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On α-points of q-analogs of the Fano plane

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Abstract
Arguably, the most important open problem in the theory of q-analogs of designs is the question regarding the existence of a q-analog D of the Fano plane. As of today, it remains undecided for every single prime power order q of the base field. A point P is called an α-point of D if the derived design of D in P is a geometric spread. In 1996, Simon Thomas has shown that there always exists a non-α-point. For the binary case $q = 2$, Olof Heden and Papa Sissokho have improved this result in 2016 by showing that the non-α-points must form a blocking set with respect to the hyperplanes. In this article, we show that a hyperplane consisting only of α-points implies the existence of a partition of the symplectic generalized quadrangle $W(q)$ into spreads. As a consequence, the statement of Heden and Sissokho is generalized to all primes q and all even values of q.

Keywords Subspace design · q-analog · Fano plane · Steiner system · Subspace code

Mathematics Subject Classification 05B05 · 05A30 · 51E10 · 51E12 · 94B60

1 Introduction

Due to the connection to network coding, the theory of subspace designs has gained a lot of interest recently. Subspace designs are the q-analogs of combinatorial designs and arise by replacing the subset lattice of the finite ambient set V by the subspace lattice of a finite ambient vector space V. Arguably the most important open problem in this field is the question regarding the existence of a q-analog of the Fano plane, which is a subspace design with the parameters $2\text{-}(7, 3, 1)_q$. This problem has already been stated in 1972 by Ray-Chaudhuri [3, Problem 28]. Despite considerable investigations, its existence remains undecided for every single order q of the base field.

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A q-analog of the Fano plane would be a $[7, 4; 3]_q$ constant dimension subspace code of size $q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1$. However, the hitherto best known sizes of such constant dimension subspace codes still leave considerable gaps, namely 333 vs. 381 in the binary case [14] and 6978 vs. 7651 in the ternary case [16].[1] Furthermore, it has been shown that the smallest instance $q = 2$, the binary q-analog of a Fano plane, can have at most a single nontrivial automorphism [5, 20].

Another approach has been the investigation of the derived designs of a putative q-analog D of the Fano plane. A derived design exists for each point $P \in \mathrm{PG}(6, q)$ and is always a q-design with the parameters $1\text{-}(6, 2, 1)_q$, which is the same as a line spread of $\mathrm{PG}(5, q)$. Following the notation of [13], a point P is called an α-point of D if the derived design in P is the geometric spread, which is the most symmetric and natural one among the line spreads of $\mathrm{PG}(5, q)$. For highest possible regularity, one would expect all points to be α-points.

However, this has been shown to be impossible, as there must always be at least one non-α-point of D [28]. For the binary case $q = 2$, this result has been improved to the statement that each hyperplane contains at least one non-α-point [13]. In other words, the non-α-points of a binary q-analog of the Fano plane form a blocking set with respect to the hyperplanes.

In this article, α-points will be investigated for general values of q, which leads to the following theorem.

Theorem 1 *Let D be a q-analog of the Fano plane and assume that there exists a hyperplane H such that all points of H are α-points of D. Then the following equivalent statements hold:*

(a) *The line set of the symplectic generalized quadrangle $W(q)$ is partitionable into spreads.*
(b) *The point set of the parabolic quadric $Q(4, q)$ is partitionable into ovoids.*

As a consequence, we get the following generalization of the result of [13].

Theorem 2 *Let D be a q-analog of the Fano plane and q be prime or even. Then each hyperplane contains a non-α-point. In other words, the non-α-points form a blocking set with respect to the hyperplanes.*

2 Preliminaries

Throughout the article, $q \neq 1$ is a prime power and V is a vector space over $\mathbb{F}_q$ of finite dimension v.

2.1 The subspace lattice

For simplicity, a subspace U of V of dimension $\dim_{\mathbb{F}_q}(U) = k$ will be called a *k-subspace*. The set of all k-subspaces of V is called the *Graßmannian* and will be denoted by $\begin{bmatrix} V \\ k \end{bmatrix}_q$. Picking the "best of two worlds", we will prefer the algebraic dimension $\dim_{\mathbb{F}_q}(U)$ over the geometric dimension $\dim_{\mathbb{F}_q}(U) - 1$, but we will otherwise make heavy use of geometric notions, such as calling the 1-subspaces of V *points*, the 2-subspaces *lines*, the 3-subspaces *planes*, the 4-subspaces *solids* and the $(v - 1)$-subspaces *hyperplanes*. In fact, the *subspace lattice* $\mathcal{L}(V)$ consisting of all subspaces of V ordered by inclusion is nothing else than the

[1] As noticed by Daniel Heinlein, the $[7, 4; 3]_3$ code of size 6977 constructed in [16] can be extended trivially by adding a further codeword.

finite projective geometry $\mathrm{PG}(v-1, q) = \mathrm{PG}(V)$.[2] There are good reasons to consider the subset lattice as a subspace lattice over the unary "field" $\mathbb{F}_1$ [11].

The number of all k-subspaces of V is given by the Gaussian binomial coefficient

$$\#\begin{bmatrix} V \\ k \end{bmatrix}_q = \begin{bmatrix} v \\ k \end{bmatrix}_q = \begin{cases} \frac{(q^v-1)\cdots(q^{v-k+1}-1)}{(q^k-1)\cdots(q-1)} & \text{if } k \in \{0, \ldots, v\}; \\ 0 & \text{otherwise.} \end{cases}$$

The Gaussian binomial coefficient $\begin{bmatrix} v \\ 1 \end{bmatrix}_q$ is also known as the q-analog of the number v and will be abbreviated as $[v]_q$.

For $S \subseteq \mathcal{L}(V)$ and $U, W \in \mathcal{L}(V)$, we will use the abbreviations

$$\begin{aligned} S|_U &= \{B \in S \mid U \leq B\}, \\ S|^W &= \{B \in S \mid B \leq W\} \quad \text{and} \\ S|_U^W &= \{B \in S \mid U \leq B \leq W\}. \end{aligned}$$

For a point P in a plane E, the set of all lines in E passing through P is known as a *line pencil*.

The subspace lattice $\mathcal{L}(V)$ is isomorphic to its dual, which arises from $\mathcal{L}(V)$ by reversing the order. Fixing a non-degenerate bilinear form β on V, a concrete isomorphism is given by $U \mapsto U^\perp$, where $U^\perp = \{\mathbf{x} \in V \mid \beta(\mathbf{x}, \mathbf{u}) = 0 \text{ for all } \mathbf{u} \in U\}$. When addressing the dual of some geometric object in $\mathrm{PG}(V)$, we mean its (element-wise) image under this map. Up to isomorphism, the image does not depend on the choice of β.

2.2 Subspace designs

Definition 2.1 Let t, v, k be integers with $0 \leq t \leq k \leq v-t$ and λ another positive integer. A set $D \subseteq \begin{bmatrix} V \\ k \end{bmatrix}_q$ is called a t-$(v, k, \lambda)_q$ *subspace design* if each t-subspace of V is contained in exactly λ elements (called *blocks*) of D. In the important case $\lambda = 1$, D is called a q*-Steiner system*.

The earliest reference for subspace designs is [10]. It is stated that "Several people have observed that the concept of a t-design can be generalised [...]", so the idea might been around before. Subspace designs have also been mentioned in a more general context in [12]. The first nontrivial subspace designs with $t \geq 2$ have been constructed in [27], and the first nontrivial Steiner system with $t \geq 2$ in [4]. An introduction to the theory of subspace designs can be found at [7], see also [25, Day 4].

Subspace designs are interlinked to the theory of network coding in various ways. To this effect we mention the recently found q-analog of the theorem of Assmus and Mattson [9], and that a t-$(v, k, 1)_q$ Steiner system provides a $(v, 2(k-t+1); k)_q$ constant dimension network code of maximum possible size.

Classical combinatorial designs can be seen as the limit case $q = 1$ of subspace designs. Indeed, quite a few statements about combinatorial designs have a generalization to subspace designs, such that the case $q = 1$ reproduces the original statement [6, 18, 19, 22].

One example of such a statement is the following [26, Lemma 4.1(1)], see also [18, Lemma 3.6]: If D is a t-$(v, k, \lambda)_q$ subspace design, then D is also an s-$(v, k, \lambda_s)_q$ subspace

[2] In established symbols like $\mathrm{PG}(v-1, q)$, the geometric dimension $v-1$ is not altered.

design for all $s \in \{0, \dots, t\}$, where

$$\lambda_s := \lambda \frac{\begin{bmatrix} v-s \\ t-s \end{bmatrix}_q}{\begin{bmatrix} k-s \\ t-s \end{bmatrix}_q}.$$

In particular, the number of blocks in D equals

$$\#D = \lambda_0 = \lambda \frac{\begin{bmatrix} v \\ t \end{bmatrix}_q}{\begin{bmatrix} k \\ t \end{bmatrix}_q}.$$

So, for a design with parameters t-$(v, k, \lambda)_q$, the numbers λ_s necessarily are integers for all $s \in \{0, \dots, t\}$ (*integrality conditions*). In this case, the parameter set t-$(v, k, \lambda)_q$ is called *admissible*. It is further called *realizable* if a t-$(v, k, \lambda)_q$ design actually exists. The smallest admissible parameters of a nontrivial q-analog of a Steiner system with $t \geq 2$ are 2-$(7, 3, 1)_q$, which are the parameters of the q-analog of the Fano plane. This explains the significance of the question of its realizability.

The numbers λ_i can be refined as follows. Let i, j be non-negative integers with $i + j \leq t$ and let $I \in \begin{bmatrix} V \\ i \end{bmatrix}_q$ and $J \in \begin{bmatrix} V \\ v-j \end{bmatrix}_q$. By [26, Lemma 4.1], see also [7, Lemma 5], the number

$$\lambda_{i,j} := \#D|_I^J = \lambda \frac{\begin{bmatrix} v-i-j \\ k-i \end{bmatrix}_q}{\begin{bmatrix} v-t \\ k-t \end{bmatrix}_q}$$

only depends on i and j, but not on the choice of I and J. Apparently, $\lambda_{i,0} = \lambda_i$. The numbers $\lambda_{i,j}$ are important parameters of a subspace design. A further generalization is given by the intersection numbers in [19].

A nice way to arrange the numbers $\lambda_{i,j}$ is the following triangle form, which may be called the *q-Pascal triangle* of the subspace design D.

$$\begin{array}{ccccccccc} & & & & \lambda_{0,0} & & & & \\ & & & \lambda_{1,0} & & \lambda_{0,1} & & & \\ & & \lambda_{2,0} & & \lambda_{1,1} & & \lambda_{0,2} & & \\ & \iddots & & \iddots & & \ddots & & \ddots & \\ \lambda_{t,0} & & \lambda_{t-1,1} & & \dots & & \lambda_{1,t-1} & & \lambda_{0,t} \end{array}$$

For a q-analog of the Fano plane, we get:

$$\begin{array}{ccccc} & & \lambda_{0,0} = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1 & & \\ & \lambda_{1,0} = q^4 + q^2 + 1 & & \lambda_{0,1} = q^5 + q^3 + q^2 + 1 & \\ \lambda_{2,0} = 1 & & \lambda_{1,1} = q^2 + 1 & & \lambda_{0,2} = q^2 + 1 \end{array}$$

The proof of the result of this article will make use of the equality $\lambda_{1,1} = \lambda_{0,2}$ in the above triangle.

As a consequence of the numbers $\lambda_{i,j}$, the *dual* design $D^\perp = \{B^\perp \mid B \in D\}$ is a subspace design with the parameters

$$t\text{-}\left(v, v-k, \frac{\begin{bmatrix} v-t \\ k \end{bmatrix}_q}{\begin{bmatrix} v-t \\ k-t \end{bmatrix}_q}\right)_q.$$

For a point $P \leq V$, the *derived* design of D in P is the set of blocks

$$\operatorname{Der}_P(D) = \{B/P \mid B \in D|_P\}$$

in the ambient vector space V/P.[3] By [18], $\mathrm{Der}_P(D)$ is a subspace design with the parameters $(t-1)$-$(v-1, k-1, \lambda)_q$. In the case of a q-analog of the Fano plane, $\mathrm{Der}_P(D)$ has the parameters 1-$(6, 2, 1)_q$.

2.3 Spreads

A 1-$(v, k, 1)_q$ Steiner system $\mathcal{S}$ is just a partition of the point set of V into k-subspaces. These objects are better known under the name $(k-1)$-*spread* and have been investigated in geometry well before the emergence of subspace designs. A 1-spread is also called a *line spread*.

A set $\mathcal{S}$ of k-subspaces is called a *partial* $(k-1)$-*spread* if each point is covered by at most one element of $\mathcal{S}$. The points not covered by any element are called *holes*. A recent survey on partial spreads is found in [17].

The parameters 1-$(v, k, 1)_q$ are admissible if and only v is divisible by k. In this case, spreads do always exist [24, Sect. VI]. An example can be constructed via field reduction: We consider V as a vector space over $\mathbb{F}_{q^k}$ and set $\mathcal{S} = \begin{bmatrix} V \\ 1 \end{bmatrix}_{q^k}$. Switching back to vector spaces over $\mathbb{F}_q$, the set $\mathcal{S}$ is a $(k-1)$-spread of V, known as the *Desarguesian* spread.

A $(k-1)$-spread $\mathcal{S}$ is called *geometric* or *normal* if for two distinct blocks $B, B' \in \mathcal{S}$, the set $\mathcal{S}|^{B+B'}$ is always a $(k-1)$-spread of $B + B'$. In other words, $\mathcal{S}$ is geometric if every $2k$-subspace of V contains either 0, 1 or $[2k]_q/[k]_q = q^k + 1$ blocks of $\mathcal{S}$. It is not hard to see that the Desarguesian spread is geometric. In fact, it follows from [2, Theorem 2] that a $(k-1)$-spread is geometric if and only if it is isomorphic to a Desarguesian spreads.

The derived designs of a q-analog of the Fano plane D are line spreads in $\mathrm{PG}(5, q)$. The most symmetric one among these spreads is the Desarguesian spread. Following the notation of [13], a point P is called an α-*point* of the q-analog of the Fano plane D if the derived design in P is the geometric spread.[4]

We remark that in the binary case $q = 2$, the line spreads of $\mathrm{PG}(5, q)$ have been classified into 131 044 isomorphism types in [21].

2.4 Generalized quadrangles

Definition 2.2 A *generalized quadrangle* is an incidence structure $Q = (\mathcal{P}, \mathcal{L}, I)$ with a non-empty set of *points* $\mathcal{P}$, a non-empty set of *lines* $\mathcal{L}$, and an incidence relation $I \subseteq \mathcal{P} \times \mathcal{L}$ such that

(i) Two distinct points are incident with at most a line.
(ii) Two distinct lines are incident with at most one point.
(iii) For each non-incident point-line-pair (P, L) there is a unique incident point-line-pair (P', L') with $P \; I \; L'$ and $P' \; I \; L$.

Generalized quadrangles have been introduced in the more general setting of generalized polygons in [29], as a tool in the theory of finite groups.

A generalized quadrangle $Q = (\mathcal{P}, \mathcal{L}, I)$ is called *degenerate* if there is a point P such that each point of Q is incident with a line through P. If each line of Q is incident with

[3] The expressions V/P and B/P are quotients of $\mathbb{F}_q$-vector spaces. In this way, $B/P = \{x + P \mid x \in B\}$ is an $\mathbb{F}_q$-subspace of $V/P = \{x + P \mid x \in V\}$ for every block $B \in D|_P$.

[4] The definition of an α-point in [13] does not use the notion of a geometric spread. Instead, the property of a geometric spread in the factor space V/P has been written down explicitly, so the definitions are equivalent.

$t + 1$ points, and each point is incident with $s + 1$ lines, we say that Q is of *order* (s, t). The *dual* $Q^\perp$ arises from Q by interchanging the role of the points and the lines. It is again a generalized quadrangle. Clearly, $(Q^\perp)^\perp = Q$, and Q is of order (s, t) if and only if $Q^\perp$ is of order (t, s).

Furthermore, Q is said to be *projective* if it is embeddable in some Desarguesian projective geometry in the following sense: There is a Desarguesian projective geometry $(\bar{\mathcal{P}}, \bar{\mathcal{L}}, \bar{I})$ such that $\mathcal{P} \subseteq \bar{\mathcal{P}}$, $\mathcal{L} \subseteq \bar{\mathcal{L}}$, for all $(P, L) \in \mathcal{P} \times \mathcal{L}$ we have $P\ I\ L$ if and only if $P\ \bar{I}\ L$, and for each point $P \in \bar{\mathcal{P}}$ with $P\ \bar{I}\ L$ for some line $L \in \mathcal{L}$ we have $P \in \mathcal{P}$.[5] The non-degenerate finite projective generalized quadrangles have been classified in [8, Theorem 1], see also [23, 4.4.8]. These are exactly the so-called *classical generalized quadrangles* which are associated to a quadratic form or a symplectic or Hermitian polarity on the ambient geometry, see [23, 3.1.1].

In this article, two of these classical generalized quadrangles will appear.

(i) The symplectic generalized quadrangle $W(q)$ consisting of the set of points of $\mathrm{PG}(3, q)$ together with the totally isotropic lines with respect to a symplectic polarity. Taking the geometry as $\mathrm{PG}(\mathbb{F}_q^4)$, the symplectic polarity can be represented by the alternating bilinear form $\beta(\mathbf{x}, \mathbf{y}) = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$. The configuration of the lines $\mathcal{L}$ in $\mathrm{PG}(3, q)$ is also known as a (general) linear complex of lines, see [23, 3.1.1 (iii)] or [15, Theorem 15.2.13]. Under the Klein correspondence, $\mathcal{L}$ is a non-tangent hyperplane section of the Klein quadric.

(ii) The second one is the parabolic quadric $Q(4, q)$, whose points $\mathcal{P}$ are the zeros of a parabolic quadratic form in $\mathrm{PG}(4, q)$, and whose lines are all the lines contained in $\mathcal{P}$. Taking the geometry as $\mathrm{PG}(\mathbb{F}_q^5)$, the parabolic quadratic form can be represented by $q(\mathbf{x}) = x_1x_2 + x_3x_4 + x_5^2$.

Both $W(q)$ and $Q(4, q)$ are of order (q, q). By [23, 3.2.1] they are duals of each other, meaning that $W(q)^\perp \cong Q(4, q)$.

Let $Q = (\mathcal{P}, \mathcal{L}, I)$ be a generalized quadrangle. As in projective geometries, a set $\mathcal{S} \subseteq \mathcal{L}$ is called a *spread* of Q if each point of Q is incident with a unique line in $\mathcal{S}$. Dually, a set $\mathcal{O} \subseteq \mathcal{P}$ is called an *ovoid* of Q if each line of Q is incident with a unique point in $\mathcal{O}$. Clearly, the spreads of Q bijectively correspond to the ovoids of $Q^\perp$. This already shows the equivalence of parts (a) and (b) in Theorem 1.

3 Proof of the theorems

For the remainder of the article, we fix $v = 7$ and assume that $D \subseteq \begin{bmatrix} V \\ 3 \end{bmatrix}_q$ is a q-analog of the Fano plane. The numbers $\lambda_{i,j}$ are defined as in Sect. 2.2.

By the design property, the intersection dimension of two distinct blocks $B, B' \in D$ is either 0 or 1. So by the dimension formula, $\dim(B + B') \in \{5, 6\}$. Therefore two distinct blocks contained in a common 5-space always intersect in a point. Moreover, a solid S of V contains either a single block or no block at all. We will call S a *rich* solid in the former case and a *poor* solid in the latter.

Remark 3.1 By [19, Remark 4.2], the poor solids form a dual 2-$(7, 3, q^4)_q$ subspace design. By the above discussion, the $\lambda_{0,2} = q^2 + 1$ blocks in any 5-subspace F form dual partial spread in F. The poor solids contained in F are exactly the holes of that partial spread.

[5] As pointed out by a referee, the latter condition is indeed necessary, as otherwise the generalized quadrangle $T_2^*(O)$ in [23, 3.1.3] would be projective.

We will call a 5-subspace F a β*-flat* with *focal point* $P \in \begin{bmatrix} F \\ 1 \end{bmatrix}_q$ if all the $\lambda_{0,2} = q^2 + 1$ blocks contained in F pass through P.

Lemma 3.2 *The focal point of a* β*-flat is uniquely determined.*

Proof Assume that $P \neq Q$ are focal points of a β-flat F. Then all $\lambda_{0,2} = q^2 + 1 > 1$ blocks in F pass through the line $P + Q$, contradicting the Steiner system property. □

Lemma 3.3 *Let* H *be a hyperplane and* P *a point in* H. *Then* P *is the focal point of at most one* β*-flat in* H.

Proof There are $\lambda_{1,1} = q^2 + 1$ blocks in H passing through P. For any β-flat $F < H$ with focal point P, all these blocks are contained in F.

Now assume that there are two such β-flats $F \neq F'$. Then the $q^2 + 1 > 1$ blocks in $D|_P^H$ are contained in $F \cap F'$. This is a contradiction, since $\dim(F \cap F') \leq 4$ and any solid contains at most a single block. □

Lemma 3.4 *Let* $F \in \begin{bmatrix} V \\ 5 \end{bmatrix}_q$ *be a* β*-flat with focal point* P.

(a) *Each point in* F *different from* P *is covered by a unique block in* F.
In other words, $D|^F/P$ *is a line spread of* $F/P \cong \mathrm{PG}(3, q)$.
(b) *A solid* S *of* F *is poor if and only if it does not contain* P.
(c) *For all poor solids* S *of* F, *the set* $\{B \cap S \mid B \in D|^F\}$ *is a line spread of* S.

Proof Part (a): As the blocks in $D|^F$ intersect each other only in the point P, the number of points in $\begin{bmatrix} F \\ 1 \end{bmatrix}_q \setminus \{P\}$ covered by these blocks is $(q^2+1)(\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q - 1) = q^4+q^3+q^2+q = \begin{bmatrix} 5 \\ 1 \end{bmatrix}_q - 1$. Therefore, each point in F that is different from P is covered by a single point in $D|^F$.

Part (b): The number of solids in F containing one of the $q^2 + 1$ blocks in F is $(q^2 + 1) \cdot \begin{bmatrix} 5-3 \\ 4-3 \end{bmatrix}_q = (q^2+1)(q+1) = q^3+q^2+q+1$.[6] These solids are rich. Moreover, the q^4 solids in F not containing P do not contain a block, so they are poor. As $q^4+(q^3+q^2+q+1) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}_q$ is already the total number of solids in F, the poor solids in F are precisely those not containing P.

Part (c): Let S be a poor solid of F. For every block B in F we have $\dim(B \cap S) \leq 2$ as S is poor, and moreover $\dim(B \cap S) \geq \dim(B) + \dim(S) - \dim(F) = 3 + 4 - 5 = 2$ by the dimension formula. So for all blocks B in F we get that $B + S = F$ and $B \cap S$ is a line. By parts (a) and (b) , every point of the poor solid S is contained in a unique block in F. Hence $\{B \cap S \mid B \in D$ and $B + S = F\}$ is a line spread of S. □

Lemma 3.5 *Let* P *be an* α*-point and* $B, B' \in D$ *two blocks with* $B \cap B' = P$. *Then* $B + B'$ *is a* β*-flat with focal point* P.

Proof Since $P = B \cap B'$ is a point, $F = B + B'$ is a 5-subspace. Since P is an α-point, we have that $\{B''/P \mid B'' \in D|_P^F\}$ is a line spread of $F/P \cong \mathbb{F}_q^4$. Such a line spread contains $[4]_q/[2]_q = q^2 + 1$ lines, so F contains $q^2 + 1$ blocks passing through P. However, the total number of blocks contained in F is only $\lambda_{0,2} = q^2 + 1$, so all the blocks contained in F pass through P. □

Lemma 3.6 *Let* F *be a* 5*-subspace such that all points of* F *are* α*-points. Then* F *is a* β*-flat.*

[6] Remember that a solid cannot contain 2 blocks.

Proof The 5-subspace F contains $\lambda_{0,2} = q^2 + 1 > 1$ blocks. Let B and B' be two distinct blocks in F. Then $P = B \cap B'$ is a point and $F = B + B'$. By assumption, P is an α-point, so by Lemma 3.5, P is the focal point of the β-flat F. □

Remark 3.7 The statement of Lemma 3.6 is still true if F contains a single non-α-point Q. Then either all blocks contained in F pass through Q, or there are two distinct blocks B, B' in F such that $P = B \cap B' \neq Q$. In the latter case, all blocks pass through the α-point P as in the proof of Lemma 3.6.

Lemma 3.8 *Let H be a hyperplane and P an α-point contained in H. Then H contains a unique β-flat whose focal point is P.*

Proof There are $\lambda_{1,1} = q^2 + 1 > 1$ blocks in H containing P. Let $B, B' \in D|_P^H$. Then $P = B \cap B'$. By Lemma 3.5, the α-point P is the focal point of the β-flat $F = B + B'$. By Lemma 3.3, the β-flat F is unique. □

Now we fix a hyperplane H of V and assume that all its points are α-points.

By Lemma 3.6, every 5-subspace F of H is a β-flat. We denote its unique focal point by $\alpha(F)$. Moreover by Lemma 3.8, each point P of H is the focal point of a unique β-flat F in H. We will denote this β-flat by $\beta(P)$. Clearly, the mappings

$$\alpha : \begin{bmatrix} H \\ 5 \end{bmatrix}_q \to \begin{bmatrix} H \\ 1 \end{bmatrix}_q \quad \text{and} \quad \beta : \begin{bmatrix} H \\ 1 \end{bmatrix}_q \to \begin{bmatrix} H \\ 5 \end{bmatrix}_q$$

are inverse to each other. So they provide a bijective correspondence between the points and the 5-subspaces of H.

Lemma 3.9 *Let B be a block in H.*

(a) *For all points P of B, $B \leq \beta(P)$.*
(b) *For all 5-subspaces F in H containing B, $\alpha(F) \leq B$.*

Proof For part (a), let P be a point on B. There are $\lambda_{1,1} = q^2+1$ blocks in H passing through P, which equals the number $\lambda_{0,2}$ of blocks in $\beta(P)$ (which all pass through P). Therefore, $B \leq \beta(P)$.

For part (b), let F be a 5-subspace containing B. All blocks in F pass through its focal point $\alpha(F)$. □

For the remainder of this article, we fix a poor solid S of H. Note that by Lemma 3.4(b), every 5-subspace of H contains a suitable solid S.[7] The set of $\begin{bmatrix} 6-4 \\ 5-4 \end{bmatrix}_q = q + 1$ intermediate 5-subspaces F with $S < F < H$ will be denoted by $\mathcal{F}$. For each $F \in \mathcal{F}$, the set $\mathcal{L}_F := \{B \cap S \mid B \in D|^F\}$ is a line spread of S by Lemma 3.4(c).

Lemma 3.10 *The line spreads $\mathcal{L}_F$ with $F \in \mathcal{F}$ are pairwise disjoint.*

Proof Let $F, F' \in \mathcal{F}$ and $L \in \mathcal{L}_F \cap \mathcal{L}_{F'}$. Then $L = B \cap S = B' \cap S$ with $B \in D|^F$ and $B' \in D|^{F'}$. So B and B' are two blocks passing through the same line L. The Steiner system property gives $B = B'$. Hence $F = B + S = B' + S = F'$. □

Now let $\mathcal{L} = \bigcup_{F \in \mathcal{F}} \mathcal{L}_F$.

[7] Using the fact that the poor solids form a dual 2-$(7, 3, q^4)_q$ subspace design in V [19, Remark 4.2], the total number of poor solids S in H is $q^4 \cdot \lambda_{1,0} = q^8 + q^6 + q^4 = q^4(q^2 + q + 1)(q^2 - q + 1)$.

Lemma 3.11 *The set $\mathcal{L}$ consists of q^3+q^2+q+1 lines of S and is partitionable into $q+1$ line spreads of S.*

Proof By Lemma 3.10, the sets $\mathcal{L}_F$ are pairwise disjoint, so $\mathcal{L}$ is a set of $\#\mathcal{F}\cdot\#D|^F = (q+1)(q^2+1) = q^3+q^2+q+1$ lines in S admitting a partition into the $q+1$ line spreads $\mathcal{L}_F$ with $F\in\mathcal{F}$. □

Lemma 3.12 *For each point P of S, $\mathcal{L}|_P$ is a line pencil in the plane $E_P=\beta(P)\cap S$.*

Proof Let P be a point in S.

By Lemma 3.4(b), the poor solid S is not contained in the 5-subspace $\beta(P)$. Therefore, $\dim(\beta(P)\cap S)\leq 3$. On the other hand, as both S and $\beta(P)$ are contained in H, we have $\dim(\beta(P)+S)\leq\dim(H)=6$ and therefore by the dimension formula $\dim(\beta(P)\cap S) = \dim(\beta(P))+\dim(S)-\dim(\beta(P)+S)\geq 3$. Hence $E_P=\beta(P)\cap S$ is a plane.

Let $L\in\mathcal{L}|_P$. Then there is a block $B\in D|^H$ with $B\cap S=L$. By Lemma 3.9(a), $B\leq\beta(P)$. So $L=B\cap S\leq\beta(P)\cap S=E_P$. As the disjoint union of $q+1$ line spreads of S, $\mathcal{L}$ contains $q+1$ lines passing through P. Therefore, these lines form a line pencil in E_P through P. □

Lemma 3.13 *The incidence structure $(\begin{bmatrix}S\\1\end{bmatrix}_q, \mathcal{L}, \subseteq)$ is a projective generalized quadrangle of order $(s,t)=(q,q)$.*

Proof Clearly, every line in $\mathcal{L}$ contains $q+1$ points in S. By Lemma 3.11, through every point in S there pass $q+1$ lines in $\mathcal{L}$. Now let P be a point in S and $L\in\mathcal{L}$ not containing P.

By Lemma 3.10, there is a unique $F\in\mathcal{F}$ with $L\in\mathcal{L}_F$, and there is a line $L''\in\mathcal{L}_F$ passing through P. By Lemma 3.12, $L''<E_P$, so we get $L\not<E_P$ as otherwise L and L'' would be distinct intersecting lines in the spread $\mathcal{L}_F$. Moreover, L and E_P are both contained in S, so they cannot have trivial intersection. Therefore $L\cap E_P$ is a point.

Now let $P'\in\begin{bmatrix}S\\1\end{bmatrix}_q$ and $L'\in\mathcal{L}$ with $L\cap L'=P'$ and $P+P'=L'$. Then L' is a line through P, so $L'<E_P$. So necessarily $P'=E_P\cap L$ and $L'=P+P'$, showing that P' and L' are unique.

By Lemma 3.12 indeed $L'\in\mathcal{L}$, as $P+P'$ is a line in E_P containing P. This shows that P' and L' do always exist and therefore, the incidence structure $(\begin{bmatrix}S\\1\end{bmatrix}, \mathcal{L})$ is a generalized quadrangle of order (q,q). □

Lemma 3.14 $(\begin{bmatrix}S\\1\end{bmatrix}_q, \mathcal{L}, \subseteq)$ *is isomorphic to $W(q)$.*

Proof By Lemma 3.13 we know that $Q=(\begin{bmatrix}S\\1\end{bmatrix}_q, \mathcal{L}, \subseteq)$ is a finite generalized quadrangle of order $(s,t)=(q,q)$ embedded in $\mathrm{PG}(S)$. By the classification in [8, Theorem 1] (see also [23, 4.4.8]), we know that Q is a finite classical generalized quadrangle which are listed in [23, 3.1.2]. Comparing the orders and the dimension of the ambient geometry, the only possibility for Q is the symplectic generalized quadrangle $W(q)$. □

Now we can prove our main result.

Proof of Theorem 1 Part (a) follows from Lemmas 3.14 and 3.11. The equivalence of parts (a) and (b) has already been discussed at the end of Sect. 2.4. □

Theorem 2 is now a direct consequence.

Proof of Theorem 2 We show that the statement in Theorem 1(b) is not satisfied.

For q prime, the only ovoids of $Q(4,q)$ are the elliptic quadrics $Q^-(3,q)$ [1, Cor. 1]. As any two such quadrics have nontrivial intersection, there is no partition of $Q(4,q)$ into ovoids.

For q even, $Q(4,q)$ does not admit a partition into ovoids by [23, 3.4.1 (i)]. □

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