The Directed Subdifferential of DC Functions

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Dedicated to Alexander Ioffe and Simeon Reich on their 70th resp. 60th birthdays.

Abstract. The space of directed sets is a Banach space in which convex compact subsets of $\mathbb{R}^n$ are embedded. Each directed set is visualized as a (nonconvex) subset of $\mathbb{R}^n$, which is comprised of a convex, a concave and a mixed-type part.

Following an idea of A. Rubinov, the directed subdifferential of a difference of convex (DC) functions is defined as the directed difference of the corresponding embedded convex subdifferentials. Its visualization is called the Rubinov subdifferential. The latter contains the Dini-Hadamard subdifferential as its convex part, the Dini-Hadamard superdifferential as its concave part, and its convex hull equals the Michel-Penot subdifferential. Hence, the Rubinov subdifferential contains less critical points in general than the Michel-Penot subdifferential, while the sharp necessary and sufficient optimality conditions in terms of the Dini-Hadamard subdifferential are recovered by the convex part of the directed subdifferential.

Furthermore, the directed subdifferential could distinguish between points that are candidates for a maximum and those for a minimum. It also allows to easily detect ascent and descent directions from its visualization. Seven out of eight axioms that A. Ioffe demanded for a subdifferential are satisfied as well as the sum rule with equality.

1. Introduction

In [**1, 2**] a linear normed and partially ordered space is introduced, in which the convex cone of all nonempty convex compact sets in $\mathbb{R}^n$ is embedded, and thus a way to subtract convex sets is opened. This space of so-called "directed sets" is a Banach and a Riesz space for dimension $n \geq 2$ and a Banach lattice for $n = 1$. It is defined without equivalence classes as the difference in [**27, 31, 34**]. Our embedding is more involved than the one using only support functions [**16**], but provides a visualization

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of differences of convex compact sets. The visualized directed differences are usually non-convex sets in $\mathbb{R}^n$ equipped with normal directions attached to their boundaries.

The idea to apply the directed differences of convex sets in order to define a subdifferential of DC (differences of convex) functions was suggested to us by A. Rubinov in 2000. It took us a long time to convince ourselves that the existence of such a new subdifferential is justified among the variety of already known subdifferentials, convex and non-convex ones. The visualization of the new subdifferential helped us to see its interesting properties and relations to other known subdifferentials.

The paper is organized as follows: After some basic notations below, we recall several well-known definitions of subdifferentials. In Section 3 we provide some basic definitions and facts on the Banach space of directed sets and in Section 4 we define the directed subdifferential and the Rubinov subdifferential and give some examples. In Section 5 we describe the relations of the Rubinov subdifferential to well-known subdifferentials, and in Section 6 we discuss conditions for optimality, saddle points, ascent and descent directions.

1.1. Basic Notations. Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all convex, compact, nonempty subsets of $\mathbb{R}^n$. We denote by S_n the unit sphere in $\mathbb{R}^n$, and by $cl(A), co(A)$ the closure and the convex hull of the set A, respectively. The following operations in $\mathcal{C}(\mathbb{R}^n)$ are well-known:

$$\begin{array}{rcll} A+B & := & \{a+b \,|\, a\in A,\ b\in B\} & \text{(Minkowski addition)},\\ \lambda\cdot A & := & \{\lambda\cdot a \,|\, a\in A\} & \text{(scalar multiplication for } \lambda\in\mathbb{R}). \end{array}$$

We call the set $-A=(-1)A$ *the pointwise negation of* A. The *support function* in the direction $l\in\mathbb{R}^n$ is $\delta^*(l,A) := \max\limits_{a\in A} \langle l,a\rangle$. The support function for $A\in\mathcal{C}(\mathbb{R}^n)$ is Lipschitz-continuous and fulfills

$$\delta^*(l,A+B)=\delta^*(l,A)+\delta^*(l,B), \qquad \delta^*(l,\lambda\cdot A)=\lambda\cdot\delta^*(l,A) \quad (\lambda\geq 0).$$

The Hausdorff distance between two sets in $\mathcal{C}(\mathbb{R}^n)$ is

$$\mathrm{d_H}(A,B)=\max_{l\in S_{n-1}} |\delta^*(l,A)-\delta^*(l,B)|.$$

The *support face* (set of supporting points) for the direction $l\in S_{n-1}$ is

$$Y(l,A) := \{y(l,A)\in A \,|\, \langle l, y(l,A)\rangle = \delta^*(l,A)\},$$

which coincides with the subdifferential of the support function.

Some known definitions of differences of two sets are listed below:

- Algebraic difference
$$A-B := \{a-b \,|\, a\in A,\ b\in B\},$$
which is not useable in our context, since in general $A-A \underset{\neq}{\supset} \{0_{\mathbb{R}^n}\}$.
- Geometric (Hadwiger-Pontryagin) difference [**13, 30**], sometimes called starshaped difference
$$A \overset{*}{-} B := \{x\in\mathbb{R}^n \,|\, x+B\subset A\}.$$
This difference has the property that $A\overset{*}{-}A=\{0\}$, but may often be empty.
- Demyanov difference [**12, 33**]

$$A \dot{-} B := \overline{\text{co}}\{y(l,A)-y(l,B) \,|\, l\in S_{n-1},\ Y(l,A) \text{ and } Y(l,B) \text{ are singletons}\}.$$

2. Preliminaries – Some Known Subdifferentials

We recall first definitions of subdifferentials which are always convex. The classical *convex (Moreau/Rockafellar) subdifferential* of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ at $x \in \mathbb{R}^n$ is:

$$\partial f(x) := \{s \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n : \ \langle s, y - x\rangle + f(x) \le f(y)\}. \tag{2.1}$$

The vector $s \in \partial f(x)$ is called the *(convex) subgradient of f at x*. This subdifferential is a convex, compact and nonempty set for convex $f : \mathbb{R}^n \to \mathbb{R}$ (see, e.g., [**32**]), and its support function is the directional derivative

$$\partial f(x) = \{s \in \mathbb{R}^n \mid \forall l \in \mathbb{R}^n : \ \langle s, l\rangle \le f'(x; l)\}, \tag{2.2}$$

where the *directional derivative of f at x in direction l* is defined as

$$f'(x; l) := \lim_{t \searrow 0} \frac{f(x + tl) - f(x)}{t}. \tag{2.3}$$

In [**20**, Chapter 2, Section 1.5], the following list of *axioms for subdifferentials* for Lipschitz functions is given.

(SD$_1$) $\partial f(x) = \emptyset$, if $x \notin \mathrm{dom}(f)$.
(SD$_2$) $\partial f(x) = \partial g(x)$, if f and g coincide in a neighborhood of x
or as Rockafellar proposed:
$\partial f(x) = \partial g(x)$, if there exists a neighborhood U of $(x, f(x))$
with $U \cap \mathrm{epi}\, f = U \cap \mathrm{epi}\, g$.
(SD$_3$) If f is convex, then $\partial f(x)$ coincides with the classical convex subdifferential.
(SD$_4$) If f satisfies the Lipschitz condition with constant L in a neighborhood of x, then $\|s\| \le L$ for all $s \in \partial f(x)$.
(SD$_5$) If x is a local minimizer of f, then $0 \in \partial f(x)$.
(SD$_6$) If $n = n_1 + n_2$ and $x^{(i)} \in \mathbb{R}^{n_i}$, $i = 1, 2$, with $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^n$ and $f(x) = f_1(x^{(1)}) + f_2(x^{(2)})$, then

$$\partial f(x) \subset \partial f_1(x^{(1)}) \times \partial f_2(x^{(2)}). \tag{2.4}$$

(SD$_7$) If $\lambda > 0$, $b \in \mathbb{R}^m$, A a bounded linear operator from $\mathbb{R}^n$ *onto* $\mathbb{R}^m$ are given and $g(x) = \lambda \cdot f(Ax + b)$, then $\partial g(x) = \lambda \cdot A^\top \partial f(Ax + b)$, where $A^\top$ denotes the transposed matrix.
(SD$_8$) If χ_S denotes the indicator function of $S \subset \mathbb{R}^n$, i.e.,

$$\chi_S(x) = \begin{cases} 0 & \text{if } x \in S, \\ \infty & \text{otherwise,} \end{cases}$$

then

$$\partial f(x) = \Big\{ s \in \mathbb{R}^n \ : (s, -1) \in \partial \chi_{\mathrm{epi}\, f}(x, f(x)) \Big\}. \tag{2.5}$$

Naturally, the classical convex subdifferential satisfies the above axioms. In addition it fulfills the following stronger form of (SD$_6$) for convex functions $g, h : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$, sometimes called the Moreau–Rockafellar theorem or the *Sum Rule* (cf. [**32**, Theorem 23.8]):

(SR) $\quad \partial(g + h)(x) = \partial g(x) + \partial h(x)$.

This strong equality is not fulfilled for all other subdifferentials of non-convex functions introduced below without additional regularity assumptions.

We now define some other *convex subdifferentials*. The *Dini-Hadamard subdifferential* is studied, e.g., in [**3, 28, 29, 19, 15**] and in [**12**, Section III.4] (under the name "Penot subdifferential"). In the framework of locally Lipschitz functions, it coincides with the radial subdifferential or Gâteaux subdifferential and is defined for a function $f : \mathbb{R}^n \to \mathbb{R}$ that has directional derivatives in $x \in \mathbb{R}^n$ for all directions $l \in S_{n-1}$:

$$\partial_{DH} f(x) := \{s \in \mathbb{R}^n \,|\, \forall l \in \mathbb{R}^n :\ \langle s, l\rangle \le f'(x;l)\}. \tag{2.6}$$

This definition is identical to (2.2) for convex functions, but the directional derivative is not necessarily a convex function w.r.t. l for a non-convex function.

If $f : \mathbb{R}^n \to \mathbb{R}$ is a DC function with $f = g - h$, it is observed in [**15**], [**12**, Chapter III, Proposition 4.1], that the Dini-Hadamard subdifferential equals the geometric difference of the two convex subdifferentials, i.e.,

$$\partial_{DH} f(x) = \partial g(x) \overset{*}{-} \partial h(x). \tag{2.7}$$

Thus, the Dini-Hadamard subdifferential may be empty (cf. [**15**, Section 2.1]), otherwise it is always compact and convex.

To introduce the *Michel-Penot subdifferential* [**23**], we recall that the Michel-Penot directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ in direction $l \in \mathbb{R}^n$ at x is

$$f'_{MP}(x;l) := \sup_{\eta\in\mathbb{R}^n} \limsup_{t\searrow 0} \frac{f(x+t(l+\eta)) - f(x+tl)}{t},$$

and the Michel-Penot subdifferential of f at x is

$$\partial_{MP} f(x) := \{s \in \mathbb{R}^n \,|\, \forall l \in \mathbb{R}^n :\ \langle s, l\rangle \le f'_{MP}(x;l)\}. \tag{2.8}$$

The following connection between the Michel-Penot subdifferential and the Demyanov difference follows from [**10**, Theorem 6.1] for any DC function $f = g - h$ (g, h convex), and $l, x \in \mathbb{R}^n$:

$$f'_{MP}(x;l) = \sup_{\eta\in\mathbb{R}^n} \big(f'(x;l+\eta) - f'(x;l)\big)$$

and the Michel-Penot subdifferential calculates as

$$\partial_{MP} f(x) = \partial g(x) \dot{-} \partial h(x). \tag{2.9}$$

The most well-known subdifferential of a non-convex function is the *Clarke's subdifferential* (cf. [**4, 5, 6, 7**]), which is also a convex set. For $f : \mathbb{R}^n \to \mathbb{R}$ and $l, x \in \mathbb{R}^n$, the Clarke directional derivative of f at x in direction l is the limit

$$f'_{Cl}(x;l) := \limsup_{\substack{t\searrow 0\\ y\to x}} \frac{f(y+tl) - f(y)}{t}.$$

The Clarke subdifferential is defined as

$$\partial_{Cl} f(x) := \{s \in \mathbb{R}^n \,|\, \forall l \in \mathbb{R}^n :\ \langle s, l\rangle \le f'_{Cl}(x;l)\}. \tag{2.10}$$

As it is well-known, cf., e.g., [**8, 10**],

$$\partial_{DH} f(x) \subseteq \partial_{MP} f(x) \subseteq \partial_{Cl} f(x),$$

and they are equal in the case of a convex function f. These inclusions may be strict as it is shown in the examples in Section 4.

Now, we recall definitions of some *non-convex subdifferentials*. The most famous non-convex subdifferential is the (basic/lower) *subdifferential of Mordukhovich*, [**24**], [**26**, Definition 1.77], $\partial_M f(x)$, which is equivalent to the approximate subdifferential of Ioffe in finite dimensions [**17, 18**], [**26**, Theorem 3.59] and may be defined as

$$\partial_M f(x) = cl\{\text{ limits of sequences of proximal subgradients of } f \text{ at } x_k \to x\,\}. \tag{2.11}$$

Recall that a vector v is a proximal subgradient of f at x, if for some $\varepsilon > 0$

$$f(y) - f(x) \geq \langle v, y - x\rangle - \varepsilon \cdot \|y - x\|^2.$$

As it is well-known, the Mordukhovich subdifferential is a compact in $\mathbb{R}^n$, and the Clarke subdifferential is its (closed) convex hull (see, e.g., [**18**], [**26**, Theorem 3.57]).

The "linear" *subdifferential of Treiman* [**35, 36**], $\partial_T f(x)$, is a subset of the Mordukhovich one, constructed as in (2.11) with only special "linear" sequences of proximal subgradients.

Finally, we mention the *quasi-differential of Demyanov-Rubinov* [**11**], [**12**, Chapter III, Section 2] of DC functions, defined as an element of a linear normed space of equivalence classes generated by pairs of convex sets, following the approach of Rådström in [**31**]. For the DC function $f(x) = g(x) - h(x)$, its quasi-differential is the equivalence class generated by the pair $[\partial g(x), -\partial h(x)]$, where the minus denotes the pointwise negation of the set.

The space of directed sets is a tool to avoid the non-uniqueness of the pairs in one equivalence class and to provide a visualization, keeping other positive features of this approach. We emphasize that the directed subdifferential does not depend on the specific DC representation of the function.

The usually non-convex Rubinov subdifferential is a superset of the Dini-Hadamard subdifferential and superdifferential as well as a subset of the Michel-Penot and Clarke subdifferentials. The sharp optimality conditions in terms of the Dini-Hadamard subdifferential are recovered by the positive part of the directed subdifferential which coincides with the Dini-Hadamard one. Distinguishing the positive from the negative part of the directed subdifferential allows to distinguish minimum conditions from maximum ones, in contrast to the Michel-Penot and Clarke subdifferentials. The directed subdifferential differs from the Mordukhovich (lower and upper) subdifferentials, but has similarly good calculus rules. In particular, the directed subdifferential satisfies the sum rule as an equality, whereas (SR) only holds with the inclusion "$\subset$" for most of the other subdifferentials. Nevertheless, one should note that advanced calculus rules and many applications are not yet proved for the Rubinov subdifferential, in comparison to, e.g., the Mordukhovich and Clarke subdifferentials.

To define our directed subdifferential of a DC function as an element of this space, we need some background on directed sets, presented in the next section.

3. Directed Sets

The directed sets are "oriented", non-convex subsets of $\mathbb{R}^n$, defined recursively with respect to their dimension and parametrized by the normal vectors of their lower-dimensional "faces". This additional information allows to construct a Banach space in which subtraction of embedded convex compact sets is well-defined. Extending the parametrization of convex compacts via their support functions, we define a directed set as a pair of mappings that associates to each unit direction an $(n-1)$-dimensional directed set ("directed supporting face") and a scalar (the value of the "support function") in this direction. This method enables us to use recursive constructions.

The definition of directed sets is inductive in the dimension $n \geq 1$ and based on the notion of directed interval for $n = 1$. Our definition of directed intervals is based on support functions, and is equivalent to the generalized and directed intervals ([**21, 22**]).

DEFINITION 3.1. A *directed interval* $\overrightarrow{A}$ is a point in $\mathbb{R}^2$ or, equivalently, a function $a_1 := \{-1, 1\} \to \mathbb{R}$, i.e.,

$$\overrightarrow{A} = (a_1(l))_{l=\pm 1} = (a_1(-1), a_1(1)) \in \mathbb{R}^2.$$

$\mathcal{D}(\mathbb{R})$ denotes the set of all directed intervals.

Denote $\overrightarrow{[\alpha, \beta]} := (-\alpha, \beta)$, where $\alpha = -a_1(-1),\ \beta = a_1(1)$.

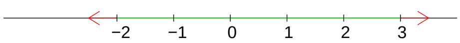
Linear combinations of directed intervals are calculated as linear combinations of vectors

$$\lambda\overrightarrow{[\alpha_1, \beta_1]} + \mu\overrightarrow{[\alpha_2, \beta_2]} = \overrightarrow{[\lambda\alpha_1 + \mu\alpha_2, \lambda\beta_1 + \mu\beta_2]}, \qquad \lambda, \mu \in \mathbb{R}.$$

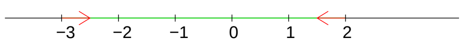
We visualize a directed interval $\overrightarrow{[\alpha, \beta]}$ attaching to each of its end points the corresponding unit normal vector, $l_1 = -1$ to the point α and $l_2 = 1$ to β. The directed interval is proper when $\alpha \leq \beta$ (then the normals are pointing outwards) and improper otherwise (with normals pointing inwards). Some proper and improper intervals are visualized in Example 3.2.

EXAMPLE 3.2. One proper and one improper interval, obtained by subtraction of directed intervals are shown here:

$$\overrightarrow{[-3, 5]} - \overrightarrow{[-1, 2]} = \overrightarrow{[-2, 3]} \quad \text{and} \quad \overrightarrow{[-1, 2]} - \overrightarrow{[-3, 5]} = \overrightarrow{[2, -3]}$$

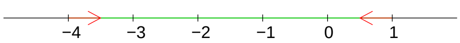


proper interval $\overrightarrow{[-2, 3]} \in \mathcal{D}(\mathbb{R})$

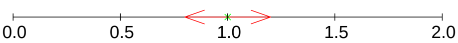


the inverse $\overrightarrow{[2, -3]}$ of $\overrightarrow{[-2, 3]}$

Another improper interval as well as the embedded scalar 1 are visualized below:



improper interval $\overrightarrow{[1, -4]} \in \mathcal{D}(\mathbb{R})$



degenerate interval $\overrightarrow{[1, 1]}$

Motivated by describing the convex, compact, nonempty set A for each direction $l \in S_{n-1}$ by its $(n-1)$-dimensional supporting face and the value of the support function, the linear normed space $\mathcal{D}(\mathbb{R}^n)$ of the directed sets in $\mathbb{R}^n$ is constructed inductively.

DEFINITION 3.3. $\overrightarrow{A}$ is called a *directed set*

(i) in $\mathbb{R}$, if it is a directed interval. Its norm is $\|\overrightarrow{A}\|_1 = \max\limits_{l=\pm 1} |a_1(l)|$.

(ii) in $\mathbb{R}^n$, $n \geq 2$, if there exist a continuous function $a_n : S_{n-1} \to \mathbb{R}$ and a uniformly bounded function $\overrightarrow{A_{n-1}} : S_{n-1} \to \mathcal{D}(\mathbb{R}^{n-1})$ with respect to $\|\cdot\|_{n-1}$.

Then we denote $\overrightarrow{A} = (\overrightarrow{A_{n-1}(l)}, a_n(l))_{l \in S_{n-1}}$ and define its norm as

$$\|\overrightarrow{A}\| := \|\overrightarrow{A}\|_n := \max\{\sup_{l \in S_{n-1}} \|\overrightarrow{A_{n-1}(l)}\|_{n-1}, \max_{l \in S_{n-1}} |a_n(l)|\}.$$

The set of all directed sets in $\mathbb{R}^n$ is denoted by $\mathcal{D}(\mathbb{R}^n)$.

The *linear operations* are defined recursively, on the two components of the directed sets $\overrightarrow{A} = (\overrightarrow{A_{n-1}(l)}, a_n(l))_{l\in S_{n-1}}$, $\overrightarrow{B} = (\overrightarrow{B_{n-1}(l)}, b_n(l))_{l\in S_{n-1}}$:

$$\boxed{\begin{aligned} \overrightarrow{A} + \overrightarrow{B} &:= (\overrightarrow{A_{n-1}(l)} + \overrightarrow{B_{n-1}(l)}, a_n(l) + b_n(l))_{l\in S_{n-1}}, \\ \lambda \cdot \overrightarrow{A} &:= (\lambda \cdot \overrightarrow{A_{n-1}(l)}, \lambda \cdot a_n(l))_{l\in S_{n-1}} \qquad (\lambda \in \mathbb{R}), \\ \overrightarrow{A} - \overrightarrow{B} &:= \overrightarrow{A} + (-\overrightarrow{B}) = (\overrightarrow{A_{n-1}(l)} - \overrightarrow{B_{n-1}(l)}, a_n(l) - b_n(l))_{l\in S_{n-1}}. \end{aligned}} \tag{3.1}$$

It is proved in [**1**] that $(\mathcal{D}(\mathbb{R}^n), +, \cdot)$ is a Banach space. The *embedding* $J_n : \mathcal{C}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ which determines for every set $A \in \mathcal{C}(\mathbb{R}^n)$ its embedded image $\overrightarrow{A} \in \mathcal{D}(\mathbb{R}^n)$ is defined as:

(i) For $n = 1$, $\overrightarrow{[a, b]} = J_1([a, b]) := (-a, b)$.

(ii) For $n \geq 2$, $\overrightarrow{A} = J_n(A) := \big(J_{n-1}(P_{n-1,l}(Y(l, A)), \delta^*(l, A)\big)_{l\in S_{n-1}}$, where $P_{n-1,l}(x) := \pi_{n-1,n}R_{n,l}(x - \delta^*(l, A)l)$ and $\pi_{n-1,n} \in \mathbb{R}^{(n-1)\times n}$ is the natural projection, $R_{n,l}$ is a fixed rotation for every $l \in S_{n-1}$ satisfying

$$R_{n,l}(l) = e^n, \qquad R_{n,l}(\text{span}\{l\}^{\perp}) = \text{span}\{e^1, e^2, \ldots, e^{n-1}\}. \tag{3.2}$$

We introduce a multiplication of a matrix $M \in \mathbb{R}^{m\times n}$ for a difference of two embedded convex compacts $\overrightarrow{A} = J_n(A)$, $\overrightarrow{B} = J_n(B)$ and $A, B \in \mathcal{C}(\mathbb{R}^n)$:

$$M \cdot (\overrightarrow{A} - \overrightarrow{B}) := J_m(M \cdot A) - J_m(M \cdot B).$$

The visualization for a directed set in $\mathcal{D}(\mathbb{R}^n)$ consists of three parts: the convex, the concave and the mixed type parts. We recall their definitions here.

Definition 3.4. Let $\overrightarrow{A} \in \mathcal{D}(\mathbb{R}^n)$. The *convex (positive) part* $P_n(\overrightarrow{A})$ and the *concave (negative) part* $N_n(\overrightarrow{A})$ of $\overrightarrow{A}$ are defined by:

$$\begin{aligned} P_n(\overrightarrow{A}) &:= \{x \in \mathbb{R}^n \,|\, \text{for every } l \in S_{n-1} : \ \langle l, x\rangle \leq a_n(l)\}, \\ N_n(\overrightarrow{A}) &:= \{x \in \mathbb{R}^n \,|\, \text{for every } l \in S_{n-1} : \ \langle l, x\rangle \geq a_n(l)\}. \end{aligned}$$

The *mixed-type part* $M_n(\overrightarrow{A})$ is defined recursively and collects all reprojected points from the visualization of the "support faces" which are not elements of the convex or concave part.

$$\begin{aligned} M_1(\overrightarrow{A}) &:= \emptyset, \qquad V_1(\overrightarrow{A}) := P_1(\overrightarrow{A}) \cup N_1(\overrightarrow{A}) && (n = 1), \\ M_n(\overrightarrow{A}) &:= \bigcup_{l\in S_{n-1}} \{x \in Q_{n,l}(V_{n-1}(\overrightarrow{A_{n-1}(l)})) \,|\, x \notin P_n(\overrightarrow{A}) \cup N_n(\overrightarrow{A})\} && (n \geq 2). \end{aligned}$$

The *visualization* $V_n : \mathcal{D}(\mathbb{R}^n) \rightrightarrows \mathbb{R}^n$ is the union of the three parts

$$V_n(\overrightarrow{A}) := P_n(\overrightarrow{A}) \cup N_n(\overrightarrow{A}) \cup M_n(\overrightarrow{A}) \qquad (n \geq 2) \tag{3.3}$$

with the reprojection $Q_{n,l}(y) := R_{n,l}^{-1}\pi_{n-1,n}^{\top}y + a_n(l)l$, $y \in \mathbb{R}^{n-1}$.
The *boundary* mapping $B_n : \mathcal{D}(\mathbb{R}^n) \rightrightarrows \mathbb{R}^n$ is defined as

$$B_n(\overrightarrow{A}) := \partial P_n(\overrightarrow{A}) \cup \partial N_n(\overrightarrow{A}) \cup M_n(\overrightarrow{A}).$$

Except the degenerate case when $V_n(\overrightarrow{A})$ consists of a single point, at least one of the convex and the concave part of A is empty. It may happen that both of them are empty and the visualization consists only of the mixed-type part. In the one-dimensional case the mixed-type part is empty, and in the non-degenerate case exactly one of $P_1(\overrightarrow{[a,b]})$ and $N_1(\overrightarrow{[a,b]})$ is empty.

The visualization of the difference of directed sets is strongly related to other differences. The following equalities are known for $\overrightarrow{A} \in \mathcal{D}(\mathbb{R}^n)$, cf. [**2**]:

$$P_n(-\overrightarrow{A}) = -N_n(\overrightarrow{A}), \quad N_n(-\overrightarrow{A}) = -P_n(\overrightarrow{A}), \quad V_n(-\overrightarrow{A}) = -V_n(\overrightarrow{A}).$$

Furthermore, if $A, B \in \mathcal{C}(\mathbb{R}^n)$, then

$$P_n(\overrightarrow{A} - \overrightarrow{B}) = A \overset{*}{-} B, \quad N_n(\overrightarrow{A} - \overrightarrow{B}) = -(B \overset{*}{-} A), \quad \overline{\mathrm{co}}\, B_n(\overrightarrow{A} - \overrightarrow{B}) = A \dot{-} B, \tag{3.4}$$

$$\bigcup_{l \in S_{n-1}} \big\{ y(l,A) - y(l,B) \,|\, Y(l,A),\, Y(l,B) \text{ are singletons} \big\} \subset B_n(\overrightarrow{A} - \overrightarrow{B}).$$

EXAMPLE 3.5. The visualization of the inverse in Fig. 1 is comprised of the pointwise negations of the boundary points, keeping the corresponding normal directions l. Note that if the set A is symmetric with respect the origin, then the

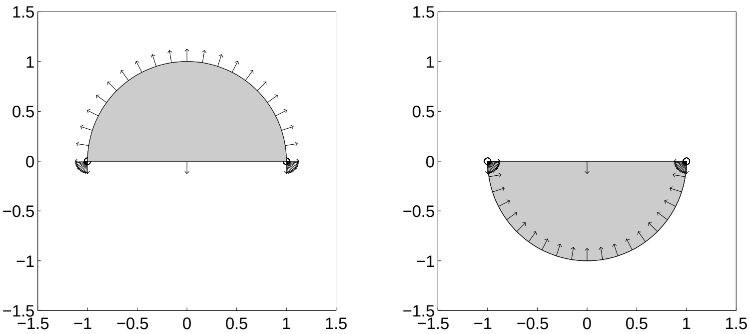


FIGURE 1. $\overrightarrow{A}$ resp. $-\overrightarrow{A}$ for $\overrightarrow{A} = J_n(A)$,
$A = \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0,\ x^2 + y^2 \leq 1\}$

visualizations of $\overrightarrow{A}$ and $-\overrightarrow{A}$ coincide, and the only way to distinguish between them is to add the normal vectors to the visualization.

The operations addition, scalar multiplication commute with the embedding and the visualization, cf. [**2**]. Namely, for $A, B \in \mathcal{C}(\mathbb{R}^n)$ and $\lambda \geq 0$:

$$\boxed{\begin{array}{r|c} J_n(A) + J_n(B) = J_n(A+B), & V_n(\overrightarrow{A} + \overrightarrow{B}) = A + B, \\ \lambda \cdot J_n(A) = J_n(\lambda \cdot A), & V_n(\lambda \cdot \overrightarrow{A}) = \lambda \cdot A. \end{array}} \tag{3.5}$$

4. The Directed Subdifferential

The following definition was suggested to us by A. M. Rubinov.

DEFINITION 4.1. Let $g, h : \mathbb{R}^n \to \mathbb{R}$ be convex and $f(x) = g(x) - h(x)$. The *directed subdifferential of f at x* is defined by

$$\overrightarrow{\partial} f(x) = J_n(\partial g(x)) - J_n(\partial h(x)). \tag{4.1}$$

We call its visualization the *Rubinov subdifferential*:

$$\partial_R f(x) = V_n\Big(\overrightarrow{\partial} f(x)\Big). \tag{4.2}$$

The vector $s \in \partial_R f(x)$ is called the *Rubinov subgradient of f at x*.

We note that the directed subdifferential is well-defined for DC functions, i.e., $\partial_R f(x)$ does not depend on the specific representation $f = g - h$, which may be easily checked, using the property (3.5) of the embedding J_n and the fact that $\mathcal{D}(\mathbb{R}^n)$ is a vector space in which the cancellation law holds.

Also, the Rubinov subdifferential is always a nonempty compact, not necessarily convex set.

The following properties of the directed subdifferential for DC functions are established (note the stronger versions of (SD_6) and (SD_7) being also valid for negative λ):

PROPOSITION 4.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a DC function and $x \in \mathbb{R}^n$. Then the directed subdifferential of f at x fulfills:*

(SD_1) *$x \notin dom(f)$ if and only if $\partial_R f(x)$ is empty.*

(SD_2) *If $\tilde{f}$ is a DC function that coincides with f in an open neighbourhood U of x, then $\overrightarrow{\partial} f(x) = \overrightarrow{\partial} \tilde{f}(x)$.*

(SD_3) *If f is convex, then $\partial_R f(x)$ coincides with the classical convex subdifferential $\partial f(x)$, i.e. $\overrightarrow{\partial} f(x) = J_n(\partial f(x))$.*

(SD_4) *If f has the local Lipschitz constant $L_f \geq 0$ for the open neighbourhood U of x, then*
$$\|s\|_2 \leq L_f \quad \text{for all } s \in \partial_R f(x).$$

(SD'_6) *Let $n = n_1 + n_2$ and $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. If $f_i : \mathbb{R}^{n_i} \to \mathbb{R}$ is a DC function, $i = 1, 2$, and $f : \mathbb{R}^n \to \mathbb{R}$ is defined by $f(x) = f_1(x^{(1)}) + f_2(x^{(2)})$, then*
$$\overrightarrow{\partial} f(x) = \Pi_1^\top \overrightarrow{\partial} f_1(x^{(1)}) + \Pi_2^\top \overrightarrow{\partial} f_2(x^{(2)}),$$
with the projection $\Pi_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ and $\Pi_i(x) = x^{(i)}$ for $x = (x^{(1)}, x^{(2)})$, $i = 1, 2$.

(SD'_7) *If $\lambda \in \mathbb{R}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}$ is DC and $f(x) = \lambda \cdot g(Ax + b)$, then*
$$\overrightarrow{\partial} f(x) = \lambda \cdot A^\top \cdot \left(\overrightarrow{\partial} g\right)(Ax + b).$$

Moreover, the sum rule holds for each DC function $\widetilde{f}$:

(SR) $\overrightarrow{\partial}(f + \widetilde{f})(x) = \overrightarrow{\partial} f(x) + \overrightarrow{\partial} \widetilde{f}(x)$.

PROOF. We prove here only (SD_3), part of (SD'_7) and (SR):
For (SD_3), $f = f - 0$ holds so that $\overrightarrow{\partial} f(x) = J_n(\partial f(x)) - J_n(0_{\mathbb{R}^n})$.

In (SD'_7) we consider only $f(x) = -\widetilde{f}(x)$ with $\widetilde{f}(x) = \widetilde{g}(x) - \widetilde{h}(x)$. Clearly, $f(x) = \widetilde{h}(x) - \widetilde{g}(x)$ is a DC representation and hence, $\overrightarrow{\partial} f(x) = J_n(\partial \widetilde{h}(x)) - J_n(\partial \widetilde{g}(x))$ which equals $-\overrightarrow{\partial} \widetilde{f}(x)$.

In (SR) with $f = g - h$, $\widetilde{f} = \widetilde{g} - \widetilde{h}$, it follows the DC representation $(g + \widetilde{g}) - (h + \widetilde{h})$ for the sum. From (SR) in Section 2 and (3.5) follows the sum rule for the directed subdifferential. □

We note that the sum rule (SR) can only be proved for the directed subdifferential and not for the Rubinov one, since the linear structure of the space $\mathcal{D}(\mathbb{R}^n)$ of directed sets is essential in the proof. For the subdifferential of Rubinov, the left-hand side is only a subset of the right-hand side, since the visualization of the sum of two directed sets is only a subset of the Minkowski sum of their visualizations, which follows easily from the definitions.

The following relations follow from (2.7), (2.9), (3.4), and Definitions 3.4 and 4.1, and clarify the position of the Rubinov subdifferential among other known subdifferentials:

THEOREM 4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a DC function and $x \in \mathbb{R}^n$. Then*

$$\partial_{DH} f(x) = P_n(\overrightarrow{\partial} f(x)) \subset V_n(\overrightarrow{\partial} f(x)) = \partial_R f(x) \subset \mathrm{co}(\partial_R f(x)) = \partial_{MP} f(x), \tag{4.3}$$

$$\partial_{MP} f(x) = co(\partial_R f(x)) = co(\partial_T f(x)) \subset co(\partial_M f(x)) = \partial_{Cl} f(x). \tag{4.4}$$

Directly from [**8**, Lemma 2.3] with the corresponding fact for the Michel-Penot subdifferential, it follows that the Rubinov subdifferential generalizes the Gâteaux differential.

COROLLARY 4.4. *Let $f = g - h$ be a DC function with Gâteaux differentiable functions g and h, then the Rubinov subdifferential consists only of the gradient of f.*

OPEN PROBLEM 4.5. The exact relation between the Rubinov subdifferential and the non-convex subdifferentials of Mordukhovich and Treiman is not yet clear to us. The following examples show that none of the subdifferentials of Rubinov and Mordukhovich is necessarily a subset of the other.

EXAMPLE 4.6 ([**12**, Ex. 9.1], [**25**, Sect. 1.2, Example 2.6], [**26**, Section 1.3.2], [**36**]). Let $f = g - h$ with $g(x) = |x_1|$, $h(x) = |x_2|$, $x = (x_1, x_2) \in \mathbb{R}^2$.

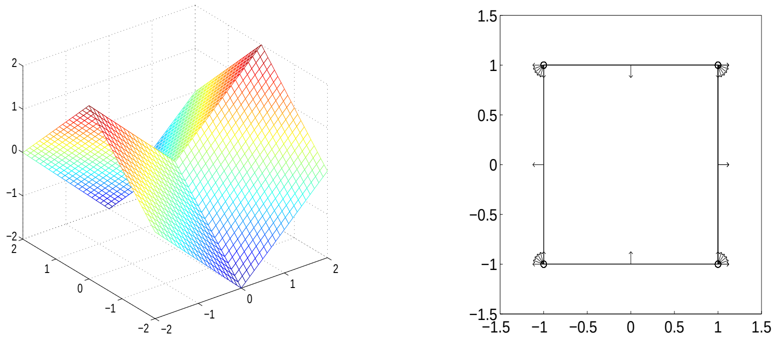


FIGURE 2. Function plot resp. directed subdifferential for Example 4.6

By (SD_6), the convex subdifferentials of g, h are

$$\partial g(0) = [-1, 1] \times \{0\}, \quad \partial h(0) = \{0\} \times [-1, 1].$$

For this function, the Dini-Hadamard subdifferential is empty, and the Michel-Penot subdifferential coincides with that of Clarke.

$$\partial_{DH} f(0) = \emptyset, \quad \partial_{MP} f(0) = \partial_{Cl} f(0) = [-1, 1]^2,$$
$$\overrightarrow{\partial} f(0) = J_2([-1, 1] \times \{0\}) - J_2(\{0\} \times [-1, 1]).$$

In this example, we have the following inclusions of the subdifferentials:

$$\partial_{DH} f(0) = \emptyset \subsetneq \partial_R f(0) \subsetneq \partial_{MP} f(0) = \partial_{Cl} f(0).$$

The Mordukhovich subdifferential of this function coincides with the Treiman one and consists of two parallel edges of the unit square:

$$\partial_M f(0) = \partial_T f(0) = [-1,1] \times \{-1,1\}.$$

The directed subdifferential is comprised of only a mixed-type part (its positive and negative part are empty), as one can see in Figure 2. For this example, the union of the Mordukhovich lower and upper subdifferential (cf. [**25**], [**26**, Definition 1.78]) in Fig. 3 gives the Rubinov subdifferential which consists of the boundary of the square $[-1,1]^2$.

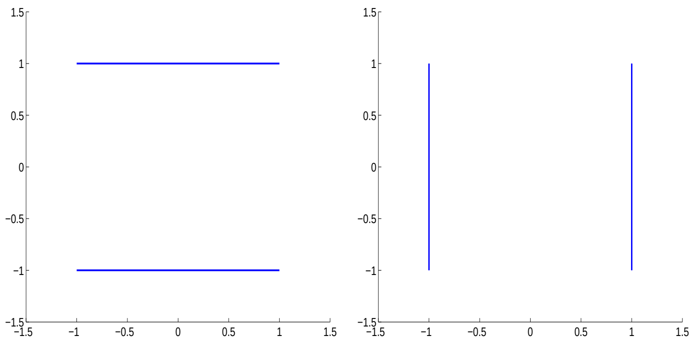


FIGURE 3. Mordukhovich lower and upper subdifferential for Example 4.6

The Mordukhovich subdifferential in the previous example is a subset of the Rubinov one, while in the next example the opposite inclusion holds.

EXAMPLE 4.7 ([**12**, Section III.4, Example 4.2]). Let $f = g - h$, $g, h : \mathbb{R}^2 \to \mathbb{R}$ with $g(x_1, x_2) = \max\{2x_2, x_1^2 + x_2\}$, $h(x_1, x_2) = \max\{0, x_1^2 + x_2\}$.

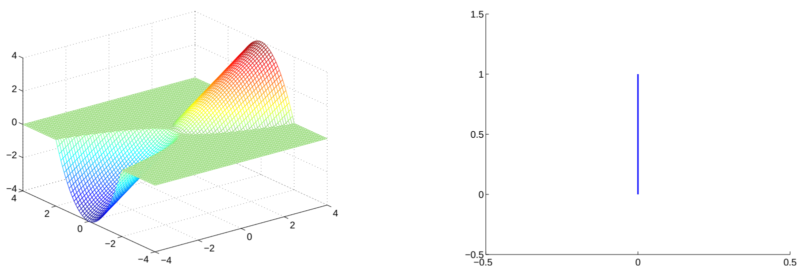


FIGURE 4. Function plot resp. Clarke's subdifferential for Example 4.7

$$\begin{aligned}
&\partial g(x) = \mathrm{co}\big\{(0,2),(2x_1,1)\big\}, \quad \partial h(x) = \mathrm{co}\big\{(0,0),(2x_1,1)\big\},\\
&\partial_{DH} f(0) = \partial_{MP} f(0) = \big\{(0,1)\big\},\\
&\partial_{Cl} f(0) = \mathrm{co}\big\{(0,0),(0,1)\big\},\\
&\overrightarrow{\partial} f(0) = J_2\big(\mathrm{co}\big\{(0,2),(0,1)\big\}\big) - J_2\big(\mathrm{co}\big\{(0,0),(0,1)\big\}\big) = J_2\big(\big\{(0,1)\big\}\big).
\end{aligned}$$

Here, we have the following inclusions of the subdifferentials:

$$\partial_{DH} f(0) = \partial_R f(0) = \partial_{MP} f(0) = \partial_T f(0) \subsetneq \partial_{Cl} f(0).$$

As it is known, the Mordukhovich subdifferential contains the extremal points $(0,0)$ and $(0,1)$ of Clarke subdifferential (cf. Fig. 4), thus

$$\partial_R f(0) = \partial_{MP} f(0) \subsetneq \partial_M f(0).$$

CONJECTURE 4.8. The following conjecture may be derived from the last examples: If the Clarke subdifferential coincides with Michel-Penot one, then the Rubinov subdifferential coincides with the Mordukhovich symmetric subdifferential (the union of the lower and the upper subdifferential). It is an open question whether a similar relation between Rubinov and the Treiman subdifferential holds.

5. Optimality Conditions, Descent and Ascent Directions

Here we translate some known necessary and sufficient conditions for an extremum into the language of directed subdifferentials. In the case when the Dini-Hadamard subdifferential is non-empty, these conditions are equivalent to those known for it. If it is empty (which means that the considered point is not a minimizer), we provide a simple criterion which says whether a given direction is a direction of ascent or descent.

Let $f : \mathbb{R}^n \to \mathbb{R}$, $l \in S_{n-1}$. We first recall some definitions. If there exists $\varepsilon > 0$ such that $f(x+tl) < f(x)$ for $t \in (0, \varepsilon]$, then l is called the *direction of descent at* x. In the case of $f(x+tl) > f(x)$ for $t \in (0, \varepsilon]$, l is called the *direction of ascent at* x. We call the point x a *strict saddle point*, if there exist a direction of ascent with a positive directional derivative and a direction of descent with a negative directional derivative at x.

Clearly, if all directions in S_{n-1} are directions of ascent at x, then x is a strict minimum, and similarly for a strict maximum.

Next, we state well-known necessary conditions for an extremum of a DC function in terms of the directed subdifferential. From Proposition 5.1(i), (SD_5) follows for the Rubinov subdifferential, since it includes the positive part by (3.3).

PROPOSITION 5.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a DC function and $x \in \mathbb{R}^n$. Then*

(i) *If x is a local minimizer of f, then $0_{\mathbb{R}^n} \in P_n(\overrightarrow{\partial} f(x)) = \partial_{DH} f(x)$.*

(ii) *If x is a local maximizer of f, then $0_{\mathbb{R}^n} \in N_n(\overrightarrow{\partial} f(x)) = -\partial_{DH}(-f)(x)$.*

(iii) *If x is a strict saddle point of f and $0_{\mathbb{R}^n} \in V_n(\overrightarrow{\partial} f(x))$, then $0_{\mathbb{R}^n} \in M_n(\overrightarrow{\partial} f(x))$.*

PROOF. (i) If $f = g - h$ has a local minimum in x, then necessarily $0_{\mathbb{R}^n} \in \partial g(x) \overset{*}{-} \partial h(x)$, cf. [**29, 11, 15**] and [**12**, Chapter V, Theorem 3.1]. But the geometric difference coincides with the positive part of the directed subdifferential $\overrightarrow{\partial} f(x) = J_n(\partial g(x)) - J_n(\partial h(x))$.

(ii) Similarly, $0_{\mathbb{R}^n} \in \partial h(x) \overset{*}{-} \partial g(x)$ is a necessary condition for a local maximum of f at x. The geometric difference could be retranslated as negative part of the visualization of $\overrightarrow{\partial} f(x)$.

(iii) Since there exist $l^1, l^2 \in S_{n-1}$ with $f'(x; l^1) < 0$ and $f'(x; l^2) > 0$, the origin could not lie in the positive or negative part of the visualization. Hence, it must lie in the mixed-type part of the directed subdifferential. □

COROLLARY 5.2.

(i) *If* x *is a local extremum point of* f, *then* $0_{\mathbb{R}^n} \in \partial_R f(x)$.

(ii) *If* $P_n(\overrightarrow{\partial} f(x)) = N_n(\overrightarrow{\partial} f(x)) = \emptyset$, *then* x *is not an extremum.*

(iii) *If* $0 \in \partial_{MP} f(x) \setminus \Big(P_n(\overrightarrow{\partial} f(x)) \cup N_n(\overrightarrow{\partial} f(x)) \Big)$, *then* x *is not an extremum.*

(iv) *If* $0 \in M_n(\overrightarrow{\partial} f(x)) \subseteq \partial_R f(x)$, *then* x *is not an extremum.*

It is known that if $f : \mathbb{R}^n \to \mathbb{R}$ is directional differentiable for $l \in S_{n-1}$, then

- if $f'(x; l) < 0$, then l is a direction of descent,
- if $f'(x; l) > 0$, then l is a direction of ascent.

The following simple criterion helps to distinguish between directions of ascent and descent by the sign of the second component of the directed subdifferential (its "support" function).

PROPOSITION 5.3. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a DC function,* $x \in \mathbb{R}^n$ *and* $\overrightarrow{\partial} f(x) = (\overrightarrow{A_{n-1}(l)}, a_n(l))_{l \in S_{n-1}}$.

(i) *If* $a_n(l) < 0$, *then* l *is a direction of descent at* x.

(ii) *If* $a_n(l) > 0$, *then* l *is a direction of ascent at* x.

PROOF. For $f = g - h$ we have, due to the embedding and (3.1), that

$$a_n(l) = \delta^*(l, \partial g(x)) - \delta^*(l, \partial h(x)) = g'(x; l) - h'(x; l) = f'(x; l).$$

Clearly, the sign of the directional derivative determines whether l is a direction of ascent or descent. □

On the visualization, this may be interpreted as follows: if for a given l there is a boundary point z of the directed subdifferential with corresponding normal direction l such that $\langle l, z \rangle < 0$, then l is a descent direction, and similarly for an ascent direction.

Next, we formulate the classical sufficient first-order condition for a minimum (see, e.g., [**9, 14**] and [**12**, Chapter V, Theorem 3.1]) for the directed subdifferential.

PROPOSITION 5.4. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a DC function with* $f = g - h$ *and* $x \in \mathbb{R}^n$.

(i) *If* $0_{\mathbb{R}^n} \in \operatorname{int} P_n(\overrightarrow{\partial} f(x))$, *then* x *is a strict local minimizer of* f.

(ii) *If* $0_{\mathbb{R}^n} \in \operatorname{int} N_n(\overrightarrow{\partial} f(x))$, *then* x *is a strict local maximizer of* f.

(iii) *If* $0_{\mathbb{R}^n} \in \operatorname{int} \partial_R f(x)$, *then* x *is a strict local extremum point of* f.

PROOF. The proof is similar to the one of Proposition 5.1 and uses the sufficient condition for optimality "$0_{\mathbb{R}^n} \in \operatorname{int} \big(\partial g(x) \overset{*}{-} \partial h(x)\big)$" mentioned above. □

For a further reference to ascent and descent directions, see, e.g., [**12**, Section V.1].

Let us now go back to the examples discussed in the last section.

EXAMPLE 5.5. Consider the function f from Example 4.6. As mentioned before, the Dini-Hadamard subdifferential at $x = 0$ is empty, thus the origin is not a minimum point (also the Dini-Hadamard subdifferential of $-f(x)$ is empty, so it is not a maximum). Clearly, the zero is outside the non-convex subdifferentials of Mordukhovich, Rubinov and Treiman, but is inside the subdifferentials of Clarke and Michel-Penot. A closer look at the visualization of the directed subdifferential

(Fig. 2) enables to conclude that the directions $(1,0),(-1,0)$ are ascent directions, and $(0,1),(0,-1)$ are descent directions. Checking the values of $f'(x;l)$ in these directions we conclude that the origin is a strict saddle point.

EXAMPLE 5.6. We consider the function $f(x)$ from Example 4.7. Here, zero is outside the Dini-Hadamard, Rubinov, Michel-Penot and Treiman subdifferentials, but belongs to the subdifferentials of Mordukhovich and Clarke. Thus the latter two subdifferentials deliver critical points for the origin being no extremum.

In the next example, the criterion in Proposition 5.3 together with Propositions 5.1 and 5.4 are used to test the optimality conditions.

EXAMPLE 5.7. Let $f = g - h$ with $g(x) = |x_1| + |x_2|$, $h(x) = r \cdot \sqrt{x_1^2 + x_2^2}$ for $x = (x_1, x_2) \in \mathbb{R}^2$. The function and the directed subdifferential are plotted in Fig. 5–8 for $r \in \{0.5, 1.0, 1.25, 1.5\}$.

$r = 0.5$: One can see in Figure 5 that the origin is a global minimum. For this

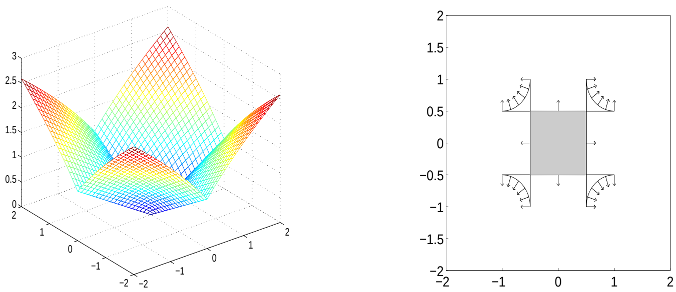


FIGURE 5. Function plot resp. directed subdifferential for Example 5.7 with $r = 0.5$

function, the sufficient condition for a minimum in Proposition 5.4 is fulfilled:

$$0_{\mathbb{R}^n} \in \operatorname{int} P_n(\overrightarrow{\partial} f(0)) = \operatorname{int} \partial_{DH} f(0) = \operatorname{int} \partial_R f(0) \subset \operatorname{int} \partial_{MP} f(0) \subset \operatorname{int} \partial_{Cl} f(0),$$

and all directions are directions of ascent (all the normals are pointing from the corresponding boundary points away from the origin).

$r = 1$: All points on the x- and y-axis are global minima, cf. Figure 6.

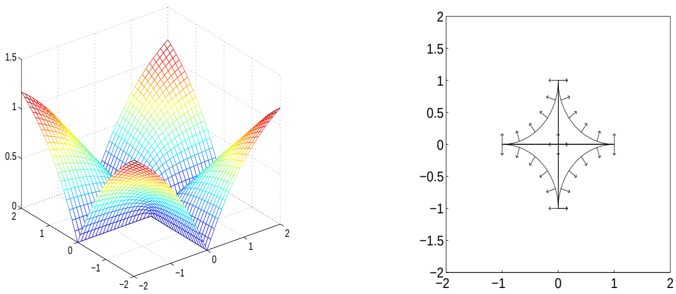


FIGURE 6. Function plot resp. directed subdifferential for Example 5.7 with $r = 1$

Here, the necessary condition for the origin being a local minimum in Proposition 5.1 is satisfied:

$$\partial_{DH} f(0) = \{0_{\mathbb{R}^n}\}, \quad 0_{\mathbb{R}^n} \in \partial_R f(0), \quad 0_{\mathbb{R}^n} \in \operatorname{int} \partial_{MP} f(0) \subset \operatorname{int} \partial_{Cl} f(0),$$

and all directions $l \in S_{n-1}$ except for $(\pm 1, 0), (0, \pm 1)$ fulfill $f'(x; l) > 0$.
$r = 1.25$: One can see in Figure 7 that the origin is a saddle point, although it is a critical point for the Michel-Penot and Clarke subdifferential.

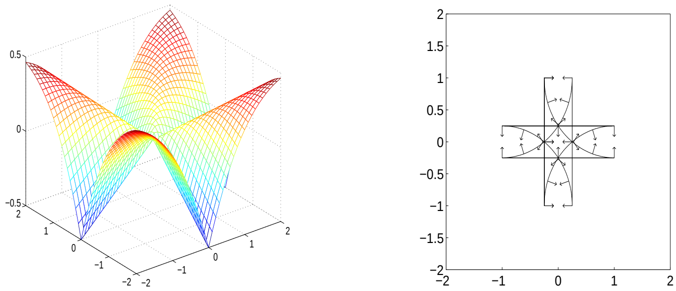


FIGURE 7. Function plot resp. directed subdifferential for Example 5.7 with $r = 1.25$

For this function,

$$\partial_{DH} f(0) = \emptyset, \quad 0_{\mathbb{R}^n} \notin \partial_R f(0), \quad 0_{\mathbb{R}^n} \in \operatorname{int} \partial_{MP} f(0) \subset \operatorname{int} \partial_{Cl} f(0).$$

The directions $(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$ are directions of ascent, while $(\pm 1, 0), (0, \pm 1)$ are directions of descent.
$r = 2$: One can see in Figure 8 that the origin is a global maximum (sufficient condition for a maximum in Proposition 5.4 is satisfied).

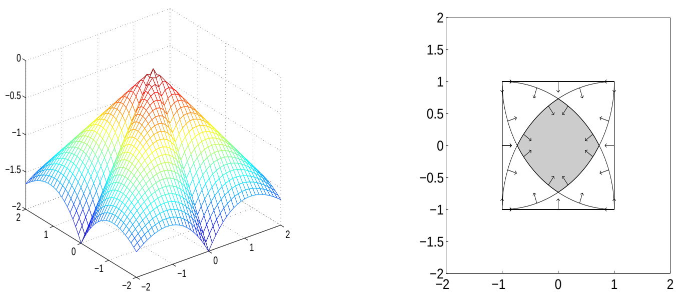


FIGURE 8. Function plot resp. directed subdifferential for Example 5.7 with $r = 2$

For this function, the Dini-Hadamard subdifferential is empty,

$$0_{\mathbb{R}^n} \in \operatorname{int} N_n(\overrightarrow{\partial} f(0)) = -\operatorname{int} \partial_{DH}(-f)(0) = \operatorname{int} \partial_R f(0) \subset \operatorname{int} \partial_{MP} f(0) \subset \operatorname{int} \partial_{Cl} f(0),$$

and all directions are directions of descent (all the normals are pointing from the corresponding boundary points towards the origin).

6. Conclusions

The directed subdifferential is a directed (oriented) set, and not a usual set in $\mathbb{R}^n$. Its non-oriented visualization, the Rubinov subdifferential, is a compact connected (generally non-convex) subset of $\mathbb{R}^n$ with its boundary parametrized by the (normal) directions of the unit sphere in $\mathbb{R}^n$: to each (unit) direction l there is a unique connected "support face" lying in a hyperplane with normal l.

As we saw in the previous section, the information of the orientation of the normals (the sign of the "support function") enables to identify directions of descent or ascent. In future research we intend to extend the calculus rules for the new subdifferential, and to introduce and study directed/Rubinov normal and tangent cones. We hope that the investigation of the relations between the directed/Rubinov subdifferential and the subdifferentials of Mordukhovich and Treiman may indicate whether it is possible to extend the definition of the Rubinov subdifferential beyond the case of DC functions. Furthermore, the property (SD_8) should be proved.

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