ECONOMIC MPC WITHOUT TERMINAL CONSTRAINTS*

LARS GRÜNE†

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EXTENDED ABSTRACT.

Problem formulation. We consider discrete time control systems with state $x \in X$ and control values $u \in U$, where X and U are normed spaces with norms denoted by $\|\cdot\|$. The control system under consideration is given by

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

with $f : X \times U \to X$. For a given control sequence $u = (u(0), \dots, u(K-1)) \in U^K$ or $u = (u(0), u(1), \dots) \in U^\infty$, by $x_u(k, x)$ we denote the solution of (1) with initial value $x = x_u(0, x) \in X$.

For given admissible sets of states $\mathbb{X} \subseteq X$ and control values $\mathbb{U} \subseteq U$ and an initial value $x \in \mathbb{X}$ we call the control sequences $u \in \mathbb{U}^K$ satisfying $x_u(k, x) \in \mathbb{X}$ for all $k = 0, \dots, K$ admissible. The set of all admissible control sequences is denoted by $\mathbb{U}^K(x)$. Similarly, we define the set $\mathbb{U}^\infty(x)$ of admissible control sequences of infinite length. For simplicity of exposition we assume $\mathbb{U}^\infty(x) \neq \emptyset$ for all $x \in \mathbb{X}$, i.e., that for each initial value $x \in \mathbb{X}$ we can find a trajectory staying inside $\mathbb{X}$ for all future times.

Given a feedback map $\mu : X \to U$, we denote the solutions of the closed loop system

$$x(k+1) = f(x(k), \mu(x(k)))$$

by $x_\mu(k)$ or by $x_\mu(k, x)$ if we want to emphasize the dependence on the initial value $x = x_\mu(0)$. We say that a feedback law μ is admissible if it renders the admissible set $\mathbb{X}$ (forward) invariant, i.e., if $f(x, \mu(x)) \in \mathbb{X}$ holds for all $x \in \mathbb{X}$. Note that $\mathbb{U}^\infty(x) \neq \emptyset$ for all $x \in \mathbb{X}$ immediately implies that such a feedback law exists.

Our goal is now to find an admissible feedback controller which yields approximately optimal average performance. To this end, for a given running cost $\ell : X \times U \to \mathbb{R}$ we define the averaged functionals and optimal value functions

$$\begin{aligned} J_N(x,u) &:= \frac{1}{N}\sum_{k=0}^{N-1} \ell(x_u(k,x), u(k)), & V_N(x) &:= \inf_{u \in \mathbb{U}^N(x)} J_N(x,u), \\ J_\infty(x,u) &:= \limsup_{N\to\infty} J_N(x,u) \quad \text{and} & V_\infty(x) &:= \inf_{u \in \mathbb{U}^\infty(x)} J_\infty(x,u). \end{aligned}$$

Here we assume that ℓ is bounded from below on $\mathbb{X}$, i.e., that $\ell_{\min} := \inf_{x\in\mathbb{X}, u\in\mathbb{U}} \ell(x,u)$ is finite. This assumption immediately yields $J_N(x,u) \geq \ell_{\min}$ and $J_\infty(x,u) \geq \ell_{\min}$ for all admissible control sequences. In order to simplify the exposition in what follows, we assume that (not necessarily unique) optimal control sequences for J_N exist which we denote by u^*.

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†Mathematical Institute, University of Bayreuth, Germany, lars.gruene@uni-bayreuth.de

Similarly to the open loop functionals, we can define the average cost of the closed loop solution for any feedback law μ by

$$J_K^{cl}(x,\mu) := \frac{1}{K}\sum_{k=0}^{K-1} \ell(x_\mu(k,x),\mu(x_\mu(k,x))) \quad \text{and} \quad J_\infty^{cl}(x,\mu) := \limsup_{K\to\infty} J_K(x,\mu).$$

In order to construct the desired feedback law, henceforth denoted by μ_N, we employ a model predictive control (MPC) approach: in each time instant k, we compute the optimal control $u^* \in \mathbb{U}^N(x_0)$ minimizing $J_N(x_0,\cdot)$ for the initial value $x_0 = x_{\mu_N}(k,x)$ and define the feedback value as $\mu_N(x_0) := u^*(0)$, i.e., as the first element of the finite horizon optimal control sequence.

Terminal constrained economic MPC. The approach just introduced is referred to as *economic MPC* in the literature since the stage cost reflects an economic criterion rather than a distance to some desired reference solution as in the more standard stabilizing or tracking MPC. In a series of papers ([2, 3, 6, 1]), a theory of economic MPC with terminal constraints has been developed. We briefly sketch some of the main results of these papers for the special case where $x^e \in \mathbb{X}$ is an equilibrium, i.e., $f(x^e,u^e) = x^e$ holds for some $u^e \in \mathbb{U}$ (some of these references also treat the case of periodic solutions which we will also briefly discuss in the talk but not in this extended abstract). For any equilibrium x^e it is shown that if we use the MPC approach with the additional terminal constraint $x_u(N,x) = x^e$ when minimizing $J_N(x,\cdot)$ (assuming that this constraint is feasible for the given initial value $x \in \mathbb{X}$), then the inequality $J_\infty^{cl}(x,\mu_N) \le \ell(x^e,u^e)$ holds. Particularly, if $\ell(x^e,u^e)$ is an optimal equilibrium (in the sense that the equilibrium cost $\ell(x^e,u^e)$ is less or equal than the infinite horizon averaged functional along any other trajectory) then optimal performance of the closed loop follows.

In general, this result does not necessarily imply convergence of the closed loop trajctories to x^e. In order to ensure convergence, the following assumption (cf. [3]) can be employed. We define a modified cost

$$\tilde{\ell}(x,u) := \ell(x,u) + \lambda(x) - \lambda(f(x,u)) \tag{2}$$

for a given function $\lambda : \mathbb{X} \to \mathbb{R}$. Then the inequality $\min_{x\in\mathbb{X},u\in\mathbb{U}} \tilde{\ell}(x,u) \le \tilde{\ell}(x^e,u^e) = \ell(x^e,u^e)$ holds. The assumption for convergence then reads as follows.

Assumption 2. *The function λ in (2) is bounded on $\mathbb{X}$ and there exists an equilibrium $(x^e,u^e) \in \mathbb{X}\times\mathbb{U}$ and $\alpha_\ell \in \mathcal{K}_\infty$ such that*

$$\min_{u\in\mathbb{U}} \tilde{\ell}(x,u) \ge \ell(x^e,u^e) + \alpha_\ell(\|x-x^e\|)$$

holds for all $x \in \mathbb{X}$ with $\tilde{\ell}$ from (2).

Economic MPC without terminal constraints. The aim of this talk is to show to what extent these results remain true if we do not impose the terminal constraint $x_u(N,x) = x^e$ when computing μ_N. The motivation for removing these constraints are, among others, a potentially larger feasible region for the problem and a simplification of the optimal control problem to be solved in each step of the MPC scheme. A first step in this direction is provided by the following proposition.

Proposition 2. *Let $N \ge 2$, abbreviate $\ell^e = \ell(x^e,u^e)$ and assume that the optimal value function V_N and the MPC feedback law μ_N satisfy the inequality*

$$V_N(f(x,\mu_N(x))) - V_N(x) \le \ell(x,\mu_N(x)) + \ell^e + \varepsilon(N) \tag{3}$$

for all $x \in \mathbb{X}$ *and a function* $\varepsilon : \mathbb{N} \to \mathbb{R}_0^+$ *Then the inequality* $J_\infty^{cl}(x, \mu_N) \le \ell^e + \varepsilon(N)$ *holds for all* $x \in \mathbb{X}$.

The proof of this proposition follows from [7, Proof of Proposition 4.1] by observing that (3) is equivalent to $V_{N+1}(x) - V_N(x) \le \ell^e + \varepsilon(N)$.

Proposition 2 means that we can prove *value convergence* for the closed loop. If, moreover, Assumption 2 holds and $N\varepsilon(N) \to 0$ as $N \to \infty$, then also convergence of the closed loop solution $x_\mu(k, x)$ to a neighborhood of x^e can be shown, where the size of this neighborhood shrinks to 0 as $N \to \infty$, cf. [7, Theorem 7.6]. Hence, under the additional condition $N\varepsilon(N) \to 0$ we can also conclude *trajectory convergence*. In the talk we will present several numerical examples in which one observes exponential decay $\varepsilon(N) \le C\theta^N$, $\theta \in (0, 1)$, which implies $N\varepsilon(N) \to 0$, see also [7].

The central question is thus whether we can ensure the inequality (3), preferably with $N\varepsilon(N) \to 0$ as $N \to \infty$. Inequality (3) can be concluded by constructing a control sequence u for initial value $f(x, \mu_N(x))$ from the optimal control sequence u^* for initial value x and using $J_N(f(x, \mu_N(x))$ as an upper bound for $V_N(f(x, \mu_N(x)))$. Details will be presented in the talk and can also be found in [7, Proof of Theorem 4.2]. Besides some continuity and boundedness conditions on f, ℓ and V_N, the main requirement for this proof to work is that the open loop optimal trajectory for horizon N satisfies $x_{u^*}(k, x) \approx x^e$ for some $k \in \{0, \dots, N\}$. In quantitative terms, this leads to the following assumption.

Assumption 3. *There exists* $\sigma : \mathbb{N} \to \mathbb{R}_0^+$ *with* $\sigma(N) \to 0$ *as* $N \to \infty$ *and* $N_1 \in \mathbb{N}$ *such that for each* $x \in \mathbb{X}$ *and each* $N \ge N_1$ *there exists an optimal trajectory* $x_{u^*}(\cdot, x)$ *satisfying* $\|x_{u^*}(k_x, x) - x^e\| \le \sigma(N)$ *for some* $k_x \in \{0, \dots, N\}$.

Assumption 3 is a particular form of a so called *turnpike property*, see, e.g., [4, Section 4.4] and [8] and the references therein.

The proof of [7, Theorem 4.2] shows that under suitable continuity and boundedness assumptions on f, ℓ and V_N in a neighborhood of x^e, the estimate $\varepsilon(N) \le p(\sigma(N))$ can be obtained, where p is a polynomial with $p(0) = 0$. Particularly, this shows that $\sigma(N) \to 0$ implies $\varepsilon(N) \to 0$ and if $\sigma(N)$ converges to 0 exponentially fast, then $\varepsilon(N)$ will do so, too.

In what follows we will sketch two ways for deriving Assumption 3 from Assumption 2. In order to simplify the computations, for the subsequent considerations we will assume $\ell(x^e, u^e) = 0$ and $\lambda(x^e) = 0$ which also implies $\tilde{\ell}(x^e, u^e) = 0$. These assumptions can be made without loss of generality by adding suitable constants to ℓ and λ. Note that adding such constants does neither change the optimal trajectories and control sequences nor does it affect the validity of Assumption 2 and the function α in this assumption. Moreover, we define the modified cost functional

$$\widetilde{J}_N(x, u) := \sum_{k=0}^{N-1} \tilde{\ell}(x_u(k, x), u(k)).$$

Observe that by definition of the modified cost $\tilde{\ell}$ the functionals J_N and $\widetilde{J}_N$ are related via

$$\widetilde{J}_N(x, u) = J_N(x, u) + \lambda(x) - \lambda(x_u(N, x)). \tag{4}$$

Version 1: Let Assumption 2 hold and let $C := 2\sup_{x \in \mathbb{X}} |\lambda(x)| < \infty$. Assume moreover that $V_N(x)$ is bounded from above on $\mathbb{X}$, i.e., $V_N(x) \le M$ holds for all $x \in \mathbb{X}$ and some $M \in \mathbb{R}$. Then from (4) we obtain

$$\widetilde{J}_N(x, u^*) \le J_N(x, u^*) + C = V_N(x) + C \le M + C.$$

for all $N \in \mathbb{N}$. This implies $\tilde{\ell}(x_{u^*}(k,x), u^*(k)) \le (M+C)/N$ for some $k \in \{0, \dots, N-1\}$. Assumption 2 then implies $\|x_{u^*}(k,x) - x^e\| \le \alpha^{-1}((M+C)/N) =: \sigma(N)$ with α from Assumption 2 which shows Assumption 3.

Unless α^{-1} happens to be very "flat" near 0 (which is an exceptional case), the proof just sketched will not yield exponential convergence $\sigma(N) \to 0$. Consequently, this proof (which follows [7, Theorem 5.3]) implies value convergence but in general no trajectory convergence. In order to improve the estimate, we present an alternative way to estimate $\sigma(N)$ which, however, needs stronger assumptions.

Version 2: Let Assumption 2 hold, assume that $\tilde{\ell}$ is bounded on $\mathbb{X} \times \mathbb{U}$ and consider the following terminal constrained optimal value function

$$\widetilde{V}_N^t(x_0, x_T) := \inf_{\substack{u \in \mathbb{U}^N(x_0) \\ x_u(N, x_0) = x_T}} \widetilde{J}_N(x_0, u).$$

Assume that there exists $\gamma \ge 1$ such that for all $x_0, x_T \in \mathbb{X}$ for which a trajectory from x_0 to x_T exists the inequality

$$\widetilde{V}_N^t(x_0, x_T) \le \gamma \min_{u \in \mathbb{U}} \ell(x_0, u) + (\gamma - 1) \min_{u \in \mathbb{U}} \ell(x_N, u) \tag{5}$$

holds. Then, exploiting that any piece of length K of an optimal trajectory for J_N is an optimal trajectory for $\widetilde{V}_K^t(x_0, x_T)$ for appropriate x_0 and x_N, and using a dynamic programming induction we obtain either

$$\tilde{\ell}_{\lfloor N/2 \rfloor} \le \gamma \left(\frac{\gamma - 1}{\gamma} \right)^{\lfloor N/2 \rfloor - 1} \tilde{\ell}_0 \qquad \text{or} \qquad \tilde{\ell}_{\lceil N/2 \rceil} \le \gamma \left(\frac{\gamma - 1}{\gamma} \right)^{\lfloor N/2 \rfloor - 1} \tilde{\ell}_N$$

for $\tilde{\ell}_k := \tilde{\ell}(x_{u^*}(k,x), u^*(k))$, details will be presented in the talk and in [5].

Since we assumed $\tilde{\ell}$ to be bounded on $\mathbb{X}$ (say, by a constant M), this implies Assumption 3 with

$$\sigma(N) = \alpha^{-1} \left(M\gamma \left(\frac{\gamma - 1}{\gamma} \right)^{\lfloor N/2 \rfloor - 1} \right).$$

If α has at least polynomial growth near 0, this σ indeed decays exponentially and thus implies the desired exponential convergence of $\varepsilon(N) \to 0$.

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