincides with the number of connected components of some subspace of $N_{\mathbb{R}}$. Before we state the theorem, note that the cohomology groups $H^p(X_\Sigma, \mathcal{O}_{X_\Sigma})$ carry a natural grading

$$H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}) = \bigoplus_{m \in M} H^p(X_\Sigma, \mathcal{O}_{X_\Sigma})_m.$$

Theorem 5.1.19 ([CLS11], Theorem 9.1.3)**.** *Let $D = \sum_\rho a_\rho D_\rho$ be a Weil divisor on X_Σ. Fix $m \in M$ and $p \geq 0$. Then*

$$H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))_m \simeq \widetilde{H}^{p-1}(V_{D,m}, \mathbb{C}),$$

where $V_{D,m} := \cup_{\sigma \in \Sigma} \operatorname{Conv}(u_\rho \mid \rho \in \sigma(1), \langle m, u_\rho \rangle > -a_\rho)$, and $\widetilde{H}^p(V_{D,m}, \mathbb{C})$ denotes the reduced cohomology of $V_{D,m}$ with coefficients in $\mathbb{C}$. In particular,

$$H^1(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))_m = 0 \iff V_{D,m} \text{ is path connected.}$$

5.2. Construction of the terminalizations and resolutions

In this section, we construct crepant terminalizations and resolutions of our quotients and compare their deformation theory with the one of the singular quotients. More precisely, we show:

Theorem 5.2.1. *All quotients X in Theorem 4.3.1 admit a crepant terminalization X_{ter} and a resolution $\hat{X}$ that fit in the diagram*

$$\begin{tikzcd} X_{\text{ter}} \arrow[r, "\psi"] & X \\ \hat{X} \arrow[u, "\eta"] \arrow[ur] & \end{tikzcd}$$

such that X_{ter} and $\hat{X}$ are infinitesimally rigid.

Note that for the quotients X with $p_g(X) = 1$, which are precisely the quotients Z_i in Theorem 4.3.1, the crepant terminalizations are in fact resolutions, and furthermore they are unique (cf. [Ogu96c, Lemma 4]).

To prove the proposition, we proceed as follows: First, we construct a crepant terminalization ψ with the properties

$$\psi_*(\Theta_{X_{\text{ter}}}) \simeq \Theta_X \qquad \text{and} \qquad R^1\psi_*(\Theta_{X_{\text{ter}}}) = 0. \tag{5.2.1}$$

Leray's spectral sequence then yields an isomorphism $H^1(X_{\text{ter}}, \Theta_{X_{\text{ter}}}) \simeq H^1(X, \Theta_X)$ (cf. Proposition 2.5.11). Thus, the rigidity of X_{ter} follows from the rigidity of X. It will turn out that the terminalizations have only cyclic quotient singularities of type $\frac{1}{d}(1,1,d-1)$, where $d = 2, 3, 4$, or 6. For varieties with such singularities, a resolution $\hat{X}$ with the properties 5.2.1 exists (cf. [BGK, Proposition 6.2]), which implies that the first cohomology with values in $\Theta_{\hat{X}}$ is trivial as well.

Since the quotients X have only isolated singularities and the terminalizations are proper, the construction of a suitable terminalization is a local problem due to the relative version of the GAGA-theorems (cf. [Oda88, Section 2.2]). By Theorem 3.0.5, the non-terminal singularities of X are all cyclic and of the following types:

$$\tfrac{1}{3}(1,1,1), \quad \tfrac{1}{7}(1,2,4), \quad \tfrac{1}{9}(1,4,7), \quad \text{or} \quad \tfrac{1}{14}(1,9,11).$$

As explained in Example 5.1.4, cyclic quotient singularities are toric. In [BG20], the authors give a crepant toric resolution of the singularities of type $\frac{1}{3}(1,1,1)$ having the properties 5.2.1. A toric crepant resolution of the Gorenstein-singularity of type $\frac{1}{7}(1,2,4)$ is constructed in [RY87], and toric crepant terminalizations of the last two singularities can be found in [Gle16]. Note that the last two singularities do not admit a crepant resolution. In the following, we prove that the terminalizations of these three singularities satisfy the conditions 5.2.1.

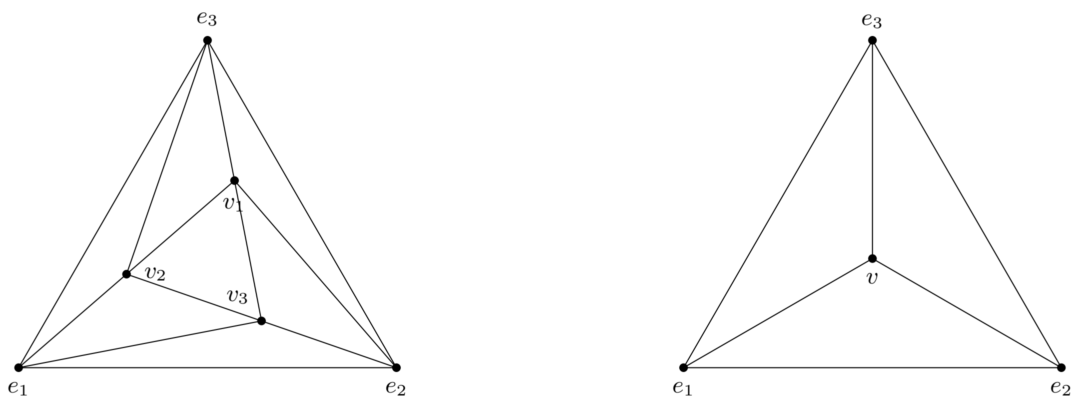
Denote the singularity of type $\frac{1}{7}(1,2,4)$ by U_1, the one of type $\frac{1}{9}(1,4,7)$ by U_2, and the one of type $\frac{1}{14}(1,9,11)$ by U_3.
As affine toric varieties, they are represented by the cone $\sigma := \operatorname{cone}(e_1,e_2,e_3)$ and the lattices

$$\begin{aligned} N_1 &:= \mathbb{Z}^3 + \mathbb{Z}\cdot \tfrac{1}{7}(1,2,4),\\ N_2 &:= \mathbb{Z}^3 + \mathbb{Z}\cdot \tfrac{1}{9}(1,4,7),\\ N_3 &:= \mathbb{Z}^3 + \mathbb{Z}\cdot \tfrac{1}{14}(1,9,11) \end{aligned}$$

in $\mathbb{R}^3$, respectively (cf. Example 5.1.4).
Subdividing the cone σ along the rays generated by

- $v_1 := \frac{1}{7}(1,2,4)$, $v_2 := \frac{1}{7}(4,1,2)$, $v_3 := \frac{1}{7}(2,4,1) \in N_1, N_3$,
- $v := \frac{1}{3}(1,1,1) \in N_2$, respectively,

yields the fans $\Sigma_1 = \Sigma_3$ and Σ_2 visualized in Figure 5.2. Note that the subdivisions, and hence the resulting fans Σ_1 and Σ_3, in the cases 1 and 3 coincide, but the lattices differ, so we get different associated toric varieties.



(a) The fan $\Sigma_1 = \Sigma_3$ (b) The fan Σ_2

Figure 5.2.: Resolution of the singularity of type $\frac{1}{7}(1,2,4)$ and terminalizations of the singularities of type $\frac{1}{9}(1,4,7)$ and $\frac{1}{14}(1,9,11)$.

The affine toric variety associated to the fan Σ_1 and lattice N_1 is smooth. Thus, we obtain a toric resolution

$$\psi_1 \colon X_{\Sigma_1} \longrightarrow U_1.$$

The affine toric varieties that correspond to the maximal cones of the fan Σ_2 are cyclic quotient singularities of type $\frac{1}{3}(1,1,2)$, and the ones that correspond to the maximal cones of Σ_3 are of type $\frac{1}{2}(1,1,1)$, hence the corresponding toric varieties X_{Σ_2} and X_{Σ_3} have only terminal singularities (cf. [Gle16, Section 4.2]). Thus, we obtain toric terminalizations

$$\psi_j \colon X_{\Sigma_j} \to U_j, \qquad j = 2, 3.$$

Since the vectors v, v_1, v_2, and v_3 belong to the plane

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1\},$$

the maps ψ_j, $j = 1, 2, 3$, are crepant by Remark 5.1.11.

It finally remains to verify that $\psi_j \colon X_{\Sigma_j} \to U_j$ actually satisfies the conditions 5.2.1 for all $j = 1, 2, 3$.

Notation 5.2.2. Denote by $D'_i \subset U_j$ and $D_i \subset X_{\Sigma_j}$, respectively, the divisors corresponding to the rays generated by e_i, $i = 1, 2, 3$, by E_k the exceptional divisors of the resolution ψ_1 and the terminalization ψ_3 corresponding to the added rays generated by v_k, $k = 1, 2, 3$, respectively, and by $E \subset X_{\Sigma_2}$ the exceptional divisor of ψ_2 corresponding to v.

Lemma 5.2.3. *The maps ψ_j fulfill $(\psi_j)_*(\Theta_{X_{\Sigma_j}}) \simeq \Theta_{U_j}$ for all $j = 1, 2, 3$.*

Proof. By Proposition 5.1.15, we have to verify for all i that $P_{D_i} \cap M_j = P_{D'_i} \cap M_j$.
Let $j = 1$ or 3. By symmetry, it is enough to consider the case $i = 1$. The polyhedrons of the divisors are given by

$$\begin{aligned} P_{D'_1} &= \{x \in \mathbb{R}^3 \mid x_1 \geq -1, \quad x_2,\, x_3 \geq 0\}, \\ P_{D_1} &= P_{D'_1} \cap \{x \in \mathbb{R}^3 \mid \langle x, v_k \rangle \geq 0,\ k = 1, 2, 3\}. \end{aligned}$$

Thus, the inclusion $P_{D_1} \cap M_j \subset P_{D'_1} \cap M_j$ is obvious. For the converse, let $x \in P_{D'_1} \cap M_j \subset \mathbb{Z}^3$. Then, since $v_1 \in N_j$, it holds that

$$-1 \leq x_1 + 2x_2 + 4x_3 \equiv 0 \mod 7.$$

This implies that the sum $x_1 + 2x_2 + 4x_3 = 7 \cdot \langle x, v_1 \rangle$ is non-negative. With a similar computation, we see that $\langle x, v_k \rangle \geq 0$ for $k = 2, 3$. Hence, x belongs to P_{D_1}.
The case $j = 2$ is analogous. □

Lemma 5.2.4. *Let $\psi \colon X_\Sigma \to X'_\Sigma$ be a proper birational map of toric varieties. Assume that Σ is simplicial, i.e., for each cone of Σ, the ray generators are linearly independent in $N_\mathbb{R}$, and that*

$\{u_\rho \mid \rho \in \Sigma(1)\}$ spans $N_{\mathbb{R}}$. Then

$$R^1\psi_*\Theta_{X_\Sigma} \simeq \bigoplus_{\rho\in\Sigma(1)} R^1\psi_*\mathcal{O}_{X_\Sigma}(D_\rho).$$

Proof. The dual of the toric Euler sequence (cf. [CLS11, Theorem 8.1.6]) on X_Σ reads

$$0 \longrightarrow \mathcal{O}_{X_\Sigma}^{\oplus r} \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_{X_\Sigma}(D_\rho) \longrightarrow \Theta_{X_\Sigma} \longrightarrow 0, \tag{5.2.2}$$

where r is the rank of $\mathrm{Cl}(X_\Sigma)$. Applying ψ_* to this sequence yields the claimed isomorphism since toric varieties have rational singularities (cf. [CLS11, Theorem 11.4.2]), thus $R^q\psi_*(\mathcal{O}_{X_\Sigma}^{\oplus r}) = 0$. $\square$

It remains to prove that the first higher direct image of the tangent sheaf vanishes in each case. Since the proof for $j = 1$ differs from the ones for $j = 2$ and 3, we split the proof into two parts.

Lemma 5.2.5. *The resolution ψ_1 of the singularity of type $\frac{1}{7}(1, 2, 4)$ fulfills $R^1(\psi_1)_*(\Theta_{X_{\Sigma_1}}) = 0$.*

Proof. Let $\psi := \psi_1$ and $\Sigma := \Sigma_1$. By Lemma 5.2.4, we have to prove that

$$R^1\psi_*\mathcal{O}_{X_\Sigma}(D_i) = 0 \qquad \text{and} \qquad R^1\psi_*\mathcal{O}_{X_\Sigma}(E_k) = 0 \qquad \text{for all} \quad j, k = 1, 2, 3.$$

Note that the exact sequence

$$0 \longrightarrow \mathcal{O}_{X_\Sigma} \longrightarrow \mathcal{O}_{X_\Sigma}(E_k) \longrightarrow \mathcal{O}_{E_k}(E_k) \longrightarrow 0$$

induces isomorphisms

$$R^1\psi_*\mathcal{O}_{X_\Sigma}(E_k) \simeq R^1\psi_*\mathcal{O}_{E_k}(E_k).$$

Thus, since U_1 is affine, it is enough to prove:

(1) $H^1(X_\Sigma, \mathcal{O}_{X_\Sigma}(D_i)) = 0$ for $i = 1, 2, 3$, and

(2) $H^1(E_k, \mathcal{O}_{E_k}(E_k)) = 0$ for $k = 1, 2, 3$.

By symmetry, it is enough to consider the cases $i = 1$ and $k = 1$. The Cartier data for the divisor D_1 are given by the elements $m_{\tau_2} = (-1, 4, 0)$ and $m_{\tau_3} = m_{\tau_{23}} = (-1, 0, 2)$, where $\tau_2 = \mathrm{cone}(e_1, e_3, v_2)$, $\tau_3 = \mathrm{cone}(e_1, e_2, v_3)$ and $\tau_{23} = \mathrm{cone}(e_1, v_2, v_3)$, and $m_\tau = 0$ for all other maximal cones τ of Σ. Since all these elements belong to the polyhedron

$$P_{D_1} = \{x \in \mathbb{R}^3 \mid x_1 \geq -1, \quad x_2,\, x_3 \geq 0, \quad \langle x, v_k\rangle \geq 0,\ k = 1, 2, 3\},$$

the divisor D_1 is basepoint free (cf. Proposition 5.1.17), hence nef. Thus $H^1(X_\Sigma, \mathcal{O}_{X_\Sigma}(D_1)) = 0$ by the theorem of Demazure (cf. Theorem 5.1.18).

Using the toric description of the divisor E_1 as explained in Remark 5.1.12, one can see that E_1 is isomorphic to the Hirzebruch surface $\mathcal{H}_2$: The lattice isomorphism

$$N(v_1) = {}^{N}\!/_{\mathbb{Z}\cdot v_1} \longrightarrow \mathbb{Z}^2, \quad [e_2] \mapsto u_1,\ [e_3] \mapsto -u_2,$$

where u_1 and u_2 denote the standard basis vectors in $\mathbb{Z}^2$, maps the fan of E_1 to the fan of $\mathcal{H}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ (cf. Example 5.1.7). Using Serre duality, the adjunction formula, and the fact that the canonical divisor of X_Σ is trivial, we finally conclude the following:

$$H^1(E_1, \mathcal{O}_{E_1}(E_1)) \simeq H^1(E_1, \omega_{E_1} \otimes \mathcal{O}_{E_1}(-E_1))^\vee \simeq H^1(E_1, \mathcal{O}_{E_1})^\vee \simeq H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^\vee = 0.$$

□

Lemma 5.2.6. *For $j = 2, 3$, the terminalizations ψ_j fulfill $R^1(\psi_j)_*(\Theta_{X_{\Sigma_j}}) = 0$.*

Proof. We only verify the assertion in the case $j = 3$, the other one is similar. For simplicity, we drop the index j in the following. By Lemma 5.2.4 and since U is affine, it is enough to prove:

(1) $H^1(X_\Sigma, \mathcal{O}_{X_\Sigma}(D_i)) = 0$ for $i = 1, 2, 3$, and

(2) $H^1(X_\Sigma, \mathcal{O}_{X_\Sigma}(E_k)) = 0$ for $k = 1, 2, 3$.

By symmetry, it is enough to consider the cases $i = 1$ and $k = 1$. Since X_Σ has only singularities of type $\frac{1}{2}(1, 1, 1)$, the divisor D_1 is $\mathbb{Q}$-Cartier with index 2. Computing the Cartier data for $2D_1$ one can show as before that $2D_1$ is nef and the vanishing of the cohomology group of the divisor D_1 follows again from the theorem of Demazure (cf. Theorem 5.1.18).

For $E_1 = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$, where $a_\rho = 1$ if $\rho = \mathrm{cone}(v_1)$ and $a_\rho = 0$, else, the situation is different: Since E_1 is not nef, we cannot apply Demazure-vanishing, and since E_1 is not smooth, we cannot use similar techniques as in the case of ψ_1. Instead, we show that the sets

$$V_{E_1,m} = \bigcup_{\tau \in \Sigma_{\max}} \mathrm{Conv}(u_\rho \mid \rho \in \tau(1), \langle m, u_\rho \rangle < -a_\rho)$$

are connected for all $m \in N^\vee$, where the sum runs over all maximal cones of the fan Σ. This implies that $H^1(X_\Sigma, \mathcal{O}_{X_\Sigma}(E_1)) = 0$ (cf. Theorem 5.1.19).

Looking at the illustration of the fan $\Sigma = \Sigma_3$ in Figure 5.2, we see that $V_{E,m}$ is disconnected if and only if there exists an i such that $e_i, v_i \in V_{E_1,m}$, but $e_j, v_j \notin V_{E_1,m}$ for all $j \neq i$. We only treat the case $i = 1$ as the other cases are analogous. The conditions $v_1 \in V_{E_1,m}$ and $v_2 \notin V_{E_1,m}$ are equivalent to $\langle m, v_1 \rangle < -1$ and $\langle m, v_2 \rangle \geq 0$, written out:

$$m_1 + 2m_2 + 4m_3 < -7 \qquad \text{and} \qquad -4m_1 - m_2 - 2m_3 \leq 0.$$

By adding the first inequality to two times the second inequality, we obtain $-7m_1 < -7$, so $m_1 > 1$. This is a contradiction because the condition $e_1 \in V_{E_1,m}$ means that

$$m_1 = \langle m, e_1 \rangle < 0.$$

□

Now that we know that all our quotients have (rigid) resolutions, the following question naturally arises: Given two quotients that are not biholomorphic (homeomorphic), do their resolutions belong to different biholomorphism (homeomorphism) classes as well?

At least if the geometric genus of the quotients equals 1 (here, the crepant resolution is unique since the quotients have only singularities of type $\frac{1}{3}(1,1,1)$ or $\frac{1}{7}(1,2,4)$, cf. [Ogu96c, Lemma 4]), we can answer this question in the biholomorphic case:

Proposition 5.2.7. *Let Z and Z' be two quotients in the list in Table 4.1, i.e., their geometric genera are equal to* 1*, that are not biholomorphic. Then their unique crepant resolutions $\hat{Z}$ and $\hat{Z}'$ belong to different biholomorphism classes as well.*

Proof. As explained in Section 4.1, the resolutions $\psi\colon \hat{Z} \to Z$ and $\psi'\colon \hat{Z}' \to Z'$, respectively, are c_2-contractions, that is, they are given by the complete linear system of divisors D and D' where the intersection product with c_2 is zero. In [OS01, Lemma-Definition 4.1], the authors show that each Calabi-Yau threefold admits a maximal c_2-contraction, which is unique up to isomorphism. In the proof of Theorem 0.4, Case B (see p. 73), they explain that the c_2-contractions given by the crepant resolutions are the maximal ones.
Now, assume that there is a biholomorphism $f\colon \hat{Z} \to \hat{Z}'$. Then first of all, Z and Z' are quotients by the same group since otherwise the Picard numbers of $\hat{Z}$ and $\hat{Z}'$ would differ (cf. [OS01, Theorem 3.4]). Hence, ψ and $\psi' \circ f$ would be two maximal contractions of $\hat{Z}$, and by the uniqueness, this would imply that Z and Z' are biholomorphic – a contradiction. $\square$

6. Semi-projective representations

In Section 4.2, we discussed how to classify quotients of complex tori by actions that are free in codimension one up to biholomorphism and homeomorphism. Let $T = \mathbb{C}^n/\Lambda$ be such a torus and $\rho\colon G \hookrightarrow \mathrm{GL}(n,\mathbb{C})$ a faithful representation such that $\rho(g)\cdot\Lambda = \Lambda$ for all $g \in G$, and let $\tau,\ \tau' \in Z^1(G,T)$ be two cocycles defining actions that are free in codimension one. We saw that the corresponding quotients X and X' are homeomorphic if and only if there exists a matrix $C \in \mathrm{GL}(2n,\mathbb{R})$ with $C\cdot\Lambda = \Lambda$ and an automorphism ψ of the group G such that

(1) $C \cdot \rho_{\mathbb{R}} \cdot C^{-1} = \rho_{\mathbb{R}} \circ \psi$,

(2) $C * \tau = C \cdot (\tau \circ \psi^{-1})$ and τ' belong to the same cohomology class in $H^1(G,T)$.

Here, the representation $\rho_{\mathbb{R}}\colon G \to \mathrm{GL}(2n,\mathbb{R})$ is the decomplexification of ρ. The quotients X and X' are biholomorphic if and only if C can be chosen as a $\mathbb{C}$-linear matrix.

Note that condition (1) says that the representations $\rho_{\mathbb{R}}$ and $\rho_{\mathbb{R}} \circ \psi$ are equivalent. In particular,

$$\psi \in \mathrm{Stab}(\chi_{\mathbb{R}}) := \{\psi \in \mathrm{Aut}(G) \mid \chi_{\mathbb{R}} = \chi_{\mathbb{R}} \circ \psi\},$$

where $\chi_{\mathbb{R}}$ is the character of $\rho_{\mathbb{R}}$.

Concretely, if the torus T and the two cocycles τ and τ' are explicitly given, one can easily check the second condition, for example by a computer, provided that the full list of candidates for C is known, as we did several times in Chapter 4.

The problem to determine the solutions C of the conjugation equation in condition (1) relates to so-called "semi-projective representations":
Assume that ρ is irreducible and of complex type[1], i.e., the Schur index fulfills $\nu(\chi) = 1$, where χ is the character of ρ. Then, by Theorem 2.3.12, for each $\psi \in \mathrm{Stab}(\chi_{\mathbb{R}})$, there exists a matrix $C_\psi \in \mathrm{GL}(2n,\mathbb{R})$ fulfilling condition (1), which is unique up to an element in the endomorphism algebra $\mathrm{End}_G(\rho_{\mathbb{R}}) \simeq \mathbb{C}$. Since $\chi_{\mathbb{R}} = \chi + \overline{\chi}$, the automorphism ψ either stabilizes χ or maps χ to $\overline{\chi}$. In the first case, the matrix C_ψ is $\mathbb{C}$-linear, whereas in the second case $\mathbb{C}$-antilinear. This yields a homomorphism

$$f\colon\ \mathrm{Stab}(\chi_{\mathbb{R}}) \longrightarrow \mathrm{PGL}(n,\mathbb{C}) \rtimes \mathrm{Aut}(\mathbb{C})$$

into the group of semi-projectivities $\mathrm{P\Gamma L}(n,\mathbb{C}) = \mathrm{PGL}(n,\mathbb{C}) \rtimes \mathrm{Aut}(\mathbb{C})$. Such homomorphisms are called *semi-projective representations*. The candidates for the linear part C of potential homeomorphisms are the elements in the group

$$\mathcal{N} := \{C \in \mathrm{GL}(n,\mathbb{C}) \rtimes \mathrm{Aut}(\mathbb{C}) \mid [C] \in \mathrm{im}(f),\ C\cdot\Lambda = \Lambda\}.$$

[1] If the action of G on T is rigid, then ρ does not contain self-conjugate subrepresentations by Proposition 2.5.8. Thus, it is always of complex type if it is irreducible.

Hence, one has to determine something like a "lift" of f to $\mathrm{GL}(n,\mathbb{C}) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Note that the image of f is obviously contained in the subgroup $\mathrm{PGL}(n,\mathbb{C}) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of $\mathrm{PGL}(n,\mathbb{C}) \rtimes \mathrm{Aut}(\mathbb{C})$.

This observation motivated us to study semi-projective representations and lifting-problems in more detail. The results are presented in this chapter.

Throughout it, K, V and G will denote a field, a non-trivial finite dimensional K-vector space and a finite group, respectively. In [Sch04], Schur developed the theory of projective representations, which are homomorphisms from a group G to the group of projective transformations $\mathrm{PGL}(V)$. We consider semi-projective representations of G; these are homomorphisms from G to the group of semi-projective transformations $\mathrm{P\Gamma L}(V)$. Here, $\mathrm{P\Gamma L}(V)$ is defined as the quotient of the group of semi-linearities

$$\Gamma\mathrm{L}(V) \simeq \mathrm{GL}(V) \rtimes \mathrm{Aut}(K)$$

modulo the action of the multiplicative group K^*. A semi-projective representation involves an action φ of G on K by automorphisms. In this way, K^* becomes a G-module and one can consider the second cohomology group $H^2_\varphi(G, K^*)$ with respect to the action. In analogy to the projective case, this group plays an important role, as it is the obstruction space of the lifting problem of semi-projective representations to semi-linear representations, i.e., homomorphisms from G to $\Gamma\mathrm{L}(V)$.

As our main result, we show that if K is algebraically closed, then for any given action φ of G, there exists a finite φ-twisted representation group Γ, which has the property that any semi-projective representation inducing the action φ admits a semi-linear lift to Γ. Despite the fact that Γ is not unique in general, it has minimal order among all groups enjoying the lifting property. This allows us to study semi-projective representations of G via semi-linear representations of Γ. We also give a cohomological characterization of a group Γ to be a φ-twisted representation group, which reduces to the classical description of a representation group in the case that the action φ is trivial. In general, it seems to be difficult to determine a φ-twisted representation group explicitly, even when φ is trivial. We approach this problem in the semi-projective case via an algorithm for the case $K = \mathbb{C}$ under the assumption that φ takes values in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$; this produces all φ-twisted representation groups of a given group G.

Note that besides the classification problem of complex tori, there is another situation where semi-projective representations arise naturally: Isaacs [Isa81] developed the concept of crossed-projective representations, which is analogous to our notion of semi-projective representations, in order to study the problem of extending G-invariant irreducible K-representations of a normal subgroup of G to G for arbitrary fields K .

6.1. Semi-projective representations and the lifting problem

In this section, we introduce semi-linear and semi-projective representations, discuss some of their basic properties, and formulate and analyze the "lifting problem".

Definition 6.1.1. A bijective map $f\colon V \to V$ is called a *semi-linear transformation* if there exists an automorphism $\varphi_f \in \mathrm{Aut}(K)$ such that

$$f(v+w) = f(v) + f(w) \qquad \text{and} \qquad f(\lambda v) = \varphi_f(\lambda) f(v)$$

for all $v, w \in V$ and all $\lambda \in K$. The set of all semi-linear transformations of V forms a group $\Gamma\mathrm{L}(V)$.

The group $\Gamma\mathrm{L}(V)$ contains $\mathrm{GL}(V)$ as a normal subgroup and sits inside the following short exact sequence

$$1 \longrightarrow \mathrm{GL}(V) \longrightarrow \Gamma\mathrm{L}(V) \longrightarrow \mathrm{Aut}(K) \longrightarrow 1.$$

This sequence splits so that $\Gamma\mathrm{L}(V) \simeq \mathrm{GL}(V) \rtimes \mathrm{Aut}(K)$.
Let $v_1, \ldots, v_n$ be a basis of V. Then we can associate to every $f \in \Gamma\mathrm{L}(V)$ an invertible matrix $A_f := (a_{ij})$ by

$$f(v_j) = \sum_{i=1}^{n} a_{ij} v_i.$$

This procedure establishes an isomorphism between $\Gamma\mathrm{L}(V)$ and $\mathrm{GL}(n, K) \rtimes \mathrm{Aut}(K)$, where the group operation of the semidirect product is given by

$$(A, \varphi) \cdot (B, \psi) := (A\varphi(B), \varphi \circ \psi).$$

Here, $\varphi(B)$ is the matrix obtained by applying the automorphism φ to the entries of B.
In analogy to the group of *projective transformations* $\mathrm{PGL}(V)$, the group of *semi-projective transformations* $\mathrm{P}\Gamma\mathrm{L}(V)$ is defined as the quotient of $\Gamma\mathrm{L}(V)$ modulo the equivalence relation

$$f \sim g \quad \text{if and only if there exists } \lambda \in K^*, \text{ such that } f = \lambda g.$$

By construction, the sequence

$$1 \longrightarrow K^* \longrightarrow \Gamma\mathrm{L}(V) \longrightarrow \mathrm{P}\Gamma\mathrm{L}(V) \longrightarrow 1$$

is exact. The structure of $\mathrm{P}\Gamma\mathrm{L}(V)$ is similar to the one of $\Gamma\mathrm{L}(V)$: The group $\mathrm{PGL}(V)$ is a normal subgroup of $\mathrm{P}\Gamma\mathrm{L}(V)$, and there is a split exact sequence

$$1 \longrightarrow \mathrm{PGL}(V) \longrightarrow \mathrm{P}\Gamma\mathrm{L}(V) \longrightarrow \mathrm{Aut}(K) \longrightarrow 1.$$

Note that the map $\mathrm{P}\Gamma\mathrm{L}(V) \to \mathrm{Aut}(K)$ is well-defined because the automorphism attached to an element in $\mathrm{P}\Gamma\mathrm{L}(V)$ is independent of the representative. After choosing a projective frame, $\mathrm{P}\Gamma\mathrm{L}(V)$ can be identified with the semidirect product

$$\mathrm{PGL}(n, K) \rtimes \mathrm{Aut}(K).$$

We now introduce our main objects of study:

Definition 6.1.2. A *semi-linear representation* is a homomorphism $F\colon G \to \Gamma\mathrm{L}(V)$ and a *semi-projective representation* is a homomorphism $f\colon G \to \mathrm{P}\Gamma\mathrm{L}(V)$.

Isaacs [Isa81] used the term *crossed-projective representation* for a semi-projective representation under the identification of $\mathrm{PGL}(n, K) \rtimes \mathrm{Aut}(K)$ with $\mathrm{P\Gamma L}(V)$.

The lifting problem: Every semi-linear representation $F: G \to \Gamma\mathrm{L}(V)$ induces a semi-projective representation $f: G \to \mathrm{P\Gamma L}(V)$ by composition with the quotient map:

$$\begin{tikzcd} \Gamma\mathrm{L}(V) \arrow[r] & \mathrm{P\Gamma L}(V) \\ G \arrow[u, "F"] \arrow[ur, "f"'] & \end{tikzcd}$$

However, it is not true that every semi-projective representation can be obtained in this way. The obstruction to the existence of a lift to $\Gamma\mathrm{L}(V)$, or, more generally, the interplay between semi-linear and semi-projective representations, can be described using group cohomology in analogy to the classical theory of projective representations.
Given a semi-linear or semi-projective representation of G, we obtain an action

$$\varphi: G \to \mathrm{Aut}(K), \quad g \mapsto \varphi_g,$$

by composition with the projection from $\Gamma\mathrm{L}(V)$ or $\mathrm{P\Gamma L}(V)$ to $\mathrm{Aut}(K)$, respectively. Via this action, the abelian group K^* obtains a G-module structure; in particular, we can consider the cohomology groups

$$H^i_\varphi(G, K^*) = Z^i_\varphi(G, K^*)/B^i_\varphi(G, K^*),$$

where the subscript φ is used to emphasize that the G-module structure of K^* is not trivial in general, and might be dropped in the trivial case. The basic observation is that we can associate to every semi-projective representation a well-defined class in the second cohomology group:

Proposition 6.1.3. *Let $f: G \to \mathrm{P\Gamma L}(V)$ be a semi-projective representation, and let $f_g \in \Gamma\mathrm{L}(V)$ be a representative of the class $f(g)$ for each $g \in G$. Then there exists a map $\alpha: G \times G \to K^*$ such that $f_{gh} = \alpha(g, h)(f_g \circ f_h)$ for all $g, h \in G$. The map α is a 2-cocycle and its cohomology class $[\alpha] \in H^2_\varphi(G, K^*)$ is independent of the chosen representatives f_g.*

Proof. Since f is a homomorphism, $[f_{gh}] = [f_g] \circ [f_h]$, which implies that $f_{gh} = \alpha(g, h)(f_g \circ f_h)$ for some $\alpha(g, h) \in K^*$. To show that α is a cocycle, we use the associativity of the multiplication in G to compute f_{ghk} in two different ways. On the one hand, we have

$$\begin{aligned} f_{g(hk)} = \alpha(g, hk)(f_g \circ f_{hk}) &= \alpha(g, hk)(f_g \circ \alpha(h, k)(f_h \circ f_k)) \\ &= \alpha(g, hk)\varphi_g(\alpha(h, k))(f_g \circ f_h \circ f_k). \end{aligned}$$

On the other hand,

$$f_{(gh)k} = \alpha(gh, k)(f_{gh} \circ f_k) = \alpha(gh, k)\alpha(g, h)(f_g \circ f_h \circ f_k).$$

Comparing the two expressions yields

$$\alpha(g, hk)\varphi_g(\alpha(h, k)) = \alpha(gh, k)\alpha(g, h).$$

If f'_g is another representative for $f(g)$, then there exists $\tau(g) \in K^*$ such that $f_g = \tau(g) f'_g$. Let α' be defined by $f'_{gh} = \alpha'(g,h)(f'_g \circ f'_h)$ for all $g, h \in G$. A computation like the one above shows that

$$\alpha'(g,h) = \varphi_g(\tau(h))\tau(gh)^{-1}\tau(g)\alpha(g,h),$$

where $\partial\tau(g,h) = \varphi_g(\tau(h))\tau(gh)^{-1}\tau(g)$ is a 2-coboundary. □

Let $f\colon G \to \mathrm{P\Gamma L}(V)$ be a semi-projective representation. Choosing id_V as a representative for $f(1)$, the 2-cocycle α is normalized. If f is induced by a semi-linear representation F, then the attached cohomology class is trivial. Conversely, assume that α is a coboundary, so $\alpha = \partial(\tau)$ for some function $\tau\colon G \to K^*$. Then the map

$$F\colon G \to \Gamma\mathrm{L}(V), \quad g \mapsto F_g := \tau(g) f_g$$

is a semi-linear representation inducing f, as the following computation shows:

$$\begin{aligned} F_g \circ F_h &= (\tau(g)f_g) \circ (\tau(h)f_h) = \tau(g)\varphi_g(\tau(h))(f_g \circ f_h) \\ &= \tau(gh)\alpha(g,h)(f_g \circ f_h) = \tau(gh)f_{gh} \\ &= F_{gh}. \end{aligned}$$

Corollary 6.1.4. *A semi-projective representation $f\colon G \to \mathrm{P\Gamma L}(V)$ is induced by a semi-linear representation if and only if its attached cohomology class in $H^2_\varphi(G, K^*)$ is trivial.*

We have assigned to every semi-projective representation an element in $H^2_\varphi(G, K^*)$. In fact, all cohomology classes arise in this way:

Let $\varphi\colon G \to \mathrm{Aut}(K)$ be an action of G on K and $\alpha \in Z^2_\varphi(G, K^*)$. In analogy to the regular representation, consider the vector space V with basis $\{e_h \mid h \in G\}$ and define for every $g \in G$ an element $R_g \in \mathrm{GL}(V)$ via $R_g(e_h) := \alpha(g,h)^{-1} e_{gh}$. Then the map

$$f\colon G \to \mathrm{PGL}(V) \rtimes \mathrm{Aut}(K), \quad g \mapsto ([R_g], \varphi_g),$$

is a semi-projective representation with assigned cohomology class $[\alpha] \in H^2_\varphi(G, K^*)$.

We have thus shown that if $H^2_\varphi(G, K^*)$ is non-trivial, then there are semi-projective representations without a semi-linear lift. In the projective case, this problem was first noticed and investigated by Schur. In order to study projective representations by means of ordinary linear representations, he and subsequent authors constructed, under certain conditions on K, a representation group Γ of G: in modern terminology, a *stem extension*

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1, \qquad \text{with } A \simeq H^2(G, K^*).$$

Stem means that A is central and contained in the commutator subgroup $[\Gamma, \Gamma]$. Such an extension has the property that for every projective representation $f\colon G \to \mathrm{PGL}(V)$, there exists an ordinary linear representation $F\colon \Gamma \to \mathrm{GL}(V)$ fitting into the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d, "F"] & G \arrow[r] \arrow[d, "f"] & 1 \\
1 \arrow[r] & K^* \arrow[r] & \mathrm{GL}(V) \arrow[r] & \mathrm{PGL}(V) \arrow[r] & 1
\end{tikzcd}$$

(see [Kar85, Chapter 3.3] for details). In this scenario, we say that F *induces* f or that f *can be lifted to* F, and we use similar terminology in the semi-projective case below.

Recall, that if

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

is an extension of G by a finite abelian group A and $s\colon G \to \Gamma$ a set-theoretic section, then there is an action of G on A defined by $g * a := s(g)as(g)^{-1}$. The equation $\beta(g,h)s(gh) = s(g)s(h)$ for some $\beta(g,h) \in A$ defines a 2-cocylcle $\beta\colon G \times G \to A$, so a cohomology class in $H^2_\varphi(G, A)$, which is uniquely determined by the extension.
Suppose now that an action $\varphi\colon G \to \mathrm{Aut}(K)$ on the field K is given. Then, by composition with the projection $\Gamma \to G$, we also obtain an action of Γ on K with kernel containing A. In this situation, the *inflation-restriction exact sequence* of Hochschild and Serre [HS53, Theorem 2] reads:

$$1 \longrightarrow H^1_\varphi(G,K^*) \xrightarrow{\mathrm{inf}} H^1_\varphi(\Gamma,K^*) \xrightarrow{\mathrm{res}} \mathrm{Hom}_G(A,K^*) \xrightarrow{\mathrm{tra}} H^2_\varphi(G,K^*) \xrightarrow{\mathrm{inf}} H^2_\varphi(\Gamma,K^*).$$

Here, inf and res are induced by inflation and restriction of cocycles and the *transgression* map tra is defined as

$$\mathrm{tra}\colon \mathrm{Hom}_G(A,K^*) \to H^2_\varphi(G,K^*), \quad \lambda \mapsto [\lambda \circ \beta].$$

Clearly, this map depends only on the cohomology class of β. Using this terminology, we obtain:

Theorem 6.1.5. *Let* $1 \to A \to \Gamma \xrightarrow{\pi} G \to 1$ *be an extension of* G *by a finite abelian group* A *with associated cohomology class* $[\beta] \in H^2_\varphi(G, A)$*. A semi-projective representation* $f\colon G \to P\Gamma L(V)$ *with class* $[\alpha] \in H^2_\varphi(G,K^*)$ *is induced by a semi-linear representation*

$$F\colon \Gamma \longrightarrow \Gamma L(V), \quad \gamma \mapsto F_\gamma,$$

if and only if $[\alpha]$ *belongs to the image of the transgression map.*

Proof. Assume that f is induced by a semi-linear representation F. By assumption, there exists a function $\lambda\colon \Gamma \to K^*$ such that $F_\gamma = \lambda(\gamma) f_{\pi(\gamma)}$ for all $\gamma \in \Gamma$. Since we may assume that $f_1 = \mathrm{id}$, it follows that $F_a = \lambda(a) f_{\pi(a)} = \lambda(a)\,\mathrm{id}$ for all $a \in A$. As a result, the restriction λ_A is a homomorphism. We claim that $\lambda \in \mathrm{Hom}_G(A,K^*)$, that is $\lambda(g * a) = \varphi_g(\lambda(a))$ for all $g \in G$ and $a \in A$, and say that λ is *G-equivariant*. This is true since

$$\varphi_g(\lambda(a))\,\mathrm{id} = F_{s(g)} \circ (\lambda(a)\,\mathrm{id}) \circ F_{s(g)^{-1}} = F_{s(g)} \circ F_a \circ F_{s(g)^{-1}} = F_{s(g)as(g)^{-1}} = \lambda(g * a)\,\mathrm{id}\,.$$

By using the definition of β, we compute

$$\begin{aligned} F_{s(gh)} = F_{\beta(g,h)s(g)s(h)} &= F_{\beta(g,h)} \circ F_{s(g)} \circ F_{s(h)} \\ &= \lambda(\beta(g,h))(\lambda(s(g))f_g) \circ (\lambda(s(h))f_h) \\ &= \lambda(\beta(g,h))\lambda(s(g))\varphi_g(\lambda(s(h))(f_g \circ f_h). \end{aligned}$$

On the other hand,

$$F_{s(gh)} = \lambda(s(gh))f_{gh} = \lambda(s(gh))\alpha(g,h)(f_g \circ f_h).$$

Comparing the results, we obtain $\alpha(g,h) = \lambda(\beta(g,h))\partial(\lambda \circ s)(g,h)$, hence

$$[\lambda \circ \beta] = [\alpha] \in H^2_\varphi(G, K^*).$$

Conversely, assume there is a function $\tau\colon G \to K^*$ and $\lambda \in \mathrm{Hom}_G(A, K^*)$ such that

$$\alpha(g,h) = \lambda(\beta(g,h))\varphi_g(\tau(h))\tau(gh)^{-1}\tau(g).$$

Define

$$F\colon \Gamma \to \Gamma L(V), \quad as(g) \mapsto \lambda(a)\tau(g)f_g;$$

then one can verify by similar computations as above that F is a homomorphism inducing f. □

A natural question arises: Is it possible to find for every finite group G, together with a given action $\varphi\colon G \to \mathrm{Aut}(K)$, an extension

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad \text{with } A \text{ finite and abelian}$$

such that every semi-projective representation $f\colon G \to \mathrm{P\Gamma L}(V)$ with action φ is induced by a semi-linear representation $F\colon \Gamma \to \Gamma\mathrm{L}(V)$?

By Theorem 6.1.5, answering this question amounts to constructing an extension with surjective transgression map

$$\mathrm{tra}\colon \mathrm{Hom}_G(A, K^*) \longrightarrow H^2_\varphi(G, K^*), \quad \lambda \mapsto [\lambda \circ \beta].$$

Clearly, this is possible only if $H^2_\varphi(G, K^*)$ is finite. In case such an extension Γ exists, its order is bounded from below:

$$|G||H^2_\varphi(G, K^*)| \leq |G||\mathrm{Hom}(A, K^*)| \leq |G||A| = |\Gamma|.$$

Unfortunately, $H^2_\varphi(G, K^*)$ is generally not finite. As an example, consider $K = \mathbb{Q}(i)$ and $G = \mathrm{Gal}(K/\mathbb{Q})$ acting naturally on K. Then the cohomology group

$$H^2(G, K^*) \simeq \mathbb{Q}^*/N_{K/\mathbb{Q}}(K^*)$$

is infinite. Indeed, an application of the *sum of two squares theorem* shows that all primes p with $p \equiv 3 \mod 4$ yield non-trivial distinct elements.

Nevertheless, in many important situations $H^2_\varphi(G,K^*)$ is finite: for example, if K is algebraically closed and $\varphi\colon G \to \mathrm{Aut}(K)$ is an arbitrary action, see [Isa76, Theorem 11.15] and note that the proof carries over to non-trivial actions and all cohomology groups $H^i_\varphi(G,K^*)$, where $i \geq 1$.

6.2. Twisted representation groups: the algebraically closed case

Throughout this section, we will assume that K is algebraically closed and that there is a fixed action $\varphi\colon G \to \mathrm{Aut}(K)$. Under these assumptions, we will mainly be dealing with a case similar to $K = \mathbb{C}$, where φ acts just by the identity and complex conjugation. Indeed, $H \coloneqq \varphi(G)$ is a finite group and $F \coloneqq K^H \subset K$ is a Galois extension with Galois group H. The Artin-Schreier Theorem [AS27] implies that if H is non-trivial, then it is isomorphic to $\mathbb{Z}_2$, $K = F(i)$ with $i^2 = -1$ and $\mathrm{char}(K) = 0$. In particular, if $\mathrm{char}(K) \neq 0$, then the action is necessarily trivial and we are in the projective setting.

The main result of this section is the following.

Theorem 6.2.1. *There exists an extension*

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

of G with A finite and abelian such that the transgression map $\mathrm{tra}\colon \mathrm{Hom}_G(A,K^*) \to H^2_\varphi(G,K^*)$ *is an isomorphism.*

Proof. By [Isa76, Lemma 11.14], we may let M' be a finite complement of $B^2_\varphi(G,K^*)$ in $Z^2_\varphi(G,K^*)$. For each $m' \in M'$, define $m \coloneqq (\partial\tau_{m'})m'$, where $\tau_{m'}(g) = \varphi_g(m'(1,1)^{-1})$. Then the set of these m forms a complement M of normalized 2-cocycles. Define an action of G on $A \coloneqq \mathrm{Hom}(M,K^*)$ via φ by $(g*a)(m) \coloneqq \varphi_g(a(m))$ for all $m \in M$. Now define $\beta\colon G\times G \to A$ by $\beta(g,h)(m) \coloneqq m(g,h)$ for all $m \in M$. A straightforward computation confirms that $\beta \in Z^2(G,A)$ under the action of G on A, and clearly, β is normalized. So, β defines an extension

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

where $\Gamma \coloneqq A\times_\beta G$ with binary operation $(a,g)\cdot(b,h) \coloneqq (a(g*b)\beta(g,h), gh)$ (cf. Theorem 2.3.36). Note that the conjugation action of G on A is given by $g*a$. Now, we claim that the transgression map

$$\mathrm{tra}\colon\ \mathrm{Hom}_G(A,K^*) \to H^2_\varphi(G,K^*), \quad \lambda \mapsto [\lambda\circ\beta],$$

is surjective. Any class in $H^2_\varphi(G,K^*)$ is represented by a (unique) element $m_0 \in M \subset Z^2_\varphi(G,K^*)$. Consider the evaluation homomorphism at m_0, that is

$$\lambda\colon A \to K^*, \quad a \mapsto a(m_0).$$

Note that λ is G-equivariant, in fact,

$$\lambda(g*a) = (g*a)(m_0) = \varphi_g(a(m_0)) = \varphi_g(\lambda(a)) \qquad \text{for all } g \in G.$$

Furthermore,

$$(\lambda \circ \beta)(g,h) = \lambda(\beta(g,h)) = \beta(g,h)(m_0) = m_0(g,h)$$

and thus, tra is surjective. Finally, tra is injective because

$$|M| = |H^2_\varphi(G,K^*)| \leq |\operatorname{Hom}_G(A,K^*)| \leq |\operatorname{Hom}(A,K^*)| \leq |A| \leq |M|. \qquad \square$$

From the above chain of inequalities, it follows that

(1) all characters of A are G-equivariant, namely $\operatorname{Hom}_G(A,K^*) = \operatorname{Hom}(A,K^*)$,

(2) $A \simeq \operatorname{Hom}(A,K^*)$,

(3) $A \simeq H^2_\varphi(G,K^*)$,

(4) the group Γ has minimal order $|\Gamma| = |G||H^2_\varphi(G,K^*)|$,

(5) $H^1_\varphi(G,K^*) \simeq H^1_\varphi(\Gamma,K^*)$ using the inflation-restriction sequence

$$0 \longrightarrow H^1_\varphi(G,K^*) \longrightarrow H^1_\varphi(\Gamma,K^*) \longrightarrow \operatorname{Hom}_G(A,K^*) \xrightarrow{\sim} H^2_\varphi(G,K^*).$$

Note that (5) is equivalent to tra being injective. If $\operatorname{char}(K) \neq 0$, the action φ is trivial. Moreover, property (2) is equivalent to $\operatorname{char}(K) \nmid |A|$, which from (3) confirms the known result that $\operatorname{char}(K) \nmid |H^2(G,K^*)|$ (see [Kar85, Theorem 2.3.2]).

The above observations motivate the following definition:

Definition 6.2.2. Let $\varphi\colon G \to \operatorname{Aut}(K)$ be an action of a finite group G on an algebraically closed field K. A group Γ is called a *φ-twisted representation group* of G if there exists an extension

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad \text{with } A \text{ finite and abelian}$$

such that the following conditions hold:

(1) $\operatorname{char}(K) \nmid |A|$,

(2) $\operatorname{Hom}_G(A,K^*) = \operatorname{Hom}(A,K^*)$,

(3) the transgression map

$$\operatorname{tra}\colon \operatorname{Hom}_G(A,K^*) \to H^2_\varphi(G,K^*)$$

is an isomorphism.

Next a numerical criterion is given to decide whether an extension is a φ-twisted representation group.

Proposition 6.2.3. *Let*

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

be an extension by a finite abelian group A. Then Γ is a φ-twisted representation group of G if and only if the following conditions are satisfied:

(1) $|A| = |H^2_\varphi(G,K^*)|$,

(2) $|\operatorname{Hom}_G(A, K^*)| = |\operatorname{Hom}(A, K^*)|$, *and*

(3) $|H^1_\varphi(G, K^*)| = |H^1_\varphi(\Gamma, K^*)|$.

Proof. Clearly, every φ-twisted representation group fulfills the three conditions. Conversely, if they hold, then the inflation-restriction sequence together with (3) implies that

$$H^1_\varphi(G, K^*) \simeq H^1_\varphi(\Gamma, K^*)$$

and hence, the transgression map is injective. Condition (1) implies $\operatorname{char}(K) \nmid |A|$. Therefore, by using condition (2), we have

$$\operatorname{Hom}_G(A, K^*) = \operatorname{Hom}(A, K^*) \simeq A.$$

Thus, the transgression map is also surjective. □

Proposition 6.2.4. *If $\varphi\colon G \to \operatorname{Aut}(K)$ is the trivial action, then an extension as in Definition 6.2.2 is a stem extension.*

Proof. Since φ is trivial, the inflation-restriction sequence is

$$1 \longrightarrow \operatorname{Hom}(G, K^*) \longrightarrow \operatorname{Hom}(\Gamma, K^*) \longrightarrow \operatorname{Hom}(A, K^*) \longrightarrow H^2(G, K^*).$$

As the transgression map is an isomorphism, restriction $\operatorname{Hom}(\Gamma, K^*) \to \operatorname{Hom}(A, K^*)$ is trivial. Let $H_\Gamma := \operatorname{Hom}(\Gamma^{\mathrm{ab}}, K^*)$ and $H_A := \operatorname{Hom}(A, K^*)$. Then

$$\bigcap_{\bar\lambda \in H_\Gamma} \ker(\bar\lambda) \subset \Gamma^{\mathrm{ab}}$$

is the Sylow p-subgroup of Γ^{ab} for $p := \operatorname{char}(K) > 0$ (cf. [Isa76, Exercise 9.17]), and is trivial for $p = 0$ (cf. [Isa76, Lemma 2.21]). Now, λ_A is trivial for each lift of $\bar\lambda$ to Γ and hence, $A \subset [\Gamma, \Gamma]$ since p does not divide the order of A. Similarly, for $a \in A$, it holds that $\lambda(s(g)as(g)^{-1}) = \lambda(a)$ for all $\lambda \in H_A$ and all $g \in G$. Thus

$$s(g)as(g)^{-1}a^{-1} \in \bigcap_{\lambda \in H_A} \ker(\lambda),$$

which is trivial. Hence, A is a subgroup of $Z(\Gamma)$. □

The next proposition shows that Definition 6.2.2 is an extension of the definition of a representation group for trivial φ.

Proposition 6.2.5. *If the G-action on K is trivial, Definition 6.2.2 is equivalent to the classical definition of a representation group (see [Isa76, Corollary 11.20]):*

1. The extension $1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$ *is stem,*

2. $|A| = |H^2(G, K^*)|$.

Proof. Suppose the extension satisfies the conditions of Definition 6.2.2, then Proposition 6.2.4 implies that it is stem and (2) is obviously true. Conversely, assume we have a stem extension

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

such that $|A| = |H^2(G, K^*)|$. Then $\mathrm{char}(K)$ does not divide $|H^2(G, K^*)|$, as previously noted. Moreover, every homomorphism $A \to K^*$ is G-equivariant since $A \subset Z(\Gamma)$. Furthermore, the inflation-restriction sequence yields that the transgression map is injective since A is contained in the commutator subgroup $[\Gamma, \Gamma]$, and hence $\mathrm{Hom}(G, K^*) \simeq \mathrm{Hom}(\Gamma, K^*)$. Finally, $\mathrm{Hom}(A, K^*)$ is isomorphic to A since $\mathrm{char}(K)$ does not divide the order of A, and so, the transgression map is also surjective by (2). $\square$

Note that the order of a φ-twisted representation group Γ is unique, whereas the group itself generally is not (see examples in Section 6.3), even in the projective case.

6.3. Examples

6.3.1. Basic examples of semi-projective representations and twisted representation groups

Example 6.3.1. Consider $K = \mathbb{C}$ as a $G = \mathbb{Z}_2$-module, where $1 \in \mathbb{Z}_2$ acts via complex conjugation $\varphi(1)(z) = \mathrm{conj}(z) = \overline{z}$. In this example, a twisted representation group Γ is of order 4 because

$$H^2_\varphi(\mathbb{Z}_2, \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\mathbb{Z}_2}/N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*) \simeq \mathbb{R}^*/\mathbb{R}^+ \simeq \mathbb{Z}_2.$$

The transgression map is required to be an isomorphism, so the extension

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \Gamma \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

has to be non-split, which implies $\Gamma \simeq \mathbb{Z}_4$. Consider the semi-projective representation

$$f\colon \mathbb{Z}_2 \to \mathrm{PGL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2, \quad 1 \mapsto \left(\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right], \mathrm{conj} \right).$$

Its cohomology class in $H^2_\varphi(\mathbb{Z}_2, \mathbb{C}^*)$ is represented by the normalized 2-cocycle α, whose only non-trivial value is $\alpha(1,1) = -1$. It has no lift to a semi-linear representation of $\mathbb{Z}_2$ but a semi-linear lift to Γ is given by

$$F\colon \mathbb{Z}_4 \to \mathrm{GL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2, \quad 1 \mapsto \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathrm{conj} \right).$$

In the following, we explain how to use a computer algebra system, such as MAGMA [BCP97], to produce all twisted representation groups of a given finite group G in the case $K = \mathbb{C}$. We assume that $\varphi\colon G \to \mathrm{Aut}(\mathbb{C})$ takes values in $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \{\mathrm{id}, \mathrm{conj}\}$.

The conditions in Proposition 6.2.3 need to be satisfied. Since we want to use a computer, it is necessary to replace the module $\mathbb{C}^*$ by a discrete module. Identifying complex conjugation with multiplication by -1, the homomorphism φ induces an action of G on $\mathbb{Z}$ that is also denoted by φ. In this way, we can consider φ as a complex character of G of degree 1 with values in $\{\pm 1\}$. Furthermore, the exponential sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2\pi i} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 1$$

becomes a sequence of G-modules. Since the cohomology groups $H^n_\varphi(G, \mathbb{C})$ vanish for $n \geq 1$, see [Bro82, III, Corollary 10.2], the corresponding long exact sequence induces isomorphisms

$$H^n_\varphi(G, \mathbb{C}^*) \simeq H^{n+1}_\varphi(G, \mathbb{Z}) \qquad \text{for all} \quad n \geq 1.$$

Similarly, these isomorphisms hold for the cohomology groups of Γ.

These considerations lead to *Algorithm 1*. It takes as inputs a finite group G and an action φ, which is given as a character with values in $\{\pm 1\}$, and it returns all φ-twisted representation groups of G.

Algorithm 1 φ-twisted representation groups

```
function TwistedRepresentationGroups(G, φ)
input: Finite group G, φ ∈ Irr(G) of degree one with values in {±1}
output: List of all φ-twisted representation groups of G

A ← H^3(G, ℤ)
(Γ_1, ..., Γ_k) ← extensions of G by A
L ← empty list
for j = 1, ..., k do
    test ← true
    for χ ∈ Irr(A) do
        if χ is not G-invariant then
            test ← false
        end if
    end for
    if test = true and #H^2(G, ℤ) = #H^2(Γ_j, ℤ) then
        L ← append(L, Γ_j)                          ▷ add Γ_j to the list L
    end if
end for
return L
```

A MAGMA-implementation can be found in the Appendix A.3.

Example 6.3.2. Running our code, we compute the φ-twisted representation groups of the dihedral group

$$\mathcal{D}_4 = \langle s, t \mid s^2 = t^4 = 1,\ sts^{-1} = t^3 \rangle$$

for all possible actions $\varphi\colon \mathcal{D}_4 \to \mathrm{Aut}(\mathbb{C})$:

$\varphi(s)$	$\varphi(t)$	$A = H^2_\varphi(\mathcal{D}_4, \mathbb{C}^*)$	φ-twisted representation groups
1	1	$\mathbb{Z}_2$	$\langle 16,7\rangle,\ \langle 16,8\rangle,\ \langle 16,9\rangle$
-1	-1	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\langle 32,14\rangle,\ \langle 32,13\rangle$
1	-1	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\langle 32,9\rangle,\ \langle 32,10\rangle,\ \langle 32,14\rangle,\ \langle 32,13\rangle$
-1	1	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\langle 32,2\rangle,\ \langle 32,10\rangle,\ \langle 32,13\rangle$

Here, the symbol $\langle n, d\rangle$ denotes the d-th group of order n in MAGMA's *Database of Small Groups*.

6.3.2. Application of semi-projective representations: homeomorphisms and biholomorphisms of torus quotients

Let us finally come back to the situation explained at the very beginning of this chapter: Let $T = \mathbb{C}^n/\Lambda$ be a complex torus and $\rho\colon G \hookrightarrow \mathrm{GL}(n,\mathbb{C})$ a faithful irreducible representation of a finite group G of complex type fixing the lattice Λ, and denote by $\rho_{\mathbb{R}}$ its decomplexification. Solving the equation

$$C \cdot \rho_{\mathbb{R}} \cdot C^{-1} = \rho_{\mathbb{R}} \circ \psi$$

for some $\psi \in \mathrm{Stab}(\chi_{\mathbb{R}})$ yields a semi-projective representation

$$f\colon \mathrm{Stab}(\chi_{\mathbb{R}}) \longrightarrow \mathrm{PGL}(n,\mathbb{C}) \rtimes \mathbb{Z}_2.$$

Since ρ is faithful, the representation f is also faithful. Given two actions of G on T with analytic representation ρ, the candidates for the linear part C of potential homeomorphisms are the elements in the group

$$\mathcal{N} := \{C \in \mathrm{GL}(n,\mathbb{C}) \rtimes \mathbb{Z}_2 \mid [C] \in \mathrm{im}(f),\ C \cdot \Lambda = \Lambda\}.$$

By construction, the group $\mathcal{N}$ sits inside the short exact sequence

$$1 \longrightarrow A \longrightarrow \mathcal{N} \longrightarrow S \longrightarrow 1,$$

where $A := \{\mu \in \mathbb{C}^* \mid \mu\Lambda = \Lambda\}$ and $S \subset \mathrm{Stab}(\chi_{\mathbb{R}})$ is the subgroup of automorphisms ψ such that $f(\psi)$ has a representative C_ψ with $C_\psi \cdot \Lambda = \Lambda$.

Proposition 6.3.3. *The group A is a finite cyclic group. In particular, $\mathcal{N}$ is finite.*

Proof. We claim that $|\mu| = 1$ for all $\mu \in A$. Suppose there exists an element $\mu \in A$ with modulus different from 1; note that we can always assume $|\mu| < 1$, otherwise we replace μ by its inverse. Let $v \in \Lambda$ be a non-zero element of minimal norm. Then $w := \mu v \in \Lambda$ has norm strictly less than v, which contradicts the minimality of v. Thus, $|\mu| = 1$ and the map defined by multiplication with μ restricts to closed balls $\overline{B}_r$ of any radius r. If r is chosen large enough, so that $\overline{B}_r$ contains a non-zero element of Λ, then the multiplication-homomorphism

$$A \to \mathrm{Sym}\left(\overline{B}_r \cap \Lambda\right), \quad \mu \mapsto (v \mapsto \mu v)$$

is injective. Since Λ is discrete, the intersection $\overline{B}_r \cap \Lambda$ is finite and it follows that A is a finite cyclic group. $\square$

Remark 6.3.4. The inclusion $i\colon \mathcal{N} \to \mathrm{GL}(n,\mathbb{C}) \rtimes \mathbb{Z}_2$ is by construction a semi-linear lift of the semi-projective representation $f_{|S}\colon S \to \mathrm{PGL}(n,\mathbb{C}) \rtimes \mathbb{Z}_2$.

Example 6.3.5. We discuss the example of the Heisenberg group

$$\mathrm{He}(3) := \langle g,h,k \mid g^3 = h^3 = k^3 = [g,k] = [h,k] = 1, \quad [g,h] = k\rangle$$

acting on the torus $T = \mathbb{C}^3/\Lambda$, where the lattice is one of the following

$$\Lambda_1 := \mathbb{Z}[\zeta_3]^3 + \langle (t,\ t,\ t)\rangle \qquad \text{or} \qquad \Lambda_2 := \Lambda_1 + \langle (t,\ -t,\ 0)\rangle,$$

where $t := \frac{1}{3}(1+2\zeta_3)$ (cf. Section 4.3.2).
The group He(3) has two irreducible complex three-dimensional representations: The first one is given by

$$\rho(g) := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \rho(h) := \begin{pmatrix} 1 & & \\ & \zeta_3^2 & \\ & & \zeta_3 \end{pmatrix}, \qquad \rho(k) = \begin{pmatrix} \zeta_3 & & \\ & \zeta_3 & \\ & & \zeta_3 \end{pmatrix},$$

and the second one is its complex conjugate $\overline{\rho}$. Note that they both have Schur index one. Furthermore, the decomplexification $\rho_{\mathbb{R}}$ of ρ is the unique irreducible 6-dimensional representation of He(3). Hence, $\mathrm{Stab}(\chi_{\mathbb{R}})$ is the full automorphism group $\mathrm{Aut}(\mathrm{He}(3)) \simeq \mathrm{AGL}(2,3)$.

In this example, $A = \langle -\zeta_3\rangle \simeq \mathbb{Z}_6$ and, for both lattices Λ_1 and Λ_2, the group $\mathcal{N}$ contains the $\mathbb{C}$-linear maps

$$C_1 := \begin{pmatrix} \zeta_3 & & \\ & \zeta_3^2 & \\ & & 1 \end{pmatrix}, \qquad C_2 := -u \cdot \begin{pmatrix} 1 & \zeta_3^2 & \zeta_3^2 \\ \zeta_3^2 & 1 & \zeta_3^2 \\ \zeta_3^2 & \zeta_3^2 & 1 \end{pmatrix}, \qquad C_3 := u \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 \end{pmatrix}$$

and the $\mathbb{C}$-antilinear map $C_4(z_1,z_2,z_3) = (\bar{z}_1,\bar{z}_2,\bar{z}_3)$. A MAGMA computation shows that the elements $C_1,\ldots,C_4$ generate a subgroup of $\mathcal{N}$ of order $2592 = |A| \cdot |\operatorname{Stab}(\chi_{\mathbb{R}})|$. Hence, this subgroup is actually equal to $\mathcal{N}$ and every class in the image of

$$f\colon \operatorname{Stab}(\chi_{\mathbb{R}}) \to \mathrm{PGL}(n,\mathbb{C}) \rtimes \mathbb{Z}_2$$

is represented by an element in $\mathcal{N}$. However, even if the semi-projective representation f lifts to $\mathcal{N}$, this group is not a φ-twisted representation group for the action $\varphi\colon \operatorname{Stab}(\chi_{\mathbb{R}}) \to \mathrm{Aut}(\mathbb{C})$ induced by f. Indeed, a MAGMA computation (cf. Appendix A.3) yields $H^1(\operatorname{Stab}(\chi_{\mathbb{R}}),\mathbb{C}^*) \simeq \mathbb{Z}_3$ and $H^1(\mathcal{N},\mathbb{C}^*) \simeq \mathbb{Z}_6$, which violates the third condition of Proposition 6.2.3.

Appendix A.

MAGMA-Codes

The MAGMA-codes used for the classifications are provided below. They can also be found on the enclosed USB stick.

A.1. Classification of the groups

We start with the functions for the proofs in Chapter 3. First, we introduce some “general” functions that will be needed in the main code.

```
/* The function "DimChar" collects all irreducible characters of a finite group "G" of degree "i".
As input, it additionally takes the character table "CT" of "G". */

function DimChar(G,CT,i)
     Set:=[];
     for x in CT do
          if Degree(x) eq i then Set:=Append(~Set,x); end if;
     end for;
     return Set;
end function;

/* The function "CharPol" determines the characterisitc polynomial of rho(g), where rho is a 3-dimensional
representation with character "x". */

function CharPol(x,g)
     L:=[x(g^i): i in[1..3]];
     return Polynomial(PowerSumToCoefficients(L));
end function;

/* The function "ElsOfOrder" determines all elements of order "m" of a group "G". */

function ElsOfOrder(G,m)
     list:=[];
     for g in G do
          if Order(g) eq m then Append(~list,g); end if;
     end for;
     return list;
end function;

/* The function "CandidatesChar" determines all characters of degree 3 of a group "G" with character
table "CT". */

function CandidatesChar(G,CT)
     listChar:=[];
     if IsAbelian (G) then
```

```
            for n1 in [1..#CT] do for n2 in [n1..#CT] do for n3 in [n2..#CT] do
                  Append(~listChar,CT[n1]+CT[n2]+CT[n3]);
            end for; end for; end for;
      else
            listChar:=listChar cat DimChar(G,CT,3);
            for x1 in DimChar(G,CT,1) do for x2 in DimChar(G,CT,2) do
                  Append(~listChar,x1+x2);
            end for; end for;
      end if;
      return listChar;
end function;

/* The function "Eigen1" checks, for a given character "x" of a group "G", whether all elements of order "i"
have eigenvalue 1. */

function Eigen1(x,G,i)
      for g in ElsOfOrder(G,i) do
      pol:=CharPol(x,g);
      if Evaluate(pol,1) ne 0 then return false; end if;
      end for;
      return true;
end function;

/* The function "TestOrder4" checks, for a character "x" of "G", whether for all elements of order 4, either
1 is an eigenvalue or the set of eigenvalues is {i,-i}. */

function TestOrder4(x,G)
      F<z>:=CyclotomicField(8);
      i:=z^2;
      for g in ElsOfOrder(G,4) do
            pol:=CharPol(x,g);
            if Evaluate(pol,1) ne 0 then
                  if Evaluate(pol,-1) eq 0 then return false; end if;
                  if {Evaluate(pol,i),Evaluate(pol,-i)} ne {0} then return false; end if;
            end if;
      end for;
      return true;
end function;
```

The next block of functions is for checking whether a finite group fulfills the standard conditions (cf. Notation 3.0.12). For the convenience of the reader, we recall them: A finite group G enjoys the standard conditions if there exists a three-dimensional representation $\rho\colon G \to \mathrm{GL}(3,\mathbb{C})$ with character χ, such that

(1) $\langle \rho, \overline{\rho} \rangle = 0$,

(2) ρ is faithful,

(3) for all $g \in G$, the characteristic polynomial of $\rho(g) \oplus \overline{\rho(g)}$ has integer coefficients,

(4) for all $g \in G$, it holds that $\mathrm{ord}(g) \in \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 12,\ 14\}$,

(5) if $\mathrm{ord}(g) \in \{5,\ 8,\ 10,\ 12\}$, then 1 is an eigenvalue of $\rho(g)$,

(6) if $\mathrm{ord}(g) \in \{7,\ 9,\ 14\}$, then 1 is not an eigenvalue of $\rho(g)$.

```
function CheckOrders(G)
      L:={1,2,3,4,5,6,7,8,9,10,12,14};
      for g in G do
            if not Order(g) in L then return false; end if;
```

```
      end for;
      return true;
end function;

/* The function "CondCharpol" checks the conditions (3), (5), (6) for a 3-dimensional character "x" of
a group "G". */

function CondCharpol(x,G)
      F<ze>:=CyclotomicField(#G);
      P<y>:=PolynomialRing(F);
      for d in {5,8,10,12} do
            if not Eigen1(x,G,d) then return false; end if;
      end for;
      for d in {7,9,14} do
            for g in ElsOfOrder(G,d) do
                  if Evaluate(CharPol(x,g),1) eq 0 then return false; end if;
            end for;
      end for;
      for g in G do
            f:=CharPol(x,g)*CharPol(ComplexConjugate(x),g);
            for a in Coefficients(f) do
                  if a notin IntegerRing() then return false; end if;
            end for;
      end for;
      return true;
end function;

function StandardProperties(G)
      if not CheckOrders(G) then return false; end if;
      CT:=CharacterTable(G);
      for x in CandidatesChar(G,CT) do
            if #Kernel(x) eq 1 and InnerProduct(x,ComplexConjugate(x)) eq 0 and CondCharpol(x,G) then
                  return true;
            end if;
      end for;
      return false;
end function;
```

The function “LinearGroups” finishes the proof of Theorem 3.0.5, where the possible stabilizer groups are characterized.

```
function LinearGroups()
      ListCandidatesC0:=[]; ListCandidatesC1:=[];
      testC1:=true;
      for b in [1..2] do
            n:=2^2*3^b;
            for G in SmallGroups(n) do
            CT:=CharacterTable(G);
            if #DimChar(G,CT,3) ge 1 then Append(~ListCandidatesC0, IdentifyGroup(G)); end if;
            end for;
      end for;
      H1:=Group("C7:C3"); H2:=Group("C7:C9");
      for b in [1..2] do for a in [0..2] do
            for G in SmallGroups(2^a*3^b*7) do
                  if #ElsOfOrder(G,2) le 1 then
                        CT:=CharacterTable(G);
                              if #DimChar(G,CT,3) ge 1 then
                                    Append(~ListCandidatesC1, IdentifyGroup(G));
                                    testG:=false;
                                    for H in AllSubgroups(G) do
```

```
                                        if IsIsomorphic(H,H1) or IsIsomorphic(H,H2) then testG:=true; end if;
                                    end for;
                                    if not testG then testC1:=false; end if;
                                end if;
                    end if;
                end for;
        end for; end for;
        return ListCandidatesC0, ListCandidatesC1, testC1;
end function;
```

The following functions rule out some cases in the proof of Proposition 3.0.14, where we discussed the groups admitting an analytic representation containing $-\mathrm{id}$ in its image.

```
function Cases3and8and11()
    list:=[];
    for a in [3..5] do
    for G in SmallGroups(2^a) do
        if not CheckOrders(G) then return false; end if;
        CT:=CharacterTable(G);
        for x in CandidatesChar(G,CT) do
            if #Kernel(x) eq 1 and InnerProduct(x,ComplexConjugate(x)) eq 0 and CondCharpol(x,G) then
                if a eq 4 then
                // if ord(g)=4 then rho(g) has to have eigenvalue 1, this is checked by "Eigen1"
                    if Eigen1(x,G,4) then Append(~list, IdentifyGroup(G)); break x; end if;
                else
                    if TestOrder4(x,G) then Append(~list, IdentifyGroup(G)); break x; end if;
                end if;
            end if;
        end for;
    end for;
    end for;
    return list;
end function;

/* The function "TestOrder6" checks the conditions about the eigenvalues of elements of order 6 as explained
in the proof of Proposition 3.0.14, cases 2 and 7. Note that for the elements of order 6 without eigenvalue
1, we only check whether all eigenvalues are of order 6, which already leads to a contradiction.*/

function TestOrder6(x,G)
    F<ze>:=CyclotomicField(24);
    z6:=ze^4;
    m6:=0;
    for g in ElsOfOrder(G,6) do
        pol:=CharPol(x,g);
        if Evaluate(pol,1) ne 0 then
            if Evaluate(pol,z6^2) eq 0 or Evaluate(pol,z6^3) eq 0 or Evaluate(pol,z6^4) eq 0 then
                return false;
            end if;
            m6:=m6+1;
        end if;
    end for;
    if m6 ne 2*#G/6 then return false; end if;
    return true;
end function;

function Cases2and7()
    list:=[];
    for n in {24,96} do
        for G in SmallGroups(n) do
            CT:=CharacterTable(G);
```

```
                if not (CheckOrders(G) and #ElsOfOrder(G,6) ge 2*n/6) then return false; end if;
                for x in CandidatesChar(G,CT) do
                     if #Kernel(x) eq 1 and InnerProduct(x,ComplexConjugate(x)) eq 0 and CondCharpol(x,G)
                      and TestOrder6(x,G) then
                          if n eq 24 and Eigen1(x,G,4) then
                               Append(~list,IdentifyGroup(G));
                          else
                               Append(~list,IdentifyGroup(G));
                          end if;
                     end if;
                end for;
           end for;
     end for;
     return list;
end function;
```

The functions “Case14” and “Case12” are used to finish the proof of Proposition 3.0.15, where we discussed the case that $\zeta_3 \cdot \mathrm{id}$ is contained in the image of the analytic representation.

```
function Case14()
     list:=[]; checklist:=[];
     for G in SmallGroups(3^5) do
          if #ElsOfOrder(G,9) eq 54 and CheckOrders(G) then
               Append(~list, IdentifyGroup(G)); Append(~checklist, StandardProperties(G));
          end if;
     end for;
     return list, checklist;
end function;

function Case12()
     list:=[]; checklist:=[];
     for G in SmallGroups(3^4) do
          if #ElsOfOrder(G,9) eq 54 and CheckOrders(G) then
               Append(~list, IdentifyGroup(G)); Append(~checklist, StandardProperties(G));
          end if;
     end for;
     return list, checklist;
end function;
```

The following code determines the possible 2-subgroups of a group G in the situation where the quotient $X = T/G$ has only singularities of type $\frac{1}{3}(1,1,2)$. In particular, all non-trivial elements in such a 2-subgroup act freely. The output is part of (the proof) of Lemma 3.0.20.

```
function CondCharpolFree(x,G)
     for g in G do
          pol:=CharPol(x,g);
          if Evaluate(pol,1) ne 0 then return false; end if;
          f:=CharPol(x,g)*CharPol(ComplexConjugate(x),g);
          for a in Coefficients(f) do
               if a notin IntegerRing() then return false; end if;
          end for;
     end for;
     return true;
end function;

function CheckGroupFree(G)
     if not CheckOrders(G) then return false; end if;
     CT:=CharacterTable(G);
     for x in CandidatesChar(G,CT) do
```

```
            if #Kernel(x) eq 1 and CondCharpolFree(x,G) then return true; end if; end for;
        return false;
end function;

function AllTwoSylow(k,SetGroups)
        n:=2^k;
        for G in SmallGroups(n: Warning := false) do
            N:=Subgroups(G);
            subs:={IdentifyGroup(N[i]`subgroup) : i in [1..#N]| N[i]`order lt n};
            if subs subset SetGroups then
                if CheckGroupFree(G) then SetGroups:=Include(SetGroups,IdentifyGroup(G)); end if;
            end if;
        end for;
        return SetGroups;
end function;

function ListTwoSylows()
        SetGroups:={<1,1>};
        k:=1;
        test:=true;
        while test do
            if k ge 10 then print "Warning: Group order becomes to big"; return false; end if;
            k;
            SetGroupsRef:=SetGroups;
            SetGroups:=AllTwoSylow(k,SetGroups);
            if SetGroups eq SetGroupsRef then test:=false; end if;
            k:=k+1;
        end while;
        return SetGroups;
end function;
```

Finally, the remaining functions finish the proof of Proposition 3.0.16, which handles the case where no central element is contained in the image of the analytic representation.

```
/* The function "TestOrder3" checks the conditions for the elements of order 3 for a character "x" of a
group "G", as explained in the proof of Proposition 3.0.16. */

function TestOrder3(x,G)
        F<ze>:=CyclotomicField(24);
        z3:=ze^8;
        m3:=0;
        for g in ElsOfOrder(G,3) do
            pol:=CharPol(x,g);
            if Evaluate(pol,1) ne 0 then
                if {Evaluate(pol,ze), Evaluate(pol,ze^2)} ne {0} then return false; end if;
                m3:=m3+1;
            end if;
        end for;
        if m3 ne 2*#G/9 then return false; end if;
        return true;
end function;

/* Since in the situation of Proposition 3.0.16, no elements of order 9 are allowed, we write a new
function to check the orders of the elemnts of group "G". */

function CheckOrdersCase15(G)
        L:={1,2,3,4,5,6,8,10,12};
        for g in G do
            if not Order(g) in L then return false; end if;
        end for;
```

```
     return true;
end function;

/* The function "WithOrd5" shows that if a group fitting in the setup of Proposition 3.0.16 has an element
of order 5, then it is the group with MAGMA ID <360, 118>. */

function WithOrd5()
     list:=[];
     for a in [0..5] do
          for G in SmallGroups(2^a*3^2*5: Warning:=false) do
               if CheckOrdersCase15(G) and #ElsOfOrder(G,3) ge 2^a*2*5 then
                    Append(~list,IdentifyGroup(G));
               end if;
          end for;
     end for;
     return list;
end function;

/* The function "NoOrd5" excludes all groups in Proposition 3.0.16 of order 2^a*3^2 with a=1,...,5. */

function NoOrd5()
     list:=[];
     for a in [1..5] do
          for G in SmallGroups(2^a*3^2: Warning:=false) do
               if not (CheckOrders(G) and #ElsOfOrder(G,3) ge 2*#G/9) then return false; end if;
               CT:=CharacterTable(G);
               for x in CandidatesChar(G,CT) do
                    if #Kernel(x) eq 1 and InnerProduct(x,ComplexConjugate(x)) eq 0 and TestOrder3(x,G)
                     and CondCharpol(x,G) then
                         Append(~list, IdentifyGroup(G));
                         break x;
                    end if;
               end for;
          end for;
     end for;
     return list;
end function;
```

A.2. Classification of the quotients

In this section, we collect all functions that we need for the classification of the quotients (cf. Section 4.3. The code mainly consists of two parts:

- Part I: general functions that we need for several groups
- Part II: specific classification functions for each group:
  (a) $G = \mathbb{Z}_3^2,\ \rho = \rho_1$ (proof of Proposition 4.3.22)
  (b) $G = \mathbb{Z}_3^2,\ \rho = \rho_2$ (proof of Proposition 4.3.25)
  (c) $G = \mathbb{Z}_3^2,\ \rho = \rho_3$ (proof of Proposition 4.3.28)
  (d) $G = \mathbb{Z}_3^3$ (proof of Proposition 4.3.32)
  (e) $G = \mathrm{He}(3)$ (proof of Proposition 4.3.37)
  (f) $G = \mathbb{Z}_9 \rtimes \mathbb{Z}_3$ (proof of Proposition 4.3.39)

```
/************************ PART I: general functions ***************************/

// We work with the 3rd cyclotomic field, ze is a 3rd primitive root of unity
F<ze>:=CyclotomicField(3);

// E3 denotes the group of 3-torsion points of the Fermat elliptic curve E,
// t is a generator of the fixed locus of ze in E.
E3:={1/3*(a+b*ze): a, b in {0,1,2}};
t:=1/3+2/3*ze;

I3:=DiagonalMatrix([1,1,1]);

/* The function "IntegralTest" checks if the entries of a 3-vector v (given as 3x1-matrix) are
Eisenstein integers, i.e., integral over Z. */

function IntegralTest(v)
     return IsIntegral(v[1][1]) and IsIntegral(v[2][1]) and IsIntegral(v[3][1]);
end function;

/* The function "InLatt" takes as input a vector "v" and a kernel "K", and decides whether "v"
belongs to the lattice "Lambda_K=Z[ze]^3+K". */

function InLatt(v,K)
     for l in K do
          if IntegralTest(v-l) then return true; end if;
     end for;
     return false;
end function;

/* The function "TestKernelEndo" decides whether the vector "d" belongs to the kernel of the ndomorphism "A"
of the torus "T=E^3/K". */

function TestKernelEndo(A,d,K)
     if InLatt(A*d,K) then return true; end if;
     return false;
end function;

/* The function "Fix" determines the fixed points of an automorphism "A" of T=E^3/K;
"candidates": list of all candidates for the fixed points. */

function FixPoints(A,K,candidates)
     list:=[];
     for d in candidates do
          if TestKernelEndo((A-I3),d,K) then Append(~list,d); end if;
     end for;
     Fix:=[];
     while not IsEmpty(list) do
          Reflist:=list;
          d:=Rep(list); Append(~Fix,d);
          for e in Reflist do
               if InLatt(d-e,K) then Exclude(~list,e); end if;
          end for;
     end while;
     return Fix;
end function;

/* With the function "TestCohom", we test if two given cocycles give the same cohomology class in
H^1(Z_3^2,E^3/K).
Input: "v1", "v2": two lists of translation vectors; "listGen": corresponding elements of the groups,
"K": kernel; "coboundaries": list of candidates for the coboundaries */

```

```
function TestCohom(v1,v2,listGen,K,coboundaries)
     for d in coboundaries do
          test:=true;
          for i in [1..#listGen] do
               if not InLatt(v1[i]-v2[i]-(listGen[i]*d-d),K) then test:=false; end if;
          end for;
          if test then return true; end if;
     end for;
     return false;
end function;

/* The function "Normal" determines the normalizers "N_C(Lambda_K)" in the cases G=Z3^2 and G=Z3^3.
Here, "N_C(Lambda_K)" is just the subgroup of "N"=normalizer of G in Aut(E^3) fixing K. */

function Normal(K,N)
     List:={};
     for C in N do
          test:= true;
          for l in K do
               if not InLatt(C*l,K) or not InLatt(C^-1*l,K) then test:=false; break l; end if;
          end for;
          if test then List:=Include(List,C); end if;
     end for;
     return List;
end function;

/* Given an affinity "f(z)=Cz + d" as a 4x4 matrix "B",
          B=[C d]
            [0 1],
the function "PartsAff" returns the 3x3 matrix "C" and the translation vector "d". */

function TransVec(B)
     return Submatrix(B,[1,2,3],[4]);
end function;

/* Given an affinity "f(z)=Cz+d", the function "MatAff" returns the affinity as a 4x4-matrix. */

function MatAff(C,d)
     Z:=ZeroMatrix(F,4,4);
     return InsertBlock(InsertBlock(InsertBlock(Z,C,1,1),d,1,4),Matrix(F,1,1,[1]),4,4);
end function;

/* Given a matrix C in "N_C(Lambda_K)" and a list of generators "listGen=[g_1,...,g_k]" of the group "G",
the function "IdAuto" determines the automorphism "phi" of "G" fullfilling equation (a) of Remark 4.2.10.
More precisely, it returns tuples of exponents such that the i-th tuple [n_1,...,n_k] of the output fulfills
phi(g_i)=g_1^(n_1)*...*g_k^(n_k). */

function IdAuto(C,listGen)
     seq:=[{0..Order(u)-1}: u in listGen]; loop:=CartesianProduct(seq);
     phi:=[];
     for i in [1..#listGen] do
          for exp in loop do
               if C*listGen[i]*C^-1 eq &*[listGen[i]^exp[i]: i in [1..#listGen]] then
                    Append(~phi,exp);
               end if;
          end for;
     end for;
     return phi;
end function;

/* The function "TestIso" checks if two given actions "v1" and "v2" (given as lists of translations parts)
```

```
lead to biholomorphic quotients. As input it takes the actions "v1" and "v2", the kernel "K", the norma-
lizer "Nor=N_C(Lambda_K)", and the list of possible translation vectors "coboundaries" of the affinities. */

function TestIso(v1,v2,K,Nor,coboundaries,listGen)
     W1:=[MatAff(listGen[i],v1[i]): i in [1..#listGen]]; W2:=[MatAff(listGen[i],v2[i]): i in [1..#listGen]];
     for C in Nor do
          phi:=IdAuto(C,listGen); phiW2:=W2;
          for i in [1..#listGen] do phiW2[i]:=&*[W2[i]^phi[i][j]: j in [1..#listGen]]; end for;
          for d in coboundaries do
               aff:=MatAff(C,d);
               test:=true;
               for i in [1..#listGen] do
                    if not InLatt(TransVec(aff*W1[i]-phiW2[i]*aff),K) then test:=false; end if;
               end for;
               if test then return true; end if;
          end for;
     end for;
     return false;
end function;
```

Next, we collect the specific functions for the classification corresponding to each group individually (PART II). For this, we recall the analytic representation and our choice for a cocycle in standard form for every case before quoting the code.

(a) $G = \mathbb{Z}_3^2, \ \rho = \rho_1$:

The analytic representation $\rho = \rho_1$ of $G = \mathbb{Z}_3^2 = \langle h, k \rangle$ is given by

$$\rho(h) = \operatorname{diag}(1, \ \zeta_3^2, \ \zeta_3), \qquad \rho(k) = \operatorname{diag}(\zeta_3, \ \zeta_3, \ \zeta_3),$$

and a cocycle in standard form is given by

$$\tau(h) = a = (a_1, \ a_2, \ a_3), \qquad \tau(k) = 0.$$

```
/* The function "GoodCondZ3xZ3_1" checks whether a cocycle in standard form is good, here "a=tau(h)". */

function GoodCondZ3xZ3_1(a,K)
     for t1 in [0,t,-t] do for t2 in [0,t,-t] do
          w1:=Matrix(F,3,1,[a[1][1],t1,t2]);
          w2:=Matrix(F,3,1,[t1,a[2][1],t2]);
          w3:=Matrix(F,3,1,[t1,t2,a[3][1]]);
          if InLatt(w1,K) or InLatt(w2,K) or InLatt(w3,K) then return false; end if;
     end for; end for;
     return true;
end function;

/* The function "ActionsZ3xZ3_1" determines all good cocycles in standard form on E^3/K for each kernel K.
The second output is a list consisting of one representative for each good cohomology class.
The actions are given as tuples of translation parts of the generators h,k.
input: kernel "K", linear parts "listGen" of the actions of the generators, list "Fix" of fixed points of
"rho(k)" (coboundaries and the candidates for "a=tau(h)"). */

function ActionsZ3xZ3_1(K,Fix,listGen)
     ListTransVec:={}; ListOfActions:={};
     for a in Fix do
          if GoodCondZ3xZ3_1(a,K) then
```

```
               ListTransVec:=Include(ListTransVec, [a,Matrix(F,3,1,[0,0,0])]);
          end if;
     end for;
     ListOfActions:=ListTransVec; GoodClasses:={}; RefList:=ListTransVec;
     while not IsEmpty(ListTransVec) do
          RefList:=ListTransVec;
          v1:=Rep(ListTransVec); Include(~GoodClasses,v1);
          for v2 in ListTransVec do
               if TestCohom(v1,v2,listGen,K,Fix) then Exclude(~ListTransVec,v2); end if;
          end for;
     end while;
     return ListOfActions, GoodClasses;
end function;

/* The function "ClassZ3xZ3_1" is the main classification function for G=Z3^2 and rho=rho_1.
The output file "Z3xZ3_rho1.txt" contains for each kernel
1) the number of actions,
2) the number of good cohomology classes,
3) the size of the normalizer N_C(Lambda_K),
4) the number of biholomorphism classes and
5) for each biholomorphism class a corresponding action on E^3/K.
*/

function ClassZ3xZ3_1(K,listGen,N,j)
     File:="Z3xZ3_rho1.txt";
     fprintf File, "Kernel %o)\n \n", j;
     IsoClasses:=[];
     Fix:=FixPoints(DiagonalMatrix([ze,ze,ze]),K,{Matrix(F,3,1,[d1,d2,d3]): d1,d2,d3 in E3});
     LA,GoodClasses:=ActionsZ3xZ3_1(K,Fix,listGen);
     fprintf File, "Number of actions with isolated fixed points: %o \n \n", #LA;
     fprintf File, "Number of good cohomology classes: %o\n\n", #GoodClasses;
     RefListAct:=GoodClasses;
     Nor:=Normal(K,N);
     fprintf File, "Size of the normalizer: %o \n \n", #Nor;
     while not IsEmpty(GoodClasses) do
          RefListAct:=GoodClasses;
          v1:=Rep(GoodClasses); Append(~IsoClasses,v1);
          for v2 in RefListAct do
               if TestIso(v1,v2,K,Nor,Fix,listGen) then Exclude(~GoodClasses,v2); end if;
          end for;
     end while;
     fprintf File, "Number of biholomorphism classes: %o \n \n", #IsoClasses;
     fprintf File, "Actions [tau(h),tau(k)]: \n %o \n \n \n \n", IsoClasses;
     return "Classification for kernel", j, "is completed!";
end function;

/* With the procedure "MainZ3xZ3_1", we run the classification for the group "G=Z3^2", where rho=rho_1. */

procedure MainZ3xZ3_1()
     h:=DiagonalMatrix([1,ze^2,ze]); k:=DiagonalMatrix([ze,ze,ze]);
     listGen:=[h,k];
     K1:={Matrix(F,3,1,[0,0,0])};
     K2:={a*Matrix(F,3,1,[t,t,0]): a in {0,1,-1}};
     K3:={a*Matrix(F,3,1,[t,t,t]): a in {0,1,-1}};
     K4:={a*Matrix(F,3,1,[t,t,t])+b*Matrix(F,3,1,[t,-t,0]): a,b in {0,1,-1}};
     Kernels:=[K1,K2,K3,K4];
     N:=MatrixGroup<3,F| DiagonalMatrix([-ze,1,1]), Matrix(F,3,3,[0,1,0,0,0,1,1,0,0]),
      Matrix(F,3,3,[0,1,0,1,0,0,0,0,1])>;
     for j in [1..#Kernels] do ClassZ3xZ3_1(Kernels[j],listGen,N,j); end for;
end procedure;
```

(b) $G = \mathbb{Z}_3^2,\ \rho = \rho_2$**:**

The analytic representation $\rho = \rho_2$ of $G = \mathbb{Z}_3^2 = \langle h, k\rangle$ is given by

$$\rho(h) = \mathrm{diag}(1,\ \zeta_3^2,\ \zeta_3^2), \qquad \rho(k) = \mathrm{diag}(\zeta_3,\ \zeta_3,\ \zeta_3^2),$$

and a cocycle in standard form is given by

$$\tau(h) = a = (a_1,\ a_2,\ a_3), \qquad \tau(k) = 0.$$

```
/* The function "GoodCondZ3xZ3_2" checks whether a cocycle in standard form is good. a=tau(h). */

function GoodCondZ3xZ3_2(a,K)
     for t1 in [0,t,-t] do for t2 in [0,t,-t] do
          w1:=Matrix(F,3,1,[a[1][1],t1,t2]);
          w2:=Matrix(F,3,1,[t1,a[2][1],t2]);
          w3:=Matrix(F,3,1,[t1,t2,a[3][1]]);
          if InLatt(w1,K) or InLatt(w2,K) or InLatt(w3,K) then return false; end if;
     end for; end for;
     return true;
end function;

/* The function "ActionsZ3xZ3_2" determines all good cocycles in standard form on E^3/K for each kernel K
and a list consisting of one representative for each good cohomology class. The actions are given as
tuples of translation parts of the generators h,k.
Input: kernel "K", linear parts "listGen" of the actions of the generators, list "Fix" of fixed points of
"rho(k)" (coboundaries and candidates for "a=tau(h)"). */

function ActionsZ3xZ3_2(K,Fix,listGen)
     ListTransVec:={}; ListOfActions:={};
     for a in Fix do
          if GoodCondZ3xZ3_2(a,K) then Include(~ListTransVec, [a,Matrix(F,3,1,[0,0,0])]); end if;
     end for;
     ListOfActions:=ListTransVec; GoodClasses:={}; RefList:=ListTransVec;
     while not IsEmpty(ListTransVec) do
          RefList:=ListTransVec;
          v1:=Rep(ListTransVec); Include(~GoodClasses,v1);
          for v2 in ListTransVec do
               if TestCohom(v1,v2,listGen,K,Fix) then Exclude(~ListTransVec,v2); end if;
          end for;
     end while;
     return ListOfActions, GoodClasses;
end function;

/* The function "ClassZ3xZ3_2" is the main classification function for G=Z3^2 and rho=rho_2.
The output file "Z3xZ3_rho2.txt" contains for each kernel
1) the number of actions,
2) the number of good cohomology classes,
3) the size of the normalizer N_C(Lambda_K),
4) the number of biholomorphism classes and
5) for each biholomorphism class a corresponding action on E^3/K.
*/

function ClassZ3xZ3_2(K,listGen,N,j)
     File:="Z3xZ3_rho2.txt";
     fprintf File, "Kernel %o)\n \n", j;
     IsoClasses:=[];
     Fix:=FixPoints(DiagonalMatrix([ze,ze,ze^2]),K,{Matrix(F,3,1,[d1,d2,d3]): d1,d2,d3 in E3});
     LA,GoodClasses:=ActionsZ3xZ3_2(K,Fix,listGen);
```

```
     fprintf File, "Number of actions with isolated fixed points: %o \n \n", #LA;
     fprintf File, "Number of good cohomology classes: %o\n\n", #GoodClasses;
     RefListAct:=GoodClasses;
     Nor:=Normal(K,N);
     fprintf File, "Size of the normalizer: %o \n \n", #Nor;
     while not IsEmpty(GoodClasses) do
          RefListAct:=GoodClasses;
          v1:=Rep(GoodClasses); Append(~IsoClasses,v1);
          for v2 in RefListAct do
               if TestIso(v1,v2,K,Nor,Fix,listGen) then Exclude(~GoodClasses,v2); end if;
          end for;
     end while;
     fprintf File, "Number of biholomorphism classes: %o \n \n", #IsoClasses;
     fprintf File, "Actions [tau(h),tau(k)]: \n %o \n \n \n \n", IsoClasses;
     return "Classification for kernel", j, "is completed!";
end function;

/* With the procedure "MainZ3xZ3_2", we run the classification for the group "G=Z3^2", where "rho=rho_2". */

procedure MainZ3xZ3_2()
     h:=DiagonalMatrix([1,ze^2,ze^2]); k:=DiagonalMatrix([ze,ze,ze^2]);
     listGen:=[h,k];
     K1:={Matrix(F,3,1,[0,0,0])};
     K2:={a*Matrix(F,3,1,[t,t,0]): a in {0,1,-1}};
     K3:={a*Matrix(F,3,1,[t,0,t]): a in {0,1,-1}};
     K4:={a*Matrix(F,3,1,[t,t,t]): a in {0,1,-1}};
     K5:={a*Matrix(F,3,1,[t,t,t])+b*Matrix(F,3,1,[t,-t,0]): a,b in {0,1,-1}};
     Kernels:=[K1,K2,K3,K4,K5];
     N:=MatrixGroup<3,F|DiagonalMatrix([-ze,1,1]), DiagonalMatrix([1,1,-ze]), Matrix([0,1,0,1,0,0,0,0,1])>;
     for j in [1..#Kernels] do ClassZ3xZ3_2(Kernels[j],listGen,N,j); end for;
end procedure;
```

(c) $G = \mathbb{Z}_3^2,\ \rho = \rho_3$**:**

The analytic representation $\rho = \rho_3$ of $G = \mathbb{Z}_3^2 = \langle h, k\rangle$ is given by

$$\rho(h) = \mathrm{diag}(\zeta_3,\ \zeta_3,\ 1), \qquad \rho(k) = \mathrm{diag}(\zeta_3,\ \zeta_3,\ \zeta_3),$$

and a cocycle in standard form is given by

$$\tau(h) = a = (a_1,\ a_2,\ a_3), \qquad \tau(k) = 0.$$

We only have to consider the torus $T = E^3/K$, where $K = \langle (t,\ t,\ t)\rangle$, because the classification for the other torus $T = E^3$ is done without MAGMA.

```
/* The function "GoodCondZ3xZ3_3" checks whether a cocycle in standard form is good. a=tau(h). */

function GoodCondZ3xZ3_3(a,K)
     for t1 in [0,t,-t] do
     if InLatt(Matrix(F,3,1,[a[1][1],a[2][1],t1]),K) then return false; end if;
          for t2 in [0,t,-t] do
               if InLatt(Matrix(F,3,1,[t1,t2,a[3][1]]),K) then return false; end if;
          end for;
     end for;
     return true;
end function;
```

```

/* The function "ActionsZ3xZ3_3" determines all good cocycles in standard form on T=E^3/K and a list
consisting of one representative for each good cohomology class. The actions are given as tuples of
translation parts of the generators h,k.
Input: kernel "K", linear parts "listGen" of the actions of the generators, list "Fix" of fixed points of
"rho(k)" (coboundaries and candidates for "a=tau(h)").
*/

function ActionsZ3xZ3_3(K,Fix,listGen)
     ListTransVec:={}; ListOfActions:={};
     for a in Fix do
          if GoodCondZ3xZ3_3(a,K) then Include(~ListTransVec, [a,Matrix(F,3,1,[0,0,0])]); end if;
     end for;
     ListOfActions:=ListTransVec; GoodClasses:={}; RefList:=ListTransVec;
     while not IsEmpty(ListTransVec) do
          RefList:=ListTransVec;
          v1:=Rep(ListTransVec); Include(~GoodClasses,v1);
          for v2 in ListTransVec do
               if TestCohom(v1,v2,listGen,K,Fix) then Exclude(~ListTransVec,v2); end if;
          end for;
     end while;
     return ListOfActions, GoodClasses;
end function;

/*
The procedure "MainZ3xZ3_3" displays one representative of each good cohomology class in H^1(Z3^2, E^3/K).
Note that these representatives may differ from the ones given in the proof of Proposition 4.3.28, but they
are cohomologous; use the function "TestCohom" with "coboundaries=fixed points of rho(k)" to see this.*/

procedure MainZ3xZ3_3()
     h:=DiagonalMatrix([ze,ze,1]); k:=DiagonalMatrix([ze,ze,ze^2]);
     listGen:=[h,k];
     K:={a*Matrix(F,3,1,[t,t,t]): a in {0,1,-1}};
     Fix:=FixPoints(k,K,{Matrix(F,3,1,[d1,d2,d3]): d1,d2,d3 in E3});
     LA, GC:=ActionsZ3xZ3_3(K,Fix,listGen);
     GC;
end procedure;
```

(d) $G = \mathbb{Z}_3^3$:

The analytic representation of $G = \mathbb{Z}_3^3 = \langle h, g, k\rangle$ is given by

$$\rho(h) = \mathrm{diag}(1,\ \zeta_3^2,\ \zeta_3), \qquad \rho(g) = \mathrm{diag}(\zeta_3,\ 1,\ 1), \qquad \rho(k) = \mathrm{diag}(\zeta_3,\ \zeta_3,\ \zeta_3),$$

and a cocycle in standard form is given by

$$\tau(h) = a = (a_1,\ a_2,\ a_3), \qquad \tau(g) = b = (b_1,\ b_2,\ b_3), \qquad \tau(k) = 0.$$

```
/* Given "a" and "b" in Fix_ze3(E^3/K), the function "WellDefinedZ3xZ3xZ3" checks whether these two vectors
define a well-defined action on T=E^3/K. */

function WellDefinedZ3xZ3xZ3(a,b,K)
     if not IsIntegral(3*a[1][1]) then return false; end if;
     if not InLatt(Matrix(F,3,1,[0,3*b[2][1],3*b[3][1]]),K) then return false; end if;
     if not TestKernelEndo(DiagonalMatrix([ze-1,1-ze^2,1-ze]),Matrix(F,3,1,[a[1][1],b[2][1],b[3][1]]),K)
          then return false;
```

```
    end if;
    return true;
end function;

/* The function "GoodCondZ3xZ3xZ3" checks whether a cocycle in standard form is good. a=tau(h), b=tau(g).*/

function GoodCondZ3xZ3xZ3(a,b,K)
    for t1 in [0,t,-t] do
        w1:=Matrix(F,3,1,[t1,b[2][1],b[3][1]]);
        w2:=Matrix(F,3,1,[ze*a[1][1]+b[1][1],t1,a[3][1]+b[3][1]]);
        w3:=Matrix(F,3,1,[2*ze*a[1][1]+b[1][1],-ze*a[2][1]+b[2][1],t1]);
        if InLatt(w1,K) or InLatt(w2,K) or InLatt(w3,K) then return false; end if;
        for t2 in [0,t,-t] do
            w4:=Matrix(F,3,1,[a[1][1],t1,t2]);
            w5:=Matrix(F,3,1,[t1,a[2][1],t2]);
            w6:=Matrix(F,3,1,[t1,t2,a[3][1]]);
            w7:=Matrix(F,3,1,[b[1][1],t1,t2]);
            w8:=Matrix(F,3,1,[t1,a[2][1]+b[2][1],t2]);
            w9:=Matrix(F,3,1,[t1,t2,-ze^2*a[3][1]+b[3][1]]);
            if InLatt(w4,K) or InLatt(w5,K) or InLatt(w6,K) or InLatt(w7,K) or
             InLatt(w8,K) or InLatt(w9,K) then
                 return false;
            end if;
        end for;
    end for;
    return true;
end function;

/* The function "ActionsZ3xZ3xZ3" determines all good cocycles in standard form on E^3/K for each kernel K
and a list consisting of one representative for each good cohomology class. The actions are given as
tuples of translation parts of the generators h,g,k.
Input: kernel "K", linear parts "listGen" of the actions of the generators, list "Fix" of fixed points of
rho(k) giving the coboundaries. */

function ActionsZ3xZ3xZ3(K,Fix,listGen)
    ListTransVec:={}; ListOfActions:={};
    for a in Fix do for b in Fix do
        if WellDefinedZ3xZ3xZ3(a,b,K) and GoodCondZ3xZ3xZ3(a,b,K) then
            ListTransVec:=Include(ListTransVec, [a,b,Matrix(F,3,1,[0,0,0])]);
        end if;
    end for; end for;
    ListOfActions:=ListTransVec; GoodClasses:={}; RefList:=ListTransVec;
    while not IsEmpty(ListTransVec) do
        RefList:=ListTransVec;
        v1:=Rep(ListTransVec); Include(~GoodClasses,v1);
        for v2 in ListTransVec do
            if TestCohom(v1,v2,listGen,K,Fix) then Exclude(~ListTransVec,v2); end if;
        end for;
    end while;
    return ListOfActions, GoodClasses;
end function;

/* The function "ClassZ3xZ3xZ3" is the main classification function for G=Z3^3.
The output file "Z3xZ3xZ3.txt" contains for each kernel
1) the number of actions,
2) the number of good cohomology classes,
3) the size of the normalizer N_C(Lambda_K),
4) the number of biholomorphism classes and
5) for each biholomorphism class a corresponding action on E^3/K.
*/

```

```
function ClassZ3xZ3xZ3(K,listGen,N,j)
      File:="Z3xZ3xZ3.txt";
      fprintf File, "Kernel %o)\n \n", j;
      IsoClasses:=[];
      Fix:=FixPoints(DiagonalMatrix([ze,ze,ze]),K,{Matrix(F,3,1,[d1,d2,d3]): d1,d2,d3 in E3});
      LA,GoodClasses:=ActionsZ3xZ3xZ3(K,Fix,listGen);
      fprintf File, "Number of actions with isolated fixed points: %o \n \n", #LA;
      fprintf File, "Number of good cohomology classes: %o\n\n", #GoodClasses;
      RefListAct:=GoodClasses;
      Nor:=Normal(K,N);
      fprintf File, "Size of the normalizer: %o \n \n", #Nor;
      while not IsEmpty(GoodClasses) do
            RefListAct:=GoodClasses;
            v1:=Rep(GoodClasses); Append(~IsoClasses,v1);
            for v2 in RefListAct do
                  if TestIso(v1,v2,K,Nor,Fix,listGen) then Exclude(~GoodClasses,v2); end if;
            end for;
      end while;
      fprintf File, "Number of biholomorphism classes: %o \n \n", #IsoClasses;
      fprintf File, "Actions [tau(h),tau(g),tau(k)]: \n %o \n \n \n \n", IsoClasses;
      return "Classification for kernel", j, "is completed!";
end function;

/* With the procedure "MainZ3xZ3xZ3", we run the classification for the group "G=Z3^3". */

procedure MainZ3xZ3xZ3()
      g:=DiagonalMatrix([ze,1,1]); h:=DiagonalMatrix([1,ze^2,ze]); k:=DiagonalMatrix([ze,ze,ze]);
      listGen:=[h,g,k];
      K1:={Matrix(F,3,1,[0,0,0])};
      K2:={a*Matrix(F,3,1,[t,t,0]): a in {0,1,-1}};
      K3:={a*Matrix(F,3,1,[t,t,t]): a in {0,1,-1}};
      K4:={a*Matrix(F,3,1,[t,t,t])+b*Matrix(F,3,1,[t,-t,0]): a,b in {0,1,-1}};
      Kernels:=[K1,K2,K3,K4];
      N:=MatrixGroup<3,F| DiagonalMatrix([-ze,1,1]), Matrix([[0,1,0],[1,0,0],[0,0,1]]),
       Matrix([[0,1,0],[0,0,1],[1,0,0]])>;
      for j in [1..#Kernels] do ClassZ3xZ3xZ3(Kernels[j],listGen,N,j); end for;
end procedure;
```

(e) $G = \mathrm{He}(3)$:

The analytic representation of $\mathrm{He}(3) = \langle g,h,k \mid g^3 = h^3 = k^3 = [g,k] = [h,k] = 1, [g,h] = k\rangle$ is given by

$$\rho(g) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \rho(h) = \mathrm{diag}(1,\ \zeta_3^2,\ \zeta_3), \qquad \rho(k) = \mathrm{diag}(\zeta_3,\ \zeta_3,\ \zeta_3),$$

and a cocycle in standardform is given by

$$\tau(g) = b = (b_1,\ b_2,\ b_3), \qquad \tau(h) = a = (a_1,\ a_2,\ a_3), \qquad \tau(k) = 0.$$

```
/* Given "a" and "b" in Fix_ze3(E^3/K), the function "WellDefinedHe3" checks whether these two vectors
define a well-defined action on T=E^3/K. */

function WellDefinedHe3(a,b,K)
```

```
     v1:=Matrix(F,3,1,[b[1][1]+b[2][1]+b[3][1],b[1][1]+b[2][1]+b[3][1],b[1][1]+b[2][1]+b[3][1]]);
     v2:=Matrix(F,3,1,[ze*a[1][1]-a[3][1]+(ze-1)*b[1][1],ze*a[2][1]-a[1][1],
      ze*a[3][1]-a[2][1]+(ze^2-1)*b[3][1]]);
     if InLatt(v1,K) and InLatt(v2,K) then return true; end if;
     return false;
end function;

/* The function "GoodCondHe3" checks whether a cocycle in standard form is good. a=tau(h), b=tau(g). */

function GoodCondHe3(a,b,K)
     v1:=b[1][1]+b[2][1]+b[3][1];
     v2:=ze^2*(b[1][1]+b[2][1])+b[3][1]+ze^2*(a[1][1]+a[3][1])+a[2][1];
     v3:=ze*(b[1][1]+b[2][1])+b[3][1]-ze*(a[1][1]+a[2][1])-a[3][1];
     if IsIntegral(v1) or IsIntegral(v2) or IsIntegral(v3) then return false; end if;
     for t1 in [0,t,-t] do for t2 in [0,t,-t] do
          if InLatt(Matrix(F,3,1,[a[1][1],t1,t2]),K) then return false; end if;
     end for; end for;
     return true;
end function;

/* The function "ActionsHe3" determines all good cocycles in standard form on E^3/K for each kernel K
and a list consisting of one representative for each good cohomology class. The actions are given as
tuples of translation parts of the generators h,g,k.
Input: kernel "K", linear parts "listGen" of the actions of the generators, list "Fix" of fixed points of
rho(k) giving the coboundaries. */

function ActionsHe3(K,Fix,listGen)
     ListTransVec:={}; ListOfActions:={};
     for a in Fix do for b in Fix do
          if WellDefinedHe3(a,b,K) and GoodCondHe3(a,b,K) then
               ListTransVec:=Include(ListTransVec, [a,b,Matrix(F,3,1,[0,0,0])]);
          end if;
     end for; end for;
     ListOfActions:=ListTransVec; GoodClasses:={}; RefList:=ListTransVec;
     while not IsEmpty(ListTransVec) do
          RefList:=ListTransVec;
          v1:=Rep(ListTransVec); Include(~GoodClasses,v1);
          for v2 in ListTransVec do
               if TestCohom(v1,v2,listGen,K,Fix) then Exclude(~ListTransVec,v2); end if;
          end for;
     end while;
     return ListOfActions, GoodClasses;
end function;

/* The function "ClassHe3" is the main classification function for G=He(3).
The output file "He3.txt" contains for each kernel
1) the number of actions,
2) the number of good cohomology classes,
3) the number of biholomorphism classes and
4) for each biholomorphism class a corresponding action on E^3/K.
*/

function ClassHe3(K,listGen,N,j)
     File:="He3.txt";
     fprintf File, "Kernel %o)\n \n", j;
     IsoClasses:=[];
     Fix:=FixPoints(DiagonalMatrix([ze,ze,ze]),K,{Matrix(F,3,1,[d1,d2,d3]): d1,d2,d3 in E3});
     LA,GoodClasses:=ActionsHe3(K,Fix,listGen);
     fprintf File, "Number of actions with isolated fixed points: %o \n \n", #LA;
     fprintf File, "Number of good cohomology classes: %o\n\n", #GoodClasses;
     RefListAct:=GoodClasses;
```

```
        while not IsEmpty(GoodClasses) do
            RefListAct:=GoodClasses;
            v1:=Rep(GoodClasses); Append(~IsoClasses,v1);
            for v2 in RefListAct do
                if TestIso(v1,v2,K,N,Fix,listGen) then Exclude(~GoodClasses,v2); end if;
            end for;
        end while;
    fprintf File, "Number of biholomorphism classes: %o \n \n", #IsoClasses;
    fprintf File, "Actions [tau(h),tau(g),tau(k)]: \n %o \n \n \n \n", IsoClasses;
    return "Classification for kernel", j, "is completed!";
end function;

/* With the procedure "MainHe3", we run the classification for the group "G=He3". */

procedure MainHe3()
    g:=Matrix(F,3,3,[0,0,1,1,0,0,0,1,0]); h:=DiagonalMatrix([1,ze^2,ze]); k:=DiagonalMatrix([ze,ze,ze]);
    listGen:=[h,g,k];
    K1:={a*Matrix(F,3,1,[t,t,t]): a in {0,1,-1}};
    K2:={a*Matrix(F,3,1,[t,t,t])+b*Matrix(F,3,1,[t,-t,0]): a,b in {0,1,-1}};
    Kernels:=[K1,K2];
    N:=MatrixGroup<3,F| DiagonalMatrix([ze,ze^2,1]),-t*Matrix([[1,ze^2,ze^2,ze^2,1,ze^2,ze^2,ze^2,1]),
     t*Matrix([[1,1,1,1,ze^2,ze,1,ze,ze^2]>;
    for j in [1..#Kernels] do ClassHe3(Kernels[j],listGen,N,j); end for;
end procedure;
```

(f) $G = \mathbb{Z}_9 \rtimes \mathbb{Z}_3$:

The analytic representation of $G = \mathbb{Z}_9 \rtimes \mathbb{Z}_3 = \langle g, h \mid h^3 = g^9 = 1,\ hgh^{-}1 = g^4 \rangle$ is given by

$$\rho(h) = \mathrm{diag}(1,\ \zeta_3^2,\ \zeta_3), \qquad \rho(g) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta_3 & 0 & 0 \end{pmatrix},$$

and a cocycle in standard form is given by

$$\tau(h) = a = (a_1,\ a_2,\ a_3), \qquad \tau(g) = 0.$$

```
/* Given "a" in E3^3, the function "WellDefinedZ9Z3" checks whether this vector defines a well-defined
action on T=E^3/K. */

function WellDefinedZ9Z3(a,K)
    if TestKernelEndo(Matrix(F,3,3,[0,1,0,0,0,1,ze,0,0])-I3,a,K) then return true; end if;
    return false;
end function;

/* The function "GoodCondZ9Z3" checks whether a cocycle in standard form is good. a=tau(h). */

function GoodCondZ9Z3(a,K)
    for t1 in [0,t,-t] do for t2 in [0,t,-t] do
        w1:=Matrix(F,3,1,[a[1][1],t1,t2]);
        w2:=Matrix(F,3,1,[t1,a[2][1],t2]);
        w3:=Matrix(F,3,1,[t1,t2,a[3][1]]);
        if InLatt(w1,K) or InLatt(w2,K) or InLatt(w3,K) then return false; end if;
    end for; end for;
    return true;
end function;
```

```

/* The function "Actions" determines all actions with only isolated fixed points with translation part in
standard form on E^3/K, for each kernel K.
*/

function Actions(K)
     list:=[];
     for a1 in E3 do for a2 in E3 do for a3 in E3 do
          a:=Matrix(F,3,1,[a1,a2,a3]);
          if WellDefinedZ9Z3(a,K) and GoodCondZ9Z3(a,K) then Append(~list,a); end if;
     end for; end for; end for;
     transpart:=[];
     while not IsEmpty(list) do
          a:=Rep(list); Append(~transpart,a);
          for b in list do
               if InLatt(a-b,K) then Exclude(~list,b); end if;
          end for;
     end while;
     return transpart;
end function;

/*
With the procedure "MainZ9Z3", we generate all actions for the group "G=Z9Z3".
*/

procedure MainZ9Z3()
     K1:={Matrix(F,3,1,[0,0,0])};
     K2:={a*Matrix(F,3,1,[t,t,t]): a in {0,1,-1}};
     K3:={a*Matrix(F,3,1,[t,t,t])+b*Matrix(F,3,1,[t,-t,0]): a,b in {0,1,-1}};
     Kernels:=[K1,K2,K3];
     for j in [1..#Kernels] do Actions(Kernels[j]); end for;
end procedure;
```

A.3. Twisted representation groups

The following code is an implementation of *Algorithm 1* of Section 6.3.1 in MAGMA.
For a given finite group G and action $\varphi\colon G \to \mathrm{Aut}(\mathbb{C})$ taking values in $\{\mathrm{id}, \mathrm{conj}\}$, we want to determine all φ-twisted representation groups Γ of G, i.e., we have to determine by Proposition 6.2.3 all extensions

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

where $A = H^2(G, \mathbb{C}^*)$, such that

(1) $|H^1(G, \mathbb{C}^*)| = |H^1(\Gamma, \mathbb{C}^*)|$ and

(2) $\mathrm{Hom}_G(A, \mathbb{C}^*) = \mathrm{Hom}(A, \mathbb{C}^*)$.

For this, we identify $H^j(G, \mathbb{C}^*) = H^{j+1}(G, \mathbb{Z})$, for $j = 1, 2$, where G acts on $\mathbb{Z}$ via φ and sending conj to -1, which gives a character χ of G with values in $\{1, -1\}$ (cf. Section 6.3.1)

The main function therefore has as input the group G and the character χ.

```
/*
The function "Phi" has as input "x=X(g)", for an element g in G, and an element "v" in C, and determines
the value phi(g)(v), which is v, if x=[1], and ComplexConjugate(v), if x=[-1].
*/

```

```
function Phi(x,v)
     Id1:=DiagonalMatrix([1]);
     if x eq Id1 then return v;
     else return ComplexConjugate(v); end if;
end function;

/*
The function "TestInvariance" has as input the group "A" with "a" generators and the group "Ga" with "m"
generators. The group A is embedded in Ga such that the generators of A equal the last a generators of Ga.
The action of Ga on Z is encoded in "actGa", which is a list where the i-th entry is the action
(as 1x1-matrix) of the i-th generator of Ga on Z.
The function checks condition (2).
We use that the first m-a generators of Ga define preimages of the generators of G under pi: Ga -> G.
*/

function TestInvariance(A,Ga,actGa,m,a)
     CT:=CharacterTable(A);
     for x in CT do
          for i in [1..m-a] do for j in [m-a+1..m] do
               if not x(Ga.i*Ga.j*Ga.i^-1) eq Phi(actGa[i],x(Ga.j)) then return false; end if;
          end for; end for;
     end for;
     return true;
end function;

/*
The function "KernelCokernelExtension" has as input an extension "Ga" (of G by A), its image "GaRef" under
the Cayley-embedding "f", the number "m" of generators of G and "a"=#A.
It returns the kernel "APer" as a subgroup of GaRef, the quotient "Quot"=Ga/APer, and the quotient map
"pi":Ga -> Quot.
Note that the kernel A is generated by the last generators of GaRef; the problem is that we don't know how
many generators we have to take (the number can differ from #Generators(A)). Therefore, the last output "i"
gives this number of generators of APer.
*/

function KernelCokernelExtension(Ga,GaRef,f,m,a)
     for i in [1..m] do
          APer:=sub<Ga | [f(GaRef.j): j in [(m-i+1)..m]]>;
          if #APer eq a then
               Quot, pi:= quo<Ga|APer>;
               return APer, Quot, pi, i;
          end if;
     end for;
end function;

/*********************** MAIN FUNCTION ***********************/

/*
INPUT: finite, solvable Group G of type GrpPerm, character X of G of degree 1 with values in {1,-1}
        representing an action phi of G on C
OUPUT: A=H^2(G,C^*)(in terms of invariants) and a list of all phi-twisted representation groups of G

(Explanation: the invariants [n_1,...n_k] correspond to the abelian group Z_{n_1} x ... x Z_{n_k})
*/

function RepGroups(G,X)
     g:=#Generators(G);
     Id1:=DiagonalMatrix([1]);
     act:=[X(G.i)*Id1: i in [1..g]];
     // The i-th element of act gives the action of the i-th generator of G on Z as a 1x1-matrix.
     CMG:= CohomologyModule(G,[0],act);
```

```
     TwistedSchurG:=CohomologyGroup(CMG,3); // TwistedSchurG=H^3(G,Z)=H^2(G,C^*)
     invarA:=Moduli(TwistedSchurG); // #invariants of the abelian Group A = #generators of A
     if invarA eq [] then // in this case, the twisted Schur multiplier is trivial.
          return invarA, G;
     end if;
     A:=AbelianGroup(GrpPerm,invarA); // A = H^2(G,C^*), of type GrpPerm
     a:=#A;
     E:=ExtensionsOfSolubleGroup(A,G); //all candidates for the phi-twisted representation groups;
                                      //each group in the list is given as GrpFP;
                                      //the last generators correspond to A
     ListRepGroups:=[];
     h1G:=#CohomologyGroup(CMG,2);
     for k in [1..#E] do
          GaRef:=E[k];
          f,Ga:= CosetAction(GaRef,sub<GaRef|>); //transforms the extension GaRef into GrpPerm using
                                                 //the Cayley-embedding f
          m:=#Generators(GaRef);
          APer, Quot, pi, genA:=KernelCokernelExtension(Ga,GaRef,f,m,a);
          test, psi:=IsIsomorphic(Quot,G); //psi: Quot -> G defines an isomorphism
          actGa:=[X(psi(pi(Ga.i)))*Id1 : i in [1..m]];
          CMGa:=CohomologyModule(Ga,[0],actGa);
          if h1G eq #CohomologyGroup(CMGa,2) and TestInvariance(APer,Ga,actGa,m,genA) then
               Append(~ListRepGroups,Ga);
          end if;
     end for;
     return invarA, ListRepGroups;
end function;
```

With the MAGMA code below, we show that the group $\mathcal{N}$ in Example 6.3.5 is not a covering group for the given action.

```
F:=CyclotomicField(12);
ze:=F.1^4;
i:=F.1^3;
t:=(1+2*ze)/3;

//The function RI returns the real and imaginary parts of a complex number "c".

function RI(c)
     return [(c+ComplexConjugate(c))/2, -i*(c-re)];
end function;

//The function "RealMat" turns a complex 3x3 matrix "D" into a real 6x6 matrix under the canonical embedding

function RealMat(D)
     return Matrix(F, 6, 6,
          [RI(D[1][1])[1],-RI(D[1][1])[2],RI(D[1][2])[1],-RI(D[1][2])[2],RI(D[1][3])[1],-RI(D[1][3])[2],
          RI(D[1][1])[2],RI(D[1][1])[1],RI(D[1][2])[2],RI(D[1][2])[1],RI(D[1][3])[2],RI(D[1][3])[1],
          RI(D[2][1])[1],-RI(D[2][1])[2],RI(D[2][2])[1],-RI(D[2][2])[2],RI(D[2][3])[1],-RI(D[2][3])[2],
          RI(D[2][1])[2],RI(D[2][1])[1],RI(D[2][2])[2],RI(D[2][2])[1],RI(D[2][3])[2],RI(D[2][3])[1],
          RI(D[3][1])[1],-RI(D[3][1])[2],RI(D[3][2])[1],-RI(D[3][2])[2],RI(D[3][3])[1],-RI(D[3][3])[2],
          RI(D[3][1])[2],RI(D[3][1])[1],RI(D[3][2])[2],RI(D[3][2])[1],RI(D[3][3])[2],RI(D[3][3])[1]]);
end function;

//These are the three C-linear matrices C1,C3,C3 which generate N, and C4 represents complex
//conjugation in each coordinate.

C1:=DiagonalMatrix([ze,ze^2,1]);
C2:=-t*Matrix([[1,ze^2,ze^2],[ze^2,1,ze^2],[ze^2,ze^2,1]]);
C3:=t*Matrix([[1,1,1],[1,ze^2,ze],[1,ze,ze^2]]);
```

```
C4:=Matrix(F, 6, 6,[1,0,0,0,0,0,0,-1,0,0,0,0,0,0,1,0,0,0,0,0,0,-1,0,0,0,0,0,0,1,0,0,0,0,0,0,-1]);

// The group of semilinearities SR=<D1,...,D4> as a subgroup of GL(6,F).

N:=sub<GL(6,F)|RealMat(C1),RealMat(C2),RealMat(C3),C4>;
a:=DiagonalMatrix([-ze,-ze,-ze]);
A:=sub<N | RealMat(a)>;

S, pi:=N/A;
I1:=DiagonalMatrix([1]);
CM_N := CohomologyModule(N,[0],[I1,I1,I1,-I1]);
CM_S := CohomologyModule(S,[0],[I1,I1,I1,-I1]);

// These are the cohomology groups H^1(S,C^*) and H^1(N,C^*). They have different orders.
CohomologyGroup(CM_S,2);
CohomologyGroup(CM_N,2);
```

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