



Affine vector space partitions

John Bamberg¹ · Yuval Filmus² · Ferdinand Ihringer³ · Sascha Kurz⁴ 

Received: 29 November 2022 / Revised: 25 May 2023 / Accepted: 5 June 2023
© The Author(s) 2023

Abstract

An affine vector space partition of $AG(n, q)$ is a set of proper affine subspaces that partitions the set of points. Here we determine minimum sizes and enumerate equivalence classes of affine vector space partitions for small parameters. We also give parametric constructions for arbitrary field sizes.

Keywords Finite geometry · Vector space partitions · Spreads · Klein quadric · Fano plane · Hitting formulas

Mathematics Subject Classification 51E23 · 05B40

1 Introduction

A *vector space partition* \mathcal{P} of the projective space $PG(n - 1, q)$ is a set of subspaces in $PG(n - 1, q)$ which partitions the set of points. For a survey on known results we refer to [12]. We say that a vector space partition \mathcal{P} has type $(n - 1)^{m_{n-1}} \dots 2^{m_2} 1^{m_1}$ if precisely m_i of its elements have dimension i , where $1 \leq i \leq n$. The classification of the possible types of a vector space partition, given the parameters n and q , is an important and difficult

This is one of several papers published in *Designs, Codes and Cryptography* comprising the “Special Issue: Finite Geometries 2022”.

✉ Sascha Kurz
sascha.kurz@uni-bayreuth.de

John Bamberg
john.bamberg@uwa.edu.au

Yuval Filmus
yuvalfi@cs.technion.ac.il

Ferdinand Ihringer
Ferdinand.Ihringer@gmail.com

¹ The University of Western Australia, Crawley, Australia

² The Henry and Marilyn Taub Faculty of Computer Science and Faculty of Mathematics, Technion — Israel Institute of Technology, Haifa, Israel

³ Ghent University, Gent, Belgium

⁴ Universität Bayreuth, Bayreuth, Germany

problem. Based on [10], the classification for the binary case $q = 2$ was completed for $n \leq 7$ in [7]. Under the assumption $m_1 = 0$ the case $(q, n) = (2, 8)$ has been treated in [6]. It seems quite natural to define a *vector space partition* \mathcal{A} of the affine space $\text{AG}(n, q)$ as a set of subspaces in $\text{AG}(n, q)$ that partitions the set of points. However, it turns out that those partitions exist for all types which satisfy a very natural numerical condition. If we impose the additional condition of *tightness*, that is that the projective closures of the elements of \mathcal{A} have an empty intersection, then the classification problem becomes interesting and challenging. This condition is natural in the context of hitting formulas as introduced in [17], that is for logical formulas in full disjunctive normal form (DNF) such that each truth assignment to the underlying variables satisfies precisely one term. For a more recent treatment and applications we refer to [26]. Here we consider the geometrical and the combinatorial point of view.

Variants of vector space partitions of $\text{PG}(n-1, q)$ have been studied in the literature. In [8] the authors study (multi-)sets of subspaces covering each point exactly λ times. The problem of covering each k -space exactly once is considered in [14]. A more general partition problem for groups is studied in [10]. Irreducible homogeneous affine vector space partitions have been studied by Agievich [1] and Tarannikov [28] motivated by the study of bent functions. However, we are not aware of any publication treating the introduced affine vector space partitions in the same generality as in the present work.

The paper is organized as follows. In Sect. 2 we formally introduce affine vector space partitions, state the preliminaries, and develop the first necessary existence conditions. Here we are guided by the published necessary conditions for vector space partitions. We also argue why tightness (see above) and *irreducibility*, that is there exists no proper subset $\mathcal{A}' \subsetneq \mathcal{A}$ such that the union of all elements of \mathcal{A}' is a subspace of $\text{AG}(n, q)$, are necessary to obtain an interesting existence question. In Sect. 3 we classify affine vector space partitions for arbitrary field sizes but small dimensions. Sect. 4 is concerned with the binary case. We completely determine the possible dimension distributions of tight irreducible affine vector space partitions of $\text{PG}(n-1, 2)$ for all $n \leq 7$. In a few cases we give theoretical or computational classifications of the corresponding equivalence classes of tight irreducible vector space partitions. A very nice example consists of eight solids in $\text{PG}(6, 2)$ whose parts at infinity live on the Klein quadric $Q^+(5, 2)$. A generalization to arbitrary finite fields of characteristic 2 is given in Sect. 5.2. Parametric constructions of tight irreducible affine vector space partitions using spreads or hitting formulas complete Sect. 5. In Sect. 6 we determine the smallest possible size of an irreducible tight affine vector space partition of $\text{PG}(7, 2)$ and give a parametric upper bound for $\text{PG}(n-1, 2)$ of size roughly $\frac{3n}{2}$, which is significantly smaller than the conjectured smallest size of an irreducible hitting formula mentioning all variables. We close with a conclusion and a list of open problems in Sect. 7. To keep the paper self-contained we present some additional material in an appendix. Section A contains details on integer linear programming formulations that we have utilized to obtain some computational results. Section B contains a few technical results that might be left to the reader or collected from the literature. Lists of hitting formulas that can be used to construct tight irreducible affine vector space partitions of the minimum possible size are given in Sect. C.

2 Preliminaries and necessary conditions

Definition 1 An *affine vector space partition* \mathcal{A} of $\text{AG}(n, q)$ is a set $\{A_1, \dots, A_r\}$ of subspaces of $\text{AG}(n, q)$ such that $0 \leq \dim(A_i) \leq n-1$ for all $1 \leq i \leq r$ and every point (element of \mathbb{F}_q^n)

is contained in exactly one element A_i . The integer r is called the *size* of the affine vector space partition.

We write $\#\mathcal{A}$ for the size of \mathcal{A} . For each affine subspace $A \in \text{AG}(n, q)$ we write \overline{A} for its projective closure. With this $\overline{\mathcal{A}} := \{\overline{A} : A \in \mathcal{A}\}$ is the natural embedding of an affine vector space partition of $\text{AG}(n, q)$ in $\text{PG}(n, q)$. Denoting the hyperplane at infinity by H_∞ , we can directly define an affine vector space partition in $\text{PG}(n, q)$:

Definition 2 An *affine vector space partition* \mathcal{U} of $\text{PG}(n - 1, q)$ is a set $\{U_1, \dots, U_r\}$ of subspaces of $\text{PG}(n - 1, q)$ such that $1 \leq \dim(U_i) \leq n - 1$ for all $1 \leq i \leq r$ and there exists a hyperplane H_∞ such that every point (1-dimensional subspace) outside of H_∞ is contained in exactly one element U_i and $U_i \not\subseteq H_\infty$ for all $1 \leq i \leq r$. The integer r is called the *size* of the affine vector space partition and also denoted by $\#\mathcal{U}$.

Here we use the algebraic dimension for subspaces in $\text{PG}(n - 1, q)$, i.e., if $\dim(U) = u$, then $\#U = \begin{bmatrix} u \\ 1 \end{bmatrix}_q := \frac{q^u - 1}{q - 1}$ and we also speak of u -spaces. Using the geometric language, we call 1-, 2-, 3-, 4-, and $n - 1$ -spaces points, lines, planes, solids, and hyperplanes, respectively. For each $1 \leq i \leq r$ the set $U_i \setminus H_\infty$ is an affine space containing $q^{\dim(U_i) - 1}$ points.

In the following we will mostly speak of an affine vector space partition, abbreviated as avsp, and will consider its embedding in $\text{PG}(n - 1, q)$. The *type* of an avsp $\mathcal{U} = \{U_1, \dots, U_r\}$ is given by $(n - 1)^{m_{n-1}} \dots 2^{m_2} 1^{m_1}$, where $m_i = \#\{U_j : 1 \leq j \leq r, \dim(U_j) = i\}$. Counting points outside of H_∞ gives

$$\sum_{i=1}^{n-1} m_i \cdot q^{i-1} = q^{n-1}. \tag{1}$$

The analog of Eq. (1) for vector space partitions of $\text{PG}(n - 1, q)$ is called the *packing condition*. While the packing condition for vector space partitions of $\text{PG}(n - 1, q)$ is just a necessary but not a sufficient condition for the existence with a given type, for avsp Equation (1) is both necessary and sufficient.

Lemma 1 For each type $(n - 1)^{m_{n-1}} \dots 2^{m_2} 1^{m_1}$ that satisfies the packing condition (1) there exists an avsp \mathcal{U} of $\text{PG}(n - 1, q)$ attaining that type.

Proof Consider a subspace K of H_∞ with $\dim(K) = n - 2$. By H_1, \dots, H_q we denote the q hyperplanes containing K that are not equal to H_∞ . Clearly, we have $0 \leq m_{n-1} \leq q$ and we can choose $H_1, \dots, H_{m_{n-1}}$ as the first elements of \mathcal{U} . The remaining elements are constructed recursively. For each index $m_{n-1} + 1 \leq j \leq q$ we consider an avsp of type $(n - 2)^{m_{n-2}^{(j)}} \dots 2^{m_2^{(j)}} 1^{m_1^{(j)}}$ where the $m_i^{(j)} \in \mathbb{N}_0$ are chosen such that the packing condition is satisfied for H_j and

$$\sum_{j=m_{n-1}+1}^q m_i^{(j)} = m_i \tag{2}$$

for all $1 \leq i \leq n - 2$. Such a decomposition can be easily constructed, see the algorithm in Sect. B. □

Definition 3 We call an avsp $\mathcal{U} = \{U_1, \dots, U_r\}$ *reducible* if there exists a subspace U and a subset $S \subsetneq \{1, \dots, r\}$ such that $\dim(U) < n$, $\#S > 1$ and $\{U_i : i \in S\}$ is an avsp of U . Otherwise \mathcal{U} is called *irreducible*.

Lemma 2 *The smallest size of an irreducible avsp \mathcal{U} of $\text{PG}(n - 1, q)$ is given by $\#\mathcal{U} = q$.*

Proof Let \mathcal{U} be an avsp of $\text{PG}(n - 1, q)$. Since there are q^{n-1} points to cover and each subspace covers at most q^{n-2} points, we have $\#\mathcal{U} \geq q$. Now consider a hyperplane K of H_∞ . By H_1, \dots, H_q we denote the q hyperplanes containing K and not being equal to H_∞ . With this, $\{H_1, \dots, H_q\}$ is an irreducible avsp of $\text{PG}(n - 1, q)$. \square

For a vector space partition \mathcal{P} of $\text{PG}(n - 1, q)$ we have $\dim(A) + \dim(B) \leq n$ for each pair $\{A, B\}$ of different elements of \mathcal{P} , which is also called *dimension condition*. Using this it can be easily shown that $\#\mathcal{P} \geq \frac{q^n - 1}{q^{n/2} - 1} = q^{n/2} + 1$ if n is even and $\#\mathcal{P} \geq q^{(n+1)/2} + 1$ if n is odd. Both bounds can be attained by spreads, i.e., vector space partitions of type $(n/2)q^{n/2+1}$, and lifted MRD codes of maximum possible rank distance, i.e., vector space partitions of type $((n + 1)/2)^1((n - 1)/2)^{q^{(n+1)/2}}$, respectively. In [25] the authors determine the minimum size $\sigma_q(n, t)$ of a vector space partition of $\text{PG}(n, q)$ whose largest subspace has dimension t .

Lemma 3 *Let \mathcal{U} be an irreducible avsp of $\text{PG}(n - 1, q)$ and $U_1, \dots, U_q \in \mathcal{U}$ be q different elements with $\dim(U_1) = \dots = \dim(U_q)$ and $\dim(\langle U_1, \dots, U_q \rangle) = \dim(U_1) + 1$. Then we have $\dim(U_1) = \dots = \dim(U_q) = n - 1$.*

Proof Let $U := \langle U_1, \dots, U_q \rangle$ and $u := \dim(U_1)$. Since $U \setminus H_\infty$ contains q^u points and $U_i \setminus H_\infty$ contains q^{u-1} points for each $1 \leq i \leq q$, the set

$$\mathcal{U} \setminus \{U_1, \dots, U_q\} \cup \{U\}$$

is an avsp unless $\dim(U) = u + 1 = n$. \square

Corollary 1 *Let \mathcal{U} be an irreducible avsp of $\text{PG}(n - 1, 2)$ and $U_1, U_2 \in \mathcal{U}$ be two different elements with $\dim(U_1) = \dim(U_2) = \dim(U_1 \cap U_2) + 1$. Then, we have $\dim(U_1) = \dim(U_2) = n - 1$.*

As an analog of the dimension condition for vector space partitions in $\text{PG}(n - 1, q)$ we have:

Lemma 4 *Let \mathcal{U} be an avsp in \mathcal{U} . For each $U, U' \in \mathcal{U}$ we have*

$$\dim(U \cap U') = \dim(U \cap U' \cap H_\infty) \geq \dim(U) + \dim(U') - n. \tag{3}$$

Proof Since $U \setminus H_\infty, U' \setminus H_\infty$ are disjoint and $U, U' \not\subseteq H_\infty$ we have $\dim(U \cap U') = \dim(U \cap U' \cap H_\infty)$. The inequality follows from

$$\dim(U_1 \cap U_2) + \dim(\langle U_1, U_2 \rangle) = \dim(U_1) + \dim(U_2)$$

and $\dim(\langle U_1, U_2 \rangle) \leq n$. \square

Due to the following general construction for (irreducible) avsp's we introduce a further condition.

Lemma 5 *Let $\mathcal{U} = \{U_1, \dots, U_r\}$ be an avsp of $\text{PG}(n - 1, q) =: V$ and P be a point outside of V (embedded in $\text{PG}(n, q)$). Then, $\mathcal{U}' := \{\langle U_1, P \rangle, \dots, \langle U_r, P \rangle\}$ is an avsp of $\langle V, P \rangle \cong \text{PG}((n + 1) - 1, q)$, where the “new” hyperplane at infinity arises via $\langle H_\infty, P \rangle$ from the “old” hyperplane at infinity H_∞ . Reducibility inherits, i.e. \mathcal{U}' is irreducible iff \mathcal{U} is irreducible.*

Definition 4 Let $\mathcal{U} = \{U_1, \dots, U_r\}$ be an avsp of $\text{PG}(n - 1, q)$. We call \mathcal{U} *tight* iff the intersection of all U_i does not contain a point, i.e. the intersection is trivial.

The same definition was proposed by Agievich [1], under the name *primitivity*, and was dubbed *A-primitivity* by Tarannikov [28]. We remark that an avsp \mathcal{A} of $\text{AG}(n, q)$ is tight iff for any $x \in \mathbb{F}_q^n$, there exists an $A \in \mathcal{A}$ such that A is not invariant under addition of x , that is $A + x \neq A$.

Lemma 6 *For each integer $n \geq 2$ there exists a tight avsp of $\text{PG}(n - 1, q)$ with size $(q - 1) \cdot (n - 1) + 1$.*

Proof Apply the following recursive construction. Start with an $(n - 2)$ -dimensional subspace K of H_∞ and consider the q hyperplanes H_1, \dots, H_q containing K but not being equal to H_∞ . Choose $q - 1$ out of these and continue the iteration with the remaining hyperplane until it becomes 2-dimensional, i.e. a line. In the final step replace the affine line by q points, so that the resulting avsp is trivially tight. \square

A classical result in computer science, attributed to Tarsi, states that a minimally unsatisfiable Boolean formula in conjunctive normal form (CNF) with m clauses mentions at most $m - 1$ variables, see e.g. [5, Theorem 8]. The proof can be slightly modified to show that for $n \geq 2$ each tight avsp of $\text{PG}(n - 1, 2)$ has size at least n . One might conjecture that Lemma 6 is tight. Some preliminary results in that direction are proven in [9, Sec. 3.2]. The determination of the minimum size of an irreducible tight avsp is quite a challenge and we will present our preliminary results in Sects. 3 and 4.

Note that tightness and irreducibility can be checked efficiently. In particular, for irreducibility it suffices to calculate the affine closure for all pairs of subspaces in the avsp. We will show efficiency formally and provide detailed algorithms in future work on hitting formulas.

Lemma 7 *Let U, K , and H_∞ be subspaces in $\text{PG}(n - 1, q)$ with $K \leq H_\infty$, $\dim(K) = n - 2$, $\dim(H_\infty) = n - 1$, and $\dim(U \cap H_\infty) = \dim(U) - 1$, i.e. $U \not\leq H_\infty$. By H_1, \dots, H_q we denote the q hyperplanes containing K but not being equal to H_∞ . Then the following statements are equivalent:*

- (1) $U \cap H_\infty \leq K$;
- (2) *there exists an index $1 \leq i \leq q$ with $U \leq H_i$;*
- (3) *there exists an index $1 \leq i \leq q$ with $U \leq H_i$ and $U \cap H_j = U \cap H_\infty = U \cap K$ for all $1 \leq j \leq q$ with $j \neq i$;*
- (4) $\dim(U \cap K) = \dim(U) - 1$.

Lemma 8 *Let U, K , and H_∞ be subspaces in $\text{PG}(n - 1, q)$ with $K \leq H_\infty$, $\dim(K) = n - 2$, $\dim(H_\infty) = n - 1$, and $\dim(U \cap H_\infty) = \dim(U) - 1$, i.e. $U \not\leq H_\infty$. By H_1, \dots, H_q we denote the q hyperplanes containing K but not being equal to H_∞ . Then the following statements are equivalent:*

- (1) $U \cap H_\infty \not\leq K$;
- (2) $\dim(U \cap H_i) = \dim(U) - 1$ for all $1 \leq i \leq q$;
- (3) *there are q $(\dim(U) - 1)$ -spaces in U containing $U \cap K$ and not being contained in H_∞ ;*
- (4) $\dim(U \cap K) = \dim(U) - 2$.

Assume that \mathcal{P} is a vector space partition of $\text{PG}(n - 1, q)$ with type $k_1^{m_1} \dots k_l^{m_l}$, where $k_1 > \dots > k_l$ and $m_i > 0$ for all $1 \leq i \leq l$. The so-called *tail* \mathcal{T} of \mathcal{P} is the set of all k_l -spaces in \mathcal{P} , i.e., the set of all elements with the smallest occurring dimension. In [11] several conditions on $\#\mathcal{T}$ have been obtained. In our situation we can also consider the *tail*

$\mathcal{T} := \{U \in \mathcal{U} : \dim(U) = k_l\}$ of an avsp of $\text{PG}(n - 1, q)$ with type $k_1^{m_1} \dots k_l^{m_l}$, where $k_1 > \dots > k_l$ and $m_i > 0$ for all $1 \leq i \leq l$. The packing condition (1) directly implies that $q^{k_l-1-k_l}$ divides $\#\mathcal{T} = m_l$ if $l \geq 2$ and that q^{n-k_l} divides $\#\mathcal{T} = m_l$ if $l = 1$. In [21] the results on the tail of a vector space partition of $\text{PG}(n - 1, q)$ were refined using the notion of Δ -divisible sets of k -spaces.

Definition 5 A (multi-)set \mathcal{S} of k -spaces in $\text{PG}(n - 1, q)$ is called Δ -divisible iff $\#\mathcal{S} \equiv \#(H \cap \mathcal{S}) \pmod{\Delta}$ for every hyperplane H , where $H \cap \mathcal{S}$ denotes the (multi-)set of elements of \mathcal{S} that are contained in H .

We use the notation $\{\{\star\}\}$ for a multiset \mathcal{S} and $\#\mathcal{S}$ for its cardinality.

Lemma 9 Let \mathcal{U} be an avsp of $\text{PG}(n - 1, q)$ of type $k_1^{m_1} \dots k_l^{m_l}$, where $k_1 > \dots > k_l > 1$ and $m_i > 0$ for all $1 \leq i \leq l$. Let $\mathcal{T} := \{U \in \mathcal{U} : \dim(U) = k_l\}$ be the tail of \mathcal{U} and $\mathcal{T}' := \{T \cap H_\infty : T \in \mathcal{T}\}$. If $l \geq 2$, then \mathcal{T}' is $q^{k_l-1-k_l}$ -divisible and $\#\mathcal{T} = \#\mathcal{T}' \equiv 0 \pmod{q^{k_l-1-k_l}}$. If $l = 1$, then \mathcal{T}' is q -divisible and $\#\mathcal{T} = \#\mathcal{T}' \equiv 0 \pmod{q^{n-k_l}}$.

Proof Clearly we have $\#\mathcal{T} = \#\mathcal{T}'$. From the packing condition (1) we directly conclude $\#\mathcal{T} \equiv 0 \pmod{q^{k_l-1-k_l}}$ if $l \geq 2$ and $\#\mathcal{T} = q^{n-k_l} \equiv 0 \pmod{q^{n-k_l}}$ if $l = 1$. Let K be an arbitrary hyperplane of H_∞ and H_1, \dots, H_q be the q hyperplanes of $\text{PG}(n - 1, q)$ not being equal to H_∞ . Call the points outside of H_∞ that are contained in some element of \mathcal{U} with dimension strictly larger than k_l covered and all others outside of H_∞ uncovered. Since each k -space covers either q^{k-1}, q^{k-2} , or 0 points of $H_i \setminus H_\infty$, the number of uncovered points in $H_i \setminus H_\infty$ is divisible by q^{k_l-1-2} if $l \geq 2$ and by q^{k_l-1} if $l = 1$, where $1 \leq i \leq q$ is arbitrary. Let a be the number of k_l -spaces in \mathcal{U} that are completely contained in H_i , so that the number of uncovered points in H_i equals

$$x := a \cdot q^{k_l-1} + (\#\mathcal{T} - a) \cdot q^{k_l-2}.$$

If $l \geq 2$ we have $x \equiv 0 \pmod{q^{k_l-1-2}}$ and $\#\mathcal{T} \equiv 0 \pmod{q^{k_l-1-k_l}}$, so that $(q - 1)a \equiv 0 \pmod{q^{k_l-1-k_l}}$ and $a \equiv 0 \pmod{q^{k_l-1-k_l}}$. If $l = 1$ we have $x \equiv 0 \pmod{q^{k_l-1}}$ and $\#\mathcal{T} \equiv 0 \pmod{q}$, so that $(q - 1)a \equiv 0 \pmod{q}$ and $a \equiv 0 \pmod{q}$. \square

Δ -divisible (multi-)sets \mathcal{S} of k -spaces in $\text{PG}(n - 1, q)$ have been studied in [21]. If we replace each k -space by its $\frac{q^k-1}{q-1}$ points we obtain a Δq^{k-1} -divisible multiset of $\#\mathcal{S} \cdot \frac{q^k-1}{q-1}$ points in $\text{PG}(n - 1, q)$. The possible cardinalities, given the divisibility constant and the field size, have been completely characterized in [18, Theorem 1]. Here we will use only a few results on the possible structure of the tail (or more precisely of \mathcal{T}') which allow more direct proofs.

Lemma 10 Let \mathcal{U} be an avsp of $\text{PG}(n - 1, 2)$ with tail \mathcal{T} . If $\#\mathcal{T} = 2$, then either \mathcal{U} is reducible or we have $\mathcal{U} = \mathcal{T}$ and $\dim(U) = n - 1$ for all $U \in \mathcal{U}$.

Proof Denote the dimension of the elements of \mathcal{T} by k . Lemma 9 yields that $\mathcal{T}' := \{T \cap H_\infty : T \in \mathcal{T}\}$ is a 2-divisible multiset of $(k - 1)$ -spaces. So, each hyperplane of H_∞ contains either all 2 or zero elements from \mathcal{T}' , so that \mathcal{T}' is a q -fold $(k - 1)$ -space. With this, the stated results follows from Lemma 3 applied to \mathcal{T} . \square

Corollary 2 Let \mathcal{U} be an irreducible avsp of $\text{PG}(n - 1, q)$ of type $k_1^{m_1} \dots k_l^{m_l}$, where $k_1 > \dots > k_l$ and $k_i > 0$ for all $1 \leq i \leq l$. If $m_l = q$, then we have $l = 1$ and $k_1 = n - 1$.

2.1 The structure of the tail for small parameters

If $\#\mathcal{T}$ is small and $q = 2$, then we can also characterize the tail using Lemma 9. To this end, let \mathcal{S} denote a set of k -spaces in $\text{PG}(n - 1, q)$. The corresponding spectrum $(a_i)_{i \in \mathbb{N}_0}$ is given by the numbers a_i of hyperplanes that contain exactly i elements from \mathcal{S} , so that

$$\sum_{i=0}^{\#\mathcal{S}} a_i = \frac{q^n - 1}{q - 1}. \tag{4}$$

The condition that \mathcal{S} is spanning, i.e. $\langle \mathcal{S} : \mathcal{S} \in \mathcal{S} \rangle = \text{PG}(n - 1, q)$, is equivalent to $a_{\#\mathcal{S}} = 0$. Double-counting the k -spaces gives

$$\sum_{i=0}^{\#\mathcal{S}} i a_i = \#\mathcal{S} \cdot \frac{q^{n-k} - 1}{q - 1}. \tag{5}$$

Lemma 11 *Let \mathcal{S} be a 2-divisible set of four k -spaces in $\text{PG}(n - 1, 2)$. Then there exists a $(k - 1)$ -space B , a plane E , and a line $L \leq E$ with $\dim(\langle E, B \rangle) = k + 2$, such that $\mathcal{S} = \{ \langle P, B \rangle : P \in E \setminus L \}$.*

Proof Assume that P is a point that is contained in at least one but not all elements from \mathcal{S} . Let x denote the number of elements of \mathcal{S} that contain P . Since all hyperplanes contain an even number of elements from \mathcal{S} we have $x \neq 3$. Assume $x = 2$ for a moment and let $S, S' \in \mathcal{S}$ be the two elements not containing P . There are 2^{n-k-1} hyperplanes that contain S but do not contain P , so that all of those hyperplanes contain S and S' . The intersection of these hyperplanes has dimension at most k and contains S as well as S' , so that $S = S'$, which is a contradiction. Thus, each point P in $\text{PG}(n - 1, 2)$ is contained in 0, 1 or 4 elements of \mathcal{S} .

By $(a_i)_{i \in \mathbb{N}_0}$ we denote the spectrum of \mathcal{S} . W.l.o.g. we assume that \mathcal{S} is spanning, i.e., we have $a_4 = 0$. From the equations (4) and (5) we conclude

$$a_0 = 2^n - 2^{n-k+1} + 1 \quad \text{and} \quad a_2 = 2^{n-k+1} - 2.$$

If there is no point P that is contained in all four elements of \mathcal{S} , then the elements of \mathcal{S} are pairwise disjoint and double-counting pairs yields

$$\binom{2}{2} a_2 = \binom{4}{2} \cdot (2^{n-2k} - 1), \tag{6}$$

so that

$$2^{n-k+1} - 2 = 6 \cdot (2^{n-2k} - 1) \iff 2^{n-k} - 3 \cdot 2^{n-2k} + 2 = 0,$$

which has the unique solution $n = 3, k = 1$.

So, by recursively quotienting out points P that are contained in all elements of \mathcal{S} we conclude the existence of a $(k - 1)$ -space B that is contained in all four elements of \mathcal{S} . Quotienting out B yields a spanning 2-divisible set of points in $\text{PG}(2, 2)$ with $a_0 = 1$ and $a_2 = 6$. Choosing E as the ambient space and L as the empty hyperplane yields the stated characterization since in $\text{PG}(2, 2)$ there are exactly four points outside a hyperplane. \square

If $k = 1$, i.e., the k -spaces are points, the Eqs. (4)–(5) and the generalization $\sum_{i=0}^{\#\mathcal{S}} \binom{i}{2} a_i = \binom{\#\mathcal{S}}{2} \cdot \frac{q^{n-2}-1}{q-1}$ of (6) are also known as “standard equations” or the first three MacWilliams equations for the corresponding (projective) linear code.

We remark that Lemma 11 is based on the fact that each 2-divisible set of 4 points is an affine plane. For $q > 2$ there are further possibilities for q -divisible sets of q^2 points over \mathbb{F}_q , see [4, 19] on the so-called cylinder conjecture. We apply Lemma 11 e.g. in Proposition 6 and Lemma 18.

3 Classification of avsp's in $\text{PG}(n - 1, q)$ for small parameters

By definition, there is no avsp in $\text{PG}(1 - 1, q)$. In $\text{PG}(2 - 1, q)$ there is a unique avsp. It has type 1^q and is irreducible and tight.

Lemma 12 *Let \mathcal{U} be an avsp of $\text{PG}(n - 1, q)$, where $n \geq 3$. If there exist pairwise different hyperplanes $U_1, \dots, U_l \in \mathcal{U}$, then there exists an $(n - 2)$ -space $K \leq H_\infty$ such that $K \leq U_i$ for all $1 \leq i \leq l$.*

Proof The statement is trivial for $l \leq 1$, so that we assume $l \geq 2$. Due to the dimensions we have $\dim(U_i \cap U_j) = n - 2$ for all $1 \leq i < j \leq l$. Since the sets of points $U_i \setminus H_\infty$ and $U_j \setminus H_\infty$ are disjoint we have $U_i \cap U_j \leq H_\infty$ and $U_i \cap U_j = U_i \cap H_\infty = U_j \cap H_\infty$. So, we set $K = U_1 \cap H_\infty$. \square

Proposition 1 *Let \mathcal{U} be an irreducible avsp of $\text{PG}(n - 1, q)$, where $n \geq 3$. If \mathcal{U} is of type $(n - 1)^{m_{n-1}} \dots 2^{m_2} 1^{m_1}$, then we have $m_{n-1} \leq q - 2$ or $m_{n-1} = q$. In the latter case \mathcal{U} is not tight.*

Proof We assume $m_{n-1} = q - 1 \geq 1$ and let $K \leq H_\infty$ as in Lemma 12. With this, let $H \neq H_\infty$ be the unique hyperplane with $K \leq H$ that is not contained as an element in \mathcal{U} and \mathcal{U}' arise from \mathcal{U} by removing the $q - 1$ $(n - 1)$ -dimensional elements. Thus, \mathcal{U}' is an avsp of H , i.e., \mathcal{U} is reducible.

If $m_{n-1} = q$, then the $(n - 2)$ -space $K \leq H_\infty$ (as in Lemma 12) is contained in all elements of \mathcal{U} , i.e., \mathcal{U} is not tight. \square

Corollary 3 *Let \mathcal{U} be an irreducible tight avsp of $\text{PG}(n - 1, 2)$ of type $(n - 1)^{m_{n-1}} \dots 1^{m_1}$, where $n \geq 3$. Then we have $m_{n-1} = 0$.*

Let $\mathcal{U} = \{U_1, \dots, U_r\}$ be an avsp of $\text{PG}(n - 1, q)$, $I \subseteq \{1, \dots, r\}$, and V be a proper subspace with $V \not\leq H_\infty$. If $\#I \geq 2$ and $\{U_i : i \in I\}$ is an avsp of V , then we say that the spaces U_i with $i \in I$ can be *joined* to V . Note that this is exactly the situation when \mathcal{U} is reducible. In $\text{PG}(n - 1, 2)$ any two points outside of H_∞ can be joined to a line, so that:

Lemma 13 *Let \mathcal{U} be an irreducible tight avsp of $\text{PG}(n - 1, 2)$ of type $(n - 1)^{m_{n-1}} \dots 1^{m_1}$, where $n \geq 3$. Then, we have $m_1 = 0$.*

Theorem 1 *Let \mathcal{U} be an avsp of $\text{PG}(3 - 1, q)$ with type $2^{m_2} 1^{m_1}$. Then, we have $0 \leq m_2 \leq q$, $m_1 = q \cdot (q - m_2)$, all lines in \mathcal{U} contain a common point $P \leq H_\infty$, and the 1-dimensional elements can be grouped into pairwise disjoint sets of size q that can be joined to a line each.*

Proof The parameterization of m_2, m_1 follows from the packing condition (1). If $m_2 > 0$, the existence of P follows from Lemma 12. If $m_2 = 0$ then choose an arbitrary point $P \leq H_\infty$. By L_1, \dots, L_q we denote the q lines containing P that are not equal to H_∞ . For each line L_i that is not an element of \mathcal{U} there exist q points in \mathcal{U} that can be joined to L_i . (Note that $L_i \cap L_j = P$ for all $1 \leq i < j \leq q$.) \square

We remark that all possibilities for $0 \leq m_2 \leq q$ can indeed be attained. In general there exist several non-isomorphic examples.

- Corollary 4** (1) *Let \mathcal{U} be an irreducible avsp of $\text{PG}(3 - 1, q)$. Then \mathcal{U} is of type 2^q and non-tight.*
 (2) *Let \mathcal{U} be an avsp of $\text{PG}(3 - 1, q)$ with type $2^{m_2}1^{m_1}$. Then, \mathcal{U} is tight iff $m_1 \geq 1$. In that case \mathcal{U} is reducible.*
 (3) *No irreducible tight avsp of $\text{PG}(3 - 1, q)$ exists.*

Let \mathcal{U} be an avsp of $\text{PG}(n - 1, q)$, where $n \geq 3$, and $K \leq H_\infty$ be an arbitrary $(n - 2)$ -space. We say that $\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(q)}$ is a K -decomposition of \mathcal{U} if the q hyperplanes containing K and not being equal to H_∞ can be labeled as H_1, \dots, H_q such that $\mathcal{U} = \cup_{i=1}^q \mathcal{U}^{(i)}$ and

$$\mathcal{U}^{(i)} = \{U \in \mathcal{U} : U \cap H_i \not\leq H_\infty\} \tag{7}$$

for all $1 \leq i \leq q$. Note that $\mathcal{U}^{(i)}$ is an avsp of H_i for each $1 \leq i \leq q$ (including the case $\mathcal{U}^{(i)} = \{H_i\}$). Moreover, any labeling of the q hyperplanes H_i induces a K -decomposition. Observe that for a fixed $(n - 2)$ -space $K \leq H_\infty$ each pair of K -decompositions arises just by relabeling, so that we also speak of *the* K -decomposition of \mathcal{U} since the actual labeling will not matter in our context.

Proposition 2 *Let \mathcal{U} be an avsp of $\text{PG}(n - 1, q)$, where $n \geq 3$, with type $(n - 1)^{m_{n-1}} \dots 2^{m_2}1^{m_1}$. If $1 \leq m_{n-1} \leq q$, then there exists an $(n - 2)$ -space $K \leq H_\infty$ such that the K -decomposition $\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(q)}$ partitions \mathcal{U} , i.e.,*

$$\bigcup_{1 \leq i \leq q} \mathcal{U}^{(i)} = \mathcal{U}.$$

Moreover, if $m_{n-1} \leq q - 1$, then \mathcal{U} is reducible. (More precisely, for each index $1 \leq i \leq q$ with $\#\mathcal{U}^{(i)} > 1$ the elements in $\mathcal{U}^{(i)}$ can be joined to H_i .)

Proof Choose some arbitrary $U \in \mathcal{U}$ with $\dim(U) = n - 1$, set $K := U \cap H_\infty$, and let $\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(q)}$ be the K -decomposition of \mathcal{U} and H_1, \dots, H_q be the corresponding hyperplanes. Due to Lemma 12 each $U \in \mathcal{U}$ with $\dim(U) = n - 1$ results in the same $(n - 2)$ -space K and the same K -decomposition $\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(q)}$ (up to relabeling). Especially we have that for each $U' \in \mathcal{U}$ with $\dim(U') = n - 1$ there exists an index $1 \leq i \leq q$ with $\mathcal{U}^{(i)} = \{U'\}$. W.l.o.g. we assume $\#\mathcal{U}^{(1)} = 1$.

Now consider an element $U \in \mathcal{U}$ with $\dim(U) < n - 1$. From Lemma 8 we conclude $U \cap H_\infty \leq K$ since otherwise $\#\mathcal{U}^{(1)} > 1$ (more precisely, U would split into q $(\dim(U) - 1)$ -spaces where one of these would be contained in $\mathcal{U}^{(1)}$ that also contains an entire hyperplane, which contradicts the packing condition (I)), which would contradict our assumption. Thus, for each $U \in \mathcal{U}$ there exists exactly one index $1 \leq i \leq q$ with $U \in \mathcal{U}^{(i)}$ and for each index $1 \leq j \leq q$ either $U \in H_j$ or $U \cap H_j \leq K \leq H_\infty$. □

Corollary 5 *Let \mathcal{U} be an avsp of $\text{PG}(n - 1, q)$ with type $(n - 1)^{m_{n-1}} \dots 2^{m_2}1^{m_1}$, where $n \geq 3$. If \mathcal{U} is irreducible, then we have $m_{n-1} \in \{0, q\}$.*

Affine vector space partitions of $\text{PG}(4 - 1, q)$ that contain at least one hyperplane as an element can be characterized easily.

Proposition 3 *Let \mathcal{U} be an avsp of $\text{PG}(4 - 1, q)$ of type $3^{m_3}2^{m_2}1^{m_1}$ with $m_3 \geq 1$. Then, we have $1 \leq m_3 \leq q$, $0 \leq m_2 \leq q \cdot (q - m_3)$, and $m_1 = q^3 - q^2m_3 - qm_2$. Moreover, there exists an $(n - 2)$ -space $K \leq H_\infty$ such that the K -decomposition $\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(q)}$ partitions \mathcal{U} , so that \mathcal{U} especially is reducible if $m_{n-1} \leq q - 1$.*

Proof The equation $m_1 = q^3 - q^2 m_3 - q m_2$ directly follows from the packing condition (1) and the ranges $0 \leq m_2 \leq q \cdot (q - m_3)$, $0 \leq m_3 \leq q$ follow from the non-negativity of m_1, m_2, m_3 . Note that we have $m_3 \geq 1$ by assumption. The remaining part follows from Proposition 2. \square

4 Classification of tight irreducible avsp's in PG($n - 1, 2$) for small dimensions n

The cases $n \leq 3$ have already been treated in Sect. 3, so that we assume $n \geq 4$ in the following. Our aim is to classify all possible types $(n - 1)^{m_{n-1}} \dots 1^{m_1}$ such that a tight irreducible avsp \mathcal{U} exists in PG($n - 1, 2$). We have $m_{n-1} = 0$ and $m_1 = 0$ due to Corollary 3 and Lemma 13. From Corollary 2 we conclude $m_l \neq 2$ for the smallest index $1 \leq l \leq n - 1$ with $m_l > 0$. The possible vectors $(m_{n-2}, \dots, m_2) \in \mathbb{N}_0^{n-3}$ are quite restricted by the packing condition (1). For $n = 4$ the only remaining possibility is type 2^4 . From Lemma 3 we conclude that the four lines are pairwise disjoint, i.e., they form a partial line spread of cardinality 4. It is well known that each partial line spread of cardinality q^2 in PG($3, q$) can be extended to a line spread, which has size $q^2 + 1$.¹ For $q = 2$ there is only the Desarguesian line spread and since it has a transitive automorphism group, there is only one equivalence class. The numbers of line spreads in PG($3, q$) are 1, 2, 3, 21, 1347 for $q = 2, 3, 4, 5, 7$.

In the three subsequent subsections we will consider tight irreducible avsp's in PG($n - 1, 2$) for $n \in \{5, 6, 7\}$. The possible types are completely determined in all cases, where realizations are computed using an integer linear programming (ILP) formulation, see Sect. A in the appendix for the details. If the sizes of the avsp's are not too large we were able to also compute all equivalence classes of avsp's using a slight modification of an algorithm from [23], see also [20, Algorithm 4.5]. A GAP implementation, based on the GAP package ‘‘FinInG’’ [2] for computations in finite incidence geometry, can be obtained from the authors upon request. In the theoretical parts we will also use classification for 2-divisible sets points that can e.g. be found in [13] or [22]. For the convenience of the reader we will also give a few selected proofs in Sect. B in the appendix.

4.1 Tight irreducible avsp's in PG(4, 2)

We may use Lemmas 9 and 11 to conclude that each avsp of PG($n - 1, 2$) of type $(n - 2)^4$ is non-tight if $n \geq 5$. However, we can further tighten the statement to:

Proposition 4 *Let $\mathcal{U} = \{U_1, \dots, U_r\}$ be an avsp of PG($n - 1, 2$), where $n \geq 4, r \geq 4$, and $\dim(U_i) = n - 2$ for all $1 \leq i \leq 3$. Then, the elements $\{U_4, \dots, U_r\}$ can be joined to an $(n - 2)$ -space B (including the case $r = 4$ and $U_4 = B$) and there exists an $(n - 4)$ -space C that is contained in all elements of $\{U_1, U_2, U_3, B\}$.*

Proof First we assume that two elements of $\{U_1, U_2, U_3\}$ can be joined to an $(n - 1)$ -space H . Without loss of generality, we assume that U_1 and U_2 can be joined to H . Let $K := H \cap H_\infty$, so that $\dim(K) = n - 2$. By H' we denote the unique hyperplane containing K that is not equal to H or H_∞ . Observe that $\{U_3, \dots, U_r\}$ is an avsp of H' and K is ‘‘the hyperplane at infinity’’ of H' . Next we set $K' := K \cap U_3$, so that $\dim(K') = n - 3$. Let B denote

¹ One argumentation is based on the fact that each q^k -divisible (multi-) set of $\frac{q^{k+1}-1}{q-1}$ points forms a $(k + 1)$ -space for each positive integer k , see e.g. [15].

the unique $(n - 2)$ -space in H' that contains K' and is not equal to U_3 or K . With this, $\{U_4, \dots, U_r\}$ is an avsp of B (including the case $r = 4, U_4 = B$). Note that the $(n - 3)$ -space K' is contained in all elements of $\{H, U_3, B\}$. Since $\{U_1, U_2\}$ forms an avsp of H and $\dim(U_1) = \dim(U_2) = n - 2$, there exists an $(n - 4)$ -space $C = K' \cap L$ that is contained in all elements of $\{U_1, U_2, U_3, B\}$, where $L = U_1 \cap U_2 \subset K$ is an $(n - 3)$ -space.

Otherwise, we assume that no two elements of $\{U_1, U_2, U_3\}$ can be joined to an $(n - 1)$ -space, so that $\dim(U_i \cap U_j) = n - 4$ for all $1 \leq i < j \leq 3$. We set $E_i := U_i \cap H_\infty$, so that $\dim(E_i) = n - 3$, for all $1 \leq i \leq 3$ and $\dim(E_i \cap E_j) = n - 4$ for all $1 \leq i < j \leq 3$. Let $K := \langle E_1, E_2, E_3 \rangle \leq H_\infty$, so that $n - 2 \leq \dim(K) \leq n - 1$. If $\dim(K) = n - 2$, then consider the K -decomposition $\mathcal{U}^{(1)}, \mathcal{U}^{(2)}$ of \mathcal{U} and let H_1, H_2 be the corresponding hyperplanes. Since $E_1, E_2, E_3 \leq K$, we have that either $U_i \leq H_1$ or $U_i \leq H_2$ for all indices $1 \leq i \leq 3$. By the pigeonhole principle two of the three $(n - 2)$ -spaces in \mathcal{U} have to be contained in the same hyperplane, which contradicts $\dim(U_i \cap U_j) = n - 4$.

Thus, we have $\dim(K) = n - 1$, i.e., $K = H_\infty$. Since $\dim(\langle E_1, E_2 \rangle) = n - 2, \dim(E_3) = n - 3$, and $\dim(K) = n - 1$, we have

$$\dim(\langle E_1, E_2 \rangle \cap E_3) = n - 4.$$

Since $\dim(E_1 \cap E_3) = \dim(E_2 \cap E_3) = n - 4$, we have $\dim(C) = n - 4$ for $C := E_1 \cap E_2 \cap E_3$. Pick three linearly independent vectors v_1, v_2, v_3 such that $E_1 = \langle C, v_1 \rangle, E_2 = \langle C, v_2 \rangle, E_3 = \langle C, v_3 \rangle$, and $H_\infty = \langle C, v_1, v_2, v_3 \rangle$. Let P_1, P_2 be two different arbitrary points outside of H_∞ that are not covered by U_1, U_2 , or U_3 . For pairwise different $i, j, h \in \{1, 2, 3\}$ consider the $(n - 2)$ -space $K_{i,j,h} := \langle C, v_i, v_j + v_h \rangle$ and let $H_{i,j,h}$ be the hyperplane that contains $K_{i,j,h}$ and U_i . Since all points in $H_{i,j,h} \setminus H_\infty$ are covered by U_1, U_2, U_3 the points P_1, P_2 have to be contained in the other hyperplane containing $K_{i,j,h}$ not equal to $H_{i,j,h}$ and H_∞ , so that $P_1 - P_2 \in \langle C, v_i, v_j + v_h \rangle$. Since

$$\langle C, v_1, v_2 + v_3 \rangle \cap \langle C, v_2, v_1 + v_3 \rangle \cap \langle C, v_3, v_1 + v_2 \rangle = \langle C, v_1 + v_2 + v_3 \rangle,$$

the 2^{n-3} points outside of H_∞ that are not covered by U_1, U_2 , or U_3 have to form an affine subspace $B \geq C$. If $\#\mathcal{U} = r = 4$, then $B = U_4$. If $\#\mathcal{U} \geq 5$, then the elements in $\{U_4, \dots, U_r\}$ form an avsp of B . □

Corollary 6 *Let \mathcal{U} be an irreducible tight avsp of $\text{PG}(n - 1, 2)$ of type $(n - 2)^{m_{n-2}} \dots 2^{m_2}$, where $n \geq 5$. Then, we have $m_{n-2} \leq 2$.*

Together with the conditions $m_{n-1} = m_1 = 0$ and the packing condition (1) we obtain:

Corollary 7 *Let \mathcal{U} be an irreducible tight avsp of $\text{PG}(5 - 1, 2)$. Then the type of \mathcal{U} is given by $3^2 2^4, 3^1 2^6$, or 2^8 .*

All types can indeed be realized and corresponding numbers of equivalence classes are given by 3, 4, and 2, respectively. I.e., for $n = 4$ we have 9 non-isomorphic examples in total. Below are representatives:

E1, 2^8 : (10000, 01000), (10100, 00010), (11100, 00001), (10010, 01100), (11010, 00101), (10001, 01010), (10011, 00110), (10111, 01110).

E2, 2^8 : (10000, 01000), (10100, 00010), (11100, 00001), (10010, 01101), (10001, 01011), (11001, 00111), (10101, 01110), (10011, 00100).

E3, $3^1 2^6$: (10000, 01000), (10100, 00010), (11100, 00001), (10010, 01100), (10001, 01010), (10111, 01101), (10011, 01010, 00110).

E4, $3^1 2^6$: (10000, 01000), (10100, 00010), (11100, 00001), (10010, 01100), (11001, 00011), (10011, 00100), (10001, 01010, 00100).

- E5, $3^1 2^6$: (10000, 01000), (10100, 00010), (11100, 00001), (10010, 01001), (10001, 01111), (10111, 01101), (10011, 01010, 00110).
- E6, $3^1 2^6$: (10000, 01000), (10100, 00010), (11100, 00001), (10010, 01100), (10001, 00100), (11001, 00011), (10011, 01000, 00100).
- E7, $3^2 2^4$: (10000, 01000), (10100, 00010), (11100, 00001), (10010, 01011), (11010, 00100, 00001), (10001, 00100, 00010).
- E8, $3^2 2^4$: (10000, 01000), (10100, 00010), (11100, 00001), (10010, 01011), (10001, 01010, 00100), (10011, 01001, 00100).
- E9, $3^2 2^4$: (10000, 01000), (10100, 00010), (11100, 00001), (10101, 01011), (10010, 01000, 00001), (10001, 01000, 00110).

We remark that the hypothetical type $3^3 2^2$ is also excluded by Corollary 2.

Similarly as we have constructed \mathcal{T}' from the tail \mathcal{T} in Lemma 9, we can consider the set $\mathcal{U}' := \{U \cap H_\infty : U \in \mathcal{U}\}$ for an avsp \mathcal{U} of $\text{PG}(n - 1, q)$. If \mathcal{U} is an irreducible tight avsp of $\text{PG}(n - 1, q)$ of type $2^{m_2} 3^{m_3}$, where $m_2 = q^{n-2} - qm_3$, then \mathcal{U}' is a configuration of m_2 points and m_3 lines in $H_\infty \cong \text{PG}(n - 2, q)$. The points are pairwise disjoint, so that Lemma 9 yields that they form a q -divisible set. Any two lines can meet in at most a point. If $n = 5$, then any two lines indeed intersect in a point. So, the maximum point multiplicity is at most $m_3 + 1$. We remark that the possibilities for \mathcal{U}' can be classified completely theoretically, i.e., without the use of computer programs. Due to space limitations we refer the interested reader to the corresponding arXiv preprint and only state two necessary criteria for \mathcal{U}' .

Lemma 14 *Let \mathcal{U} be an irreducible avsp of $\text{PG}(n - 1, q)$ not of type $(n - 1)^q$ and $\mathcal{U}' := \{U \cap H_\infty : U \in \mathcal{U}\}$. Then \mathcal{U}' is spanning, i.e., \mathcal{U}' spans H_∞ .*

Proof Assume that K is a hyperplane of H_∞ that contains all elements of \mathcal{U}' . From Lemma 7 we can conclude that the K -decomposition $\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(q)}$, with corresponding hyperplanes H_1, \dots, H_q , is a partition of \mathcal{U} , i.e., the elements of $\mathcal{U}^{(i)}$ can be joined to H_i for all $1 \leq i \leq q$. Since we have assumed that \mathcal{U} is not of type $(n - 1)^q$ we obtain a contradiction. \square

Lemma 15 *Let \mathcal{U} be an avsp of $\text{PG}(n - 1, q)$ of type $(n - 1)^{m_{n-1}} \dots 2^{m_2}$ and $\mathcal{U}' := \{U \cap H_\infty : U \in \mathcal{U}\}$. For each hyperplane K of H_∞ let a_i^K denote the number of i -dimensional elements of \mathcal{U}' that are contained in K and $b_i^K = m_{i+1} - a_i^K$ the number of i -dimensional elements of \mathcal{U}' that are not contained in K , where $1 \leq i \leq n - 2$. Then there exist $c_{i,j}^K \in \mathbb{N}_0$ for all $1 \leq j \leq q, 1 \leq i \leq n - 2$ such that*

$$\sum_{j=1}^q c_{i,j}^K = a_i^K \quad \forall 1 \leq i \leq n - 2, \tag{8}$$

$$\sum_{i=1}^{n-2} \left(c_{i,j}^K \cdot q^i + b_i^K \cdot q^{i-1} = q^{n-2} \right) \quad \forall 1 \leq j \leq q. \tag{9}$$

Proof For an arbitrary but fixed hyperplane K of H_∞ let $\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(q)}$ be the K -decomposition of \mathcal{U} with corresponding hyperplanes H_1, \dots, H_q . From Lemma 7 we conclude that for each element $U \in \mathcal{U}$ with $U \cap H_\infty \leq K$ there exists an index $1 \leq j \leq q$ such that $U \leq H_j$. The integers $c_{i,j}^K$ just count how many $(i + 1)$ -dimensional elements of \mathcal{U} are contained in H_j (which depends on K). Since the hyperplanes H_1, \dots, H_q are pairwise disjoint, we obtain Equation (8). From Lemma 8 we conclude that for each element $U \in \mathcal{U}$ such that $U \cap H_\infty \not\leq K$ we have $\#((U \cap H_j) \setminus H_\infty) = q^{\dim(U)-2}$, so that the packing condition for H_j yields Equation (9). \square

We call the process of moving from \mathcal{U}' to \mathcal{U} the *extension problem*. An integer linear programming formulation is given in Sect. A in the appendix. Note that the extension problem comprises additional symmetry given by the pointwise stabilizer of H_∞ of order q^{n-1} .

Given a set \mathcal{U}' satisfying all of the necessary conditions mentioned so far it is neither clear that an extension to a corresponding avsp \mathcal{U} always exists nor that it is, in the case of existence, unique up to symmetry. Indeed, we will give counter examples later on. However, for the nine classified configurations \mathcal{U}' in $\text{PG}(3, 2)$ it turns out that there always is an up to symmetry unique extension.

4.2 Tight irreducible avsp in $\text{PG}(5, 2)$

Lemma 16 *For $n \geq 6$ no tight irreducible avsp of type $(n - 2)^2(n - 3)^4$ in $\text{PG}(n - 1, 2)$ exists.*

Proof Assume that such an avsp \mathcal{U} exists and consider the intersections of the elements with the hyperplane H_∞ at infinity, i.e., $\mathcal{U}' := \{U \cap H_\infty : U \in \mathcal{U}\}$. By E_1, E_2 we denote the two $(n - 3)$ -spaces and by L_1, \dots, L_4 the four $(n - 4)$ -spaces. The intersection of E_1 and E_2 is an $(n - 4)$ -space L' and $\dim(E_i \cap L_j) \geq n - 5$ for all $i = 1, 2$ and $j = 1, \dots, 4$. From Lemma 9 we conclude that $\mathcal{T}' = \{L_1, \dots, L_4\}$ is a 2-divisible set of four $(n - 4)$ -spaces, so that Lemma 11 implies the existence of a plane $E \leq H_\infty$, a line $L \leq E$, and an $(n - 5)$ -space $B \leq H_\infty$ with $B \cap E = \emptyset$ and

$$\{L_1, \dots, L_4\} = \{\langle Q, B \rangle : Q \in E \setminus L\}.$$

Since \mathcal{U} is tight we have $B \cap L' = \emptyset$. However, $\dim(E_i \cap L_j) \geq n - 5$ implies $\dim(L', L_j) \geq n - 6$ for all $1 \leq j \leq 4$. So, we clearly have $n \leq 7$.

For $n = 7$ we have $\dim(E_i \cap B) \geq \dim(E_i \cap L_j) - 1 \geq n - 6 = 1$, where $1 \leq i \leq 2$ and $1 \leq j \leq 4$, so that $\dim(L' \cap B) = 0$ implies $E_i \leq K := \langle B, L' \rangle$ and $\dim(E_i \cap B) = 1$. With this, $\dim(E_i \cap L_j) \geq n - 5 = 2$ yields the existence of a point $Q_{i,j} \notin B$ with $Q_{i,j} \leq E_i \cap L_j$, so that $L_j = \langle B, Q_{i,j} \rangle$, i.e., $L_j \leq K$. However, this implies that \mathcal{U}' is not spanning, which is a contradiction with Lemma 14.

For $n = 6$ we have $\dim(B) = 1$, i.e., B is a point. Since \mathcal{U} is tight we have $B \not\leq L'$. W.l.o.g. we assume $B \not\leq E_1$. Let $S := \langle E_1, B \rangle$, so that $\dim(S) = 4$. Since E_1 intersects each of the lines L_j in at least a point not equal to B , we have $L_j \leq S$ for all $1 \leq j \leq 4$. Let $1 \leq h \leq 4$ be a suitable index with $L_h \neq L'$ and let $Q \leq E_2 \cap L_h$ be a point, so that $Q \not\leq L'$. Thus $E_2 = \langle L', Q \rangle \leq S$, so that $\langle \mathcal{U}' \rangle \leq S$, i.e., \mathcal{U}' is not spanning, which is a contradiction with Lemma 14. □

Proposition 5 *Let \mathcal{U} be a tight irreducible avsp of $\text{PG}(5, 2)$, then \mathcal{U} has one of the following types:*

- $4^2 3^i 2^{8-2i}$ for $i \in \{0, 1, 2\}$;
- $4^1 3^i 2^{12-2i}$ for $i \in \{0, \dots, 6\} \setminus \{5\}$; and
- $3^i 2^{16-2i}$ for $i \in \{0, \dots, 8\} \setminus \{7\}$.

All types are realizable.

Proof Let the type of \mathcal{U} be $5^{m_5} \dots 1^{m_1}$. From Corollary 3 and Lemma 13 we conclude $m_5 = 0$ and $m_1 = 0$, so that the packing condition (1) gives $4m_4 + 2m_3 + m_2 = 16$. Corollary 6 gives $m_4 \leq 2$ and Lemma 16 excludes $(m_4, m_3, m_2) = (2, 4, 0)$. Moreover, Corollary 2 implies $m_2 \neq 2$. All remaining possibilities $(m_4, m_3, m_2) \in \mathbb{N}_0^3$ are listed in the statement and for each type we found a realization using ILP computations. □

Corollary 8 *If \mathcal{U} is a tight irreducible avsp of $\text{PG}(5, 2)$ of minimum possible size, then $\#\mathcal{U} = 7$ and \mathcal{U} has type $4^1 3^6$.*

Table 1 Number of isomorphism types of tight irreducible avsp's in $PG(5, 2)$

type	$4^1 3^6$	$4^2 3^2 2^4$	3^8	$4^2 3^1 2^6$	$4^1 3^4 2^4$	$4^2 2^8$	$4^1 3^3 2^6$
#	6	38	32	55	827	83	8096
in H_∞	6	38	32	55	811	50	6686

For small sizes we have enumerated the isomorphy types of tight irreducible avsp's in $PG(5, 2)$, see Table 1. The last row concerns the parts U' at the hyperplane H_∞ at infinity w.r.t. the avsp's U counted up to isomorphy in the second row. So, for e.g. types $4^1 3^4 2^4$ and $4^2 2^8$ there exist configurations U' that allow more than one extension up to symmetry.

For the minimum possible size of a tight irreducible avsp in $PG(5, 2)$ we can write down all implications of the stated necessary conditions for the part U' at infinity. So, for type $4^1 3^6$ configuration U' consists of one plane E and six lines $\mathcal{L} = \{L_1, \dots, L_6\}$ satisfying the following conditions:

- (1) the configuration is spanning, i.e., $\langle E, L_1, \dots, L_6 \rangle = PG(4, 2)$;
- (2) the configuration is tight, i.e., there does not exist a point P that is contained in E and all lines in \mathcal{L} ;
- (3) the lines in \mathcal{L} form a 2-divisible set of lines, i.e., each hyperplane contains an even number of lines;
- (4) each line L_i intersects E in at least a point;
- (5) hyperplanes that contain E also contain at least two lines.

Up to symmetry ten such configurations exist:

- E1: $\langle 10000, 01000, 00100 \rangle, \langle 10000, 01000 \rangle, \langle 10000, 00100 \rangle, \langle 01000, 00010 \rangle, \langle 01000, 00110 \rangle, \langle 10100, 00001 \rangle, \langle 10100, 01101 \rangle$
- E2: $\langle 10000, 01000, 00100 \rangle, \langle 10000, 01000 \rangle, \langle 10000, 00010 \rangle, \langle 10000, 00110 \rangle, \langle 01100, 00010 \rangle, \langle 01100, 00001 \rangle, \langle 10011, 01100 \rangle$
- E3: $\langle 10000, 01000, 00100 \rangle, \langle 10000, 01000 \rangle, \langle 10000, 00010 \rangle, \langle 10000, 00001 \rangle, \langle 01000, 00011 \rangle, \langle 10100, 01011 \rangle, \langle 01011, 00111 \rangle$
- E4: $\langle 10000, 01000, 00100 \rangle, \langle 10000, 01000 \rangle, \langle 10000, 00010 \rangle, \langle 01000, 00110 \rangle, \langle 00100, 00010 \rangle, \langle 11100, 00001 \rangle, \langle 10111, 01011 \rangle$
- E5: $\langle 10000, 01000, 00100 \rangle, \langle 10000, 00010 \rangle, \langle 10000, 01010 \rangle, \langle 00100, 00001 \rangle, \langle 01100, 00011 \rangle, \langle 11001, 00100 \rangle, \langle 10111, 01100 \rangle$
- E6: $\langle 10000, 01000, 00100 \rangle, \langle 10000, 00010 \rangle, \langle 10000, 00001 \rangle, \langle 10000, 01011 \rangle, \langle 01000, 00010 \rangle, \langle 01000, 00001 \rangle, \langle 10011, 01000 \rangle$
- E7: $\langle 10000, 01000, 00100 \rangle, \langle 10000, 00010 \rangle, \langle 10000, 00001 \rangle, \langle 10000, 01011 \rangle, \langle 01000, 00010 \rangle, \langle 01000, 00101 \rangle, \langle 10111, 01000 \rangle$
- E8: $\langle 10000, 01000, 00100 \rangle, \langle 10000, 00010 \rangle, \langle 10000, 00001 \rangle, \langle 01000, 00010 \rangle, \langle 01000, 00001 \rangle, \langle 10100, 01111 \rangle, \langle 10111, 01100 \rangle$
- E9: $\langle 10000, 01000, 00100 \rangle, \langle 10000, 00010 \rangle, \langle 10000, 00001 \rangle, \langle 01000, 00110 \rangle, \langle 01000, 00101 \rangle, \langle 10100, 01111 \rangle, \langle 10111, 01100 \rangle$
- E10: $\langle 10000, 01000, 00100 \rangle, \langle 10000, 00010 \rangle, \langle 01000, 00010 \rangle, \langle 00100, 00001 \rangle, \langle 10100, 01011 \rangle, \langle 11001, 00101 \rangle, \langle 10011, 01100 \rangle$

It turns out that E2, E4, E7, and E9 are not extendable to an avsp while the other six cases are. Moreover, the extension is unique up to symmetry in these cases.

4.3 Tight irreducible avsp's in $PG(6, 2)$

Lemma 17 *In $PG(6, 2)$ no tight irreducible avsp of type $5^2 4^2 3^4$ or $5^1 4^6$ exists.*

Proof All two possibilities are excluded using ILP computations, see Sect. A. They are also excluded using GAP computations. □

Proposition 6 *Let U be a tight irreducible avsp of $PG(6, 2)$, then U has one of the following types:*

- $5^2 4^i 3^j 2^{16-2j-4i}$ for $i \in \{0, 1, 2\}$ and $0 \leq j \leq 8 - 2i$, where $j + 2i \neq 7$ and $(i, j) \neq (2, 4)$;

- $5^1 4^i 3^j 2^{24-2j-4i}$ for $0 \leq i \leq 4$ and $0 \leq j \leq 12 - 2i$, where $j + 2i \neq 11$;
- $4^i 3^j 2^{32-2j-4i}$ for $0 \leq i \leq 8$ and $0 \leq j \leq 16 - 2i$, where $j + 2i \neq 15$ and $i \neq 7$.

All types are realizable.

Proof Let the type of \mathcal{U} be $6^{m_6} \dots 1^{m_1}$. From Corollary 3 and Lemma 13 we conclude $m_6 = 0$ and $m_1 = 0$, so that the packing condition (1) gives $8m_5 + 4m_4 + 2m_3 + m_2 = 32$. Corollary 6 yields $m_5 \leq 2$ and Lemma 16 excludes $(m_5, m_4, m_3, m_2) = (2, 4, 0, 0)$. Moreover, Corollary 2 implies $m_l \neq 2$ for the smallest index with $m_l > 0$, which excludes the types $5^2 4^j 3^i 2^2$ with $j + 2i = 7$, $5^2 4^3 2^2$, $5^1 4^j 3^i 2^2$ with $j + 2i = 11$, $5^1 4^5 3^2$, $4^i 3^i 2^2$ with $j + 2i = 15$, and $4^7 3^2$. The two hypothetical types $5^2 4^2 3^4$ and $5^1 4^6$ are excluded in Lemma 17. For the three hypothetical types $5^2 4^3 2^4$, $5^1 4^5 2^4$, and $4^7 2^4$ we apply Lemma 9 to conclude that the set of 2-spaces is 4-divisible. However, Lemma 11 characterizes the 2-divisible sets of cardinality 4 and we can easily check that it is not 4-divisible. All remaining possibilities $(m_5, m_4, m_3, m_2) \in \mathbb{N}_0^4$ are listed in the statement and for each type we found a realization using an ILP formulation, see Sect. A. □

Corollary 9 *If \mathcal{U} is a tight irreducible avsp of $\text{PG}(6, 2)$ of minimum possible size, then $\#\mathcal{U} = 8$ and \mathcal{U} has type 4^8 .*

Here we describe all four isomorphism types of homogeneous irreducible tight avsp \mathcal{U} of $\text{PG}(6, 2)$ of type 4^8 , where we call an avsp \mathcal{U} homogeneous if all of its elements have the same dimension. Geometrically each \mathcal{U} is given by eight solids S_1, \dots, S_8 in $\text{PG}(6, 2)$ intersecting a hyperplane H_∞ in a plane (plus some extra conditions). Here we directly consider the part \mathcal{U}' at infinity, i.e. the eight planes $\pi_1, \dots, \pi_8 \in H_\infty \cong \text{PG}(5, 2)$ given by $\pi_i = S_i \cap H_\infty$. The conditions for the pairwise intersections are

$$1 \leq \dim(\pi_i \cap \pi_j) \leq 2 \quad \forall 1 \leq i < j \leq 8. \tag{10}$$

Since the planes form a spanning 2-divisible set we have

$$\#\{1 \leq i \leq 8 : \pi_i \not\subseteq H\} \in \{2, 4, 6, 8\} \tag{11}$$

for every hyperplane H of $H_\infty \cong \text{PG}(5, 2)$.

Let e_i denote the i th unit vector, i.e., the vector with a 1 at the i -th position and zeros everywhere else. If the pairwise intersection of the planes π_i is a line in all cases then they span a solid, which contradicts the condition that not all eight planes can be contained in a hyperplane. W.l.o.g. we assume $\pi_1 = \langle e_1, e_2, e_3 \rangle$ and $\pi_2 = \langle e_3, e_4, e_5 \rangle$, i.e., the intersection point between π_1 and π_2 is $\langle e_3 \rangle$. Since the intersection of all eight planes is empty we assume w.l.o.g. that π_3 does not contain $\pi_1 \cap \pi_2 = e_3$. Up to symmetry we have the following three cases for π_3 :

- (a) $\dim(\pi_1 \cap \pi_3) = 2, \dim(\pi_2 \cap \pi_3) = 1: \pi_3 = \langle e_1, e_2, e_4 \rangle$;
- (b) $\dim(\pi_1 \cap \pi_3) = \dim(\pi_2 \cap \pi_3) = 1, \dim(\langle \pi_1, \pi_2, \pi_3 \rangle) = 5: \pi_3 = \langle e_1, e_4, e_2 + e_5 \rangle$; and
- (c) $\dim(\pi_1 \cap \pi_3) = \dim(\pi_2 \cap \pi_3) = 1, \dim(\langle \pi_1, \pi_2, \pi_3 \rangle) = 6: \pi_3 = \langle e_1, e_4, e_6 \rangle$.

Starting from the three possibilities for π_1, π_2, π_3 we build up a graph whose vertices consist of the planes that have intersection dimension 1 or 2 with π_i for $1 \leq i \leq 3$, cf. Condition (10). Two vertices π and π' are connected by an edge if $1 \leq \dim(\pi \cap \pi') \leq 2$, cf. Condition (10). For these graphs we determine all cliques of size five and check Condition (11) afterwards:

Table 2 Irreducible tight avspss of PG(6, 2) of type 4⁸

γ_0	2	3	4	5
point incidences:	2 ²⁸	1 ²¹ 2 ⁷ 3 ⁷	1 ¹⁶ 2 ¹² 4 ⁴	1 ²⁰ 2 ⁶ 3 ² 4 ² 5 ²
line incidences:	1 ⁵⁶	1 ⁵⁶	1 ⁴⁸ 2 ⁴	1 ⁴⁶ 2 ² 3 ²
solid incidences:	1 ⁵⁶	1 ⁵⁶	1 ⁴⁸ 2 ⁴	1 ⁴⁰ 2 ⁸
hyperplane incidences:	2 ²⁸	2 ²⁸	2 ²⁴ 4 ²	2 ²³ 4 ¹ 6 ¹
triples of planes π_i :	c ¹⁶⁸	c ¹⁶⁸	a ⁴⁸ c ⁹⁶	a ⁷² c ⁴⁸

- (a) 3,014,435,152 cliques → 432 cases;
- (b) 2,198,293,872 cliques → 0 cases;
- (c) 1,218,975,648 cliques → 320 cases.

The overall computation took just a few minutes. Note that the constructed 752 cases are just candidates for the extension problem to eight solids. Up to symmetry they decompose into just four non-isomorphic examples. It turns out that they can be distinguished by the maximum number γ_0 of incidences of a point and the eight planes, where $2 \leq \gamma_0 \leq 5$. In Table 2 we summarize incidence counts, i.e., for $X \in \{\text{point, line, solid, hyperplane}\}$ the stated vector $a_1^{b_1} \dots a_r^{b_r}$ says that b_i of the X s have exactly a_i incidences with the eight planes, given the isomorphy type characterized by γ_0 . The last row states how often the “third plane” is of type (a), (b), or (c) after fixing a pair of planes π_1, π_2 .

For $\gamma_0 = 2$ we consider an arbitrary plane π contained in the hyperbolic quadric $\mathcal{Q} = \mathcal{Q}^+(5, 2)$, which form a single orbit under its collineation group $\text{PGO}^+(6, 2) = C_2 \times \text{PGL}(3, 2) = S_8$ of order 40,320. From the 35 points on \mathcal{Q} the points in π have no incidences with the eight planes while all other 28 points on \mathcal{Q} have exactly two incidences. This example is obtained in 16 cases. The symmetry group of the eight planes has order 1344 and type $C_2^3 : \text{PGL}(3, 2)$.

For $\gamma_0 = 3$ choose a projective base of PG(5, 2), i.e., put $f_i = e_i$ for $1 \leq i \leq 6$ and $f_7 = \sum_{i=1}^6 e_i$. Consider a Fano plane on the set $\{1, 2, 3, 4, 5, 6, 7\}$:

$$\begin{aligned} \ell_1 &= \{1, 2, 3\}, & \ell_2 &= \{1, 4, 5\}, & \ell_3 &= \{1, 6, 7\}, & \ell_4 &= \{2, 4, 6\}, \\ \ell_5 &= \{3, 4, 7\}, & \ell_6 &= \{2, 5, 7\}, & \ell_7 &= \{3, 5, 6\}. \end{aligned}$$

Choose seven planes $\pi_i := \langle f_j : j \in \ell_i \rangle$ for $1 \leq i \leq 7$ and an eight plane. $\pi_8 = K := \langle \sum_{j \in \ell_i} f_j : 1 \leq i \leq 7 \rangle$. Note that K itself is also a Fano plane (of course with a different embedding). The points with three incidences with the eight planes are the f_i for $1 \leq i \leq 7$ and the points with two incidences with the eight planes are the points of K . This example is obtained in 112 cases. The symmetry group of the eight planes has order 168 and type $\text{PGL}(3, 2)$.

For $\gamma_0 = 4$ let $\{Q_1, Q_2, Q_3, Q_4, R_1, R_2\}$ be a basis of H_∞ . With this, we construct the eight planes as

$$\begin{aligned} &\langle Q_{i+j}, Q_{i+j+1}, R_i \rangle \text{ for } i \in \{1, 2\}, j \in \{0, 2\}, \\ &\langle Q_{i+j}, Q_{i+j+1}, R_i + A \rangle \text{ for } i \in \{1, 2\}, j \in \{0, 2\}, \end{aligned}$$

where $A = Q_1 + Q_2 + Q_3 + Q_4$ and $Q_5 = Q_1$. The points with four incidences with the eight planes are Q_1, \dots, Q_4 . The lines with two incidences with the eight planes are $\langle Q_i, Q_{i+1} \rangle$ for $1 \leq i \leq 4$ (again setting $Q_5 = Q_1$; so this is some kind of a cyclic construction). This

example is obtained in 192 cases. The symmetry group of the eight planes has order 128 and type $D_8^2 : C_2$.

For $\gamma_0 = 5$ let $\{Q_1, Q_2, R_1, R_2, S_1, T_1\}$ be a basis of H_∞ . With this, we set $S_2 := S_1 + Q_1 + Q_2, T_2 := T_1 + Q_1 + Q_2 + R_1 + R_2$ and construct the eight planes as

$$\begin{aligned} &\langle Q_1, Q_2, R_i \rangle \text{ for } i \in \{1, 2\}, \\ &\langle R_1, R_2, S_i \rangle \text{ for } i \in \{1, 2\}, \text{ and} \\ &\langle Q_i, R_i, T_j \rangle \text{ for } i, j \in \{1, 2\}, \end{aligned}$$

which also reflects the three orbits of the eight planes w.r.t. the action of their automorphism group. The points with five incidences with the eight planes are R_1 and R_2 . The points with four incidences with the eight planes are Q_1 and Q_2 . The points with three incidences with the eight planes are $R_1 + Q_1$ and $R_2 + Q_2$. The lines with three incidences with the eight planes are $\langle R_1, Q_1 \rangle$ and $\langle R_2, Q_2 \rangle$. The lines with two incidences with the eight planes are $\langle R_1, R_2 \rangle$ and $\langle Q_1, Q_2 \rangle$. This example is obtained in 432 cases. The symmetry group of the eight planes has order 1024.

5 Constructions of tight irreducible avsp

In this section we collect a few general constructions for tight irreducible avsp using different combinatorial objects. We use spreads, the Klein quadric, and hitting formulas in Sects. 5.1, 5.2, and 5.3, respectively.

5.1 Constructions from projective spreads

A k -spread in $PG(n - 1, q)$ is a disjoint set of k -spaces that partitions $PG(n - 1, q)$. It is well known that k -spreads exist iff k divides n .

Proposition 7 *For each positive even integer n there exists a tight irreducible avsp \mathcal{U} of $PG(n - 1, q)$ of type $(n/2)^m$, where $m = q^{n/2}$.*

Proof Let $k = n/2$ and \mathcal{P} be a k -spread of $PG(n - 1, q)$, which has size $q^k + 1$. Now choose an arbitrary element $K \in \mathcal{P}$ and an arbitrary hyperplane H containing K . With this we set $\mathcal{U} = \mathcal{P} \setminus \{K\}$ where we choose H as the hyperplane at infinity. By construction \mathcal{U} is an avsp of $PG(n - 1, q)$. Since all elements are pairwise disjoint \mathcal{U} is tight and since any two elements span $PG(n - 1, q)$ \mathcal{U} is irreducible. \square

We have seen that in $PG(5, 2)$ there exist tight irreducible avsp of types 3^8 and 2^{16} . Starting from a 2-spread of $PG(5, q)$ we can clearly obtain a tight avsp \mathcal{U} by removing all lines that are completely contained in an arbitrarily chosen hyperplane H . However, it may happen that \mathcal{U} is reducible. This is indeed the case if we start with the Desarguesian line spread. In $PG(5, 2)$ there exist 131,044 non-isomorphic line spreads [24].

Conjecture 1 *For each integer $1 < k < n$ that divides n there exists a tight irreducible avsp \mathcal{U} of $PG(n - 1, q)$ of type k^m , where $m = q^{n-k}$.*

If n is odd no $\lfloor (n - 1)/2 \rfloor$ -spread exists, but we can construct tight irreducible avsp from some special large partial spreads.

Proposition 8 *For each odd integer $n \geq 5$ there exists a tight irreducible avsp \mathcal{U} of $PG(n - 1, q)$ of type $((n - 1)/2)^m$, where $m = q^{(n+1)/2}$.*

Proof Let $k = (n - 1)/2$ and \mathcal{P} be a vector space partition of $\text{PG}(n - 1, q)$ of type $(k + 1)^1 k^m$, where $m = q^{k+1}$. Now choose an arbitrary hyperplane H containing the unique $(k + 1)$ -dimensional element K of \mathcal{P} . With this we set $\mathcal{U} = \mathcal{P} \setminus \{K\}$ where we choose H as the hyperplane at infinity. By construction \mathcal{U} is an avsp of $\text{PG}(n - 1, q)$. Since all elements are pairwise disjoint \mathcal{U} is tight. Any two elements of \mathcal{U} span a hyperplane of $\text{PG}(n - 1, q)$. Since the elements of $\mathcal{U}' := \{U \cap H_\infty : U \in \mathcal{U}\}$ span H_∞ , not all elements of \mathcal{U}' can be contained in a hyperplane of H_∞ and \mathcal{U} is irreducible. \square

Vector space partitions of the used type can be obtained from lifted MRD codes, see e.g. [27] for a survey on MRD codes. They also occur as extendible partial k -spreads, where $k = (n - 1)/2$, of the second largest size q^{k+1} and are the main building block in the construction of partial k -spreads of size $q^{k+1} + 1$ as described by Beutelspacher [3]. For more details on the relations between these different geometrical objects we refer e.g. to [16].

For each $n \geq 5$ there also exist a vector space partition \mathcal{P} of $\text{PG}(n - 1, q)$ of type $(n - 2)^1 2^m$, where $m = q^{n-2}$. Choosing a hyperplane that contains the unique $(n - 2)$ -space as the hyperplane at infinity we can obtain a tight avsp \mathcal{U} of $\text{PG}(n - 1, q)$ of type $2q^{n-2}$. The remaining question is whether we can choose \mathcal{P} in such a way that \mathcal{U} becomes irreducible.

5.2 Constructions from the Klein quadric

It seems very likely that the avsp of $\text{PG}(6, 2)$ of type 4^8 with maximum point multiplicity 2, see Sect. 4.2, can be generalized to arbitrary field sizes.

Theorem 2 *There exists a tight irreducible avsp of type $4q^3$ in $\text{PG}(6, q)$ for q even.*

Proof We will use the following finite field model of $\text{AG}(6, q)$. Let $V = \mathbb{F}_{q^3} \times \mathbb{F}_{q^3} \times \mathbb{F}_q$ and let H_∞ be the hyperplane $X_3 = 0$. So we identify $\text{AG}(6, q)$ with the elements of V of the form (a, b, c) , where $c \neq 0$. Consider the following quadratic form on H_∞ :

$$Q(x, y, 0) := \text{Tr}_{q^3/q}(xy).$$

Then Q defines the points of a hyperbolic quadric \mathcal{Q} . Next, let π be the plane $\{(0, y, 0) : y \in \mathbb{F}_{q^3}^*\}$. Then π is totally singular with respect to Q . Let $S_0 := \{(x, 0, 1) : x \in \mathbb{F}_{q^3}\}$ and $S_1 := \{(y, y^{q^2} + y^q + 1, 1) : y \in \mathbb{F}_{q^3}\}$. Let α be a primitive element of \mathbb{F}_{q^3} , let σ be the map

$$\sigma : (x, y, z) \mapsto (\alpha^{-1}x, \alpha y, z),$$

and let $G := \langle \sigma \rangle$. We will show that $\mathcal{S} := \{S_0\} \cup S_1^G$ is a tight irreducible avsp of size q^3 in $\text{AG}(6, q)$.

First note that σ has order $q^3 - 1$. Let $(a, b, 1)$ be a point P of $\text{AG}(6, q)$. We show that P lies in a unique element of \mathcal{S} . If $b = 0$, then P lies in S_0 . The condition that P lies in $S_1^{\sigma^m}$ (where $1 \leq m \leq q^3 - 1$) can be restated as

$$a = \alpha^{-m}y, \quad b = \alpha^m(y^{q^2} + y^q + 1).$$

for some $y \in \mathbb{F}_{q^3}$. We have

$$ab = y^{q^2+1} + y^{q+1} + y$$

and the polynomial $y^{q^2+1} + y^{q+1} + y$ is a permutation on \mathbb{F}_{q^3} , by [29, Theorem 4] (using our assumption q even). Hence, y and, thus, m are determined by a and b . Therefore, \mathcal{S} is an avsp.

Note that $\pi_1 := \overline{S_1} \cap H_\infty = \{(y, y^{q^2} + y^q, 0) : y \in \mathbb{F}_{q^3}\}$ and $\pi_0 := \overline{S_0} \cap H_\infty = \{(x, 0, 0) : x \in \mathbb{F}_{q^3}\}$. To compute the image of π_1 under σ^m , notice that

$$\begin{aligned} (\alpha^{-m}y, \alpha^m(y^{q^2} + y^q), 0) &= (\alpha^{-m}y, \alpha^{(q^2+1)m}(\alpha^{-m}y)^{q^2} + \alpha^{(q+1)m}(\alpha^{-m}y)^q, 0) \\ &= (w, \zeta w^{q^2} + \zeta^q w^q, 0) \end{aligned}$$

where $w = \alpha^{-m}y$ and $\zeta = \alpha^{(q^2+1)m}$. Therefore, upon application of G ,

$$S_\infty := \{\overline{S} \cap H_\infty : S \in \mathcal{S}\} = \{\pi_0\} \cup \pi_1^G = \{\pi_\zeta : \zeta \in \mathbb{F}_{q^3}^*\}$$

where $\pi_\zeta := \{(y, \zeta y^{q^2} + \zeta^q y^q, 0) : y \in \mathbb{F}_{q^3}^*\}$. Note that $|S_\infty| = q^3$ and that \mathcal{S} consists of totally singular planes of \mathcal{Q} disjoint from π . As these are all totally singular planes of \mathcal{Q} disjoint from π , these cover the points of \mathcal{Q} uniformly and their common intersection is empty and, thus, \mathcal{S} is tight. As these pairwise meet in a point, any two elements of \mathcal{S} span $\text{PG}(6, q)$. This shows irreducibility. \square

Let \mathcal{P} be the set of planes in the Klein quadric $\mathcal{Q} = Q^+(5, q)$ that is disjoint to an arbitrary but fixed plane π in \mathcal{Q} . One can verify that \mathcal{P} is a spanning q -divisible set of q^3 planes in $\text{PG}(5, q)$ such that the intersection of a pair of planes is a point, i.e., all known conditions for the part \mathcal{U}' at infinity of a tight irreducible avsp of $\text{PG}(6, q)$ of type $4q^3$ are satisfied. The remaining question is whether a solution of the extension problem for \mathcal{P} exists.

Conjecture 2 The extension problem for \mathcal{P} admits a solution for all prime powers q .

Theorem 2 shows the conjecture for q even. By computer we showed Conjecture 2 for $q = 3, 5$.

5.3 Constructions using hitting formulas

A *hitting formula* is a DNF such that each truth assignment to the underlying variables satisfies precisely one term [17]. For example:

$$(x \wedge y \wedge z) \vee (\bar{x} \wedge \bar{y} \wedge \bar{z}) \vee (\bar{x} \wedge y) \vee (\bar{y} \wedge z) \vee (\bar{z} \wedge x).$$

We say that a variable *appears* in the DNF if one of the two corresponding literals appears in one of the terms. The variables mentioned in the above DNF are x, y, z . We can represent hitting formulas over x_1, \dots, x_n as collections of strings in $\{0, 1, *\}^n$, where 0 in the i 'th position represents \bar{x}_i , 1 in the i 'th position represents x_i , and $*$ in the i 'th position represents the absence of x_i in the term. For example, the above hitting formula corresponds to the strings 111, 000, 01*, *01, 1*0.

This notion describes subcubes of affine points. Taking the projective closure we end up with the list

$$\langle 1111 \rangle, \langle 1000 \rangle, \langle 1010, 0001 \rangle, \langle 1001, 0100 \rangle, \langle 1100, 0010 \rangle$$

of subspaces of $\text{PG}(3, 2)$ that form an avsp, which obviously is not irreducible. However, we can join the first two elements to $\langle 1000, 0111 \rangle$ and obtain a tight irreducible avsp. While every string corresponds to an affine subspace, not every affine subspace corresponds to a

string. It turns out that any two strings having its stars at the same positions can be joined to an affine subspace. For brevity, we speak of *compression*. Interestingly enough, several tight irreducible avsp's of $\text{PG}(n - 1, 2)$ of the minimum possible size can be obtained by compression, see Sect. C in the appendix. More theoretical insights on the relations between hitting formulas and avsp's can be found in [9]—focusing on irreducible hitting formulas.

6 The minimum possible size of tight irreducible avsp's

We have discussed the minimum possible size of a (tight) avsp of $\text{PG}(n - 1, q)$ in Sect. 2. Before we consider the minimum possible size $\sigma_q(n)$ of a tight irreducible avsp \mathcal{U} of $\text{PG}(n - 1, q)$ we remark that Lemma 13 implies the upper bound $\#\mathcal{U} \leq 2^{n-2}$ for $q = 2$. The constructions mentioned in Sect. 5 suggest that this upper bound can be attained. In Sects. 3 and 4 we have determined the exact values $\sigma_q(2) = q, \sigma_q(3) = \infty, \sigma_2(4) = 4, \sigma_2(5) = 6, \sigma_2(6) = 7$, and $\sigma_2(7) = 8$.

Lemma 18 *In $\text{PG}(7, 2)$ no tight irreducible avsp of type $6^1 5^4 4^4$ exists.*

Proof Assume that \mathcal{U} is a tight irreducible avsp of type $6^1 5^4 4^4$ in $\text{PG}(7, 2)$. Consider $\mathcal{U}' := \{U \cap H_\infty \mid U \in \mathcal{U}\}$. From Lemmas 9 and 11 we conclude that the four planes in \mathcal{U}' share a common line L and that there is a unique configuration up to symmetry. Since the 5-space in \mathcal{U}' intersects each of the four planes in dimension at least 2, it also intersects L in dimension at least 1. We enumerate the possible configurations of the 5-space and the four planes in \mathcal{U}' up to symmetry. For each such configuration we build up a list of candidates for the four solids using the facts that the intersect the planes in dimension 1 or 2 and the 5-space in dimension at least 3 or 4. Next we consider a 4-subsets of those candidates whose dimensions of the pairwise intersections are contained in $\{2, 3\}$. We end up with a list of candidates for \mathcal{U}' . Here we can eliminate those which a common point or are not spanning, cf. Lemma 14. For each hyperplane H of H_∞ let $s := (s_3, s_4, s_5)$ be given by $s_i := \#\{U \in \mathcal{U}' \mid \dim(U) = i, U \leq H\}$. From Lemma 15 we can conclude that the following cases cannot occur:

- $s = (0, 1, 0)$;
- $s = (0, 3, 0)$;
- $s = (0, 0, 1)$;
- $s = (2, 3, 1)$;
- $s = (0, 1, 1)$.

For the remaining cases we have checked computationally that the extension problem does not admit a solution. □

Lemma 19 *In $\text{PG}(6, 2)$ every configuration \mathcal{U}' of type $5^2 4^2 3^4, 5^2 4^1 3^6, 5^1 4^6$, or 4^8 that satisfies the conditions of Lemmas 14, 15, and the dimension condition, cf. Lemma 4, admits a point P that is contained in all elements of \mathcal{U}' .*

Proof All cases have been excluded by ILP computations, cf. Sect. A for general model formulations. □

Corollary 10 *In $\text{PG}(7, 2)$ no tight irreducible avsp of the following types exist: $6^2 5^2 4^4, 6^2 5^1 4^6, 6^1 5^6, 6^1 5^4 4^4, 5^8$.*

Corollary 11 *The minimum size $\sigma_2(8)$ of an irreducible tight avsp of $\text{PG}(7, 2)$ is given by 10.*

For attaining examples we refer to Sect. C in the appendix.

Our next aim is a recursive construction which implies an asymptotic upper bound of roughly $\frac{3n}{2}$ for $\sigma_2(n)$.

Theorem 3 *Let $\mathcal{U} = \{U_1, \dots, U_r\}$ be an irreducible tight avsp of $\text{PG}(n - 1, 2)$ with $\dim(U_1) = n - 2$ and $n \geq 3$. Then, there exists an irreducible tight avsp \mathcal{U}' of $\text{PG}(n + 2 - 1, 2)$ of size $\#\mathcal{U} + 3 = r + 3$ that contains an element of dimension n .*

Proof Let $V = \text{PG}(n + 2 - 1, 2)$, H_∞ be the hyperplane at infinity, and $K \leq H_\infty$ be an arbitrary subspace with $\dim(K) = n$. With this, denote the two hyperplanes containing K and not being equal to H_∞ by H_1 and H_2 . Choose an arbitrary point $P \leq K$ and a subspace $K' \leq K$ such that $\dim(K') = n - 1$ and $\langle P, K' \rangle = K$. Now choose an irreducible tight avsp $\mathcal{U} = \{U_1, \dots, U_r\}$ of H_1/P such that $\dim(U_1) = n - 2$. We set $A_i := \langle U_i, P \rangle$ for all $1 \leq i \leq r$. Choose an n -space B with $B \cap H_1 = A_1$ and $B \not\leq H_\infty$, so that $C_1 := B \cap H_2$ is an $(n - 1)$ -space in H_2 with $C_1 \not\leq H_\infty$ and $P \leq C_1$. In H_2 choose three further $(n - 1)$ -spaces C_2, C_3, C_4 such that $\dim(C_i \cap C_j) = \dim(C_1 \cap C_2 \cap C_3 \cap C_4) = n - 3$ for all $1 \leq i < j \leq 4$, $C_1 \cap C_2 \cap C_3 \cap C_4 \leq K'$, and that $\{C_1, C_2, C_3, C_4\}$ forms an avsp of H_2 . (This boils down to an avsp of $\text{PG}(4 - 1, 2)$ of type 2^4 , which is a union of four disjoint lines.) Then,

$$\mathcal{U}' := \{A_2, \dots, A_r, B, C_2, C_3, C_4\}$$

is an irreducible tight avsp of V of size $\#\mathcal{U} + 3 = r + 3$. The size follows directly from the construction and $\dim(B) = n$. Since $B \cap H_1 \cap H_\infty = B \cap H_2 \cap H_\infty$ we have $B \cap C_2 \cap C_3 \cap C_4 = C_1 \cap C_2 \cap C_3 \cap C_4 \leq K'$ and $B \cap A_2 \cap \dots \cap A_r = A_1 \cap \dots \cap A_r = P$, so that \mathcal{U}' is tight. Noting that $\mathcal{U}'' := \{A_1, \dots, A_r\}$ is an avsp of H_1 , $\{C_1, \dots, C_4\}$ is an avsp of H_2 , and $\{A_1, C_1\}$ is an avsp of B , we conclude that \mathcal{U}' is indeed an avsp of V .

It remains to show that \mathcal{U}' is irreducible. So, assume that there exists a proper subset $\tilde{\mathcal{U}} \subsetneq \mathcal{U}'$ that can be joined to an x -space X . If $\tilde{\mathcal{U}} \cap \{B, C_2, C_3, C_4\} = \emptyset$, then we have $\tilde{\mathcal{U}} \subseteq \mathcal{U}''$ contradicting the fact that \mathcal{U}'' is irreducible. So, especially we have $x \in \{n, n + 1\}$. Noting that any two elements in $\{C_1, C_2, C_3, C_4\}$ span H_2 , we conclude $\#(\tilde{\mathcal{U}} \cap \{B, C_2, C_3, C_4\}) = 1$.

- (i) If $x = n$, then let $2 \leq i \leq 4$ be the unique index such $C_i \in \tilde{\mathcal{U}}$. Clearly, $B \notin \tilde{\mathcal{U}}$. Let \tilde{C} be the the other $(n - 1)$ space in X not contained in H_∞ and not equal to C_i with $\tilde{C} \cap H_\infty = C_i \cap \tilde{C}$, so that the elements of $\tilde{\mathcal{U}} \setminus \{C_i\}$ form a vector space partition of \tilde{C} . However, since $P \not\leq C_i$ and all elements in $\mathcal{U} \setminus \{B, C_1, C_2, C_3\}$ contain P , this is impossible.
- (ii) If $x = n + 1$ and $\#(\tilde{\mathcal{U}} \cap \{B, C_2, C_3, C_4\}) = 1$, then we have $\dim(X') = n$ for $X' := X \cap H_1$. If $B \in \tilde{\mathcal{U}}$, then $\tilde{\mathcal{U}} \setminus \{B\} \cup \{A_1\}$ can be joined to X' in H_1 , which is a contradiction. If $C_i \in \tilde{\mathcal{U}}$, then $\tilde{\mathcal{U}} \setminus \{C_i\}$ can be joined to X' in H_1 , which is also a contradiction.

Thus, \mathcal{U}' is irreducible. □

Corollary 12 *For each $n \geq 4$ an irreducible tight avsp \mathcal{U} of $\text{PG}(n - 1, 2)$ of size $\lfloor \frac{3n-3}{2} \rfloor$ exists.*

Proof For $n = 4$ there exists such an example with type 2^4 and for $n = 5$ there exists such an example with type $3^2 2^4$. Then, iteratively apply the construction from Theorem 3. □

We remark that the constructive upper bound for $q_2(n)$ is tight for $n \in \{4, 5, 6, 8\}$.

7 Conclusion

We have introduced the geometrical object of affine vector space partitions. To make their study interesting we need the additional conditions of tightness and irreducibility, which are natural in the context of hitting formulas. A very challenging problem is the determination of the minimum possible size of an irreducible tight avsp of $\text{PG}(n-1, q)$. To this end we have obtained some preliminary results for arbitrary field sizes but small dimensions and for the binary case with medium sized dimensions. We also gave a parametric construction that matches the known exact values in many cases. That irreducible tight avsp's are nice geometric objects can be e.g. seen at their sometimes large automorphism groups as well as the mentioned connection to the hyperbolic quadric $Q^+(5, q)$. While we have obtained a few insights, many questions remain open. So, we would like to close with a list of a few open problems:

1. Consider tight irreducible avsp's of $\text{PG}(4, q)$ of type $3^{m_3}2^{m_2}$. What is the largest possible value for m_3 ?
2. Determine a solution of the extension problem for the set \mathcal{P} of q^3 planes in $\text{PG}(5, q)$ obtained from the hyperbolic quadric $Q^+(5, q)$ for q odd, cf. Conjecture 2.
3. Determine further constructions for tight irreducible avsp's of $\text{PG}(n-1, q)$ with large automorphism groups.
4. Construct a tight irreducible avsp of $\text{PG}(n-1, q)$ of type $2^{q^{n-2}}$ for all $n \geq 5$.
5. Is it possible that a tight irreducible avsp of $\text{PG}(n-1, q)$ contains 1-dimensional elements if $n \geq 4$ and $q \geq 3$?
6. Determine further exact values of the minimum size $\sigma_q(n)$ of a tight irreducible avsp of $\text{PG}(n-1, q)$.
7. Determine $\lim_{n \rightarrow \infty} \sigma_q(n)/n$.
8. Is $\sigma_q(n)$ strictly increasing in n ?

Acknowledgements First of all we would like to thank the two anonymous referees for their careful reading and the many useful remarks that improved the presentation of the paper a lot. We thank Esmeralda Năstase, Artur Riazanov, and Yuriy V. Tarannikov for helpful discussions. Further thanks go to Tomáš Peitl and Stefan Szeider for sharing with us the results of the computer search reported in [26]. Ferdinand Ihringer and Sascha Kurz would like to thank the organizers of the Sixth Irsee Conference on Finite Geometries for their invitation. During that conference the idea of analyzing and introducing avsp's slowly evolved, being triggered by some open problems for hitting formulas. This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 802020-ERC-HARMONIC. Ferdinand Ihringer is supported by a postdoctoral fellowship of the Research Foundation – Flanders (FWO).

Funding Open Access funding enabled and organized by Projekt DEAL.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

A Integer linear programming formulations

Let \mathcal{U}' be an arbitrary set of subspaces of H_∞ in $\text{PG}(n - 1, q)$. For the question whether \mathcal{U}' can be extended to an avsp \mathcal{U} of $\text{PG}(n - 1, q)$ we utilize binary variables x_C for all subspaces C of $\text{PG}(n - 1, q)$ such that $C \not\leq H_\infty$ and $C \cap H_\infty \in \mathcal{U}'$ with the meaning $x_C = 1$ iff $C \in \mathcal{U}$. We denote the set of all of these subspaces by \mathcal{C} . For each point P in $\text{PG}(n - 1, q) \setminus H_\infty$ the equation

$$\sum_{C \in \mathcal{C}: P \leq C} x_C = 1 \tag{12}$$

and for each $U \in \mathcal{U}'$ the inequality

$$\sum_{C \in \mathcal{C}: U \leq C} x_C \geq 1 \tag{13}$$

has to be satisfied. (If we are only interested in irreducible avsp's, then we can require “=” in Inequality (13).) The 0/1 solutions of this equation system are in one-to-one correspondence to extensions of \mathcal{U}' to avsp's \mathcal{U} in $\text{PG}(n - 1, q)$.

Searching a tight irreducible avsp \mathcal{U} in $\text{PG}(n - 1, q)$ directly can be achieved by a similar model. Now let \mathcal{C} be the set of subspaces of $\text{PG}(n - 1, q)$ that are not incident with H_∞ . Again we use binary variables x_C for all $C \in \mathcal{C}$ with the meaning $x_C = 1$ iff $C \in \mathcal{U}$. Partitioning the affine points is modeled by

$$\sum_{C \in \mathcal{C}: P \leq C} x_C = 1 \tag{14}$$

for all points P not contained in H_∞ . The condition that \mathcal{U} is tight can be written as

$$\sum_{C \in \mathcal{C}: Q \leq C} x_C + 1 \leq \sum_{C \in \mathcal{C}} x_C \tag{15}$$

for all points $Q \leq H_\infty$. In order to model the condition that \mathcal{U} is irreducible we say that a subspace A escapes a subspace B if A has both points that are contained and points that are not contained in B . So, for each $B \in \mathcal{C}$ we require

$$x_B + \sum_{C \in \mathcal{C} \text{ such that } C \text{ escapes } B} x_C \geq 1, \tag{16}$$

i.e., either $B \in \mathcal{U}$ or there exists an element $C \in \mathcal{U}$ certifying that no subset of \mathcal{U} can be joined to B .

Of course we can fix the type of \mathcal{U} by additional equations. Using a target function we can minimize or maximize $\#\mathcal{U}$ as well as the number of i -dimensional elements. We have to mention that this ILP formulation comprises a lot of symmetry, so that it can be solved in reasonable time for small parameters n and q only. However, we can use the inherent symmetry to fix some of the x_C variables. I.e. the symmetry group acts transitively on the set of a -spaces that are not contained in H_∞ . For pairs of an a -space A and a b -space B that both are not contained in H_∞ , the different orbits under the action of the symmetry group are characterized by the invariant $\dim(A \cap B)$.

B Technical details

In order to keep the paper more readable, we have moved some technical details, that may also be left to the reader, to this section. The proof of Lemma 1 uses the numbers $m_i^{(j)}$ satisfying certain constraints. For completeness we state how those number can be computed in:

Input: $m_{n-1}, \dots, m_1 \in \mathbb{N}_0$ with $\sum_{i=1}^{n-1} m_i \cdot q^{i-1} = q^{n-1}$
Output: $m_i^{(j)} \in \mathbb{N}_0$ with $\sum_{i=1}^{n-2} m_i^{(j)} \cdot q^{i-1} = q^{n-2}$ for all $m_{n-1} + 1 \leq j \leq q$
 and $\sum_{j=m_{n-1}+1}^q m_i^{(j)} = m_i$ for all $1 \leq i \leq n-2$
 $h \leftarrow n-2$
for $m_{n-1} + 1 \leq j \leq q$ **do**
 $r \leftarrow q^{n-2}$
 while $r > 0$ **do**
 $t \leftarrow \min \{r/q^{h-1}, m_h\}$
 $m_h \leftarrow m_h - t$
 $r \leftarrow r - t \cdot q^{h-1}$
 if $t = 0$ **then**
 $h \leftarrow h - 1$
 end if
 end while
end for
return $m_i^{(j)}$

In the three subsequent lemmas we characterize 2-divisible sets in $\text{PG}(3, 2)$ of cardinality $s \in \{3, 6, 8\}$.

Lemma 20 *Let \mathcal{P} be a 2-divisible set of three points in $\text{PG}(3, 2)$ then \mathcal{P} forms a line.*

Proof Let $\mathcal{P} = \{P_1, P_2, P_3\}$ and $L := \langle P_1, P_2 \rangle$. Since all hyperplanes containing L have to contain \mathcal{P} , we have $P_3 \in L$. □

Lemma 21 *Let \mathcal{P} be a 2-divisible set of six points in $\text{PG}(3, 2)$ then \mathcal{P} is the disjoint union of two lines.*

Proof If H is a hyperplane containing all points of \mathcal{P} , then there is a unique point $P \leq H$ with $P \notin \mathcal{P}$. Since every hyperplane H' that does not contain P intersects \mathcal{P} in cardinality 3, so that this case cannot occur, i.e., \mathcal{P} is spanning. From the standard equations we compute $a_0 = 0$, $a_2 = 9$, and $a_4 = 6$ for the spectrum. From the MacWilliams transform for the corresponding linear code we conclude the existence of a triple of points \mathcal{P}' forming a line. Since $\mathcal{P} \setminus \mathcal{P}'$ is also 2-divisible the statement follows from Lemma 20. □

We remark that there exists a second 2-divisible set of six points—a projective base of dimension 5, which clearly cannot be embedded in $\text{PG}(3, 2)$.

Lemma 22 *Let \mathcal{P} be a 2-divisible set of eight points in $\text{PG}(3, 2)$ then \mathcal{P} is either an affine solid or given by the points of a plane and an intersecting line without the intersection point.*

Proof Assume that π is a hyperplane, which is a plane in our situation, containing six of the eight points and denote the unique uncovered point of π by P . Each hyperplane that is incident with P contains either two or six of the points in π . Thus, the remaining two points

form a line L containing P . Clearly, there is a unique example up to symmetry. Otherwise each hyperplane contains either 0, 2, or 4 points, so that the standard equations yield that there is a unique empty hyperplane and all other hyperplanes contain exactly four points, i.e., the point set is given by an affine solid. \square

We remark that both point sets can also be described as unions of two 2-divisible point sets, i.e., the union of two affine planes in the first case and the union of a line and a projective basis of size five in the second case.

C Tight irreducible affine vector space partitions of minimum size that can be obtained by compression

In Sect. 5.3 we have shown how avsp's of $PG(n - 1, 2)$ can be obtained from hitting formulas by compression. In [26] irreducible hitting formulas of minimum possible mentioning all variables where enumerated up to seven variables. Going over their list we obtain the following examples of tight irreducible avsp's that can be obtained by compression and that have the minimum possible size $\sigma_2(n)$, see Sect. 6. The pairs of strings that can be compressed to an affine subspace are separated by horizontal lines.

Examples for $n = 5$:

00 * *	00 * *
1 * 0*	1 * 0*
010*	01 * 0
1 * 10	1 * 10
<u>*111</u>	<u>*111</u>
0110	0101
1011	1011

Examples for $n = 6$:

00 * **	00 * **	00 * **
100 * *	1 * 00*	100 * *
* 100*	1 * 1 * 0	01 * 0*
1 * 1 * 0	* 101*	1 * 1 * 0
<u>*1 * 11</u>	<u>*11 * 1</u>	<u>*1 * 11</u>
101 * 1	0100*	01 * 10
<u>011 * 0</u>	<u>1001*</u>	<u>11 * 01</u>
* 1101	011 * 0	101 * 1
* 1010	101 * 1	110 * 0

For $n = 7$ there is a unique example:

000 * **
 10 * 0 * **
 * 1 * * 00
 * * 111*
*10 * * 1
 * 0110*
1101
 * 11 * 01
*10 * 10
 0010 * *
 1001 * *

For $n = 8$, there are 26 irreducible hitting formulas of size 13 mentioning all $n - 1 = 7$ variables. Curiously enough, compression was always successful. Moreover, we can also obtain tight irreducible avsp's of $PG(7, 2)$ of minimum size $\sigma_2(8) = 10$ by compression starting from an irreducible hitting formulas with strictly more than 13 terms:

* * * 1 * * 0	* 10 * * * *	1 * 10 * 1 *
* * * 10 * 1	* 000 * 0 *	0 * 11 * * 0
00 * 0 * 1 *	100 * * 1 *	111 * * 0 *
0 * 00 * 0 *	00 * 11 * *	<u>001 * * * 1</u>
1 * * * 1 * 1	<u>1 * 100 * *</u>	0010 * * 0
<u>*110 * * 0</u>	1001 * 0 *	<u>0111 * * 1</u>
* 010 * 00	<u>0000 * 1 *</u>	110 * 1 * *
<u>*100 * 10</u>	0 * 10 * * 1	<u>000 * 0 * *</u>
0 * * 11 * 1	<u>1 * 11 * * 0</u>	01 * 0 * * *
<u>1 * * 00 * 1</u>	0 * 1 * 0 * 0	<u>10 * 1 * * *</u>
1 * 00 * 00	<u>1 * 1 * 1 * 1</u>	0 * 011 * *
<u>0 * 10 * 01</u>	01111 * *	<u>1 * 000 * *</u>
10 * 0 * 10	<u>00010 * *</u>	1010 * 0 *
01 * 0 * 11	* * 101 * 0	<u>1111 * 1 *</u>
	* * 110 * 1	* 0001 * *
		* 1010 * *

References

1. Agievich S.V.: Bent rectangles. In: Boolean Functions in Cryptology and Information Security, volume 18 of NATO Science for Peace and Security Series D: Information and Communication Security (2008).
2. Bamberg J., Betten A., Cara P., De Beule J., Lavrauw M., Neunhöffer M.: FinInG—Finite Incidence Geometry, Version 1.4.1 (2018). <http://www.fining.org>.

3. Beutelspacher A.: Partial spreads in finite projective spaces and partial designs. *Mathematische Zeitschrift* **145**(3), 211–229 (1975).
4. De Beule J., Demeyer J., Mattheus S., Sziklai P.: On the cylinder conjecture. *Des. Codes Cryptogr.* **87**(4), 879–893 (2019).
5. Davydov G., Davydova I., Büning H.K.: An efficient algorithm for the minimal unsatisfiability problem for a subclass of CNF. *Ann. Math. Artif. Intell.* **23**(3–4), 229–245 (1998).
6. El-Zanati S.I., Heden O., Seelinger G.F., Sissokho P.A., Spence L.E., Eynden C.V.: Partitions of the 8-dimensional vector space over $\text{GF}(2)$. *J. Comb. Des.* **18**(6), 462–474 (2010).
7. El-Zanati S.I., Seelinger G.F., Sissokho P.A., Spence L.E., Eynden C.V.: On partitions of finite vector spaces of low dimension over $\text{GF}(2)$. *Discret. Math.* **309**(14), 4727–4735 (2009).
8. El-Zanati S.I., Seelinger G.F., Sissokho P.A., Spence L.E., Eynden C.V.: On λ -fold partitions of finite vector spaces and duality. *Discret. Math.* **311**(4), 307–318 (2011).
9. Filmus Y., Hirsch E., Kurz S., Ihringer F., Riazanov A., Smal A., Vinyals M.: Irreducible subcube partitions. arXiv preprint [arXiv:2212.14685](https://arxiv.org/abs/2212.14685) (2022).
10. Heden O.: Partitions of finite abelian groups. *Eur. J. Comb.* **7**(1), 11–25 (1986).
11. Heden O.: On the length of the tail of a vector space partition. *Discret. Math.* **309**(21), 6169–6180 (2009).
12. Heden O.: A survey of the different types of vector space partitions. *Discret. Math. Algorithms Appl.* **4**(01), 1250001 (2012).
13. Heinlein D., Honold T., Kiermaier M., Kurz S., Wassermann A.: Projective divisible binary codes. In: *The Tenth International Workshop on Coding and Cryptography 2017: WCC Proceedings*. IEEE Information Theory Society, Saint-Petersburg, September (2017). <https://eref.uni-bayreuth.de/40887/>.
14. Heinlein D., Honold T., Kiermaier M., Kurz S.: Generalized vector space partitions. *Australas. J. Comb.* **73**(1), 162–178 (2019).
15. Honold T., Kiermaier M., Kurz S.: Partial spreads and vector space partitions. In: *Network Coding and Subspace Designs*, pp. 131–170. Springer, Berlin (2018).
16. Honold T., Kiermaier M., Kurz S.: Classification of large partial plane spreads in $\text{PG}(6, 2)$ and related combinatorial objects. *J. Geom.* **110**(1), 1–31 (2019).
17. Iwama K.: CNF-satisfiability test by counting and polynomial average time. *SIAM J. Comput.* **18**(2), 385–391 (1989).
18. Kiermaier M., Kurz S.: On the lengths of divisible codes. *IEEE Trans. Inf. Theory* **66**(7), 4051–4060 (2020).
19. Kurz S., Mattheus S.: A generalization of the cylinder conjecture for divisible codes. *IEEE Trans. Inf. Theory* **68**(4), 2281–2289 (2021).
20. Kaski P., RJ Östergård P.: *Classification Algorithms for Codes and Designs*, vol. 15. Springer, Berlin (2006).
21. Kurz S.: Heden’s bound on the tail of a vector space partition. *Discret. Math.* **341**(12), 3447–3452 (2018).
22. Kurz S.: Divisible codes. arXiv preprint [arXiv:2112.11763](https://arxiv.org/abs/2112.11763) (2021).
23. Linton S.: Finding the smallest image of a set. In: *Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation*, pp 229–234 (2004).
24. Mateva Z.T., Topalova S.T.: Line spreads of $\text{PG}(5, 2)$. *J. Comb. Des.* **17**(1), 90–102 (2009).
25. Năstase E.L., Sissokho P.A.: The minimum size of a finite subspace partition. *Linear Algebra Appl.* **435**(6), 1213–1221 (2011).
26. Peitl T., Szeider S.: Are hitting formulas hard for resolution? (2022). [arXiv:2206.15225](https://arxiv.org/abs/2206.15225).
27. Sheekey J., Schmidt K.-U., Winterhof A.: MRD codes: constructions and connections. In: *Combinatorics and finite fields: difference sets, polynomials, pseudorandomness and applications*, vol. 23, pp. 255–286. de Gruyter (2019).
28. Tarannikov Y.V.: On the existence of Agievich-primitive partitions. *Diskretn. Anal. Issled. Oper. Ser. 1* **29**(4), 104–123 (2022).
29. Ziran T., Zeng X., Lei H.: Several classes of complete permutation polynomials. *Finite Fields Appl.* **25**, 182–193 (2014).