Optimal additive quaternary codes of dimension 3.5

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ABSTRACT. After the optimal parameters of additive quaternary codes of dimension $k \le 3$ have been determined in [2], there is some recent activity to settle the next case of dimension $k = 3.5$ [11, 12]. Here we complete dimension $k = 3.5$ and give partial results for dimension $k = 4$. We also solve the problem of the optimal parameters of additive quaternary codes of arbitrary dimension when assuming a sufficiently large minimum distance.

Keywords: additive codes, linear codes, quaternary codes, Galois geometry
Mathematics Subject Classification: 94Bxx, 51E22

1. INTRODUCTION

A quaternary block code C of length n is a subset of $\mathbb{F}_4^n$. If C is closed under componentwise addition then C is called additive. If C is additive and closed under $\mathbb{F}_4$ scalar multiplication then C is called linear. The parameter k such that the number of codewords $|C|$ equals 4^k is called the dimension of C. Clearly, k is an integer if C is linear and a half-integer if C is additive. For each integer s let $n_k(s)$ denote the maximal length n such that an additive quaternary code of length n, dimension k, and minimum Hamming distance $n - s$ exists. For $k \le 3$ the function $n_k(s)$ was completely determined in [2]. In the sequence of papers [11, 12] the determination of $n_{3.5}(s)$ was narrowed down to $s \in \{6, 7, 12\}$.[1] Geometrically, $n_k(s)$ is the maximum number of lines in the projective space $\operatorname{PG}(2k-1, 2)$ such that each hyperplane contains at most s lines, see e.g. [2]. The aim of this paper is to completely determine $n_{3.5}(s)$ and $n_k(s)$ for all sufficiently large s. For $n_4(s)$ we give partial results.

The remaining part of the paper is structured as follows. In Section 2 we introduce the necessary preliminaries. The problem of the optimal parameters of additive quaternary codes of arbitrary dimension, assuming a sufficiently large minimum distance, is solved in Section 3. The determination of $n_{3.5}(s)$ and the partial determination of $n_4(s)$ is obtained in Section 4.

2. PRELIMINARIES

The set of all subspaces of $\mathbb{F}_2^r$, ordered by the incidence relation $\subseteq$, is called $(r-1)$-*dimensional projective geometry over* $\mathbb{F}_2$ and denoted by $\operatorname{PG}(r-1, 2)$. Employing this algebraic notion of dimension instead of the geometric one, we will use the term i-space to denote an i-dimensional subspace of $\mathbb{F}_q^r$. To highlight the important geometric interpretation of subspaces we will call 1-, 2−, and $(r-1)$-spaces points, lines, and hyperplanes, respectively. Every arbitrary i-space in $\operatorname{PG}(r-1, 2)$, where $r \ge i$, contains exactly $2^i - 1$ points. For two subspaces S and S' we write $S \le S'$ if S is contained in S'. Moreover, we say that S and S' are *incident* iff $S \le S'$ or $S \ge S'$.

Definition 2.1. An (n, r, s) *system* is a multiset $\mathcal{S}$ of n lines in $\operatorname{PG}(r-1, 2)$ such that each hyperplane contains at most s elements from $\mathcal{S}$ and some hyperplane contains exactly s elements of $\mathcal{S}$. We say that $\mathcal{S}$ is *spanning* iff $s < n$.

By $n_k(s)$ we denote the maximum n such that a spanning $(n, 2k, s)$ system exists, which is the same as the maximal length n of an additive quaternary code with dimension k and minimum Hamming distance $n - s$, see e.g. [2]. So, we will always assume $2(s + 1) \ge k$ when considering $n_k(s)$.

Definition 2.2. For an (n, r, s) system $\mathcal{S}$ let $\mathcal{P}(\mathcal{S})$ denote the multiset of points that we obtain by replacing each element of $\mathcal{S}$ by its contained three points.

We also call a multiset of points spanning iff no hyperplane contains all points. If C is a binary linear code with length n and minimum Hamming distance d, then we say that C is an $[n, k, d]_2$ code, where $2^k = |C|$. Given a generator matrix G for C we can construct a multiset of points from C by considering the span of each column of G as a point in $\operatorname{PG}(k-1, 2)$. And indeed, it is well known that a spanning multiset of points $\mathcal{P}$ in $\operatorname{PG}(k-1, 2)$, such that at most s elements are contained in a hyperplane and some hyperplane contains exactly s elements, is in one-to-one correspondence to a linear $[n, k, n-s]_2$ code C, see e.g. [20, §1.1.2] or [7]. Let us write $\mathcal{P} = \mathcal{X}(C)$ and $C = \mathcal{X}^{-1}(\mathcal{P})$ for this correspondence. The elements of a code are called *codewords*. The *weight* of a codeword $c \in C$ of a linear code is the number of non-zero entries in c. So, the minimum occurring non-zero weight of a linear code coincides with its minimum distance. We call a linear code Δ-*divisible* if the weights of all codewords are divisible by Δ.

[1] The example for $s = 13$ refers to [16].

Lemma 2.3. *(Cf.* [2, Lemma 1]*) Let $\mathcal{S}$ be a spanning (n,r,s) system. Then, $C := \mathcal{X}^{-1}(\mathcal{P}(\mathcal{S}))$ is a 2-divisible $[3n, r, 2(n-s)]$ code with maximum weight at most $2n$.*

Proof. Since each line consists of $2^2 - 1 = 3$ points the cardinality of $\mathcal{P}(\mathcal{S})$ equals $3n$, so that C has length $3n$. Since $\mathcal{S}$ is spanning also $\mathcal{P}(\mathcal{S})$ is spanning and C has dimension r. Given an arbitrary hyperplane H in $\operatorname{PG}(r-1,2)$ and an arbitrary line L we have that either L is completely contained in H or intersects the hyperplane in exactly a point. Since each hyperplane H contains $0 \le i \le s$ out of the n lines in $\mathcal{S}$ we have that H contains $3i + (n-i) = n + 2i$ points from $\mathcal{P}(\mathcal{S})$, so that the codeword $c \in C$ that corresponds to H has weight $(3n) - (n+2i) = 2(n-i)$. □

So, bounds on the parameters of a binary linear code yield upper bounds for $n_k(s)$. E.g. the so-called *Griesmer bound* [10]

$$n \ge \sum_{i=0}^{k-1} \left\lceil \frac{d}{2^i} \right\rceil =: g(k,d) \tag{1}$$

relates the parameters of an $[n,k,d]_2$ code. Interestingly enough, this bound can always be attained with equality if the minimum distance d is sufficiently large and a nice geometric construction was given by Solomon and Stiffler [19].

Lemma 2.4. *(Cf.* [1, Theorem 12], [2, Lemma 1]*, and* [11, Lemma 3]*) Let $r > 2$ and $\mathcal{S}$ be an (n,r,s) system. Then, we have $g(r, 2(n-s)) \le 3n$.*

Proof. Combine Lemma 2.3 with Inequality (1). □

Definition 2.5. The *Griesmer upper bound* for $n_k(s)$ is the largest integer n such that $g(2k, 2(n-s)) \le 3q \cdot n$. The *weak coding upper bound* for $n_k(s)$ is the largest integer n such that an $[3n, 2k, 2(n-s)]_2$ code C exists. The *(strong) coding upper bound* for $n_k(s)$ is the largest integer n such that a $2-$divisible $[3n, 2k, 2(n-s)]_2$ code C with maximum weight at most $2n$ exists.

We remark that the minimal possible length of an $[n,k,d]_2$ code is known for all $d \in \mathbb{N}$ when $k \le 8$ [4], so that the weak coding upper bound can be evaluated in all cases for $n_4(s)$. This is different for the strong coding upper bound for $n_4(s)$ and we present partial results in Section 4 and Subsection 4.1.

s	Griesmer upper bound	weak coding upper bound	$n_4(s)$
3	9	7	5
4	12		10
5	17		17
6	22	18	18
7	25	23	23
8	30	28	28
9	33		33
10	38	36	35–36
11	43	40	40
12	44		44
13	49		49
14	54		54
15	59	57	55–57
24	94		93–94
28	110		108–110
29	115		113–115

TABLE 1. The Griesmer and the weak coding upper bound for $n_4(s)$.

Example 2.6. The Griesmer upper bound for $n_4(8)$ is 30 and the weak coding upper bound is 28. I.e., the Griesmer bound implies that no $[93,8,46]_2$ code exists but cannot rule out the existence of a $[90,8,44]_2$ code, so that $n_4(8) \le 30$ is the sharpest upper bound we can deduce from the Griesmer bound (for linear codes). However, since the existence of a $[84,8,40]_2$ code and the non-existence of a $[87,8,42]_2$ code is known, we obtain $n_4(8) \le 28$. In Table 1 we list the Griesmer and the weak coding upper bound for $n_4(s)$ for $3 \le s \le 15$ and all cases were either the weak coding upper bound is strictly less than the Griesmer upper bound or the value of $n_4(s)$ is still unknown. For $s \in \{3,4\}$ we refer to [3]. Note that the cases $s \in \{1,2\}$ cannot occur for a spanning $(n,8,s)$ system. We do not display the weak coding upper bound when it coincides with the Griesmer upper bound.

In order to partially evaluate the strong coding upper bound we present a few tools from coding theory. Let C be a $[n,k,d]_2$ code with generator matrix G and $c \in C$ be a codeword of weight w. The *residual code of C with respect to c*, denoted by $\operatorname{Res}(C;c)$, is the code generated by the restriction of G to the columns where c has a zero entry. If only the weight w of c is relevant we will denote it by $\operatorname{Res}(C;w)$. The following statement on the residual code is well-known:

Lemma 2.7. *Let C be an $[n,k,d]_2$ code and let $d>\frac{w}{2}$. Then $\operatorname{Res}(C;w)$ is an $[n-w,k-1,\ge d-\lfloor w/2\rfloor]_2$ code.*

Lemma 2.8. ([21, Theorem 1]) *Let C be an $[n,k,d]_2$ code with $n=g(k,d)$. If 2^e divides d, then C is 2^e-divisible.*

For the constructive lower bound we have:

Lemma 2.9. *For $k>1$ we have $n_k(s_1+s_2)\ge n_k(s_1)+n_k(s_2)$ and $n_k(s+1)\ge n_k(s)+1$.*

Proof. Let $\mathcal{S}_i$ be spanning $(n_i,2k,s_i)$ systems for $i=1,2$ and $\mathcal{S}$ a spanning $(n,2k,s)$ system. With this, the multiset union of $\mathcal{S}_1$ and $\mathcal{S}_2$ is a spanning $(n_1+n_2,2k,\le s_1+s_2)$ system. Adding an arbitrary line to $\mathcal{S}$ gives a spanning $(n+1,2k,\le s+1)$ system. □

Definition 2.10. A *vector space partition* of $\mathrm{PG}(r-1,2)$ is a multiset $\mathcal{V}$ of subspaces with dimension at most $(r-1)$ such that every point of $\mathrm{PG}(r-1,2)$ is contained in exactly one element of $\mathcal{V}$. We say that $\mathcal{V}$ has type $1^{t_1}2^{t_2}\dots(r-1)^{t_{r-1}}$ if exactly t_i elements of $\mathcal{V}$ have dimension i for all $1\le i\le r-1$.

A set of matrices $M\subseteq\mathbb{F}_2^{m\times n}$ with $\operatorname{rk}(A-B)\ge\delta$ for all $A,B\in M$ with $A\ne B$ is called a *rank metric code* with minimum rank distance δ. A Singleton-type upper bound gives $\#M\le 2^{\max\{m,n\}\cdot(\min\{m,n\}-\delta+1)}$. Rank metric codes attaining this bound are called *MRD* codes. They exist for all parameters with $\delta\le\min\{m,n\}$, even if one additionally requires that M is linearly closed, see e.g. [18] for a survey.

Lemma 2.11. *For $r>4$ there exists a vector space partition $\mathcal{V}$ of $\mathrm{PG}(r-1,2)$ of type $2^{t_2}(r-2)^1$ where $t_2=2^{r-2}$.*

Proof. Let $M\subseteq\mathbb{F}_2^{h\times(r-2)}$ be an MRD code with minimum rank distance h and cardinality 2^{r-2}. Prepending a 2×2 unit matrix to the elements of M gives generator matrices of h-spaces in $\mathrm{PG}(r-1,q)$ that are pairwise disjoint and disjoint to an $(r-2)$-space S. □

Lemma 2.12. *For $r>a>2$ with $r\equiv a \pmod 2$ there exists a vector space partition $\mathcal{V}$ of $\mathrm{PG}(r-1,2)$ of type $2^{t_2}(a)^1$ where $t_2=2^a\cdot\frac{2^{r-a}-1}{3}$.*

Proof. We prove by induction over r. Let $\mathcal{V}$ be the vector space partition obtained from Lemma 2.11 and let $S\in\mathcal{V}$ be the unique $(r-2)$-dimensional element. If $a=r-2$, which is indeed the case for all $r<6$, then $\mathcal{V}$ is the desired vector space partition. Otherwise we identify S with $\mathrm{PG}(r-3,2)$ and replace S by a vector space partition of $\mathrm{PG}(r-3,2)$ of type $2^{t_2'}a^1$, which exists by induction. □

Lemma 2.13. *For $r>a>2$ with $r\equiv a \pmod 2$ let $\mathcal{S}$ be the set of 2-dimensional elements of a vector space partition $\mathcal{V}$ of $\mathrm{PG}(r-1,2)$ of type $2^{t_2}a^1$ and A be the unique a-dimensional element in $\mathcal{V}$. Then, $\mathcal{S}$ is a (t_2,r,s) system where $t_2=2^a\cdot\frac{2^{r-a}-1}{3}$ and $s=2^{a-2}\cdot\frac{2^{r-a}-1}{3}$. Moreover, each hyperplane that contains A contains $s-q^{a-h}$ elements from $\mathcal{S}$.*

Proof. Let H be an arbitrary hyperplane of $\mathrm{PG}(r-1,2)$. Note that every i-space intersects H in either 2^i-1 or $2^{i-1}-1$ points and that the elements of $\mathcal{S}$ partition the points outside of A. Counting points yields that H contains

$$\frac{2^{r-1}-2^{a-1}-t_2\cdot\left(2^{h-1}-1\right)}{2}=2^{a-2}\cdot\frac{2^{r-a}-1}{3}=s$$

elements from $\mathcal{S}$ if $A\not\le H$ and

$$\frac{2^{r-1}-2^{a}-t_2\cdot\left(2^{h-1}-1\right)}{2}=\frac{2^{r-1}-2^{a-1}-2^{a-1}-t_2\cdot\left(2^{h-1}-1\right)}{2}=s-2^{a-2}$$

elements from $\mathcal{S}$ if $A\le H$. □

3. A generalization of the Solomon–Stiffler construction

In [19] Solomon and Stiffler constructed $[n,k,d]_2$ codes with $n=g(k,d)$ for all parameters with sufficiently large minimum distance d. Here we want to show the generalization that the Griesmer upper bound for $n_{k/2}(s)$ can always be attained if s is sufficiently large. Using a specific parameterization of the minimum distance d the Griesmer bound in Inequality (1) can be written more explicitly:

Lemma 3.1. *Let k and d be positive integers. Write d as*

$$d=\sigma\cdot 2^{k-1}-\sum_{i=1}^{k-1}\varepsilon_i\cdot 2^{i-1},\tag{2}$$

where $\sigma\in\mathbb{N}_0$ and $\varepsilon_i\in\{0,1\}$ for all $1\le i\le k-1$. Then, Inequality (1) is satisfied with equality iff

$$n=\sigma\cdot\left(2^k-1\right)-\sum_{i=1}^{k-1}\varepsilon_i\cdot\left(2^i-1\right),\tag{3}$$

which is equivalent to

$$n-d=\sigma\cdot\left(2^{k-1}-1\right)-\sum_{i=1}^{k-1}\varepsilon_i\cdot\left(2^{i-1}-1\right). \quad (4)$$

Given k and d Equation (2) always determines σ and the ε_i uniquely. This is different for Equation (4) given k and $n-d=s$. Here it may happen that no solution with $0\le\varepsilon_i\le 1$ exists. By relaxing to $0\le\varepsilon_i\le 2$ we can ensure existence and uniqueness is enforced by additionally requiring $\varepsilon_j=0$ for all $j<i$ where $\varepsilon_i=2$ for some i. The same is true for Equation (3) given k and n. For more details we refer to [9, Chapter 2] which also gives pointers to Hamada's work on minihypers.

Definition 3.2. Let $\sigma\in\mathbb{N}$, $\varepsilon_1,\dots,\varepsilon_{r-1}\in\mathbb{Z}$, and let V denote the r-dimensional ambient space $\operatorname{PG}(r-1,2)$. We say that an (n,r,s) system $\mathcal{S}$ has *type* $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ if there exist subspaces $S_1\le\dots\le S_{r-1}$ with $\dim(S_i)=i$ and

$$\sum_{S\in\mathcal{S}}\chi_S=\sigma\cdot\chi_V-\sum_{i=1}^{r-1}\varepsilon_i\cdot\chi_{S_i}. \quad (5)$$

We say that $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is *partitionable* if an (n,r,s) system with type $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ exists for suitable parameters n and s.

Note that all chains $S_1\le\dots\le S_{r-1}$ are isomorphic, so that the notion of being h-partitionable does not depend on the choice of the subspaces $S_1,\dots,S_{r-1}$. If $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is partitionable, then also $0[r']-(-\sigma)[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ is partitionable for all $r'>r$. The parameters of an (n,r,s) system can be computed from the parameters of a partition:

Lemma 3.3. *If $\mathcal{S}$ is an (n,r,s) system with type $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$, then we have*

$$n=\left(\sigma\cdot(2^r-1)-\sum_{i=1}^{r-1}\varepsilon_i\cdot\left(2^i-1\right)\right)/3, \quad (6)$$

$$s=\max_{1\le j\le r}\left(s_1-\sum_{i=1}^{j-1}\varepsilon_i\cdot 2^{i-2}\right), \quad (7)$$

where

$$s_1=\left(\sigma\cdot\left(2^{r-2}-1\right)-\sum_{i=2}^{r-1}\varepsilon_i\cdot\left(2^{i-2}-1\right)+\frac{1}{2}\cdot\varepsilon_1\right)/\,3. \quad (8)$$

Moreover, ε_1 is divisible by 2 and

$$\sum_{i=1}^{r-1}\varepsilon_i\cdot\left(2^i-1\right)\equiv\sigma\cdot(2^r-1)\pmod 3, \quad (9)$$

where the right hand side is congruent to zero modulo 3 if r is even.

Proof. Let $\mathcal{M}$ be the multiset of points covered by the elements of $\mathcal{S}$ and $S_1\le\dots\le S_{r-1}$ be subspaces as in Definition 3.2. Since $\mathcal{M}$ has cardinality

$$\sigma\cdot(2^r-1)-\sum_{i=1}^{r-1}\varepsilon_i\cdot\left(2^i-1\right)$$

and one line contains 3 points, we conclude Equation (6).

For an arbitrary hyperplane H let $1\le j\le r$ denote the minimal integer such that $S_j\not\le H$, where we set $j=r$ if $H=S_{r-1}$. Counting points gives

$$\mathcal{M}(H)=\sigma\cdot\left(2^{r-1}-1\right)-\sum_{i=1}^{j-1}\varepsilon_i\cdot\left(2^i-1\right)-\sum_{i=j}^{r-1}\varepsilon_i\cdot\left(2^{i-1}-1\right)=\sigma\cdot\left(2^{r-1}-1\right)-\sum_{i=1}^{r-1}\varepsilon_i\cdot\left(2^{i-1}-1\right)-\sum_{i=1}^{j-1}\varepsilon_i\cdot 2^{i-1}.$$

The number s_j of elements of $\mathcal{S}$ contained in H is given by $\left(\mathcal{M}(H)-n\right)/2$, so that

$$\begin{aligned}
s_j &= \left(\sigma\cdot\left(2^{r-1}-1\right)-\sum_{i=1}^{r-1}\varepsilon_i\cdot\left(2^{i-1}-1\right)-\sum_{i=1}^{j-1}\varepsilon_i\cdot 2^{i-1}-\left(\sigma\cdot(2^r-1)-\sum_{i=1}^{r-1}\varepsilon_i\cdot\left(2^i-1\right)\right)\cdot\frac{1}{3}\right)/2\\
&= \left(\sigma\cdot\left(2^{r-1}-2\right)-\sum_{i=1}^{r-1}\varepsilon_i\cdot\left(2^{i-1}-2\right)\right)/\,6-\sum_{i=1}^{j-1}\varepsilon_i\cdot 2^{i-2}\\
&= \left(\sigma\cdot\left(2^{r-2}-1\right)-\sum_{i=2}^{r-1}\varepsilon_i\cdot\left(2^{i-2}-1\right)+\frac{1}{2}\cdot\varepsilon_1\right)/\,3-\sum_{i=1}^{j-1}\varepsilon_i\cdot 2^{i-2}.
\end{aligned}$$

This verifies Equation (8) and yields

$$s_j = s_1 - \sum_{i=1}^{j-1} \varepsilon_i \cdot 2^{i-2} \tag{10}$$

for $2 \le j \le r$, which implies Equation (7). From $s_2 \in \mathbb{N}$ we conclude that ε_1 is divisible by 2. Equation (6) implies Equation (9) and $2^r - 1$ is divisible by 3 iff r is even. . □

Corollary 3.4. *If all ε_i are nonnegative, then $s = s_1$ (using the notation from Lemma 3.3).*

Corollary 3.5. *If $\mathcal{S}_t$ is an (n_t, r, s_t) system with type $\left(\sigma + t \cdot \frac{3}{2^{\gcd(r,2)}-1}\right) \cdot [r] - \sum_{i=2}^{r-1} \varepsilon_i[i]$, where $\varepsilon_2, \dots, \varepsilon_{r-1} \in \mathbb{N}$, then we have*

$$n_t = t \cdot \frac{2^r - 1}{2^{\gcd(r,2)} - 1} + \left(\sigma \cdot (2^r - 1) - \sum_{i=2}^{r-1} \varepsilon_i \cdot \left(2^i - 1\right)\right)/3, \tag{11}$$

$$s_t = t \cdot \frac{2^{r-2} - 1}{2^{\gcd(r,2)} - 1} + \left(\sigma \cdot \left(2^{r-2} - 1\right) - \sum_{i=2}^{r-1} \varepsilon_i \cdot \left(2^{i-2} - 1\right)\right)/3, \tag{12}$$

and

$$n_t - s_t = t \cdot \frac{3}{2^{\gcd(r,2)} - 1} \cdot 2^{r-2} + \sigma \cdot 2^{r-2} - \sum_{i=2}^{r-1} \varepsilon_i \cdot 2^{i-2}. \tag{13}$$

Next we state a few constructions.

Lemma 3.6. *If $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ and $\sigma'[r] - \sum_{i=1}^{r-1} \varepsilon_i'[i]$ are partitionable, then $(\sigma + \sigma') \cdot [r] - \sum_{i=1}^{r-1} (\varepsilon_i + \varepsilon_i') \cdot [i]$ is partitionable.*

Proof. Fix some subspaces $S_1 \le \cdots \le S_{r-1}$ as in Definition 3.2. Let $\mathcal{S}$ be an (n, r, s) system with type $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ and $\mathcal{S}'$ be an (n', r, s') system with type $\sigma'[r] - \sum_{i=1}^{r-1} \varepsilon_i'[i]$, then the multiset union of the elements of $\mathcal{S}$ and $\mathcal{S}'$ is an (n'', r, s'') system with type $(\sigma + \sigma') \cdot [r] - \sum_{i=1}^{r-1} (\varepsilon_i + \varepsilon_i') \cdot [i]$. □

Lemma 3.7. *For $r > a \ge 2$ with $r \equiv a \pmod 2$ and $\sigma \in \mathbb{N}_{\ge 1}$ we have that $\sigma[r] - \sigma[a]$ is partitionable.*

Proof. If $a > 2$, then Lemma 2.13 yields the existence of an (n, r, s) system $\mathcal{S}$ with type $[r] - [a]$ and we can use σ copies thereof. For $a = 2$ we replace $\mathcal{S}$ by a spread of lines of $\mathrm{PG}(r-1, 2)$ where we remove an arbitrary element. □

Theorem 3.8. *(Cf.* [13, page 83], [8, Corollary 8], *or* [14, Lemma 2]*) For each $r \ge 2$ we have that $[r]$ is partitionable if r is even and that $3[r]$ is partitionable if r is odd.*

Proof. The statement is obvious for $r = 2$ and for $r = 3$ we consider the set of all seven lines in $\mathrm{PG}(2, 2)$. From Lemma 3.7 we deduce that $[r] - [2]$ is partitionable for all even $r \ge 4$ and that $3[r] - 3[3]$ is partitionable for all odd $r \ge 5$, so that the statement follows from Lemma 3.6. □

A set of lines that partitions $\mathrm{PG}(r-1, 2)$ is called a *line spread*. They do exist iff r is even.

Lemma 3.9. *If $x[r] - \sum_{i=2}^{r-1} \varepsilon_i[i]$ is partitionable for $x \in \{\sigma, \sigma'\}$ then*

$$\left(\sigma + t \cdot \frac{3}{2^{\gcd(r,2)} - 1}\right) \cdot [r] - \sum_{i=2}^{r-1} \varepsilon_i[i]$$

is partitionable for all $t \ge 0$ and we have $\sigma \equiv \sigma' \pmod{\frac{3}{2^{\gcd(r,h)}-1}}$.

Proof. Note that $2^{\gcd(r,2)} - 1$ equals 3 iff r is even and 1 iff r is odd, so that Theorem 3.8 and Lemma 3.6 imply the first statement. For even r the statement $\sigma \equiv \sigma' \pmod{\frac{3}{2^{\gcd(r,h)}-1}}$ is true since $\sigma, \sigma' \in \mathbb{N}$. For odd r we use Equation (9). □

Definition 3.10. We say that $\star[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ is partitionable if there exists an integer σ such that $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ is partitionable.

Theorem 3.11. *Let $r \ge 3$, $g := \gcd(r, 2)$, and $\varepsilon_2, \dots, \varepsilon_{r-1} \in \mathbb{Z}$ such that*

$$\sum_{i=2}^{r-1} \varepsilon_i \cdot \left(2^i - 1\right) \equiv 0 \pmod{2^g - 1}. \tag{14}$$

Then $\star[r] - \sum_{i=2}^{r-1} \varepsilon_i[i]$ is partitionable.

Proof. Due to Theorem 3.8 and Lemma 3.6 it suffices to consider the case $\varepsilon_i \in \mathbb{N}$ for $2 \le i \le r-2$ while Equation (14) still holds. From Lemma 3.7 we conclude that $\varepsilon_i[r] - \varepsilon_i[i]$ is partitionable for all $i \equiv r \pmod 2$ and $2 \le i < r$ as well as that $\varepsilon_i[r-1] - \varepsilon_i[i]$ is partitionable for all $i \equiv r-1 \pmod 2$ and $2 \le i < r-1$. So using Lemma 3.6 we can assume $\varepsilon_i = 0$ for all $2 \le i \le r-2$ while Equation (14) still holds. Reusing our first argument again we can additionally assume $\varepsilon_{r-1} \in \mathbb{N}$. If r is odd, then let $\mathcal{S}$ be a line spread of S_{r-1} (using the notation from Definition 3.2 and the construction from Theorem 3.8). Choosing each line in $\operatorname{PG}(r-1,2)$ ε_{r-1} times and removing an ε_i-fold copy of $\mathcal{S}$ shows that $\star[r] - \varepsilon_{r-1}[r-1]$ is partitionable. If r is even, then Equation (14) yields $\varepsilon_{r-1} \equiv 0 \pmod 3$. Now let $\mathcal{S}$ be a multiset of lines partitioning the 3-fold copy of the points of S_{r-1}. Choosing each line in $\operatorname{PG}(r-1,2)$ $\varepsilon_{r-1}/3$ times and removing an $\varepsilon_i/3$-fold copy of $\mathcal{S}$ shows that $\star[r] - \varepsilon_{r-1}[r-1]$ is partitionable. □

Definition 3.12. For integers $n > s \ge 1$ and $r > 2$ let the *surplus* be defined by

$$\theta(n,r,s) := 3n - g(r, 2(n-s)). \tag{15}$$

So the surplus is negative iff n is larger than the Griesmer upper bound for $n_{r/2}(s)$.

Lemma 3.13. *Let $n > s \ge 1$ and $r > 2$ be integers. If $\theta(n,r,s) \ge 0$, then there exists an*

$$\left(n + t \cdot \frac{2^r - 1}{2^{\gcd(r,2)} - 1}, r, s + t \cdot \frac{2^{r-h} - 1}{2^{\gcd(r,2)} - 1}\right)$$

system $\mathcal{S}_t$ for all sufficiently large t.

Proof. Setting $d' := 2(n-s)$ and $n' := g(r, d')$ we have $3 > \theta(n,r,s) = 3n - n' \ge 0$. Due to Lemma 3.1 we can choose integers $\sigma, \varepsilon_1, \ldots, \varepsilon_{r-1}$, with $\sigma \ge 0$ and $0 \le \varepsilon_i < 2$ for all $1 \le i \le r-1$, such that

$$d' = \sigma \cdot 2^{r-1} - \sum_{i=1}^{r-1} \varepsilon_i \cdot 2^{i-1} \tag{16}$$

and

$$n' = \sigma \cdot (2^r - 1) - \sum_{i=1}^{r-1} \varepsilon_i \cdot \left(2^i - 1\right). \tag{17}$$

Since d' is divisible by 2 we have $\varepsilon_1 = 0$. Let $\tau := \theta(n,r,s)$, $\sigma' := \sigma + \tau$, $\varepsilon'_{r-1} = \varepsilon_{r-1} + 2\tau$, and $\varepsilon'_i = \varepsilon_i$ for all $1 \le i \le r-2$, so that $\varepsilon'_i \in \mathbb{N}$ for all $1 \le i \le r-1$ and $\varepsilon'_1 = 0$. Note that

$$d' = \sigma' \cdot 2^{r-1} - \sum_{i=2}^{r-1} \varepsilon'_i \cdot 2^{i-1} \tag{18}$$

and

$$3n = n' + \tau = \sigma' \cdot (2^r - 1) - \sum_{i=2}^{r-1} \varepsilon'_i \cdot \left(2^i - 1\right), \tag{19}$$

so that

$$\sum_{i=2}^{r-1} \varepsilon'_i \cdot \left(2^i - 1\right) \equiv 0 \pmod{2^{\gcd(r,2)} - 1}. \tag{20}$$

From Theorem 3.11 and Lemma 3.9 we conclude that $\left(\sigma' + t \cdot \frac{3}{2^{\gcd(r,h)}-1}\right) \cdot [r] - \sum_{i=2}^{r-1} \varepsilon'_i[i]$ is partitionable for all sufficiently large t. From Corollary 3.5, Equation (18), and Equation (19) we compute the stated parameters of $\mathcal{S}_t$. □

Theorem 3.14. *For all sufficiently large s we have that $n_{r/2}(s)$ attains the Griesmer upper bound, see Definition 2.5.*

Proof. Let $s_i := \frac{2^{r-2}-1}{2^{\gcd(r,2)}-1} - i$ for $0 \le i < \frac{2^{r-2}-1}{2^{\gcd(r,2)}-1}$ and n_i be the Griesmer upper bound for $n_{r/2}(s_i)$, i.e. $3n_i \ge g(r, 2(n_i - s_i))$ while $3(n_i + 1) < g(r, 2(n_i + 1 - s_i))$. Let $\sigma_i, \varepsilon_{1,i}, \ldots, \varepsilon_{r-1,i} \in \mathbb{N}$ with $\varepsilon_{j,i} < 2$ for all $1 \le j \le r-1$ be uniquely defined by

$$d_i := 2 \cdot (n_i - s_i) = \sigma_i \cdot 2^{r-1} - \sum_{j=1}^{r-1} \varepsilon_{j,i} \cdot 2^{j-1}, \tag{21}$$

so that

$$g(r, d_i) = \sigma_i \cdot (2^r - 1) - \sum_{j=1}^{r-1} \varepsilon_{j,i} \cdot \left(2^j - 1\right), \tag{22}$$

using Lemma 3.1, and $\theta(n_i, r, s_i,) \ge 0$. Similarly, let $\sigma'_i, \varepsilon'_{1,i}, \ldots, \varepsilon'_{r-1,i} \in \mathbb{N}$ with $\varepsilon'_{j,i} < 2$ for all $1 \le j \le r-1$ be uniquely defined by

$$d'_i := 2 + d_i = \sigma'_i \cdot 2^{r-1} - \sum_{j=1}^{r-1} \varepsilon'_{j,i} \cdot 2^{j-1}, \tag{23}$$

so that

$$g(r,d_i') = \sigma_i' \cdot (2^r-1) - \sum_{j=1}^{r-1} \varepsilon_{j,i}' \cdot \left(2^j-1\right) = \qquad (24)$$

and $\theta(n_i+1,r,s_i,) < 0$. Now, let $s_{i,t} := s_i + t \cdot \frac{2^{r-2}-1}{2^{\gcd(r,2)}-1}$ and $n_{i,t} := n_i + t \cdot \frac{2^r-1}{2^{\gcd(r,2)}-1}$, so that Lemma 3.1 implies

$$d_{i,t} := 2\cdot(n_{i,t}-s_{i,t}) = \left(\sigma_i + t\cdot\frac{3}{2^{\gcd(r,2)}-1}\right)\cdot 2^{r-1} - \sum_{i=1}^{r-i}\varepsilon_i \cdot 2^{i-1}, \qquad (25)$$

$$g(r,d_{i,t}) = t\cdot\frac{2^r-1}{2^{\gcd(r,2)}-1}\cdot 3 + \sigma_i\cdot(2^r-1) - \sum_{j=1}^{r-1}\varepsilon_{j,i}\cdot\left(2^j-1\right), \qquad (26)$$

$$d_{i,t}' := 2 + d_{i,t} = \left(\sigma_i' + t\cdot\frac{3}{2^{\gcd(r,2)}-1}\right)\cdot 2^{r-1} - \sum_{i=1}^{r-i}\varepsilon_i'\cdot 2^{i-1}, \qquad (27)$$

and

$$g\left(r,d_{i,t}'\right) = t\cdot\frac{2^r-1}{2^{\gcd(r,2)}-1}\cdot 3 + \sigma_i'\cdot(2^r-1) - \sum_{j=1}^{r-1}\varepsilon_{j,i}'\cdot\left(2^j-1\right). \qquad (28)$$

Thus we have

$$\theta(n_{i,t},r,s_{i,t}) = \theta(n_i,r,s_i) \ge 0 \text{ and } \theta(n_{i,t}+1,r,s_{i,t}) = \theta(n_i+1,r,s_i) < 0,$$

i.e. the Griesmer upper bound for $n_{r/2}(s_{i,t})$ is given by $n_{i,t}$ for all $t \in \mathbb{N}$. Lemma 3.13 yields the existence of an $(n_{i,t},r,s_{i,t})$ system $\mathcal{S}_{i,t}$ for all sufficiently large t. □

We remark that the statement of Theorem 3.14 was generalized to arbitrary additive codes over $\mathbb{F}_q$ in [15]. The proof of Theorem 3.14 suggest the following algorithm to determine explicit formulas for $n_{r/2}(s)$ assuming that s is sufficiently large. For all $0 \le i < \frac{2^{r-2}-1}{2^{\gcd(r,2)}-1}$ compute the Griesmer upper bound n_i for $n_{r/2}(s_i)$ where $s_i = \frac{2^{r-2}-1}{2^{\gcd(r,2)}-1} - i$. Then we have

$$n_{r/2}\left(t\cdot\frac{2^{r-2}-1}{2^{\gcd(r,2)}-1} - i\right) = t\cdot\frac{2^r-1}{2^{\gcd(r,2)}-1} - \left(\frac{2^r-1}{2^{\gcd(r,2)}-1} - n_i\right) \qquad (29)$$

for all sufficiently large t. As an example we mention:

Proposition 3.15. *(Cf.* [11, Table I],[12, Table II]*) For all sufficiently large t we have*

- $n_{3.5}(31t) = 127t$;
- $n_{3.5}(31t-1) = 127t-5$;
- $n_{3.5}(31t-2) = 127t-10$;
- $n_{3.5}(31t-3) = 127t-15$;
- $n_{3.5}(31t-4) = 127t-20$;
- $n_{3.5}(31t-5) = 127t-21$;
- $n_{3.5}(31t-6) = 127t-26$;
- $n_{3.5}(31t-7) = 127t-31$;
- $n_{3.5}(31t-8) = 127t-36$;
- $n_{3.5}(31t-9) = 127t-41$;
- $n_{3.5}(31t-10) = 127t-42$;
- $n_{3.5}(31t-11) = 127t-47$;
- $n_{3.5}(31t-12) = 127t-52$;
- $n_{3.5}(31t-13) = 127t-55$;
- $n_{3.5}(31t-14) = 127t-60$;
- $n_{3.5}(31t-15) = 127t-63$;
- $n_{3.5}(31t-16) = 127t-68$;
- $n_{3.5}(31t-17) = 127t-73$;
- $n_{3.5}(31t-18) = 127t-76$;
- $n_{3.5}(31t-19) = 127t-81$;
- $n_{3.5}(31t-20) = 127t-84$;
- $n_{3.5}(31t-21) = 127t-87$;
- $n_{3.5}(31t-22) = 127t-92$;
- $n_{3.5}(31t-23) = 127t-95$;
- $n_{3.5}(31t-24) = 127t-100$;
- $n_{3.5}(31t-25) = 127t-105$;
- $n_{3.5}(31t-26) = 127t-108$;

- $n_{3.5}(31t-27) = 127t-113$;
- $n_{3.5}(31t-28) = 127t-116$;
- $n_{3.5}(31t-29) = 127t-121$;
- $n_{3.5}(31t-30) = 127t-126$.

In [12] the stated formulas of Proposition 3.15 were indeed shown to be true for all $t \ge 2$ and $n_2(7,2;31-i)$ was determined for all $i \in \{0,\dots,31\}\backslash\{19,24,25\}$, referring to [16] for $i=18$ and [11] for the previous state of the art.

4. Exact values of $n_{3.5}(s)$ and $n_4(s)$

(n,r,s) systems can be easily modeled as ILPs. To reduce the search space we prescribe subgroups of the automorphism group. Alternatively we can try to partition suitable multisets of points. Those multisets of points can again be modeled as ILPs and we may prescribe subgroups of the automorphism group. Alternatively we use the database of *best known linear codes* (BKLC) in `Magma` or enumerate suitable linear codes using `LinCode` [5]. For the (known) conditions of the binary codes we refer to Lemma 2.3.

Theorem 4.1. *(Cf.* [11, Table I],[12, Table II]*) We have*

- $n_{3.5}(31t) = 127t$ *for* $t \ge 1$;
- $n_{3.5}(31t-1) = 127t-5$ *for* $t \ge 1$;
- $n_{3.5}(31t-2) = 127t-10$ *for* $t \ge 1$;
- $n_{3.5}(31t-3) = 127t-15$ *for* $t \ge 1$;
- $n_{3.5}(31t-4) = 127t-20$ *for* $t \ge 1$;
- $n_{3.5}(31t-5) = 127t-21$ *for* $t \ge 1$;
- $n_{3.5}(31t-6) = 127t-26$ *for* $t \ge 1$;
- $n_{3.5}(31t-7) = 127t-31$ *for* $t \ge 1$;
- $n_{3.5}(31t-8) = 127t-36$ *for* $t \ge 1$;
- $n_{3.5}(31t-9) = 127t-41$ *for* $t \ge 1$;
- $n_{3.5}(31t-10) = 127t-42$ *for* $t \ge 1$;
- $n_{3.5}(31t-11) = 127t-47$ *for* $t \ge 1$;
- $n_{3.5}(31t-12) = 127t-52$ *for* $t \ge 1$;
- $n_{3.5}(31t-13) = 127t-55$ *for* $t \ge 1$;
- $n_{3.5}(31t-14) = 127t-60$ *for* $t \ge 1$;
- $n_{3.5}(31t-15) = 127t-63$ *for* $t \ge 1$;
- $n_{3.5}(31t-16) = 127t-68$ *for* $t \ge 1$;
- $n_{3.5}(31t-17) = 127t-73$ *for* $t \ge 1$;
- $n_{3.5}(31t-18) = 127t-76$ *for* $t \ge 1$;
- $n_{3.5}(31t-19) = 127t-81$ *for* $t \ge 1$;
- $n_{3.5}(31t-20) = 127t-84$ *for* $t \ge 1$;
- $n_{3.5}(31t-21) = 127t-87$ *for* $t \ge 1$;
- $n_{3.5}(31t-22) = 127t-92$ *for* $t \ge 1$;
- $n_{3.5}(31t-23) = 127t-95$ *for* $t \ge 1$;
- $n_{3.5}(31t-24) = 127t-100$ *for* $t \ge 1$;
- $n_{3.5}(31t-25) = 127t-105$ *for* $t \ge 1$;
- $n_{3.5}(31t-26) = 127t-108$ *for* $t \ge 2$ *and* $n_{3.5}(5) = 17$;
- $n_{3.5}(31t-27) = 127t-113$ *for* $t \ge 2$ *and* $n_{3.5}(4) = 12$;
- $n_{3.5}(31t-28) = 127t-116$ *for* $t \ge 2$ *and* $n_{3.5}(3) = 7$;
- $n_{3.5}(31t-29) = 127t-121$ *for* $t \ge 2$;
- $n_{3.5}(31t-30) = 127t-126$ *for* $t \ge 2$.

Proof. We can assume $s \ge 3$. Theorem 3.8 yields $n_2(7,2;31t) = 127t$ for $t \ge 1$. In [3] $n_2(7,2;3) \le 7$ was shown. The coding upper bound implies $n_2(7,2;4) \le 12$ and $n_2(7,2;5) \le 17$. All other upper bounds follow from the Griesmer upper bound. Due to Theorem 3.8 and Lemma 2.9 it suffices to gives constructions for $s \in \{3,\dots,13,15,21,25,26,30\}$. Using ILP searches we have found the following explicit constructions:[2]
$s=3: \binom{0100100}{0010101}, \binom{0101100}{0011111}, \binom{0101000}{0011010}, \binom{1000111}{0110101}, \binom{1001110}{0111111}, \binom{1001001}{0111010}, \binom{1100000}{0010000}$;
$s=4: \binom{1000111}{0010110}, \binom{1001110}{0011101}, \binom{1001001}{0011011}, \binom{1010110}{0111110}, \binom{1011101}{0111001}, \binom{1011011}{0110111}, \binom{1110100}{0000001}, \binom{1111100}{0000011}, \binom{1111000}{0000010}, \binom{0001010}{0000101}, \binom{0100000}{0010000}, \binom{1000000}{0100000}$;
$s=5: \binom{1000010}{0010101}, \binom{1000001}{0011101}, \binom{1001111}{0011001}, \binom{1000100}{0011010}, \binom{1001000}{0011011}, \binom{1000101}{0110111}, \binom{1001101}{0111100}, \binom{1001011}{0110011}, \binom{1001010}{0111110}, \binom{1001001}{0110110}, \binom{1010101}{0110010}, \binom{1011101}{0110001}, \binom{1011010}{0110100}, \binom{1011001}{0111111}, \binom{1011011}{0111000}, \binom{0100000}{0010000}, \binom{1000000}{0100000}$;

[2] We remark that constructions for $s=3,4$ were given in [3] and for $s=5,21$ we can use quaternary linear codes. For $s=9$ an example is given by a vector space partition of $\operatorname{PG}(6,2)$ of type $2^{35}3^14^1$. For $s=15$ an example is given in [12, Example 2]. More constructions can e.g. be found in [12].

$s=6$: $\binom{0001001}{0000010}$, $\binom{0100011}{0000110}$, $\binom{0100010}{0010100}$, $\binom{0100110}{0011110}$, $\binom{0101101}{0010010}$, $\binom{0101100}{0010111}$, $\binom{1001100}{0000001}$, $\binom{1000011}{0011000}$, $\binom{1001011}{0011011}$, $\binom{1001110}{0010001}$, $\binom{1011010}{0000100}$, $\binom{1000110}{0100001}$, $\binom{1000001}{0110001}$, $\binom{1000010}{0110000}$, $\binom{1010111}{0100000}$, $\binom{1011101}{0110101}$, $\binom{1011000}{0111100}$, $\binom{1100010}{0001011}$, $\binom{1100101}{0001000}$, $\binom{1101010}{0011111}$, $\binom{1110001}{0001110}$, $\binom{1110100}{0001100}$;

$s=7$: $\binom{0010100}{0001010}$, $\binom{0011001}{0000101}$, $\binom{0010111}{0001111}$, $\binom{0100111}{0011010}$, $\binom{0101110}{0010101}$, $\binom{0101001}{0011111}$, $\binom{1000100}{0010001}$, $\binom{1001100}{0010011}$, $\binom{1001000}{0010010}$, $\binom{1000101}{0101011}$, $\binom{1001111}{0100110}$, $\binom{1001010}{0101101}$, $\binom{1010111}{0101000}$, $\binom{1011110}{0100100}$, $\binom{1010110}{0101010}$, $\binom{1011101}{0100101}$, $\binom{1011011}{0101111}$, $\binom{1011001}{0101100}$, $\binom{1010100}{0110001}$, $\binom{1011100}{0110011}$, $\binom{1011000}{0110010}$, $\binom{1100011}{0000111}$, $\binom{1100010}{0001110}$, $\binom{1100001}{0001001}$, $\binom{0001000}{0000100}$;

$s=8$: $\binom{0100001}{0010011}$, $\binom{0100011}{0010010}$, $\binom{0100010}{0010001}$, $\binom{0110101}{0001011}$, $\binom{0110111}{0001010}$, $\binom{0110110}{0001001}$, $\binom{1000001}{0011110}$, $\binom{1000011}{0011101}$, $\binom{1000010}{0011111}$, $\binom{1001101}{0010110}$, $\binom{1001111}{0010101}$, $\binom{1001110}{0010111}$, $\binom{1010101}{0101110}$, $\binom{1010111}{0101101}$, $\binom{1010110}{0101111}$, $\binom{1010001}{0111010}$, $\binom{1010011}{0111001}$, $\binom{1010010}{0111011}$, $\binom{1100101}{0011010}$, $\binom{1100111}{0011001}$, $\binom{1100110}{0011011}$, $\binom{1110001}{0000111}$, $\binom{1110011}{0000110}$, $\binom{1110010}{0000101}$, $\binom{0011000}{0000100}$, $\binom{0100000}{0010100}$, $\binom{1010100}{0001000}$, $\binom{1000000}{0110000}$, $\binom{1001100}{0111000}$, $\binom{1010000}{0101100}$, $\binom{1011000}{0111100}$, $\binom{1101000}{0010000}$;

$s=9$: $\binom{0100010}{0010111}$, $\binom{0100001}{0011100}$, $\binom{0101111}{0010110}$, $\binom{0100100}{0011110}$, $\binom{0101000}{0010011}$, $\binom{1000011}{0010010}$, $\binom{1001110}{0010001}$, $\binom{1000110}{0010100}$, $\binom{1000111}{0011111}$, $\binom{1001100}{0011000}$, $\binom{1000010}{0100101}$, $\binom{1000001}{0101101}$, $\binom{1001111}{0101001}$, $\binom{1000100}{0101010}$, $\binom{1001000}{0101011}$, $\binom{1000101}{0110010}$, $\binom{1001101}{0110001}$, $\binom{1001010}{0110100}$, $\binom{1001001}{0111111}$, $\binom{1001011}{0111000}$, $\binom{1010111}{0100011}$, $\binom{1011100}{0101110}$, $\binom{1010110}{0100111}$, $\binom{1010011}{0101100}$, $\binom{1011110}{0100110}$, $\binom{1010101}{0111100}$, $\binom{1011101}{0110110}$, $\binom{1011001}{0110011}$, $\binom{1011011}{0111110}$, $\binom{1011010}{0110111}$, $\binom{1100001}{0011010}$, $\binom{1101111}{0010101}$, $\binom{1100010}{0011011}$, $\binom{1100100}{0011001}$, $\binom{1101000}{0011101}$;

$s=10$: $\binom{0100001}{0010110}$, $\binom{0101111}{0010011}$, $\binom{0100101}{0010001}$, $\binom{0101101}{0011111}$, $\binom{0100100}{0010111}$, $\binom{0100010}{0011100}$, $\binom{0101010}{0010010}$, $\binom{0101011}{0010100}$, $\binom{0101000}{0011110}$, $\binom{0101001}{0011000}$, $\binom{1000111}{0010101}$, $\binom{1001100}{0011101}$, $\binom{1000011}{0011011}$, $\binom{1001110}{0011010}$, $\binom{1000110}{0011001}$, $\binom{1010101}{0000010}$, $\binom{1011100}{0000001}$, $\binom{1010011}{0001000}$, $\binom{1011010}{0000100}$, $\binom{1010110}{0001111}$, $\binom{1000101}{0100111}$, $\binom{1001101}{0101100}$, $\binom{1001011}{0100011}$, $\binom{1001010}{0101110}$, $\binom{1001001}{0100110}$, $\binom{1000010}{0110101}$, $\binom{1000001}{0111101}$, $\binom{1001111}{0111001}$, $\binom{1000100}{0111010}$, $\binom{1001000}{0111011}$, $\binom{1100101}{0000011}$, $\binom{1100011}{0001110}$, $\binom{1101001}{0000111}$, $\binom{1100111}{0001100}$, $\binom{1101010}{0000110}$, $\binom{1110001}{0001010}$, $\binom{1111010}{0000101}$, $\binom{1110010}{0001011}$, $\binom{1110100}{0001001}$, $\binom{1110101}{0001101}$;

$s=11$: $\binom{0100010}{0010101}$, $\binom{0100001}{0011101}$, $\binom{0100101}{0010111}$, $\binom{0101101}{0011100}$, $\binom{0101111}{0011001}$, $\binom{0100100}{0011010}$, $\binom{0101011}{0010011}$, $\binom{0101010}{0011110}$, $\binom{0101001}{0010110}$, $\binom{0101000}{0011011}$, $\binom{1000101}{0000010}$, $\binom{1001100}{0000001}$, $\binom{1000011}{0001000}$, $\binom{1001010}{0000100}$, $\binom{1000110}{0001111}$, $\binom{1010010}{0000101}$, $\binom{1010001}{0001101}$, $\binom{1010011}{0001011}$, $\binom{1010100}{0001010}$, $\binom{1010110}{0001001}$, $\binom{1000001}{0111010}$, $\binom{1001111}{0110101}$, $\binom{1000010}{0111011}$, $\binom{1000100}{0111001}$, $\binom{1001000}{0111101}$, $\binom{1010101}{0100111}$, $\binom{1011101}{0101100}$, $\binom{1011011}{0100011}$, $\binom{1011010}{0101110}$, $\binom{1011001}{0100110}$, $\binom{1100001}{0000110}$, $\binom{1101100}{0000011}$, $\binom{1100011}{0000111}$, $\binom{1100010}{0001100}$, $\binom{1100110}{0001110}$, $\binom{1100101}{0010010}$, $\binom{1101101}{0010001}$, $\binom{1101010}{0010100}$, $\binom{1101001}{0011111}$, $\binom{1101011}{0011000}$, $\binom{0100000}{0010000}$, $\binom{1000000}{0010000}$, $\binom{1000000}{0110000}$;

$s=12$: $\binom{0010001}{0000110}$, $\binom{0010011}{0000101}$, $\binom{0010010}{0000111}$, $\binom{0100001}{0011110}$, $\binom{0100011}{0011101}$, $\binom{0100010}{0011111}$, $\binom{0101101}{0011011}$, $\binom{0101111}{0011010}$, $\binom{0101110}{0011001}$, $\binom{1000001}{0001011}$, $\binom{1000011}{0001010}$, $\binom{1000010}{0001001}$, $\binom{1000101}{0110010}$, $\binom{1000111}{0110001}$, $\binom{1000110}{0110011}$, $\binom{1001101}{0110011}$, $\binom{1001111}{0110010}$, $\binom{1001110}{0110001}$, $\binom{1010101}{0100111}$, $\binom{1010111}{0100110}$, $\binom{1010110}{0100101}$, $\binom{1010001}{0101010}$, $\binom{1010011}{0101001}$, $\binom{1010010}{0101011}$, $\binom{1011101}{0110111}$, $\binom{1011111}{0110110}$, $\binom{1011110}{0110101}$, $\binom{1011001}{0111010}$, $\binom{1011011}{0111001}$, $\binom{1011010}{0111011}$, $\binom{1100101}{0011111}$, $\binom{1100111}{0011110}$, $\binom{1100110}{0011101}$, $\binom{1101101}{0011011}$, $\binom{1101111}{0011010}$, $\binom{1101110}{0011001}$, $\binom{0010000}{0001000}$, $\binom{0100100}{0010100}$, $\binom{0110100}{0001000}$, $\binom{1001100}{0010100}$, $\binom{1000100}{0100000}$, $\binom{1000000}{0111000}$, $\binom{1001000}{0111100}$, $\binom{1010100}{0100000}$, $\binom{1011100}{0100100}$, $\binom{1010000}{0111000}$;

$s=13$: $\binom{0011010}{0000001}$, $\binom{0101110}{0000001}$, $\binom{0011000}{0000010}$, $\binom{1110001}{0000101}$, $\binom{0011001}{0000110}$, $\binom{1100011}{0001001}$, $\binom{1100110}{0001001}$, $\binom{1000101}{0001010}$, $\binom{1010011}{0001010}$, $\binom{1100101}{0001101}$, $\binom{1110011}{0001101}$, $\binom{1010101}{0001110}$, $\binom{1110110}{0001111}$, $\binom{0101111}{0010000}$, $\binom{1000011}{0010001}$, $\binom{1000111}{0010010}$, $\binom{1100111}{0010010}$, $\binom{1001000}{0010100}$, $\binom{1100010}{0010101}$, $\binom{1000001}{0010110}$, $\binom{1001000}{0010111}$, $\binom{0100111}{0011001}$, $\binom{0101000}{0011100}$, $\binom{0101100}{0011100}$, $\binom{0101101}{0011101}$, $\binom{0100101}{0011110}$, $\binom{0101101}{0011110}$, $\binom{1000100}{0100000}$, $\binom{1010000}{0100001}$, $\binom{1001010}{0100011}$, $\binom{1010001}{0100011}$, $\binom{1011000}{0100100}$, $\binom{1000000}{0100110}$, $\binom{1000110}{0101000}$, $\binom{1001101}{0101001}$, $\binom{1000110}{0101011}$, $\binom{1010000}{0101011}$, $\binom{1011101}{0101010}$, $\binom{1010011}{0110010}$, $\binom{1000000}{0110101}$, $\binom{1001110}{0110111}$, $\binom{1010110}{0110110}$, $\binom{1000010}{0111000}$, $\binom{1000100}{0111001}$, $\binom{1010100}{0111000}$, $\binom{1001011}{0111011}$, $\binom{1011011}{0111010}$, $\binom{1000011}{0111100}$, $\binom{1010111}{0111101}$, $\binom{1011110}{0111100}$, $\binom{1001100}{0111110}$;

$s=15$: $\binom{0010101}{0001000}$, $\binom{0011001}{0000100}$, $\binom{0010101}{0001111}$, $\binom{0011001}{0000010}$, $\binom{0011010}{0000001}$, $\binom{0100011}{0000101}$, $\binom{0100011}{0001101}$, $\binom{0100111}{0001001}$, $\binom{0100111}{0001011}$, $\binom{0100110}{0001010}$, $\binom{0100101}{0011000}$, $\binom{0101101}{0010100}$, $\binom{0101001}{0010010}$, $\binom{0101011}{0010001}$, $\binom{0101010}{0011111}$, $\binom{1000010}{0010111}$, $\binom{1000001}{0011100}$, $\binom{1001111}{0010110}$, $\binom{1000100}{0011110}$, $\binom{1001000}{0010011}$, $\binom{1000110}{0100001}$, $\binom{1000011}{0101111}$, $\binom{1000111}{0100100}$, $\binom{1001100}{0100010}$, $\binom{1000111}{0100101}$, $\binom{1001100}{0101101}$, $\binom{1000011}{0101011}$, $\binom{1001110}{0101010}$, $\binom{1001110}{0101000}$, $\binom{1000110}{0101001}$, $\binom{1000101}{0110110}$, $\binom{1001101}{0110011}$, $\binom{1001011}{0110111}$, $\binom{1001010}{0111100}$, $\binom{1001001}{0111110}$, $\binom{1010100}{0100001}$, $\binom{1010010}{0101111}$, $\binom{1010001}{0101000}$, $\binom{1011111}{0100100}$, $\binom{1011000}{0100010}$, $\binom{1010010}{0110101}$, $\binom{1010001}{0111101}$, $\binom{1010111}{0110010}$, $\binom{1011100}{0110001}$, $\binom{1010011}{0111000}$, $\binom{1011110}{0110100}$, $\binom{1011111}{0111001}$, $\binom{1010100}{0111010}$, $\binom{1010110}{0111111}$, $\binom{1011000}{0111011}$, $\binom{1110001}{0000011}$, $\binom{1110001}{0001110}$, $\binom{1110010}{0000110}$, $\binom{1111000}{0000111}$, $\binom{1110100}{0001100}$, $\binom{1000000}{0010000}$, $\binom{1000000}{0110000}$, $\binom{1010000}{0110000}$, $\binom{1100000}{0010000}$;

$s=21$: $\binom{0010011}{0000100}$, $\binom{0011100}{0000010}$, $\binom{0010110}{0000001}$, $\binom{0010011}{0001111}$, $\binom{0010110}{0001000}$, $\binom{0010101}{0001100}$, $\binom{0011011}{0000110}$, $\binom{0010101}{0001110}$, $\binom{0011010}{0000111}$, $\binom{0011001}{0000011}$, $\binom{0110001}{0000101}$, $\binom{0110010}{0001101}$, $\binom{0110011}{0000101}$, $\binom{0110011}{0001101}$, $\binom{0110001}{0001001}$, $\binom{0110100}{0001011}$, $\binom{0110010}{0001010}$, $\binom{0110111}{0001001}$, $\binom{0110111}{0001011}$, $\binom{0110110}{0001010}$, $\binom{1000101}{0001000}$, $\binom{1001001}{0000100}$, $\binom{1000101}{0001111}$, $\binom{1001001}{0000010}$, $\binom{1001010}{0000001}$, $\binom{1000011}{0010010}$, $\binom{1001110}{0010001}$, $\binom{1000110}{0010100}$, $\binom{1000111}{0011111}$, $\binom{1001100}{0011000}$, $\binom{1000010}{0100101}$, $\binom{1000001}{0101101}$, $\binom{1000001}{0101011}$, $\binom{1001111}{0101010}$, $\binom{1000010}{0101001}$, $\binom{1001111}{0101001}$, $\binom{1000100}{0101010}$, $\binom{1000100}{0101101}$, $\binom{1001000}{0100101}$, $\binom{1001000}{0101011}$, $\binom{1000111}{0110101}$, $\binom{1001100}{0111101}$, $\binom{1000011}{0111011}$, $\binom{1001110}{0111010}$, $\binom{1000110}{0111001}$, $\binom{1010101}{0100010}$, $\binom{1011101}{0100001}$, $\binom{1010110}{0100011}$, $\binom{1010011}{0101110}$, $\binom{1010111}{0100010}$, $\binom{1011100}{0100001}$, $\binom{1010101}{0100111}$, $\binom{1011101}{0101100}$, $\binom{1010011}{0101000}$, $\binom{1011110}{0100100}$, $\binom{1011110}{0100111}$, $\binom{1010111}{0101100}$, $\binom{1011100}{0100110}$, $\binom{1010110}{0101111}$, $\binom{1011011}{0100011}$, $\binom{1011010}{0101110}$, $\binom{1011001}{0100110}$, $\binom{1011010}{0100100}$, $\binom{1011001}{0101111}$, $\binom{1011011}{0101000}$, $\binom{1010010}{0110101}$, $\binom{1010001}{0111101}$, $\binom{1011111}{0111001}$, $\binom{1010100}{0111010}$, $\binom{1011000}{0111011}$, $\binom{1100001}{0000011}$, $\binom{1100001}{0001110}$, $\binom{1100010}{0000110}$, $\binom{1101000}{0000111}$, $\binom{1100100}{0001100}$, $\binom{1100101}{0010010}$, $\binom{1101101}{0010001}$, $\binom{1101010}{0010100}$, $\binom{1101001}{0011111}$, $\binom{1101011}{0011000}$, $\binom{0100000}{0010000}$, $\binom{1000000}{0100000}$, $\binom{1010000}{0100000}$, $\binom{1010000}{0110000}$, $\binom{1100000}{0010000}$;

$s=25$: $\binom{0010011}{0000101}$, $\binom{0010011}{0001101}$, $\binom{0010111}{0001001}$, $\binom{0010111}{0001011}$, $\binom{0010110}{0001010}$, $\binom{0010101}{0001100}$, $\binom{0011011}{0000110}$, $\binom{0010101}{0001110}$, $\binom{0011010}{0000111}$, $\binom{0011001}{0000011}$, $\binom{0100001}{0010010}$, $\binom{0101111}{0010001}$, $\binom{0100010}{0010100}$, $\binom{0100011}{0011001}$, $\binom{0101110}{0011011}$, $\binom{0100100}{0011000}$, $\binom{0100111}{0011010}$, $\binom{0101100}{0010101}$, $\binom{0100110}{0011101}$, $\binom{0101000}{0011111}$, $\binom{1000011}{0000100}$, $\binom{1001100}{0000010}$, $\binom{1000110}{0000001}$, $\binom{1000011}{0001111}$, $\binom{1000110}{0001000}$, $\binom{1000010}{0010111}$, $\binom{1000001}{0011100}$, $\binom{1001111}{0010110}$, $\binom{1000100}{0011110}$, $\binom{1001000}{0010011}$, $\binom{1000011}{0100111}$, $\binom{1001110}{0101100}$, $\binom{1000100}{0100101}$, $\binom{1000010}{0101101}$, $\binom{1000111}{0100110}$, $\binom{1001100}{0100011}$, $\binom{1000001}{0101001}$, $\binom{1001111}{0101011}$, $\binom{1000101}{0101010}$, $\binom{1001101}{0100101}$, $\binom{1000110}{0101110}$, $\binom{1001001}{0101101}$, $\binom{1001000}{0101010}$, $\binom{1001011}{0101001}$, $\binom{1001010}{0101011}$, $\binom{1000010}{0110101}$, $\binom{1000001}{0111101}$, $\binom{1000101}{0110010}$, $\binom{1001101}{0110001}$, $\binom{1000101}{0110111}$, $\binom{1001101}{0111100}$, $\binom{1001111}{0111001}$, $\binom{1000100}{0111010}$, $\binom{1001011}{0110011}$, $\binom{1001010}{0111110}$, $\binom{1001001}{0110110}$, $\binom{1001010}{0110100}$, $\binom{1001001}{0111111}$, $\binom{1001000}{0111011}$, $\binom{1001011}{0111000}$, $\binom{1010011}{0100010}$, $\binom{1011110}{0100001}$, $\binom{1010110}{0100100}$, $\binom{1010111}{0101111}$, $\binom{1011100}{0101000}$, $\binom{1010010}{0110101}$, $\binom{1010001}{0111101}$, $\binom{1010010}{0110111}$, $\binom{1010001}{0111100}$, $\binom{1010101}{0110010}$, $\binom{1011101}{0110001}$, $\binom{1010110}{0110001}$, $\binom{1010011}{0111111}$, $\binom{1010111}{0110100}$, $\binom{1011100}{0110010}$, $\binom{1011111}{0110110}$, $\binom{1011111}{0111001}$, $\binom{1011110}{0111000}$, $\binom{1010100}{0111010}$, $\binom{1010100}{0111110}$, $\binom{1011000}{0110011}$, $\binom{1011010}{0110100}$, $\binom{1011001}{0111111}$, $\binom{1011000}{0111011}$, $\binom{1011011}{0111000}$, $\binom{1100101}{0011000}$, $\binom{1101101}{0010100}$, $\binom{1100101}{0011111}$, $\binom{1101101}{0011000}$, $\binom{1101001}{0010010}$, $\binom{1101011}{0010001}$, $\binom{1101010}{0010001}$, $\binom{1101010}{0011111}$, $\binom{1101011}{0010010}$, $\binom{1101001}{0010100}$, $\binom{0100000}{0010000}$, $\binom{1000000}{0010000}$, $\binom{1000000}{0100000}$, $\binom{1000000}{0110000}$, $\binom{1010000}{0110000}$, $\binom{1100000}{0010000}$;

$s=26$: $\binom{0010101}{0001000}$, $\binom{0011001}{0000100}$, $\binom{0010101}{0001111}$, $\binom{0011001}{0000010}$, $\binom{0011010}{0000001}$, $\binom{0100011}{0000101}$, $\binom{0100011}{0001101}$, $\binom{0100111}{0001001}$, $\binom{0100111}{0001011}$, $\binom{0100110}{0001010}$, $\binom{0100101}{0001100}$, $\binom{0101011}{0000110}$, $\binom{0100101}{0001110}$, $\binom{0101010}{0000111}$, $\binom{0101001}{0000011}$, $\binom{0100101}{0011000}$, $\binom{0101101}{0010100}$, $\binom{0101001}{0010010}$, $\binom{0101011}{0010001}$, $\binom{0101010}{0011111}$, $\binom{0110011}{0000101}$, $\binom{0110011}{0001101}$, $\binom{0110111}{0001001}$, $\binom{0110111}{0001011}$, $\binom{0110110}{0001010}$, $\binom{1000011}{0000100}$, $\binom{1001100}{0000010}$, $\binom{1000110}{0000001}$, $\binom{1000011}{0001111}$, $\binom{1000110}{0001000}$, $\binom{1000101}{0100011}$, $\binom{1001101}{0101110}$, $\binom{1001001}{0100111}$, $\binom{1001011}{0101100}$, $\binom{1001010}{0100110}$, $\binom{1000001}{0110010}$, $\binom{1001111}{0110001}$, $\binom{1000010}{0110001}$, $\binom{1000001}{0111111}$, $\binom{1000010}{0110100}$, $\binom{1000100}{0110010}$, $\binom{1001111}{0111000}$, $\binom{1000100}{0111000}$,

$\binom{1000101}{0111000}, \binom{1001101}{0110100}, \binom{1001001}{0110010}, \binom{1001011}{0110001}, \binom{1001010}{0111111}, \binom{1001000}{0110100}, \binom{1001000}{0111111}, \binom{1010011}{0100100}, \binom{1011110}{0100010}, \binom{1010101}{0100001}, \binom{1011101}{0101111},$
$\binom{1010111}{0100001}, \binom{1011100}{0101111}, \binom{1010101}{0100100}, \binom{1011101}{0100010}, \binom{1010110}{0101000}, \binom{1011001}{0100001}, \binom{1011011}{0101111}, \binom{1011010}{0100010}, \binom{1011011}{0100100}, \binom{1011001}{0101000}, \binom{1011010}{0101000},$
$\binom{1010010}{0110101}, \binom{1010001}{0111101}, \binom{1010010}{0110111}, \binom{1010001}{0111100}, \binom{1010111}{0110101}, \binom{1011100}{0111101}, \binom{1010011}{0111011}, \binom{1011110}{0111010}, \binom{1011111}{0110110}, \binom{1011111}{0111001}, \binom{1010100}{0111010},$
$\binom{1010110}{0111001}, \binom{1010100}{0111110}, \binom{1011000}{0110011}, \binom{1011000}{0111011}, \binom{1100001}{0000011}, \binom{1100001}{0001110}, \binom{1100010}{0000110}, \binom{1101000}{0000111}, \binom{1100100}{0001100}, \binom{1100011}{0010010}, \binom{1101110}{0010001},$
$\binom{1100110}{0010100}, \binom{1100101}{0011100}, \binom{1101101}{0010110}, \binom{1100101}{0011110}, \binom{1101101}{0010111}, \binom{1100111}{0011111}, \binom{1101100}{0011000}, \binom{1101001}{0010011}, \binom{1101011}{0011110}, \binom{1101010}{0010011}, \binom{1101010}{0010111},$
$\binom{1101011}{0010110}, \binom{0101001}{0000010}, \binom{1001000}{0100000}, \binom{1000000}{0100000}, \binom{1000000}{0110000}, \binom{1010000}{0100000}, \binom{1010000}{0110000}, \binom{1100000}{0010000};$
$s = 30$: $\binom{0010001}{0000101}, \binom{0010010}{0001101}, \binom{0010011}{0000101}, \binom{0010011}{0001101}, \binom{0010001}{0001001}, \binom{0010100}{0001011}, \binom{0010010}{0001010}, \binom{0010111}{0001001}, \binom{0010111}{0001011}, \binom{0010110}{0001010},$
$\binom{0100001}{0010101}, \binom{0101111}{0011101}, \binom{0100011}{0010100}, \binom{0101110}{0010010}, \binom{0100111}{0010001}, \binom{0101100}{0011111}, \binom{0100010}{0011010}, \binom{0100100}{0011011}, \binom{0100110}{0011000}, \binom{0101000}{0011001}, \binom{0110101}{0000010},$
$\binom{0111100}{0000001}, \binom{0110011}{0001000}, \binom{0111010}{0000100}, \binom{0110110}{0001111}, \binom{1000011}{0000100}, \binom{1001100}{0000010}, \binom{1000110}{0000001}, \binom{1000011}{0001111}, \binom{1000110}{0001000}, \binom{1000101}{0001100}, \binom{1001011}{0000110},$
$\binom{1000101}{0001110}, \binom{1001010}{0000111}, \binom{1001001}{0000011}, \binom{1000101}{0011001}, \binom{1001101}{0011011}, \binom{1001011}{0010101}, \binom{1001010}{0011101}, \binom{1001001}{0011010}, \binom{1000001}{0101010}, \binom{1001111}{0100101}, \binom{1000001}{0101011},$
$\binom{1001111}{0101010}, \binom{1000010}{0101001}, \binom{1000010}{0101011}, \binom{1000100}{0101001}, \binom{1000100}{0101101}, \binom{1001000}{0100101}, \binom{1001000}{0101101}, \binom{1000011}{0110001}, \binom{1001110}{0111111}, \binom{1000010}{0110101}, \binom{1000001}{0111101},$
$\binom{1000110}{0110010}, \binom{1001111}{0111001}, \binom{1000100}{0111010}, \binom{1000111}{0111000}, \binom{1001100}{0110100}, \binom{1001000}{0111011}, \binom{1010001}{0100010}, \binom{1011111}{0100001}, \binom{1010010}{0100100}, \binom{1010101}{0100010}, \binom{1011101}{0100001},$
$\binom{1010110}{0100011}, \binom{1010011}{0101110}, \binom{1010101}{0100111}, \binom{1011101}{0101100}, \binom{1010111}{0100101}, \binom{1011100}{0101101}, \binom{1010011}{0101011}, \binom{1011110}{0101010}, \binom{1011110}{0100111}, \binom{1010100}{0101000}, \binom{1010110}{0101001},$
$\binom{1010111}{0101100}, \binom{1011100}{0100110}, \binom{1011011}{0100011}, \binom{1011010}{0101110}, \binom{1011001}{0100110}, \binom{1011010}{0100100}, \binom{1011001}{0101111}, \binom{1011000}{0101111}, \binom{1011011}{0101000}, \binom{1010010}{0110001}, \binom{1010001}{0111111},$
$\binom{1010010}{0110101}, \binom{1010001}{0111101}, \binom{1010100}{0110010}, \binom{1011111}{0111001}, \binom{1011111}{0111000}, \binom{1010100}{0111010}, \binom{1010101}{0111100}, \binom{1011101}{0110110}, \binom{1011001}{0110011}, \binom{1011011}{0111110}, \binom{1011000}{0110100},$
$\binom{1011010}{0110111}, \binom{1011000}{0111011}, \binom{1100001}{0000011}, \binom{1100001}{0001110}, \binom{1100010}{0000110}, \binom{1101000}{0000111}, \binom{1100100}{0001100}, \binom{1100010}{0010111}, \binom{1100001}{0011100}, \binom{1100011}{0011010}, \binom{1101110}{0010101},$
$\binom{1101111}{0010110}, \binom{1100110}{0011011}, \binom{1100100}{0011110}, \binom{1100111}{0011101}, \binom{1101100}{0011001}, \binom{1101000}{0010011}, \binom{0100000}{0010000}, \binom{1000000}{0010000}, \binom{1000000}{0100000}, \binom{1000000}{0110000}, \binom{1010000}{0100000},$
$\binom{1010000}{0110000}, \binom{1100000}{0010000}.$

□

Theorem 4.2. *For $s \ge 30$ the Griesmer upper bound for $n_4(s)$ can always be attained.*

- $n_4(21) = 85t$ *for* $t \ge 1$*;*
- $n_4(21t-1) = 85t-5$ *for* $t \ge 1$*;*
- $n_4(21t-2) = 85t-10$ *for* $t \ge 1$*;*
- $n_4(21t-3) = 85t-15$ *for* $t \ge 1$*;*
- $n_4(21t-4) = 85t-20$ *for* $t \ge 1$*;*
- $n_4(21t-5) = 85t-21$ *for* $t \ge 1$*;*
- $n_4(21t-6) = 85t-26$ *for* $t \ge 2$ *and* $n_4(15) \in \{55, 56, 57\}$*;*
- $n_4(21t-7) = 85t-31$ *for* $t \ge 1$*;*
- $n_4(21t-8) = 85t-36$ *for* $t \ge 1$*;*
- $n_4(21t-9) = 85t-41$ *for* $t \ge 1$*;*
- $n_4(21t-10) = 85t-42$ *for* $t \ge 2$ *and* $n_4(11) = 40$*;*
- $n_4(21t-11) = 85t-47$ *for* $t \ge 2$ *and* $n_4(10) \in \{35, 36\}$*;*
- $n_4(21t-12) = 85t-52$ *for* $t \ge 1$*;*
- $n_4(21t-13) = 85t-55$ *for* $t \ge 3$*,* $n_4(8) = 28$*, and* $n_4(29) \in \{113, 114, 115\}$*;*
- $n_4(21t-14) = 85t-60$ *for* $t \ge 3$*,* $n_4(7) = 23$*, and* $n_4(28) \in \{108, 109, 110\}$*;*
- $n_4(21t-15) = 85t-63$ *for* $t \ge 2$ *and* $n_4(6) = 18$*;*
- $n_4(21t-16) = 85t-68$ *for* $t \ge 1$*;*
- $n_4(21t-17) = 85t-73$ *for* $t \ge 2$ *and* $n_4(4) = 10$*;*
- $n_4(21t-18) = 85t-76$ *for* $t \ge 3$*,* $n_4(3) = 5$*,* $n_4(24) \in \{92, 93, 94\}$*;*
- $n_4(21t-19) = 85t-81$ *for* $t \ge 2$*;*
- $n_4(21t-20) = 85t-84$ *for* $t \ge 2$*.*

Proof. We can assume $s \ge 3$. Theorem 3.8 yields $n_4(21t) = 85$ for $t \ge 1$. In [3] $n_4(3) \le 5$ and $n_4(4) \le 10$ were shown. The coding upper bound implies $n_4(6) \le 18$, $n_4(7) \le 23$, $n_4(8) \le 28$, $n_4(10) \le 36$, $n_4(11) \le 40$, and $n_4(15) \le 57$. All other upper bounds follow from the Griesmer upper bound. For all

$$s \in \{5, \ldots, 40\} \backslash \{9, 10, 11, 14, 15, 23, 24, 27, 28, 29, 44, 45, 49, 50\}$$

the mentioned upper bound for $n_4(s)$ is matched by a quaternary linear code. We have $n_4(44) \ge n_4(21) + n_4(23) = 85 + 89 = 174$. The following explicit examples were obtained using ILP searches:

$s = 9$: $\binom{01001011}{00010110}, \binom{01000110}{00111101}, \binom{01010101}{00000010}, \binom{01111110}{00000001}, \binom{01001101}{00101011}, \binom{01010010}{00100101}, \binom{01101110}{00011111}, \binom{01011001}{00111010}, \binom{01101001}{00000011},$
$\binom{10010000}{00000111}, \binom{10110000}{00001110}, \binom{10010001}{00101010}, \binom{10100110}{00010101}, \binom{10010101}{00101101}, \binom{10100100}{00011011}, \binom{10011100}{00110110}, \binom{10011101}{00111111}, \binom{10100000}{00001001}, \binom{10001000}{01011111},$
$\binom{10000100}{01111010}, \binom{10001100}{01100101}, \binom{10011000}{01100010}, \binom{10110100}{01010001}, \binom{10101100}{01110011}, \binom{11000110}{00011110}, \binom{11001101}{00111001}, \binom{11001011}{00100111}, \binom{11010000}{00100001}, \binom{11100011}{00010011},$
$\binom{11011100}{00100000}, \binom{11101000}{00010000}, \binom{11010010}{00110010}, \binom{11010100}{00110000};$
$s = 10$: $\binom{00101111}{00011011}, \binom{11100010}{00011000}, \binom{00111000}{00000010}, \binom{11100100}{00000001}, \binom{01000011}{00010010}, \binom{11011111}{00101110}, \binom{01000001}{00101001}, \binom{01000010}{00110111}, \binom{11001101}{00100001},$
$\binom{01011101}{00100010}, \binom{11001000}{00010001}, \binom{01011110}{00111100}, \binom{11010111}{00110001}, \binom{01101011}{00000100}, \binom{01101001}{00011111}, \binom{11000011}{00111011}, \binom{01110010}{00001111}, \binom{11000001}{00001000}, \binom{10001111}{00010111},$
$\binom{10110100}{01001000}, \binom{10000110}{00100100}, \binom{10001000}{00110010}, \binom{10100101}{01000100}, \binom{10010001}{00111111}, \binom{10111111}{01010111}, \binom{10000010}{01001101}, \binom{10010110}{01010100}, \binom{10110011}{01100001}, \binom{10000100}{01101110},$
$\binom{10000101}{01110001}, \binom{10011011}{01100100}, \binom{10011100}{01110011}, \binom{10110110}{01111000}, \binom{10101000}{01110100}, \binom{10100011}{01100111};$
$s = 11$: $\binom{00010011}{00001001}, \binom{11110101}{00001011}, \binom{00100110}{00001101}, \binom{00110110}{00001000}, \binom{11100011}{00000100}, \binom{01001001}{00000101}, \binom{01000100}{00011111}, \binom{11011010}{00100010}, \binom{01000011}{00110011},$
$\binom{11000000}{00101110}, \binom{01010100}{00110001}, \binom{11010010}{00111101}, \binom{01010111}{00110111}, \binom{11011100}{00110000}, \binom{01100011}{00001111}, \binom{01100000}{00010111}, \binom{11001100}{00110000}, \binom{01111100}{00000010}, \binom{11000110}{00000001},$

$\binom{10000111}{00001010}, \binom{10000100}{00010001}, \binom{10111101}{01000010}, \binom{10001000}{00100001}, \binom{10010100}{00100101}, \binom{10101111}{00011101}, \binom{10011101}{00101000}, \binom{10101101}{00010100}, \binom{10101011}{01011001}, \binom{10100001}{01011000},$
$\binom{10000001}{01011010}, \binom{10011010}{01001111}, \binom{10001011}{01011111}, \binom{10010010}{01001010}, \binom{10000110}{01100110}, \binom{10010110}{01110110}, \binom{10110011}{01110000}, \binom{10010001}{01111000}, \binom{10111111}{01111011}, \binom{10101000}{01100010},$
$\binom{10101110}{01101110};$

$s = 14$: $\binom{01010000}{00100010}, \binom{01100001}{00010001}, \binom{01010011}{00100000}, \binom{01100010}{00010000}, \binom{01010001}{00110000}, \binom{01010011}{00110011}, \binom{10010110}{00001001}, \binom{11111000}{00000111}, \binom{10010101}{00100110},$
$\binom{11100010}{00011111}, \binom{10010001}{00111000}, \binom{11010111}{00100100}, \binom{10100100}{00001100}, \binom{11010100}{00001010}, \binom{10100010}{00011110}, \binom{11010001}{00111011}, \binom{10111011}{00000101}, \binom{11100110}{00001101}, \binom{10000001}{01011101},$
$\binom{10111010}{01111001}, \binom{10001010}{01010101}, \binom{10111000}{01111101}, \binom{10001001}{01011110}, \binom{10111100}{01111011}, \binom{10001110}{01011011}, \binom{10111101}{01110110}, \binom{10010011}{01000101}, \binom{10111111}{01001101}, \binom{10010100}{01000111},$
$\binom{10110010}{01001100}, \binom{10010000}{01001000}, \binom{10110100}{01000100}, \binom{10011011}{01001011}, \binom{10110000}{01000110}, \binom{10010111}{01011000}, \binom{10001000}{01110100}, \binom{10010101}{01011111}, \binom{10000101}{01111000}, \binom{10011110}{01010111},$
$\binom{10000111}{01111100}, \binom{10011101}{01011100}, \binom{10000011}{01111010}, \binom{10000010}{01100111}, \binom{10000110}{01101111}, \binom{10001100}{01100110}, \binom{10001101}{01101001}, \binom{10100111}{01100101}, \binom{10101001}{01100100}, \binom{10100110}{01101010},$
$\binom{10101010}{01101110}, \binom{10100001}{01001001}, \binom{10100000}{01001110}, \binom{10101011}{01001010}, \binom{10101111}{01001111};$

$s = 23$: $\binom{00100001}{00010010}, \binom{00100000}{00010101}, \binom{00100101}{00010001}, \binom{00100110}{00010000}, \binom{00100011}{00011001}, \binom{00100010}{00011010}, \binom{00100100}{00011010}, \binom{00100111}{00011011}, \binom{01000001}{00010000},$
$\binom{01000111}{00010011}, \binom{01000100}{00010100}, \binom{01000101}{00010110}, \binom{01000011}{00011001}, \binom{01000000}{00011101}, \binom{01000000}{00011110}, \binom{01001001}{00011011}, \binom{01001000}{00011101}, \binom{01010110}{00001101}, \binom{01001010}{00111000},$
$\binom{01001100}{00111001}, \binom{01011001}{00110000}, \binom{01011000}{00110010}, \binom{01100110}{00011110}, \binom{10000010}{00001110}, \binom{10000100}{00001001}, \binom{10000101}{00001010}, \binom{10000110}{00001100}, \binom{10000111}{00001111}, \binom{10000001}{00011100},$
$\binom{10001011}{00011100}, \binom{10010100}{00001000}, \binom{10010000}{00110001}, \binom{10100100}{00011111}, \binom{10101101}{00010111}, \binom{10101001}{00011111}, \binom{10110111}{00001011}, \binom{10001001}{01000010}, \binom{10010011}{01000110}, \binom{10011000}{01001000},$
$\binom{10000011}{01101011}, \binom{10000000}{01101101}, \binom{10001110}{01110001}, \binom{10011010}{01100000}, \binom{10011011}{01100101}, \binom{10011111}{01100100}, \binom{10010000}{01110110}, \binom{10010010}{01110111}, \binom{10010110}{01110010}, \binom{10010001}{01111111},$
$\binom{10010101}{01111010}, \binom{10011000}{01110001}, \binom{10011001}{01110011}, \binom{10011110}{01111001}, \binom{10100010}{01001110}, \binom{10100110}{01001101}, \binom{10101011}{01001011}, \binom{10101111}{01001111}, \binom{10110101}{01010111}, \binom{10111111}{01011100},$
$\binom{10111110}{01011111}, \binom{10100000}{01100011}, \binom{10100111}{01100001}, \binom{10100011}{01101111}, \binom{10101000}{01101100}, \binom{10101110}{01101110}, \binom{10100101}{01110011}, \binom{10101010}{01111110}, \binom{10101100}{01111101}, \binom{10110001}{01100010},$
$\binom{10110010}{01100000}, \binom{10110000}{01100111}, \binom{10110000}{01110101}, \binom{10110011}{01111011}, \binom{10110100}{01111100}, \binom{10111001}{01110100}, \binom{10111000}{01110110}, \binom{10111000}{01110111}, \binom{10111101}{01110100}, \binom{11000010}{00111011},$
$\binom{11000001}{00111101}, \binom{11000111}{00111111}, \binom{11001010}{00110111}, \binom{11011011}{00101010}, \binom{11011001}{00101101}, \binom{11011010}{00101111}, \binom{11011100}{00101011}, \binom{11011111}{00101001}, \binom{11011101}{00101110}, \binom{11011110}{00101100};$

$s = 24$: $\binom{00010011}{00001100}, \binom{00101000}{00010100}, \binom{00101001}{00010100}, \binom{00101001}{00011110}, \binom{00101111}{00011010}, \binom{00110011}{00001101}, \binom{00111000}{00000110}, \binom{01001000}{00000100}, \binom{01000010}{00011000},$
$\binom{01000000}{00011101}, \binom{01010101}{00001001}, \binom{01000011}{00101010}, \binom{01001001}{00110001}, \binom{01010011}{00100111}, \binom{01011011}{00101100}, \binom{01011110}{00101111}, \binom{01010010}{00110101}, \binom{01010100}{00110110}, \binom{01010010}{00111111},$
$\binom{01101010}{00000101}, \binom{01100000}{00010000}, \binom{01100110}{00011001}, \binom{01100110}{00011100}, \binom{01101011}{00010010}, \binom{10000110}{00011011}, \binom{10001011}{00011011}, \binom{10010010}{00001000}, \binom{10011110}{00000001}, \binom{10000011}{00101101},$
$\binom{10001110}{00100100}, \binom{10001101}{00111111}, \binom{10010101}{00100010}, \binom{10010110}{00100110}, \binom{10010011}{00101011}, \binom{10011000}{00100000}, \binom{10011100}{00111001}, \binom{10100100}{00000011}, \binom{10101110}{00011101}, \binom{10110001}{00000101},$
$\binom{10110110}{00001100}, \binom{10111011}{00000111}, \binom{10000010}{01000011}, \binom{10001001}{01000010}, \binom{10001010}{01000111}, \binom{10001000}{01001010}, \binom{10001100}{01001010}, \binom{10001111}{01010110}, \binom{10001100}{01011101}, \binom{10010110}{01000100},$
$\binom{10011110}{01001000}, \binom{10011101}{01001101}, \binom{10011001}{01010000}, \binom{10011100}{01011000}, \binom{10001111}{01101000}, \binom{10000001}{01111011}, \binom{10000000}{01111110}, \binom{10000111}{01111010}, \binom{10000101}{01111100}, \binom{10010111}{01100101},$
$\binom{10010100}{01111000}, \binom{10100001}{01001110}, \binom{10101010}{01000001}, \binom{10101101}{01001100}, \binom{10100010}{01010001}, \binom{10101000}{01011001}, \binom{10110101}{01001011}, \binom{10111011}{01011111}, \binom{10100100}{01101110}, \binom{10100110}{01101100},$
$\binom{10101100}{01101001}, \binom{10101111}{01101111}, \binom{10100000}{01110011}, \binom{10101011}{01110000}, \binom{10101001}{01110101}, \binom{10111101}{01100100}, \binom{10111001}{01111110}, \binom{11010011}{00001011}, \binom{11010111}{00001010}, \binom{11011000}{00000010},$
$\binom{11000011}{00100011}, \binom{11001000}{00100001}, \binom{11001111}{00100000}, \binom{11001100}{00101110}, \binom{11001110}{00110001}, \binom{11010101}{00111011}, \binom{11011011}{00110000}, \binom{11100011}{00001110}, \binom{11100001}{00010111}, \binom{11101000}{00010101},$
$\binom{11101010}{00010110}, \binom{11110000}{00000111}, \binom{11110101}{00001101}, \binom{11110100}{00001111};$

$s = 27$: $\binom{00110101}{00000011}, \binom{01000011}{00011010}, \binom{01001010}{00010000}, \binom{01001000}{00011100}, \binom{01011011}{00000100}, \binom{01000001}{00101001}, \binom{01000110}{00110010}, \binom{01001011}{00111101}, \binom{01001100}{00111111},$
$\binom{01001101}{00111110}, \binom{01010110}{00101111}, \binom{01011101}{00100000}, \binom{01010110}{00110100}, \binom{01100000}{00001111}, \binom{01100101}{00010010}, \binom{01100111}{00011011}, \binom{01101011}{00010100}, \binom{01101101}{00011111}, \binom{01101110}{00011110},$
$\binom{10000100}{00011001}, \binom{10001000}{00101100}, \binom{10000001}{00111000}, \binom{10001101}{00110011}, \binom{10001010}{00111100}, \binom{10010100}{00100100}, \binom{10010101}{00100100}, \binom{10010101}{00100110}, \binom{10010111}{00101000}, \binom{10011111}{00100011},$
$\binom{10010001}{00111111}, \binom{10010011}{00111110}, \binom{10010100}{00111000}, \binom{10011000}{00110000}, \binom{10011010}{00110010}, \binom{10100000}{00000001}, \binom{10100110}{00011101}, \binom{10101010}{00011000}, \binom{10110101}{00000010}, \binom{10110011}{00001111},$
$\binom{10110101}{00001000}, \binom{10111000}{00000101}, \binom{10001110}{01000010}, \binom{10001101}{01001110}, \binom{10000101}{01010001}, \binom{10000000}{01011111}, \binom{10001111}{01011110}, \binom{10010011}{01001111}, \binom{10011011}{01000101}, \binom{10011100}{01000100},$
$\binom{10010010}{01011100}, \binom{10011000}{01010111}, \binom{10011111}{01011101}, \binom{10000111}{01101001}, \binom{10000110}{01110100}, \binom{10001001}{01110110}, \binom{10001011}{01111101}, \binom{10010000}{01100001}, \binom{10010010}{01100110}, \binom{10011001}{01100011},$
$\binom{10010110}{01111011}, \binom{10011001}{01110000}, \binom{10100100}{01010101}, \binom{10100110}{01011011}, \binom{10101000}{01010000}, \binom{10101011}{01011000}, \binom{10110010}{01000111}, \binom{10111000}{01000000}, \binom{10111001}{01001001}, \binom{10110001}{01010001},$
$\binom{10110110}{01010101}, \binom{10110000}{01011000}, \binom{10111110}{01010010}, \binom{10101110}{01100100}, \binom{10100011}{01110001}, \binom{10100101}{01110101}, \binom{10100011}{01111110}, \binom{10100101}{01111010}, \binom{10101001}{01111010}, \binom{10111011}{01101100},$
$\binom{10111111}{01101010}, \binom{10110111}{01111110}, \binom{11000100}{00010111}, \binom{11001000}{00010001}, \binom{11001101}{00010110}, \binom{11000000}{00100010}, \binom{11000110}{00101101}, \binom{11000111}{00110011}, \binom{11000001}{00111010}, \binom{11000101}{00111011},$
$\binom{11001011}{00110100}, \binom{11010111}{00100001}, \binom{11011101}{00100111}, \binom{11011001}{00101110}, \binom{11010001}{00110101}, \binom{11010100}{00110001}, \binom{11010110}{00110111}, \binom{11100110}{00001001}, \binom{11100111}{00010101}, \binom{11101010}{00010011},$
$\binom{10000011}{00111001}, \binom{10011110}{00101010}, \binom{10101111}{01010011}, \binom{10100010}{01111000}, \binom{10000011}{00111001}, \binom{10011110}{00101010}, \binom{10101111}{01010011}, \binom{10100010}{01111000};$

$s = 45$: $\binom{01001011}{00010010}, \binom{01001111}{00010001}, \binom{01001001}{00010101}, \binom{01001110}{00010100}, \binom{01001010}{00010111}, \binom{01001100}{00010011}, \binom{01001101}{00010110}, \binom{01010001}{00001010}, \binom{01010110}{00001001},$
$\binom{01010010}{00001100}, \binom{01010011}{00001110}, \binom{01010111}{00001011}, \binom{01010101}{00001111}, \binom{01010100}{00001101}, \binom{01000011}{00101001}, \binom{01000111}{00101110}, \binom{01000001}{00101100}, \binom{01000110}{00101010}, \binom{01000010}{00101101},$
$\binom{01000101}{00101011}, \binom{01000100}{00101111}, \binom{01100001}{00010011}, \binom{01100110}{00010111}, \binom{01100011}{00010101}, \binom{01100111}{00010100}, \binom{01100010}{00010110}, \binom{01100100}{00010001}, \binom{01100101}{00010010}, \binom{10000011}{00001010},$
$\binom{10000111}{00001001}, \binom{10000001}{00001101}, \binom{10000110}{00001100}, \binom{10000010}{00001111}, \binom{10000100}{00001011}, \binom{10000101}{00001110}, \binom{10010001}{00000010}, \binom{10010110}{00000001}, \binom{10010001}{00000101}, \binom{10010010}{00000100},$
$\binom{10010010}{00000111}, \binom{10010011}{00000110}, \binom{10010100}{00000011}, \binom{10011001}{00100010}, \binom{10011110}{00100001}, \binom{10011001}{00100011}, \binom{10011110}{00100111}, \binom{10011010}{00100100}, \binom{10011011}{00100101}, \binom{10011111}{00100100},$
$\binom{10011010}{00100110}, \binom{10011011}{00100110}, \binom{10011111}{00100011}, \binom{10011100}{00100001}, \binom{10011101}{00100010}, \binom{10011101}{00100111}, \binom{10011100}{00100101}, \binom{10011001}{00111010}, \binom{10011110}{00111001}, \binom{10011010}{00111100},$
$\binom{10011011}{00111110}, \binom{10011111}{00111011}, \binom{10011101}{00111111}, \binom{10011100}{00111101}, \binom{10100001}{00011010}, \binom{10100110}{00011001}, \binom{10100010}{00011100}, \binom{10100011}{00011110}, \binom{10100111}{00011011}, \binom{10100101}{00011111},$
$\binom{10100100}{00011101}, \binom{10001010}{01001011}, \binom{10001001}{01001111}, \binom{10001110}{01001101}, \binom{10001011}{01001100}, \binom{10001111}{01001010}, \binom{10001101}{01001001}, \binom{10001100}{01001110}, \binom{10010011}{01000001}, \binom{10010111}{01000110},$
$\binom{10010001}{01000100}, \binom{10010110}{01000010}, \binom{10010010}{01000101}, \binom{10010101}{01000011}, \binom{10010100}{01000111}, \binom{10000001}{01101010}, \binom{10000110}{01101001}, \binom{10000010}{01101100}, \binom{10000011}{01101110}, \binom{10000111}{01101011},$
$\binom{10000101}{01101111}, \binom{10000100}{01101101}, \binom{10000001}{01111010}, \binom{10000110}{01111001}, \binom{10000010}{01111100}, \binom{10000011}{01111110}, \binom{10000111}{01111011}, \binom{10000101}{01111111}, \binom{10000100}{01111101}, \binom{10101001}{01100010},$
$\binom{10101110}{01100001}, \binom{10101001}{01100011}, \binom{10101110}{01100111}, \binom{10101010}{01100100}, \binom{10101011}{01100101}, \binom{10101111}{01100100}, \binom{10101010}{01100110}, \binom{10101011}{01100110}, \binom{10101111}{01100011}, \binom{10101100}{01100001},$
$\binom{10101101}{01100010}, \binom{10101101}{01100111}, \binom{10101100}{01100101}, \binom{10110010}{01110001}, \binom{10110001}{01110110}, \binom{10110110}{01110011}, \binom{10110011}{01110111}, \binom{10110111}{01110101}, \binom{10110100}{01110010}, \binom{10110101}{01110100},$
$\binom{10110001}{01111011}, \binom{10110110}{01111111}, \binom{10110011}{01111101}, \binom{10110111}{01111100}, \binom{10110010}{01111110}, \binom{10110100}{01111001}, \binom{10110101}{01111010}, \binom{11000001}{00110010}, \binom{11000110}{00110001}, \binom{11000010}{00110100},$
$\binom{11000011}{00110110}, \binom{11000111}{00110011}, \binom{11000101}{00110111}, \binom{11000100}{00110101}, \binom{11010001}{00101010}, \binom{11010110}{00101001}, \binom{11010010}{00101100}, \binom{11010011}{00101110}, \binom{11010111}{00101011}, \binom{11010101}{00101111},$
$\binom{11010100}{00101101}, \binom{11010010}{00110001}, \binom{11010001}{00110110}, \binom{11010110}{00110011}, \binom{11010011}{00110111}, \binom{11010111}{00110101}, \binom{11010100}{00110010}, \binom{11010101}{00110100}, \binom{11011001}{00110010}, \binom{11011110}{00110001},$
$\binom{11011010}{00110100}, \binom{11011011}{00110110}, \binom{11011111}{00110011}, \binom{11011101}{00110111}, \binom{11011100}{00110101}, \binom{11011001}{00111010}, \binom{11011110}{00111001}, \binom{11011010}{00111100}, \binom{11011011}{00111110}, \binom{11011111}{00111011},$
$\binom{11011101}{00111111}, \binom{11011100}{00111101}, \binom{11101001}{00011010}, \binom{11101110}{00011001}, \binom{11101010}{00011100}, \binom{11101011}{00011110}, \binom{11101111}{00011011}, \binom{11101101}{00011111}, \binom{11101100}{00011101}, \binom{01000000}{00100000},$
$\binom{10000000}{00001000}, \binom{10000000}{00110000}, \binom{10011000}{00111000}, \binom{10010000}{01001000}, \binom{10010000}{01100000}, \binom{10011000}{01110000}, \binom{10111000}{01011000}, \binom{11001000}{00011000}, \binom{11000000}{00101000}, \binom{11001000}{00110000};$

$s = 49$: $\binom{00010001}{00001010}, \binom{00010110}{00001001}, \binom{00010010}{00001100}, \binom{00010011}{00001110}, \binom{00010111}{00001011}, \binom{00010101}{00001111}, \binom{00010100}{00001101}, \binom{00100010}{00001011}, \binom{00100001}{00001111},$
$\binom{00100110}{00001101}, \binom{00100011}{00001100}, \binom{00100111}{00001010}, \binom{00100101}{00001001}, \binom{00100100}{00001110}, \binom{00100010}{00011011}, \binom{00100001}{00011111}, \binom{00100110}{00011101}, \binom{00100011}{00011100}, \binom{00100111}{00011010},$

$\binom{00100101}{00011001}, \binom{00100100}{00011110}, \binom{01000011}{00100001}, \binom{01000111}{00100110}, \binom{01000001}{00100100}, \binom{01000110}{00100010}, \binom{01000010}{00100101}, \binom{01000101}{00100011}, \binom{01000100}{00100111}, \binom{01010001}{00101010},$
$\binom{01010110}{00101001}, \binom{01010010}{00101100}, \binom{01010011}{00101110}, \binom{01010111}{00101011}, \binom{01010101}{00101111}, \binom{01010100}{00101101}, \binom{01011011}{00110010}, \binom{01011111}{00110001}, \binom{01011001}{00110101}, \binom{01011110}{00110100},$
$\binom{01011010}{00110111}, \binom{01011100}{00110011}, \binom{01011101}{00110110}, \binom{10000001}{00111011}, \binom{10000110}{00111111}, \binom{10000011}{00111101}, \binom{10000111}{00111100}, \binom{10000010}{00111110}, \binom{10000100}{00111001}, \binom{10000101}{00111010},$
$\binom{10010001}{00101010}, \binom{10010110}{00101001}, \binom{10010010}{00101100}, \binom{10010011}{00101110}, \binom{10010111}{00101011}, \binom{10010101}{00101111}, \binom{10010100}{00101101}, \binom{10001001}{01000010}, \binom{10001110}{01000001}, \binom{10001010}{01000100},$
$\binom{10001011}{01000110}, \binom{10001111}{01000011}, \binom{10001101}{01000111}, \binom{10001100}{01000101}, \binom{10010011}{01001001}, \binom{10010111}{01001110}, \binom{10010001}{01001100}, \binom{10010110}{01001010}, \binom{10010010}{01001101}, \binom{10010101}{01001011},$
$\binom{10010100}{01001111}, \binom{10011011}{01001001}, \binom{10011111}{01001110}, \binom{10011001}{01001100}, \binom{10011110}{01001010}, \binom{10011010}{01001101}, \binom{10011101}{01001011}, \binom{10011100}{01001111}, \binom{10011001}{01011010}, \binom{10011110}{01011001},$
$\binom{10011010}{01011100}, \binom{10011011}{01011110}, \binom{10011111}{01011011}, \binom{10011101}{01011111}, \binom{10011100}{01011101}, \binom{10000001}{01101010}, \binom{10000110}{01101001}, \binom{10000010}{01101100}, \binom{10000011}{01101110}, \binom{10000111}{01101011},$
$\binom{10000101}{01101111}, \binom{10000100}{01101101}, \binom{10001001}{01100010}, \binom{10001110}{01100001}, \binom{10001010}{01100100}, \binom{10001011}{01100110}, \binom{10001111}{01100011}, \binom{10001101}{01100111}, \binom{10001100}{01100101}, \binom{10001010}{01110001},$
$\binom{10001001}{01110110}, \binom{10001110}{01110011}, \binom{10001011}{01110111}, \binom{10001111}{01110101}, \binom{10001100}{01110010}, \binom{10001101}{01110100}, \binom{10011001}{01100010}, \binom{10011110}{01100001}, \binom{10011010}{01100100}, \binom{10011011}{01100110},$
$\binom{10011111}{01100011}, \binom{10011101}{01100111}, \binom{10011100}{01100101}, \binom{10110001}{01010010}, \binom{10110110}{01010001}, \binom{10110010}{01010100}, \binom{10110011}{01010110}, \binom{10110111}{01010011}, \binom{10110101}{01010111}, \binom{10110100}{01010101},$
$\binom{10101001}{01101011}, \binom{10101110}{01101111}, \binom{10101011}{01101101}, \binom{10101111}{01101100}, \binom{10101010}{01101110}, \binom{10101100}{01101001}, \binom{10101101}{01101010}, \binom{10100001}{01110010}, \binom{10100110}{01110001}, \binom{10100010}{01110100},$
$\binom{10100011}{01110110}, \binom{10100111}{01110011}, \binom{10100101}{01110111}, \binom{10100100}{01110101}, \binom{10100001}{01111010}, \binom{10100110}{01111001}, \binom{10100010}{01111100}, \binom{10100011}{01111110}, \binom{10100111}{01111011}, \binom{10100101}{01111111},$
$\binom{10100100}{01111101}, \binom{10101001}{01110010}, \binom{10101110}{01110001}, \binom{10101010}{01110100}, \binom{10101011}{01110110}, \binom{10101111}{01110011}, \binom{10101101}{01110111}, \binom{10101100}{01110101}, \binom{10110001}{01111011}, \binom{10110110}{01111111},$
$\binom{10110011}{01111101}, \binom{10110111}{01111100}, \binom{10110010}{01111110}, \binom{10110100}{01111001}, \binom{10110101}{01111010}, \binom{11000001}{00110010}, \binom{11000110}{00110001}, \binom{11000010}{00110100}, \binom{11000011}{00110110}, \binom{11000111}{00110011},$
$\binom{11000101}{00110111}, \binom{11000100}{00110101}, \binom{11001001}{00111010}, \binom{11001110}{00111001}, \binom{11001010}{00111100}, \binom{11001011}{00111110}, \binom{11001111}{00111011}, \binom{11001101}{00111111}, \binom{11001100}{00111101}, \binom{11010010}{00110001},$
$\binom{11010001}{00110110}, \binom{11010110}{00110011}, \binom{11010011}{00110111}, \binom{11010111}{00110101}, \binom{11010100}{00110010}, \binom{11010101}{00110100}, \binom{01001000}{00100000}, \binom{01001000}{00101000}, \binom{01000000}{00110000}, \binom{01000000}{00111000},$
$\binom{01110000}{00001000}, \binom{10000000}{00011000}, \binom{10001000}{00101000}, \binom{10010000}{00100000}, \binom{10101000}{00011000}, \binom{10001000}{01010000}, \binom{10011000}{01011000}, \binom{10001000}{01100000}, \binom{10000000}{01110000}, \binom{10111000}{01101000},$
$\binom{10111000}{01111000}, \binom{11001000}{00010000}, \binom{11001000}{00100000}, \binom{11010000}{00101000}, \binom{11010000}{00110000}, \binom{11011000}{00111000};$
$s = 50$: $\binom{00100010}{00011011}, \binom{00100001}{00011111}, \binom{00100110}{00011101}, \binom{00100011}{00011100}, \binom{00100111}{00011010}, \binom{00100101}{00011001}, \binom{00100100}{00011110}, \binom{00101010}{00010001}, \binom{00101001}{00010110},$
$\binom{00101110}{00010011}, \binom{00101011}{00010111}, \binom{00101111}{00010101}, \binom{00101100}{00010010}, \binom{00101101}{00010100}, \binom{01000010}{00001001}, \binom{01000001}{00001110}, \binom{01000110}{00001011}, \binom{01000011}{00001111}, \binom{01000111}{00001101},$
$\binom{01000100}{00001010}, \binom{01000101}{00001100}, \binom{01001010}{00111001}, \binom{01001001}{00111110}, \binom{01001110}{00111011}, \binom{01001011}{00111111}, \binom{01001111}{00111101}, \binom{01001100}{00111010}, \binom{01001101}{00111100}, \binom{01010001}{00101010},$
$\binom{01010110}{00101001}, \binom{01010010}{00101100}, \binom{01010011}{00101110}, \binom{01010111}{00101011}, \binom{01010101}{00101111}, \binom{01010100}{00101101}, \binom{01100001}{00001010}, \binom{01100110}{00001001}, \binom{01100010}{00001100}, \binom{01100011}{00001110},$
$\binom{01100111}{00001011}, \binom{01100101}{00001111}, \binom{01100100}{00001101}, \binom{10001001}{00011010}, \binom{10001110}{00011001}, \binom{10001010}{00011100}, \binom{10001011}{00011110}, \binom{10001111}{00011011}, \binom{10001101}{00011111}, \binom{10001100}{00011101},$
$\binom{10010001}{00001010}, \binom{10010110}{00001001}, \binom{10010010}{00001100}, \binom{10010011}{00001110}, \binom{10010111}{00001011}, \binom{10010101}{00001111}, \binom{10010100}{00001101}, \binom{10011001}{00101010}, \binom{10011110}{00101001}, \binom{10011010}{00101100},$
$\binom{10011011}{00101110}, \binom{10011111}{00101011}, \binom{10011101}{00101111}, \binom{10011100}{00101101}, \binom{10101001}{00010010}, \binom{10101110}{00010001}, \binom{10101010}{00010100}, \binom{10101011}{00010110}, \binom{10101111}{00010011}, \binom{10101101}{00010111},$
$\binom{10101100}{00010101}, \binom{10000010}{01000011}, \binom{10000001}{01000111}, \binom{10000110}{01000101}, \binom{10000011}{01000100}, \binom{10000111}{01000010}, \binom{10000101}{01000001}, \binom{10000100}{01000110}, \binom{10001010}{01001011}, \binom{10001001}{01001111},$
$\binom{10001110}{01001101}, \binom{10001011}{01001100}, \binom{10001111}{01001010}, \binom{10001101}{01001001}, \binom{10001100}{01001110}, \binom{10000010}{01011001}, \binom{10000001}{01011110}, \binom{10000110}{01011011}, \binom{10000011}{01011111}, \binom{10000111}{01011101},$
$\binom{10000100}{01011010}, \binom{10000101}{01011100}, \binom{10010001}{01101010}, \binom{10010110}{01101001}, \binom{10010010}{01101100}, \binom{10010011}{01101110}, \binom{10010111}{01101011}, \binom{10010101}{01101111}, \binom{10010100}{01101101}, \binom{10100001}{01011010},$
$\binom{10100110}{01011001}, \binom{10100010}{01011100}, \binom{10100011}{01011110}, \binom{10100111}{01011011}, \binom{10100101}{01011111}, \binom{10100100}{01011101}, \binom{10110001}{01010010}, \binom{10110110}{01010001}, \binom{10110010}{01010100}, \binom{10110011}{01010110},$
$\binom{10110111}{01010011}, \binom{10110101}{01010111}, \binom{10110100}{01010101}, \binom{10111001}{01011010}, \binom{10111110}{01011001}, \binom{10111010}{01011100}, \binom{10111011}{01011110}, \binom{10111111}{01011011}, \binom{10111101}{01011111}, \binom{10111100}{01011101},$
$\binom{10100001}{01101010}, \binom{10100110}{01101001}, \binom{10100010}{01101100}, \binom{10100011}{01101110}, \binom{10100111}{01101011}, \binom{10100101}{01101111}, \binom{10100100}{01101101}, \binom{10100001}{01111010}, \binom{10100110}{01111001}, \binom{10100010}{01111100},$
$\binom{10100011}{01111110}, \binom{10100111}{01111011}, \binom{10100101}{01111111}, \binom{10100100}{01111101}, \binom{10101001}{01111010}, \binom{10101110}{01111001}, \binom{10101010}{01111100}, \binom{10101011}{01111110}, \binom{10101111}{01111011}, \binom{10101101}{01111111},$
$\binom{10101100}{01111101}, \binom{10110001}{01100010}, \binom{10110110}{01100001}, \binom{10110010}{01100100}, \binom{10110011}{01100110}, \binom{10110111}{01100011}, \binom{10110101}{01100111}, \binom{10110100}{01100101}, \binom{11001010}{00100001}, \binom{11001001}{00100110},$
$\binom{11001110}{00100011}, \binom{11001011}{00100111}, \binom{11001111}{00100101}, \binom{11001100}{00100010}, \binom{11001101}{00100100}, \binom{11000001}{00110010}, \binom{11000110}{00110001}, \binom{11000010}{00110100}, \binom{11000011}{00110110}, \binom{11000111}{00110011},$
$\binom{11000101}{00110111}, \binom{11000100}{00110101}, \binom{11010010}{00110001}, \binom{11010001}{00110110}, \binom{11010110}{00110011}, \binom{11010011}{00110111}, \binom{11010111}{00110101}, \binom{11010100}{00110010}, \binom{11010101}{00110100}, \binom{11101001}{00011010},$
$\binom{11101110}{00011001}, \binom{11101010}{00011100}, \binom{11101011}{00011110}, \binom{11101111}{00011011}, \binom{11101101}{00011111}, \binom{11101100}{00011101}, \binom{00010000}{00001000}, \binom{00110000}{00001000}, \binom{01001000}{00100000}, \binom{01001000}{00110000},$
$\binom{01011000}{00111000}, \binom{01101000}{00010000}, \binom{10000000}{00011000}, \binom{10010000}{00101000}, \binom{10011000}{00101000}, \binom{10100000}{00011000}, \binom{10000000}{01011000}, \binom{10001000}{01010000}, \binom{10001000}{01011000}, \binom{10010000}{01110000},$
$\binom{10010000}{01111000}, \binom{10100000}{01000000}, \binom{10100000}{01001000}, \binom{10101000}{01010000}, \binom{10110000}{01000000}, \binom{10101000}{01101000}, \binom{10110000}{01100000}, \binom{11000000}{00001000}, \binom{11000000}{00100000}, \binom{11001000}{00111000},$
$\binom{11010000}{00101000}.$

For $s \in \{15, 28, 29\}$ the best known lower bound is still given by quaternary linear codes. With this, all remaining constructions can be obtained using Theorem 3.8 and Lemma 2.9. □

In Table 2 we state the known bounds for $n_4(s)$ when $s \leq 60$. Lower bounds based on quaternary linear codes are stated in columns headed with "L". Upper bounds, based on [3] for $s \leq 4$ and on binary linear codes for $s > 4$, i.e.. the weak coding upper bound, are stated in columns headed with "U". Values of improved constructions are given in columns headed with "I". Open cases are marked in bold font and we remark that we have $n_4(s) = n_4(s-21)+85$ for $n > 60$. For $s > 60$ there are improvements over the linear case iff s is congruent to 2, 3, 7, or 8 modulo 21. Generator matrices of the improvements are given in the proof of Theorem 4.2. We observe that $n_4(44) \geq n_4(23) + n_4(21)$ is attained with equality.

4.1. **Existence and non-existence of binary linear codes related to quaternary additive codes.** In this subsection we want to summarize the current knowledge on the putative binary linear codes corresponding to the open cases in Table 2.

Lemma 4.3. *If $\mathcal{S}$ is a $(36, 8, 10)$ system, then $C := \mathcal{X}^{-1}(\mathcal{P}(\mathcal{S}))$ is a 2-divisible $[108, 8, 52]_2$ code with maximum weight at most 72. Moreover, the maximum point multiplicity of $\mathcal{P}(\mathcal{S})$ is at most 2.*

Proof. The statements for C follow from Lemma 2.3. Since no $[104, 7, 52]_2$ code exists the maximum column multiplicity of a generator matrix of C is at most 3, so that $\mathcal{P}(\mathcal{S})$ has a maximum point multiplicity of at most 3. Now assume that C is a $[108, 8, 52]_2$ code with maximum column multiplicity 3 and $\mathcal{P}$ a corresponding multiset of points. Projection through a point P_3 with multiplicity 3 gives a multiset of points $\mathcal{P}'$ corresponding to a $[105, 7, 52]$ code C'. Since no $[103, 6, 52]$ code exists the maximum point multiplicity of $\mathcal{P}'$ has to be equal to one. However, in $\mathcal{P}$ the point

s	L	I	U	s	L	I	U	s	L	I	U
1	–		–	21	85		85	41	165		165
2	–		–	22	86		86	42	170		170
3	5		5	23	87	89	89	43	171		171
4	10		10	**24**	**92**	**93**	**94**	44	172	174	174
5	17		17	25	97		97	45	177	179	179
6	18		18	26	102		102	46	182		182
7	23		23	**27**	103	107	107	47	187		187
8	28		28	**28**	**108**		**110**	48	192		192
9	31	33	33	**29**	**113**		**115**	49	193	195	195
10	**34**	**35**	**36**	30	118		118	50	198	200	200
11	39	40	40	31	123		123	51	203		203
12	44		44	32	128		128	52	208		208
13	49		49	33	129		129	53	213		213
14	50	54	54	34	134		134	54	214		214
15	**55**		**57**	35	139		139	55	219		219
16	64		64	36	144		144	56	224		224
17	65		65	37	149		149	57	229		229
18	70		70	38	150		150	58	234		234
19	75		75	39	155		155	59	235		235
20	80		80	40	160		160	60	240		240

TABLE 2. Bounds for $n_4(s)$.

P_3 has to be contained in three lines so that projection through P_3 yields a point of multiplicity at least two in $\mathcal{P}'$ – contradiction. □

With respect to a point of multiplicity 2 we remark that there exist at least 23 even $[106, 7, 52]$ codes with maximum weight at most 72 and maximum column multiplicity 2. Several of these can be partitioned into 34 lines and two double points. Trying to produce a configuration of 36 lines in $\mathrm{PG}(7, 2)$ failed so far.

Lemma 4.4. *If $\mathcal{S}$ is a $(57, 8, 15)$ system, then $C := \mathcal{X}^{-1}(\mathcal{P}(\mathcal{S}))$ is a 2-divisible $[171, 8, 84]_2$ code with maximum weight at most 112 and no weight in $\{86, 94, 98, 102\}$. Moreover, the maximum point multiplicity of $\mathcal{P}(\mathcal{S})$ is at most 2.*

Proof. From Lemma 2.3 we conclude that C is a 2-divisible $[171, 8, 84]_2$ code with maximum weight at most 114. From Lemma 2.7 and the non-existence of the corresponding residual codes we conclude that C does not have a codeword with a weight in $\{86, 94, 98, 102\}$. Since no $[168, 7, 84]_2$ code exists the maximum column multiplicity of a generator matrix of C is at most 2, so that $\mathcal{P}(\mathcal{S})$ has a maximum point multiplicity of at most 2. □

If C is a 2-divisible $[171, 8, 84]_2$ code with maximum weight at most 112 that is not 4-divisible then we can use the MacWilliams Identities [17] and [6, Proposition 5]) to deduce that C has many codewords of weight 90. However, 5-dimensional subcodes with 20 codewords of weight 90 do indeed exist.

Lemma 4.5. *If $\mathcal{S}$ is a $(94, 8, 24)$ system, then $C := \mathcal{X}^{-1}(\mathcal{P}(\mathcal{S}))$ is a 4-divisible $[282, 8, 140]_2$ code with maximum weight at most 188. Moreover, the maximum point multiplicity of $\mathcal{P}(\mathcal{S})$ equals 2 and C does not contain a codeword with a weight 148.*

Proof. From Lemma 2.3 we conclude that C is a 2-divisible $[282, 8, 188]_2$ code with maximum weight at most 188. Lemma 2.8 yields the 4-divisibility of C. Since no $[134, 7, 66]_2$ code exists, Lemma 2.7 implies that C does not contain a codeword of weight 148. Since no $[279, 7, 140]_2$ code exists the maximum column multiplicity of a generator matrix of C is at most 2, so that $\mathcal{P}(\mathcal{S})$ has a maximum point multiplicity of at most 2. Since $\mathrm{PG}(7, 2)$ contains only $255 < 282$ points the maximum column multiplicity of $\mathcal{P}(\mathcal{S})$ indeed equals 2. □

Lemma 4.6. *If $\mathcal{S}$ is a $(110, 8, 28)$ system, then $C := \mathcal{X}^{-1}(\mathcal{P}(\mathcal{S}))$ is a 4-divisible $[330, 8, 164]_2$ code with maximum weight at most 220. Moreover, the maximum point multiplicity of $\mathcal{P}(\mathcal{S})$ equals 2.*

Proof. From Lemma 2.3 we conclude that C is a 2-divisible $[330, 8, 164]_2$ code with maximum weight at most 220. Lemma 2.8 yields the 4-divisibility of C. Since no $[327, 7, 164]_2$ code exists the maximum column multiplicity of a generator matrix of C is at most 2, so that $\mathcal{P}(\mathcal{S})$ has a maximum point multiplicity of at most 2. Since $\mathrm{PG}(7, 2)$ contains only $255 < 282$ points the maximum column multiplicity of $\mathcal{P}(\mathcal{S})$ indeed equals 2. □

Lemma 4.7. *If $\mathcal{S}$ is a $(115, 8, 29)$ system, then $C := \mathcal{X}^{-1}(\mathcal{P}(\mathcal{S}))$ is a 4-divisible $[345, 8, 172]_2$ code with maximum weight at most 228. Moreover, the maximum point multiplicity of $\mathcal{P}(\mathcal{S})$ equals 2 and C does not contain a codeword with a weight in $\{180, 196, 204, 208, 212\}$.*

Proof. From Lemma 2.3 we conclude that C is a 2-divisible $[345, 8, 182]_2$ code with maximum weight at most 230. Lemma 2.8 yields the 4-divisibility of C. From Lemma 2.7 and the non-existence of the corresponding residual codes we conclude that C does not have a codeword with a weight in $\{180, 196, 204, 208, 212\}$. Since no $[342, 7, 182]_2$ code exists the maximum column multiplicity of a generator matrix of C is at most 2, so that $\mathcal{P}(\mathcal{S})$ has a maximum point multiplicity of at most 2. Since $\operatorname{PG}(7, 2)$ contains only $255 < 282$ points the maximum column multiplicity of $\mathcal{P}(\mathcal{S})$ indeed equals 2. □

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