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# On the Evolution Equations of Nonlinearly Permissible, Coherent Hole Structures Propagating Persistently in Collisionless Plasmas

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**A fundamental study describing nonlinear plasma wave propagation is presented. Elementary linear wave theory describes small-amplitude random waves, but lacks information about coherent structures. This improved wave model arises from the fact that structure formation is inevitably associated with particle trapping, which can only be properly addressed by the pseudo-potential method instead of Bernstein, Greene, and Kruskal (BGK) - likemethods. Only by using this method can legitimate nonlinear dispersion relations be obtained and reconciled with trapping scenarios. This privilege is used to derive evolution equations for five structures, the derivation being simplified by the acoustic nature of the permitted modes. The focus is on a special structure, the solitary electron hole of negative polarity, with which it can explain a spacecraft observation for the first time. Furthermore, it is shown that an intrinsically nonlinear structure can become macroscopically linear and thus harmonic by suitably adjusting the trapping scenario. An example is the monochromatic ion acoustic wave that propagates at ion sound velocity without dispersion. In this literature research, it also takes a critical look at a recently awarded work.**

## 1. Introduction

A look behind the scenes of pattern formation in collisionless plasmas, more precisely in Vlasov-Poisson (VP) systems, still reveals some surprises and contradictions despite decades of research.[1,2] On the one hand, satellite measurements in near-Earth regions of space[3] show a wealth of coherent, long-lived structures, the recording and interpretation of which being the more advanced the better the current understanding of solitary stuctures is. On the other hand, many theoretical model builders who make the linear Vlasov dynamics the starting point of their investigations try in vain to come close to these patterns. Not only do they overlook the fact that in the process of structure formation, coherence, and particle trapping form an alliance without which persistent equilibria cannot exist, but also that the trapping of individual particles by a coherent wave is in principle stochastic in nature. The trajectories of resonant particles near the separatrix become ergodic, and with it the bundle of characteristics of the Vlasov equation. It follows immediately that Cauchy's initial value problem is ill-posed for VP systems, a dilemma the plasma community has yet to learn how to deal with.

In a recent review,[2] one of the authors of the present work achieved some clarification by putting coherence and particle trapping on the same footing and reflecting the chaotic nature of particle capture through different trapping scenarios. Only then can diversity and abundance, as observed, be fully taken into account. His investigations suggest a double paradigm shift: i) the transition from the linearized Vlasov spectra (discrete: Landau; continuous: van Kampen/Case) to the continuous spectra of hole equilibria and ii) the transition from the BGK[4] to the pseudopotential method.

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## 2. The Pseudo-Potential Method

The question we address in this article is how to use the second part of Schamel's pseudo-potential method[5] to set up evolution equations for long-lived electrostatic structures of the form $\phi(x, t)$ that self-consistently propagate in collision-free plasmas. This is of paramount interest for two reasons:

(i) such structures are observed ubiquitously in space and measured in laboratories, e.g.,[6] and
(ii) the standard wave theory, which seems to have been generally accepted since van Kampen and Landau, cannot give a satisfactory, i.e., complete answer.

Several requirements determine the path for such an undertaking.

1. The structure must be embedded in a plasma, i.e., when it disappears, the undisturbed plasma must come out.
2. Regardless of its strength, it must satisfy the Vlasov-Poisson system.
3. It must in some sense represent and take care of the chaotic behavior of the particle trajectories in the resonance range.
4. The multiplicity of solutions must also include the case of a stationary propagating wave $\phi(x - v_0 t)$, where $v_0$ is the phase velocity.
5. The coherence of the wave is directly related to the concept of particle trapping in the resonance range, i.e., both aspects are mutually dependent and cannot be treated independently.
6. The absence of Landau damping and, more generally, of linear Vlasov dynamics caused by corresponding distortions in the resonance range of $f_e$ and $f_i$, respectively, is a necessary prerequisite.
7. In general, the electric potential $\phi(x,t)$ can no longer be expressed by known functions and identified experimentally because of its almost unlimited form.

In the review article[2] a comprehensive theory of hole equilibria and several associated evolution equations were presented including the evolution equations of the KdV- or Schamel type.[7–10]

In[2] it was shown how a suitable equilibrium structure can be derived: starting from distributions that satisfy the corresponding Vlasov equation, one can get the densities and then derive a pseudo-potential $\mathcal{V}(\phi)$ with which one obtains self-consistent solutions of the VP system by solving Poisson's equation.

Whereas $\mathcal{V}(\phi)$ stands for the shape of the structure, the phase velocity $v_0$ satisfies a nonlinear dispersion relation (NDR).

An important aspect of finding an appropriate Vlasov solution is to consider different trapping scenarios to represent the many different ways a particle can be captured, a situation that reflects the chaotic resonant region.

The most general solution is in the small amplitude limit given in [2] by Equation (6), extended by an appropriate ionic expression derived in Section VII, for the NDR and by (7), again extended by an ionic term, for the pseudo-potential $\mathcal{V}(\phi)$.

In general, one has to deal with various trapping scenarios for each species, but for simplicity, we only consider the $\beta$ (and $\alpha$, respectively) trapping effect here, as in Schamel's earlier publications. It is assumed that $0 \leq \phi(x) \leq \psi << 1$, where $\psi$ is the small amplitude of the electric potential $\phi(x)$.

In the case that all trapping terms vanish except $(B_e(\beta), B_i(\alpha))$, the NDR and the pseudo-potential are given by

$$k_0^2 - \frac{1}{2} Z_r'\left(\frac{v_0}{\sqrt{2}}\right) - \frac{\theta}{2} Z_r'\left(\frac{u_0}{\sqrt{2}}\right) = B_e + \frac{3}{2}\theta^{\frac{3}{2}} B_i + q \tag{1}$$

$$-\frac{\mathcal{V}(\varphi)}{\psi^2} = \frac{k_0^2}{2}\varphi(1-\varphi) + B_e \frac{\varphi^2}{2}(1-\sqrt{\varphi})$$
$$+ B_i \frac{\theta^{\frac{3}{2}}}{2}\left[1 - (1-\varphi)^{5/2} - \frac{1}{2}\varphi(5-3\varphi)\right] + q\frac{\varphi^2}{2}(1-\varphi) \tag{2}$$

where $\varphi = \phi/\psi$, $u_0 = v_0\sqrt{\theta/\delta}$, $\theta = T_e/T_i$, $\delta = m_e/m_i$, $Z_r'(x)$ is the first derivative of the real part of the complex plasma dispersion function $Z(z)$ and $q = [\theta^2 Z_r'''(u_0/\sqrt{2}) - Z_r'''(v_0/\sqrt{2})]\psi/24$. The trapping parameters $B_e$, $B_i$ are defined later in the text. In the case of vanishing $q$ both Equations are identical with (51),(52) of [2]. The parameter $k_0$ is connected with the actual wavenumber $k$ by (9) in [2]. The normalized phase velocity is given by $v_0$. It has to be replaced by $\tilde{v}_D := v_D - v_0$ in case of a current-carrying plasma with a non-zero drift velocity $v_D$.

## 3. A New Approach to Establishing Evolution Equations

One way to find an evolution equation for acoustic wave structures that contains such an equilibrium solution is to combine two ZEROS, namely $\phi_t + v_0\phi_x = 0$ and $-\mathcal{V}''(\phi)\phi_x - \phi_{xxx} = 0$. The latter is derived from Poisson's equation $\phi''(x) = -\mathcal{V}'(\phi)$ and both are connected by a coupling constant c. The rationale for c and the addition of the two zeros is as follows: For a coherent wave one has phase locking, i.e., all Fourier harmonics must propagate in lowest order with the same phase velocity given by c, requiring an acoustic wave. The deviations due to dispersion and nonlinear effects are small and thus of a higher order which must be compensated for in order to arrive at a consistent theory in the next order. In the case of mathematically disclosed functions such as $n_e(x,t), n_i(x,t)$, or $\phi(x,t)$ with known scaling properties, this method is equivalent to the reductive perturbation method (see for instance Appendix A of [11]). However, for undisclosed functions, which is the typical situation when more trapping scenarios are involved, this is the only method available to get an evolution equation. The concept of undisclosed functions $\phi(x)$ was first introduced in [11] and reflects the fact that in the pseudo-potential method for inverting $x(\phi)$ to $\phi(x)$ an integral appears, which can no longer be solved by mathematically known functions. It is a consequence of the many trapping scenarios involved. For more details see [2, 11].

In addition to the arguments already mentioned, their legitimacy also results from the fact that only slow processes are admitted to appear on the macroscopic level. This excludes fast transient processes such as filamentation, folding, detrapping, and retrapping, separatrix crossings, or other violent relaxations since they are kinetic in nature. Furthermore, one must be aware that determinism is challenged due to the non-integrability of the single particle - coherent wave interaction problem. A secure way is therefore to concentrate on acoustic kinetic equilibrium solutions that are characterized by acoustic velocities $v_0$ and shapes given by $\mathcal{V}(\phi)$.

This method is therefore justified because it reproduces the result of the reductive perturbation method in cases where the latter is applicable. However, it is superior to reductive perturbation theory in that it applies to cases where the latter can no longer be used.

At the end we briefly go into the riddle that there are also kinetic non-acoustic VP solutions that cannot be covered by an evolution equation at least not by our method.

We hence have generally

$$\phi_t + v_0\phi_x + c[-\mathcal{V}''(\phi)\phi_x - \phi_{xxx}] = 0 \tag{3}$$

The selection of several trapping scenarios allows a large number of very diverse structures, which, however, are mostly undisclosed mathematically, since the potential $\phi(x)$ cannot be expressed anymore in mathematically known functions.[11]

## 4. The Five Most Relevant Evolution Equations

We restrict ourselves to the five most relevant special cases.

### 4.1. The Solitary Electron Hole

This branch is obtained by assuming $k_0 = 0$, $q = 0$, $B_i = 0$, $v_0 \approx O(1)$ and $|B_e| << 1$. Making use of the Taylor expansion of the $Z_r'(x)$ function: $-\frac{1}{2}Z_r'(v_0/\sqrt{2}) \approx\ (1.307 - |v_0|)/1.307$ and of $\frac{1}{2}Z_r'(u_0/\sqrt{2}) \approx \delta/\theta v_0^2 << 1$ we get $|v_0| = 1.307(1 - B_e)$, the slow electron acoustic branch. This mode is placed on both sides of the electron Maxwellian at a distance of 1.307 and it holds

$$-\mathcal{V}(\phi) = B_e\frac{\phi^2}{2}\left(1 - \sqrt{\frac{\phi}{\psi}}\right),$$

$$x(\phi) = \frac{4}{\sqrt{B_e}}\tanh^{-1}\left(\sqrt{1 - \sqrt{\frac{\phi}{\psi}}}\right),$$

$$\phi(x) = \psi\,\mathrm{sech}^4\left(\frac{\sqrt{B_e}}{4}x\right) \tag{4}$$

In this case c=1.307 and the evolution equation is found to be given by the Schamel equation[8–10]

$$\phi_t + 1.307\left(1 - B_e\frac{15}{8}\sqrt{\frac{\phi}{\psi}}\right)\phi_x - 1.307\phi_{xxx} = 0 \tag{5}$$

### 4.2. The Ion Acoustic Soliton

Utilizing the expansion $-\frac{1}{2}Z_r'(x) = 1 - 2x^2 + \cdots$ for $|x| << 1$ and $-\frac{1}{2}Z_r'(x) = -\frac{1}{2x^2}(1 + \frac{3}{2x^2} + \cdots)$ for $|x| >> 1$ we get in the solitary wave limit $k_0 = 0$ and with $B_e = 0 = B_i$ and $\theta >> 1$ a NLD of the form $1 - v_0^2 - \theta\frac{1}{u_0^2}(1 + \frac{3}{u_0^3} + \cdots .) = q$. It is solved in lowest order by $u_0 = \sqrt{\theta}[1 + \frac{1}{2}(q + \sqrt{\delta} + \frac{3}{\theta})] >> 1$ and $v_0 = \sqrt{\delta} << 1$ where $q = (\frac{\delta^2}{v_0^4} - \frac{1}{3})\psi = 2\psi/3 > 0$. In this case $c = \sqrt{\delta}$ and the evolution equation is found to be

$$\frac{1}{\sqrt{\delta}}\phi_t + \left(1 + \phi\right)\phi_x + \frac{1}{2}\phi_{xxx} = 0 \tag{6}$$

which becomes after the renormalization of the time ($\sqrt{\delta}t \to t$) the famous KdV equation.[7]

This soliton is therefore characterized by

$$-\mathcal{V}(\phi) = q\frac{\phi^2}{2}\left(1 - \frac{\phi}{\psi}\right), \qquad x(\phi) = \frac{2}{\sqrt{q}}\tanh^{-1}\sqrt{1 - \frac{\phi}{\psi}},$$

$$\phi(x) = \psi\,\mathrm{sech}^2\left(\frac{\sqrt{q}x}{2}\right) \tag{7}$$

For this wave to exist, the pair $(B_e, B_i)$ must vanish. The absence of $(B_e, B_i)$ (and of course of Landau damping) is guaranteed by setting the two quantities $B_e := \frac{16}{15}\frac{1}{\sqrt{\pi}}(1 - \beta - v_0^2)e^{-v_0^2/2}\sqrt{\psi}$ and $B_i := \frac{16}{15}\frac{1}{\sqrt{\pi}}(1 - \alpha - u_0^2)e^{-u_0^2/2}\sqrt{\psi}$, respectively, equal to zero (for more details see [2]). With $v_0 = \sqrt{\delta}$ and $u_0 = \sqrt{\theta}$ the trapped region is therefore distorted by $\beta = 1 - \delta \approx 1$. The electron distribution in the trapped region is only slightly distorted from the Maxwell distribution, while the ion distribution shows a remarkable dip near the ion sound velocity in the high-energy tail.[12]

Next we consider the simple but nonetheless interesting single wave case.

### 4.3. The Monochromatic Ion Acoustic Wave Pattern

In this case $k_0 = k$, q=0 and $(B_e = 0, B_i = 0)$. The wave solution is then given by

$$-\mathcal{V}(\phi) = \frac{k^2}{2}\phi(\psi - \phi) \qquad x(\phi) = \frac{1}{k}\left[\frac{\pi}{2} + \sin^{-1}\left(1 - \frac{2\phi}{\psi}\right)\right]$$

$$\phi(x) = \frac{\psi}{2}[1 + \cos(kx)] \tag{8}$$

The coupling constant is in the ion acoustic range again $c = \sqrt{\delta}$ and the evolution equation degenerates to

$$\frac{1}{\sqrt{\delta}}\phi_t + (1 - k^2)\phi_x - \phi_{xxx} = 0 \tag{9}$$

which is identically satisfied by $\phi(x - v_0t)$ of (8) since it holds $v_0 = \sqrt{\delta}$.

Note that a nonlinearly correct, monochromatic ion acoustic wave structure moves with exactly $v_0 = \sqrt{\delta}$, i.e., it shows no dispersion. It satisfies the “thumb-teardrop” dispersion relation, which is strictly nonlinear. The fact that Landau and van Kampen/Case equilibria actually obey the same equation and thus could be equally well used oversees that they only obey the linear Vlasov but not the full Vlasov equation.[13,14] To say that this mode is the ordinary ion acoustic wave structure and therefore well-known and not new misjudges the nonlinear background. A monochromatic wave structure is subject to the nonlinear particle-wave interaction as well, which leads to ergodic particle trajectories (see also [15]) in the resonance region of phase space, i.e., near the separatrix, which is manifest in the various trapping scenarios.

As shown in [2], and earlier papers cited therein, this harmonic mode is marginally stable with respect linear perturbations and hence lacks Landau damping (and Landau growth, respectively, in case of a nonzero linearly supercritical drift velocity or current).[16]

Note that $(B_e = 0, B_i = 0)$ does not mean the absence of trapping but only the absence of these parameters.

Macroscopically the evolution equation is linear, but microscopically, i.e., on the kinetic level, the nonlinearity is in the spirit of the trapping nonlinearity mandatory. Long-lived structures, as seen in collision-free plasmas, can only be constructed by suitable distortions of $f_e$ and $f_i$ in the resonance range. This implies

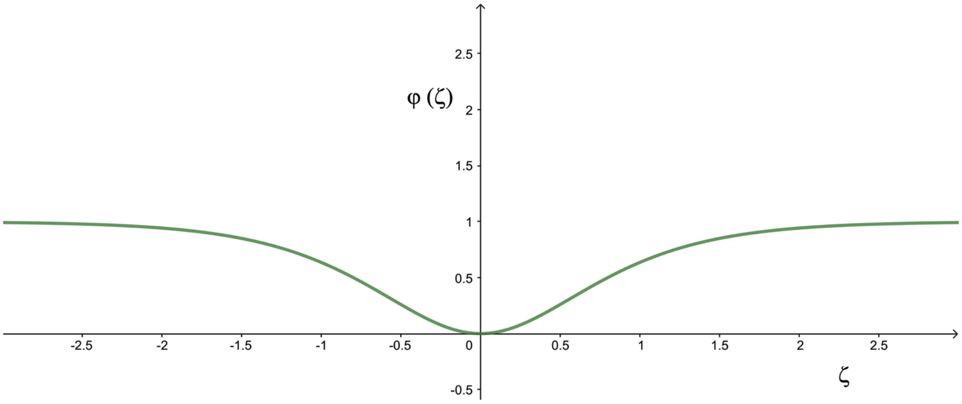


**Figure 1.** Solitary electron hole of negative polarity $\varphi(\zeta)$ as a function of $\zeta$

that only fully nonlinear solutions of the Vlasov-Poisson system are allowed.

### 4.4. The Solitary Electron Hole of Opposite Polarity

A remarkable structure, almost unknown in the literature, is a single solitary electron hole in which $\phi(x)$ is dip rather hump like. This special structure is obtained in the limit of $q = 0 = B_i$ by setting $2k_0^2 = -B_e > 0$ (see Section 4.1 of [2]). The pseudo-potential then becomes

$$-\frac{\mathcal{V}(\varphi)}{\psi^2} = -\frac{B_e}{4}\varphi(1-\sqrt{\varphi})(1+\sqrt{\varphi}-2\varphi) \tag{10}$$

This expression shows that $\mathcal{V}(\varphi)$ has a double zero at $\varphi = 1$ and a single zero at $\varphi = 0$. It hence represents an inverted solitary wave structure with $\varphi = 0$ at $x = 0$ and $\varphi = 1$ for $|x| \to \infty$ (see **Figure 1** of [2]). Its shape is given by

$$\varphi(\zeta) = \frac{1}{4}[3\tanh^2(|\zeta| + \zeta_0) - 1]^2 \tag{11}$$

where $\zeta := \frac{-\sqrt{3B_e}}{8}x$ and $\zeta_0 = \tanh^{-1}(1/\sqrt{3}) = 0.65848$. In Figure 1 we plot $\varphi(\zeta)$ as a function of $\zeta$.

The phase velocity at the slow electron acoustic speed is $v_0 = 1.307(1-\frac{3}{2}B_e)$ and the evolution equation is identical to (5) but with a different sign of $B_e$. Consequently, the electron distribution function in the resonant trapped particle range is now less depressed or even hump-like ($\beta > -0.71$).

As an example, we present in **Figure 2** the relevant distribution function of electrons at $\phi = \psi$ in the wave frame for the consistent set of parameters $B_e = -1/4, k_0^2 = 1/8, v_0 = 1.8, \psi = 0.1, \beta = 4.35, v_D = 0$. As before, the remaining trapping parameters $\gamma, \chi_1, \chi_2$, and $\zeta$ have been set zero. The underlying distribution is (1) of [2] and is given by

$$f_e(v) = \frac{1+\frac{k_0^2\psi}{2}}{\sqrt{2\pi}}\left\{\begin{array}{ll} e^{-(\sqrt{v^2-2\psi}+v_0)^2/2} & \text{for } v > \sqrt{2\psi} \\ e^{-v_0^2/2}[1-\beta(\frac{v^2}{2}-\psi)] & \text{for } -\sqrt{2\psi} \le v \le +\sqrt{2\psi} \\ e^{-(-\sqrt{v^2-2\psi}+v_0)^2/2} & \text{for } v < -\sqrt{2\psi} \end{array}\right\} \tag{12}$$

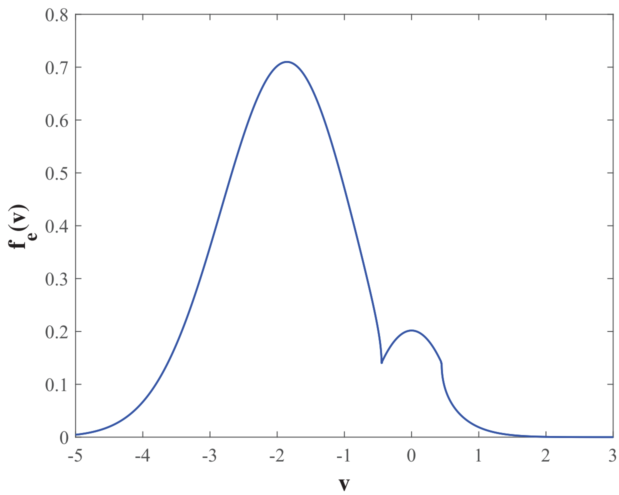


**Figure 2.** The distribution $f_e(v)$ at $\varphi = 1$ or $|x| = \infty$ as a function of $v$.

We see that a hump-like trapped electron distribution is responsible for this particular solution. An analogous mode structure is obtained on the opposite side of the Maxwellian peak, since the solution is independent of the sign of $v_0$.

As an isolated structure propagating in a quiescent plasma, it may be less relevant since the perturbation must extend to infinity, which in the case of asymmetric potentials would even force different asymptotic regions.[17] However, as the experimental observations[3] suggest, immersed in a more turbulent environment, it can meet the conditions for its existence. It can also be the origin of incoherent Langmuir wave trapping and give rise to a so-called envelope soliton ([18–20]), which is another intricate coherent plasma wave structure involving both particle and wave trapping.

### 4.5. The Solitary Ion Hole

This negatively polarized potential structure, which is well localized, is obtained by assuming $k_0^2 = \frac{1}{2}(\theta^{3/2}B_i - B_e)$,[21] $q = 0$, and $v_0 << 1$. In case of Boltzmann electrons, when $B_e = 0, \beta = 1$, we get $u_0 = 1.307(1+\frac{1}{\theta}-\sqrt{\theta}B_i)$. Moreover applying the shift

$\varphi \to \hat{\varphi} := \varphi$ - 1, where $-1 \leq \hat{\varphi} \leq 0$, we find

$$-\frac{\mathcal{V}(\hat{\varphi})}{\psi^2} = \frac{B_i}{2}\theta^{3/2}\hat{\varphi}^2(1-\sqrt{-\hat{\varphi}}),$$

$$\hat{\varphi}(x) = -\mathrm{sech}^4\left(\frac{\sqrt{B_i\theta^{3/2}}}{4}x\right) \tag{13}$$

which agrees with (8) of [22] or (23) of [23]. The coupling constant $c$ is in the slow ion acoustic range, $c = 1.307\sqrt{\delta/\theta}$, and the corresponding evolution equation reads

$$\sqrt{\frac{\theta}{\delta}}\hat{\varphi}_t + 1.307\Big(\mathcal{A} - B_i\theta^{3/2}\frac{15}{8}\sqrt{-\hat{\varphi}}\Big)\hat{\varphi}_x - 1.307\hat{\varphi}_{xxx} = 0 \tag{14}$$

which is again of Schamel type. The constant $\mathcal{A}$ is given by $\mathcal{A} := 1 + \frac{1}{\theta} + B_i\sqrt{\theta}(\theta - 1)$. It can easily be checked, e.g., by means of the pseudo-potential formalism, that the stationary solutions of (14), $\hat{\varphi}(x - v_0 t)$, coincide with $\hat{\varphi}$ of (13) and the corresponding phase velocity $v_0 = \sqrt{\frac{\delta}{\theta}}u_0 = \sqrt{\frac{\delta}{\theta}}1.307(1 + \frac{1}{\theta} - \sqrt{\theta}B_i)$.

Using this procedure, it is straightforward to construct for acoustic modes further evolution equations e.g., for ion holes of positive polarity and/or for solitary holes that are based on the $D_1$ (and/or $D_2$, respectively) trapping mechanism that is responsible for the well-known Gaussian (or second order Gaussian, respectively) solitary mode (see [2]). For the latter case, as shown in (A4) of Appendix B of [11], the nonlinear term in the evolution equation is extended by a $\ln\varphi$ and a $\ln^2\varphi$ term in addition to the $\sqrt{\varphi}$ term (see also (42) of [11]).

## 5. A First Application: The Negative Potential Pulses Observed in Space

Last but not least, as a direct application, we offer an explanation of a series of electrostatic solitary waves found by multispacecraft observations in the magnetosheath.[3] These structures, propagating close to the thermal electron velocity and below, exhibited a negative electrostatic potential for which no consistent description could be presented by the authors. Their expectation was that "a kinetic pseudo-potential method used with a specific distribution function to solve for a steady-state BGK solution" might be a viable approach.

Here, we confirm their expectation by the following consideration.

We use the negative solitary potential found in Section IV.4, but include the further trapping scenario $\Gamma_e$ of [2]. A negative potential is then obtained by shifting $\varphi$ by -1, which implies a replacing of $\varphi$ through $\varphi := 1 + \hat{\varphi}$ by 1+$\hat{\varphi}$ in (10) so that we get

$$-\frac{\mathcal{V}(\hat{\varphi})}{\psi^2} = -\frac{B_e}{4}(1+\hat{\varphi})(1-\sqrt{(1+\hat{\varphi})})(1+\sqrt{(1+\hat{\varphi})}-2(1+\hat{\varphi})) \tag{15}$$

where $-1 \leq \hat{\varphi} \leq 0$. Its phase speed, complemented by the $\Gamma_e$ effect, becomes

$$v_0 = 1.307(1 + \Gamma_e - \frac{3}{2}B_e) \tag{16}$$

(see Section 4.1 of [2], Equation (28), in which $2k_0^2$ has to be replaced by $-B_e$). We see a high degree of flexibility in the phase velocity since $\Gamma_e$ can take either sign. The evolution equation underlying the structure is again identical to (5), where 1 is to be replaced by 1+$\Gamma_e$ and $\sqrt{\varphi}$ by $\sqrt{(1+\hat{\varphi})}$ and where $B_e$ takes now negative values. It can be re-derived by applying our method and using (15) and (16).

In our last step we prove that the electron density in the center of such a structure is increased corresponding to a negativley charged core.[3]

To show this, we return to our first notation $\varphi$, with $0 \leq \varphi \leq 1$, in which the electron density is expressed by

$$\frac{n_e(\varphi) - 1}{\psi} = \frac{k_0^2}{2} + \left[\Gamma_e - \frac{1}{2}Z_r'(\frac{v_0}{\sqrt{2}})\right]\varphi - \frac{5}{4}B_e\varphi^{3/2} \tag{17}$$

(see (2) of [24] or (2) in [2]). To get a useful expression for the big bracket in (17) we use the NDR, which reads

$$k_0^2 + \left[\Gamma_e - \frac{1}{2}Z_r'(\frac{v_0}{\sqrt{2}})\right] - \frac{\theta}{2}Z_r'(\frac{u_0}{\sqrt{2}}) = B_e \tag{18}$$

It is solved in the SEAW limit (and immobile ions: $\theta = 0$) by (16). By use of $2k_0^2 = -B_e$ we can express the big bracket in this limit by

$$\left[\Gamma_e - \frac{1}{2}Z_r'(\frac{v_0}{\sqrt{2}})\right] = \frac{3}{2}B_e \tag{19}$$

and insert it into (17) to get

$$\frac{n_e(\varphi) - 1}{\psi} = \frac{-B_e}{4}(1 - 6\varphi + 5\varphi^{3/2}) \tag{20}$$

Since $-B_e > 0$ it follows that this expression is positive for $\varphi = 0$ corresponding to a centrally enhanced density peak and zero for $\varphi = 1$, which stands for the constant density in the solitary wave limit. A negative solitary potential has of course a positive curvature at the minimum ($\varphi'' = \frac{n_e - 1}{\psi} > 0$) and a zero curvature at infinity.

This is true as long as electron trapping dominates.

For an even lower $v_0$ in the ion thermal range, $B_i$ is no longer negligible and one then has to look for a two-parametric ($B_e$, $B_i$) solution (see Section 4 of [2]), which in the limit of $B_e = 0$ becomes the ordinary solitary ion hole Section 4.5 ).

Unfortunately, the experimental data are too sparse to perform a more detailed analysis. At least a measurement of the distribution of trapped particles along with an available phase velocity related to the plasma rather than the spacecraft would be a necessary requirement. Also, to what extent ions contribute to the trapping nonlinearity should be an important issue. Experimenters are therefore encouraged to take this next step and dig deeper into the kinetic world to contribute to a better understanding of pattern formation in collisionless plasmas.

## 6. Finite Amplitude Extensions

Often, finite amplitudes are encountered. In this case, the present theory loses its validity because the underlying Taylor expansions are no longer possible. Instead, one has to go back to the early work of Bujarbarua and Schamel,[17,23,25–27] where the appropriate extensions in the case of $\beta$ trapping scenario for electrons and $\alpha$ trapping scenario for ions were worked out. Although the formalism remains the same, new functions found in these papers are involved. As seen in (5a),(5b) of [25] both densities, $n_e$ and $n_i$, can be expressed by the functions $I(x)$, $\mathcal{K}(x,y)$, and $T_{+,-}(\beta,y)$ that are the appropriate extensions of our earlier functions. The subscript $+,-$ in the trapped particle contribution $T_{+,-}(\beta,y)$ refers to whether the first argument $\beta$ is positive or negative, where a positive $\beta$ indicates a hump-like distribution of the trapped electrons. By use of hump-like trapped particles, $\beta>0, \alpha>0$, the authors in [27] were able to achieve a new regime for double layers, the so-called strong double layer, which doesn't possess a small amplitude limit being entirely restricted to finite amplitudes. It is clear that such an extension to finite amplitudes, supplemented by the additional trapping scenarios, will increase the variety of solutions immeasurably and definitely keep up with the experimental variety.

It goes without saying that the spatiotemporal behavior of structures of finite amplitude cannot be addressed by an evolution equation, although they also are characterized by a pseudo-potential $\mathcal{V}(\phi)$ and a nonlinear dispersion relation NDR (see (14),(17) of [25] or (8),(10) of [23]). In the existence diagram of solitary electron holes of finite amplitude (see Figure 3 of [23] or Figure 8 of [17]) hence only the region around the line $v_0 = 1.307$ and $\psi << 1$ can be directly linked with an evolution equation.

## 7. A Selected Literature Search

Finally, after being asked to update the manuscript to reflect recent publications, we are responding in two ways. We cover articles dealing with the pseudo-potential method and the Schamel equation (S equation),[8] and also take a slightly more critical look at work that has received a lot of attention lately. First of all, we note that the number of publications on electrostatic structures has become unmanageably large, making a serious study of them difficult, if not impossible, at least for us. In our focus, we can see some trends and relate the work of some other groups to the current work.

The effect of an inhomogeneous magnetic field on an electron hole was investigated in [28] using the pseudo-potential method. They show that if a hole propagates into the area with a stronger magnetic field, it grows. If it propagates along a positive plasma density gradient, it will instead be accelerated[29] and constricted.

Kar et al.[30] reported on a possible excitation of solitary electron holes in a laboratory plasma, whose propagation speed agrees with the theoretical one. When using a dielectric covered metallic (instead of a purely metallic) electrode,[31] solitary ion holes could also be observed. Mathematical and background plasma aspects related to the S-equation have been treated in a number of papers ([32–35]). The pseudo-potential method was used to derive the electron hole stucture in a superthermal plasma having a $\kappa$ distribution with singularities in [36], and a regularized $\kappa$ distribution in [37], the latter of which presenting an illustration of the Schamel distribution.

Most publications, however, focus on the BGK method and use mathematically known profiles $\phi(x)$, which limits the solution space. In particular, the phase velocity $v_0$ remains an often unsolved or wrongly solved problem.

This brings us to part two and a critical look at Hutchinson's work. In his award-winning review article[38] he compares the pseudo-potential method with the BGK method for single electron holes ($k_0 = 0$), but does not adequately acknowledge their differences. In particular, he sees no need for an independent derivation of a NDR depending on the trapping scenarios, which must be carried within a BGK analysis in a second step using the pseudo-potential method. Of course, it would be better to start directly with the pseudo-potential method. His work therefore focuses mainly on the structural part of a pattern, essentially ignoring the second, equally important aspect, the treatment of phase velocity as a function of trapping scenarios. Apart from that, he keeps making mistakes in his calculations.[2]

The real situation is much more delicate because a macroscopic measurement e.g., of $\phi(x)$ is insufficient for what is happening in the microscopic phase space and even a measurement of the trapped particles does not enforce unambiguousness, as shown in [39, 40]. Moreover, undisclosed solitary electron holes $\phi(x)$, which can no longer be expressed mathematically,[41] as well as periodic, cnoidal waves inclusively the present ($k_0 \neq 0$) solitary wave are also not discussed by him.

His error in [42] is particularly striking, when he actually claims that the conditions for the existence of ion sound waves and ion holes, respectively, are not $\theta > 3.5$, as is found in the literature, but the opposite $\theta < 3.5$. That hot electrons, not hot ions, favor the existence of these waves can be easily seen by looking at the NDR in its simplest form. We neglect for simplicity in (1) the right hand side and get: $k_0^2 - \frac{1}{2}Z_r'(\frac{v_0}{\sqrt{2}}) - \frac{\theta}{2}Z_r'(\frac{u_0}{\sqrt{2}}) = 0$, which is sometimes called the "on-dispersion case".[43] For ion acoustic waves it holds $v_0^2 \approx \delta << 1$ and $u_0^2 = \theta$ such that, using $-\frac{1}{2}Z_r'(\frac{v_0}{\sqrt{2}}) \approx 1$, one gets:$-\frac{1}{2}Z_r'(\frac{u_0}{\sqrt{2}}) \approx \frac{-1}{\theta}(k_0^2+1)$. This is essentially (14) of his paper and becomes in the long wave limit: $-\frac{1}{2}Z_r'(\frac{u_0}{\sqrt{2}}) \approx \frac{-1}{\theta}$, which has to be solved. This quantity has to be negative and be placed between -0.285, the minimum of the left hand side, and 0 (see Figure 1 of [2]). It then follows $-\theta^{-1} > -0.285$ or $\theta^{-1} < 0.285$ or $\theta > \frac{1}{0.285} = 3.5$. In an equal temperature plasma $\theta = 1$ the propagation of these waves is not possible, which every plasma physicist learns at the beginning of his training. And of course the same holds for ion holes with $u_0 = 1.307$.[22]

Moreover, asymmetric solitary holes have already been mentioned by [17], and ultra-slow solitary electron holes, $u_0 << 1$, have already been found in many earlier papers of Schamel et al. without the requirement of a second hump in the ion distribution as claimed by him in [44]. By the way, in a plasma with two ion peaks, the hole can settle even in the high-energy tail of the second hump because high velocities often lower its energy and increase its attractive property as a negative energy hole.[2,24,45–49] Therefore, neither linear wave theories nor BGK-like nonlinear theories can adequately treat this phenomenon, which is controlled by the different trapping scenarios and therefore the pseudo-potential method is a necessary requisite.

To mention also a possibly positive innovation, we note that Hutchinson in [42] generalized the trapping scenario of (1) in [2] by another term, namely $(-\epsilon)^{1/2+2/l}$, l integer, e.g., l= 2, 4, 8, in the trapped ion (electron) distribution, where $\epsilon = u^2/2 - \theta(\psi - \phi(x))$ ($\epsilon = v^2/2 - \phi(x)$) that correlates with the potential $|\varphi(x)| = \mathrm{sech}^l(x)$.

For $l = 4$ we can see under which circumstances his solution is obtained by the pseudo-potential method, namely by setting $\Gamma_e + \Gamma_i = \frac{3}{2}\theta^{\frac{3}{2}} B_i$, where we have included for the most general case (see [2]) the two Gamma trapping terms on the left hand side of the NDR (1) and where we have set $B_e = 0 = q$. However, the fact that there is also an "off-dispersion excitation" with $B_i > 0$ and $\Gamma_{i,e}$ arbitrary but with the same electric potential remains unrecognized or is at least not commented by him. Such a discussion could potentially shed more light on the limitations of the BGK method used.

In summary, many of his ideas are still influenced by linear wave theory, which, however, is misplaced in this area. Even using a BGK-like theory is not enough. He sees no need for the pseudo-potential method to fully exploit the spectrum of phase velocities. No wonder, then, when invalid results such as "ultraslow velocities exist only at the center of a double-humped ion distribution" are obtained.

We also regret that there are many other publications worth reading that we could not include in our literature research due to time constraints, and also point out that the earlier but less precise notation "modified KdV equation", introduced by HS in [8], is still used in current publications instead of the more explicit notation "S equation".

## 8. Conclusion

A key issue remains finally to be discussed. Are there also evolutionary equations for the countless number of small amplitude equilibrium structures apart from the acoustic limit? A simple example is the harmonic (single) mode that satisfies the "Thumb-Teardrop" dispersion relation for $k = O(1)$ rather than for $|k| << 1$. An answer to this question is left to future research. However, there are good reasons to be suspicious of the existence and validity of such evolutionary equations. One reason is that the macroscopic and even the kinetic-microscopic description is overwhelmed to deal with such structural kinetic events, since correlations in the separatrix region of phase space become important to control and stabilize the trapping dynamics.[2]

In summary, we have studied dynamical evolution equations in Vlasov-Poisson systems that are centered around equilibria or, in the language of nonlinear dynamics, around fixed points. This new, rather quiescent plasma state is nonlinear and either appears after the passage of a more violent linear evolution (e.g., linear two-stream instability or nonlinear Landau damping) or is directly initiated by local nuclei that are ubiquitous in driven, actual plasmas.[24,47–49] In this state, particle trapping and coherence have entered during phase locking into a mutually dependent relationship that controls a dynamics. It lets you completely forget the linear world and comes up with an unlimited wealth of structures and phase velocities. Under certain circumstances - more precisely under special trapping conditions - this can lead to wave phenomena that are macroscopically harmonic and linear, but nevertheless microscopically, i.e., intrinsically nonlinear.

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## Conflict of Interest

The authors declare no conflict of interest.

## Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

## Keywords

anomalous transport, electron holes, intermittent turbulence, phase space structures, ion holes

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[1] This article is an abridged version of the review article,[2] enriched by an application and two further sections. It is intended for readers with a basic knowledge of the pseudo-potential method who wish to become familiar with the modern theory of pattern formation in collisionless plasmas. It focuses on the derivation of evolutionary equations, but also serves to guide through the wide variety of electrostatic structures and the proper treatment of phase velocities.

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