Analysis of unconstrained NMPC schemes with incomplete optimization ⋆

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Abstract: We analyze nominal NMPC schemes without stabilizing terminal constraints in which the online optimization is terminated prior to convergence to the optimum. We state a new stability based termination criterion for nonlinear optimization methods and give conditions for stability and performance estimates. Additionally we present a numerical simulation to illustrate our results.

1. INTRODUCTION

Model predictive control (MPC) is a well established method for the optimal control of linear and nonlinear systems, cf. Allgöwer and Zheng [2000], Qin and Badgwell [2003] or Rawlings and Mayne [2009]. It relies on the iterative online solution of finite horizon optimal control problems where in each sampling interval the first element of the resulting optimal control sequence is applied, resulting in a sampled data feedback law, see, e.g., Diehl et al. [2009], Zavala and Biegler [2009b,a]. MPC can be interpreted as an approximate solution method for in general computationally intractable infinite horizon optimal control problems. This interpretation led to a general stability and performance analysis of nonlinear MPC (NMPC) schemes without stabilizing terminal constraints in Grüne [2009], Grüne et al. [2009a] and Grüne and Rantzer [2008], which was already extended in different directions in Grüne and Pannek [2009] and Grüne et al. [2009b].

(N)MPC is popular because of its conceptual simplicity and its ability to handle both state and input constraints. Its main drawback, on the other hand, is the computational effort needed to solve the underlying optimal control subproblems in real time. Hence, much effort has been spent to reduce this computational burden. One approach in this direction is to relax the condition that the optimization algorithm computes an optimal solution for the subproblems. Since for this purpose usually Newton-like iterative optimization algorithms like the SQP method are used, a natural way to implement such a relaxation is to use incomplete optimization iterations. This means that we stop the iterative optimization after a small number of iteration steps prior to the convergence to the optimal solution.

For (N)MPC schemes with stabilizing terminal constraints this method was investigated for instance in Diehl et al. [2005] and Scokaert et al. [1999]. The main idea in Diehl et al. [2005] is to use an upper bound on the sampling periods which allows to prove that, starting from the shifted optimal control function of the previous sampling instant, a single Newton-step is sufficient to arrive at a sufficently accurate approximation for the optimal control for the current sampling instant. In contrast to this, the approach by Scokaert et al. [1999] works for arbitrary sampling times by ensuring that solutions which may be far from optimal still satisfy the stabilizing terminal constraints from which closed loop stability can be obtained.

Both references heavily rely on the fact that stabilizing terminal constraints are considered, which allow to decouple the stability investigation from optimality considerations. In contrast to this, here we investigate this problem for so called unconstrained nominal NMPC schemes, i.e., schemes in which no additional terminal constraints or terminal costs are added to the finite horizon problem in order to enforce stability properties for undisturbed systems. These schemes are appealing in many ways, cf. the discussion at the end of the introductory Section 2.

Since in unconstrained schemes stability is derived from optimality, in general we cannot expect stability when we use incomplete optimization. For this reason, this paper investigates conditions — theoretically and numerically — for termination of the optimization algorithm which, in controst to the usual approach, is based on a stability instead of an optimality criterion. Here, we show that stability and guaranteed performance can be maintained for the closed–loop using such an algorithm.

After defining the setting in Section 2 and summarizing the results from Grüne [2009], Grüne et al. [2009a] and Grüne and Rantzer [2008] in a simplified setting in Section 3, we define a first condition of this type in Section 4. This condition relies on the online check of a suitable relaxed dynamic programming inequality and is thus well suited to be implemented numerically. A respective algorithm is presented in Section 4 and numerically illustrated in Section 5. In Section 6 we further investigate this condition and show that with incomplete optimization we cannot in general guarantee its feasibility. As a consequence, two ideas on how feasibility can be ensured are discussed and illustrated by a simple example. Finally, Section 7 gives some conclusions.

2. SETUP AND PRELIMINARIES

We consider a nonlinear discrete time control system given by

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$$x(n+1) = f(x(n), u(n)), \quad x(0) = x_0 \tag{1}$$

with $x(n) \in X$ and $u(n) \in U$ for $n \in \mathbb{N}_0$. Here the state space X and the control value space U are arbitrary metric spaces. We denote the space of control sequences $u : \mathbb{N}_0 \to U$ by $\mathcal{U}$ and the solution trajectory for given $u \in \mathcal{U}$ by $x_u(\cdot)$. State and control constraints can be incorporated by replacing X and U by appropriate subsets of the respective spaces, however, for brevity of exposition we will not address this aspect in this paper.

A typical class of such discrete time systems are sampled–data systems induced by a controlled — finite or infinite dimensional — differential equation with sampling period $T > 0$ where the discrete time control value $u(n)$ corresponds to the constant control value $u_c(t)$ applied in the sampling interval $[nT, (n+1)T)$.

Our goal is to minimize the infinite horizon cost functional $J_\infty(x_0, u) = \sum_{n=0}^{\infty} \ell(x_u(n), u(n))$ with running cost $\ell : X \times U \to \mathbb{R}_0^+$ by a static state feedback control law $\mu : X \to U$ which is applied according to the rule

$$x_\mu(0) = x_0, \quad x_\mu(n+1) = f(x_\mu(n), \mu(x_\mu(n))). \tag{2}$$

We denote the optimal value function for this problem by $V_\infty(x_0) := \inf_{u \in \mathcal{U}} J_\infty(x_0, u)$. The motivation for this problem stems from stabilizing the system (1) at a fixed point, i.e., at a point $x^\star \in X$ for which there exists a control value $u^\star \in U$ with $f(x^\star, u^\star) = x^\star$ and $\ell(x^\star, u^\star) = 0$. Under mild conditions on ℓ it is known that the optimal feedback for J_∞ indeed asymptotically stabilizes the system with V_∞ as a Lyapunov function.

Since infinite horizon optimal control problems are in general computationally infeasible, we use a receding horizon NMPC method in order to compute an approximately optimal feedback law. To this end, we consider the finite horizon functional

$$J_N(x_0, u) = \sum_{k=0}^{N-1} \ell(x_u(k), u(k)) \tag{3}$$

with *optimization horizon* $N \in \mathbb{N}_{\geq 2}$ and optimal value function $V_N(x_0) := \inf_{u \in \mathcal{U}} J_N(x_0, u)$. By minimizing (3) over $u \in \mathcal{U}$ we obtain an optimal control sequence [1] $u^*(0), u^*(1), \ldots, u^*(N-1)$ depending on the initial value x_0. Implementing the first element of this sequence, i.e., $u^*(0)$, yields a new state $x_{u^\star}(1, x_0)$ for which we redo the procedure, i.e., at the next time instant we minimize (3) for $x_0 := x_{u^\star}(1, x_0)$. Iterative application of this procedure provides a control sequence on the infinite time interval. A corresponding closed loop representation of the type (2) is obtained as follows.

Definition 1. For $N \geq 2$ we define the MPC feedback law $\mu_N(x_0) := u^\star(0)$, where $u^\star$ is a minimizing control for (3) with initial value x_0.

In many papers in the (N)MPC literature additional stabilizing terminal constraints or terminal costs are added to the optimization objective (3) in order to ensure asymptotic stability of the NMPC closed loop despite the truncation of the horizon (see, e.g., the monograph Rawlings and Mayne [2009] for a recent account of this theory). In contrast to this approach, here we investigate (3) without any changes. This is motivated by the fact that this "plain" NMPC scheme is the most easy one to implement and appears to be predominant in practical applications, cf. Qin and Badgwell [2003]. Another reason appears when looking at the infinite horizon performance of the NMPC feedback law μ_N given by

$$V_\infty^{\mu_N}(x_0) := \sum_{n=0}^{\infty} l(x_{\mu_N}(n), \mu_N(x_{\mu_N}(n))).$$

As we will see in the following section, under a suitable controllability condition for NMPC without stabilizing constraints we can establish an upper bound for this value in terms of the optimal value function $V_\infty(x_0)$, which is in general not possible for schemes with stabilizing constraints.

[1] For simplicity of exposition we assume that a minimizing control sequence u^* exists for (3).

3. ANALYSIS FOR COMPLETE OPTIMIZATION

In this section we summarize the main steps of the stability and suboptimality analysis of unconstrained NMPC schemes from Grüne [2009], Grüne et al. [2009a], Grüne and Rantzer [2008]. Here, we assume that the optimization algorithm delivers an optimal solution in each sampling instant. The cornerstone of our analysis is the following proposition which uses ideas from relaxed dynamic programming.

Proposition 2. Assume there exists $\alpha \in (0, 1]$ such that for all $x \in X$ the inequality

$$V_N(x) \geq V_N(f(x, \mu_N(x)) + \alpha \ell(x, \mu_N(x)) \tag{4}$$

holds. Then for all $x \in X$ the estimate

$$\alpha V_\infty(x) \leq \alpha V_\infty^{\mu_N}(x) \leq V_N(x) \leq V_\infty(x) \tag{5}$$

holds. If, in addition, there exist $x^\star \in X$ and $\mathcal{K}_\infty$-functions α_1, α_2 such that the inequalities

$$\begin{aligned} \ell^\star(x) := \min_{u \in U} \ell(x, u) \;&\geq\; \alpha_1(d(x, x^\star)) \quad \text{and} \\ V_N(x) \;&\leq\; \alpha_2(d(x, x^\star)) \end{aligned} \tag{6}$$

hold for all $x \in X$, then $x^\star$ is a globally asymptotically stable equilibrium for (2) with $\mu = \mu_N$ with Lyapunov function V_N.

Proof: The first part follows from [Grüne and Rantzer, 2008, Proposition 2.2] or [Grüne, 2009, Proposition 2.4] and the second from [Grüne, 2009, Theorem 5.2] observing that the definition of V_N implies $V_N(x) \geq \ell^\star(x) \geq \alpha_1(d(x, x^\star))$. □

In order to compute α in (4) we use the following controllability property: we call the system (1) *exponentially controllable* with respect to the running cost ℓ if there exist constants $C \geq 1$ (overshoot bound) and $\sigma \in [0, 1)$ (decay rate) such that

$$\begin{aligned} &\text{for each } x \in X \text{ there exists } u_x \in \mathcal{U} \text{ with} \\ &\ell(x_u(n, x), u_x(n)) \leq C\sigma^n \ell^\star(x) \;\text{ for all }\; n \in \mathbb{N}_0. \end{aligned} \tag{7}$$

This condition implies

$$V_N(x) \leq J_N(x, u_x) \leq \sum_{n=0}^{N-1} C\sigma^n \ell^\star(x) =: B_N(\ell^\star(x)). \tag{8}$$

Hence, in particular (6) follows for $\alpha_2 = B_N \circ \alpha_3$ if the inequality

$$\alpha_1(d(x, x^\star)) \leq \ell^\star(x) \leq \alpha_3(d(x, x^\star)) \tag{9}$$

holds for some $\alpha_1, \alpha_3 \in \mathcal{K}_\infty$ and all $x \in X$.

In order to compute α in (4), consider an arbitrary $x \in X$ and let $u^\star \in \mathcal{U}$ be an optimal control for $J_N(x, u)$, i.e.,

$J_N(x, u^\star) = V_N(x)$. Note that by definition of μ_N the identity $x_{u^\star}(1, x) = f(x, \mu_N(x))$ follows.
For the following lemma we abbreviate

$$\begin{aligned} \lambda_n &= \ell(x_{u^\star}(n, x), u^\star(n)), \quad n = 0, \ldots, N-1 \quad \text{and} \\ \nu &= V_N(x_{u^\star}(1, x)). \end{aligned} \tag{10}$$

Lemma 3. Assume (7) holds. Then the inequalities

$$\sum_{n=k}^{N-1} \lambda_n \le B_{N-k}(\lambda_k) \quad \text{and} \quad \nu \le \sum_{n=0}^{j-1} \lambda_{n+1} + B_{N-j}(\lambda_{j+1}) \tag{11}$$

hold for $k = 0, \ldots, N-2$ and $j = 0, \ldots, N-2$.

Proof: The first inequalities follow from (8) since by Bellman's optimality principle $x_{u^\star}(k, x), \ldots, x_{u^\star}(N-1, x)$ is an optimal trajectory for the functional $J_{N-k}(x_{u^\star}(k, x), u)$ for $k = 0, \ldots, N-2$. The second inequalities follow from $V_N(x_{u^\star}(1, x)) \le J_N(x_{u^\star}(1, x), u_j)$ for each $j \in \{0, \ldots, N-2\}$ with control function

$$u_j(n) = \begin{cases} u^\star(n+1), & n = 0, \ldots, j-1 \\ u_{x_{u^\star}(j+1,x)}(n+j), & n = j, \ldots, N-1 \end{cases}$$

and $u_{x_{u^\star}(j+1,x)}$ from (7). For details see [Grüne, 2009, Section 3 and Proposition 4.1]. □

The inequalities from Lemma 3 now lead to the following theorem.

Theorem 4. Assume that the system (1) and ℓ satisfy the controllability condition (7). Then inequality (4) holds for all $x \in X$ with

$$\alpha = \inf_{\lambda_1, \ldots, \lambda_{N-1}, \nu} 1 - \nu + \sum_{n=0}^{N-1} \lambda_n \tag{12}$$

subject to the constraints (11) with $\lambda_0 = 1$ and $\lambda_1, \ldots, \lambda_{N-1}, \nu \ge 0$.

Proof: Inequality (4) is equivalent to

$$\sum_{n=0}^{N-1} \lambda_n \ge \nu + \alpha \lambda_0 \tag{13}$$

for all $x \in X$, the corresponding optimal trajectories $x_{u^\star}(n, x)$ and the values $\lambda_0, \ldots, \lambda_{N-1}, \nu$ from (10). Using the linearity of all expressions in (11), (12) it follows that for α from (12) inequality (13) holds for all $\lambda_0, \ldots, \lambda_{N-1}, \nu$ satisfying (11). Since by Lemma 3 this set contains all values of the form (10) for all possible optimal trajectories $x_{u^\star}(n, x)$ of the system, inequality (4) follows. For details see [Grüne, 2009, Section 4]. □

The consequence of this theorem for the performance of the NMPC closed loop, i.e., (2) with $\mu = \mu_N$, is as follows: if (1) and ℓ satisfy (7) and (9), then global asymptotic stability of $x^\star$ and the suboptimality estimate (5) are guaranteed whenever α from (12) is positive. In fact, regarding stability we can show more: by construction of an explicit example it can be shown that whenever α from (12) is negative, then there exists a system (1) and an ℓ satisfying (7) and (9) but for which (2) with $\mu = \mu_N$ is not asymptotically stable, cf. [Grüne, 2009, Theorem 5.3].
The key observation for computing an explicit expression for α in (4) is that the linear program defined by (12) can be solved explicitly.

Theorem 5. Under the assumptions of Theorem 4 the value α from (12) is given by

$$\alpha = 1 - \frac{(\gamma_N - 1) \prod_{i=2}^{N} (\gamma_i - 1)}{\prod_{i=2}^{N} \gamma_i - \prod_{i=2}^{N} (\gamma_i - 1)} \quad \text{with} \quad \gamma_i = C \frac{1 - \sigma^{i-1}}{1 - \sigma}. \tag{14}$$

Proof: See [Grüne et al., 2009a, Theorem 5.3]. □

The explicit formula thus derived for α allows us to visualize the impact of the parameters C, σ in (7) on the value of α in (4). As an example, Figure 1 shows the regions in the C,σ-plane for which $\alpha > 0$ and thus asymptotic stability holds [2] for optimization horizons $N = 2, 4, 8$, and 16. Note that since α is increasing in N the stability region for N is always contained in the stability region for all $\widetilde{N} > N$.

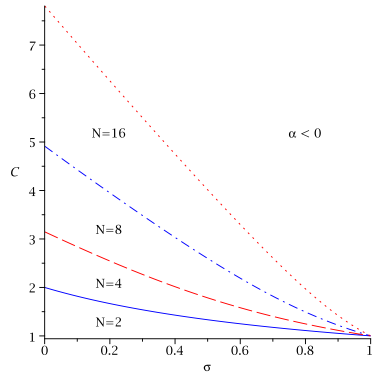

Fig. 1. Stability regions for various optimization horizons N depending on C and σ from (7)

Figure 1 clearly shows the different roles of the parameters C and σ in (7): While for fixed C the minimal stabilizing N for varying σ is usually larger than 2, for fixed σ it is always possible to achieve stability with $N = 2$ by reducing C. Thus, the overshoot bound C plays a decisive role for the stability and performance of NMPC schemes.

4. INCOMPLETE OPTIMIZATION

In order to deal with incomplete optimization in which we terminate the iterative optimization algorithm prior to convergence, we extend Proposition 2. To this end, we first introduce some notation.
The resulting NMPC feedback law will be denoted by $\tilde{\mu}_N$ and the corresponding NMPC closed loop trajectory will consequently be denoted by $x_{\tilde{\mu}_N}(n)$. At each sampling instant, $\tilde{u}_n(k)$, $k = 0, \ldots, N-1$ corresponds to the control sequence computed by the (incomplete) optimization algorithm for initial value $x_0 = x_{\tilde{\mu}_N}(n)$ and $x_{\tilde{u}_n}(k)$, $k = 0, \ldots, N-1$ denotes the corresponding trajectory. This implies the equalities

$$\tilde{\mu}_N(x_{\tilde{\mu}_N}(n)) = \tilde{u}_n(0),$$

and

$$x_{\tilde{\mu}_N}(n+1) = x_{\tilde{u}_n}(1) = x_{\tilde{u}_{n+1}}(0).$$

With

$$\widetilde{V}_N(n) := J_N(x_{\tilde{\mu}_N}(n), \tilde{u}_n) = \sum_{k=0}^{N-1} \ell(x_{\tilde{u}_n}(k), \tilde{u}_n(k))$$

[2] The analogous regions for $\alpha \ge \alpha_0 \in (0, 1)$ look qualitatively similar.

we denote the value of functional J_N along the trajectory $x_{\tilde u_n}$.

If the optimization algorithm yields a globally optimal control then we obtain the usual NMPC scheme discussed in the previous sections, i.e., $\tilde\mu_N = \mu_N$ and $\widetilde V_N(n) = V_N(x_{\tilde\mu_N}(n))$. In general, however, if we do not assume that $\tilde u_n$ is optimal, we obtain

$$\widetilde V_N(n) \ge V_N(x_{\tilde\mu_N}(n)).$$

The following proposition, which has some similarities with Proposition 3 in Grüne and Pannek [2009], generalizes Proposition 2 to our non-optimal setting.

Proposition 6. Consider a closed loop trajectory $x_{\tilde\mu_N}$ and assume there exists $\alpha \in (0,1]$ such that for all $n \in \mathbb{N}_0$ the inequality

$$\widetilde V_N(n) \ge \widetilde V_N(n+1) + \alpha\ell(x_{\tilde\mu_N}(n), \tilde\mu_N(x_{\tilde\mu_N}(n))) \qquad (15)$$

holds. Then we obtain the estimate

$$\alpha V_\infty^{\tilde\mu_N}(x_{\tilde\mu_N}(0)) \le \widetilde V_N(0) \qquad (16)$$

If, in addition, there exist $x^\star \in X$ and a $\mathcal{K}_\infty$-function α_1 such that the inequality

$$\ell^\star(x) := \min_{u\in U} \ell(x,u) \ge \alpha_1(d(x,x^\star)) \qquad (17)$$

holds for all $x \in X$, then $x_{\tilde\mu_N}(n)$ converges to $x^\star$ as $n \to \infty$.

Proof: Rearranging (15) and summing over n we obtain the upper bound

$$\alpha \sum_{k=0}^{K-1} \ell(x_{\tilde\mu_N}(k), \tilde\mu_N(x_{\tilde\mu_N}(k)) \le \widetilde V_N(0) - \widetilde V_N(K) \le \widetilde V_N(0).$$

Hence, taking $K \to \infty$ and using the definition of $V_\infty^{\tilde\mu_N}$ gives the first assertion.

From (17) we immediately obtain the inequality

$$\widetilde V_N(n) \ge \alpha_1(d(x_{\tilde\mu_N}(n), x^\star)) \ge 0 \qquad (18)$$

for all $n \ge 0$. Furthermore, (17) in conjunction with (15) yields

$$\widetilde V_N(n+1) \le \widetilde V_N(n) - \alpha\alpha_1(d(x_{\tilde\mu_N}(n), x^*)), \qquad (19)$$

which in particular implies that $n \mapsto \widetilde V_N(n)$ is monotone decreasing. If we now assume $x_{\tilde\mu_N}(n) \not\to x^\star$ then we find $\varepsilon > 0$ and a sequence $n_j \to \infty$ such that $d(x_{\tilde\mu_N}(n_j), x^\star) > \varepsilon$. By induction over (19) using the fact that $\widetilde V_N(n)$ is decreasing in n this implies

$$\widetilde V_N(n_j) \le \widetilde V_N(0) - j\alpha\alpha_1(\varepsilon).$$

Thus, for j sufficiently large we get $\widetilde V_N(n_j) < 0$ which contradicts (18). □

Note that using the techniques from the proof of Theorem 5.2 in Grüne [2009] we could also construct a $\mathcal{KL}$-function β for which the inequality

$$d(x_{\tilde\mu_N}(n_j), x^\star) \le \beta(d(x_{\tilde\mu_N}(0), x^\star), n)$$

holds. However, the precise shape of this function depends on $\widetilde V_N(0)$ which in turn depends on $x_{\tilde\mu_N}(0)$ and the outcome of the optimization algorithm at the first sampling instant. Thus, unless we assume some uniform bound on the map $x_{\tilde\mu_N}(0) \mapsto \widetilde V_N(0)$, the resulting function β will depend on $x_{\tilde\mu_N}(0)$. Therefore, we will not get the desirable uniform upper bounds with respect to the initial value usually imposed in the definition of asymptotic stability. Still, for notational simplicity we will refer to the convergence property ensured by Proposition 6 as "stability".

Proposition 6 immediately motivates the following algorithm which gives a criterion for the number of steps we should perform in the iterative optimization algorithm in each sampling period. To this end, we assume that the optimization algorithm for minimizing $J_N(x_{\tilde\mu_N}(0), u)$ over the control sequences $u = u(\cdot) \in \mathcal{U}$ works iteratively. More precisely, at time n starting from some initial guess $u_n^{(0)}(\cdot)$ the algorithm iteratively produces control sequences $u_n^{(i)}(\cdot)$, $i = 1, 2, \ldots$, of length N which converge to an optimal control sequence $u_n^\star(\cdot)$. In the following algorithm we assume that for the initial time $n = 0$ the control sequence $\tilde u_0(\cdot)$ and thus the feedback law $\tilde\mu_N(x_{\tilde\mu_N}(0))$ are already computed, e.g., by optimization with a fixed number of iteration steps. Furthermore, we fix a desired value $\alpha \in (0,1)$.

Algorithm: At each sampling instant $n = 1, 2, 3, \ldots$:

(1) Obtain an initial guess $u_n^{(0)}(0), \ldots, u_n^{(0)}(N-1)$, e.g., by using the shifted values $u_n^{(0)}(k) = \tilde u_{n-1}(k+1)$ of the control sequence from the previous sampling instant for $k = 0, \ldots, N-2$ and extending it by some default value $u_n^{(0)}(N-1)$

(2) Set $x_0 = x_{\tilde\mu_N}(0)$ and use the optimization algorithm to compute iteratively control sequences $u_n^{(i)}(0), \ldots, u_n^{(i)}(N-1)$, $i = 1, 2, \ldots$ until the condition

$$\begin{aligned} J_N(x_0, u_n^{(i)}) \;\le\; & J_N(x_{\tilde\mu_N}(n-1), \tilde u_{n-1}) \\ & + \alpha\ell(x_{\tilde\mu_N}(n-1), \tilde u_{n-1}(0)) \end{aligned} \qquad (20)$$

holds and set $\tilde u_n = u_n^{(i)}$.

Note that within the algorithm it is a priori unclear how many iteration steps have to be performed. To maintain applicability the computation of $u_n^{(i)}$ is required to terminate before the time instant n. From the definitions in this section we immediately obtain that (20) implies (15). Thus, if at each sampling instant n the algorithm is successful in finding $u_n^{(i)}$ for which (20) holds then Proposition 6 is applicable and the respective assertion holds. Hence, condition (20) gives a condition under which it is save to terminate the iterative optimization without loosing the stability and performance estimate (16).

In Section 6 we will address the question whether the termination condition (20) is feasible. Before we do this, we illustrate our algorithm by a numerical example.

5. NUMERICAL EXAMPLE

To illustrate the algorithm displayed above we consider the nonlinear pendulum

$$\begin{aligned} \dot x_1 &= x_2 \\ \dot x_2 &= -\frac{g}{l}\sin(x_1) - u\cos(x_1) - \frac{F_a}{l}x_2|x_2| - F_r\,\mathrm{sgn}(x_2) \\ \dot x_3 &= x_4 \\ \dot x_4 &= u \end{aligned}$$

with gravitational constant $g = 9.81$, length of the pendulum $l = 1.25$, air drag $F_a = 0.007$ and friction $F_r = 0.197$. Here, x_1 denotes the angle of the pendulum, x_2 the angular velocity, x_3 the position of the cart and x_4 the velocity of the cart. Starting in the position $t_0 = 0$, $x_0 = (10, 0, 0, 0)$, our aim is to stabilize the origin $(0, 0, 0, 0)$ for this system which corresponds to a stable downward equilibrium.

Within the Figures 2 and 3 below, we display sections of different closed loop trajectories of x_1 and x_3. These solutions are the outcome of the MPC algorithm for the cost functional

$$J_N(x_0, u) = \sum_{k=0}^{N-1} \int_{kT}^{(k+1)T} 100.0 \sin^2(0.5 x_{u1}(t)) + x_{u2}^2(t) + 10.0 x_{u3}^2(t) + x_{u4}^2(t) + u(kT)^2 dt$$

with $N = 17$, $T = 0.15$, constraints $U = [-1, 1]$ and different levels of the parameter α used within the suboptimality based termination criterion (20). One can clearly

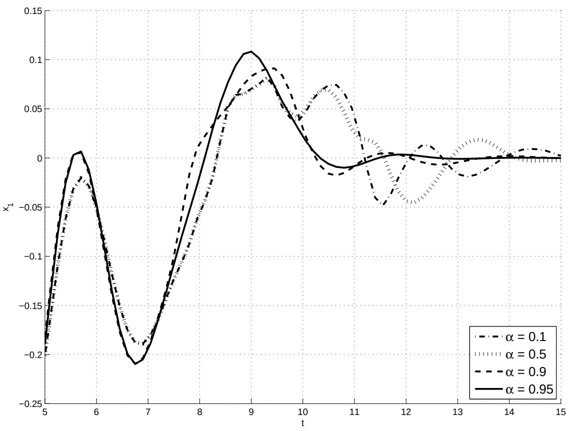

Fig. 2. Trajectory of the angle of the pendulum x_1

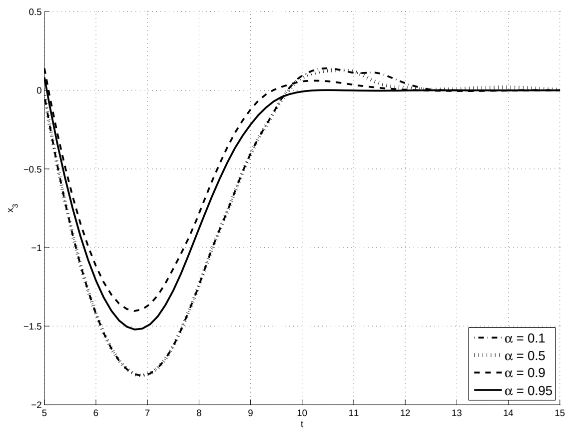

Fig. 3. Trajectory of the position of the cart x_3

see from Figures 2 and 3 that the closed loop system is stable for all values of α. Moreover, one can nicely observe the improvement of the closed loop behaviour visible in the smaller overshoot in both components if the value of α is increased.

This is also reflected in the total closed loop costs: While for small values of α the costs sum up to $V_\infty^{\tilde{\mu}_N}(x_0) \approx 2512.74$, we obtain a total cost of $V_\infty^{\tilde{\mu}_N}(x_0) \approx 2485.83$ for the largest value of α. Note that the majority of the costs, i.e. approximately 2435, is accumulated on the interval $[0, 5]$ on which the trajectories for different α are almost identical and which is therefore not displayed in Figures 2 and 3. Hovever, the chosen α exhibits a large impact on the closed loop performance in the remaining part of the interval.

Regarding the computational cost, the total number of SQP steps which are executed during the run of the MPC process reduces from 455 for $\alpha = 0.95$ and 407 for $\alpha = 0.9$, to 267 and 246 for $\alpha = 0.5$ and $\alpha = 0.1$ respectively. Hence, we obtain an average of approximately $2.5 - 4.5$ optimization iterations per MPC step over the entire interval $[0, 15]$ while using standard termination criteria 9.5 optimization iterations per MPC step are required.

6. FEASIBILITY OF (20)

A closer look at the numerical simulation in the last section reveals that for each α there were some sampling instants n at which it was not possible to satisfy the suboptimality based termination criterion (20). In this case in our algorithm we simply iterated the SQP optimization routine until convergence.

While this fact is not visible in Figures 2 and 3 and obviously does not affect stability and performance in our example, this observation raises the question whether (20) can be satisfied, i.e., whether this condition is feasible in the n-th step regardless of how $\tilde{u}_{n-1}$ was chosen in the previous step. One obvious limitation for this property is that even if $\tilde{u}_{n-1}$ is an optimal control sequence, in general the value α cannot be chosen larger than α from (14). However, even if we choose α smaller than (14) condition (20) may not be feasible for arbitrary control functions $\tilde{u}_{n-1}$.

In order to understand why this is the case we investigate how Lemma 3 changes if the optimal control sequence $u^\star$ in (10) is replaced by the non-optimal control function $\tilde{u}_{n-1}$. To this end, we simplify the notation by setting $x = x_{\tilde{\mu}_N}(n)$ and $\tilde{u} = \tilde{u}_{n-1}$. Now, first observe that the second set of inequalities in (11) remains valid regardless of the optimality of $u^\star$ in (10). All inequalities in the first set of inequalities in (11), however, require optimality of the control function $u^\star$ generating the λ_n in (10). In order to maintain at least some of these inequalities we can pick an optimal control function $\tilde{u}^\star$ for $x_{\tilde{u}}(1, x)$ and horizon length $N - 1$ and define the control sequence $\bar{u}$ via $\bar{u}(0) = \tilde{u}(0)$, $\bar{u}(n) = \tilde{u}^\star(n-1)$, $n = 1, \ldots, N-1$. Then, abbreviating

$$\begin{aligned} \bar{\lambda}_n &= \ell(x_{\bar{u}}(n, x), \bar{u}(n)), \quad n = 0, \ldots, N-1 \quad \text{and} \\ \bar{\nu} &= V_N(x_{\tilde{u}}(1, x)) = V_N(x_{\bar{u}}(1, x)), \end{aligned} \tag{21}$$

we arrive at the following version of Lemma 3.

Lemma 7. Assume (7) holds. Then the inequalities

$$\sum_{n=k}^{N-1} \bar{\lambda}_n \leq B_{N-k}(\bar{\lambda}_k) \quad \text{and} \quad \bar{\nu} \leq \sum_{n=0}^{j-1} \bar{\lambda}_{n+1} + B_{N-j}(\bar{\lambda}_{j+1}) \tag{22}$$

hold for $k = 1, \ldots, N-2$ and $j = 0, \ldots, N-2$.

Proof: Analogous to the proof of Lemma 3. □

The subtle but crucial difference of (22) to (11) is that the left inequality is not valid for $k = 0$. As a consequence, $\bar{\lambda}_0$ does not appear in any of the inequalities, thus for any $\bar{\lambda}_1, \ldots, \bar{\lambda}_n$ and $\bar{\nu}$ satisfying (22) and any $\omega > 0$ the values

$\omega\bar{\lambda}_1, \ldots, \omega\bar{\lambda}_n$ and $\omega\bar{\nu}$ satisfy (22), too. Hence, unless (22) implies $\bar{\nu} \leq \sum_{n=0}^{N-1} \bar{\lambda}_n$ — which is a very particular case — the value α in (12) will be $-\infty$ and consequently feasibility of (20) cannot be concluded for any positive α.

The following example shows that this undesirable result is not simply due to an insufficient estimate for α but that infeasibility of (20) can actually happen.

Example 8. Consider the 1d system

$$x(n+1) = x(n)/2 + u(n) \tag{23}$$

with $\ell(x, u) = |x|$ and input constraint $u \geq 0$. A simple computation using $u_x \equiv 0$ shows that for this system (7) is satisfied with $C = 1$ and $\sigma = 1/2$. Furthermore, for initial value $x_0 \geq 0$ it is obvious that the control $u^* \equiv 0$ is optimal. Using the non-optimal control given by $\tilde{u}_0(0) = \varepsilon > 0$ and $\tilde{u}_0(k) = 0$ for $k = 1, \ldots, N-1$ for the initial value $x_0 = 0$ yields the trajectory $x_{\tilde{u}_0}(0) = x_0 = 0$, $x_{\tilde{u}_0}(k) = \varepsilon 2^{-k+1}$, $k = 1, \ldots, N$, which implies

$$J_N(x_0, \tilde{u}_0) = \sum_{k=0}^{N-2} \varepsilon 2^{-k}.$$

On the other hand, for the initial value $x_{\tilde{u}}(1, x_0) = \varepsilon$ it is easily seen that for each control $\tilde{u}$ the inequality

$$J_N(x_{\tilde{u}_0}(1), \tilde{u}) \geq \sum_{k=0}^{N-1} \varepsilon 2^{-k} > J_N(x_0, \tilde{u}_0)$$

holds. Hence, for $x_{\tilde{\mu}_N}(0) = x_0$ the inequality (20) is indeed not feasible for any $\alpha > 0$ and any $i \in \mathbb{N}$.

Clearly, in order to rigorously ensure stability and guaranteed performance one should derive conditions which exclude these situations and we briefly discuss two possible approaches for this purpose.

One way to guarantee feasibility of (20) is to add the missing inequality in (22) (i.e., the left inequality for $k = 0$) as an additional constraint in the optimization. This guarantees feasibility of (20) for any α smaller than the value from (14). The drawback of this appraoch is that an additional constraint in the optimization is needed which needs to be ensured for all $i \geq 1$. Furthermore, the value $B_N(\bar{\lambda}_0)$ depends on the in general unknown parameters C and σ in (7) and thus needs to be determined by a try-and-error procedure.

Another way to guarantee feasibility is to choose ℓ in such a way that there exists $\gamma > 0$ for which

$$\gamma\ell(x, u) \geq \ell^*(f(x, u)) \tag{24}$$

holds for all $x \in X$ and all $u \in U$ with ℓ^* from (6). Then from (24) and from the controllability condition (7) for $x = f(x, \bar{u}(0))$ we get

$$\sum_{n=0}^{N-1} \bar{\lambda}_n \leq \bar{\lambda}_0 + B_{N-1}(\ell^\star(f(x, \bar{u}(0)))) \leq \bar{\lambda}_0 + \gamma B_{N-1}(\bar{\lambda}_0).$$

Replacing C by $(1+\gamma)C$ this right hand side is $\leq B_N(\bar{\lambda}_0)$ which again yields the left inequality in (22) for $k = 0$ and thus feasibility of (20). Note that (24) holds for our example (23) if we change $\ell(x, u) = |x|$ to $\ell(x, u) = |x| + |u|/\gamma$. The advantage of this method is that no additional constraints have to be imposed in the optimization. Its disadvantages are that constructing ℓ satisfying (24) may be complicated for more involved dynamics and that C and σ may increase for the re-designed ℓ. In turn, this may lower the NMPC closed loop performance and cause the need for larger optimization horizons N in order to obtain stability.

7. CONCLUSIONS

We have investigated unconstrained nominal NMPC schemes with incomplete optimization and have presented a condition which is easily implemented and ensures stability and a performance estimate of the closed loop. Despite the fact that this condition produces good numerical results, in general its feasibility cannot be guaranteed. As a remedy, two approaches ensuring feasibility of this condition have been presented and briefly discussed.

Future research will include an in depth study of these approaches and in particular their algorithmic implementation and numerical evaluation.

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