Feedback stabilization methods for the numerical solution of ordinary differential equations

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Abstract: In this work we study the problem of step size selection for numerical schemes, which guarantees that the numerical solution presents the same qualitative behavior as the original system of ordinary differential equations. We apply tools from nonlinear control theory, specifically Lyapunov function and small-gain based feedback stabilization methods for systems with a globally asymptotically stable equilibrium point. Proceeding this way, we derive conditions under which the step size selection problem is solvable (including a nonlinear generalization of the well-known A-stability property for the implicit Euler scheme) as well as step size selection strategies for several applications.

Keywords: asymptotic stability of numerical approximations, feedback stabilization, nonlinear systems.

1 Introduction

It is well-known that step size control can enhance the performance of numerical schemes for solving ordinary differential equations (ODEs). In fact, the use of the word "control" suggests that methods and techniques from mathematical control theory can in principle be used in order to achieve certain objectives for the numerical solution. For example, in [16] the authors use a "Proportional-Integral" technique which is similar to the "Proportional-Integral" controller used in Linear Systems Theory in order to keep the local discretization error within certain bounds, see also [14, 15, 19]. Theoretical results on the behavior of adaptive time stepping methods have been presented in [27, 29] and the control theoretic notion of input-to-state stability (ISS) has been successfully used in [11, 12] in order to explain the behavior of attractors under discretization.

In this work, we develop tools for numerical schemes which are similar to methods used in nonlinear control theory. We consider the problem of selecting the step size for numerical schemes so that the numerical solution presents the same qualitative behavior as the original nonlinear ODE. It is well-known that any consistent and stable numerical scheme for ODEs inherits the asymptotic stability of the original equation in a practical sense, even for more general attractors than equilibria, see for instance [11, 12, 26] and [35, Chapter 7] for fixed step size and [5, 27] for schemes with variable step size. Practical asymptotic stability means that the system exhibits an asymptotically stable set close to the original attractor, i.e., in our case a small neighbourhood around the equilibrium point, which shrinks down to the attractor as the time step h tends to 0. In contrast to these results, in this paper we investigate the case in which the numerical approximation is asymptotically stable in the usual sense, i.e., not only practically.

Here, we concentrate on nonlinear systems for which an equilibrium point is the global attractor. In Section 2 of the present work it is shown how the problem of appropriate step size selection can be converted to a rigorous abstract feedback stabilization problem for a particular hybrid system. The idea of representing numerical schemes as hybrid systems goes back to [22] and the reader should notice that the standard stability analysis of numerical schemes uses discrete-time system, see, e.g., [19, 17, 21, 28, 35], rather than hybrid systems. With this approach, we are in the position to use all methods of feedback design for nonlinear systems. Specifically, we consider methods based on small-gain theorems and methods based on Lyapunov functions.

Both methods have been used widely in nonlinear systems theory for the solution of feedback stabilization problems, see [1, 4, 20, 23, 25, 33, 34] and references therein. In the present work, the above methods are used for the step size selection for numerical schemes for ODEs, see Section 3 and Section 4. While the small-gain method allows for the design of novel numerical schemes for nonlinear systems with specific structures, cf. Theorem 3.1 and Theorem 3.3, the Lyapunov function based method allows for results for general nonlinear systems. It applies to arbitrary consistent Runge-Kutta schemes (see Theorem 4.5, Theorem 4.9 and Theorem 4.12) as well as to specific Runge-Kutta schemes, see Corollary 4.7 and Theorem 4.17. Some of our results constitute nonlinear extensions of well-known properties of numerical schemes like, e.g., A-stability, cf. Corollary 4.18. While the idea behind this Lyapunov based approach is conceptually similar to the geometric integration method recently proposed in [10], our methodology relies on the appropriate selection of the time step rather than on the modification of the numerical scheme.

The kea idea used in small-gain approach is to formulate numerical schemes in such a way that small-gain criteria from the hybrid control systems literature become applicable. These criteria then induce an upper bound on the time step for which stability of the numerically computed solutions can be guaranteed. In the Lyapunov based approach, the basic idea is to use a Lyapunov function for the continuous time system as a Lyapunov function for the numerical approximation, which in turn implies the desired stability property by Lemma 4.1. Conditions under which this is possible and corresponding bounds on the time step are derived either from estimates on the discretization error as in Theorem 4.5, Theorem 4.9 and Theorem 4.12, or from structural properties of the scheme and the Lyapunov function as in Theorem 4.17.

A number of applications of the obtained results is developed in Sections 5 and 6. For instance, in Section 6 we consider the possibility of using explicit schemes for stiff linear systems of ODEs. An application of the stabilization method based on the small-gain analysis for systems described by partial differential equations (PDEs) is presented in Section 5.

Thus, the contribution of the paper is twofold. On the one hand, our control theoretic approach yields new insight into the stability properties of numerical schemes and as such it adds another means to the toolbox for stability investigations of numerical schemes. On the other hand, our method leads to the design of new discretization schemes and step size control algorithms, which instead of the usual control of the local discretization error take care of the global qualitative behaviour.

Notation Throughout this paper we adopt the following notation:

Let $A \subseteq \mathbb{R}^n$ be a set. By $C^0(I\ ;\ \Omega)$, we denote the class of continuous functions on I, which take values in Ω. By $C^k(I\ ;\ \Omega)$, where $k \geq 1$ is an integer, we denote the class of differentiable functions on A with continuous derivatives up to order k, which take values in Ω. By $C^\infty(A;\Omega)$, we denote the class of differentiable functions on A having continuous derivatives of all orders, which take values in Ω, i.e., $C^\infty(A;\Omega) = \bigcap_{k\geq 1} C^k(A;\Omega)$.

For a vector $x \in \mathbb{R}^n$ we denote by $|x|$ its usual Euclidean norm and by x' its transpose. By $B_\varepsilon(x)$, where $\varepsilon > 0$ and $x \in \mathbb{R}^n$, we denote the ball of radius $\varepsilon > 0$ centered at $x \in \mathbb{R}^n$, i.e., $B_\varepsilon(x) := \{\, y \in \mathbb{R}^n \,:\, |y - x| < \varepsilon \,\}$. For a real matrix $A \in \mathbb{R}^{n\times m}$ we denote by $|A|$ its induced norm, i.e., $|A| := \max\{\, |Ax| \,:\, x \in \mathbb{R}^m \,,\, |x| = 1 \,\}$ and by $A' \in \mathbb{R}^{m\times n}$ its transpose.

$\mathbb{R}^+$ denotes the set of non-negative real numbers and $\mathbb{Z}^+$ the set of non-negative integer numbers. $\mathbb{C}$ denotes the set of complex numbers. By $\mathcal{K}_\infty$ we denote the set of all increasing and continuous functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho(0) = 0$ and $\lim_{s\to+\infty} \rho(s) = +\infty$.

For every scalar continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, $\nabla V(x)$ denotes the gradient of V at $x \in \mathbb{R}^n$, i.e., $\nabla V(x) = (\frac{\partial V}{\partial x_1}(x), \ldots, \frac{\partial V}{\partial x_n}(x))$. We say that a function $V : \mathbb{R}^n \to \mathbb{R}^+$ is positive definite if $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$. We say that a continuous function $V : \mathbb{R}^n \to \mathbb{R}^+$ is radially unbounded if for every $M > 0$ the set $\{\, x \in \mathbb{R}^n \,:\, V(x) \leq M \,\}$ is compact.

For a sufficiently smooth function $V : \mathbb{R}^n \to \mathbb{R}$ we denote by $L_f V(x) := \nabla V(x) f(x)$ the Lie derivative of V along f and we define recursively $L_f^{(i+1)} V(x) = L_f(L_f^{(i)} V(x))$ for $i \geq 1$.

2 Setup, preliminaries and problem formulation

Consider the autonomous system

$$\dot{z}(t) = f(z(t)) \;,\; z(t) \in \mathbb{R}^n \tag{2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz vector field with $f(0) = 0$. For every $z_0 \in \mathbb{R}^n$ and $t \geq 0$, the solution of (2.1) with initial condition $z(0) = z_0$ will be denoted by $z(t, z_0)$.

There are several ways of formalizing numerical approximations of system (2.1) schemes with varying step-sizes as dynamical systems. In this paper we will use hybrid systems for this purpose. After introducing this class of systems, establishing its relation to numerical schemes and deriving some of its properties, we will discuss in Remark 2.1 why we prefer to use this formulation. The hybrid system we are considering is given by

$$\begin{aligned}
&\dot{x}(t) = F(h_i, x(\tau_i)) \;,\; t \in [\tau_i, \tau_{i+1}) \\
&\tau_0 = 0 \;,\; \tau_{i+1} = \tau_i + h_i \\
&h_i = \varphi(x(\tau_i)) \exp(-u(\tau_i)) \\
&x(t) \in \mathbb{R}^n \,,\, u(t) \in [0, +\infty)
\end{aligned} \tag{2.2}$$

where $\varphi \in C^0(\mathbb{R}^n; (0, r])$, $r > 0$ is a constant, $F : \bigcup_{x\in\mathbb{R}^n} ([0, \varphi(x)] \times \{x\}) \to \mathbb{R}^n$ is a (not necessarily continuous) vector field with $F(h, 0) = 0$ for all $h \in [0, \varphi(0)]$, $\lim_{h\to 0^+} F(h, z) = f(z)$, for all $z \in \mathbb{R}^n$. More specifically, the solution $x(t)$ of the hybrid system (2.2) is obtained for every locally bounded $u : \mathbb{R}^+ \to \mathbb{R}^+$ and $x_0 \in \mathbb{R}^n$ by setting $\tau_0 = 0$, $x(0) := x_0$ and then proceeding iteratively for $i = 0, 1, \ldots$ as follows (cf. [22]):

(i) Given τ_i and $x(\tau_i)$, calculate τ_{i+1} using the equation $\tau_{i+1} = \tau_i + \varphi(x(\tau_i)) \exp(-u(\tau_i))$

(ii) Compute the state trajectory $x(t)$, $t \in (\tau_i, \tau_{i+1}]$ as the solution of the differential equation $\dot{x}(t) = F(h_i, x(\tau_i))$, i.e., $x(t) = x(\tau_i) + (t - \tau_i) F(h_i, x(\tau_i))$ for $t \in (\tau_i, \tau_{i+1}]$.

We denote the resulting trajectory by $x(t, x_0, u)$ or briefly $x(t)$ when x_0 snd u are clear from the context.

We will further assume that there exists a continuous, non-decreasing function $M : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|F(h, x)| \leq |x| \, M\left(|x|\right) \text{ for all } x \in \mathbb{R}^n \text{ and } h \in [0, \varphi(x)] \tag{2.3}$$

It should be noticed that the hybrid system (2.2) under hypothesis (2.3) is an autonomous system, which satisfies the "Boundedness-Implies-Continuation" property and for each locally bounded input $u : \mathbb{R}^+ \to \mathbb{R}^+$ and $x_0 \in \mathbb{R}^n$ there exists a unique absolutely continuous function $[0, +\infty) \ni t \to x(t) \in \mathbb{R}^n$ with $x(0) = x_0$ which satisfies (2.2), see [22]. Some remarks are needed in order to explain the name "numerical approximation of system (2.1)" for the hybrid system (2.2).

(i) The condition $\lim_{h\to 0^+} F(h, z) = f(z)$ is the usual consistency condition for the numerical scheme applied to (2.1).

(ii) The sequence $\{h_i\}_0^\infty$ is the sequence of step sizes used in order to obtain the numerical solution. Notice that for the case $\varphi(x) \equiv r$, constant inputs $u(t) \equiv u \geq 0$ will produce constant step sizes with $h_i \equiv r\exp(-u)$. Arbitrary variable step size sequences $h_i \in (0, \varphi(x(\tau_i)]$ can be represented easily by selecting appropriate inputs $u : \mathbb{R}^+ \to \mathbb{R}^+$.

(iii) The constant $r > 0$ is the maximal allowable step size.

(iv) The function $\varphi \in C^0(\mathbb{R}^n; (0, r])$ determines the maximum allowable step size $\varphi(x(\tau_i))$ for each $x(\tau_i) \in \mathbb{R}^n$. This is important for implicit numerical schemes as shown below.

All consistent s-stage Runge-Kutta methods can be represented by the hybrid system (2.2). More specifically, let $x_0 \in \mathbb{R}^n$ and consider a consistent s-stage Runge-Kutta method for (2.1):

$$Y_i = x_0 + h\sum_{j=1}^{s} a_{ij} f(Y_j), \quad i = 1, \ldots, s \tag{2.4}$$

$$x = x_0 + h\sum_{i=1}^{s} b_i f(Y_i) \tag{2.5}$$

with $\sum_{i=1}^{s} b_i = 1$. If the scheme is explicit, i.e., if $a_{ij} = 0$ for $j \geq i$, then there always exists a unique solution to equations (2.4). If the scheme is implicit, then in order to be able to guarantee that equations (2.4) admit a unique solution it may be necessary to restrict the step size to $h \in [0, \varphi(x_0)]$ for some maximal step size $\varphi(x_0)$ depending on the state $x_0 \in \mathbb{R}^n$. In all subsequent statements on implicit schemes, we will tacitly assume that such a step size restriction is imposed if necessary.

A suitable choice for $\varphi(x)$ may be obtained in the following way. Let $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, non-decreasing function with $|f(x)| \leq |x| \, \gamma\left(|x|\right)$ for all $x \in \mathbb{R}^n$ (such a function always exists since $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz vector field with $f(0) = 0$). Let $L_\lambda : \mathbb{R}^n \to (0, +\infty)$ be a continuous function with $L_\lambda(x_0) \geq \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|} \;:\; x, y \in N_\lambda(x_0)\,,\, x \neq y \right\}$ for all $x_0 \in (\mathbb{R}^n \backslash \{0\})$, with $N_\lambda(x_0) := \{x \in \mathbb{R}^n \,:\, |x - x_0| \leq \lambda\, |x_0|\,\}$, $\lambda \in (0, 1)$. The continuous function $\varphi(x) := \frac{\lambda}{|A|\,(\,L_\lambda(x)+\gamma(|x|))}$, where $|A| := \max_{i=1,\ldots,s} \sum_j^s |a_{ij}|$, guarantees that for all $x_0 \in \mathbb{R}^n$ and $h \in [0, \varphi(x_0)]$ the equations (2.4) have a unique solution satisfying $Y_i \in N_\lambda(x_0)$, $i = 1, \ldots, s$.

Note, however, that this bound may be conservative. For instance, we may apply the implicit Euler scheme ($s = 1, a_{11} = b_1 = 1$) to an asymptotically stable linear ODE of the form $\dot{x} = Qx$ with a matrix $Q \in \mathbb{R}^{n\times n}$, i.e., all eigenvalues of Q have negative real part. Then (2.4) becomes

$$Y_1 = x_0 + hQY_1 \quad \Leftrightarrow \quad (I - hQ)\,Y_1 = x_0$$

which always has a unique solution because all eigenvalues of $-Q$ and thus of $I - hQ$ have positive real parts for all $h \geq 0$; hence $I - hQ$ is invertible for all $h \geq 0$.

In order to obtain the hybrid system (2.2) from (2.4), (2.5), we define

$$F(h, x_0) := h^{-1}\,(x - x_0) = \sum_{i=1}^{s} b_i f(Y_i) \tag{2.6}$$

A moment's thought reveals that for every locally bounded $u : \mathbb{R}^+ \to \mathbb{R}^+$ and $x_0 \in \mathbb{R}^n$ the solution of (2.2) with (2.6) coincides at each τ_i, $i \geq 0$ with the numerical solution of (2.1) with $x(0) = x_0$ obtained by using the Runge-Kutta numerical scheme (2.4), (2.5) and using the discretization step sizes $h_i = \varphi(x(\tau_i))\exp(-u(\tau_i))$, $i \geq 0$. The reader should notice that other ways (besides (2.6)) of defining the vector field $F : \bigcup_{x\in\mathbb{R}^n} ([0, \varphi(x)] \times \{x\}) \to \mathbb{R}^n$ may be possible; here we have selected the simplest way of obtaining a piecewise linear numerical solution.

Appropriate step size restriction can always guarantee that (2.3) holds for F from (2.6). For example, if $\varphi(x) := \frac{\lambda}{|A|\,(\,L_\lambda(x)+\gamma(|x|))}$ is the step size restriction described above, then F from (2.6) satisfies $|F(h, x)| \leq |x|\,[1 + r(1 + \lambda)\,(\sum_{i=1}^{s} |b_i|)\ \gamma\,((1 + \lambda)\,|x|)]$ for all $x \in \mathbb{R}^n$ and $h \in [0, \varphi(x)]$. Thus (2.3) holds with $M(y) := 1 + r(1 + \lambda)\,(\sum_{i=1}^{s} |b_i|)\ \gamma\,((1 + \lambda)y)$.

Before we turn to the problem formulation, we collect some further estimates on Runge-Kutta schemes which will be useful in the following sections.

If the Runge-Kutta scheme (2.4), (2.5) is of order $p \geq 1$, we will occasionally further assume that $f \in C^p(\mathbb{R}^n; \mathbb{R}^n)$ and for each fixed $x \in \mathbb{R}^n$ the mapping $[0, \varphi(x)] \ni h \to F(h, x)$ is p times continuously differentiable with

$$|F(h, x)| + \sum_{j=1}^{p} \left| \frac{\partial^j}{\partial\, h^j} F(h, x) \right| \leq G\,(|x|)\ \max\left\{\, |f(y)|\ :\ y \in \mathbb{R}^n\,,\ |y - x| \leq |x|\,\varphi(x)M(|x|)\,\right\} \tag{2.7}$$

for all $x \in \mathbb{R}^n$ and $h \in [0, \varphi(x)]$ and some continuous, non-decreasing function $G : \mathbb{R}^+ \to \mathbb{R}^+$, where $M : \mathbb{R}^+ \to \mathbb{R}^+$ is the function involved in (2.3). Again, appropriate step size restriction can always guarantee that (2.7) holds for F from (2.6). Notice that the implicit function theorem for (2.4) guarantees for each fixed $x \in \mathbb{R}^n$ the existence of $\varphi(x) > 0$ such that the mapping $[0, \varphi(x)] \ni h \to F(h, x)$ is p times continuously differentiable. A suitable choice for $\varphi(x)$ may be obtained by the formula $\varphi(x) := \frac{\lambda}{1+2|A|\max\{\,|Df(z)|\,:\,|z|\leq(1+\lambda)|x|\,\}}$, where $\lambda \in (0, 1)$, $|A| := \max_{i=1,\ldots,s} \sum_{j}^{s} |a_{ij}|$. However, again this step size restriction may be conservative, e.g., for explicit schemes.

Using Theorem II.3.1 in [18], (2.7), the fact that $f \in C^p(\mathbb{R}^n; \mathbb{R}^n)$ and the fact that $g_k(z(h, x)) = \frac{\partial^k}{\partial\, h^k} z(h, x)$ for $k \geq 1$, where $g_k : \mathbb{R}^n \to \mathbb{R}^n$ for $k = 1, \ldots, p + 1$ are vector fields obtained by the recursive formulae $g_1(z) = f(z)$, $g_{i+1}(z) = Dg_i(z)f(z)$, we may conclude that there exist continuous functions $N : \mathbb{R}^n \to (0, +\infty)$, $C : \mathbb{R}^n \to \mathbb{R}^+$ such that the inequalities

$$\begin{aligned} C(x) \leq N\,(x)\ \Big[&\max\left\{\, |f(y)|\ :\ y \in \mathbb{R}^n\,,\ |y - x| \leq |x|\,\varphi(x)M(|x|)\,\right\} \\ &+\ \max\left\{\, |f(z(h, x))|\ :\ h \in [0, \varphi(x)]\,\right\} \Big] \end{aligned} \tag{2.8}$$

and

$$|z(h,x) - x - hF(h,x)| \le h^{p+1} C(x) \tag{2.9}$$

hold for all $x \in \mathbb{R}^n$ and $h \in [0, \varphi(x)]$.

If we further assume that there exists a neighborhood $\mathcal{N} \subseteq \mathbb{R}^n$ with $0 \in \mathcal{N}$ satisfying

(i) there exists a constant $\Lambda > 0$ and an integer $q \ge 1$ such that $|f(x)| \le \Lambda\,|x|^q$ for all $x \in \mathcal{N}$

(ii) there exists a constant $Q > 0$ such that $|z(h,x)| \le Q\,|x|$ for all $x \in \mathcal{N}$ and $h \in [0, \varphi(x)]$

then it follows from (2.8) that there exists a neighborhood $\widetilde{\mathcal{N}} \subseteq \mathcal{N}$ with $0 \in \widetilde{\mathcal{N}}$ and a constant $K > 0$ such that

$$C(x) \le K h^{p+1}\,|x|^q \text{ for all } x \in \widetilde{\mathcal{N}}. \tag{2.10}$$

Remark 2.1 Modelling numerical schemes as hybrid systems is nonstandard since usually numerical approximations are represented as discrete time dynamical systems. In this context, varying time steps can either be handled as part of an extended state space, cf. [29], or by defining the discrete time system on the nonuniform time grid $\{\tau_0, \tau_1, \tau_2, \ldots\}$ induced by the time steps, cf. [27] or [5]. In particular, the formulation in [5] in which the time steps h_i are included as additional arguments in the solution maps is very similar to our approach and we conjecture that with this setting one could obtain similar results as in this paper. Still, we believe that for our purposes hybrid systems have some advantages over the alternative discrete time approaches as summarized in the following points.

(i) In our problem formulation, below, we aim at stability statements for all step size sequences $(h_i)_{i\in\mathbb{N}_0}$ with $h_i > 0$ and $h_i \le \varphi(x(\tau_i))$, cf. the discussion after Definition 2.3. Once φ is fixed, for the hybrid system (2.2) this is equivalent to ensuring the desired stability property for all locally bounded functions $u : \mathbb{R}^+ \to \mathbb{R}^+$. Hence, our hybrid approach leads to an explicit condition ("for all u") while the discrete time approach leads to a more technical implicit condition ("for all h_i satisfying $h_i \le \varphi(x(\tau_i))$").

(ii) The interpolation of the solution in between the grid points τ_i as induced by the definition of F in (2.6) does not complicate our analysis. Indeed, it is well known that any meaningful interpolation of numerical solutions does not change the stability behavior of the resulting solution. We have decided to include the interpolation in order to make our definition of hybrid systems compatible with the literature we are using. While on the one hand this makes the definition of the numerical approximation somewhat more technical, on the other hand we do not have to keep track of the grid points τ_n or time steps h_i in formulating our results which enhances the readability of these statements.

(iii) Last but not least, the formulation via hybrid models enables us to use readily available stability results from the hybrid control systems literature, while for other formulations we would have to rely on ad hoc arguments in several places in this paper. ◁

Let us now turn to the formulation of the problem we will consider in this paper. We assume that (2.1) satisfies the following property, cf. [30] (see also [22, 25]).

Definition 2.2 We say that the origin $0 \in \mathbb{R}^n$ is *uniformly globally asymptotically stable* (UGAS) for (2.1) if it is

(i) *Lyapunov stable*, i.e., for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|z(t,z_0)| \le \varepsilon$ for all $t \ge 0$ and all $z_0 \in \mathbb{R}^n$ with $|z_0| \le \delta$ and

(ii) *uniformly attractive*, i.e., for each $R > 0$ and $\varepsilon > 0$ there exists $T > 0$ such that $|z(t,z_0)| \le \varepsilon$ for all $t \ge T$ and all $z_0 \in \mathbb{R}^n$ with $|z_0| \le R$.

Furthermore, we say that $0 \in \mathbb{R}^n$ is *locally exponentially stable* if there exists $C > 0$, $\sigma > 0$ and $\delta > 0$ such that $|z(t,z_0)| \le C\exp(-\sigma t)|z_0|$ holds for all $t \ge 0$ and all $z_0 \in \mathbb{R}^n$ with $|z_0| \le \delta$. ◁

Given an ordinary differential equation (2.1) for which the origin is UGAS, our goal is to be able to produce numerical solutions which inherit this qualitative property. That is, we would like to know a continuous function $\varphi : \mathbb{R}^n \to (0, r]$ such that the numerical solution produced by (2.2) has the correct qualitative behavior, i.e., that $x(t, x_0, u)$ (instead of $z(t, z_0)$) satisfies Definition 2.2(i) and (ii). Continuity of the function $\varphi : \mathbb{R}^n \to (0, r]$ is essential because without continuity it may happen that $\liminf_{x\to 0} \varphi(x) = 0$. This would require discretization step sizes of vanishing magnitude as $t \to +\infty$ which we would like to avoid.

More specifically, we would like to be able to guarantee the correct behavior for the numerical solution uniformly for arbitrary positive discretization step sizes $h_i \le \varphi(x(\tau_i))$. By means of our choice of the step size as $h_i = \varphi(x(\tau_i)) \exp(-u(\tau_i))$, this leads to the following definition, cf. [22].

Definition 2.3 We say that the origin $0 \in \mathbb{R}^n$ is *uniformly robustly globally asymptotically stable* (URGAS) for (2.2) if it is

(i) *robustly Lagrange stable*, i.e., for each $\varepsilon > 0$ it holds that $\sup\{|x(t, x_0, u)| \mid t \ge 0,\, |x_0| \le \varepsilon,\, u : \mathbb{R}^+ \to \mathbb{R}^+ \text{ locally bounded}\} < \infty$.

(ii) *robustly Lyapunov stable*, i.e., for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|x(t, x_0, u)| \le \varepsilon$ for all $t \ge 0$, all $x_0 \in \mathbb{R}^n$ with $|x_0| \le \delta$ and all locally bounded $u : \mathbb{R}^+ \to \mathbb{R}^+$ and

(iii) *robustly uniformly attractive*, i.e., for each $R > 0$ and $\varepsilon > 0$ there exists $T > 0$ such that $|x(t, x_0, u)| \le \varepsilon$ for all $t \ge T$, all $x_0 \in \mathbb{R}^n$ with $|x_0| \le R$ and all locally bounded $u : \mathbb{R}^+ \to \mathbb{R}^+$. $\triangleleft$

Contrary to the ordinary differential equation (2.1), for the hybrid system (2.2) Lyapunov stability and attraction do not necessarily imply Lagrange stability. This is why — in contrast to Definition 2.2 — we explicitly included this property in Definition 2.3.

Ensuring asymptotic stability for all (positive) step sizes $h_i \le \varphi(x(\tau_i))$ is important because it allows us to couple our method with other step size selection schemes. For instance, we could use the step size $\min\{\varphi(x(\tau_i)), \tilde{h}_i\}$ where $\tilde{h}_i$ is chosen such that a local error bound is guaranteed. Such methods are classical, cf. [18] or any other textbook on numerical methods for ODEs and also Example 2.4, below. Proceeding this way results in a numerical solution which is asymptotically stable and at the same time maintains a pre-defined accuracy. Note that our approach will not incorporate error bounds, hence the approximation may deviate from the true solution, at least in the transient phase, i.e., away from 0. On the other hand, as Example 2.4, below, shows, local error based step size control does in general not guarantee asymptotic stability of the numerical approximation. Thus, a coupling of both approaches may be needed in order to ensure both accuracy and asymptotic stability.

The precise formulation of the problems we consider in this paper is as follows.

(P1) Existence Problem *Is there a continuous function* $\varphi : \mathbb{R}^n \to (0, r]$*, such that* $0 \in \mathbb{R}^n$ *is URGAS for system* (2.2)*?*

(P2) Design Problem *Construct a continuous function* $\varphi : \mathbb{R}^n \to (0, r]$*, such that* $0 \in \mathbb{R}^n$ *is URGAS for system* (2.2)*.*

Since φ in these problems can be interpreted as a stabilizing feedback for the hybrid system (2.2), this leads to studying a feedback stabilization problem. Consequently, for answering (P1) and (P2) we will use methods from nonlinear control theory.

It is well known that any consistent and stable numerical scheme for ODEs inherits the asymptotic stability of the original equation in a practical sense, even for more general attractors than equilibria see for instance [11, 12] or [35, Chapter 7]. Practical asymptotic stability means that the system exhibits an asymptotically stable set close to the original attractor, i.e., in our case a small neighbourhood around the equilibrium point, which shrinks down to the attractor as the time step h tends to 0.

Here, the property we are looking for, i.e., "true" asymptotic stability, is a stronger property which cannot in general be deduced from practical stability. In [35, Chapter 5], several results for our problem for specific classes of ODEs are derived using classical numerical stability concepts like A-stability, B-stability and the like. In contrast to this reference, in the sequel we use nonlinear control theoretic analysis and feedback design techniques; more precisely small-gain and Lyapunov function techniques in Sections 3 and 4, respectively, for solving Problems (P1) and (P2). This allows us to obtain asymptotic stability results under different structural assumptions and for more general classes of systems as in [35, Chapter 5].

The following example illustrates that in general standard step size control algorithms based on estimating the local error do not solve problem (P2).

Example 2.4 Consider the linear planar system

$$\dot{x}_1 = -0.005x_1 + x_2, \qquad \dot{x}_2 = -x_1 - 0.005x_2 \tag{2.11}$$

The standard local discretization error based step size control method relies on the comparison of the solutions for two method with different consistency orders, cf. [18, pages 167–169]. Here we use the explicit Euler and the Heun scheme. For these schemes, the new step size is given by the formula

$$h_{new} = h \min\left\{ P\,,\, 0.8\sqrt{\frac{1}{err}} \right\}, \tag{2.12}$$

where

$$err = \sqrt{\frac{1}{2}\left(\frac{x_{1,EULER} - x_{1,HEUN}}{sc_1}\right)^2 + \frac{1}{2}\left(\frac{x_{2,EULER} - x_{2,HEUN}}{sc_2}\right)^2}$$

and

$$sc_i = Atol + Rtol\ \max\left\{ |x_i|\ ,\ |x_{i,HEUN}| \right\}, i = 1, 2.$$

Here $Atol > 0$ is the tolerance for absolute errors, $Rtol > 0$ is the tolerance for relative errors, $P \geq 1$ is a constant factor which determines the magnitude of a (possible) increase of the step size, $x_{i,EULER}$ and $x_{i,HEUN}$, $i = 1, 2$, are the approximations of the components of the solution by the respective schemes. We applied this method to (2.4) with initial condition $(x_1, x_2) = (1, 0)$, parameter $P = 2$ and different error tolerances

Figure 2.1(left) shows the phase portrait for $Atol = Rtol = 10^{-2}$: the numerical solution exhibits an asymptotically stable limit cycle of radius $r = 0.17195$. Figure 2.1(right) shows the corresponding step sizes over time which take values in the interval $[0.347, 0.351]$ for large times.

The limit cycle shrinks to the origin as $Atol, Rtol \to 0$, but exists for all $Atol, Rtol > 0$. This is also visible from Figure 2.2, which shows the logarithm of the squared Euclidean norm along the numerical solution for $Atol = Rtol = 10^{-2}$ on the left and for $Atol = Rtol = 10^{-3}$ on the right. Obviously, the numerical solutions are not asymptotically stable.

We will reconsider system (2.4) in Example 4.16, below, where we apply one of the methods proposed in this paper. $\triangleleft$

3 Small-Gain Methodology

One of the tools used in mathematical control theory for nonlinear feedback design is the methodology based on small-gain results. The method was first used in [20] where a nonlinear small-gain result based on the notion of input-to-state stability (ISS, see [32]) was presented. Since then it has been applied successfully to many feedback stabilization problems. Recently, the small-gain

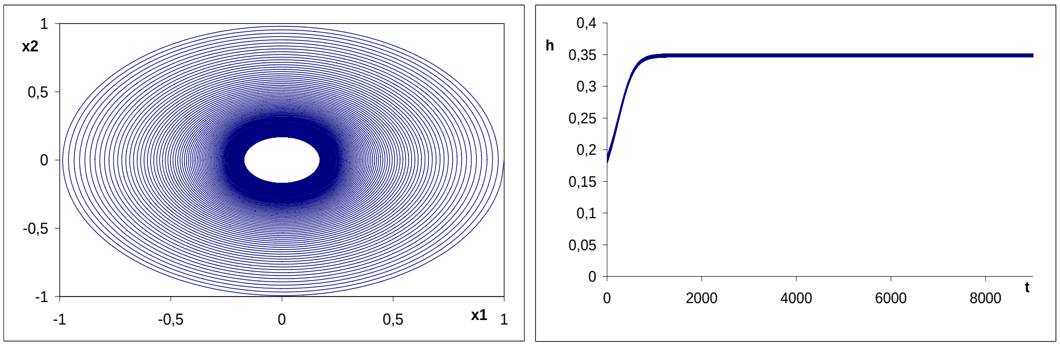

Figure 2.1: Phase portrait of the numerical solution (left) and time steps (right) for $Atol = Rtol = 10^{-2}$

theorem was extended to general control systems including hybrid systems (see [23]) and is thus applicable for the solution of problem (P2) for certain classes of nonlinear systems (2.1). Here we apply the method to two types of systems. The first is a system in triangular form which is called cascade in the control literature. In Section 5, below, we will see that this particular structure is suitable for handling discretizations of certain PDEs.

We consider the system

$$\dot{z} = f_0(z) \tag{3.1}$$

$$\begin{aligned} \dot{x}_1 &= -a_1(x_1)x_1 + f_1(z) \\ \dot{x}_i &= -a_i(x_i)x_i + f_i(z, x_1, \dots, x_{i-1}), \quad i = 2, \dots, n \end{aligned} \tag{3.2}$$

with $z \in \mathbb{R}^m$ and $x = (x_1, \dots, x_n)' \in \mathbb{R}^n$. Here $f_0 : \mathbb{R}^m \to \mathbb{R}^m$, $f_1 : \mathbb{R}^m \to \mathbb{R}$, $f_i : \mathbb{R}^m \times \mathbb{R}^{i-1} \to \mathbb{R}$ and $a_i : \mathbb{R} \to \mathbb{R}$, $i = 2, \dots, n$ are locally Lipschitz mappings with $f_0(0) = 0$, $f_1(0) = \dots = f_n(0, 0, \dots, 0) = 0$. We assume that there exist constants $L_i > 0$, $i = 1, \dots, n$ such that

$$a_i(y) \geq L_i \text{ for all } y \in \mathbb{R} \tag{3.3}$$

We also assume that $0 \in \mathbb{R}^m$ is UGAS for (3.1). Under these assumptions, using the fact that system (3.1), (3.2) has a cascade structure, we may prove by induction over n that the system is UGAS.

The proof for $n = 1$ is based on the fact that for every $x_{10} \in \mathbb{R}$ and for every measurable $u : \mathbb{R}^+ \to \mathbb{R}$ the solution of $\dot{x}_1 = -a_1(x_1)x_1 + u$ with initial condition $x_1(0) = x_{10}$ satisfies

$$|x_1(t)| \leq \exp\left(-\frac{L_1}{2}t\right)|x_{10}| + \frac{1}{L_1}\sup_{0 \leq s \leq t}|u(s)| \qquad \text{for all } t \geq 0 \tag{3.4}$$

Consequently, the solution of $\dot{x}_1 = -a_1(x_1)x_1 + f_1(z)$ satisfies $|x_1(t)| \leq \exp\left(-\frac{L_1}{2}t\right)|x_{10}| + \frac{1}{L_1}\sup_{0 \leq s \leq t}|f_1(z(s))|$, i.e., it is uniformly ISS with respect to the input $z \in \mathbb{R}^m$. Since $0 \in \mathbb{R}^m$ is UGAS for (3.1), a well-known corollary of the small-gain theorem for systems in cascade guarantees UGAS for the composite system. For $n \geq 2$ this argument is used inductively.

Now suppose that a stable numerical scheme is available for (3.1), i.e., there exist functions $\varphi \in C^0(\mathbb{R}^m; (0, r])$, $r > 0$ and $F_0 : \bigcup_{z \in \mathbb{R}^m} ([0, \varphi(z)] \times \{z\}) \to \mathbb{R}^m$ with $F_0(h, 0) = 0$ for all

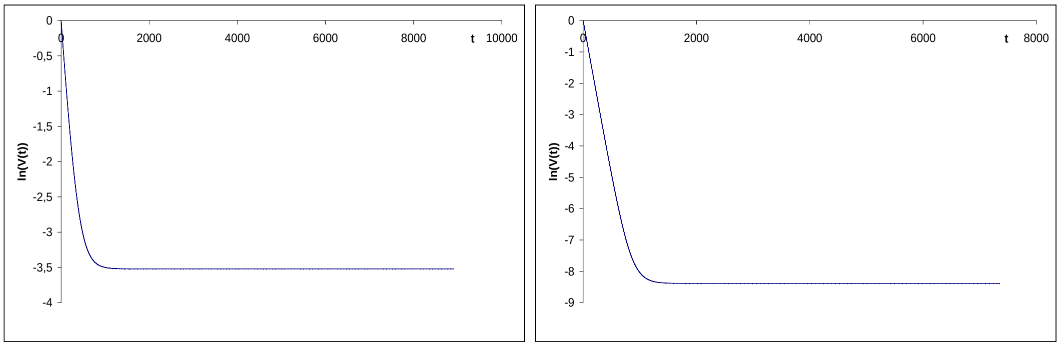

Figure 2.2: Logarithm of the squared Euclidean norm $V(t) = |x(t)|^2$ of the numerical solution for $Atol = Rtol = 10^{-2}$ (left) and $Atol = Rtol = 10^{-3}$ (right)

$h \in [0, \varphi(0)]$ and $\lim_{h\to 0^+} F_0(h,z) = f_0(z)$, for all $z \in \mathbb{R}^m$ such that $0 \in \mathbb{R}^m$ is URGAS for the hybrid system (2.2) with $F = F_0$. Then we propose the following first order numerical scheme for the subsystem (3.2).

$$\begin{aligned} x_1(t+h) &= x_1(t) - h a_1(x_1(t)) x_1(t+h) + h f_1(z(t)) \\ x_i(t+h) &= x_i(t) - h a_i(x_i(t)) x_i(t+h) + h f_i(z(t), x_1(t), \ldots, x_{i-1}(t)), \quad i = 2, \ldots, n \end{aligned} \tag{3.5}$$

The above scheme is a partitioned scheme which treats the states $z, x_1, \ldots, x_{i-1}$ in different ways. The continuous dynamics of the resulting hybrid system are

$$\begin{aligned} \dot z(t) &= F_0(h_i, z(\tau_i)) \\ \dot x_1(t) &= \frac{-a_1\left(x_1(\tau_i)\right)}{1 + h_i a_1\left(x_1(\tau_i)\right)} x_1(\tau_i) + \frac{1}{1 + h_i a_1\left(x_1(\tau_i)\right)} f_1(z(\tau_i)) \\ \dot x_j(t) &= \frac{-a_j\left(x_j(\tau_i)\right)}{1 + h_i a_j\left(x_j(\tau_i)\right)} x_j(\tau_i) + \frac{1}{1 + h_i a_j\left(x_j(\tau_i)\right)} f_j(z(\tau_i), x_1(\tau_i), \ldots, x_{j-1}(\tau_i)) \end{aligned} \tag{3.6}$$

for $j = 2, \ldots, n$. For this scheme the following theorem holds.

Theorem 3.1 The origin $0 \in \mathbb{R}^m \times \mathbb{R}^n$ is URGAS for system (3.6).

The proof of this theorem, which can be found at the end of this section, relies on the following technical lemma which is based on the variations of constants formula.

Lemma 3.2 Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous function with $L = \inf_{y\in\mathbb{R}} a(y) > 0$ and $r > 0$ be a constant. Then for every sequence $\{h_i\}_0^\infty$ with $h_i \in (0, r]$ for all $i \geq 0$, for every locally bounded function $v : \mathbb{R}^+ \to \mathbb{R}$ and for every $x_0 \in \mathbb{R}$ the solution of

$$\begin{aligned} &\dot x(t) = \frac{-a\left(x(\tau_i)\right)}{1 + h_i a\left(x(\tau_i)\right)} x(\tau_i) + \frac{1}{1 + h_i a\left(x(\tau_i)\right)} v(\tau_i), \quad t \in [\tau_i, \tau_{i+1}) \\ &\tau_{i+1} = \tau_i + h_i\,,\ h_i \in (0, r]\,,\ x(t) \in \mathbb{R} \end{aligned} \tag{3.7}$$

with initial condition $x(0) = x_0 \in \mathbb{R}$, $\tau_0 = 0$ satisfies

$$|x(t)| \leq \exp\left(\sigma\, r\right) |x_0| \exp\left(-\sigma\, t\right) + \frac{1}{\sigma\, L} \sup_{0\leq s\leq t} |v(s)| \quad \text{for all } t \in [0, \sup_{i\geq 0} \tau_i) \tag{3.8}$$

where $\sigma > 0$ is any constant such that $\frac{1}{1+s} \le \exp(-\sigma\, s)$ for all $s \in [0, rL]$, i.e., $\sigma \le \frac{\ln(1+rL)}{rL}$.

Proof: For every $i \ge 0$ the variations of constants formula implies

$$x(\tau_{i+1}) = x_0 \prod_{j=0}^{i} (1 + h_j a\,(x(\tau_j)))^{-1} + \sum_{j=0}^{i} \left[h_j v(\tau_j) \left(\prod_{k=j}^{i} (1 + h_k a\,(x(\tau_k)))^{-1} \right) \right] \tag{3.9}$$

Using the definition of L, we obtain the following bound from (3.9)

$$|x(\tau_{i+1})| \le |x_0| \prod_{j=0}^{i} (1 + h_j L)^{-1} + \max_{j=0,\dots,i} |v(\tau_j)| \sum_{j=0}^{i} \left[h_j \left(\prod_{k=j}^{i} (1 + h_k L)^{-1} \right) \right] \tag{3.10}$$

Now the definition of σ implies

$$\begin{aligned} \sum_{j=0}^{i} \left[h_j \left(\prod_{k=j}^{i} (1 + h_k L)^{-1} \right) \right] \ &\le\ \sum_{j=0}^{i} \left[h_j \left(\prod_{k=j}^{i} \exp(-\sigma\, L h_k) \right) \right] \\ = \ \sum_{j=0}^{i} [h_j \exp(-\sigma\, L(\tau_{i+1} - \tau_j))] \ &= \ \exp(-\sigma\, L\tau_{i+1}) \sum_{j=0}^{i} \left[\exp(\sigma\, L\tau_j) \int_{\tau_j}^{\tau_{j+1}} ds \right] \\ \le \ \exp(-\sigma\, L\tau_{i+1}) \sum_{j=0}^{i} \left[\int_{\tau_j}^{\tau_{j+1}} \exp(\sigma\, Ls) ds \right] \ &= \ \exp(-\sigma\, L\tau_{i+1}) \int_0^{\tau_{i+1}} \exp(\sigma\, Ls) ds \ \le \ \frac{1}{\sigma\, L} \end{aligned}$$

which in conjunction with (3.10) implies

$$|x(\tau_{i+1})| \le |x_0| \exp\left(-\sigma\, \tau_{i+1}\right) + \frac{1}{\sigma\, L} \max_{0 \le j \le i} |v(\tau_j)| \tag{3.11}$$

for all $i \ge 0$. Now for every $i \ge 0$ and $t \in [\tau_i, \tau_{i+1})$ it holds that

$$|x(t)| \le \max\{|x(\tau_i)|,\, |x(\tau_{i+1})|\}. \tag{3.12}$$

Combining (3.11) and (3.12) finishes the proof. □

Proof of Theorem 3.1: We proceed by induction over n. For $n = 0$, the assertion follows immediately from the assumption on F_0. For $n \to n+1$, Lemma 3.2 guarantees

$$|x_{n+1}(t)| \le \exp\left(\sigma\, r\right) |x_{n+1}(0)| \exp\left(-\sigma\, t\right) + \frac{1}{\sigma\, L_{n+1}} \sup_{0 \le s \le t} |f_{n+1}(z(s), \dots, x_n(s))|\,,$$

where $\sigma > 0$ is a constant with $\frac{1}{1+s} \le \exp(-\sigma\, s)$ for all $s \in [0, r \max_{i=1,\dots,n+1}(L_i)]$. Now Remark 3.2(b) in [23] (for systems in cascade) guarantees URGAS. □

In Theorem 3.1 we use the special triangular cascade structure of (3.6). Indeed, due to this cascade structure we could also have derived the result from the discrete time Gronwall lemma. The following application shows that with small-gain arguments we can also handle more complex coupling structures. Consider the equation

$$\dot{x}_i = -a_i(x_i)x_i + f_i(x_{-i}), \quad i = 1, \dots, n \tag{3.13}$$

with $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ and $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)^T \in \mathbb{R}^{n-1}$. Here $f_i : \mathbb{R}^{n-1} \to \mathbb{R}$ and $a_i : \mathbb{R} \to \mathbb{R}$ are supposed to be locally Lipschitz for $i = 1, \dots, n$. We assume the existence of constants $L_i > 0$ and $G_{ij} > 0$, $i, j = 1, \dots, n$, with

$$a_i(x_i) \ge L_i \quad \text{and} \quad |f_i(x_{-i})| \le \max_{j \ne i} G_{ij} |x_j| \quad \text{for all } x \in \mathbb{R}^n. \tag{3.14}$$

Systems of the form (3.13) under the assumption (3.14) are frequently found in the neural networks literature, in particular for Hopfield neural networks, see [31] and the references therein.

Again we consider a partitioned first order numerical scheme which is here of the form

$$x_i(t+h) = x_i(t) - ha_i(x_i(t))x_i(t+h) + hf_i(x_{-i}(t)), \quad i = 1, \ldots, n. \tag{3.15}$$

The resulting hybrid system can be written in explicit form as

$$\dot{x}_j(t) = \frac{-a_j(x_j(\tau_i))}{1+h_i a_j(x_j(\tau_i))} x_j(\tau_i) + \frac{1}{1+h_i a_j(x_j(\tau_i))} f_j(x_{-j}(\tau_i)), \quad j = 1, \ldots, n \tag{3.16}$$

with τ_i and h_i as in (2.2) where we use the constant step size selection $\varphi \equiv r > 0$. by virtue of Lemma 3.2 and recent small-gain results in [24] the following result follows. Observe that the resulting scheme is explicit and does not require an iterative solution of nonlinear equations for its implementation.

Theorem 3.3 The origin $0 \in \mathbb{R}^n$ is URGAS for system (3.15) provided that for each $p = 2, \ldots, n$ the inequality

$$G_{i_1 i_2} G_{i_2 i_3} \cdots G_{i_p i_1} < \left(\frac{\ln(1 + r\max\{L_1, \ldots, L_n\})}{r\max\{L_1, \ldots, L_n\}} \right)^p L_{i_1} L_{i_2} \cdots L_{i_p} \tag{3.17}$$

holds for all $i_j \in \{1, \ldots, n\}$ with $i_j \neq i_k$ for $j \neq k$.

Condition (3.17) is termed a cyclic or cycle small-gain condition in mathematical systems theory, cf. [3], [24] or [36]. For $r \to 0$ we obtain

$$\frac{\ln(1 + r\max\{L_1, \ldots, L_n\})}{r\max\{L_1, \ldots, L_n\}} \to 1$$

and we recover the cyclic small-gain condition $G_{i_1 i_2} G_{i_2 i_3} \cdots G_{i_p i_1} < L_{i_1} L_{i_2} \cdots L_{i_p}$ which guarantees that $0 \in \mathbb{R}^n$ is UGAS for the continuous time system (3.13). Provided that this inequality holds, (3.17) gives a condition on the upper bound on the time step $\varphi \equiv r$ such that the asymptotic stability carries over to the numerical approximation.

Finally, note that Theorem 3.3 can easily be adapted to other classes of large scale systems which can be decomposed into smaller subsystems.

4 Lyapunov function based Step Selection

While the small-gain methodology is suitable for systems of differential equations with particular structures, it cannot be applied to general systems in a systematic way. On the other hand, Lyapunov-based feedback design methods can be applied to general nonlinear systems of differential equations and yield explicit formulas for the feedback law (see [33]). In this section we apply the Lyapunov-based feedback design methodology for the solution of Problems (P1) and (P2). It is well known that Lyapunov functions exist for every asymptotically stable ODE and in many applications one can even give explicit formulas for these functions (some examples can be found in Section 6). However, even if a Lyapunov function is not exactly known, under suitable assumptions on the ODE, certain structural properties of the Lyapunov function can be obtained (cf., e.g., Proposition 4.4, below) and used in our context. Hence, the main task of this section is to derive conditions under which the Lyapunov function for the ODE system can be used in order to conclude stability for the hybrid system (2.2) and thus for the numerical approximation of system (2.1).

The results will be developed in the following way. First we provide some background material needed for the derivation of the main results in Section 4.1. In Section 4.2 we consider general consistent Runge-Kutta schemes and provide sufficient conditions for the solvability of Problem (P1) and (P2). The results are specialized for the explicit Euler method. Finally, in Section 4.3, we present special results for the implicit Euler scheme.

4.1 Background Material

The crucial technical result that allows the use of Lyapunov functions for hybrid systems of the form (2.2) is the following lemma.

Lemma 4.1 Consider system (2.2) and suppose that there exist a continuous, positive definite and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}^+$ and a continuous, positive definite function $W : \mathbb{R}^n \to \mathbb{R}^+$ such that for every $x \in \mathbb{R}^n$ the following inequality holds for all $h \in [0, \varphi(x)]$.

$$V(x + hF(h, x)) \le V(x) - h\,W(x) \tag{4.1}$$

Then the origin $0 \in \mathbb{R}^n$ is URGAS for system (2.2).

Proof: Notice first that by virtue of (2.3) there exist a function $\bar{a} \in \mathcal{K}_\infty$ such that for each $x_0 \in \mathbb{R}^n$ and $h \in [0, \varphi(x_0)]$ the solution $y(t)$ of $\dot{y}(t) = F(h, x_0)$, $y(0) = x_0$ exists for all $t \in [0, h]$ and satisfies

$$|y(t)| \le \bar{a}\left(|x_0|\right) \quad \text{for all } t \in [0, h]. \tag{4.2}$$

This $\bar{a}$ can be chosen, e.g., as $\bar{a}(s) = s(1 + rM(s))$ for M from (2.3).

Now consider $R \ge 0$ and the solution $x(t, x_0, u)$ of (2.2) with initial condition $x(0) = x_0$ satisfying $|x_0| \le R$. Since $V : \mathbb{R}^n \to \mathbb{R}^+$ is continuous, positive definite and radially unbounded, it follows from Lemma 3.5 in [25] that there exist functions $a_1, a_2 \in \mathcal{K}_\infty$ with

$$a_1\left(|x|\right) \le V(x) \le a_2\left(|x|\right) \quad \text{for all } x \in \mathbb{R}^n. \tag{4.3}$$

Using induction over i and (4.1) we obtain

$$V(x(\tau_i, x_0, u)) \le V(x_0) \quad \text{for all } i \ge 0. \tag{4.4}$$

Inequality (4.4) in conjunction with (4.3) and (4.2) shows that

$$|x(t, x_0, u)| \le \bar{a}\left(a_1^{-1}\left(a_2\left(|x_0|\right)\right)\right) \quad \text{for all } t \in [0, \sup \tau_i). \tag{4.5}$$

Moreover, inequality (4.4) implies that the sequence $x(\tau_i, x_0, u)$ is bounded, which combined with the fact that $u : \mathbb{R}^+ \to \mathbb{R}^+$ is locally bounded, implies that $t_{\max} = \sup \tau_i = +\infty$. Consequently, estimate (4.5) guarantees both robust Lagrange and robust Lyapunov stability, i.e., Definition 2.3(i) and (ii). In order to prove URGAS it remains to show uniform robust global attractivity, i.e., Definition 2.3(iii). To this end, we next establish that for every $\varepsilon > 0$ the inequality

$$V(x(\tau_i, x_0, u)) \le a_1(\bar{a}^{-1}(\varepsilon)) \quad \text{for all } i \in \mathbb{Z}^+ \text{ with } \tau_i \ge \frac{a_2\left(R\right)}{w(\varepsilon, R)}, \tag{4.6}$$

holds with

$$w(\varepsilon, R) := \min\left\{\, W(x) \,:\, a_2^{-1}\left(a_1(\bar{a}^{-1}(\varepsilon))\right) \le |x| \le \bar{a}\left(a_1^{-1}\left(a_2\left(R\right)\right)\right) \right\} > 0. \tag{4.7}$$

Using (4.3), (4.6) and (4.2) this property implies $|x(t, x_0, u)| \le \varepsilon$ for all $t \ge T = r + \frac{a_2(R)}{w(\varepsilon,R)}$. Since T is independent of u, this shows Uniform Robust Global Attractivity.

It remains to prove (4.6) which we do by contradiction. Let $\varepsilon > 0$ be arbitrary. Suppose that (4.6) does not hold, i.e., that there exists $i \geq 0$ with $\tau_i \geq \frac{a_2(R)}{w(\varepsilon,R)}$ such that $V(x(\tau_i)) > a_1(\bar{a}^{-1}(\varepsilon))$. By virtue of (4.1) it follows that $V(x(\tau_k, x_0, u)) > a_1(\bar{a}^{-1}(\varepsilon))$, for all $k = 0, \ldots, i$. The previous inequality in conjunction with inequalities (4.1), (4.5) and definition (4.7) implies $V(x(\tau_{k+1}, x_0, u)) \leq V(x(\tau_k, x_0, u)) - h_k w(\varepsilon, R)$ for all $k = 0, \ldots, i-1$. Thus, we obtain $V(x(\tau_i, x_0, u)) \leq V(x_0) - w(\varepsilon, R) \sum_{k=0}^{i-1} h_k$. Notice that inequality (4.3) implies that $V(x_0) \leq a_2(R)$. Since $\tau_i = \sum_{k=0}^{i-1} h_k$, we obtain $a_1(\bar{a}^{-1}(\varepsilon)) < a_2(R) - \tau_i w(\varepsilon, R) \leq 0$, a contradiction. This finishes the proof. $\square$

The essential problem with the use of Lemma 4.1 is the knowledge of the Lyapunov function V. In the sequel, we will use a Lyapunov function for the continuous-time system (2.1) in order to construct a Lyapunov function for its hybrid numerical approximation. To this end we use the following definition.

Definition 4.2 A positive definite, radially unbounded function $V \in C^1(\mathbb{R}^n; \mathbb{R}^+)$ is called a *Lyapunov function* for system (2.1) if the inequality

$$\nabla V(x) f(x) < 0 \tag{4.8}$$

holds for all $x \in \mathbb{R}^n \backslash \{0\}$. $\triangleleft$

In the following subsections, we show that under certain assumptions a Lyapunov function V for the original system (2.1) can be used as a control Lyapunov function (see [1, 4, 33, 34]) for its numerical approximation (2.2) in order to design the step size function $\varphi : \mathbb{R}^n \to (0, r]$ in problems (P1) and (P2). For this purpose we need the following technical results whose proofs are provided in the appendix.

Lemma 4.3 Let $V \in C^1(\mathbb{R}^n; \mathbb{R}^+)$ be a Lyapunov function for system (2.1). Then the following statements hold.

(i) There exists a locally Lipschitz, positive definite function $W : \mathbb{R}^n \to \mathbb{R}^+$ such that the inequality

$$W(x) \leq -\nabla V(x) f(x) \tag{4.9}$$

holds for all $x \in \mathbb{R}^n$.

(ii) Let $l_f : \mathbb{R}^n \to (0, +\infty)$ be a continuous function satisfying

$$l_f(x) \geq \sup \left\{ \frac{|f(y) - f(z)|}{|y - z|} \,:\, y, z \in \mathbb{R}^n \,,\, y \neq z \,,\, \max\{V(z), V(y)\} \leq V(x) \right\}$$

for all $x \in (\mathbb{R}^n \backslash \{0\})$. Then for every positive constant $b > 0$ there exists a continuous, positive definite function $\widetilde{W} : \mathbb{R}^n \to \mathbb{R}^+$ such that the inequality

$$V(z(h, x)) \leq V(x) - h\widetilde{W}(x) \tag{4.10}$$

holds for all $x \in \mathbb{R}^n$ and $h \in [0, \varphi(x)]$ with

$$\varphi(x) := \frac{b}{l_f(x)}. \tag{4.11}$$

(iii) Let $b > 0$, $W : \mathbb{R}^n \to \mathbb{R}^+$ be the function from statement (i), above, and let $l_W^b : \mathbb{R}^n \to \mathbb{R}^+$ be a continuous positive definite function satisfying

$$l_W^b(x) \geq \sup \left\{ \frac{|W(y) - W(z)|}{|y - z|} \,:\, y, z \in \mathbb{R}^n \,,\, y \neq z \,,\, \max\{\,|y|\,,\,|z|\,\} \leq \exp(b)\,|x| \right\}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. If there exist constants $\varepsilon, c > 0$ such that

$$|x|\; l_W^b(x) \le c\, W(x) \tag{4.12}$$

holds for all $x \in B_\varepsilon(0)$, then for each $\lambda \in (0,1)$ inequality (4.10) holds for all $x \in \mathbb{R}^n$ and $h \in [0, \varphi(x)]$ with $\widetilde{W}(x) := \lambda\, W(x)$ where $\varphi \in C^0(\mathbb{R}^n; (0,+\infty))$ is any function satisfying

$$\varphi(x) \le \min\left\{ \frac{b}{l_f(x)}\,,\ \frac{(1-\lambda)\exp(-b)W(x)}{|x|\; l_W^b(x) l_f(x)} \right\} \quad \text{for all } x \in \mathbb{R}^n \backslash \{0\}. \tag{4.13}$$

Proposition 4.4 Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable vector field, $0 \in \mathbb{R}^n$ is UGAS and locally exponentially stable for (2.1). Then there exist a Lyapunov function $V \in C^1(\mathbb{R}^n; \mathbb{R}^+)$ for (2.1), a symmetric, positive definite matrix $P \in \mathbb{R}^{n\times n}$ and constants $\varepsilon, \mu > 0$ such that the following inequalities hold.

$$V(x) = x'Px \quad \text{for all } x \in B_\varepsilon(0) \tag{4.14}$$

$$\nabla V(x) f(x) \le -\mu\, |x|^2 \quad \text{for all } x \in \mathbb{R}^n \tag{4.15}$$

4.2 General Runge-Kutta Schemes

In this section we will provide two theorems giving different sufficient conditions for the solvability of the problems (P1) and (P2) for general Runge-Kutta schemes based on a Lyapunov function V for the continuous dynamical system (2.1). Since the expressions involved in these theorems can be quite involved, in addition we present a simple computational method based on our approach in Algorithm 4.14. Our first result uses information on the derivatives of V as formulated in the following theorem.

Theorem 4.5 Suppose that there exists an integer $p \ge 1$ and a Lyapunov function $V \in C^{(p+1)}(\mathbb{R}^n; \mathbb{R}^+)$ for system (2.1). Consider system (2.2) corresponding to a Runge-Kutta scheme for (2.1) and suppose that

(i) for each fixed $x \in \mathbb{R}^n$ the mapping $[0, \varphi(x)] \ni h \to V(x + hF(h,x))$ is $(p+1)$ times continuously differentiable

(ii) the Runge-Kutta scheme is consistent with order $p \ge 1$, i.e., for every $x \in \mathbb{R}^n$ and $h \in [0, \varphi(x)]$ there exists constant $K > 0$ such that $|z(h,x) - x - hF(h,x)| \le Kh^{p+1}$

(iii) there exists a constant $\lambda \in (0,1)$ such that the inequality $\varphi(x) \min_{j=1,\dots,p} K_j(x) \le (\lambda - 1) L_f V(x)$ holds for every $x \in \mathbb{R}^n$, where

$$K_j(x) := \max\left\{ \sum_{i=2}^{j} \frac{s^{i-2}}{i!} L_f^i V(x) + \frac{s^{j-1}}{(j+1)!} \frac{\partial^{j+1}}{\partial\, h^{j+1}} V(x + hF(h,x))\ :\ h, s \in [0, \varphi(x)] \right\}$$

for $j \ge 2$ and $K_1(x) := \frac{1}{2} \max\left\{ \frac{\partial^2}{\partial\, h^2} V(x + hF(h,x))\ :\ h \in [0, \varphi(x)] \right\}$.

Then $0 \in \mathbb{R}^n$ is URGAS for system (2.2).

Proof: Since for each fixed $x \in \mathbb{R}^n$ the mapping $[0, \varphi(x)] \ni h \to g(h) = V(x + hF(h,x))$ is $(p+1)$ times continuously differentiable, by Taylor's theorem for all $j = 1, \ldots, p$ and $h \in [0, \varphi(x)]$ we have

$$V(x + hF(h,x)) = g(h) \le g(0) + \sum_{i=1}^{j} \frac{h^i}{i!} \frac{d^i g}{dh^i}(0) + \frac{h^{j+1}}{(j+1)!} \max_{0 \le \xi \le h} \frac{d^{j+1} g}{dh^{j+1}}(\xi). \tag{4.16}$$

Since the Runge-Kutta scheme is of order $p \ge 1$, we have

$$\frac{d^i g}{dh^i}(0) = L_f^i V(x) \quad \text{for all } i = 1, \ldots, p. \tag{4.17}$$

Consequently, for all $j = 1, \ldots, p$ and $h \in [0, \varphi(x)]$ we obtain

$$V(x + hF(h,x)) \le V(x) + hL_f V(x) + h^2 K_j(x) \tag{4.18}$$

or, equivalently, for all $h \in [0, \varphi(x)]$

$$V(x + hF(h,x)) \le V(x) + hL_f V(x) + h^2 \min_{j=1,\ldots,p} K_j(x) \tag{4.19}$$

The inequality $\varphi(x) \min_{j=1,\ldots,p} K_j(x) \le (\lambda - 1) L_f V(x)$ in conjunction with (4.19) implies $V(x + hF(h,x)) \le V(x) + \lambda h L_f V(x)$. Thus, Lemma 4.1 implies that $0 \in \mathbb{R}^n$ is URGAS for system (2.2). □

Remark 4.6 **(a)** Theorem 4.5 implies the following property for a Runge-Kutta scheme with order $p \ge 1$ satisfying (2.7) and a system of ODEs (2.1) with $f \in C^{(p+1)}(\mathbb{R}^n; \mathbb{R}^n)$ for which $0 \in \mathbb{R}^n$ is UGAS:

If a Lyapunov function $V \in C^{(p+1)}(\mathbb{R}^n; \mathbb{R}^+)$ for (2.1) is available for which there exist constants $K, \Lambda > 0$, an integer $q \ge 1$ and a neighborhood $\mathcal{N} \subset \mathbb{R}^n$ with $0 \in \mathcal{N}$ such that $\nabla V(x) f(x) \le -K |x|^{q+1}$ and $|f(x)| \le \Lambda |x|^q$ for all $x \in \mathcal{N}$, then for every $\lambda \in (0,1)$ and every compact $S \subset \mathbb{R}^n$ we can find $h > 0$ sufficiently small such that $V(x + hF(h,x)) \le V(x) + \lambda h \nabla V(x) f(x)$ for all $x \in S$.

This fact follows from (2.7) and the observation that $K_1(x) = O(|x|^{q+1})$ for x close to zero. Consequently, the numerical solution of (2.1) with sufficiently small step size will give the correct dynamic behavior.

(b) The functions K_j, $j \ge 1$ involved in hypothesis (iii) of Theorem 4.5 are in general difficult to be computed for higher order Runge-Kutta schemes. However, for the explicit Euler scheme $F(h,x) = f(x)$ the function $K_1(x)$ can be computed without difficulty using the formula $K_1(x) := \frac{1}{2} \max \left\{ f'(x) \nabla^2 V(x + hf(x)) f(x) \,:\, h \in [0, \varphi(x)] \right\}$. Consequently, we obtain the following corollary. ◁

Corollary 4.7 (Explicit Euler method) Suppose that there exists a Lyapunov function $V \in C^2(\mathbb{R}^n; \mathbb{R}^+)$ for system (2.1) where $f \in C^0(\mathbb{R}^n; \mathbb{R}^n)$ is locally Lipschitz and that there exist constants $r \ge \delta > 0$, $\lambda \in (0,1)$ and a neighborhood $\mathcal{N} \subset \mathbb{R}^n$ with $0 \in \mathcal{N}$ and

$$\delta\, q(x) \le -2(1-\lambda) \nabla V(x) f(x) \quad \text{for all } x \in \mathcal{N}, \tag{4.20}$$

where $q(x) := \max \left\{ f'(x) \nabla^2 V(x + hf(x)) f(x) \,:\, h \in [0, r] \right\}$. Then Problem (P1) is solvable for system (2.2) with $F(h,x) := f(x)$ and Problem (P2) is solved for any $\varphi \in C^0(\mathbb{R}^n; (0, r])$ satisfying

$$\varphi(x) q(x) \le -2(1-\lambda) \nabla V(x) f(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{4.21}$$

Proof: Inequality (4.20) guarantees the existence of $\varphi \in C^0(\mathbb{R}^n;(0,r])$ satisfying (4.21), e.g., we may define $\varphi(x) := \delta$ if $x \in \mathcal{N}$, $\varphi(x) := \delta$ if $x \notin \mathcal{N}$ and $q(x) \leq 0$, and $\varphi(x) := \min\left\{ -\frac{2(1-\lambda)\nabla V(x)f(x)}{q(x)}\ ,\ \delta \right\}$ else. The rest is a consequence of Theorem 4.5 and the fact that $2K_1(x) \leq q(x)$ for all $x \in \mathbb{R}^n$. □

Remark 4.8 Corollary 4.7 implies the following property for a system of ODEs (2.1) with $f \in C^0(\mathbb{R}^n;\mathbb{R}^n)$ being locally Lipschitz for which $0 \in \mathbb{R}^n$ is UGAS:

If a Lyapunov function $V \in C^2(\mathbb{R}^n;\mathbb{R}^+)$ for (2.1) is available for which there exist constants $K, \Lambda > 0$, an integer $q \geq 1$ and a neighborhood $\mathcal{N} \subset \mathbb{R}^n$ with $0 \in \mathcal{N}$ such that $\nabla V(x)f(x) \leq -K\,|x|^{2q}$ and $|f(x)| \leq \Lambda\,|x|^q$ for all $x \in \mathcal{N}$, then for every $\lambda \in (0,1)$ and every compact $S \subset \mathbb{R}^n$ we can find $h > 0$ sufficiently small such that $V(x + hf(x)) \leq V(x) + \lambda\, h \nabla V(x) f(x)$ for all $x \in S$.

This fact follows from (2.7) and the observation that $q(x) = O(|x|^{2q})$ for x close to zero. Note the difference to Remark 4.6(a): due to the particular structure of the Euler method here we only need to require $\nabla V(x)f(x) \leq -K\,|x|^{2q}$ instead of $\nabla V(x)f(x) \leq -K\,|x|^{q+1}$.

The following second theorem for general Runge-Kutta schemes provides alternative sufficient conditions for the solvability of problem (P2) based on a Lyapunov function for the ODE (2.1). The conditions are different from those in Theorem 4.5, in particular they do not require the Lyapunov function to be smoother than C^1. ◁

Theorem 4.9 Consider system (2.2) corresponding to a Runge-Kutta scheme for (2.1) of order $p \geq 1$ satisfying (2.7), (2.8), (2.9) for certain $\varphi \in C^0(\mathbb{R}^n;(0,+\infty))$. Suppose that

(i) there exist a Lyapunov function $V \in C^1(\mathbb{R}^n;\mathbb{R}^+)$ for system (2.1) and a continuous, positive definite function $\widetilde{W} : \mathbb{R}^n \to \mathbb{R}^+$ such that (4.10) holds for all $x \in \mathbb{R}^n$ and $h \in [0,\varphi(x)]$

(ii) there exists $b \geq 0$ such that $|z(h,x)| \leq \exp(b)\,|x|$ and $|x + hF(h,x)| \leq \exp(b)\,|x|$ for all $x \in \mathbb{R}^n$ and $h \in [0,\varphi(x)]$

(iii) there exists a constant $\lambda \in (0,1)$ such that

$$\varphi(x) \leq \left(\frac{(1-\lambda)\,\widetilde{W}(x)}{l_V^b(x)C(x)} \right)^{\frac{1}{p}} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}, \tag{4.22}$$

where $l_V^b(x) := \max\{|\nabla V(z)| \,:\, z \in \mathbb{R}^n\,,\ |z| \leq \exp(b)\,|x|\,\}$ for all $x \in \mathbb{R}^n$ and $C : \mathbb{R}^n \to \mathbb{R}^+$ is a continuous positive definite function with $|z(h,x) - x - hF(h,x)| \leq C(x)h^{p+1}$ for all $x \in \mathbb{R}^n$ and $h \in [0,\varphi(x)]$.

Then $0 \in \mathbb{R}^n$ is URGAS for system (2.2).

Proof: Utilizing hypotheses (i) and (ii) and

$$l_V^b(x) \geq \sup\left\{ \frac{|V(z) - V(y)|}{|z-y|} \,:\, z, y \in \mathbb{R}^n\,,\ \max\{|y|\,,|z|\} \leq \exp(b)\,|x|\ ,\ z \neq y \right\},$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ and $h \in [0,\varphi(x)]$ we obtain

$$\begin{aligned} V(x + hF(h,x)) &\leq |V(x + hF(h,x)) - V(z(h,x))| + V(z(h,x)) \\ &\leq l_V^b(x)\,|x + hF(h,x) - z(h,x)| + V(x) - h\tilde{W}(x) \end{aligned}$$

For all $x \in \mathbb{R}^n$ and $h \in [0, \varphi(x)]$ this inequality in conjunction with (2.9) gives

$$V(x + hF(h,x)) \leq V(x) - h\left(\tilde{W}(x) - h^p l_V^b(x) C(x)\right),$$

where $C : \mathbb{R}^n \to \mathbb{R}^+$ is the continuous positive definite function with $|z(h,x) - x - hF(h,x)| \leq C(x)h^{p+1}$ for all $x \in \mathbb{R}^n$ and $h \in [0, \varphi(x)]$. Together with (4.22) this implies

$$V(x + hF(h,x)) \leq V(x) - \lambda\, h\tilde{W}(x) \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\} \text{ and all } h \in [0, \varphi(x)]. \tag{4.23}$$

Observing that (2.3) guarantees that (4.23) holds for $x = 0$ as well, Lemma 4.1 implies that $0 \in \mathbb{R}^n$ is URGAS for system (2.2). □

Remark 4.10 The proof of Lemma 4.3 (see formula (A.3) in the appendix), inequality (2.10) and Theorem 4.8 imply the following fact for a Runge-Kutta scheme with order $p \geq 1$ and a system of ODEs (2.1) with $f \in C^p(\mathbb{R}^n; \mathbb{R}^n)$ for which $0 \in \mathbb{R}^n$ is UGAS:

If a Lyapunov function $V \in C^2(\mathbb{R}^n; \mathbb{R}^+)$ for (2.1) is available for which there exist constants $K, \Lambda, c > 0$, an integer $q \geq 1$ and a neighborhood $\mathrm{N} \subset \mathbb{R}^n$ with $0 \in \mathrm{N}$ such that $\nabla V(x) f(x) \leq -K\,|x|^{q+1}$, $|f(x)| \leq \Lambda\,|x|^q$ and (4.12) with $W(x) := -\nabla V(x) f(x)$ holds for all $x \in \mathrm{N}$, then for every $\lambda \in (0,1)$ and every compact $S \subset \mathbb{R}^n$ we can find $h > 0$ sufficiently small such that $V(x + hF(h,x)) \leq V(x) + \lambda\, h\nabla V(x) f(x)$ for all $x \in S$.

This property follows from (2.10) and the observation that

$$l_V^b(x) := \max\left\{|\nabla V(z)| \,:\, z \in \mathbb{R}^n\,,\, |z| \leq \exp(b)\,|x|\,\right\} = O\left(|x|\right)$$

for x close to zero. The reader should notice that in contrast to Remark 4.6(a) we need less smoothness of V here. ◁

The following example illustrates Remarks 4.6 and 4.10.

Example 4.11 Consider the three planar systems with $\dot{x} = f_k(x)$, $k = 1, 2, 3$, $x = (x_1, x_2)' \in \mathbb{R}^2$ with

$$f_1(x) := \begin{bmatrix} -x_1 + x_2 \\ -x_1 - x_2 \end{bmatrix}, \quad f_2(x) := \begin{bmatrix} -|x|^2 x_1 + x_2 \\ -x_1 - |x|^2 x_2 \end{bmatrix}, \quad f_3(x) := |x|^2 \begin{bmatrix} -x_1 + x_2 \\ -x_1 - x_2 \end{bmatrix}.$$

For each of the systems we can use the Lyapunov function $V(x) = |x|^2$. We obtain

$$\nabla V(x) f_1(x) = -2\,|x|^2\,, \quad \nabla V(x) f_2(x) = -2\,|x|^4\,, \quad \nabla V(x) f_3(x) = -2\,|x|^4\,.$$

Clearly, for $k = 1, 2, 3$ there exist constants $\Lambda_k > 0$, integers $q_k \geq 1$ and a neighborhood $\mathcal{N} \subset \mathbb{R}^2$ with $0 \in \mathcal{N}$ such that $|f_k(x)| \leq \Lambda_k\,|x|^{q_k}$ for all $x \in \mathcal{N}$ with $q_1 = q_2 = 1$ and $q_3 = 3$. Remark 4.6(a) shows that for $k = 1$ and $k = 3$ we can apply any consistent Runge-Kutta numerical scheme with sufficiently small step size and produce a qualitatively correct numerical solution. The same conclusion is derived from Remark 4.10 (notice that (4.12) holds for each of the systems with $W_k(x) := -\nabla V(x) f_k(x)$, $l_{W_1}^b(x) = 4\exp(b)\,|x|$, $l_{W_2}^b(x) = l_{W_3}^b(x) = 8\exp(3b)\,|x|^3$ for a neighborhood $\mathcal{N} \subset \mathbb{R}^2$ with $0 \in \mathcal{N}$). This is confirmed by the numerical simulations for the Euler and Heun scheme shown in Figure 4.1 for constant step size $h = 0.2$.

On the other hand, the requirements presented in Remark 4.6(a) or Remark 4.10 are not fulfilled for $k = 2$. Similarly, the requirements presented in Remark 4.8 are not fulfilled for $k = 2$. Consequently, we cannot conclude that the application of any consistent Runge-Kutta numerical scheme with sufficiently small step size will produce a qualitatively correct numerical solution. Numerical solutions with the explicit Euler and the Heun scheme confirm these results, cf.

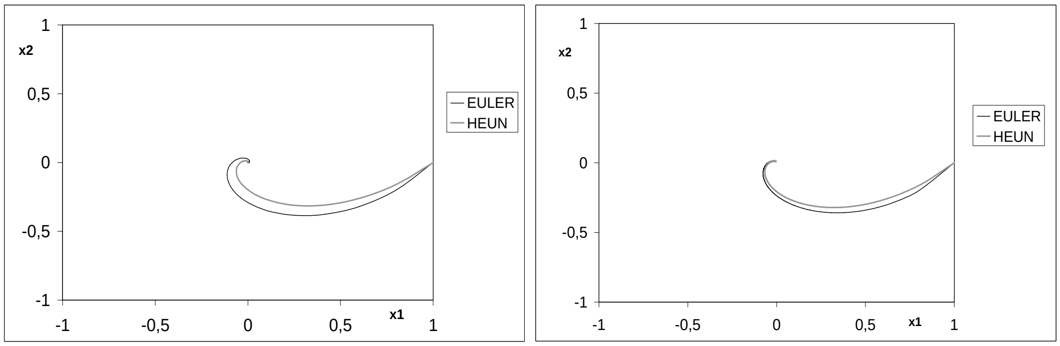

Figure 4.1: Numerical solution for $\dot{x} = f_1(x)$ (left) and $\dot{x} = f_3(x)$ (right) with initial condition $x = (1, 0)$ using the explicit Euler and the Heun method

Figure 4.2. For the system $\dot{x} = f_2(x)$ both schemes applied with constant $h > 0$ exhibit an asymptotically stable limit cycle, which shrinks to the origin as $h \to 0$, but exists for all $h > 0$.

Observe that a local error based step size control does not resolve the lack of asymptotic stability of the origin for our second example. Figure 4.3 shows the solutions for this method using the Euler and Heun schemes outlined in Example 2.4 for parameters $P = 1$, $Rtol = 0$ and $Atol = 10^{-4}$. Again, the numerical solution exhibits an asymptotically stable limit cycle which shrinks to the origin as $Atol \to 0$, but exists for all $Atol > 0$.

This behavior is expected from our theoretical results, since the fact that the requirements of Remarks 4.6(a), 4.8 and 4.10 are not satisfied indicates that for this system and the chosen explicit methods any step size control method will fail to provide an asymptotically stable solution. Note that this is a different situation as in Example 2.4 as we will see in Example 4.16, below.

We would like to emphasize that Theorem 4.5 and Theorem 4.9 and the respective Remarks 4.6(a), 4.8 and 4.10 derived from these theorems do not state that there does not exist a Runge-Kutta scheme which produces an asymptotically stable approximation for system $\dot{x} = f_2(x)$, since the conditions in these results are merely sufficient but not necessary. In fact, for instance the implicit Euler scheme produces an asymptotically stable approximation, which we will rigorously show in Example 4.21, below. $\triangleleft$

Based on the general Theorem 4.9, the following theorem shows that for the special case of a locally exponentially stable ODE system, problem (P1) is always solvable.

Theorem 4.12 Consider system (2.1), a consistent Runge-Kutta scheme with order $p \geq 1$ and $f \in C^p(\mathbb{R}^n; \mathbb{R}^n)$. Assume that $0 \in \mathbb{R}^n$ is UGAS and locally exponentially stable for (2.1). Then Problem (P1) is solvable.

Proof: We are going to show that there exists $\varphi \in C^0(\mathbb{R}^n; (0, +\infty))$ satisfying all requirements of Theorem 4.9.

Since $0 \in \mathbb{R}^n$ is UGAS and locally exponentially stable for (2.1), by virtue of Proposition 4.4, there exist a Lyapunov function $V \in C^1(\mathbb{R}^n; \mathbb{R}^+)$ for (2.1), a symmetric, positive definite matrix $P \in \mathbb{R}^{n \times n}$ and constants $\varepsilon, \mu > 0$ such that (4.14), (4.15) hold. It follows from (4.15) that statement (i) of Lemma 4.3 holds with $W(x) := \mu |x|^2$.

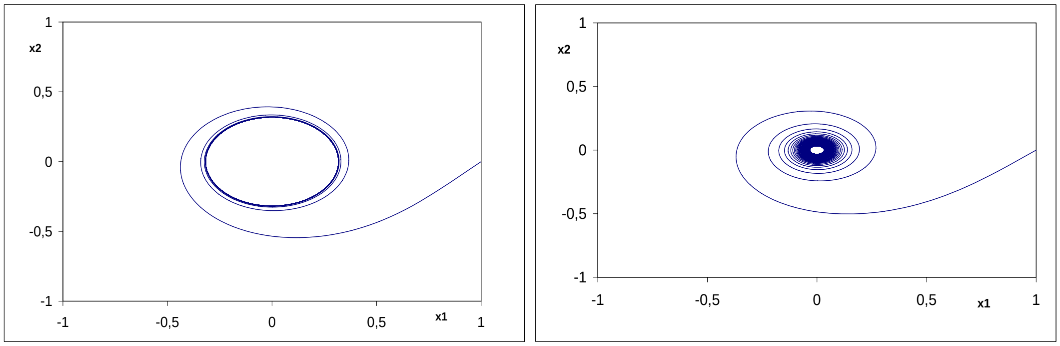

Figure 4.2: Numerical solution for $\dot{x} = f_2(x)$ with initial condition $x = (1, 0)$ using the explicit Euler (left) and the Heun method (right)

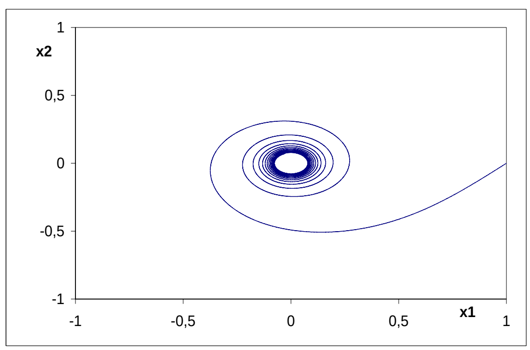

Figure 4.3: Numerical solution for $\dot{x} = f_2(x)$ with local error based step size control

Let $b > 0$. Then for all $x \neq 0$ the inequality

$$l_W^b(x) := 2\exp(b)\,|x| \geq \sup\left\{ \frac{|W(y) - W(z)|}{|y - z|} \;:\; y, z \in \mathbb{R}^n\,,\; y \neq z\,,\; \max\{\,|y|\,,\,|z|\,\} \leq \exp(b)\,|x| \right\}$$

holds. Notice that (4.12) holds for all $x \in \mathbb{R}^n$ with $c := 2\mu^{-1}\exp(b)$. By virtue of statement (iii) of Lemma 4.3, for each $\lambda \in (0, 1)$ inequality (4.10) holds for all $x \in \mathbb{R}^n$ and $h \in [0, \varphi(x)]$ with $\widetilde{W}(x) := \lambda\,\mu\,|x|^2$, where $\varphi \in C^0(\mathbb{R}^n; (0, +\infty))$ is any function satisfying

$$\varphi(x) \leq \frac{1}{1 + 2l_f(x)} \min\{\,2b\,,\,(1-\lambda)\exp(-2b)\mu\,\} \tag{4.24}$$

and $l_f(x) := \{\,|Df(z)| \,:\, z \in \mathbb{R}^n\,,\, V(z) \leq V(x)\,\}$. Moreover, formula (A.3) from the proof of Lemma 4.3 in the appendix shows that for all $x \in \mathbb{R}^n$ and $h \in [0, \varphi(x)]$ it holds that $|z(h, x)| \leq \exp(b)\,|x|$. Let $\bar{\varphi} \in C^0(\mathbb{R}^n; (0, +\infty))$ the function for which (2.7), (2.8), (2.9) hold for all $x \in \mathbb{R}^n$ and $h \in [0, \bar{\varphi}(x)]$. We notice that the inequality $|x + hF(h, x)| \leq \exp(b)\,|x|$ holds for all $x \in \mathbb{R}^n$

and $h \in [0, \varphi(x)]$, where $\varphi \in C^0(\mathbb{R}^n; (0, +\infty))$ is any function satisfying

$$\varphi(x) \le \min\left\{ \bar{\varphi}(x)\,,\ \frac{\exp(b)}{1+\bar{\varphi}(x)M\left(|x|\right)} \right\} \quad \text{for all } x \in \mathbb{R}^n \tag{4.25}$$

and $M : \mathbb{R}^+ \to \mathbb{R}^+$ is the continuous, non-decreasing function involved in (2.3).

Next we show the existence of $\varphi \in C^0(\mathbb{R}^n; (0, +\infty))$ satisfying (4.22). It suffices to show that there exist constants $\delta > 0$, $\lambda \in (0, 1)$ and a neighborhood $\mathcal{N} \subset \mathbb{R}^n$ with $0 \in \mathrm{N}$ such that

$$\delta^p\, l_V^b(x) C(x) \le (1-\lambda)\mu\,\lambda\, |x|^2 \quad \text{for all } x \in \mathcal{N}, \tag{4.26}$$

where $C : \mathbb{R}^n \to \mathbb{R}^+$ is a continuous positive definite function with $|z(h, x) - x - hF(h, x)| \le C(x)h^{p+1}$ for all $x \in \mathbb{R}^n$ and $h \in [0, \varphi(x)]$. Let $\mathcal{N} = B_\rho(0)$, where $\rho := \varepsilon \exp(-b)$ and $\varepsilon > 0$ is the constant involved in (4.14). Clearly, (4.14) implies

$$l_V^b(x) \le 2\,|P| \exp(b)\,|x| \quad \text{for all } x \in \mathcal{N}, \tag{4.27}$$

where $P \in \mathbb{R}^{n\times n}$ is the symmetric, positive definite matrix involved in (4.14). Notice that without loss of generality we may assume that there exists constant $K > 0$ such that (2.10) holds with $q = 1$ for all $x \in \mathcal{N}$ and $h \in [0, \bar{\varphi}(x)]$ (the existence of $Q > 0$ with $|z(h, x)| \le Q\ |x|$ for all $x \in \mathcal{N}$ and $h \ge 0$ is a consequence of local exponential stability). Consequently, by virtue of (2.10), (4.27), we can guarantee that (4.26) holds for every $\lambda \in (0, 1)$ with $\delta := \left(\frac{(1-\lambda)\mu\,\lambda}{2K|P|\exp(b)}\right)^{\frac{1}{p}}$. Therefore, from all the above we conclude that we may define

$$\varphi(x) := \begin{cases} \min\left\{ \delta\,,\ \bar{\varphi}(x)\,,\ \frac{\exp(b)}{1+\bar{\varphi}(x)M(|x|)}\,,\ \frac{\kappa}{1+2l_f(x)} \right\}, & x \in \mathcal{N} \\ \min\left\{ \delta\,,\ \bar{\varphi}(x)\,,\ \frac{\exp(b)}{1+\bar{\varphi}(x)M(|x|)}\,,\ \frac{\kappa}{1+2l_f(x)}\,,\ \left(\frac{(1-\lambda)\,\lambda\mu|x|^2}{l_V^b(x)C(x)}\right)^{\frac{1}{p}} \right\}, & x \notin \mathcal{N}, \end{cases}$$

where $\kappa := \min\{\,2b\,,\ (1-\lambda)\exp(-2b)\mu\,\}$, so that all requirements of Theorem 4.9 are fulfilled. The proof is complete. □

Remark 4.13 Theorem 4.12 is an existence result which does not give an explicit estimate for $\varphi(x)$, i.e., it answers (P1) but does not solve (P2). However, similar to Remark 4.8 and 4.10 we can conclude that the numerical approximation is asymptotically stable on each compact set S for sufficiently small step size h. Note that local exponential stability is not a necessary condition for asymptotic stability of explicit Runge-Kutta schemes, as Example 4.11 shows: there $0 \in \mathbb{R}^2$ is UGAS but not locally exponentially stable for $\dot{x} = f_3(x)$. ◁

The calculations needed in order to verify whether a map φ meets the assumptions of Theorem 4.5 or Theorem 4.9 are rather complex. However, given that the assumptions of one of these theorems are satisfied, an appropriate time step can be obtained by the following straightforward algorithm. Here we assume that we are given a Runge-Kutta schemes and a parameter $\lambda \in (0, 1)$.

Algorithm 4.14 In each step of the computation:

(1) Set $h := 2h$ (where $h > 0$ on the right hand side is the time step from the previous step)

(2) If $V(x + hF(h, x)) \le V(x) + \lambda\, hL_f V(x)$ then the time step $h > 0$ is accepted; otherwise set $h := h/2$ and go to (2)

Here $V \in C^2(\mathbb{R}^n; \mathbb{R}^+)$ is a Lyapunov function for (2.1) for which there exist a constant $K > 0$ and a neighborhood $\mathcal{N} \subset \mathbb{R}^n$ with $0 \in \mathcal{N}$ such that $\nabla V(x)f(x) \le -K\,|x|^2$ for all $x \in \mathcal{N}$. Using this algorithm, we do not have to compute the step size function $\varphi(x)$ that guarantees robust global asymptotic stability of the numerical approximation. The following example illustrates this point.

Example 4.15 We consider four different explicit numerical schemes: the explicit Euler scheme, Heun's scheme, the 2nd order improved polygonal scheme and Kutta's 3rd order scheme. The numerical schemes are applied to the planar system

$$\dot{x}_1 = -x_1 + x_2^2, \qquad \dot{x}_2 = -x_2 - x_1 x_2 \tag{4.28}$$

using the Lyapunov function $V(x) = (x_1^2 + x_2^2)/2$. For all numerical schemes (except the explicit Euler method) the calculation of the maximum allowed time step by using Theorem 4.5 or Theorem 4.9 is very complicated. However, using Algorithm 4.14, for each x we can determine the maximum $h > 0$ for which the inequality $V(x + hF(h,x)) \leq V(x) + \lambda\, hL_f V(x)$ with $\lambda = \frac{1}{2}$ holds. Figure 4.4 shows the graph of the maximum allowable time step for the four numerical methods with $x = (x_1, 1)' \in \mathbb{R}^2$ and varying $x_1 \in \mathbb{R}$.

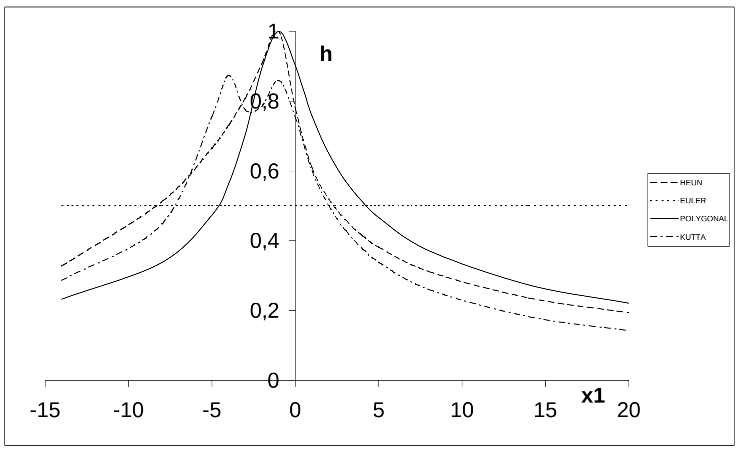

Figure 4.4: Maximum allowable time step determined by Algorithm 4.14 with $\lambda = 1/2$ for various numerical schemes and (4.28) with $x = (x_1, 1)' \in \mathbb{R}^2$

It should be noticed that for x_1 close to zero all higher order schemes allow greater time steps than the the explicit Euler method (notice that for $x = (x_1, 1)' \in \mathbb{R}^2$ and $\lambda = \frac{1}{2}$ the maximum allowable time step for which the inequality $V(x + hF(h,x)) \leq V(x) + \lambda\, hL_f V(x)$ holds for the explicit Euler method is $h = \frac{1}{2}$). However, for large values of $|x_1|$ the maximum allowable time step for higher order schemes are considerably smaller than the time step allowed by the explicit Euler method. This shows that a higher order method does not necessarily allow higher values for the maximum allowable time step for which the inequality $V(x+hF(h,x)) \leq V(x)+\lambda\, hL_f V(x)$ holds. $\triangleleft$

Example 4.16 We apply Algorithm 4.14 to the planar system (2.11). Figure 4.5 shows the logarithm of the value of the Lyapunov function $V(x) = x_1^2 + x_2^2$ for the numerical solution obtained by Heun's 2nd order scheme with $\lambda = 0.5$ or $\lambda = 0.9$ and initial condition $(x_1, x_2) = (1, 0)$: the numerical solution exhibits convergence to the globally exponentially stable equilibrium point $0 \in \mathbb{R}^2$. In both cases the step is selected once and remains constant thereafter ($h = 0.277$ for $\lambda = 0.5$ and $h = 0.165$ for $\lambda = 0.9$). Observe that in contrast to Example 2.4 here asymptotic stability of the numerical approximation is achieved.

Since $\dot{V}(t) = -0.01V(t)$ for system (2.11), it is clear that the logarithm of the Lyapunov function

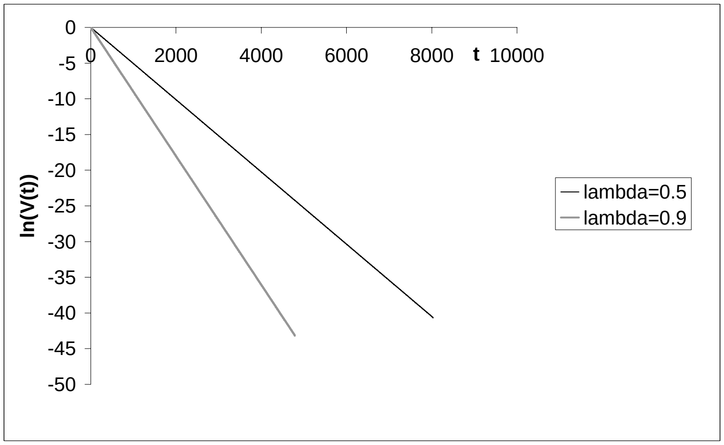

Figure 4.5: Logarithm of the Lyapunov function $V(x) = x_1^2 + x_2^2$ along the numerical solution of (2.11) using Algorithm 4.14 with $\lambda = 0.5$ or $\lambda = 0.9$.

$V(x) = x_1^2 + x_2^2$ along the exact solution will be a straight line with slope -0.01. As $\lambda \to 1$ our numerical result approaches this line at the cost of using smaller step sizes. $\triangleleft$

4.3 Implicit Runge-Kutta schemes

In this section we show how Lyapunov function based arguments can be used for implicit schemes. In order to keep the presentation technically simple, we restrict ourselves to the implicit Euler scheme for which we can prove the following result.

Theorem 4.17 (Implicit Euler method) Suppose that there exists a convex Lyapunov function for (2.1), where $f \in C^0(\mathbb{R}^n;\mathbb{R}^n)$ is locally Lipschitz. Let $\bar{\varphi} \in C^0(\mathbb{R}^n;(0,+\infty))$ be such that the equation $Y = x + hf(Y)$ has a unique solution $Y \in \mathbb{R}^n$ for all $h \in [0, \bar{\varphi}(x)]$ and $x \in \mathbb{R}^n$. Then for each $r > 0$ the origin $0 \in \mathbb{R}^n$ is URGAS for the corresponding system (2.2) with $F(h,x) := f(Y)$, $\varphi(x) := \min\{\bar{\varphi}(x)\ ,\ r\}$, where $Y = x + hf(Y)$.

Proof: Define the functions

$$W_1(x) := \min\left\{ -\nabla V(y)f(y)\ :\ y \in \mathbb{R}^n\ ,\ V(x) \geq V(y) \geq \frac{1}{2}V(x) \right\}, W_2(x) := \frac{1}{2r}V(x). \quad (4.29)$$

By virtue of (4.8) both functions are continuous and positive definite. Since $V \in C^1(\mathbb{R}^n;\mathbb{R}^+)$ is convex the following inequality holds for all $x_1, x_2 \in \mathbb{R}^n$:

$$V(x_1) + \nabla V(x_1)x_2 \leq V(x_1 + x_2) \quad (4.30)$$

Applying (4.30) with $x_1 = Y$ and $x_2 = -hf(Y)$, where $Y = x + hf(Y)$ and $h \in [0, \bar{\varphi}(x)]$, we get

$$V(x) = V(Y - hf(Y)) \geq V(Y) - h\nabla V(Y)f(Y) \quad (4.31)$$

By virtue of (4.8), (4.31) implies that $V(Y) \leq V(x)$. Now we distinguish the following cases.

Case 1: $V(Y) \geq \frac{1}{2}V(x)$. In this case from (4.31) in conjunction with definition (4.29) of W_1 we obtain

$$V(Y) + hW_1(x) \leq V(x). \tag{4.32}$$

Case 2: $V(Y) < \frac{1}{2}V(x)$. In this case definition (4.29) of W_2 implies

$$V(Y) + hW_2(x) \leq V(x) \tag{4.33}$$

for all $h \in [0, r]$.

Consequently, in both cases we obtain

$$V(Y) \leq V(x) - hW(x) \tag{4.34}$$

for all $h \in [0, \varphi(x)]$ and all $x \in \mathbb{R}^n$, where $Y = x + hf(Y)$ and $W(x) := \min\{W_1(x), W_2(x)\}$ is a positive definite function. Thus, Lemma 4.1 yields the assertion. □

The following corollary shows that Theorem 4.17 can be seen as a nonlinear generalization of the well-known A-stability property of the implicit Euler method.

Corollary 4.18 Consider the system of ODEs $\dot{x} = Ax$, $x \in \mathbb{R}^n$ where $A \in \mathbb{R}^{n \times n}$ is a matrix whose eigenvalues have negativereal parts. Then the implicit Euler method is URGAS for arbitrary step size $h > 0$.

Proof: As pointed out before (2.6), the implicit Euler method is well defined for each step size $h > 0$. Furthermore, the system $\dot{x} = Ax$, $x \in \mathbb{R}^n$ admits the quadratic Lyapunov function $V(x) = x'Px$, where $P \in \mathbb{R}^{n \times n}$ is a symmetric, positive definite matrix, see [34, Theorem 5.7.18]. This Lyapunov function is obviously convex and thus Theorem 4.17 yields the assertion. □

Remark 4.19 The main result in [13] guarantees that if $n \neq 4, 5$ then there exists a homeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ with $\Phi(0) = 0$, being a diffeomorphism on $\mathbb{R}^n \setminus \{0\}$ and C^1 on $\mathbb{R}^n$ such that the transformed system (2.1) $\dot{y} = D\Phi(\Phi^{-1}(y)) f\left(\Phi^{-1}(y)\right)$ admits the convex Lyapunov function $V(y) := \frac{1}{2}|y|^2$. Consequently, the implicit Euler can be applied to the transformed system, see [22]. However, for numerical purposes the method is not practical, since the homeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is usually not available. On the other hand, for certain classes of systems Theorem 4.17 is directly applicable. One such class are the so called gradient systems, as shown in the following example. ◁

Example 4.20 Consider the following class of systems

$$\dot{x} = f(x) = -(P(x) + G(x))(\nabla V(x))', \quad x \in \mathbb{R}^n, \tag{4.35}$$

where $V \in C^2(\mathbb{R}^n; \mathbb{R}^+)$ is a positive definite, radially unbounded function with positive definite Hessian and $\nabla V(0) = 0$, $P(x) \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix with locally Lipschitz elements and $G(x) \in \mathbb{R}^{n \times n}$ is a matrix with locally Lipschitz elements with $G'(x) = -G(x)$ for all $x \in \mathbb{R}^n$. The class of systems of the form (4.35) is a generalization of the class of the so-called gradient systems, see [35].

Under our assumptions, V is a convex Lyapunov function for (4.35). Hence, it follows from Theorem 4.17 that the implicit Euler scheme produces asymptotically stable numerical solutions of of (4.35) for every $r > 0$, $\lambda \in (0, 1)$ with $\varphi(x) := \min\left\{\frac{\lambda}{L_\lambda(x) + \gamma(x)}, r\right\}$, where $\gamma : \mathbb{R}^n \to \mathbb{R}^+$ is a continuous function with $|f(x)| \leq |x|\, \gamma(x)$ for all $x \in \mathbb{R}^n$, $L_\lambda : \mathbb{R}^n \to (0, +\infty)$ is a continuous function with $L_\lambda(x) \geq \sup\left\{\frac{|f(z) - f(y)|}{|z - y|} \;:\; z, y \in N_\lambda(x),\, z \neq y\right\}$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $N_\lambda(x) := \{y \in \mathbb{R}^n \;:\; |y - x| \leq \lambda\, |x|\,\}$. ◁

We end this section by noting that Theorem 4.17 also applies to all systems considered in Example 4.11.

Example 4.21 Consider again systems $\dot{x} = f_k(x)$, $k = 1, 2, 3$ from Example 4.11. Since these systems admit the convex Lyapunov function $V(x) = |x|^2$, it follows from Theorem 4.17 that the implicit Euler scheme produces asymptotically stable solutions for all systems $\dot{x} = f_k(x)$, $k = 1, 2, 3$ of Example 4.11. $\triangleleft$

5 Application of the small-gain step selection

In this section we show an application of the small-gain based step selection method developed in Section 3 to a discretization of a PDE. Consider the infinite-dimensional dynamical system

$$\frac{\partial x}{\partial t}(t,z) + c\frac{\partial x}{\partial z}(t,z) = b(x(t,z))x(t,z), \quad z \in (0,1], \quad x(t,0) = 0 \tag{5.1}$$

with $x(t,z) \in \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$ being locally Lipschitz, $c > 0$ and initial condition $x(0,z) = x_0(z)$, where $x_0 \in C^1([0,1];\mathbb{R})$ with $x_0(0) = \frac{dx_0}{dz}(0) = 0$, under the following hypothesis

(H) There exists constant $K \geq 0$ such that $b(x) \leq K$ for all $x \in \mathbb{R}$.

Using the method of characteristics and hypothesis (H), it can be shown that the PDE (5.1) admits a unique classical solution $x(t,\cdot) \in C^1([0,1];\mathbb{R})$ with $x(t,0) = \frac{\partial x}{\partial z}(t,0) = 0$ for all $t \geq 0$. Moreover, the zero solution is globally asymptotically stable, since for every $x_0 \in C^1([0,1];\mathbb{R})$ with $x_0(0) = \frac{dx_0}{dz}(0) = 0$, the unique classical solution $x(t,\cdot) \in C^1([0,1];\mathbb{R})$ of (5.1) with initial condition $x(0,z) = x_0(z)$ satisfies $x(t,z) = 0$ for all $t \geq c^{-1}z$.

Using a uniform space grid of $n+1$ points with space discretization step $\Delta z = \frac{1}{n}$, setting $x_i(t) = x(t, i\Delta z)$, $i = 0, 1, \ldots, n$ and approximating the spatial derivative by the backward difference scheme

$$\frac{\partial x}{\partial z}(t, i\Delta z) \approx \frac{x(t,i\Delta z) - x(t,(i-1)\Delta z)}{\Delta z} = \frac{x_i(t) - x_{i-1}(t)}{\Delta z}$$

for $i = 1, \ldots, n$, we obtain the following set of ordinary differential equations.

$$\begin{aligned} \dot{x}_1 &= -\left(\frac{c}{\Delta z} - b(x_1)\right)x_1 \\ \dot{x}_i &= -\left(\frac{c}{\Delta z} - b(x_i)\right)x_i + \frac{c}{\Delta z}x_{i-1}, \quad i = 2, \ldots, n \end{aligned} \tag{5.2}$$

It is clear that system (5.2) has the structure of system (3.1), (3.2) with $a_i(x_i) = \frac{c}{\Delta z} - b(x_i)$ for $i = 1, \ldots, n$. Moreover, if the space discretization step is selected so that

$$K\Delta z < c \tag{5.3}$$

holds for $K \geq 0$ from Hypothesis (H), then inequalities (3.3) hold as well with $L_i = \frac{c}{\Delta z} - K$ for $i = 1, \ldots, n$. Theorem 3.1 allows us to conclude that for every $h > 0$ the numerical scheme

$$\begin{aligned} x_1(t+h) &= \frac{x_1(t)}{1 + h\left(\frac{c}{\Delta z} - b(x_1(t))\right)} \\ x_i(t+h) &= \frac{x_i(t) + \frac{ch}{\Delta z}x_{i-1}(t)}{1 + h\left(\frac{c}{\Delta z} - b(x_i(t))\right)}, \quad i = 2, \ldots, n \end{aligned} \tag{5.4}$$

will give the correct qualitative behavior.

The reader should notice that for the case $b(x) \equiv 0$ inequality (5.3) is automatically satisfied (with $K = 0$) and the numerical scheme (5.4) reduces to the so-called implicit upwind scheme for the advection equation, which is unconditionally stable.

6 Applications of the Lyapunov-based step selection

In this section we present some applications for the Lyapunov-based step selection method provided in Section 4. It should be emphasized that this method can in principle be applied to all dynamical systems for which a Lyapunov function is known with a globally asymptotically stable and locally exponentially stable equilibrium, cf. Theorem 4.12. However, as the following applications show, there are certain classes of systems for which we can guarantee more properties or which deserve special attention.

6.1 Solution of Nonlinear Programming Problems

There are many nonlinear programming problems which can be solved by constructing a dynamical system with a globally asymptotically stable equilibrium point which coincides with the minimizer of the nonlinear programming problem, see [9, 37, 38, 39]. A special feature for such methods is that a Lyapunov function is available, however, the position of the equilibrium point is not known (this is what we seek). Consider the following nonlinear programming problem.

$$\begin{array}{l} \min f(x)\,,\ x \in \mathbb{R}^n \\ \text{s.t. } \ Ax = b, \end{array} \tag{P}$$

where $f \in C^3(\mathbb{R}^n;\mathbb{R})$ is strictly convex and radially unbounded with positive definite Hessian and $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ with $m < n$ satisfies $\det(AA') \neq 0$. Under these hypotheses there is a global minimum $x^* \in \mathbb{R}^n$ of problem (P). Moreover, there exists a vector $z^* \in \mathbb{R}^m$ such that $(x^*, z^*) \in \mathbb{R}^{n+m}$ is the unique solution of the equations

$$\begin{array}{rcl} \nabla f(x) + z'A &=& 0 \\ Ax &=& b. \end{array} \tag{6.1}$$

Problem (P) may be solved by means of differential equations if we further assume that the function $G(x) = \left|\,\nabla f(x)\left(I - A'(AA')^{-1}A\right)\right|^2 + |Ax-b|^2$ is radially unbounded. Indeed, the system

$$\begin{array}{rcl} \dot{x} &=& -\left(\nabla^2 f(x)\,(\nabla f(x) + z'A)' + A'(Ax-b)\right) \\ \dot{z} &=& -A\,(\nabla f(x) + z'A)' \end{array} \tag{6.2}$$

has the unique equilibrium point $(x^*, z^*) \in \mathbb{R}^{n+m}$, which is UGAS for (6.2). This fact can be proved by using the Lyapunov function $V(x,z) = \frac{1}{2}\left|\nabla f(x) + z'A\right|^2 + \frac{1}{2}\left|Ax-b\right|^2$, which is radially unbounded. Notice that $\dot{V} = -\left|\dot{x}\right|^2 - \left|\dot{z}\right|^2$ for all $(x,z) \in \mathbb{R}^{n+m}$. Thus, the dynamical system (6.2) can be solved by means of Runge-Kutta methods with a Lyapunov-based step selection methodology: each Runge-Kutta method applied to the dynamical system (6.2) will yield a method for the solution of the nonlinear programming problem (P).

Here we will discuss the explicit Euler method. Indeed, the requirements of Corollary 4.7 are fulfilled. In order to see this, let $r > 0$, $\lambda \in (0,1)$ and notice that the function $q : \mathbb{R}^{n+m} \to (0,+\infty)$ involved in (4.20), (4.21) satisfies

$$q(x,z) \leq \left|(\dot{x},\dot{z})\right|^2 p(x,z), \tag{6.3}$$

where $p(x,z) := \max\left\{\left|\nabla^2 V(y,\xi)\right| \,:\, |(y-x,\xi-z)| \leq r\ |(\dot{x},\dot{z})|\right\}$ is a continuous function which can be evaluated without knowledge of the equilibrium point $(x^*,z^*) \in \mathbb{R}^{n+m}$. Let $\mathcal{N} \subset \mathbb{R}^{n+m}$ be defined by $\mathcal{N} := \{(x,z) \in \mathbb{R}^{n+m} \,:\, |(x-x^*, z-z^*)| < c\}$, where $c > 0$ is any positive constant. Then condition (4.20) is implied by the inequality

$$\delta\, p(x,z) \leq 2(1-\lambda) \quad \text{for all } (x,z) \in \mathcal{N} \tag{6.4}$$

and it is clear that (6.4) holds with $\delta > 0$ sufficiently small. Notice that inequality (4.21) is satisfied with $\varphi(x,z) \le \min\left\{ \frac{2(1-\lambda)}{p(x,z)},\ r \right\}$. Consequently, Corollary 4.7 guarantees that for every $(x_0, z_0) \in \mathbb{R}^{n+m}$, the sequence $\{(x_k, z_k) \in \mathbb{R}^{n+m}\}_0^\infty$ generated by the recursive formulae

$$\begin{aligned} x_{k+1} &= x_k - h_k \left(\nabla^2 f(x_k) \left(\nabla f(x_k) + z_k' A \right)' + A'(Ax_k - b) \right) \\ z_{k+1} &= z_k - h_k A \left(\nabla f(x_k) + z_k' A \right)' \end{aligned} \tag{6.5}$$

will achieve convergence to the (unknown) equilibrium point $(x^*, z^*) \in \mathbb{R}^{n+m}$ of (6.2), provided that the discretization step size $h_k > 0$ satisfies $h_k \le \min\left\{ \frac{2(1-\lambda)}{p(x_k,z_k)},\ r \right\}$.

6.2 Control Systems under Feedback Control

A class of dynamical systems for which a Lyapunov function is known is the class of control systems for which a continuous feedback stabilizer is designed by using a Lyapunov function based methodology, see [1, 4, 25, 33]. This is evident for the class of so-called triangular control systems, cf. [4]. Consider the triangular control system

$$\begin{aligned} \dot{x}_i &= f_i(x_1, \ldots, x_i) + g_i(x_1, \ldots, x_i) x_{i+1}, \quad i = 1, \ldots, n-1 \\ \dot{x}_n &= f_n(x) + g_n(x) u, \end{aligned} \tag{6.6}$$

where $x = (x_1, \ldots, x_n)' \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $f_i : \mathbb{R}^i \to \mathbb{R}$, $g_i : \mathbb{R}^i \to \mathbb{R}$, $i = 1, \ldots, n$ are locally Lipschitz functions with $f_i(0) = 0$ and $g_i(y) > 0$ for all $y \in \mathbb{R}^i$.

Using backstepping [4], we can construct a smooth function $k : \mathbb{R}^n \to \mathbb{R}$ with $k(0) = 0$ and a positive definite and radially unbounded smooth function $V : \mathbb{R}^n \to \mathbb{R}^+$ such that

$$\nabla V(x) \begin{bmatrix} f_1(x_1) + g_1(x_1) x_2 \\ \vdots \\ f_n(x) + g_n(x) k(x) \end{bmatrix} \le -\sigma V(x), \quad \text{for all } x \in \mathbb{R}^n \tag{6.7}$$

for an appropriate constant $\sigma > 0$. Moreover, $0 \in \mathbb{R}^n$ is locally exponentially stable for the closed-loop system (6.6) with $u = k(x)$ and for every $\Delta \ge 0$ there exist constants $K_1, K_2 > 0$ with

$$K_1 |x|^2 \le V(x) \le K_2 |x|^2 \quad \text{for all } x \in \mathbb{R}^n \text{ with } |x| \le \Delta. \tag{6.8}$$

Consequently, Corollary 4.7 guarantees that the explicit Euler method can be used for the numerical approximation of the closed-loop system (6.6) with $u = k(x)$. Furthermore, Corollary 4.7 can be used in order to obtain an explicit estimate of the allowable discretization time step for the explicit Euler method. Indeed, notice that all requirements of Corollary 4.7 hold: (6.7) in conjunction with (6.8) show that $\nabla V(x) \begin{bmatrix} f_1(x_1) + g_1(x_1) x_2 \\ \vdots \\ f_n(x) + g_n(x) k(x) \end{bmatrix} \le -c\,|x|^2$ for appropriate $c > 0$ for every bounded neighborhood of the origin. Formula (4.21) combined with (6.7) provides an explicit upper bound for the function $\varphi \in C^0(\mathbb{R}^n; (0, r])$ given by

$$\varphi(x) \le \min\left\{ -\frac{2(1-\lambda)\sigma V(x)}{p(x)\,|F(x)|^2},\ r \right\} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}, \tag{6.9}$$

where $F(x) := \begin{bmatrix} f_1(x_1) + g_1(x_1) x_2 \\ \vdots \\ f_n(x) + g_n(x) k(x) \end{bmatrix}$ and $p : \mathbb{R}^n \to (0, +\infty)$ is defined by

$$p(x) := \max\left\{ \left| \nabla^2 V(y) \right| \,:\, |y - x| \le r\,|F(x)| \right\} \tag{6.10}$$

Other Runge-Kutta numerical schemes can be used as well. Notice that the backstepping procedure achieves the construction of the Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}^+$ by constructing a diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ with $\Phi(0) = 0$ such that $V(x) = \Phi'(x)P\Phi(x)$, where $P \in \mathbb{R}^{n\times n}$ is a symmetric, positive definite matrix. Then Theorem 4.17 guarantees that the implicit Euler can be used as well for the transformed closed-loop system (6.6) with $u = k(x)$, i.e.,

$$\dot{z} = \tilde{F}(z) := D\Phi(x)F(x)|_{x=\Phi^{-1}(z)} \quad \text{with} \quad F(x) := \begin{bmatrix} f_1(x_1) + g_1(x_1)x_2 \\ \vdots \\ f_n(x) + g_n(x)k(x) \end{bmatrix}. \tag{6.11}$$

It follows that for every $r > 0$, $\lambda \in (0, 1)$, the implicit Euler scheme can be applied to (6.11) with $\varphi(z) := \min\left\{ \frac{\lambda}{L_\lambda(z)+\gamma(z)} \ , r \right\}$, where $\gamma : \mathbb{R}^n \to \mathbb{R}^+$ is a continuous function with $\left|\tilde{F}(z)\right| \leq |z|\,\gamma(z)$ for all $z \in \mathbb{R}^n$, $L_\lambda : \mathbb{R}^n \to (0, +\infty)$ is a continuous function with

$$L_\lambda(z) \geq \sup\left\{ \frac{\left|\tilde{F}(x) - \tilde{F}(y)\right|}{|x-y|} \ : \ x, y \in N_\lambda(z)\,,\, x \neq y \right\}$$

for all $z \in \mathbb{R}^n \setminus \{0\}$ and $N_\lambda(z) := \{y \in \mathbb{R}^n \ : \ |y - z| \leq \lambda\,|z|\,\}$. This fact was observed in [22].

6.3 Explicit Methods for Stiff Linear Systems

Even for linear stiff systems the results provided by Theorems 4.5, 4.9 and 4.12 have important consequences. Consider the linear system

$$\dot{x} = Ax, \quad x = (x_1, \dots, x_n)' \in \mathbb{R}^n, \tag{6.12}$$

where $A \in \mathbb{R}^{n\times n}$ is a diagonalizable matrix whose eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ have negative real part. The standard criterion used in numerical analysis for the stability of a Runge-Kutta scheme requires that for all $i = 1, \dots, n$, the complex number $h\lambda_i$ lies inside the region $S = \{\, z \in \mathbb{C} \,:\, |R(z)| \leq 1\,\}$, where $R(z)$ is the stability function of the scheme and h is the (constant) discretization step size. The possibility of using larger discretization step size for explicit Runge-Kutta methods than the one allowed by the classical analysis was recently considered in [7, 6]. There it was shown that after a sequence of "small" time steps a "big" time step can be allowed.

Here for simplicity, we consider the explicit Euler scheme. The fact that the eigenvalues of A have negative real part guarantees the existence of a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ so that $PA + A'P < 0$. Then Corollary 4.7 implies that for every $\lambda \in (0, 1)$, $r > 0$ the step-size function $\varphi \in C^0(\mathbb{R}^n; (0, r])$ satisfying the inequality

$$\varphi(x) \leq \min\left\{ -\frac{(1-\lambda)x'(A'P + PA)x}{x'A'PAx} \ ; \ r \right\}, \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\} \tag{6.13}$$

guarantees that the numerical solution produced by the explicit Euler scheme has the correct qualitative behavior. Notice that the quantity $-\frac{x'(A'P+PA)x}{x'A'PAx}$ depends heavily on the direction of the vector $x \in \mathbb{R}^n$ and can allow greater discretization step sizes than the one produced by classical stability analysis.

Example 6.1 Consider the stiff linear system obtained by space discretization of the heat equation on the unit interval

$$\dot{x}_i = \frac{1}{(\Delta z)^2}\left(x_{i-1} - 2x_i + x_{i+1}\right), i = 1, \dots, n \tag{6.14}$$

with Dirichlet boundary conditions $x_0 = x_{n+1} = 0$ (in this case $\Delta z = \frac{1}{n+1}$). Classical results demand $h < h_0 = \frac{1}{2}(\Delta z)^2$ when the explicit Euler method with constant step size is applied to system (6.14). Systems of ordinary differential equations obtained by semi-discretization of parabolic partial differential equations were recently studied in [7]. Here we will apply Corollary 4.7. For this problem we consider the Lyapunov function

$$V(x) = \frac{1}{2}\sum_{i=1}^{n} P_i x_i^2 \quad \text{with} \quad P_i = \cos(i\omega) + \cos\left((N+1-i)\omega\right), \quad i = 1, \ldots, n, \tag{6.15}$$

where $\omega \in \left(0, \frac{\pi}{2n}\right)$. Notice that for this problem we have $\dot{V} \leq -\frac{2(1-\cos(\omega))}{(\Delta z)^2} V$ for all $x \in \mathbb{R}^n$. Figure 6.1(left) shows the step sizes for the explicit Euler method with

$$h = \varphi(x) = \min\left\{ -\frac{(1-\lambda)x'(A'P + PA)x}{\varepsilon + x'A'PAx} \; ; \; r \right\}, P = diag(P_1, \ldots, P_n), \omega = \frac{\pi}{40} \tag{6.16}$$

$r = 1$, $\varepsilon = 10^{-6}$, $\lambda = 0.8$, $n = 10$ and initial condition $x_i(0) = 1$, $i = 2, \ldots, 9$, $x_1(0) = x_{10}(0) = 4$. Figure 6.1(right) shows the exponential decrease of the value of the Lyapunov function for this simulation.

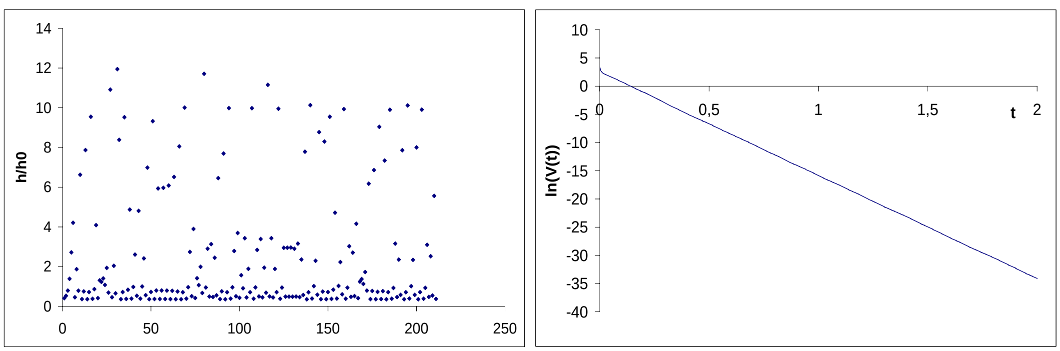

Figure 6.1: Sequence of step sizes for the explicit Euler scheme for (6.14) with (6.16), $n = 10$, $\lambda = 0.8$, $r = 1$ (left) and corresponding value of the Lyapunov function (right)

The reader should notice that in many cases the applied time step was many times higher than the maximum allowable time step $h_0 = \frac{1}{2}(\Delta z)^2 = 0.004132$ for constant step size. In order to counteract the effect of large step sizes there are also many cases where the applied time step was less than h_0. However, after 200 steps the value of time was $t = 2.109$ while that constant step size would give $t < 1$. Figure 6.1(left) shows that the resulting step size policy resulting from the feedback law (6.16) can be described as “many small steps—one large step”.

When the value of λ increases, we get more accurate results. Figure 6.2 shows the step sizes for the explicit Euler method with (6.16) for $n = 10$, $\lambda = 0.95$ and $r = 1$.

Figure 6.2 shows that after a transient phase the feedback law (6.16) results in the policy “one small step—one large step”. Therefore, the feedback law (6.16) can give different step size policies for different values of λ. As expected, a trade-off between the allowable time steps and the accuracy of the numerical solution exists. For $\lambda = 0.95$, after 200 steps the value of time was $t = 1.1956$. $\triangleleft$

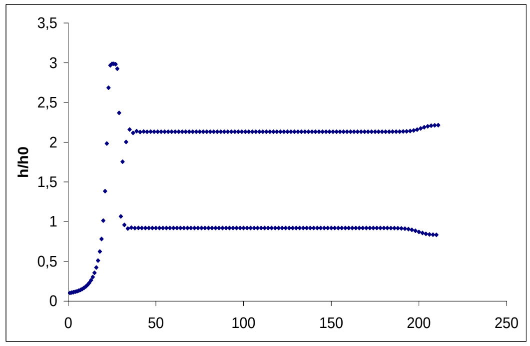

Figure 6.2: Sequence of step sizes for the Explicit Euler scheme for (6.14) with (6.16), $n = 10$, $\lambda = 0.95$, $r = 1$

7 Conclusions

In this work, we considered the problem of step size selection for numerical schemes such that the numerical solution presents the same qualitative behavior as the original system of ODEs. Specifically, we developed tools for nonlinear systems with a globally asymptotically stable equilibrium point which are similar to methods used in nonlinear control theory. It is shown how the problem of appropriate step size selection can be converted to a rigorous abstract feedback stabilization problem for a particular hybrid system. Feedback stabilization methods based on Lyapunov functions and Small-Gain results were employed. The obtained results have been applied to several examples of applications including ODEs and semi-discretized PDEs.

The methodology presented in the present work can be used for more complicated numerical problems such as the step size selection problem for

(i) the numerical approximation of the solution of infinite-dimensional systems, i.e., systems governed by partial differential equations or systems described by retarded functional differential equations

(ii) systems with more complicated attractors,

(iii) time-varying systems

(iv) systems with inputs.

Future work will address these problems.

A Appendix

In this appendix we provide the proofs of Lemma 4.3 and Proposition 4.4.

Proof of Lemma 4.3: **(i)** Define $Q(x) := -\nabla V(x) f(x)$. The function $Q : \mathbb{R}^n \to \mathbb{R}^+$ is continuous and by virtue of (4.8) it is positive definite, too. Standard results on inf-convolutions (see, e.g., [2, Section 3.5]) guarantee that the function $W(x) := \inf \{ Q(y) + |y - x| \, : \, y \in \mathbb{R}^n \}$ is globally Lipschitz on $\mathbb{R}^n$ with Lipschitz constant $L = 1$, positive definite and satisfies (4.9).

(ii) Since $f(0) = 0$, for all $x \in \mathbb{R}^n$ it follows

$$|f(z)| \leq l_f(x)\,|z| \quad \text{for all } z \in \mathbb{R}^n \text{ with } V(z) \leq V(x). \tag{A.1}$$

Inequality (A.1) in conjunction with the fact that $V(z(t,x)) \leq V(x)$ for all $t \geq 0$ and Gronwall's inequality implies

$$\exp\left(-l_f(x)\,t\right)|x| \leq |z(t,x)| \leq \exp\left(l_f(x)\,t\right)|x| \quad \text{for all } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \tag{A.2}$$

Therefore (4.11) and inequality (A.2) imply

$$\exp\left(-b\right)|x| \leq |z(h,x)| \leq \exp\left(b\right)|x| \quad \text{for all } h \in [0, \varphi(x)]. \tag{A.3}$$

Let $W : \mathbb{R}^n \to \mathbb{R}^+$ be the locally Lipschitz, positive definite function which satisfies inequality (4.9) and define

$$\widetilde{W}(x) := \min\left\{W(y)\ :\ y \in \mathbb{R}^n\,,\ \exp(-b)\,|x| \leq |y| \leq \exp(b)\,|x|\,\right\}. \tag{A.4}$$

Clearly, definition (A.4) guarantees that $\widetilde{W} : \mathbb{R}^n \to \mathbb{R}^+$ is a continuous, positive definite function. Moreover, by virtue of inequalities (4.9), (A.3) and definition (A.4) we obtain for all $h \in [0, \varphi(x)]$ and $x \in \mathbb{R}^n$

$$V(z(h,x)) - V(x) = \int_0^h \nabla V(z(s,x)) f(z(s,x))ds \leq -\int_0^h W(z(s,x))ds \leq -h\widetilde{W}(x), \tag{A.5}$$

i.e., (4.10).

(iii) Notice first that inequality (4.12) guarantees that there exists $\varphi \in C^0(\mathbb{R}^n; (0, +\infty))$ satisfying (4.13). Define

$$M_f^b(x) := \max\left\{\,|f(y)|\ :\ y \in \mathbb{R}^n\,,\ |y| \leq \exp(b)\,|x|\,\right\}. \tag{A.6}$$

Inequality (A.1) and definition (A.6) imply

$$M_f^b(x) \leq l_f(x)\exp(b)\,|x|\,. \tag{A.7}$$

Taking into account inequalities (A.3), (A.7) and (4.13) in conjunction with definition (A.6) we obtain for all $h \in [0, \varphi(x)]$ and $x \in \mathbb{R}^n$

$$|W(x) - W(z(h,x))| \leq l_W^b(x)\,|x - z(h,x)| \leq l_W^b(x) l_f(x)\exp(b)\,|x|\,\varphi(x). \tag{A.8}$$

Inequalities (A.8) and (4.13) imply

$$-W(z(h,x)) \leq -\lambda\, W(x) \quad \text{for all } h \in [0, \varphi(x)] \text{ and all } x \in \mathbb{R}^n. \tag{A.9}$$

Moreover, by virtue of inequalities (4.9), (A.3) and definition (A.4) we obtain

$$V(z(h,x)) - V(x) = \int_0^h \nabla V(z(s,x)) f(z(s,x))ds \leq -\int_0^h W(z(s,x))ds \leq -\lambda\, hW(x) \tag{A.10}$$

for all $h \in [0, \varphi(x)]$ and all $x \in \mathbb{R}^n$, i.e., the assertion. This finishes the proof. ∎

Proof of Proposition 4.4: Since the origin $0 \in \mathbb{R}^n$ is locally exponentially stable for (2.1), it follows that the matrix $A := Df(0)$ has only eigenvalues with negative real part. Consequently there exists a symmetric, positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a constant $\mu > 0$ such that

$$x'(A'P + PA)x \leq -2\mu\,|x|^2 \quad \text{for all } x \in \mathbb{R}^n, \tag{A.11}$$

see [34]. Consequently, for sufficiently small $\delta > 0$ we obtain

$$2x'Pf(x) \leq -\mu\,|x|^2 \quad \text{for all } x \in \mathbb{R}^n \text{ with } |x| \leq 2\delta. \tag{A.12}$$

Let $W : \mathbb{R}^n \to \mathbb{R}^+$ a continuously differentiable function with

$$W(x) := \begin{cases} -2x'Pf(x), & |x| < \delta \\ \mu\,|x|^2 \left(1 + |f(x)|^2\right), & |x| > 2\delta \end{cases} \quad \text{and} \quad W(x) \geq \mu\,|x|^2 \quad \text{for all } x \in \mathbb{R}^n \tag{A.13}$$

and define the function

$$V(x) := \int_0^{+\infty} W(z(t,x))dt. \tag{A.14}$$

By virtue of Theorem 2.46 in [8], V as defined by (A.14) is a Lyapunov function for (2.1) satisfying

$$\nabla V(x) f(x) = -W(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{A.15}$$

From Proposition 2.48 in [8] it follows that V is the unique function satisfying (A.15) with $V(0) = 0$. An inspection of the proof of this proposition yields that if equation (A.15) holds on a forward invariant set for (2.1) then uniqueness holds on this set, because uniqueness is established by looking at trajectories in forward time. Furthermore, by virtue of (A.15) and definition (A.13) it follows that (4.15) holds.

Now we pick a forward invariant neighborhood $\mathcal{N} \subset B_\delta(0)$ of zero which exists because $0 \in \mathbb{R}^n$ is asymptotically stable. Then we observe by (A.13) that the function $\widetilde{V}(x) = x'Px$ satisfies (A.15) as well on $\mathcal{N} \subset B_\delta(0)$ and $\widetilde{V}(0) = 0$. Consequently, $V(x) \equiv \widetilde{V}(x) = x'Px$ on $\mathcal{N} \subset B_\delta(0)$. Thus, for $\varepsilon > 0$ sufficiently small such that $B_\varepsilon(0) \subseteq \mathcal{N}$, it follows that (4.14) holds. □

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