

# Differential games and Zubov's Method

Lars Grüne \*      Oana Silvia Serea †

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## Abstract

In this paper we provide generalizations of Zubov's equation to differential games without Isaacs' condition. We show that both generalizations of Zubov's equation (which we call min-max and max-min Zubov equation, respectively) possess unique viscosity solutions which characterize the respective controllability domains. As a consequence, we show that under the usual Isaacs condition the respective controllability domains as well as the local controllability assumptions coincide.

Keywords: asymptotic null controllability, differential games, Lyapunov functions, Hamilton-Jacobi-Bellman equation, viscosity solutions, Zubov's method

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## 1 Introduction

Zubov's method [39] is a classical method for computing Lyapunov functions and domains of attraction for differential equations

$$\dot{x} = f(x), \quad x \in \mathbb{R}^N$$

with a locally asymptotically stable equilibrium  $x^* \in \mathbb{R}^N$ . Zubov's main result states that under appropriate conditions and for a suitable function  $g : \mathbb{R}^N \rightarrow \mathbb{R}$  the Zubov equation

$$(1) \quad \nabla W(x)f(x) = -g(x)(1 - W(x))\sqrt{1 + \|f(x)\|^2},$$

a first order partial differential equation, has a unique differentiable solution  $W : \mathbb{R}^N \rightarrow [0, 1]$  with  $W(x^*) = 0$ , which characterizes the domain of attraction  $\mathcal{D}$  of  $x^*$  via  $\mathcal{D} = \{x \in \mathbb{R}^N \mid W(x) < 1\}$  and which is a Lyapunov function on  $\mathcal{D}$ .

In recent years, Zubov's method has been generalized into various directions. In [9] and [28] it was extended to systems with time varying deterministic perturbation and in [8] to systems with stochastic perturbation. A first application to control systems, in which asymptotic stability is replaced by asymptotic controllability was given in [32], in which Zubov's equation (1) was used in an integral form. The original differential version of (1) was investigated for control systems in [23], [21, Section 7.2] and — under more general assumptions — in [12]. A variant of Zubov's method for systems with both controls and stochastic perturbations was studied in [7]. Furthermore, [10] introduced a numerical method based on the Zubov equation which was originally developed for the deterministically perturbed setting from [9] but was subsequently successfully applied to other settings, see for instance [11] for a controlled inverted pendulum or [22] for a stochastic control problem from mathematical economy. An alternative numerical approach which is, however, so far

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\*Mathematisches Institut, Fakultät für Mathematik und Physik, Universität Bayreuth, 95440 Bayreuth, Germany  
lars.gruene@uni-bayreuth.de

†CMAP Ecole Polytechnique, Palaiseau and University of Perpignan, France oana-silvia.serea@univ-perp.fr

only applicable to Zubov’s original setting without perturbation or control has been presented in [20].

When dealing with Zubov’s equation (1) in its original differential form for perturbed or controlled systems, one immediately realizes that classical solutions, i.e. differentiable functions  $W : \mathbb{R}^N \rightarrow \mathbb{R}$  satisfying (1), can hardly be obtained. Since (1) and its generalizations are Hamilton-Jacobi equations, the concept of viscosity solutions turns out to be the right generalized solution concept, cf., e.g., [1].

In this paper we provide a further generalization of Zubov’s equation, namely to differential games in which two (deterministic) players influence the differential equation. While the first player wants to control the solutions asymptotically towards an equilibrium  $x^*$  (which we always assume to be the origin  $x^* = 0$  in this paper), the second player wants the solutions to stay away from  $x^*$ . Although the interpretation of these two players is in principle arbitrary, in light of the earlier papers on Zubov’s method we find it intuitive to refer to the first player as “control” and to the second player as “perturbation”. In contrast to the case when only one player is present, in the case of two players the interplay between the players is crucial in order to obtain a well defined problem. To this end, the concept of nonanticipative strategies [1, Section VIII.1] (sometimes also called Elliot-Kalton-Varayia strategies, cf. [15]) has turned out to be a concept which provides well defined solutions and is compatible with the Hamilton-Jacobi calculus. In this concept, one player plays with open loop controls while the other player can choose in advance his or her “optimal answer” to the open loop player’s control (hence *strategy*). However, in selecting this strategy, he or she is only allowed to take into account the past of the other players control function (hence *nonanticipative*). In general, this gives a small advantage to the player playing strategies and thus the notions of asymptotic controllability and the values of the corresponding Zubov equation depend on whether the control or the perturbation uses these nonanticipative strategies, cf. also the discussion after Definition 4, below. Consequently, in this paper we investigate both situations in parallel. In particular, we investigate the (uniform) domains of controllability for both situations and define two versions of the local asymptotic controllability condition which extends the local stability condition in Zubov’s original approach to the game theoretic setting. With this setting we will show that both generalizations of Zubov’s equation (which we call min-max and max-min Zubov equation, respectively) possess unique solutions which characterize the respective controllability domains. Furthermore, we will show that under the usual Isaacs condition both situations yield the same solutions in the min-max and in the max-min case and that hence the respective controllability domains as well as the local controllability assumptions coincide. As in the earlier extensions discussed above, we will work both with a generalized Zubov equation and with an equivalent rescaled equation which allows for simpler proofs in some occasions.

When studying uniqueness for Zubov’s equation, one faces two difficulties. One difficulty stems from the fact that the problem belongs to the class of “free boundary problems”. This kind of problems was studied for instance in [2, 36, 37] and the results from [37] play a crucial role in our analysis in order to cope with this problem, cf. Proposition 12. The second difficulty arises because the Zubov PDE has a singularity at the equilibrium  $x^*$ , which is due to the fact that the function  $g$  in (1) and also in our generalized Zubov equation vanishes at the equilibrium, i.e.,  $g(x^*) = 0$ . In most of the results in [2, 36] this problem is circumvented by studying a target problem instead of the Zubov equation which allows for imposing a positive lower bound  $g_0 > 0$  on  $g$ . While this approach also yields a complete characterization of asymptotic controllability in terms of viscosity solutions and will also turn out to be useful as an auxiliary problem in our analysis in this paper, the resulting value functions for the target problem are typically discontinuous unless suitable controllability properties at the boundary of the target are imposed. In contrast to this, as we will show in this paper, the game theoretic generalization of the Zubov PDE always has a continuous viscosity solution. From a systems theoretic point of view, this is a desirable property, for instance because it implies the existence of a continuous Lyapunov function for the system from which

robustness properties of the asymptotic controllability with respect to unmodelled disturbances can be concluded, cf. e.g. [34]. Continuity is also a desirable property from a numerical point of view, since rigorous convergence results for numerical approximations of solutions to Zubov's equation as in, e.g., [10] or [20] typically require continuity of the solution. In [36], the case that  $g$  vanishes at the equilibrium (or on the target in the more general setting of this reference) has been discussed in Remark 3.12, cf. the discussion before Theorem 13. However, in this reference only special choices of  $g$  and only the lower value of the game is considered. Here we provide uniqueness results analysis for the upper and the lower value of the game and for more general choices of  $g$ , which we consider as one of the central contributions of this paper.

The paper is organized as follows. In the ensuing Section 2 we describe the precise setting and the necessary definitions. In Section 3 we define the domains of controllability and prove some properties for these sets. In Section 4 we introduce the Zubov-type differential game and the respective min-max and max-min Zubov equations and show the relation of the value functions to the controllability domains. In Section 5 we show that these Zubov equations possess unique viscosity solutions which hence coincide under Isaacs' condition. Finally, in Section 6 we provide a couple of results implied by our main results and illustrate our approach by a simple example.

## 2 General facts on differential games

In this section we introduce the definitions and notations which are employed in the rest of the paper. We consider a differential game with dynamics given by

$$(2) \quad \begin{cases} i) & \dot{x}(s) = f(x(s), u(s), v(s)) \text{ for almost all } s \geq t \\ ii) & x(0) = x. \end{cases}$$

Here  $f : \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}^n$  is a continuous and bounded vector field and we assume that  $f(x, u, v)$  is locally Lipschitz in  $x$  uniformly in  $u$  and  $v$ . The sets  $U$  and  $V$  are compact subsets of finite dimensional spaces and the controls  $u(\cdot) : [0, \infty) \rightarrow U$  and  $v(\cdot) : [0, \infty) \rightarrow V$  are measurable functions. We define

$$\begin{aligned} \mathcal{U} &= \{u(\cdot) : [0, \infty) \rightarrow U, \text{ measurable}\}, \\ \mathcal{V} &= \{v(\cdot) : [0, \infty) \rightarrow V, \text{ measurable}\} \end{aligned}$$

as the respective sets of control functions.

Note that the boundedness assumption on  $f$  can be made without loss of generality because we could always replace  $f$  by

$$(3) \quad \tilde{f}(x, u, v) = \frac{f(x, u, v)}{\sqrt{1 + \|f(x, u, v)\|^2}}.$$

In fact, Zubov included the factor in the denominator of (3) in his original equation (1) precisely to avoid the problem of unboundedness of  $f$ . In this paper, in order to keep the exposition less technical we prefer to avoid the use of this factor and instead impose the boundedness assumption.

Throughout the paper we will investigate the situation in which the first player wants to control the system asymptotically to the origin  $x = 0$  while the second player tries to avoid this. For this reason, we will usually interpret  $u(\cdot)$  as a control function while we consider  $v(\cdot)$  as a (time varying) perturbation.

In order to obtain a Zubov type characterization of this situation, to any solution  $x(\cdot, x, u(\cdot), v(\cdot))$  of (2) with initial value  $x$  we associate a payoff which depends on  $u(\cdot)$  and  $v(\cdot)$  and is denoted by

$$(4) \quad J(x, u(\cdot), v(\cdot))$$

We study the differential game associated to (2) and (4) in which the first player acting on the control  $u$  tries to minimize the cost  $J(x, u(\cdot), v(\cdot))$  while the second player acting on the control  $v$  tries to maximize it. The precise definition of  $J$  will be given in Section 4.

In order to model the interplay of the actions of the players, we introduce two notions of strategies.

**Definition 1** *We say that a map  $\alpha : \mathcal{V} \rightarrow \mathcal{U}$  is a nonanticipative strategy (for the first player) if it satisfies the following condition:*

*For any  $v_1, v_2 \in \mathcal{V}$  which coincide almost everywhere on  $[0, s]$  for an  $s \geq 0$ , the images  $\alpha(v_1)$  and  $\alpha(v_2)$  also coincide almost everywhere on  $[0, s]$ .*

*Nonanticipative strategies  $\beta : \mathcal{U} \rightarrow \mathcal{V}$  (for the second player) are defined similarly.*

*The set of nonanticipating strategies  $\alpha$  for the first player are denoted by  $\Gamma$  and the respective set for the second player is denoted by  $\Delta$ .*

Nonanticipative strategies (Elliot-Kalton-Varayia strategies, cf. [15]) are very classical in the literature of differential games. With the help of this concept we define the following value functions for the game.

**Definition 2** *The lower value function for the game (2), (4) is given by*

$$(5) \quad V^-(x) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} J(x, \alpha(v), v)$$

*and the upper value function is defined by*

$$(6) \quad V^+(x) = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} J(x, u, \beta(u))$$

*for all  $x \in \mathbb{R}^N$ .*

Nonanticipative strategies are a suitable tool for modeling the fact that the players act simultaneously and for showing that these value functions solve suitable PDEs of Hamilton Jacobi type in the viscosity sense.

One of the main questions in differential games is whether  $V^+$  and  $V^-$  coincide, in which case one says that there exists a value for the game. For the case of nonanticipative strategies this question is investigated for instance in [19], [3], [5], [6], [14], [15], [17], [30], [31]. Typically, Isaacs' condition is needed in order to ensure the existence of a value. In Sections 5 and 6 we will see that the same is true for our generalizations of Zubov's equation.

### 3 The lower and upper domains of null controllability

The main goal of this paper is to formulate and analyze a differential game whose solutions characterize domains of controllability. In this section we introduce these domains and show some of their properties.

To this end, we assume that  $x = 0$  is an equilibrium (or fixed point) for some control  $u_0 \in U$  and all  $v \in V$ , i.e., there exists  $u_0 \in U$  such that  $f(0, u_0, v) = 0$  for any  $v \in V$ . Analogous to [36] or [8], the equilibrium 0 could be replaced by a more general compact set  $A \subset \mathbb{R}^n$ . For simplicity of exposition, in this paper we restrict ourselves to the equilibrium case.

**Definition 3** *i) We call a point  $x \in \mathbb{R}^n$  lower asymptotically controllable to 0 if there exists a nonanticipative strategy  $\alpha_x(\cdot) \in \Gamma$  such that for any perturbation  $v(\cdot) \in \mathcal{V}$  the corresponding solution  $x(t, x, \alpha_x(v), v)$  of (2) satisfies  $x(t, x, \alpha_x(v), v) \rightarrow 0$  for  $t \rightarrow \infty$ . The domain of lower asymptotic null-controllability  $\mathcal{D}^-$  is the collection of all points that are lower asymptotically controllable to 0.*

ii) We call a point  $x \in \mathbb{R}^n$  upper asymptotically controllable to 0 if for any nonanticipative strategy  $\beta(\cdot) \in \Delta$  there exists a control  $u_{x,\beta}(\cdot) \in \mathcal{U}$  such that the corresponding solution  $x(t, x, u_{x,\beta}, \beta(u_{x,\beta}))$  of (2) satisfies  $x(t, x, u_{x,\beta}, \beta(u_{x,\beta})) \rightarrow 0$  for  $t \rightarrow \infty$ . The domain of upper asymptotic null-controllability  $\mathcal{D}^+$  is the collection of all points that are lower asymptotically controllable to 0.

The following definition strengthens Definition 3 by requiring uniformity of the convergence with respect to the perturbation  $v(\cdot)$  in (i) or  $\beta(\cdot)$  in (ii), respectively.

**Definition 4** i) We call a point  $x \in \mathbb{R}^n$  uniformly lower asymptotically controllable to 0 if there exist a function  $\theta(t) \rightarrow 0$  as  $t \rightarrow \infty$  and a nonanticipative strategy  $\alpha_x(\cdot) \in \Gamma$  such that for any perturbation  $v(\cdot) \in \mathcal{V}$  we have that the corresponding solution  $x(t, x, \alpha_x(v), v)$  of (2) satisfies  $\|x(t, x, \alpha_x(v), v)\| \leq \theta(t)$  for all  $t > 0$ . The domain of uniform lower asymptotic null-controllability  $\mathcal{D}_0^-$  is the collection of all points that are lower uniformly asymptotically controllable to 0.

ii) We call a point  $x \in \mathbb{R}^n$  uniformly upper asymptotically controllable to 0 if there exists a function  $\theta(t) \rightarrow 0$  as  $t \rightarrow \infty$  such that for any nonanticipative strategy  $\beta(\cdot) \in \Gamma$  there exists a control  $u_{x,\beta}(\cdot) \in \mathcal{U}$  with the property that the corresponding solution  $x(t, x, u_{x,\beta}, \beta(u_{x,\beta}))$  of (2) satisfies  $\|x(t, x, u_{x,\beta}, \beta(u_{x,\beta}))\| \leq \theta(t)$  for all  $t > 0$ . The domain of uniform upper asymptotic null-controllability  $\mathcal{D}_0^+$  is the collection of all points that are upper uniformly asymptotically controllable to 0.

In these definitions, the goal of the  $u$ -player is to control the system asymptotically to 0 while the goal of the  $v$ -player is to keep the system trajectories away from 0. In each of the formulations, one of the players chooses a nonanticipating strategy  $\alpha$  or  $\beta$  and the other player chooses an “open loop” control  $v$  or  $u$  which depends on  $\alpha$  or  $\beta$ , respectively, and also on the initial state  $x$ . This is in accordance with the Elliott-Kalton-Varayia definition of the upper and lower value of a differential game, cf. Definition 2, which motivates the notions of upper and lower asymptotic controllability; more details about this relation will be given in Proposition 10 and Corollary 14, below.

However, this abstract definition conceals the intuition which lies behind these concepts. This intuition is easily made precise for discrete time dynamic games: suppose that at times  $i = 1, 2, \dots$  the minimizing player chooses  $u_i$  and the maximizing player chooses  $v_i$ . Then, in the case of the upper value, at each step  $i$  the  $u$ -player knows  $v_j$  for all  $j < i$  while the  $v$ -player knows  $u_j$  for all  $j \leq i$ . For the lower value this relation is reversed. For continuous time differential games, for the upper value at an intuitive level the  $u$ -player should know  $v(s)$  for all  $s < t$  when  $u(t)$  is chosen. The  $v$ -player should choose  $v(t)$  knowing  $u(s)$  for all  $s \leq t$ . Since in continuous time these decisions are to be taken continuously for an uncountable set of times  $t$ , one needs the formal concept of nonanticipative strategies in order to make this intuition mathematically precise.

This intuitive interpretation explains why the player using strategies has a slight advantage, from which the inequality  $V^+(x) \geq V^-(x)$  follows under appropriate conditions, although its proof is in general far from trivial, cf. the discussion in [1, Section VIII.1]. This inequality poses a number of interesting questions, e.g., whether there are conditions under which  $V^+ = V^-$  or  $\mathcal{D}_0^+ = \mathcal{D}_0^-$  holds. A condition for the second equality to hold will be derived in Corollary 17, below. Another interesting question is how to find a subclass  $\Gamma_1$  of the class  $\Gamma$  of nonanticipative strategies, such that

$$(7) \quad V^+(x) = \inf_{\alpha \in \Gamma_1} \sup_{v \in \mathcal{V}} J(x, \alpha(v), v)$$

holds. In [18, Section 11.9] property 7 is shown to hold for games on a finite time interval, if  $\Gamma_1$  is the class of “strictly progressive” strategies. Another possible choice for  $\Gamma_1$  is the class of “slightly delayed strategies” defined in [1, Section VIII.2].

In terms of feedback control policies, at an intuitive level the open loop player should choose  $u(t)$  or  $v(t)$  as a function of  $(t, x(t))$  while the strategy player should choose  $u(t)$  as a function of

$(t, x(t), v(t))$  or  $v(t)$  as a function of  $(t, x(t), u(t))$ , respectively, see [18, Setcion 11.3]. Again, in the continuous time case rigorous theorems using this feedback control formalism are difficult to obtain, because the resulting feedback laws may be highly degenerate such that existence and uniqueness of the corresponding closed loop solutions may not hold. However, it can be a useful guide when considering numerical approximations to the Hamilton-Jacobi-Isaacs PDEs for differential games.

The viewpoint taken in the introduction, i.e., that  $u(t)$  is a control which tries to steer the system towards the equilibrium and that  $v(t)$  is a perturbation trying to avoid this is the same as in nonlinear  $H_\infty$ -control theory, see [4]. In this case, for the lower game or lower asymptotic controllability the  $u$ -player should know the perturbation value  $v(t)$  before  $u(t)$  is chosen. This is unreasonable in most applications, hence with this interpretation the upper value is usually the more realistic ones. This situation is similar in applications of dynamic games different from controllability questions. For instance, in math finance models the control  $u(t)$  can correspond to an allocation of assets and the disturbance  $v(t)$  may correspond to a fluctuation in risky asset prices. An investor who knows  $v(t)$  before choosing  $u(t)$  can become infinitely rich. However, one may also consider the situation in which  $u(t)$  is interpreted as a perturbation and the goal of the control  $v(t)$  is to avoid that the trajectories approach 0. In this case, typically the lower value is more appropriate for most applications.

Similar to [9] it will turn out that in our game theoretical setting the respective uniform domains  $\mathcal{D}_0^\pm$  from Definition 4 are the domains which are characterized by Zubov's method. We will henceforth mainly work with these sets. The following local asymptotic controllability assumptions ensure that  $\mathcal{D}_0^-$  and  $\mathcal{D}_0^+$ , respectively, contain a neighborhood of the origin  $x = 0$ .

$(H^-)$  There exists an open ball  $B(0, r)$  and  $\eta \in \mathcal{KL}^1$  such that for any  $x \in B(0, r)$  there is a nonanticipative strategy  $\alpha_x(\cdot) \in \Gamma$  such that for any perturbation  $v(\cdot) \in \mathcal{V}$  the solution  $x(t, x, \alpha_x(v), v)$  exists for all  $t \geq 0$  and satisfies

$$\|x(t, x, \alpha_x(v), v)\| \leq \eta(\|x\|, t), \quad \forall t \geq 0.$$

$(H^+)$  There exists an open ball  $B(0, r)$  and  $\eta \in \mathcal{KL}$  such that for any  $x \in B(0, r)$  and for any nonanticipative strategy  $\beta \in \Delta$  there is a control  $u_{x,\beta}(\cdot) \in \mathcal{U}$  for which the solution  $x(t, x, u_{x,\beta}, \beta(u_{x,\beta}))$  exists for all  $t \geq 0$  and satisfies

$$\|x(t, x, u_{x,\beta}, \beta(u_{x,\beta}))\| \leq \eta(\|x\|, t), \quad \forall t \geq 0.$$

Note that  $(H^\pm)$  immediately imply the inclusions  $B(0, r) \subseteq \mathcal{D}_0^\pm$ .

There are several ways to ensure  $(H^\pm)$ . For instance, it can be checked that  $(H^+)$  respectively  $(H^-)$  holds whenever  $\min_u \max_v \langle f(x, u, v), x \rangle < 0$  or, respectively,  $\max_v \min_u \langle f(x, u, v), x \rangle < 0$  for all  $x \in B(0, r)$ ,  $x \neq 0$ , because in this case the Euclidean norm of the solutions is strictly decreasing.

Another way to ensure  $(H^+)$  is to assume that the linearization of  $f$  from (2) in the equilibrium  $x = 0$  is of the form

$$Ax + \sum_{k=1}^m v_k A_k x + Bu$$

in which  $v = (v_1, \dots, v_m)^T \in V \subset \mathbb{R}^m$ ,  $A$ ,  $A_k$  and  $B$  are matrices of appropriate dimensions and the pair  $(A, B)$  is stabilizable. In this case, we can find an exponentially stabilizing linear feedback law, i.e., a matrix  $K$  of appropriate dimension such that  $\dot{x} = Ax + BKx$  is exponentially stable. Using the results from [38] it then follows that for any sufficiently small set of perturbation values  $V \subset \mathbb{R}^m$  the system

$$(8) \quad \dot{x}(t) = f(x(t), Kx(t), v(t))$$

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<sup>1</sup>As usual we call a function  $\alpha$  of class  $\mathcal{K}_\infty$  if it is a homeomorphism of  $[0, \infty)$ , a continuous function  $\eta$  in two real nonnegative arguments is called of class  $\mathcal{KL}$  if it is of class  $\mathcal{K}_\infty$  in the first and monotonely decreasing to zero in the second argument.

is locally exponentially stable uniformly for all measurable functions  $v : \mathbb{R} \rightarrow V$ . This means that for all such functions the solutions  $x_K(t, x, v)$  of (8) satisfy the inequality

$$\|x_K(t, x, v)\| \leq Ce^{-\sigma t}\|x\|$$

for suitable  $C, \sigma > 0$  and all initial values  $x \in \mathbb{R}^n$  with  $\|x\| > 0$  sufficiently small. If we now set  $u_{x,\beta}(t) = Kx_K(t, x, \beta(u_{x,\beta}))$  (note that this is well defined since the right hand side of this definition does not depend on  $\beta(u_{x,\beta})(t)$  and thus not on  $u_{x,\beta}(t)$ ), then we get

$$x(t, x, u_{x,\beta}, \beta(u_{x,\beta})) = x_K(t, x, \beta(u_{x,\beta}))$$

and thus  $(H^+)$  follows with  $\eta(r, t) = Ce^{-\sigma t}r$ .

The following example shows that  $\mathcal{D}_0^-$  may be strictly larger than  $\mathcal{D}_0^+$ .

**Example 5** Consider the 1d control system

$$\dot{x}(t) = -x(t) + u(t)v(t)x(t)^3 =: f(x(t), u(t), v(t))$$

with  $x \in \mathbb{R}$  and  $U = V = \{-1, 1\}$ . For simplicity we directly investigate this unbounded vector field which could be rendered bounded using the transformation (3) which does, however, not change the domains of controllability.

For  $|x| \leq 1/2$  one easily sees that  $f$  satisfies  $f(x, u, v) \leq -3x/4$  if  $x \geq 0$  and  $f(x, u, v) \geq -3x/4$  if  $x \leq 0$ , regardless of how  $u$  and  $v$  are chosen. Thus, for all  $u(\cdot) \in \mathcal{U}$  and  $v(\cdot) \in \mathcal{V}$  we obtain  $|x(t, x, u(\cdot), v(\cdot))| \leq e^{-3t/4}|x|$ . This implies both  $(H^-)$  and  $(H^+)$  with  $\eta(r, t) = e^{-3t/4}r$ . Furthermore, for all  $|x| < 1$  one sees that  $f(x, u, v) < 0$  if  $x > 0$  and  $f(x, u, v) > 0$  for  $x < 0$  for all  $u \in U$  and  $v \in V$ . Hence, all solutions starting in some  $|x| < 1$  converge to 0 which immediately implies the inclusions  $(-1, 1) \subseteq \mathcal{D}_0^+$  and  $(-1, 1) \subseteq \mathcal{D}_0^-$ .

Now we investigate  $\mathcal{D}_0^-$ . We define a nonanticipating strategy  $\alpha \in \Gamma$  by

$$\alpha(v)(t) = -v(t).$$

This implies

$$(9) \quad f(x, \alpha(v)(t), v(t)) = -x - x^3$$

for all  $v(\cdot) \in \mathcal{V}$  and all  $t \geq 0$ . Since (9) is a globally asymptotically stable vector field, choosing  $\alpha_x = \alpha$  for all  $x$  implies  $\mathcal{D}_0^- = \mathbb{R}$ .

On the other hand, defining  $\beta \in \Delta$  via

$$\beta(u)(t) = u(t)$$

implies

$$(10) \quad f(x, u(t), \beta(u)(t)) = -x + x^3$$

for all  $u(\cdot) \in \mathcal{U}$ . Thus, for all  $x > 1$  the corresponding solution  $x(t, x, u, \beta(u))$  diverges to  $\infty$  and for all  $x < -1$  the solution diverges to  $-\infty$ . Thus, all these points cannot belong to  $\mathcal{D}_0^+$ . Since  $x = 1$  and  $x = -1$  are equilibria of (10) the corresponding solutions satisfy  $x(t, \pm 1, u, \beta(u)) = \pm 1$  and thus do not converge to 0, either, hence  $\pm 1 \notin \mathcal{D}_0^+$ . Since above we already showed  $(-1, 1) \subseteq \mathcal{D}_0^+$  we thus obtain  $\mathcal{D}_0^+ = (-1, 1)$ .

Hence, summarizing we obtain that both  $(H^-)$  and  $(H^+)$  are satisfied for this example and that  $\mathcal{D}_0^- = \mathbb{R} \neq (-1, 1) = \mathcal{D}_0^+$ .

Under the assumptions  $(H^\pm)$  we can obtain a different way of characterizing  $\mathcal{D}_0^\pm$  by looking at the minimal time to reach the ball  $B(0, r)$  associated to any solution  $x(\cdot; x, u(\cdot), v(\cdot))$  of (2). For each  $u(\cdot) \in \mathcal{U}$  and  $v(\cdot) \in \mathcal{V}$  this time is defined by

$$(11) \quad t(x, u(\cdot), v(\cdot)) = \inf \{t \geq 0 \text{ such that } x(t; x, u(\cdot), v(\cdot)) \in B(0, r)\}.$$

For this time we define

$$(12) \quad t^-(x) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} t(x, \alpha(v), v) \quad \text{and} \quad t^+(x) = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} t(x, u, \beta(u))$$

for all  $x$ . These functions have been studied in the context of domains of controllability, e.g., in [36]. In our context they will turn out to be useful as auxiliary functions for our analysis of Zubov's equation. The next lemma shows how the domain  $\text{dom}(t^\pm)$ , i.e., the set of points  $x$  for which  $t^\pm(x) < \infty$  holds, is related to the controllability domains  $\mathcal{D}^\pm$  and  $\mathcal{D}_0^\pm$ .

**Lemma 6** *Assume  $(H^\pm)$ . Then the following identities hold*

- i)  $\mathcal{D}_0^- = \{x \in \mathbb{R}^n \mid t^-(x) < \infty\} =: \text{dom}(t^-)$ ,
- ii)  $\mathcal{D}^- = \{x \in \mathbb{R}^n \mid \exists \alpha_x(\cdot) \in \Gamma \text{ such that } t(x, \alpha_x(v), v) < \infty, \forall v(\cdot) \in \mathcal{V}\}$ ,
- iii)  $\mathcal{D}_0^+ = \{x \in \mathbb{R}^n \mid t^+(x) < \infty\} =: \text{dom}(t^+)$ ,
- iv)  $\mathcal{D}^+ = \{x \in \mathbb{R}^n \mid \forall \beta(\cdot) \in \Delta, \exists u_\beta(\cdot) \in \mathcal{U} \text{ such that } t(x, u_\beta, \beta(u_\beta)) < \infty\}$ .

**Proof.** i) Consider  $x \in \text{dom}(t^-)$ , i.e.,  $t^-(x) = T < \infty$ . Then there exists  $\alpha_x(\cdot) \in \Gamma$  such that  $t(x, \alpha_x(v), v) < T+1$  for all  $v(\cdot) \in \mathcal{V}$ . So, for any  $v(\cdot)$  there exists  $t_v \leq T+1$  with  $x(t_v; x, \alpha_x(v), v) \in B(0, r)$ .

Moreover, by  $(H^-)$  there exists  $\tilde{\alpha}_{t_v, \alpha_x, v}(\cdot) \in \Gamma$  such that for any perturbation  $\tilde{v} \in \mathcal{V}$  we have

$$\|x(t; x(t_v; x, \alpha_x(v), v), \tilde{\alpha}_{t_v, \alpha_x, v}(\tilde{v}), \tilde{v})\| \leq \eta(\|x(t_v; x, \alpha_x(v), v)\|, t), \forall t \geq 0.$$

Consider now a fixed perturbation  $v(\cdot) \in \mathcal{V}$ . We define the nonanticipative strategy

$$\hat{\alpha}(v)(t) := \begin{cases} \alpha_x(v)(t), & t \in [0, t_v) \\ \tilde{\alpha}_{t_v, \alpha_x, v}(v(\cdot + t_v))(t - t_v), & t \in [t_v, \infty) \end{cases}$$

which yields  $x(t + t_v; x, \hat{\alpha}(v), v) = x(t; x(t_v; x, \alpha_x(v), v), \tilde{\alpha}_{t_v, \alpha_x, v}(\tilde{v}), \tilde{v})$  for  $\tilde{v} = v(\cdot + t_v)$  and all  $t \geq 0$ . Thus, defining the function

$$\theta(t) := \begin{cases} \max_{v, t \in [0, T+1]} \|x(t_v; x, \alpha_x(v), v)\|, & t \in [0, T+1) \\ \eta(r, t - T - 1), & t \in [T+1, \infty) \end{cases}$$

which satisfies  $\theta(t) \rightarrow 0$  as  $t \rightarrow \infty$ , for each  $v(\cdot) \in \mathcal{V}$  we obtain

$$\|x(t; x, \hat{\alpha}(v), v)\| \leq \theta(t), \forall t \geq 0.$$

which shows  $x \in \mathcal{D}_0^-$ .

For the converse inclusion, we consider  $x \in \mathcal{D}_0^-$ . By definition, there exist a function  $\theta(t) \rightarrow 0$  as  $t \rightarrow \infty$  and a nonanticipative strategy  $\alpha_x(\cdot) \in \Gamma$  such that for any perturbation  $v(\cdot)$  we have that the corresponding solution  $x(t, x, \alpha_x(v), v)$  of (2) satisfies  $\|x(t, x, \alpha_x(v), v)\| \leq \theta(t)$  for all  $t > 0$ . Consequently, there exists a finite time  $T$  such that

$$\|x(T; x, \alpha_x(v), v)\| \leq \theta(T) < r.$$

Since this inequality holds for all  $v(\cdot) \in \mathcal{V}$ , we obtain  $t^-(x) \leq T < \infty$ .

ii) The proof for  $\mathcal{D}^-$  is similar and we omit it.

iii) Consider  $x \in \text{dom}(t^+)$ , i.e.,  $t^+(x) = T < \infty$ . Then for all  $\beta \in \Delta$  there exists a control  $u_{x,\beta} \in \mathcal{U}$  such that  $t(x, u_{x,\beta}, \beta(u_{x,\beta})) < T + 1$ . So, there exists  $t_{\beta, u_{x,\beta}} \leq T + 1$  with  $\tilde{x} := x(t_{\beta, u_{x,\beta}}; x, u_{x,\beta}, \beta(u_{x,\beta})) \in B(0, r)$ .

Moreover, for all  $\tilde{\beta} \in \Delta$  and all these  $\tilde{x}$  by  $(H^+)$  there exists  $\tilde{u}_{\tilde{x}, \tilde{\beta}}$  such that we have

$$\|x(t; x(t_{\beta, u_{x,\beta}}; x, u_{x,\beta}, \beta(u_{x,\beta})), \tilde{u}_{\tilde{x}, \tilde{\beta}}, \tilde{\beta}(\tilde{u}_{\tilde{x}, \tilde{\beta}}))\| \leq \eta(\|x(t_{\beta, u_{x,\beta}}; x, u_{x,\beta}, \beta(u_{x,\beta}))\|, t), \forall t \geq 0.$$

Now we define

$$\theta(t) := \begin{cases} \max_{\beta \in \Delta, t \in [0, T+1]} \|x(t, x, u_{x,\beta}, \beta(u_{x,\beta}))\|, & t \in [0, T+1) \\ \eta(r, t - T - 1), & t \in [T+1, \infty) \end{cases}$$

and for all  $\beta \in \Delta$  we define the control

$$\bar{u}_{x,\beta}(t) := \begin{cases} u_{x,\beta}(t), & t \in [0, t_{\beta, u_{x,\beta}}) \\ \tilde{u}_{\tilde{x}, \tilde{\beta}}(t - t_{\beta, u_{x,\beta}}), & t \in [t_{\beta, u_{x,\beta}}, \infty) \end{cases}$$

with  $\tilde{x} = x(t_{\beta, u_{x,\beta}}; x, u_{x,\beta}, \beta(u_{x,\beta}))$  and  $\tilde{\beta}(u)(t) = \beta(\bar{u})(t + t_{\beta, u_{x,\beta}})$  where  $\bar{u}$  is defined analogously to  $\bar{u}_{x,\beta}$  with  $u$  in place of  $\tilde{u}_{\tilde{x}, \tilde{\beta}}$ . With these definitions we obtain

$$\|x(t; x, \bar{u}_{x,\beta}, \beta(\bar{u}_{x,\beta}))\| \leq \theta(t), \forall t \geq 0$$

which implies  $x \in \mathcal{D}_0^+$ .

For the converse inclusion, we consider  $x \in \mathcal{D}_0^+$ . By definition, there exist a function  $\theta(t) \rightarrow 0$  as  $t \rightarrow \infty$  such that for any nonanticipative strategy  $\beta(\cdot)$  there exists a control  $u_\beta(\cdot)$  with the property that the corresponding solution  $x(t, x, u_{x,\beta}, \beta(u_{x,\beta}))$  of (2) satisfies  $\|x(t, x, u_{x,\beta}, \beta(u_{x,\beta}))\| \leq \theta(t)$  for all  $t > 0$ . Consequently, there exists a finite time  $T$  such that

$$\|x(T, x, u_{x,\beta}, \beta(u_{x,\beta}))\| \leq \theta(T) < r$$

which implies  $t^+(x) \leq T < \infty$  since  $T$  is independent of  $\beta \in \Delta$ .

iv) The proof for  $\mathcal{D}^+$  is similar and we omit it. ■

We continue with characterizing properties of  $\mathcal{D}_0^-$  and  $\mathcal{D}_0^+$ . For doing this we introduce the following definition (see [26, 13] for details).

**Definition 7** *i) A set  $M$  is called discriminant domain for the dynamics (2) if for any  $x \in M$  there exists a nonanticipative strategy  $\alpha_x(\cdot) \in \Gamma$  such that for any perturbation  $v(\cdot) \in \mathcal{V}$  the corresponding solution  $x(t, x, \alpha_x(v), v)$  of (2) satisfies  $x(t, x, \alpha_x(v), v) \in M$  for all  $t \geq 0$ .*

*ii) A set  $N$  is called leadership domain for the dynamics (2) if for any  $x \in N$  and for any nonanticipative strategy  $\beta(\cdot) \in \Delta$  there exists a control  $u_{x,\beta}(\cdot) \in \mathcal{U}$  such that the corresponding solution  $x(t; x, u_{x,\beta}, \beta(u_{x,\beta}))$  of (2) satisfies  $x(t; x, u_{x,\beta}, \beta(u_{x,\beta})) \in N$  for all  $t \geq 0$ .*

**Proposition 8** *Under the Assumptions  $(H^\pm)$  the following properties hold.*

- 1)  $\text{cl}B(0, r) \subseteq \mathcal{D}_0^-$
- 2) The set  $\mathcal{D}_0^-$  is open, connected and a discriminant domain for (2)
- 3) The set  $\mathcal{D}^-$  is pathwise connected and a discriminant domain for (2)
- 4)  $\text{cl}\mathcal{D}^-$  and  $\text{cl}\mathcal{D}_0^-$  are discriminant domains for (2)
- 5)  $t^-(x_n) \rightarrow \infty$  for any sequence of points with  $x_n \rightarrow x \in \partial\mathcal{D}_0^-$  or  $\|x_n\| \rightarrow \infty$  as  $n \rightarrow \infty$
- 6)  $\text{cl}B(0, r) \subseteq \mathcal{D}_0^+$
- 7) The set  $\mathcal{D}_0^+$  is open, connected and a leadership domain for (2)
- 8) The set  $\mathcal{D}^+$  is pathwise connected and a leadership domain for (2)

9)  $cl\mathcal{D}^+$  and  $cl\mathcal{D}_0^+$  are leadership domains for (2)

10)  $t^+(x_n) \rightarrow \infty$  for any sequence of points with  $x_n \rightarrow x \in \partial\mathcal{D}_0^+$  or  $\|x_n\| \rightarrow \infty$  as  $n \rightarrow \infty$

**Proof.** 1) It is obvious that  $B(0, r) \subseteq \mathcal{D}_0^-$ . Consider now  $x \in \partial B(0, r)$ . There exists a sequence  $x_n \in B(0, r)$  such that  $x_n \rightarrow x$ . Moreover, for every  $n$  there exists a nonanticipative strategy  $\alpha_n(\cdot)$  such that for any perturbation  $v(\cdot)$  we have that  $\|x(t, x_n, \alpha_n(v), v)\| \leq \eta(\|x_n\|, t)$ . We have the following estimation

$$\begin{aligned} \|x(t, x, \alpha_n(v), v)\| &\leq \|x(t, x_n, \alpha_n(v), v)\| + \|x(t, x_n, \alpha_n(v), v) - x(t, x, \alpha_n(v), v)\| \\ &\leq \eta(\|x_n\|, t) + \|x(t, x_n, \alpha_n(v), v) - x(t, x, \alpha_n(v), v)\| \end{aligned}$$

Fixing  $\tilde{t}$  such that  $\eta(r, \tilde{t}) < r/2$  and using  $\|x_n\| < r$  and the fact that the uniform Lipschitz property implies  $\lim_{n \rightarrow \infty} \|x(\tilde{t}, x_n, \alpha_n(v), v) - x(\tilde{t}, x, \alpha_n(v), v)\| = 0$  uniformly for all  $v \in \mathcal{V}$ , we can conclude that  $t^-(x) \leq \tilde{t}$ . Hence Lemma 6 yields  $x \in \mathcal{D}_0^-$ .

2) Consider  $x \in \mathcal{D}_0^-$ . By definition there exists a nonanticipative strategy  $\alpha_x(\cdot)$  and a function  $\theta(t) \rightarrow 0$  such that for any perturbation  $v(\cdot) \in \mathcal{V}$  we have that  $\|x(t, x, \alpha_x(v), v)\| \leq \theta(t)$ . Hence, there exists  $\tilde{t} > 0$  such that  $x(t_v, x, \alpha_x(v), v) \in B(0, r/2)$ . By continuous dependence on the initial point we obtain that  $x(t_v, y, \alpha_x(v), v) \in B(0, r)$  for all  $y$  in a neighborhood of  $x$ . Thus  $t(y, \alpha_x(v), v) < \tilde{t}$  for all  $v(\cdot) \in \mathcal{V}$  which implies  $t^-(y) \leq \tilde{t} < \infty$  and thus by Lemma 6  $y \in \mathcal{D}_0^-$  for all  $y$  in this neighborhood of  $x$ . This shows that  $\mathcal{D}_0^-$  is open.

We observe that by definition every  $x \in \mathcal{D}_0^-$  is connected by a trajectory with  $B(0, r)$ . Consequently,  $\mathcal{D}_0^-$  is a connected set.

Finally, in order to show that  $\mathcal{D}_0^-$  is a discriminant domain, let us consider  $x \in \mathcal{D}_0^-$ . By Lemma 6 there exists a nonanticipative strategy  $\alpha_x(\cdot)$  and a time  $T > 0$  such that for any perturbation  $v(\cdot) \in \mathcal{V}$  we have that  $t(x, \alpha_x(v), v) \leq T$ . Now for each point of the form  $\tilde{x} = x(\tilde{t}, \alpha_x(v), v)$  for some  $\tilde{t} > 0$  and each  $\tilde{v} \in \mathcal{V}$  we can define  $\alpha_{\tilde{x}}(\tilde{v})(t) = \alpha_x(\tilde{v})(t + \tilde{t})$  where  $\tilde{v}(t) = v(t)$  for  $t \in [0, \tilde{t}]$  and  $\tilde{v}(t) = \tilde{v}(t - \tilde{t})$  else. This implies  $t(\tilde{x}, \alpha_{\tilde{x}}(\tilde{v}), \tilde{v}) \leq T - \tilde{t}$  for all  $\tilde{v} \in \mathcal{V}$  and thus  $x \in \mathcal{D}_0^-$  by Lemma 6.

3) This follows by arguments similar to 2).

4) The proof is similar to 2).

5) Consider a sequence  $x_n \rightarrow x \in \partial\mathcal{D}_0^-$  and suppose that c. Then there exist nonanticipative strategies  $\alpha_n(\cdot) \in \Gamma$  such that  $t(x_n, \alpha_n(v), v) \leq T + 1$  for all  $v \in \mathcal{V}$ . From  $(H^-)$  there exists  $\tilde{t} > 0$  such that we can choose  $\alpha_n$  in such a way that  $x(\tilde{t}_{v,n}, x_n, \alpha_n(v), v) \in B(0, r/2)$  holds for all  $n$  and all  $v$  for some time  $t_{n,v} \leq \tilde{t}$  (cf. the construction of  $\hat{\alpha}$  in the proof of Lemma 6(i)). By continuous dependence on the initial value we obtain  $x(\tilde{t}_{v,n}, x, \alpha_n(v), v) \in B(0, r)$  for all  $n$  sufficiently large and all  $v \in \mathcal{V}$ . This implies  $x \in \mathcal{D}_0^-$  which is not possible since  $x \in \partial\mathcal{D}_0^-$  and  $\mathcal{D}_0^-$  is open.

The assertion for  $\|x_n\| \rightarrow \infty$  follows since by assumption the vector field  $f$  is bounded.

6)–10) These properties follow by similar arguments as in 1)–5) using the definitions of  $\mathcal{D}_0^+$ ,  $t^+$  and  $(H^+)$ . For the sake of brevity we omit the details of the proof. ■

**Remark 9** If  $f$  does not depend on  $u$ , then the identities  $\mathcal{D}^- = \mathcal{D}^+ =: \mathcal{D}$  and  $\mathcal{D}_0^- = \mathcal{D}_0^+ =: \mathcal{D}_0$  hold. In this case, the inclusion  $\mathcal{D} \subseteq cl\mathcal{D}_0$  was shown in [9, Proposition 2.3(iv)]. The proof of this inclusion relies on [35, Lemma III.2] which shows that  $t(x) = \infty$  implies the existence of  $y$  arbitrarily close to  $x$  and  $v_y \in \mathcal{V}$  such that  $t(y, v_y) = \infty$  holds. The proof of this lemma, however, does not carry over to our case in which we have an additional dependence on  $\alpha_x$  in  $t^-$  and have  $\beta(u)$  instead of  $v$  in  $t^+$ . Thus, it is an open question whether the inclusions  $\mathcal{D}^- \subseteq cl\mathcal{D}_0^-$  and  $\mathcal{D}^+ \subseteq cl\mathcal{D}_0^+$  hold in our game theoretical setting.

## 4 Characterization of $\mathcal{D}_0^\pm$ using Zubov type differential Games

In the last section we showed that  $\mathcal{D}_0^\pm$  can be characterized via the optimal hitting times  $t^\pm$ . In this section we show that we can alternatively establish a characterization via an integral cost  $J(x, u, v)$ .

We formulate the Hamilton-Jacobi equations corresponding to the respective upper and lower value functions — the min-max and max-min Zubov equations — and show the relation between these value functions and the domains of controllability  $\mathcal{D}_0^\pm$ . Uniqueness of the viscosity solutions will then be addressed in the following section.

In order to define the integral cost, consider a bounded, continuous function  $g : \mathbb{R}^N \times U \times V \rightarrow \mathbb{R}$  satisfying the following three conditions. Here, in the first condition  $u_0 \in U$  is the control value for which  $x = 0$  is an equilibrium, i.e., for which  $f(0, u_0, v) = 0$  holds for all  $v \in V$ , cf. the beginning of Section 3.

$$(13) \quad \left\{ \begin{array}{l} \text{i) } g(0, u_0, v) = 0 \text{ and } g(x, u, v) > 0 \text{ for all } u \in U \text{ and } v \in V \text{ if } x \neq 0; \\ \text{ii) } \text{there exists a constant } g_0 > 0 \text{ such that } \inf_{x \notin B(0, r), u \in U, v \in V} g(x, u, v) \geq g_0; \\ \text{iii) } \text{For every } R > 0 \text{ there exists a constant } L_R \text{ such that} \\ \qquad \|g(x, u, v) - g(y, u, v)\| \leq L_R \|x - y\| \\ \qquad \text{holds for all } \|x\|, \|y\| \leq R \text{ and all } u \in U, v \in V. \end{array} \right.$$

Using this  $g$  we define the integral cost

$$J(x, u, v) := \int_0^\infty g(x(s, x, u, v), u(s), v(s)) ds$$

For this cost, we define the lower value function

$$V^-(x) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} J(x, \alpha(v), v)$$

and its Kruzkov transformation

$$W^-(x) := 1 - e^{-V^-(x)} = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \left\{ 1 - e^{-J(x, \alpha(v), v)} \right\}$$

Since  $g$  is nonnegative, it is immediate that  $V^-(x) \geq 0$  and  $W^-(x) \in [0, 1]$  for all  $x \in \mathbb{R}^N$ . Furthermore, standard results from differential games imply that  $V^-$  and  $W^-$  satisfy the dynamic programming principle. Note that  $V^-$  and  $W^-$  do not need to be continuous for this purpose; the proof of the principle can be obtained by appropriate modifications of the proofs in, e.g., [16] or [31]. This principle states that for each  $t > 0$  we have

$$V^-(x) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \left\{ J_t(x, \alpha(v), v) + V^-(x(t, x, \alpha(v), v)) \right\}$$

and

$$W^-(x) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \left\{ 1 - G_t(x, \alpha(v), v) + G_t(x, \alpha(v), v) W^-(x(t, x, \alpha(v), v)) \right\}$$

where we used the abbreviations

$$(14) \quad J_t(x, u, v) := \int_0^t g(x(s, x, u, v), u(s), v(s)) ds \quad \text{and} \quad G_t(x, u, v) := e^{-J_t(x, u, v)}.$$

Analogously, we define the upper value function

$$V^+(x) = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} J(x, u, \beta(u))$$

and its Kruzkov transformation, i.e.

$$W^+(x) := 1 - e^{-V^+(x)} = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} \left\{ 1 - e^{-J(x,u,\beta(u))} \right\}$$

Since  $g$  is nonnegative, it is again immediate that  $V^+(x) \geq 0$  and  $W^+(x) \in [0, 1]$  for all  $x \in \mathbb{R}^N$ .

Again, standard results from differential games imply that  $V^+$  and  $W^+$  satisfy the dynamic programming principle, i.e., for each  $t > 0$  we have

$$V^+(x) = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} \left\{ J_t(x, u, \beta(u)) + V^+(x(t, x, u, \beta(u))) \right\}$$

and

$$W^+(x) = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} \left\{ G_t(x, u, \beta(u)) + G_t(x, u, \beta(u))W^+(x(t, x, u, \beta(u))) \right\}$$

with  $J_t(x, u, v)$  and  $G_t(x, u, v)$  from (14).

The following relations between  $V^\pm$  and  $W^\pm$  are immediate: we have

$$\begin{aligned} V^\pm(x) = 0 &\Leftrightarrow W^\pm(x) = 0, \\ V^\pm(x) \in (0, \infty) &\Leftrightarrow W^\pm(x) \in (0, 1), \\ V^\pm(x) = \infty &\Leftrightarrow W^\pm(x) = 1. \end{aligned}$$

In the next proposition we investigate the relation between  $\mathcal{D}_0^\pm$  and  $V^\pm$  (and thus also  $W^\pm$ ) and the continuity of  $V^\pm$  and  $W^\pm$ . To this end, we make the following additional assumption on  $g$ .

$$(15) \quad \left\{ \begin{array}{l} \text{there exists } \gamma \in \mathcal{K}_\infty \text{ such that for each } x \in B(0, r) \text{ and } \alpha_x \text{ and } u_x \text{ from} \\ (H^-) \text{ and } (H^+), \text{ respectively, the inequalities} \\ \quad g(x(t, x, \alpha_x(v), v), \alpha_x(v)(t), v(t)) \leq e^{-t}\gamma(\|x\|) \\ \text{and} \\ \quad g(x(t, x, u_x, \beta(u_x)), u_x(t), \beta(u_x)(t)) \leq e^{-t}\gamma(\|x\|) \\ \text{hold for all } v \in \mathcal{V} \text{ and } \beta \in \Delta, \text{ respectively, and almost all } t \geq 0. \end{array} \right.$$

This assumption can always be satisfied by choosing  $g$  sufficiently “flat” around the origin. More precisely, by Sontag’s  $\mathcal{KL}$ -Lemma [33] the function  $\eta(r, t)$  from  $(H^\pm)$  can be bounded from above by  $\eta(r, t) \leq \alpha_2(\alpha_1(r)e^{-t})$  for suitable functions  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ . Then, one can check that  $(H^\pm)$  imply (15) with  $\gamma(r) = \alpha_1(r)$  if  $g$  is chosen such that  $g(x, u, v) \leq \alpha_2^{-1}(\|x\|)$  holds for all  $x \in B(0, \eta(r, 0))$ , all  $v \in V$  and all values  $u \in U$  appearing in (15). For details of this construction see [12, Assumption (H4) and proof of Proposition 3.3(i)].

**Proposition 10** *Assume  $(H^\pm)$  and Assumptions (13) and (15) for  $g$ . Then the following properties hold.*

- 1)  $V^-(x) < \infty$  iff  $x \in \mathcal{D}_0^-$
- 2)  $V^-(x) = 0$  iff  $x = 0$
- 3)  $V^-$  is continuous on  $\mathcal{D}_0^-$
- 4)  $V^-(x_n) \rightarrow \infty$  for any sequence of points with  $x_n \rightarrow x \in \partial\mathcal{D}_0^-$  or  $\|x_n\| \rightarrow \infty$  as  $n \rightarrow \infty$
- 5)  $W^-(x) < 1$  iff  $x \in \mathcal{D}_0^-$
- 6)  $W^-(x) = 0$  iff  $x = 0$
- 7)  $W^-$  is continuous on  $\mathbb{R}^N$
- 8)  $W^-(x_n) \rightarrow 1$  for any sequence of points with  $x_n \rightarrow x \in \partial\mathcal{D}_0^-$  or  $\|x_n\| \rightarrow \infty$  as  $n \rightarrow \infty$
- 9)  $V^+(x) < \infty$  iff  $x \in \mathcal{D}_0^+$

- 10)  $V^+(x) = 0$  iff  $x = 0$
- 11)  $V^+$  is continuous on  $\mathcal{D}_0^+$
- 12)  $V^+(x_n) \rightarrow \infty$  for any sequence of points with  $x_n \rightarrow x \in \partial\mathcal{D}_0^+$  or  $\|x_n\| \rightarrow \infty$  as  $n \rightarrow \infty$
- 13)  $W^+(x) < 1$  iff  $x \in \mathcal{D}_0^+$
- 14)  $W^+(x) = 0$  iff  $x = 0$
- 15)  $W^+$  is continuous on  $\mathbb{R}^N$
- 16)  $W^+(x_n) \rightarrow 1$  for any sequence of points with  $x_n \rightarrow x \in \partial\mathcal{D}_0^+$  or  $\|x_n\| \rightarrow \infty$  as  $n \rightarrow \infty$

**Proof.** 1) First observe that for each  $x \in B(0, r)$  the assumptions  $(H^-)$  and (15) imply the existence of  $\alpha_x \in \Gamma$  such that for each  $v \in \mathcal{V}$  we get the inequality

$$\begin{aligned} J(x, \alpha_x(v), v) &= \int_0^\infty g(x(t, x, \alpha_x(v), v), \alpha_x(v)(t), v(t)) dt \\ &\leq \int_0^\infty e^{-t} \gamma(\|x\|) dt = \gamma(\|x\|). \end{aligned}$$

which implies

$$(16) \quad V^-(x) \leq \gamma(\|x\|).$$

Now consider  $x \in \mathcal{D}_0^-$ . By definition, there exists a nonanticipative strategy  $\alpha_x(\cdot)$  and  $\theta(\cdot)$  such that for any perturbation  $v(\cdot) \in \mathcal{V}$  we have that  $\|x(t, x, \alpha_x(v), v)\| \leq \theta(t)$ . So, there exists  $T > 0$  such that  $x(t, x, \alpha(v), v) \in B(0, r)$  for all  $v(\cdot)$  and  $t \geq T$ . Thus,

$$\begin{aligned} V^-(x) &= \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} J(x, \alpha(v), v) \\ &= \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \{J_T(x, \alpha(v), v) + V^-(x(T, x, \alpha(v), v))\} \\ &\leq \sup_{v \in \mathcal{V}} J_T(x, \alpha_x(v), v) + \gamma(\|x(T, x, \alpha_x(v), v)\|) < \infty, \end{aligned}$$

where we used (16) in the second last inequality.

Now consider  $x \notin \mathcal{D}_0^- = \text{dom}(t^-)$ . By definition of  $t^-$ , for every nonanticipative strategy  $\alpha(\cdot) \in \Gamma$  there exists a sequence of perturbations  $v_n(\cdot) \in \mathcal{V}$  such that we have  $\lim_{n \rightarrow \infty} t^-(x, \alpha(v_n), v_n) = \infty$ . So, by (13)(ii) for every nonanticipating strategy  $\alpha(\cdot) \in \Gamma$  we obtain

$$\begin{aligned} \sup_{v \in \mathcal{V}} J(x, \alpha(v), v) &\geq \lim_{n \rightarrow \infty} \int_0^\infty g(x(s, x, \alpha(v_n), v_n), \alpha(v_n)(s), v_n(s)) ds \\ &\geq \lim_{n \rightarrow \infty} \int_0^{t^-(x, \alpha(v_n), v_n)} g_0 ds = \lim_{n \rightarrow \infty} t^-(x, \alpha(v_n), v_n) g_0 = \infty. \end{aligned}$$

This implies  $V^-(x) = \infty$ .

2) The fact that  $V^-(x) = 0$  iff  $x = 0$  is an easy consequence of the identities  $f(0, u_0, v) = 0$  and  $g(0, u_0, v) = 0$  and the inequality  $g(x, u, v) > 0$  for all  $x \neq 0$  which are included in our assumptions on  $f$  and (13)(i).

3) We first use that (13)(i) and (ii) together with the continuity of  $g$  and the compactness of  $U$  and  $V$  imply that for each  $\varepsilon > 0$  there exists a constant  $g_\varepsilon > 0$  such that the inequality

$$\inf_{\|x\| \geq \varepsilon, u \in U, v \in V} g(x, u, v) \geq g_\varepsilon$$

holds. This immediately implies that for each trajectory satisfying  $x(t, x, u, v) \notin B(0, \varepsilon)$  for all  $t \in [0, T]$  and some  $T > 0$  we get  $J(x, u, v) \geq J_T(x, u, v) \geq Tg_\varepsilon$ .

Consequently, for each  $R > 0$  and  $\varepsilon > 0$  there exists  $T_{R,\varepsilon} > 0$  such that for each  $x \in \mathcal{D}_0^-$  and each  $\alpha \in \Gamma$  with

$$\sup_{v(\cdot) \in \mathcal{V}} J(x, \alpha(v), v) \leq R$$

we can choose times  $t_v \in [0, T_{R,\varepsilon}]$  such that  $x(t_v, x, \alpha(v), v) \in B(0, \varepsilon)$  for each  $v(\cdot) \in \mathcal{V}$ . Note that  $T_{R,\varepsilon}$  does not depend on  $x$  but only on  $R$  and  $\varepsilon$ .

Now pick  $R > 0$  and  $\varepsilon > 0$  with  $\varepsilon < r/2$ . Then there exists a nonanticipative strategy  $\alpha_\varepsilon(\cdot) \in \Gamma$  such that for any perturbation  $v(\cdot) \in \mathcal{V}$  we have

$$V^-(x) \geq J(x, \alpha_\varepsilon(v), v) - \varepsilon.$$

By the above consideration we can conclude the existence of  $T = T_{R,\varepsilon} > 0$  such that for each  $x \in \mathcal{D}_0^-$  with  $V^-(x) \leq R - r$  and  $x \in B(0, R)$  and each  $v(\cdot) \in \mathcal{V}$  there exists  $t_v \in [0, T]$  with  $x(t_v, x, \alpha_\varepsilon(v), v) \in B(0, \varepsilon)$ . We set  $\eta = \varepsilon/T$ , assuming without loss of generality  $T \geq 1$ . By our continuity and boundedness assumptions there exists  $\delta > 0$  such that

$$\sup_{u \in \mathcal{U}, v \in \mathcal{V}} |g(x(t, x, u, v), u(t), v(t)) - g(x(t, y, u, v), u(t), v(t))| < \eta$$

holds for all  $t \in [0, T]$  and all  $y \in B(x, \delta)$ . In particular, since  $\eta \leq \varepsilon < r/2$  we get  $x(t_v, x, \alpha_\varepsilon(v), v) \in B(0, 2\varepsilon) \subset B(0, r)$  for all  $y \in B(x, \delta)$  and all  $v \in \mathcal{V}$ . Furthermore, this inequality implies

$$(17) \quad |J_t(x, \alpha_\varepsilon(v), v) - J_t(y, \alpha_\varepsilon(v), v)| \leq T\eta = \varepsilon$$

for all  $t \in [0, T]$  and all  $y \in B(0, \delta)$ . Observe that  $\delta$  only depends on  $R, T$  and  $\eta$  (and thus  $\varepsilon$ ) but not on  $x$  since we used continuity on the bounded set  $(t, x) \in [0, T] \times B(0, R)$ .

Now recall the dynamic programming principle

$$V^-(y) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \{J_\tau(y, \alpha(v), v) + V^-(x(\tau, y, \alpha(v), v))\}$$

and let  $v_{y,\alpha,\tau} \in \mathcal{V}$  be such that the inequality

$$V^-(y) \leq J_\tau(y, \alpha_\varepsilon(v_{y,\alpha,\tau}), v_{y,\alpha,\tau}) + V^-(x(\tau, y, \alpha_\varepsilon(v_{y,\alpha,\tau}), v_{y,\alpha,\tau})) + \varepsilon$$

holds. For  $\tau = t_{v_{y,\alpha,\tau}}$ , using (16) and  $x(t_{v_{y,\alpha,\tau}}, y, \alpha_\varepsilon(v_{y,\alpha,\tau}), v_{y,\alpha,\tau}) \in B(0, 2\varepsilon) \subset B(0, r)$  as well as (17) and  $J_t \leq J$  we can continue

$$\begin{aligned} V^-(y) &\leq J_{t_{v_{y,\alpha,\tau}}}(y, \alpha_\varepsilon(v_{y,\alpha,\tau}), v_{y,\alpha,\tau}) + \varepsilon + \gamma(2\varepsilon) \leq J_{t_{v_{y,\alpha,\tau}}}(x, \alpha_\varepsilon(v_{y,\alpha,\tau}), v_{y,\alpha,\tau}) + 2\varepsilon + \gamma(2\varepsilon) \\ &\leq \sup_{v \in \mathcal{V}} J(x, \alpha_\varepsilon(v), v) + 2\varepsilon + \gamma(2\varepsilon) \leq V^-(x) + 3\varepsilon + \gamma(2\varepsilon). \end{aligned}$$

Since  $T$  only depends on  $R$  and  $\varepsilon$ , and  $\delta$  only depends on  $R, T$  and  $\varepsilon$  but not on  $x$  we can exchange the roles of  $x$  and  $y$  in order to obtain

$$|V^-(x) - V^-(y)| \leq 3\varepsilon + \gamma(2\varepsilon)$$

for all  $x, y \in \mathcal{D}_0^-$  with  $V^-(x) \leq R - r$ ,  $V^-(y) \leq R - r$ ,  $x, y \in B(0, R)$  and  $\|x - y\| \leq \delta$ . This shows the desired continuity, since by (1) all  $x, y \in \mathcal{D}_0^-$  satisfy these conditions for  $R > 0$  sufficiently large.

4) We have that  $t^-(x_n) \rightarrow \infty$  for  $x_n \rightarrow x \in \partial\mathcal{D}_0^-$  or  $\|x_n\| \rightarrow \infty$ . So,

$$\begin{aligned} V^-(x) &= \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \int_0^\infty g(x(s, y, \alpha(v), v), \alpha(v(s)), v(s)) ds \\ &\geq \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \int_0^{t^-(y, \alpha(v), v)} g_0 ds \geq t^-(x_n) g_0 \rightarrow \infty. \end{aligned}$$

Consequently,  $V^-(x_n) \rightarrow \infty$ .

The proof of 5)–8) is a direct consequence of 1)–4), observing that continuity of  $W^-$  on the whole  $\mathbb{R}^N$  follows from 3) and 4).

The proof of 9)–16) follows by analogous arguments. ■

Having established continuity, from the dynamic programming principles we can derive the associated Hamilton-Jacobi-Isaacs (HJI) PDEs, see [1]. Firstly,  $V^-$  and  $W^-$  are viscosity solutions of the equations

$$(18) \quad H^-(x, \nabla V(x)) = 0 \text{ for } x \in \mathcal{D}_0^-$$

and, respectively,

$$(19) \quad \tilde{H}^-(x, W(x), \nabla W(x)) = 0 \text{ for } x \in \mathbb{R}^N.$$

Here,  $H^-, \tilde{H}^- : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ , are given by

$$H^-(x, p) = \min_{v \in V} \max_{u \in U} \{-\langle p, f(x, u, v) \rangle - g(x, u, v)\}$$

and, respectively,

$$\tilde{H}^-(x, r, p) = \min_{v \in V} \max_{u \in U} \{-\langle p, f(x, u, v) \rangle - (1-r)g(x, u, v)\}.$$

Secondly,  $V^+$  and  $W^+$  are viscosity solutions of the equations

$$(20) \quad H^+(x, \nabla V(x)) = 0 \text{ for } x \in \mathcal{D}_0^+$$

and, respectively,

$$(21) \quad \tilde{H}^+(x, W(x), \nabla W(x)) = 0 \text{ for } x \in \mathbb{R}^N.$$

Here,  $H^+, \tilde{H}^+ : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ , are given by:

$$H^+(x, p) = \max_{u \in U} \min_{v \in V} \{-\langle p, f(x, u, v) \rangle - g(x, u, v)\}$$

and, respectively,

$$\tilde{H}^+(x, r, p) = \max_{u \in U} \min_{v \in V} \{-\langle p, f(x, u, v) \rangle - (1-r)g(x, u, v)\}.$$

**Remark 11** Comparing (18)–(21) with (1) one sees that (19) and (21) are the direct generalizations of Zubov's original equation (1) — except for the square root factor in (1) which we do not need here because of our boundedness assumption on  $f$ . Hence, we will refer to (19) and (21) as the min-max and max-min Zubov equation, respectively. However, since (18) and (20) are only rescaled versions of these generalized Zubov equations, we may also refer to them as Zubov-type equations.

## 5 Uniqueness of the solution

In the last section we saw that  $V^\pm$  and  $W^\pm$  characterize the uniform controllability domains  $\mathcal{D}_0^\pm$  and are viscosity solutions of the min-max and max-min Zubov equations (18), (20), (19) and (21), respectively. In this section we show that these functions are the unique viscosity solutions of these equations.

In order to obtain uniqueness, we use results from [37]. To this end we define the Hamilton-Jacobi equations

$$(22) \quad \max_{u \in U} \min_{v \in V} \{-\langle \nabla W(x), f(x, u, v) \rangle - h(x, u, v) + k(x, u, v)W(x)\} = 0$$

and

$$(23) \quad \min_{v \in V} \max_{u \in U} \{-\langle \nabla W(x), f(x, u, v) \rangle - h(x, u, v) + k(x, u, v)W(x)\} = 0$$

for locally Lipschitz functions  $k, h : \mathbb{R}^N \times U \times V \rightarrow \mathbb{R}$  with  $k$  being nonnegative. Observe that (21) and (19) are special cases of (22) and (23) with  $h = -g$  and  $k = g$ .

For these equations the following proposition follows from [37].

**Proposition 12** *Abbreviate*

$$\begin{aligned} L_T(x_0, u, v) &= \int_0^T h(x(t), u(t), v(t)) \exp\left(-\int_0^t k(x(\tau), u(\tau), v(\tau))d\tau\right) dt \\ &\quad + \exp\left(-\int_0^T k(x(\tau), u(\tau), v(\tau))d\tau\right) W(x(T)) \end{aligned}$$

with  $x(t) = x(t, x_0, u, v)$ . Let  $\Omega \subset \mathbb{R}^N$  be an open and bounded set and define

$$t_\Omega(x_0, u, v) := \inf\{t \geq 0 \mid x(t) \notin \Omega\}.$$

i) Let  $W$  be a continuous supersolution of (22) in  $\mathbb{R}^N$ . Then the equation

$$W(x_0) = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} \sup_{t \in [0, t_\Omega(x_0, u, \beta(u))]} L_t(x_0, u, \beta(u))$$

holds.

ii) Let  $W$  be a continuous subsolution of (22) in  $\mathbb{R}^N$ . Then the equation

$$W(x_0) = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} \inf_{t \in [0, t_\Omega(x_0, u, \beta(u))]} L_t(x_0, u, \beta(u))$$

holds.

iii) Let  $W$  be a continuous supersolution of (23) in  $\mathbb{R}^N$ . Then the equation

$$W(x_0) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \sup_{t \in [0, t_\Omega(x_0, \alpha(v), v)]} L_t(x_0, \alpha(v), v)$$

holds.

iv) Let  $W$  be a continuous subsolution of (23) in  $\mathbb{R}^N$ . Then the equation

$$W(x_0) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \inf_{t \in [0, t_\Omega(x_0, \alpha(v), v)]} L_t(x_0, \alpha(v), v)$$

holds.

**Proof.** We first show (iii). Formula (4.6) in [37] implies (iii) provided  $f$  is globally Lipschitz. It remains to be shown that we can replace this assumption by our local Lipschitz condition. In order to achieve this, observe that Formula (4.6) in [37] also holds if  $W$  is a viscosity supersolution on  $\Omega$  only. Since  $\Omega$  is bounded, we can change  $f$  (e.g. by multiplication with a smooth function which is constant to one on  $\Omega$  and constant to zero for all  $x$  sufficiently large) such that  $f$  becomes globally Lipschitz and  $W$  remains a viscosity supersolution on  $\Omega$ . Then Formula (4.6) in [37] is applicable and implies (iii).

iv) is obtained similarly using [37, Remark 4.2].

i) is obtained from iv) by observing that if  $W$  satisfies i) then  $-W$  satisfies iv) for  $-h$  instead of  $h$  and  $u$  and  $v$  changing their respective roles. Then the inequality obtained from iv) for  $-W$  implies i). Finally, ii) is obtained similarly from iii). ■

Using this proposition we can now state the following existence and uniqueness results for (21) and (19). The uniqueness part of the proof essentially consists of proving comparison results for viscosity sub- and supersolutions. Recall from the introduction that the main difficulty to overcome is that our Lagrangian  $g$  is not bounded from below by a positive constant  $g_0 > 0$  but only by 0. Comparison results for this case have been obtained before in [36, Remark 3.12] and these results could also be used in order to derive a uniqueness result. However, in this reference only the lower value of the game and Lagrangians of the form  $g(x, u, v) = \mu(\|x\|)$  are considered. In contrast to this, here we provide statements for both the upper and the lower value and general choices of  $g$ .

**Theorem 13** *Assume that  $(H^\pm)$  and (13), (15) hold.*

*i) Let  $O$  be an open set and let  $V : O \rightarrow \mathbb{R}$  be a continuous viscosity solution of (18) or, respectively, (20). Suppose that  $V$  satisfies*

$$V(0) = 0 \quad \text{and} \quad V(y) \rightarrow \infty \text{ for } y \rightarrow x \in \partial O \text{ and for } \|y\| \rightarrow \infty.$$

*Then  $O = \mathcal{D}_0^-$  and  $V = V^-$  or, respectively,  $O = \mathcal{D}_0^+$  and  $V = V^+$ .*

*ii) The function  $W^-$  is the unique continuous and bounded viscosity solution of (19) on  $\mathbb{R}^N$  with  $W^-(0) = 0$ .*

*iii) The function  $W^+$  is the unique continuous and bounded viscosity solution of (21) on  $\mathbb{R}^N$  with  $W^-(0) = 0$ .*

*Moreover, under Isaacs' condition, i.e.,  $H^- = H^+$  for  $H^-$  from (18) and  $H^+$  from (20), we have that  $V^- = V^+$ ,  $W^- = W^+$  and consequently  $\mathcal{D}_0^- = \mathcal{D}_0^+$ .*

**Proof.** We prove ii) and iii). The assertion for i) can be obtained by an appropriate modification of the arguments used in the proof which we skip for sake of brevity.

ii) We already know that  $W^-$  is a continuous and bounded viscosity solution of (19) with  $W^-(0) = 0$ . Abbreviating  $x(t) = x(t, x_0, \alpha(v), v)$  and using  $G_t$  from (14) and the equality

$$1 - G_t(x_0, u, v) = \int_0^t G_s(x, u, v) g(x(s), u(s), v(s)) ds$$

one sees that  $W^-$  satisfies

$$\begin{aligned} W^-(x_0) &= 1 - e^{-V^-(x_0)} = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \lim_{t \rightarrow \infty} \{1 - G_t(x_0, \alpha(v), v)\} \\ &= \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \left\{ \int_0^\infty G_t(x_0, \alpha(v), v) g(x(t), \alpha(v)(t), v(t)) dt \right\} \end{aligned}$$

In order to prove ii), it remains to show that every other continuous and bounded viscosity solution  $W$  of (19) on  $\mathbb{R}^N$  with  $W(0) = 0$  coincides with  $W^-$ . To this end, we pick an arbitrary  $x_0 \in \mathbb{R}^N$  and show  $W(x_0) \geq W^-(x_0)$  and  $W(x_0) \leq W^-(x_0)$ .

“ $\geq$ ”: Since  $W$  is a continuous viscosity supersolution of (19) we can apply Proposition 12(i) with  $h = -g$  and  $k = g$ . Note that using the definition of  $G_t$  from (14),  $L_t$  from Proposition 12 can be written as

$$L_t(x_0, u, v) = 1 + G_t(x_0, u, v)(W(x(t)) - 1).$$

Thus, Proposition 12(i) yields

$$(24) \quad W(x_0) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \sup_{t \in [0, t_\Omega(x_0, \alpha(v), v)]} \{1 + G_t(x_0, \alpha(v), v)(W(x(t)) - 1)\}$$

with  $\Omega \subset \mathbb{R}^N$  being an arbitrary open and bounded set. Now we distinguish two cases.

a)  $x_0 \notin \mathcal{D}_0^-$ : In this case we have  $W^-(x_0) = 1$ , hence we need to show  $W(x_0) \geq 1$ . To this end, recall that by definition of  $\mathcal{D}_0^-$  for every  $\alpha \in \Gamma$  there exists  $v_\alpha \in \mathcal{V}$  such that  $x(t, x_0, \alpha(v_\alpha), v_\alpha) \notin B(0, r)$  for all  $t \geq 0$ . Thus the inequality  $g(x(t, x_0, \alpha(v_\alpha), v_\alpha), \alpha(v_\alpha)(t), v_\alpha(t)) \geq g_0$  holds and consequently  $G_t(x_0, \alpha(v_\alpha), v_\alpha) \rightarrow 0$  for  $t \rightarrow \infty$  uniformly for all  $\alpha \in \Gamma$ . Because  $W$  is bounded this implies the convergence  $1 + G_t(x_0, \alpha(v_\alpha), v_\alpha)(W(x(t)) - 1) \rightarrow 1$  as  $t \rightarrow \infty$  for all  $\alpha \in \Gamma$  which is again uniform for all  $\alpha \in \Gamma$ . Now since  $f$  is bounded, for  $\Omega = B(x_0, R)$  we obtain  $t_\Omega \rightarrow \infty$  as  $R \rightarrow \infty$ . Thus, the right hand side of (24) converges to a value  $\geq 1$  as  $R \rightarrow \infty$  implying  $W(x_0) \geq 1$ .

b)  $x_0 \in \mathcal{D}_0^-$ : In this case we have  $W^-(x_0) < 1$ , hence if  $W(x_0) = 1$  then there is nothing to show. Thus, we may assume  $W(x_0) = 1 - \delta$  for some  $\delta > 0$ . Let  $M$  be a bound on  $\|f\|$ .

Now pick  $\varepsilon > 0$ ,  $\varepsilon < \delta/4$  and  $\alpha_\varepsilon \in \Gamma$  such that the infimum in (24) is attained up to this  $\varepsilon$  and  $v_\varepsilon \in \mathcal{V}$  such that the supremum up to this  $\varepsilon$  is attained in the definition of  $W^-(x_0)$  for  $\alpha_\varepsilon \in \Gamma$ . Furthermore, set  $\Omega = B(0, R)$ . We have that

$$(25) \quad W(x_0) + \varepsilon \geq \sup_{t \in [0, t_\Omega(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon)]} \{1 + G_t(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon)(W(x(t)) - 1)\}$$

holds for all  $v \in \mathcal{V}$ . If  $t_\Omega(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon) < \infty$  then from the boundedness of  $f$  it follows that  $x(t) \notin B(r, 0)$  for  $t \in [t_\Omega(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon) - (R-r)/M, t_\Omega(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon)]$  implying  $G_t(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon) \leq \exp(-(R-r)/Mg_0)$ . Thus, for sufficiently large  $R$ , from (25) and the boundedness of  $W$  we obtain

$$W(x_0) + \varepsilon \geq 1 - \delta/2$$

which contradicts  $W(x_0) = 1 - \delta$ . Thus, choosing  $R$  sufficiently large we obtain  $t_\Omega(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon) = \infty$ .

Now for each  $\eta > 0$  we find arbitrary large times  $t_\eta > 0$  such that  $\|x(t_\eta)\| \leq \eta$  because otherwise from (13)(i) and (ii) we obtain  $G_t(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon) \rightarrow 0$  as  $t \rightarrow \infty$  which inserted into (25) again contradicts  $W(x_0) = 1 - \delta$ . For  $\eta > 0$  sufficiently small, continuity and  $W(0) = 0$  imply

$$W(x(t_\eta)) \geq -\varepsilon.$$

Using (25) once again yields

$$\begin{aligned} W(x_0) &\geq \{1 + G_{t_\eta}(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon)(W(x(t_\eta)) - 1)\} - \varepsilon \\ &\geq \{1 - G_{t_\eta}(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon)\} - 2\varepsilon. \end{aligned}$$

Since  $t_\eta$  can be chosen arbitrary large, we can let  $t_\eta \rightarrow \infty$  and continue

$$\begin{aligned} W(x_0) &\geq \lim_{t \rightarrow \infty} \{1 - G_t(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon)\} - 2\varepsilon \\ &\geq \sup_{v \in \mathcal{V}} \lim_{t \rightarrow \infty} \{1 - G_{t_\eta}(x_0, \alpha_\varepsilon(v), v)\} - 3\varepsilon \\ &\geq \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \lim_{t \rightarrow \infty} \{1 - G_{t_\eta}(x_0, \alpha(v), v)\} - 3\varepsilon \\ &= W^-(x_0) - 3\varepsilon \end{aligned}$$

where we used the choice of  $v_\varepsilon$  in the last step. Now the assertion follows since  $\varepsilon$  was arbitrarily small.

“ $\leq$ ”: Since  $W$  is a continuous viscosity subsolution of (19) we can apply Proposition 12(iv) with  $h = -g$  and  $k = g$ . Using the notation introduced above, Proposition 12(iv) yields

$$(26) \quad W(x_0) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \inf_{t \in [0, t_\Omega(x_0, \alpha(v), v)]} \{1 + G_t(x_0, \alpha(v), v)(W(x(t)) - 1)\}$$

We first show the following property: For each  $\varepsilon > 0$  there exists a time  $T > 0$  such that for every  $u \in \mathcal{U}$  and  $v \in \mathcal{V}$  and every  $\tau \geq 0$  there exists a time  $t^* \in [\tau, \tau + T]$  such that either

$$(27) \quad G_{t^*}(x_0, u, v) \leq \varepsilon$$

or

$$(28) \quad W(x(t^*), x_0, u, v) \leq \varepsilon.$$

holds. In order to show this property assume that for each  $T > 0$  there exist  $u$  and  $v$  and  $\tau$  such that (28) does not hold for all  $t \in [\tau, \tau + T]$ . Since  $W$  is continuous with  $W(0) = 0$  this implies  $\|x(t, x_0, u, v)\| > \eta$  for all  $t \in [\tau, \tau + T]$  for some  $\eta > 0$ . By (13)(i) and (ii) this, in turn, implies the existence of  $g_\eta > 0$  such that  $g(x(t, x_0, u, v), u(t), v(t)) \geq g_\eta$  holds for all  $t \in [\tau, \tau + T]$ . Thus  $G_T(x_0, u, v) \leq e^{-g_\eta T}$  follows. Now (27) follows with  $t^* = \tau + T$  if we choose  $T$  so large that  $e^{-g_\eta T} < \varepsilon$  holds.

Now we distinguish two cases.

(a)  $x_0 \notin \mathcal{D}_0^-$ : In this case we need to show  $W(x_0) \leq 1$ . Let  $M_W \geq 1$  be a bound for  $W$ , pick  $\varepsilon \in (0, 1)$  and  $T > 0$  such that (27) or (28) holds for some  $t^* \in [0, T]$ , i.e., for  $\tau = 0$ . In case (27) holds we get

$$1 + G_{t^*}(x_0, \alpha(v), v)(W(x(t^*)) - 1) \leq 1 + \varepsilon(W(x(t^*)) - 1) \leq 1 + \varepsilon(M_W - 1)$$

and if (28) holds we obtain

$$1 + G_{t^*}(x_0, \alpha(v), v)(W(x(t^*)) - 1) \leq 1 + G_{t^*}(x_0, \alpha(v), v)(\varepsilon - 1) \leq 1 \leq 1 + \varepsilon(M_W - 1).$$

Now let  $\Omega$  be so large that  $t_\Omega(x_0, u, v) > T$  for all  $u$  and  $v$  which is possible since  $f$  is bounded. Then we get

$$\inf_{t \in [0, t_\Omega(x_0, \alpha(v), v)]} \{1 + G_t(x_0, \alpha(v), v)(W(x(t)) - 1)\} \leq 1 + \varepsilon(M_W - 1)$$

for all  $v$  and  $\alpha$  and thus by (26) also

$$W(x_0) \leq 1 + \varepsilon(M_W - 1).$$

Since  $\varepsilon > 0$  was arbitrary this shows the claim.

(b)  $x_0 \in \mathcal{D}_0^-$ : Pick  $M_W, \varepsilon, T$  and  $\Omega$  as in (a). This implies

$$\begin{aligned} & \inf_{t \in [0, t_\Omega(x_0, \alpha(v), v)]} \{1 + G_t(x_0, \alpha(v), v)(W(x(t)) - 1)\} \\ & \leq 1 + G_{t^*}(x_0, \alpha(v), v)(W(x(t^*)) - 1) \\ & \leq 1 - G_{t^*}(x_0, \alpha(v), v) + \varepsilon M_W \end{aligned}$$

for all  $\alpha \in \Gamma$  and  $v \in \mathcal{V}$ . Now, we pick  $\alpha_\varepsilon \in \Gamma$  such that

$$\sup_{v \in \mathcal{V}} \lim_{t \rightarrow \infty} \{1 - G_t(x_0, \alpha_\varepsilon(v), v)\} \leq \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \lim_{t \rightarrow \infty} \{1 - G_t(x_0, \alpha(v), v)\} + \varepsilon$$

and  $v_\varepsilon \in \mathcal{V}$  such that

$$\begin{aligned} & \sup_{v \in \mathcal{V}} \inf_{t \in [0, t_\Omega(x_0, \alpha_\varepsilon(v), v)]} \{1 + G_t(x_0, \alpha_\varepsilon(v), v)(W(x(t)) - 1)\} \\ & \leq \inf_{t \in [0, t_\Omega(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon)]} \{1 + G_t(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon)(W(x(t)) - 1)\} + \varepsilon. \end{aligned}$$

These choices imply

$$\begin{aligned} W(x_0) & \leq \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \inf_{t \in [0, t_\Omega(x_0, \alpha(v), v)]} \{1 + G_t(x_0, \alpha(v), v)(W(x(t)) - 1)\} \\ & \leq \sup_{v \in \mathcal{V}} \inf_{t \in [0, t_\Omega(x_0, \alpha_\varepsilon(v), v)]} \{1 + G_t(x_0, \alpha_\varepsilon(v), v)(W(x(t)) - 1)\} \\ & \leq \inf_{t \in [0, t_\Omega(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon)]} \{1 + G_t(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon)(W(x(t)) - 1)\} + \varepsilon \\ & \leq 1 - G_{t^*}(x_0, \alpha_\varepsilon(v_\varepsilon), v_\varepsilon) + \varepsilon + \varepsilon M_W \end{aligned}$$

Since  $t^*$  is arbitrary large (by letting  $\tau \rightarrow \infty$ ) this estimate also holds for  $t^* \rightarrow \infty$  and thus by choice of  $v_{\varepsilon, \alpha}$  we can continue

$$\begin{aligned} W(x_0) & \leq \lim_{t \rightarrow \infty} \{1 - G_t(x_0, \alpha_\varepsilon(v_{\varepsilon, \alpha}), v_{\varepsilon, \alpha}) + \varepsilon + \varepsilon M_W\} \\ & \leq \sup_{v \in \mathcal{V}} \lim_{t \rightarrow \infty} \{1 - G_t(x_0, \alpha_\varepsilon(v), v) + \varepsilon + \varepsilon M_W\} \\ & \leq \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \lim_{t \rightarrow \infty} \{1 - G_t(x_0, \alpha(v), v) + 2\varepsilon + \varepsilon M_W\} \\ & = W^-(x_0) + 2\varepsilon + \varepsilon M_W. \end{aligned}$$

Now the assertion follows because  $\varepsilon > 0$  is arbitrary small.

iii) We already know that  $W^+$  is a continuous and bounded viscosity solution of (21) with  $W^+(0) = 0$ . As in ii) one sees that  $W^+$  satisfies

$$\begin{aligned} W^+(x_0) & = 1 - e^{V^+(x_0)} = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} \lim_{t \rightarrow \infty} \{1 - G_t(x_0, u, \beta(u))\} \\ & = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} \left\{ \int_0^\infty G_t(x_0, u, \beta(u)) g(x(t), u(t), \beta(u)(t)) dt \right\} \end{aligned}$$

In order to prove iii), it remains to show that every other continuous and bounded viscosity solution  $W$  of (21) on  $\mathbb{R}^N$  with  $W(0) = 0$  coincides with  $W^+$ . To this end, we pick an arbitrary  $x_0 \in \mathbb{R}^N$  and show  $W(x_0) \geq W^+(x_0)$  and  $W(x_0) \leq W^+(x_0)$ .

“ $\geq$ ”: Since  $W$  is a continuous viscosity supersolution of (21) we can apply Proposition 12(i) with  $h = -g$  and  $k = g$ . Similar to ii), Proposition 12(i) yields

$$(29) \quad W(x_0) = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} \sup_{t \in [0, t_\Omega(x_0, u, \beta(u))]} \{1 + G_t(x_0, u, \beta(u))(W(x(t)) - 1)\}.$$

Again we distinguish two cases.

a)  $x_0 \notin \mathcal{D}_0^+$ : In this case we have  $W^+(x_0) = 1$ , hence we need to show  $W(x_0) \geq 1$ . To this end, recall that by definition of  $\mathcal{D}_0^+$  there exists  $\beta \in \Delta$  such that  $x(t, x_0, u, \beta(u)) \not\rightarrow 0$  for all  $u \in \mathcal{U}$ . For this  $\beta$  we thus get  $x(t, x_0, u, \beta(u)) \notin B(0, r)$  for all  $t \geq 0$  and all  $u \in \mathcal{U}$ , thus  $g(x(t), u, \beta(u)) \geq g_0$  and consequently  $G_t(x_0, u, \beta(u)) \rightarrow 0$  for  $t \rightarrow \infty$ . Since  $W$  is bounded this implies  $1 + G_t(x_0, u, \beta(u))(W(x(t)) - 1) \rightarrow 1$  as  $t \rightarrow \infty$  uniformly for all  $u \in \mathcal{U}$ . Now since  $f$  is bounded, for  $\Omega = B(x_0, R)$  we obtain  $t_\Omega \rightarrow \infty$  as  $R \rightarrow \infty$ . Thus, the right hand side of (29) converges to a value  $\geq 1$  as  $R \rightarrow \infty$  implying  $W(x_0) \geq 1$ .

b)  $x_0 \in \mathcal{D}_0^+$ : In this case we have  $W^+(x_0) < 1$ , hence if  $W(x_0) = 1$  then there is nothing to show. Thus, we may assume  $W(x_0) = 1 - \delta$  for some  $\delta > 0$ . Let  $M$  be a bound on  $\|f\|$ .

Now pick  $\varepsilon > 0$ ,  $\varepsilon < \delta/4$ ,  $\beta_\varepsilon \in \Delta$  such that the supremum in the definition of  $W^+(x_0)$  is attained up to this  $\varepsilon$  and  $u_\varepsilon$  such that the infimum in (29) is attained up to this  $\varepsilon$  for  $\beta = \beta_\varepsilon$ . Furthermore, we set  $\Omega = B(0, R)$ . Then

$$(30) \quad W(x_0) + \varepsilon \geq \sup_{t \in [0, t_\Omega(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon))]} \{1 + G_t(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon))(W(x(t)) - 1)\}$$

holds. If  $t_\Omega(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon)) < \infty$  then from the boundedness of  $f$  it follows that  $x(t) \notin B(r, 0)$  for  $t \in [t_\Omega(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon)) - (R-r)/M, t_\Omega(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon))]$  implying  $G_t(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon)) \leq \exp(-(R-r)/Mg_0)$ . Thus, for sufficiently large  $R$  from (30) and the boundedness of  $W$  we obtain

$$W(x_0) + \varepsilon \geq 1 - \delta/2$$

which contradicts  $W(x_0) = 1 - \delta$ . Thus, choosing  $R$  sufficiently large we obtain  $t_\Omega(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon)) = \infty$ .

Now for each  $\eta > 0$  we find arbitrary large times  $t_\eta > 0$  such that  $\|x(t_\eta)\| \leq \eta$  because otherwise  $G_t(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon)) \rightarrow 0$  as  $t \rightarrow \infty$  which inserted into (30) again contradicts  $W(x_0) = 1 - \delta$ . For  $\eta > 0$  sufficiently small, continuity and  $W(0) = 0$  imply

$$W(x(t_\eta)) \geq -\varepsilon.$$

Using (30) once again yields

$$\begin{aligned} W(x_0) &\geq \{1 + G_{t_\eta}(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon))(W(x(t_\eta)) - 1)\} - \varepsilon \\ &\geq \{1 - G_{t_\eta}(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon))\} - 2\varepsilon. \end{aligned}$$

Since  $t_\eta$  can be chosen arbitrary large, we can let  $t_\eta \rightarrow \infty$  and continue

$$\begin{aligned} W(x_0) &\geq \lim_{t \rightarrow \infty} \{1 - G_t(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon))\} - 2\varepsilon \\ &\geq \inf_{u \in \mathcal{U}} \lim_{t \rightarrow \infty} \{1 - G_t(x_0, u, \beta_\varepsilon(u))\} - 2\varepsilon \\ &\geq \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} \lim_{t \rightarrow \infty} \{1 - G_t(x_0, u, \beta(u))\} - 3\varepsilon \\ &= W^+(x_0) - 3\varepsilon \end{aligned}$$

where we used the choice of  $\beta_\varepsilon$  in the last step. Now the assertion follows since  $\varepsilon$  was arbitrarily small.

“ $\leq$ ”: Since  $W$  is a continuous viscosity subsolution of (21) we can apply Proposition 12(ii) with  $h = -g$  and  $k = g$ . Using the notation introduced above, Proposition 12(ii) yields

$$(31) \quad W(x_0) = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} \inf_{t \in [0, t_\Omega(x_0, u, \beta(u))]} \{1 + G_t(x_0, u, \beta(u))(W(x(t)) - 1)\}.$$

We first recall the following property from the proof of part ii): For each  $\varepsilon > 0$  there exists a time  $T > 0$  such that for every  $u \in \mathcal{U}$  and  $v \in \mathcal{V}$  and every  $\tau \geq 0$  there exists a time  $t^* \in [\tau, \tau + T]$  such that either

$$(32) \quad G_{t^*}(x_0, u, v) \leq \varepsilon$$

or

$$(33) \quad W(x(t^*), x_0, u, v) \leq \varepsilon.$$

holds.

Now we distinguish two cases.

(a)  $x_0 \notin \mathcal{D}_0^+$ : In this case we need to show  $W(x_0) \leq 1$ . Let  $M_W \geq 1$  be a bound for  $W$ , pick  $\varepsilon \in (0, 1)$  and  $T > 0$  such that (32) or (33) holds for some  $t^* \in [0, T]$ , i.e., for  $\tau = 0$ . In case (32) holds we get

$$1 + G_{t^*}(x_0, u, \beta(u))(W(x(t)) - 1) \leq 1 + \varepsilon(W(x(t^*)) - 1) \leq 1 + \varepsilon(M_W - 1)$$

and if (33) holds we obtain

$$1 + G_{t^*}(x_0, u, \beta(u))(W(x(t^*)) - 1) \leq 1 + G_{t^*}(x_0, u, \beta(u))(\varepsilon - 1) \leq 1 \leq 1 + \varepsilon(M_W - 1).$$

Now let  $\Omega$  be so large that  $t_\Omega(x_0, u, v) > T$  for all  $u$  and  $v$  which is possible since  $f$  is bounded. Then we get

$$\inf_{t \in [0, t_\Omega(x_0, u, \beta(u))]} \{1 + G_t(x_0, u, \beta(u))(W(x(t)) - 1)\} \leq 1 + \varepsilon(M_W - 1)$$

for all  $u$  and  $\beta$  and thus by (31) also

$$W(x_0) \leq 1 + \varepsilon(M_W - 1).$$

Since  $\varepsilon > 0$  was arbitrary this shows the claim.

(b)  $x_0 \in \mathcal{D}_0^+$ : Pick  $M_W, \varepsilon, T$  as in (a). This implies

$$\begin{aligned} & \inf_{t \in [0, t_\Omega(x_0, u, \beta(u))]} \{1 + G_t(x_0, u, \beta(u))(W(x(t)) - 1)\} \\ & \leq 1 + G_{t^*}(x_0, u, \beta(u))(W(x(t^*)) - 1) \\ & \leq 1 - G_{t^*}(x_0, u, \beta(u)) + \varepsilon M_W \end{aligned}$$

Now pick  $\beta_\varepsilon \in \Delta$  such that the supremum on the right hand side of (31) is attained up to  $\varepsilon$  and pick  $u_\varepsilon \in \mathcal{U}$  such that

$$\lim_{t \rightarrow \infty} \{1 - G_t(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon))\} \leq \inf_{u \in \mathcal{U}} \lim_{t \rightarrow \infty} \{1 - G_t(x_0, u, \beta_\varepsilon(u))\} + \varepsilon.$$

These choices imply

$$\begin{aligned} W(x_0) & \leq \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} \inf_{t \in [0, t_\Omega(x_0, u, \beta(u))]} \{1 + G_t(x_0, u, \beta(u))(W(x(t)) - 1)\} \\ & \leq \inf_{u \in \mathcal{U}} \inf_{t \in [0, t_\Omega(x_0, u, \beta_\varepsilon(u))]} \{1 + G_t(x_0, u, \beta_\varepsilon(u))(W(x(t)) - 1)\} + \varepsilon \\ & \leq \inf_{t \in [0, t_\Omega(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon))]} \{1 + G_t(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon))(W(x(t)) - 1)\} + \varepsilon \\ & \leq 1 - G_{t^*}(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon)) + \varepsilon + \varepsilon M_W \end{aligned}$$

Since  $t^*$  is arbitrary large (by letting  $\tau \rightarrow \infty$ ) this estimate also holds for  $t^* \rightarrow \infty$  and thus by choice of  $u_\varepsilon$  we can continue

$$\begin{aligned} W(x_0) & \leq \lim_{t \rightarrow \infty} \{1 - G_t(x_0, u_\varepsilon, \beta_\varepsilon(u_\varepsilon))\} + \varepsilon + \varepsilon M_W \\ & \leq \inf_{u \in \mathcal{U}} \lim_{t \rightarrow \infty} \{1 - G_t(x_0, u, \beta_\varepsilon(u))\} + 2\varepsilon + \varepsilon M_W \\ & \leq \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} \lim_{t \rightarrow \infty} \{1 - G_t(x_0, u, \beta(u))\} + 2\varepsilon + \varepsilon M_W \\ & = W^+(x_0) + 2\varepsilon + \varepsilon M_W. \end{aligned}$$

Now the assertion follows because  $\varepsilon > 0$  is arbitrary small. ■

## 6 Implications of our main result

In this section we state and prove some implications of our main result. Furthermore, we illustrate our results by re-considering the system from Example 5 at the end of the section.

Our first corollary combines Proposition 10 and Theorem 13 into a characterization result for the uniform controllability domains via our min-max and max-min Zubov equations.

**Corollary 14** *Assume  $(H^-)$  and  $(H^+)$ , respectively, as well as (13) and (15). Then the unique continuous and bounded viscosity solutions  $W^-$  and  $W^+$  of (19) and (21) characterize  $\mathcal{D}_0^-$  and  $\mathcal{D}_0^+$  via*

$$\mathcal{D}_0^- = \{x \in \mathbb{R}^N \mid W^-(x) < 1\} \quad \text{and} \quad \mathcal{D}_0^+ = \{x \in \mathbb{R}^N \mid W^+(x) < 1\}.$$

**Proof.** The proof follows immediately from Theorem 13 ii) and iii) and Proposition 10 5) and 13).

■

In our next result we investigate the relation between the Assumptions  $(H^-)$  and  $(H^+)$ .

**Theorem 15** *The Assumptions  $(H^-)$  and  $(H^+)$  satisfy the following properties.*

(i) *Assumption  $(H^+)$  implies  $(H^-)$ .*

(ii) *If the condition*

$$(34) \quad \max_{v \in V} \min_{u \in U} \langle p, f(x, u, v) \rangle = \min_{u \in U} \max_{v \in V} \langle p, f(x, u, v) \rangle \quad \text{for all } p \in \mathbb{R}^N$$

*holds, then  $(H^+)$  is equivalent to  $(H^-)$ .*

**Proof.** (i) Assume that  $(H^+)$  holds for some  $r > 0$  and some  $\eta \in \mathcal{KL}$ . Then, using the construction from [12, Assumption (H4) and proof of Proposition 3.3(i)] one sees that  $g$  satisfying (13) and (15) for  $(H^+)$  can always be found. Furthermore, it is easy to see that  $\tilde{g}(x) := \min_{u \in U, v \in V} g(x, u, v)$  also satisfies (13) and (15) for  $(H^+)$ . We can thus assume without loss of generality that  $g$  is independent of  $u$  and  $v$ .

Since  $V^+$  is a continuous viscosity solution of (20) defined on  $O = \mathcal{D}_0^+$  and since the Hamiltonians  $H^-$  and  $H^+$  from (18) and (20) satisfy

$$\begin{aligned} H^-(x, p) &= \min_{v \in V} \max_{u \in U} \{-\langle p, f(x, u, v) \rangle - g(x)\} \\ &\geq \max_{u \in U} \min_{v \in V} \{-\langle p, f(x, u, v) \rangle - g(x)\} = H^+(x, p), \end{aligned}$$

we obtain that  $V^+$  is a continuous viscosity supersolution of  $H^-$  on  $O = \mathcal{D}_0^+$ . Thus, applying Proposition 12(iii) with  $h = g$  and  $k \equiv 0$  for each open and bounded set  $\Omega \subseteq \mathcal{D}_0^+$  we obtain

$$(35) \quad V^+(x_0) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \sup_{t \in [0, t_\Omega(x_0, \alpha(v), v)]} \{J_t(x_0, \alpha(v), v) + V^+(x(t, x_0, \alpha(v), v))\}.$$

The function  $V^+$  satisfies  $V^+(0) = 0$ ,  $V^+(x) > 0$  for  $x \neq 0$  and  $V^+(x_n) \rightarrow \infty$  for  $x_n \rightarrow x \in \partial \mathcal{D}_0^+$  or  $\|x_n\| \rightarrow \infty$ . This implies the existence of  $\bar{\gamma} \in \mathcal{K}_\infty$  such that the inequality  $V^+(x) \geq \bar{\gamma}(\|x\|)$  holds for all  $x \in \mathcal{D}_0^+$ . Furthermore, similar to (16) we get  $V^+(x) \leq \gamma(\|x\|)$  for all  $x \in B(0, r)$  from  $(H^+)$ .

Since this in particular implies that  $\rho := \sup_{y \in B(0, r)} V^+(y)$  is finite, fixing an arbitrary  $\delta > 0$  the sublevel set

$$\Omega := \{x \in \mathbb{R}^N \mid V^+(x) < (1 + \delta)\rho\}$$

is open and bounded, contains  $B(0, r)$  and is contained in  $\mathcal{D}_0^+$ . Thus, (35) is applicable for this  $\Omega$ .

Now for each  $x \in B(0, r)$  we pick  $\alpha_x \in \Gamma$  such that

$$(1 + \delta/2)V^+(x) \geq \sup_{t \in [0, t_\Omega(x, \alpha_x(v), v)]} \{J_t(x, \alpha_x(v), v) + V^+(x(t, x, \alpha_x(v), v))\}$$

holds for all  $v \in \mathcal{V}$ . If  $V^+(x) > 0$  this is possible by (35) and if  $V^+(x) = 0$  (implying  $x = 0$ ) this is possible by choosing  $\alpha_x \equiv 0$  implying  $x(t, x, \alpha_x(v), v) = 0$  for all  $t \geq 0$  and  $v \in \mathcal{V}$  by the properties of  $f$ . Since  $J_t \geq 0$ , this implies  $V^+(x(t, x, \alpha_x(v), v)) \leq (1 + \delta/2)V^+(x) \leq (1 + \delta/2)\rho$  for all  $t \in [0, t_\Omega(x, \alpha_x(v), v)]$ . In particular, if  $t_\Omega(x, \alpha_x(v), v)$  is finite we obtain the inequality  $V^+(x(t_\Omega(x, \alpha_x(v), v), x, \alpha_x(v), v)) \leq (1 + \delta/2)\rho$  and thus  $x(t_\Omega(x, \alpha_x(v), v), x, \alpha_x(v), v) \in \Omega$  which contradicts the definition of  $t_\Omega$ . Hence,  $t_\Omega(x, \alpha_x(v), v) = \infty$  and consequently for all  $t \geq 0$  we get

$$(36) \quad (1 + \delta/2)V^+(x) \geq J_t(x, \alpha_x(v), v) + V^+(x(t, x, \alpha_x(v), v)).$$

Now Property (13)(i) and (ii) of  $g$  together with the continuity imply that one can find  $\tilde{\mu} \in \mathcal{K}_\infty$  such that for all  $x \in \mathbb{R}^N$  the inequality  $g(x) \geq \tilde{\mu}(\|x\|)$  holds. Using the bound  $V^+(x) \leq \gamma(\|x\|)$  which implies  $\|x\| \geq \gamma^{-1}(V^+(x))$ , for  $\mu = \tilde{\mu} \circ \gamma^{-1} \in \mathcal{K}_\infty$  we obtain

$$g(x) \geq \tilde{\mu}(\|x\|) \geq \tilde{\mu}(\gamma^{-1}(V^+(x))) = \mu(V^+(x)).$$

Inserting this inequality into  $J_t$  in (36) we get

$$V^+(x(t, x, \alpha_x(v), v)) \leq (1 + \delta/2)V^+(x) - \int_0^t \mu(V(x(t, x, \alpha_x(v), v)))dt$$

for all  $x \in B(0, r)$ , all  $v \in \mathcal{V}$  and all  $t \geq 0$ .

By [27, Theorem 1.9.2] this implies the inequality  $V^+(x(t, x, \alpha_x(v), v)) \leq \varphi(t, (1 + \delta/2)V^+(x))$  where  $\varphi(t, r)$  solves  $\dot{\varphi}(t, r) = -\mu(\varphi(t, r))$  with initial value  $\varphi(0, r) = r$ . Since  $\mu \in \mathcal{K}_\infty$ , it immediately follows that  $\varphi(t, r)$  converges to 0 monotonically as  $t \rightarrow \infty$  for each  $r \geq 0$ . From this inequality,  $\eta$  satisfying  $(H^-)$  can now be constructed similar to [25, pp. 146–147].

(ii) Again we can without loss of generality assume that  $g$  is independent of  $u$  and  $v$ . In this case, (34) implies Isaacs' condition for (18) and (20), i.e.,  $H^- = H^+$ . Hence  $V^-$  is also a viscosity solution of  $H^+$ , in particular it is a subsolution of  $H^+$ . Thus, we can follow the arguments of Part (i), above, with the obvious modifications in order to obtain  $(H^+)$ . ■

**Remark 16** (i) *The proof uses the fact that  $V^+$  is a Lyapunov functions for (2). Indeed, besides the characterization of the controllability domains, its ability to deliver Lyapunov functions is another main feature of Zubov's method. More precisely, using the same arguments as in the previous proof one can show the following Lyapunov function properties for  $V^\pm$  and  $W^\pm$ , each  $x \in \mathcal{D}_0^\pm$  and each  $\delta > 0$ .*

(a) *There exists  $\alpha_x \in \Gamma$  such that for all  $v \in \mathcal{V}$  and all  $t \geq 0$  the inequality*

$$V^-(x(t, x, \alpha_x(v), v)) \leq \varphi(t, (1 + \delta)V^-(x))$$

*holds.*

(b) *There exists  $\alpha_x \in \Gamma$  such that for all  $v \in \mathcal{V}$  and all  $t \geq 0$  the inequality*

$$W^-(x(t, x, \alpha_x(v), v)) \leq \theta(t, (1 + \delta)W^-(x))$$

*holds.*

(c) *For each  $\beta \in \Delta$  there exists  $u_{x, \beta} \in \mathcal{U}$  such that for all  $t \geq 0$  the inequality*

$$V^+(x(t, x, u_{x, \beta}, \beta(u_{x, \beta}))) \leq \varphi(t, (1 + \delta)V^+(x))$$

*holds.*

(d) For each  $\beta \in \Delta$  there exists  $u_{x,\beta} \in \mathcal{U}$  such that for all  $t \geq 0$  the inequality

$$W^+(x(t, x, u_{x,\beta}, \beta(u_{x,\beta}))) \leq \theta(t, (1 + \delta)W^+(x))$$

holds.

Here the function  $\varphi(t, r)$  is constructed as in Part (i) of the proof of Theorem 15 and  $\theta(t, s) = 1 - e^{-\varphi(t, -\ln(1-s))}$ . These functions are independent of  $\delta$  and converge to 0 monotonically for  $t \rightarrow \infty$  for all  $r \geq 0$  and all  $s \in [0, 1)$ , respectively. If the infimum in the definition of  $V^\pm$  or  $W^\pm$  is a minimum, then these inequalities also hold for  $\delta = 0$ .

(ii) The equality (34) is also called Isaacs' condition. It implies the Isaacs' condition for the Hamiltonians used in Theorem 13 if  $g$  does not depend on  $u$  and  $v$ . Recall that the study of two person zero-sum differential games was initiated by Isaacs (see, for instance, [24]).

(iii) Under Isaacs' condition (34) the existence and the regularity of the value was studied in [29] under the following "transversality or Petrov" condition

$$H^+(x, n_x) = H^-(x, n_x) < 0 \quad \text{for all } x \in B(0, r) \text{ and all } n_x \text{ exterior normal to } B(0, r) \text{ at } x$$

which is stronger than our assumption ( $H^+$ ).

(iv) The function  $\eta$  for ( $H^-$ ) obtained in Part (i) of the proof will in general be different from the function  $\eta$  in ( $H^+$ ), and vice versa in Part (ii). This is due to the fact that in Zubov's method the the function  $V^+$  does not contain information about the rate of controllability  $\eta$  from ( $H^+$ ). We conjecture that a game theoretic extension of alternative Lyapunov function constructions (like the one in [21, Remark 3.5.4]) may be used to obtain identical  $\eta$  in both assumptions.

Our final corollary shows that under Isaacs' condition (34) not only ( $H^-$ ) and ( $H^+$ ) are equivalent but that also  $\mathcal{D}_0^-$  and  $\mathcal{D}_0^+$  coincide.

**Corollary 17** Assume that Isaacs' condition (34) holds and that ( $H^+$ ) and thus also ( $H^-$ ) is satisfied. Then  $\mathcal{D}_0^- = \mathcal{D}_0^+$  holds.

**Proof.** As in the proof of Theorem 15 we may assume without loss of generality that  $g$  is independent of  $u$  and  $v$ . Then (34) implies Isaacs' condition for the Hamiltonians, i.e.,  $\tilde{H}^- = \tilde{H}^+$  for (19) and (21), respectively. Hence, Theorem 13 implies  $W^- = W^+$  and thus Corollary 14 yields the assertion. ■

We end this section by illustrating our results for the system from Example 5.

**Example 18** In Example 5 we showed by direct arguments that for the vector field

$$f(x, u, v) = -x + uvx^3$$

with  $x \in \mathbb{R}$  and  $U = V = \{-1, 1\}$  that the controllability domains satisfy  $\mathcal{D}_0^- = \mathbb{R} \neq \mathcal{D}_0^+ = (-1, 1)$ . Since in Example 5 we showed that both ( $H^-$ ) and ( $H^+$ ) hold, Corollary 17 implies that Isaacs' condition (34) must be violated. Indeed, e.g. for  $p = 1$  we get

$$\langle p, f(x, u, v) \rangle = -x + uvx^3$$

implying

$$\max_{v \in V} \min_{u \in U} \langle p, f(x, u, v) \rangle = -x - x^3$$

but

$$\min_{u \in U} \max_{v \in V} \langle p, f(x, u, v) \rangle = -x + x^3,$$

hence (34) does not hold.

In fact, for this example and the choice  $g(x) = x^2$  the solutions to our equations are readily computable (e.g., using MAPLE) and we obtain

$$W^-(x) = \frac{\sqrt{1+x^2} - 1}{\sqrt{1+x^2}}$$

and

$$W^+(x) = \begin{cases} 1 - \sqrt{1-x^2}, & |x| \leq 1 \\ 1, & |x| > 1 \end{cases}$$

For  $x \in [-5, 5]$ , these functions are plotted in Figure 1. Using Corollary 14, these explicit formulas and the plots confirm our computations from Example 5, i.e.,  $\mathcal{D}_0^- = \mathbb{R}$  and  $\mathcal{D}_0^+ = (-1, 1)$ .

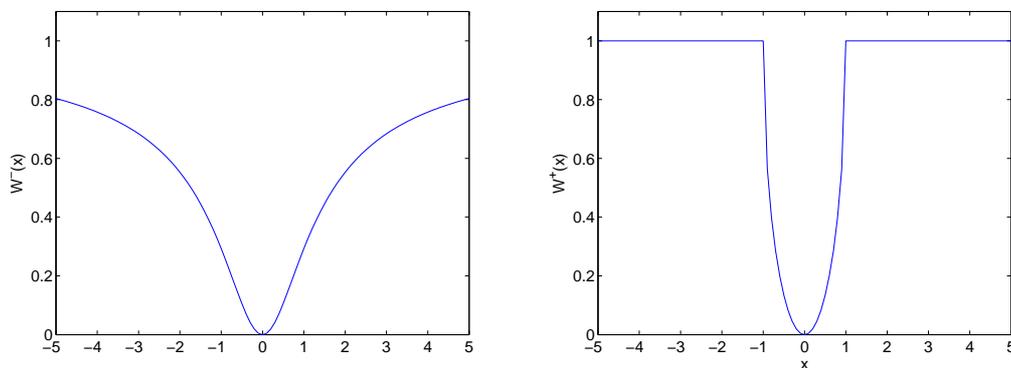


Figure 1:  $W^-(x)$  (left) and  $W^+(x)$  (right) for Example 5.

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